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Foliations by complex manifolds involving the complex Hessian

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CONTENTS

Summary

In 1979 the second named author proved, in a joint paper with J. Ławrynowicz, the existence of a foliation of a bounded domain in $\mathbb{C}^n$ by complex submanifolds of codimension $k+p-1$, connected in some sense with a real $(1,1)$ C^3-form of rank k and the pth power of the complex Hessian of a C^3-function u with $\operatorname{im} u$ plurisubharmonic and the property that for every leaf of this foliation the restricted functions $\operatorname{im} u$, $\operatorname{re} u$ and $(\partial/\partial z_j)\operatorname{im} u$, $(\partial/\partial z_j)\operatorname{re} u$ are pluriharmonic and holomorphic, respectively.

Now the theorem is extended in two directions: to holomorphically decomposable (k,k)-forms, $k<n$, of class C^3, and to exterior products of the complex Hessians of p plurisubharmonic C^3-functions. This vast generalization gives rise to considerable extensions of the existence theorem of J. Ławrynowicz and M. Okada on a natural Markov process associated with the foliation as well as to the study of some of its properties. The main result is that the diffusion X_t^θ uniquely determined by a foliation has the property that the sample paths of X_t^θ remain to diffuse on leaves. Next, the convex case is examined and some examples depending on special and arbitrary holomorphic functions are presented. Since the foliations and canonical diffusions can be constructed in those cases effectively, we arrive at some properties of holomorphic functions on hypersurfaces, eliminating the inconvenient notions of foliations and canonical diffusions.

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1. Introduction and an outline of results

The paper is based on suitable extensions of a result on foliations by complex manifolds, mentioned in the Summary and proved in [20, 21].

A real (k,k)-form Ω on a complex manifold M is called *holomorphically decomposable* if for every $x \in M$ there is an $\mathbb{R}$-linearly independent system of forms a_j and Ja_j, $a_j \in T_x^*M$, $j = 1, \ldots, k$, and J being the complex structure on M, with the property

$$\bigwedge_{j=1}^{k} a_j \wedge Ja_j = \pm\Omega(x). \tag{1}$$

In Section 4, after some preliminaries on the generalized complex Monge–Ampère equations (Section 2) and foliations (Section 3), the following result will be proved:

THEOREM 1. *Suppose that D is an arbitrary domain in $\mathbb{C}^n$ and $u : D \to \mathbb{C}$ is a function of class $C^3(D)$ with $\operatorname{im} u$ plurisubharmonic in D. Let further Ω be a holomorphically decomposable (k,k)-form, $k \le n$, of class $C^3(D)$ such that, at every point of D, the following conditions hold*:

$$\Omega \wedge (dd^c u)^p = 0, \qquad 2 \le k + p \le n, \tag{2}$$

$$\Omega \wedge (dd^c u)^{p-1} \neq 0, \quad d\Omega \in \operatorname{ideal}(\Omega, dd^c \operatorname{re} u, dd^c \operatorname{im} u) \quad \textit{if } k = 1, \tag{3}$$

$$d\Omega = 0 \quad \textit{if } k \neq 1. \tag{4}$$

Then there exists a foliation $\mathcal{L}_{k+p-1}$ of D by complex submanifolds of D of codimension $k+p-1$ with the property that for every leaf $M \in \mathcal{L}_{k+p-1}$ the functions $\operatorname{im} u|M$ and $\operatorname{re} u|M$ are pluriharmonic, but $\partial(\operatorname{im} u)/\partial z_j|M$ and $\partial(\operatorname{re} u)/\partial z_j|M$ are holomorphic on M for each $j = 1, \ldots, n$.

Remark 1. Let M be a C^∞ manifold, $\bigwedge T^*M$ the Grassmann bundle over M, I an ideal in the sheaf of sections of $\bigwedge T^*M$, and I_x the ideal in the fibre $\bigwedge T_x^*M$ of $\bigwedge T^*M$ over $x \in M$, defined by I. Let further $O(x)$ be the subspace of T_xM consisting of all vectors $X_x \in T_xM$ such that the inner product $i(X_x)$ in the exterior algebra $\bigwedge T_x^*M$, applied to I_x, is included in I_x. It is known that O is completely integrable. Assume that I is generated by a set of differential forms $\{\omega_1, \ldots, \omega_k\}$ of the same degree. In this case

$$O(x) = \{X_x \in T_xM : X_x \rfloor \omega_s = 0 \text{ for } s = 1, \ldots, k\}.$$

If the forms ω_s have different degrees, then, in general,

$$\{X_x \in T_x M : x, X_x \rfloor \omega_s = 0 \text{ for } s = 1, \ldots, k\} \not\subseteq O(x).$$

Throughout the paper we assume that

$$\operatorname{Ann} I = O(x) := \{X_x \in T_x M : X_x \rfloor \omega_s = 0 \text{ for } s = 1, \ldots, k\}.$$

If the forms have different degrees, we assume, in general, that the ω_s are closed, which implies that O is completely integrable provided that it has a constant dimension. In fact, let X and Y belong to O, i.e., let

$$X \rfloor \omega_s = Y \rfloor \omega_s = 0 \quad \text{for } s = 1, \ldots, k.$$

We have to show that

$$[X, Y] \rfloor \omega_s = 0 \quad \text{for } s = 1, \ldots, k.$$

Let $\mathcal{L}_X$ be the Lie derivative with respect to the vector field X. Since

$$[X, Y] \rfloor \omega_s = [\mathcal{L}_X, (Y \rfloor \cdot)] \omega_s = -Y \rfloor \mathcal{L}_X \omega_s, \quad \mathcal{L}_X = (X \rfloor \cdot) \circ d + d \circ (X \rfloor \cdot),$$

we see that $\mathcal{L}_X \omega_s = 0$ and, consequently, we arrive at the desired relation which proves the complete integrability of O.

In connection with Theorem 1, or rather with its particular case mentioned in the Summary, it is natural to ask whether the hypotheses: ω of type $(1,1)$ and C^3, u plurisubharmonic, $\omega^k \wedge (dd^c u)^p = 0$, and $d\omega \in \operatorname{ideal}(\omega, dd^c u)$ imply

$$d(\omega^k \wedge (dd^c u)^{p-1}) \in \operatorname{ideal}(\omega^k \wedge (dd^c u)^{p-1}). \tag{5}$$

It is clear that (5) is satisfied if

$$d\omega \in \operatorname{ideal}(\omega). \tag{6}$$

Thus a counter-example has to be looked for among the situations where (6) fails to hold. Let $p = k = 1$ and $n = 2$. The function $f(y_1, y_2) = \log(y_1/y_2)$ satisfies $df \wedge d\overline{f} \wedge \partial\overline{\partial} f = 0$ and is plurisubharmonic in $\{z \in \mathbb{C}^2 : y_1/y_2 > 0\}$. In order to prove that in our case, for $\omega = \partial f \wedge \overline{\partial} f$, the condition (6) fails to hold, we consider the distribution

$$\operatorname{Ann} \omega = \operatorname{span}\{X, Y\},$$

where

$$X = (\partial/\partial x_1) + [\log(y_1/y_2) + 1](y_2/y_1)(\partial/\partial y_1),$$
$$Y = (\partial/\partial x_2) + [\log(y_1/y_2) + 1](y_2/y_1)(\partial/\partial y_2).$$

Since $X, Y \in \operatorname{Ann} \omega$ and $[X, Y] \notin \operatorname{Ann} \omega$, the distribution is not integrable, and this suffices to conclude the proof.

In Section 5, we modify Theorem 1 so that $(dd^c u)^p$ in (2) can be replaced by

$$(dd^c u)^p \begin{bmatrix} j_1, \ldots, j_p \\ u_1, \ldots, u_p \end{bmatrix} := (dd^c u_1)^{j_1} \wedge \ldots \wedge (dd^c u_p)^{j_p}, \quad j_1 + \ldots + j_p = p, \tag{7}$$

with $0 \le j_k \le p$, $k = 1, \ldots, p$, $2 \le p \le n$, and $(dd^c u)^{p-1}$ in (3) by

$$(dd^c)_r^{p-1}[u_1, \ldots, u_p] := dd^c u_1 \wedge \ldots \wedge dd^c u_{r-1} \wedge dd^c u_{r+1} \wedge \ldots \wedge dd^c u_p. \tag{8}$$

Unfortunately, we have to suppose that $u_1, \ldots, u_p$ are plurisubharmonic themselves. The modified Theorem 1 reads as follows:

THEOREM 2. *Suppose that D is an arbitrary domain in $\mathbb{C}^n$ and $u_j : D \to \mathbb{R}$, $j = 1, \ldots, p$, $2 \le p \le n$, are plurisubharmonic functions of class $C^3(D)$ satisfying in D, for every sequence $(j_1, \ldots, j_p)$ of integers, $0 \le j_k \le p$, $j_1 + \ldots + j_p = p$, the conditions*

$$(dd^c)^p \begin{bmatrix} j_1, \ldots, j_p \\ u_1, \ldots, u_p \end{bmatrix}(z) = 0, \qquad (dd^c)^{p-1}_{r(z)}[u_1, \ldots, u_p](z) \neq 0 \tag{9}$$

with the notation (7) *and* (8), *where the integer $r(z)$ may, in general, depend on $z \in D$. Then there exists a foliation $\mathcal{L}_{p-1}$ of D by complex submanifolds of D of codimension $p - 1$ with the property that for every leaf $M \in \mathcal{L}_{p-1}$ the functions $u_k|M$ are pluriharmonic, but $(\partial/\partial z_j)u_k|M$ are holomorphic on M for each $j = 1, \ldots, n$ and $k = 1, \ldots, p$.*

By combining Theorem 1 with Theorem 2 we obtain a corollary which, because of its importance, will be called Theorem 3. Generally speaking, we consider D and u_j, $j = 1, \ldots, p$, as in Theorem 2, but we multiply (9) by an arbitrary holomorphically decomposable (k,k)-form Ω, $k < n$, of class $C^3(D)$, restricted only by the generalized conditions (9). If

$$u_1 = \ldots = u_p \equiv u, \tag{10}$$

then we adopt the hypotheses of Theorem 1. Our result reads as follows:

THEOREM 3. *Suppose that D is an arbitrary domain in $\mathbb{C}^n$ and*

$$u_j : D \to \mathbb{C}, \qquad j = 1, \ldots, p, \;\; 2 \le p \le n, \tag{11}$$

are functions of class $C^3(D)$ with $\operatorname{im} u$ plurisubharmonic in D. Apart from the case (10) *we assume that $u_j(D) \subset \mathbb{R}$ and u_j, $j = 1, \ldots, p$, are themselves plurisubharmonic functions of class $C^3(D)$. Let further Ω be a holomorphically decomposable (k,k)-form, $k < n$, of class $C^3(D)$, satisfying in D, for every sequence $(j_1, \ldots, j_p)$ of integers, $0 \le j_l \le p$, $l = 1, \ldots, p$, $j_1 + \ldots + j_p = p$, the conditions*

$$\Omega \wedge (dd^c)^p \begin{bmatrix} j_1, \ldots, j_p \\ u_1, \ldots, u_p \end{bmatrix}(z) = 0, \qquad 2 \le k + p \le n, \tag{12}$$

$$\Omega \wedge (dd^c)^{p-1}_{r(z)}[u_1, \ldots, u_p](z) \neq 0, \tag{13}$$

$$\begin{aligned} &d\Omega \in \operatorname{ideal}(\Omega, dd^c \operatorname{re} u_1, dd^c \operatorname{im} u_1, \ldots, dd^c \operatorname{re} u_p, dd^c \operatorname{im} u_p) \qquad \text{if } k = 1, \\ &d\Omega = 0 \qquad \text{if } k \neq 1 \end{aligned} \tag{14}$$

with the notation (7) *and* (8), *where the integer $r(z)$ may, in general, depend on $z \in D$. Then there exists a foliation $\mathcal{L}_{k+p-1}$ of D by complex submanifolds of D of codimension $k + p - 1$ with the property that for every leaf $M \in \mathcal{L}_{k+p-1}$ the functions $u_l|M$ are pluriharmonic, but $(\partial/\partial z_j)u_l|M$ are holomorphic on M for each $j = 1, \ldots, n$ and $l = 1, \ldots, p$.*

In Section 6, Theorems 1–3 are used to prove the existence of natural Markov processes associated with foliations. They are of the form of special integrals which, within the probabilistic potential theory, are interpreted as canonical diffusions: in Theorem 4 we state not only their existence, but also uniqueness. Section 7 is devoted to the properties of the diffusions; the main result, formulated in Theorem 5, is that sample paths of the diffusion stay in each leaf of the foliation. After some preliminaries on the Laplace–Beltrami operator on Riemannian manifolds (Section 8) and an account of the harmonic theory on compact manifolds (Section 9), the Laplace–Beltrami operator of the foliation is defined and investigated in Section 10. Under suitable hypotheses, in Theorem 6 the existence of a canonical diffusion is proved, whose generator is the Laplace–Beltrami operator on each leaf and the sample paths of the diffusion remain to diffuse on each leaf associated with the admissible positive current in question. Theorem 6 is then used to construct the symmetric conformal diffusion. Sections 6–10 give a considerable generalization of the results obtained in [28].

In Section 11 it is remarked that the concept of the Laplace–Beltrami operator can naturally be extended to a more general context of Clifford analysis [23, 7], for instance in the cases of the sphere and hyperboloid.

Next, Theorem 3 is applied in Section 12 to study complex Hessians involving convex functions. The main result is formulated as Theorem 7 which is also illustrated in Section 13 by two examples. The (final) Section 14 is devoted to the investigation of a particular case depending on two holomorphic functions in $D \subset \mathbb{C}^3$. Since the foliations and canonical diffusions can be constructed in those cases effectively, we arrive at some properties of holomorphic functions on hypersurfaces, eliminating the inconvenient notions of foliations and canonical diffusions. The most impressive case, considered in Section 14, gives rise to the formulation of Theorem 8.

The present research can be considered, in some sense, a continuation of the papers [24, 25, 12, 8].

2. Capacities on hermitian manifolds and the generalized complex Monge–Ampère equations

Let M be a complex manifold of complex dimension n endowed with an hermitian metric h and a C^1 tensor field H of type $(1,1)$. In particular, we may let H depend on h or take as H an almost complex structure of the tangent bundle TM, for instance the complex structure of M. Let further D be a *condenser* on M, i.e. a domain whose complement consists of two distinguished disjoint closed sets C_0 and C_1 (the *condenser plates*), $q : M \to C$ a continuous mapping (the *inhomogeneity function*), and p a real number ≥ 1. Consider the class $\operatorname{adm} D$ of all plurisubharmonic C^2-functions u on $\operatorname{cl} D$ satisfying the conditions $0 < u(z) < 1$

for $z \in D$, $u|\partial C_0 = 0$ and $u|\partial C_1 = 1$. Let (see [26])

$$\text{(15)}\quad \operatorname{Cap}_p(D,q) = \inf_{\tilde{u}\in \operatorname{adm} D} \Big| \int_D q[h(d^c\widetilde{u}, d^c\widetilde{u})]^{p/2-1} \det H d\widetilde{u} \wedge d^c\widetilde{u} \wedge (dd^c\widetilde{u})^{n-1}\Big|,$$

where $h(d^cu, d^cu) = h^{j\overline{k}} u_{|j} u_{|\overline{k}}$, $u_{|j} = (u \circ \mu^{-1})_{|j} \circ \mu$ in any local coordinate system $\mu = (\mu^j)$ on M. Let further Γ be a homology class of D with real coefficients and $\dim \Gamma = r$. Consider all currents of Γ (more precisely: corresponding to the elements of Γ) in the sense of de Rham, and a locally finite open covering $U = \{U_j : j \in I\}$ of M. Denote by $\operatorname{adm}(D,U)$ the family of all plurisubharmonic C^2-functions u_j on $U_j \in D$ defined in each member of the covering which satisfy the following conditions:

(i) the oscillation of u_j in $U_j \in D$ is less than one,
(ii) $du_j = du_k$ in $U_j \cap U_k \cap D \neq \emptyset$.

Condition (ii) describes a closed real one-form in D. Similarly D^cu_j and dd^cu_j are also well defined in D. Without ambiguity, we can write them omitting the indices. Let (see [23])

$$\text{(16)}\qquad \operatorname{Cap}_p(D,q,\Gamma,U) = \sup_{u\in \operatorname{adm}(D,U)} \inf_{T\in\Gamma} |T[q\{h(d^cu, d^cu)\}^{p/2-1} \det H D^r u]|,$$

where

$$D^r u = \begin{cases} d^cu \wedge (dd^cu)^{(r-1)/2} & \text{for } r \text{ odd},\\ du \wedge d^cu \wedge (dd^cu)^{r/2-1} & \text{for } r \text{ even}. \end{cases}$$

For a detailed description of the capacities (15) and (16) we refer to [25], and for an example of their application to [27].

When looking for a complex analogue of the principle of Dirichlet (cf. e.g. [26]) a natural procedure is to take in (16) for Γ the $(2n-1)$-dimensional homology class of level hypersurfaces $\{z \in \operatorname{cl} D : u(z) = \text{const}\}$. (In analogy, for a complex counterpart of the principle of Thomson (cf. e.g. [26]) we had to take in (16) for Γ the orthogonal 1-dimensional homology class.) One should expect that under some reasonable conditions, in particular if we take in (16) for admissible functions only u defined globally with $0 < u(z) < 1$ for $z \in D$ (we write $u \in \operatorname{adm} D$), both capacities (16) and (15) will coincide.

The above idea, as well as both definitions in the case where $H = J$ (the complex structure of M), $p = 2$, and $q = \text{const}$, is due to Chern, Levine and Nirenberg [6], but the affirmative answer is known only in very special subcases [6, 19, 4]. In the case mentioned the functional minimized attains its minimum for $\widetilde{u} = u$ if and only if u satisfies the complex Monge–Ampère equation

$$(dd^cu)^n \equiv 4^n n! \det[u_{|j\bar{k}}] dV = 0$$

involving the *complex Hessian* $[u_{|j\bar{k}}] = [(\partial^2/\partial z_j \partial \overline{z}_k)u]$, where dV is the volume element. This equation is a special case of the *generalized complex Monge–Ampère equations*

$$dd^c(Fu) \wedge (dd^c u)^{n-1} = 0, \qquad F \in C^2(\mathrm{cl}\, D),$$

or

$$d(Gd^c u) \wedge (dd^c u)^{n-1} = 0, \qquad G \in C^1(\mathrm{cl}\, D), \tag{17}$$

which play an analogous role for the general capacity (15) with

$$d^c(Fu) = Gd^c u, \qquad G = q[h(d^c u, d^c u)]^{p/2-1} \det H.$$

In the general case the function u is replaced by a system satisfying the condition (ii). We quote the following two propositions due to Andreotti and Ławrynowicz [1, 2]:

PROPOSITION 1. *Suppose that*:

(α) *D has a piecewise C^1-smooth boundary and compact closure*,
(β) *$n \geq 2$, $p = 2$, and q is of class C^1*,
(γ) *u belongs to* adm$D$ *and satisfies* (17), *where $G = q \det H$*,
(δ) *$d(Gd^c u) = f dd^c u$, $f \geq (n-1)^{-1} G$, f being of class C^1.*

Then the infimum in (15) *is attained for the u in question.*

Remark 2. Proposition 1 remains valid for $n = 1$. In this case the condition (δ) is superfluous.

PROPOSITION 2. *Suppose that* (α) *and* (β) *hold and that*:

(ε) *the infimum in* (15) *is attained for some u*,
(η) *$d(Gd^c u) = f dd^c u$, $G = q \det H$, f being continuous.*

Then u satisfies (17).

Remark 3. Proposition 2 remains valid for $n = 1$. In this case the condition (η) is superfluous.

3. Foliations

We give some preliminaries on foliations. Here we refer to [5] and [29].

By a p-dimensional *C^r-foliation* of an m-dimensional C^r-differentiable manifold M we mean a decomposition of M into a union of disjoint connected subsets $\{L_j : j \in I\}$ (I always uncountable) called the *leaves* of the foliation, with the following property: Every point of M has a neighbourhood U and a system of local C^r-differentiable coordinates $x = (x^1, \ldots, x^m) : U \to \mathbb{R}^m$ such that for each leaf L_j the components of $U \cap L_j$ are described by the equations $x^{p+1} = \mathrm{const}, \ldots, x^m = \mathrm{const}$.

Foliations arise naturally in various situations in mathematics and it is instructive to give some examples.

EXAMPLE 1. *Submersions*. Let M and N be C^r-differentiable manifolds of dimension m and n, $m \geq n$, respectively, and let $f : M \to N$ be a submersion, that is, suppose that $\operatorname{rank}(df) \equiv n$. It follows from the Implicit Function Theorem that f induces on M a C^r-foliation of codimension n whose leaves are defined to be the components of $f^{-1}(\{y\})$ for $y \in N$. Also differentiable fibre bundles are examples of this sort.

EXAMPLE 2. *Subbundles of the tangent bundle of a C^r-differentiable manifold M.* We say that a smooth subbundle $E \subset TM$ is *integrable* if and only if for any two smooth sections X and Y of E the section $[X, Y]$ is also a section of E. By the Frobenius theorem the set of all maximum *integrals* E_0 of E (i.e. submanifolds of M such that $T_x E_0$ is contained in the fibre over x of the subbundle E for every $x \in E_0$) forms a foliation of M.

A foliation also appears as the family of solutions for some nonsingular system of differential equations [30] (cf. Example 2). The study of foliations is relevant to the global behaviour of solutions. For instance, a nonsingular system of ordinary differential equations, when reduced to a first order system, becomes a nonvanishing vector field. The local solutions (orbits of the local flow generated by the vector field) form a 1-dimensional foliation.

One can consider analogously ordinary differential equations in the complex case (where dependence on the variables is holomorphic). One obtains nonsingular holomorphic vector fields and the corresponding foliations by complex curves.

Let now M be a $2n$-dimensional C^∞-differentiable manifold and let TM be its tangent bundle. Let J denote an almost complex structure on M. The spaces $TM^{1,0}$ and $TM^{0,1}$ may be defined by the splitting $TM \otimes_{\mathbb{R}} \mathbb{C} \simeq TM^{1,0} + TM^{0,1}$, where $\alpha = \frac{1}{2}(\alpha - iJ\alpha) + \frac{1}{2}(\alpha + iJ\alpha)$. Hence also $T^*M \otimes_{\mathbb{R}} \mathbb{C} \simeq T^*M^{1,0} + T^*M^{0,1}$. Under this splitting, $d = \partial + \overline{\partial}$. We shall use the notation $\partial_j = \partial/\partial z_j$ and $\overline{\partial}_j = \partial/\partial \overline{z}_j$. We denote by

$$A^{p,q}M = \Lambda^p(T^*M^{1,0}) \wedge \Lambda^q(T^*M^{0,1})$$

the space of forms of type (p, q) on M, and the space of k-forms on M is given by

$$A^k M = \bigoplus_{p+q=k} A^{p,q}M.$$

We also extend J^* (the adjoint of J) to

$$A(M) = \bigoplus_k A^k M$$

by the rule $J^* f = f$ if f is a 0-form and, in general, by $J^*(\xi \wedge \eta) = J^*\xi \wedge J^*\eta$. If $X \in TM \otimes_{\mathbb{R}} \mathbb{C}$ is any tangent vector, then $X\rfloor : A^k \to A^{k-1}$ is the *contraction* by X defined by

$$(X \rfloor \omega)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}),$$

where $\omega \in A^k M$ and $\rfloor$ stands for the inner product. If $F \subset A(M)$ is an ideal, then

$$\operatorname{Ann} F = \{X \in TM : X\rfloor\omega \in F \text{ for all } \omega \in F\}.$$

The following lemma can easily be established [3]:

LEMMA 1. *Let* $F = (\omega_1, \ldots, \omega_k)$ *be the ideal of* AM *generated by* q*-forms* $\omega_1, \ldots, \omega_k$. *If* $dF \subset F$, *then* $\operatorname{Ann} F$ *is involutive*, *i.e. for* $X, Y \in \operatorname{Ann} F$ *we have* $[X, Y] \in \operatorname{Ann} F$.

Proof. The assertion easily follows from the identities $[X, Y]\rfloor = [\mathcal{L}_X, Y\rfloor]$ and $\mathcal{L}_X = d \circ X\rfloor + X\rfloor \circ d$.

We also have (see [3])

LEMMA 2. *If* $F = (\omega_1, \ldots, \omega_k)$ *is an ideal of* A^*D, $D \subset \mathbb{C}^n$, *generated by real* $(1,1)$*-forms*, *then* $\operatorname{Ann} F$ *is* J*-invariant.*

Proof. The conclusion follows from the observation that every real $(1,1)$-form ω may be diagonalized into the form $\omega = \pm b_1 \wedge Jb_1 \pm \ldots \pm b_{n'} \wedge Jb_{n'}$, where $b_1, \ldots, b_{n'}$ are real 1-forms.

The following lemma is important for our considerations:

LEMMA 3. *Let* ω_1 *and* ω_2 *be real* $(1,1)$*-forms with* $\omega_2 \geq 0$, *and let there exist* $a_j, b_j \in T^*D$, $D \subset \mathbb{C}^n$, *such that* $\omega_1 = \sum \pm a_j \wedge Ja_j$, $\omega_2 = -\sum b_j \wedge Jb_j$. *Then* $\dim \operatorname{span}\{a_j, Ja_j, b_j, Jb_j\} = 2p$ *if and only if* $(\omega_1 + i\omega_2)^p \neq 0$ *and* $(\omega_1 + i\omega_2)^{p+1} = 0$.

For the proof we refer to [3].

Remark 4. There are situations in which the condition $\omega_2 \geq 0$ is unnecessary. For instance, if ω_1 and ω_2 can be simultaneously diagonalized, then the conclusion remains valid.

4. Proof of the existence theorem in the holomorphically decomposable case

We prove Theorem 1. The ideal in question,

$$F = \operatorname{ideal}(\Omega, dd^c \operatorname{re} u, dd^c \operatorname{im} u),$$

is d-closed and invariant under J, the complex structure of the tangent bundle TD. It follows from (3) that the annihilator of F,

$$\operatorname{Ann} F = \{X \in TD : X\rfloor\Omega = X\rfloor dd^c \operatorname{re} u = X\rfloor dd^c \operatorname{im} u = 0\},$$

has complex codimension at least $k + p - 1$. To show that it is exactly $k + p - 1$, for every $z \in D$ we select forms $c_\kappa, a_\kappa, b_\kappa \in T_z^* D$ such that

$$\Omega = \pm \bigwedge_{j=1}^{k} c_j \wedge Jc_j, \quad dd^c \operatorname{re} u = \sum_{\kappa=1}^{r} \pm a_\kappa \wedge Ja_\kappa, \quad dd^c \operatorname{im} u = \sum_{\lambda=1}^{s} \pm b_\lambda \wedge Jb_\lambda.$$

We claim that the real dimension of the real span of

$$\{c_j, Jc_j, a_\kappa, Ja_\kappa, b_\lambda, Jb_\lambda\} = V^*$$

is exactly $2(k+p-1)$; we already know that it is at least $2(k+p-1)$. Let us choose a J-invariant complementary subspace of

$$V_0^* = \operatorname{span}\{c_j, Jc_j\}, \quad \text{i.e.} \quad V^* = V_1^* \oplus V_0^*.$$

Thus we have

$$a_\kappa = \alpha_\kappa + \gamma_\kappa, \quad b_\lambda = \beta_\lambda + \delta_\lambda, \quad \alpha_\kappa, \beta_\lambda \in V_1^*, \;\; \gamma_\kappa, \delta_\lambda \in V_0^*.$$

By definition of V^*, $\operatorname{span}\{\alpha_\kappa, J\alpha_\kappa, \beta_\lambda, J\beta_\lambda\} = V_1^*$. Therefore

$$dd^c \operatorname{re} u = \sum_{\kappa=1}^{r} \pm\alpha_\kappa \wedge J\alpha_\kappa + \sum_{j=1}^{k} (c_j \wedge s_j + Jc_j \wedge \sigma_j), \tag{18}$$

$$dd^c \operatorname{im} u = \sum_{\lambda=1}^{s} -\beta_\lambda \wedge J\beta_\lambda + \sum_{j=1}^{k} (c_j \wedge t_j + Jc_j \wedge \tau_j). \tag{19}$$

We have $\dim_{\mathbb{R}} V^* \geq 2(k+p-1)$. If $\dim_{\mathbb{R}} V^* \geq 2(k+p)$, then we would obtain $\dim_{\mathbb{R}} V_1^* \geq 2p$. On the other hand, Lemma 2.3 in [3] states the following: Let ω_1 and ω_2 be real $(1,1)$-forms with $\omega_2 \geq 0$, and let there exist $\alpha_\kappa, \beta_\lambda \in T^*D$, $D \subset \mathbb{C}^n$, such that

$$\omega_1 = \sum \pm\alpha_\kappa \wedge J\alpha_\kappa, \qquad \omega_2 = -\sum \beta_\lambda \wedge J\beta_\lambda. \tag{20}$$

Then $\dim \operatorname{span}\{\alpha_\kappa, J\alpha_\kappa, \beta_\lambda, J\beta_\lambda\} = 2p$ if and only if

$$(\omega_1 + i\omega_2)^p \neq 0 \quad \text{and} \quad (\omega_1 + i\omega_2)^{p+1} = 0.$$

Hence, in our case we would get, in particular,

$$\Big(\sum \pm\alpha_\kappa \wedge J\alpha_\kappa - i\sum \beta_\lambda \wedge J\beta_\lambda\Big)^p \neq 0$$

since $\{\alpha_\kappa, J\alpha_\kappa, \beta_\lambda, J\beta_\lambda\}$ spans V_1^*. But this would imply, by (18) and (19), that

$$\begin{aligned} \Omega \wedge (dd^c u)^p = {} & \Big(\pm \bigwedge_{j=1}^{k} c_j \wedge Jc_j\Big) \wedge \Big[\sum \pm\alpha_\kappa \wedge J\alpha_\kappa - i\sum \beta_\lambda \wedge J\beta_\lambda \\ & + \sum_{j=1}^{k} (c_j \wedge s_j + Jc_j \wedge \sigma_j + ic_j \wedge t_j + iJc_j \wedge \tau_j)\Big]^p \\ = {} & \Big(\pm \bigwedge_{j=1}^{k} c_j \wedge Jc_j\Big) \wedge \Big(\sum \pm\alpha_\kappa \wedge J\alpha_\kappa - i\sum \beta_\lambda \wedge J\beta_\lambda\Big)^p \neq 0, \end{aligned}$$

which contradicts (2). Thus indeed $\dim_{\mathbb{R}} V^* = 2(k+p-1)$.

Consequently, the complex codimension of $\operatorname{Ann} F$ is exactly $k+p-1$. On the other hand, by (4), F is d-closed, so it is integrable and, by the classical theorem of Frobenius, this gives the required foliation $\mathcal{L}_{p+k-1}$.

If we denote by $\imath : M \to D$ the inclusion mapping (the holomorphic mapping of the leaf M in question into D), then $\imath^* u = u|M$ and hence

$$\begin{aligned}(21)\qquad \partial_M \overline{\partial}_M (u|M) &= \partial_M \overline{\partial}_M (\imath^* \operatorname{re} u) + i \partial_M \overline{\partial}_M (\imath^* \operatorname{im} u) \\ &= \imath^* \partial \overline{\partial} \operatorname{re} u + i(\imath^* \partial \overline{\partial} \operatorname{im} u) = 0,\end{aligned}$$

since $TM \subset \operatorname{Ann}(dd^c \operatorname{re} u, dd^c \operatorname{im} u)$. This proves that the functions $\operatorname{im} u|M$ and $\operatorname{re} u|M$ are pluriharmonic.

Finally, we have to show that the functions

$$(22)\qquad (\operatorname{im} u)_{|j}|M \quad \text{and} \quad (\operatorname{re} u)_{|j}|M, \quad \text{where} \quad f_{|j} = \partial f / \partial z_j,$$

are holomorphic. Consider an arbitrary vector field

$$(23)\qquad X = \sum_{j=1}^{n} C^j (\partial / \partial z_j) \in TM^{1,0}.$$

The bundles TM and JTM are contained in $\operatorname{Ann}(dd^c \operatorname{re} u, dd^c \operatorname{im} u)$. Hence $X \rfloor dd^c \operatorname{re} u = X \rfloor dd^c \operatorname{im} u = 0$; but this is equivalent to

$$\sum_{j=1}^{n} C^j \operatorname{re} u_{|j\bar{k}} = \sum_{j=1}^{n} C^j \operatorname{im} u_{|j\bar{k}} = 0 \qquad \text{for every system } (C^1, \ldots, C^n),$$

C^j being real numbers, so $\operatorname{re} u_{|j\bar{k}}|M = \operatorname{im} u_{|j\bar{k}}|M = 0$ for each j. This proves that the functions (22) are indeed holomorphic, and so concludes the proof.

Remark 5. There are situations in which the condition $\omega_2 \geq 0$ in Lemma 2.3 in [3] is unnecessary. As is noticed there, if the forms (20) can be simultaneously diagonalized, then the conclusion remains valid.

Remark 6. If the domain D in Theorem 1 is bounded and u is, in addition, continuous on $\operatorname{cl} D$, then the functions $\operatorname{re} u$, $\operatorname{im} u$, $|(\operatorname{re} u)_{|j}|$ and $|(\operatorname{im} u)_{|j}|$ satisfy the *weak maximum principle* in $\operatorname{cl} D$, i.e. the maximum of $\operatorname{re} u$ on $\operatorname{cl} D$ is equal to the maximum of $\operatorname{re} u$ on ∂D etc.

5. Proof of the existence theorem in the exterior product case

Now we prove Theorem 2. As mentioned before, the method is essentially different from that applied for Theorem 1. Consider the ideal

$$F = \operatorname{ideal}(dd^c u_1, \ldots, dd^c u_p).$$

It is d-closed and invariant under J, the complex structure of TD. Unfortunately, in contrast to Theorem 1, it is not obvious now that

$$\operatorname{Ann} F = \{X \in TD : X \rfloor dd^c u_j = 0 \text{ for all } j = 1, \ldots, p\}$$

has complex codimension at least $p-1$. To show that it is exactly $p-1$, for every $z \in D$ we select forms $a^s_{j_s}$, $j_s = 1, \dots, k_s$, $s = 1, \dots, p$, such that

$$dd^c u_s = -\sum_{j_s=1}^{k_s} a^s_{j_s} \wedge J a^s_{j_s}, \qquad s = 1, \dots, p.$$

The possibility of decomposing each $dd^c u_s$ in the form as above follows from the plurisubharmonicity of u_s and, in particular, from its positive definiteness.

We claim that the real dimension of the real span of

$$\{a^j_{j_1}, Ja^1_{j_1}, \dots, a^p_{j_p}, Ja^p_{j_p} : j_1 = 1, \dots, k_1; \dots; j_p = 1, \dots, k_p\} = V^*$$

is at least $2(p-1)$. In order to prove this, consider, at every $z \in D$, the $2p$-vector

$$\omega = a^{l_1}_{j_1} \wedge Ja^{l_1}_{j_1} \wedge \dots \wedge a^{l_p}_{j_p} \wedge Ja^{l_p}_{j_p}, \quad 1 \le l_s \le p,\ 1 \le j_s \le k_{l_s},\ s = 1, \dots, p. \tag{24}$$

If it is not zero, then it is built from linearly independent vectors which form a basis of a J-invariant subspace of $T_z D$ of (complex) dimension $p-1$. If ω' is another $2p$-vector of the form (24) generating the same subspace, then

$$\omega' = t\omega \quad \text{with } t \text{ of the form } t = \det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} > 0. \tag{25}$$

Now, from the second relation in (9) it follows that at the z in question, there is a $2(p-1)$-vector

$$\omega_r = a^1_{j_1} \wedge Ja^1_{j_1} \wedge \dots \wedge a^{r-1}_{j_{r-1}} \wedge Ja^{r-1}_{j_{r-1}} \wedge a^{r+1}_{j_{r+1}} \wedge Ja^{r+1}_{j_{r+1}} \wedge \dots \wedge a^p_{j_p} \wedge Ja^p_{j_p}$$

(with $r = r(z)$) which is not zero, so it is built from linearly independent vectors that form a basis of a J-invariant subspace of $T_z D$ of (complex) dimension $p-1$, in general different from the previous $(p-1)$-dimensional subspace of $T_z D$. Hence $\dim_{\mathbb{R}} V^* \ge 2(p-1)$.

Now we claim that $\dim_{\mathbb{R}} V^* = 2(p-1)$. Suppose the contrary. In view of the previous conclusion this would mean that $\dim_{\mathbb{R}} V^* = 2m \ge 2p$, m being a positive integer. Thus there would exist a system of linearly independent vectors

$$(a^1_{t^1_1}, Ja^1_{t^1_1}, \dots, a^1_{t^1_{s_1}}, Ja^1_{t^1_{s_1}}, \dots, a^p_{t^p_{s_p}}, Ja^p_{t^p_{s_p}}) \tag{26}$$

with $s_1 + \dots + s_p = m \ge p$ and $0 \le s_j \le m$, $j = 1, \dots, p$. Consider the differential form

$$\varphi = (dd^c)^p \begin{bmatrix} s_1, \dots, s_p \\ u_1, \dots, u_p \end{bmatrix}.$$

Since the vectors of the system (26) are linearly independent, the $2m$-vector ω_0, defined as the wedge product of all the vectors of (26), would not be zero. Next, regarding φ as a vector, we would observe that the totality of its addends different from zero would form a $\mathbb{C}$-basis of the vector space V^*, so according to (25) they would differ from ω_0 by a factor $t > 0$. Therefore we would have $\varphi = (\sum t_j)\omega_0$, where each $t_j > 0$. This would mean that $\varphi \ne 0$, but $\varphi = 0$ by the first relation in (9), where we can take $j_1 = s_1, \ldots, j_p = s_p$ because of the arbitrariness of $(j_1, \ldots, j_p)$. Thus indeed $\dim_{\mathbb{R}} V^* = 2(p-1)$.

Consequently, the complex codimension of $\operatorname{Ann} F$ is exactly $p-1$. On the other hand, however, F is d-closed, so it is integrable and, by the classical theorem of Frobenius, this gives the required foliation $\mathcal{L}_{p-1}$.

If we denote by $\imath : M \to D$ the inclusion mapping (the holomorphic mapping of the leaf M in question into D), then $\imath^* u = u_k|M$ for $k = 1, \ldots, p$ and hence, in analogy to (14), we get

$$\partial_M \overline{\partial}_M (u_k|M) = \partial_M \overline{\partial}_M (\imath^* u_k) = \imath^* \partial\overline{\partial} u_k = 0, \qquad k = 1, \ldots, p,$$

since $TM \subset \operatorname{Ann}(dd^c u_1, \ldots, dd^c u_p)$. This proves that the functions $u_k|M$ are pluriharmonic.

Finally, we have to show that the functions $u_{k|j}|M$ are holomorphic. Consider an arbitrary vector field (23). The bundles TM and JTM are contained in $\operatorname{Ann}(dd^c u_1, \ldots, dd^c u_p)$. Hence $X \rfloor dd^c u_k = 0$, but this is equivalent to

$$\sum_{l=1}^{n} C^l u_{k|j\bar{l}} \equiv \sum_{l=1}^{n} C^l u_{k|\bar{l}j} = 0 \qquad \text{for every system } (C^1, \ldots, C^n),$$

C^j being real numbers, so $u_{k|j\bar{l}}|M = 0$ for each j. This proves that the functions $u_{k|j}|M$ are indeed holomorphic, and so the proof is complete.

Remark 7. If the domain D in Theorem 2 is bounded and the functions u_j, $j = 1, \ldots, p$, are in addition continuous on the closure $\operatorname{cl} D$, then the functions u_j and $|u_{k|j}|$ satisfy the weak maximum principle in $\operatorname{cl} D$ in the sense of Remark 6.

Proof of Theorem 3 is a minor modification of Theorem 1 in the case (10), and of that of Theorem 2 in other cases.

Remark 8. Theorem 3 can still be generalized by using the complex variant of the theorem of Frobenius, due to Nirenberg [30, p. 175, Th. 1], instead of the classical theorem. This will be the subject of a separate study of the second named author.

6. Natural Markov processes connected with the foliation $\mathcal{L}_{k+p-1}$

Theorems 1–3 may give complex foliations that generate some natural Markov processes. From now on we suppose the form Ω in Theorem 3 to be nonnegative. We consider D and u_j, $j = 1, \ldots, p$, as in that theorem. Let θ be the positive current of type $(n-1, n-1)$ defined by

$$\theta := (dd^c|z|^2)^{n-k-p} \wedge \Omega \wedge (dd^c)_r^{p-1}[u_1, \ldots, u_p] \tag{27}$$

where $(dd^c)_r^{p-1}$ is given by (8) and the integer r is supposed to be independent of $z \in D$.

We associate with (27) the Dirichlet form E_θ defined by

$$E_\theta(\varphi,\psi) := \int_D d\varphi \wedge d^c\psi \wedge \theta \quad \text{for } \varphi,\psi \in C_0^2(D). \tag{28}$$

We note that, by Stokes' theorem, we have

$$E_\theta(\varphi,\psi) = -\int_D \varphi dd^c\psi \wedge \theta + \int_D \varphi d^c\psi \wedge d\theta, \tag{29}$$

which means that ψ is E_θ-harmonic if and only if

$$(L_\theta\psi)dV := dd^c\psi \wedge \theta - d^c\psi \wedge d\theta = 0, \tag{30}$$

where dV is the canonical Kähler form of $\mathbb{C}^n$.

LEMMA 4. *E_θ is closable on $L^2(D,dV)$, which implies that L_θ has a self-adjoint extension on $L^2(D,dV)$.*

Proof. We proceed in analogy to the proof of Lemma 1 in [11]. By [10], pp. 4–5, it is sufficient to show that for any sequence $\psi_k \in C_0^2(D)$ with

$$E_\theta(\psi_j-\psi_k,\psi_j-\psi_k) \to 0 \quad \text{as } j,k\to\infty$$

and

$$\int_D \psi_k^2 dV \to 0 \quad \text{as } k\to\infty,$$

we get

$$E_\theta(\psi_k,\psi_k) \to 0 \quad \text{as } k\to\infty;$$

yet this is obvious since θ and $d\theta$ have continuous coefficients.

Now we follow the general line of correspondence of the diffusion, a specific Markov process, to our Dirichlet form; cf. [10–12]. Firstly, for any $\varphi,\psi \in C_0^2(D)$, we have

$$E_\theta(\varphi,\psi) = -(\varphi, L_\varphi\psi)_{L^2}. \tag{31}$$

Secondly, by Lemma 4, L_θ can be considered a self-adjoint operator on $L^2(D,dV)$ in the sense that L_θ has a closed extension which is self-adjoint on $L^2(D,dV)$. Thirdly, as in [28], for any complex number γ with $|\arg\gamma| \le \pi-\varepsilon$, ε being fixed and positive, we get

$$\|(\lambda I - L_\theta)^{-1}\| \le \text{const}/(|\lambda|+1),$$

where I is the identity operator. By the Hille–Yosida theory of semi-groups, especially Corollary 2 in [34], p. 249, $L_\theta - \lambda I$ is the infinitesimal generator of an equicontinuous semi-group $\{T_t : t\ge 0\}$ of class (C_0) such that

$$T_0 = I, \quad T_t = \exp(tL_\theta) \quad \text{for} \quad t>0.$$

Summing up, since, by construction, T_t inherits the Markov property (cf. e.g. [10], p. 88), it yields a family of transition functions

$$\{p_t^\theta(x,y) : t>0,\ x,y\in D\} \tag{32}$$

with the property

$$T_t f(x) = \int_D p_t^\theta(x,y) f(y)\, dV(y) \quad \text{for } f \in L^2(D, dV), \tag{33}$$

which gives the desired *diffusion* X_t^θ, that is, a Markov process with continuous sample paths, associated in a natural way with the corresponding foliation $\mathcal{L}_{k+p-1}$ whenever this foliation exists by Theorem 3. Therefore we have proved

THEOREM 4. *Consider an arbitrary domain* D, *arbitrary functions* (11), *and an arbitrary nonnegative form* Ω, *satisfying all the hypotheses of Theorem* 3. *In addition, suppose that the integer* $r = r(z)$ *in* (13) *is independent of* $z \in D$. *Then to any positive current* θ *of type* $(n-1, n-1)$, *defined by* (27), *there corresponds a unique diffusion* X_t^θ *determined by a family of transition functions* (32) *with the property* (33).

Moreover, we have the following corollary to this theorem, which is obtained immediately, by Theorem 3:

COROLLARY 1. *Under the hypotheses of Theorem* 4, *to any foliation* $\mathcal{L}_{k+p-1}$ *described in Theorem* 3 *there corresponds a unique diffusion* X_t^θ, *that is*, *a Markov process with continuous sample paths*, *determined by a family of transition functions* (32) *with the property* (33).

Remark 9. The current θ given by (27) is not always d-closed. For instance, as proved in [28] (Proposition 1), if u and ϕ are real smooth functions such that we have, on D,

$$du \wedge (dd^c u)^p = 0, \quad \phi \neq 0, \quad d\phi \wedge du \wedge d^c u \wedge (dd^c u)^{p-1} \neq 0,$$

then u and $\Omega = \phi du \wedge d^c u$ satisfy the hypotheses of Theorem 1.

Remark 10. If we take u and Ω as in Remark 9, then

$$\theta = (dd^c|z|^2)^{n-1-p} \wedge \phi du \wedge d^c u \wedge (dd^c u)^{p-1}.$$

Let us normalize θ taking $\theta_0 = \phi^{-1}\theta$. Then, as noticed in [28],

$$(L_\theta \psi) dV = dd^c\psi \wedge \theta - d^c\psi \wedge d\theta = \phi[(L_{\theta_0}\psi) dV - d^c\psi \wedge d\log\phi \wedge \theta_0],$$

so X_t^θ is a diffusion which differs from $X_t^{\theta_0}$ by a drift force due to $-d^c\psi \wedge d\log\phi \wedge \theta_0$, up to time change caused by ϕ.

7. Properties of canonical diffusions

Consider the sample paths [18] of the diffusion X_t^θ, determined uniquely in Theorem 4. We prove the following generalization of Theorem 3 of [28]:

THEOREM 5. *Consider any foliation* $\mathcal{L}_{k+p-1}$ *determined in Theorem* 3 *and let* M *be its arbitrary leaf. Then the sample paths of the diffusion* X_t^θ, *determined uniquely in Theorem* 4 *by that foliation*, *remain invariant on* M.

The proof is based upon four lemmas. Firstly, we have

LEMMA 5. *With any foliation* $\mathcal{L}_{k+p-1}$ *determined in Theorem* 3 *we can associate the annihilator*

$$\operatorname{Ann} F = \{X \in TD : X \rfloor \alpha = 0 \quad \text{for } \alpha = \Omega,\ dd^c \operatorname{re} u_j,\ dd^c \operatorname{im} u_j,\ \ j = 1, \dots, p\} \tag{34}$$

of the corresponding ideal

$$F = \operatorname{ideal}(\Omega, dd^c \operatorname{re} u_1, dd^c \operatorname{im} u_1, \dots, dd^c \operatorname{re} u_p, dd^c \operatorname{im} u_p); \tag{35}$$

(34) *is the set of* J*-invariant involutive tangent vector fields of constant real dimension* $2(n-k-p+1)$, *where* J *is the complex structure of* TD.

The proof is obvious. Secondly, by the classical theorem of Frobenius, we get

LEMMA 6. *Under the hypotheses of Theorem* 3, *there locally exist* $q := 2(k+p-1)$ *real-valued functions* f_j, $j = 1, \dots, q$, *such that each leaf* M *of* $\mathcal{L}_{k+p-1}$ *is given by*

$$\{z \in D : f_j(z) = C_j,\ \ j = 1, \dots, q\}, \tag{36}$$

where the C_j *are real constants and*

$$\operatorname{span}\{df_j : j = 1, \dots, q\} = (\operatorname{Ann} F)^{\perp}. \tag{37}$$

Further, we have

LEMMA 7. *Fix a point of the domain* D *in question*, $D \subset \mathbb{C}^n$, *and its sufficiently small neighbourhood* $D_0 \subset D$, $D_0 \neq D$, *where all the functions* f_j *of* (36) *are defined. Select forms*

$$c_j, a^s_{j_s}, b^s_{j_s}, \qquad j = 1, \dots, k,\ \ j_s = 1, \dots, k_s,\ \ s = 1, \dots, p, \tag{38}$$

such that

$$\Omega = \pm \bigwedge_{j=1}^{k} c_j \wedge Jc_j, \qquad dd^c \operatorname{re} u_s = -\sum_{j_s=1}^{k_s} a^s_{j_s} \wedge Ja^s_{j_s},$$

$$dd^c \operatorname{im} u_s = -\sum_{j_s=1}^{k_s} b^s_{j_s} \wedge Jb^s_{j_s}, \qquad s = 1, \dots, p,$$

where J *denotes the complex structure of* TD *and*, *for any* $r \neq r'$, c_r *and* $c_{r'}$ *resp.* a_r *and* $a_{r'}$ *resp.* b_r *and* $b_{r'}$ *are mutually orthogonal with respect to the Kähler metric of* $\mathbb{C}^n$. *Suppose* (*without any loss of generality*) *that*

$$\{c_j, Jc_j, a^s_{j_s}, Ja^s_{j_s}, b^s_{j_s}, Jb^s_{j_s} : j, j_s \text{ and } s \text{ as in } (38),\ \ s < p\}$$

is a basis of

$$V_1^* := \operatorname{span}\{c_j, Jc_j, a^s_{j_s}, Ja^s_{j_s}, b^s_{j_s}, Jb^s_{j_s} : j, j_s \text{ and } s \text{ as in } (38),\ \ s \leq p\}.$$

Choose V_3^* *as a* J*-invariant complementary subspace of*

$$V_2^* := \operatorname{span}\{a^s_{j_s}, Ja^s_{j_s}, b^s_{j_s}, Jb^s_{j_s} : j, j_s \text{ and } s \text{ as in } (38),\ \ s < p\},$$

i.e. $V_1^* = V_2^* \oplus V_3^*$ *so that*

$$c_s = c'_s + a'^s_{j_s} \quad \textit{or} \quad c_s = c'_s + b'^s_{j_s}, \qquad \textit{where } a'^s_{j_s} \textit{ resp. } b'^s_{j_s} \in V_2^* \textit{ and } c'_s \in V_3^*;$$

if $k < p$, *we set* $c'_s = 0$ *for* $s > k$; *if* $k \geq p$, *we set*

$$a'^s_{j_s} = 0 \quad \textit{or} \quad b'^s_{j_s} = 0 \qquad \textit{for } s > p.$$

Next, take k *mutually orthogonal elements* $c''_1, \ldots, c''_k$ *of* V_3^* *such that*

$$V_3^* = \operatorname{span}\{c''_1, Jc''_1, \ldots, c''_k, Jc''_k\}.$$

Then

$$\begin{aligned} (39) \qquad & (\operatorname{Ann} F)^{\perp} \\ & = \operatorname{span}\{c''_1, Jc''_1, \ldots, c''_k, Jc''_k, a_1, Ja_1, b_1, Jb_1, \ldots, a_{p-1}, Ja_{p-1}, b_1, Jb_{p-1}\}. \end{aligned}$$

Proof. The verification of (39) is straightforward. Note that $\dim_{\mathbb{R}} V_1^* = q$.

LEMMA 8. *Fix* $z_0 \in D$ *and its sufficiently small neighbourhood* $D_0 \subset D$, $D_0 \neq D$, *where all the functions* f_j *of* (36) *are defined. Then*

$$(40) \qquad \begin{aligned} df_j \wedge \Omega \wedge (dd^c)^{p-1}_{r(z)}[u_1, \ldots, u_p](z) = 0, \\ d^c f_j \wedge \Omega \wedge (dd^c)^{p-1}_{r(z)}[u_1, \ldots, u_p](z) = 0, \end{aligned} \qquad z \in D_0, \ \ j = 1, \ldots, p,$$

with the notation (8), *where the integer* $r(z)$ *may, in general, depend on* $z \in D_0$.

Proof. By Lemmas 6 and 7, in particular the formulae (32) and (39), we have, on D_0, for each $j = 1, \ldots, p$,

$$\begin{aligned} & df_j \wedge \Omega \wedge ((dd^c)^{p-1}_{r(z)}[u_1, \ldots, u_p](z)) \\ & \quad = \text{const}\, df_j \wedge \Big(\pm \bigwedge_{j=1}^{k} c_j \wedge Jc_j \Big) \bigwedge_{s=1}^{p-1} \Big(\sum_{j_s=1}^{k_s} a^s_{j_s} \wedge Ja^s_{j_s} \Big) \wedge \Big(\sum_{j'_s=1}^{k_s} b^s_{j'_s} \wedge b^s_{j'_s} \Big) \\ & \quad = \text{const}\, df_j \wedge \Big(\pm \bigwedge_{j=1}^{k} c''_j \wedge Jc''_j \Big) \bigwedge_{s=1}^{p-1} \Big(\sum_{j_s=1}^{k_s} a^s_{j_s} \wedge Ja^s_{j_s} \Big) \wedge \Big(\sum_{j'_s=1}^{k_s} b^s_{j'_s} \wedge b^s_{j'_s} \Big) = 0. \end{aligned}$$

Moreover, $d^c f_j = -J(df_j)$ and, by J-invariance of (37),

$$\operatorname{span}\{J(df_j) : j = 1, \ldots, q\} = J[(\operatorname{Ann} F)^{\perp}] = (\operatorname{Ann} F)^{\perp},$$

so we also get the latter equality in (40).

Proof of Theorem 5. By applying the operator d to the latter equality in (40), we have, on D,

$$(41) \qquad dd^c f_j \wedge \theta - d^c f_j \wedge d\theta = 0, \qquad j = 1, \ldots, q,$$

that is, by (30), f_j is E_θ-harmonic, i.e. $L_\theta f_j = 0$. Next, we observe that, by (40) and (41),

$$L_\theta(f_j^2) = 2 f_j L_\theta(f_j) + 2(df_j \wedge d^c f_j \wedge \theta)/dV = 0.$$

Hence, by applying the Dynkin formula (cf., e.g., [18], p. 232) to $[f_j - f_j(z_0)]^2(X_t^\theta)$ with $X_0^\theta = z_0$, we get

$$
\begin{aligned}
E_{z_0}\{[f_j(X^\theta_{t\wedge\zeta}) - f_j(z_0)]^2\} \\
&= E_{z_0}[f_j^2(X^\theta_{t\wedge\zeta})] - 2f_j(z_0)E_{z_0}[f_j(X^\theta_{t\wedge\zeta})] + f_j^2(z_0)E_{z_0}[1] \\
&= f_j^2(z_0) - 2f_j^2(z_0) + f_j^2(z_0) = 0
\end{aligned}
$$

with ζ being the first hitting time ([10], p. 91) of ∂D_0, by virtue of the Dynkin formula, applied again to $f_j^2(X_t^\theta)$ and $f_j(X_t^\theta)$. Therefore $f_j(X^\theta_{t\wedge\zeta}) = f_j(X_t^\theta)$ for any $t > 0$, $j = 1, \ldots, q$, that is, $X^\theta_{t\wedge\zeta}$ does not quit the level surface of f_j. Since this fact holds locally at any point of D, the sample paths of the diffusion X_t^θ, uniquely determined in Theorem 4 by the foliation $\mathcal{L}_{k+p-1}$, remain indeed invariant on M.

Theorem 5 yields

COROLLARY 2. *If a positive current θ of type $(n-1, n-1)$, of the form* (27), *is d-closed*, *i.e. $d\theta = 0$*, *then holomorphic functions are E_θ-harmonic*, *that is*, *X_t^θ is a symmetric conformal diffusion.*

8. Laplace–Beltrami operator on Riemannian manifolds

Let M be a Riemannian manifold of dimension m with a Riemannian scalar product g. There exists a unique connection ∇ on M, the so-called Riemannian connection for g, which is torsion free and for which g is parallel ($\nabla g = 0$).

The curvature tensor R of ∇ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for arbitrary vector fields X, Y, Z. The Ricci tensor is the trace

$$\mathrm{Ric}(X, Y) = \mathrm{trace}(Z \to R(Z, X)Y).$$

The scalar product g may be extended onto the whole tensor algebra of M, in particular, onto the exterior algebra of smooth differential forms on M,

$$A(M) = \bigoplus_{p=1}^{m} A^p M.$$

For example, if $x \in M$, and $v, w \in T_x^* M$, we get

$$g(v, w) = g(v^\sharp, w^\sharp),$$

where $v^\sharp$, $w^\sharp$ are tangent vectors dual to v and w, respectively. If, now, $v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p \in \bigwedge^p T_x^* M$, then

$$g(v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p) = \sum_\pi \varepsilon(\pi) g(v_1, w_{\pi_1}) \ldots g(v_p, w_{\pi_p}),$$

where the sum is taken over all permutations $\pi = (\pi_1, \ldots, \pi_p)$ of $\{1, \ldots, p\}$. Assuming that the spaces A^p and A^q are orthogonal for $p \neq q$ we extend q onto the whole $A(M)$. The use of the same letter q for the extended product should not be confusing.

For any two smooth forms $\lambda, \mu \in A(M)$ we define the global scalar product $\langle\, ,\, \rangle$ by

$$\langle \lambda, \mu \rangle = \int_M g(\lambda, \mu) \mathrm{Vol}_M, \tag{42}$$

where Vol_M is the volume of the metric g (M is assumed to be compact and oriented).

Now, take one of the simplest and most natural first order differential operators $d : A(M) \to A(M)$. We are interested in finding its formal adjoint d^*. To this end let us introduce the Hodge-star homomorphism $\star : A^p \to A^{n-p}$. It is uniquely determined by the condition

$$g(\lambda, \mu)\mathrm{Vol}_M = \lambda \wedge \star\mu, \qquad \lambda, \mu \in A^p. \tag{43}$$

The homomorphism $\star$ may be extended onto the whole algebra A.

The properties given below follow directly from the definition (43):

$$\star\star_{|A^p} = (-1)^{p(n-p)} \tag{44}$$

and

$$g(\star\lambda, \star\mu) = g(\lambda, \mu). \tag{45}$$

By (42) and (43) we get

$$\langle \lambda, \mu \rangle = \int_M \lambda \wedge \star\mu, \qquad \lambda, \mu \in A^p. \tag{46}$$

When applying the Stokes theorem to $\langle \lambda, \nu \rangle$ we get, by (44)–(46),

$$\langle d\lambda, \nu \rangle = \langle \lambda, \overline{\omega} \star d \star \omega\nu \rangle,$$

where ω and $\overline{\omega}$ restricted to A^p denote multiplication by $(-1)^p$ and $(-1)^{p(n-p)}$, respectively.

Consider the operator

$$d^* = \overline{\omega} \star d \star \omega.$$

Then d^* is formally adjoint to d in the sense that

$$\langle d\lambda, \nu \rangle = \langle \lambda, d^*\nu \rangle. \tag{47}$$

It is a first-order linear differential operator on M.

By the definition (47), the properties (43) and (44) of $\star$ and the well-known properties of d one can easily check the following properties of d^*:

(i) $d^* : A^p \to A^{p-1}$, in particular $d^* f = 0$ on functions f,

(ii) $d^* d^* = 0$,

(iii) $d^*\lambda = (-1)^{n(p+1)+1} \star d \star \lambda$, $\lambda \in A^p$,
in particular, if n is even,

(iv) $d^* = - \star d \star$.

Now, using d and d^* we can build the *Laplace–Beltrami operator* as follows:

$$\Delta = d^* d + dd^*. \tag{48}$$

PROPOSITION 3. *The Laplace–Beltrami operator Δ is a second-order linear differential operator and it has the following properties*:

1° $\Delta : A^p \to A^p$, *while* $\Delta = -\sum_i \partial^2/(\partial x^i)^2$ *on functions in* $M = \mathbb{R}^n$,
2° $\star\Delta = \Delta\star$,
3° $\Delta = (d + d^*)^2$,
4° $\langle\Delta\lambda, \mu\rangle = \langle\lambda, \Delta\mu\rangle$,
5° *if* $\Delta\mu = 0$, *then* $\langle\Delta\lambda, \mu\rangle = 0$,
6° $\Delta\lambda = 0$ *if and only if* $d\lambda = 0$ *and* $d^*\lambda = 0$.

Proof. 1° is evident. 2° is numerical. 3° is a consequence of the fact that $d^2 = 0$ and $d^{*2} = 0$.

4°: By (47) and (48), we get $\langle\Delta\lambda, \mu\rangle = \langle(d^*d + dd^*)\lambda, \mu\rangle = \langle d^*d\lambda, \mu\rangle = \langle dd^*\lambda\mu\rangle = \langle\lambda, dd^*\mu\rangle + \langle\lambda, dd\mu\rangle = \langle\lambda, \Delta\mu\rangle$. 5° is a consequence of 4°.

6°: The implication "$\Leftarrow$" is evident, so if $\Delta\lambda = 0$ then $0 = \langle\Delta\lambda, \mu\rangle = \langle(d^*d + dd^*)\lambda, \lambda\rangle + \langle d\lambda, d\lambda\rangle + \langle d^*\lambda, d^*\lambda\rangle$. This implies that $d\lambda = 0$ and $d^*\lambda = 0$, and completes the proof.

9. Harmonic theory on compact complex manifolds

Let M be a compact connected hermitian manifold of complex dimension $n = 2m$. We wish to find canonical representatives for the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(M)$. To do this, we copy the development in the real case replacing d by $\overline{\partial}$ and adding complex conjugate signs in various places.

First, the hermitian metric on M induces a hermitian inner product $(\ ,\ )_x$ on each space $\bigwedge^{p,q} T^{*c}_x(M)$ and hence it induces a "global" hermitian inner product on $A^{p,q}(M)$ by

$$\langle\psi, \eta\rangle = \int_M (\psi(x), \eta(x))_x \, dV(x) \qquad \text{for } \psi, \eta \in A^{p,q}$$

making $A^{p,q}$ a complex pre-Hilbert space. The star operator $\star : A^{p,q} \to A^{m-p,m-q}$ is defined by the requirement that $\psi(x) \wedge \star\eta(x) = (\psi(x), \eta(x))_x dV(x)$. By the above it is easy to verify that $\star\star\eta = (-1)^{p+q}\eta$ for every $\eta \in A^{p,q}$. We recall the following

PROPOSITION 4. *The formal adjoint* $\overline{\partial}^* : A^{p,q} \to A^{p,q-1}$ *of* $\overline{\partial} : A^{p,q-1} \to A^{p,q}$ *is given by*

$$\overline{\partial}^* = -\star\overline{\partial}\star.$$

The $\overline{\partial}$-*Laplacian* $\Delta_{\overline{\partial}} : A^{p,q} \to A^{p,q}$ is defined by $\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$. A (p, q)-form ψ is said to be $\overline{\partial}$-*harmonic* if $\Delta_{\overline{\partial}}\psi = 0$.

Let us denote by $H^{p,q} = Z^{p,q}/B^{p,q}$ the (p,q)-cohomology group in the sense of Dolbeault. Now let a Dolbeault cohomology class $[\psi] \in H^{p,q}$ be represented by a (p,q)-form ψ. Then each $\overline{\partial}$-closed (p,q)-form in this cohomology class is of the form $\psi + \overline{\partial}\eta$, where $\eta \in A^{p,q-1}$. Thus, the set of $\overline{\partial}$-closed (p,q)-forms in a given cohomology class is an affine subspace $S \subset A^{p,q}$. A natural question is whether there exists a form which has a minimum norm in S. Since $A^{p,q}$ is not complete there may not be such an element. To study this we find a criterion for ψ to have a minimum norm. First note that $\|\psi + \overline{\partial}\eta\|^2 = \|\overline{\psi}\|^2 + \|\overline{\partial}\eta\|^2 + 2\,\mathrm{re}\langle\psi, \overline{\partial}\eta\rangle$. Then we again recall a few well-known propositions.

Proposition 5. *A $\overline{\partial}$-closed form ψ is of minimum norm within its Dolbeault cohomology class if and only if* $\overline{\partial}^*\psi = 0$.

Proposition 6. $\Delta_{\partial}\psi = 0$ *if and only if* $\overline{\partial}\psi = 0$.

Define $H^{p,q}_{\overline{\partial}}(M) = \{\psi \in A^{p,q}(M) : \Delta_{\overline{\partial}}\psi = 0\}$. We have

Proposition 7 (the Hodge decomposition). *For any integers p and q with $0 \le p \le m, 0 \le q \le m$, $H^{p,q}_{\overline{\partial}}$ is finite-dimensional and the following orthogonal direct decomposition holds*:

$$A^{p,q} = H^{p,q}_{\overline{\partial}} \oplus \Delta_{\overline{\partial}}(A^{p,q}) = H^{p,q}_{\overline{\partial}} \oplus \overline{\partial}(A^{p,q-1}) \oplus \partial(A^{p-1,q}).$$

If we denote by $\pi_{(H^{p,q}_{\overline{\partial}})^\perp}$ the orthogonal projection on $(H^{p,q}_{\overline{\partial}})^\perp$ then by the Hodge decomposition we can define the operator $G = (\Delta_{\overline{\partial}}|(H^{p,q}_{\overline{\partial}})^\perp) - \pi_{(H^{p,q}_{\overline{\partial}})^\perp} : A^{p,q} \to (H^{p,q}_{\overline{\partial}})^\perp$. It is bounded, self-adjoint and compact. Moreover, it has the following properties:

$$\overline{\partial} \circ G = G \circ \overline{\partial} \quad \text{and} \quad \overline{\partial}^* \circ G = \overline{\partial}^* \circ G,$$
$$\mathrm{id} = \pi_{H^{p,q}_{\overline{\partial}}} + \Delta_{\overline{\partial}} \circ G \qquad \text{on } A^{p,q}.$$

G is called the *Green operator* for $\Delta_{\overline{\partial}}$ and, by the definition, $G(\alpha)$ is the unique solution of $\Delta_{\overline{\partial}}\omega = \alpha - \pi_{H^{p,q}_{\overline{\partial}}}\alpha$ in $(H^{p,q}_{\overline{\partial}})^\perp$.

Corollary 3. *Each Dolbeault cohomology class contains a unique harmonic representative*, *i.e.*

$$\ker\Delta_{\overline{\partial}} \simeq H^{p,q}_{\overline{\partial}}.$$

On a compact Hermitian manifold we define a number of operators on the space A such as d, ∂, $\overline{\partial}$, their adjoints δ, ∂^*, $\overline{\partial}^*$ and the associated Laplacians $\Delta_d = d\delta + \delta d$, Δ_∂ and $\Delta_{\overline{\partial}}$, respectively. We define three more operators:

1) $d^c = (i/4\pi)(\overline{\partial} - \partial)$;

2) $L : A^{p,q} \to A^{p+1,q+1}$ by the formula $L(\eta) = \eta \wedge \omega$, where ω is the fundamental form;

3) $\Lambda = L^* : A^{p,q} \to A^{p+1,q+1}$, formally adjoint to L.

Note that d^c (like d) is a real operator and

$$dd^c = -d^c d.$$

On a general Hermitian manifold there are no simple relations between these operators. In the Kähler case, however, we shall establish some of Hodge identities joining them together.

LEMMA 9.

(i) $[\Lambda, d] = \Lambda \circ d - d \circ \Lambda = -4\pi d^{c*}$, (iii) $[\Lambda, \overline{\partial}] = -i\partial^*$,

(ii) $[L, d^*] = 4\pi d^c$, (iv) $[\Lambda, \partial] = i\overline{\partial}^*$.

Proof. For the proof see [17].

From the above lemma we easily get the following

LEMMA 10. *On a compact Kähler manifold*

$$[L, \Delta_d] = 0 \quad \text{and} \quad [\Lambda, \Delta_d] = 0.$$

Proof. Since ω is closed,

$$d(\omega \wedge \eta) = \omega \wedge d\eta \quad \text{or} \quad [L, d] = 0 \text{ and } [\Lambda, d^*] = 0.$$

Then $\Lambda(dd^* + d^*d) = (d\Lambda d^* - 4\pi d^{c*}d^*) + d^*\Lambda d = d\Lambda d^* + 4\pi d^* + d^{c*} + d^*\Lambda d = (dd^* + d^*d)\Lambda$.

Lemmas 9 and 10 enable us to prove the following fundamental fact about the complex Laplace–Beltrami operator.

PROPOSITION 8. *On a compact Kähler manifold*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial.$$

Proof. First, we show that $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. Since

$$\Lambda\partial - \partial\Delta = i\overline{\partial}^*,$$

it follows that

$$i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0.$$

Then

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}).$$

Finally, we show that $\Delta_\partial = \Delta_{\overline{\partial}}$:

$$-i\Delta_\partial = \overline{\partial}(\Lambda\overline{\partial} - \overline{\partial}\Lambda) + (\Lambda\overline{\partial} - \overline{\partial}\Lambda)\overline{\partial} = \overline{\partial}\Lambda\overline{\partial} - \overline{\partial}\overline{\partial}\Lambda + \Lambda\overline{\partial}\overline{\partial} + \overline{\partial}\Lambda\overline{\partial}$$

and

$$i\Delta_\partial = \overline{\partial}(\Lambda\overline{\partial} - \overline{\partial}\Lambda) + (\Lambda\partial - \partial\Lambda)\overline{\partial} = \overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} + \partial\Lambda\overline{\partial} = i\Delta_{\overline{\partial}},$$

which proves our proposition.

COROLLARY 4. *On a compact Kähler manifold*, Δ_d *preserves bidegree.*

Now let us see what this implies on cohomology. To avoid confusion, set

$$H_d^{p,q} = Z_d^{p,q}/B_d^{p,q}, \quad \mathbf{H}_d^{p,q} = \{\eta \in A^{p,q} : \Delta_d \eta = 0\}, \quad \mathbf{H}_d^r = \{\eta \in A^r : \Delta_d \eta = 0\},$$

and, similarly, for ∂ and $\overline{\partial}$. Since $\Delta_d = 2\Delta_{\overline{\partial}}$, we immediately see that

$$\mathbf{H}_d^{p,q} = \mathbf{H}_{\overline{\partial}}^{p,q}.$$

We also have

$$\mathbf{H}_d^r = \bigoplus_{p+q=r} \mathbf{H}_d^{p,q}.$$

Indeed, all (p,q)-components of a harmonic form are harmonic since $[\Delta_d, \pi_{p,q}] = 0$ (Δ_d preserves type). Since Δ_d is real we also have $\mathbf{H}_d^{p,q} = \mathbf{H}_d^{q,p}$. If η is a closed form of type (p,q) then

$$\eta = \mathbf{H}(\eta) + dd^* G(\eta),$$

where $\mathbf{H}(\eta)$ is the harmonic part of η which is also of type (p,q). Hence $H_d^{p,q} \simeq \mathbf{H}_d^{p,q}$. When combining them with the Hodge isomorphism $H_{DR}^* \simeq \mathbf{H}^*$, where DR stands for *de Rham*, we get

Hodge decomposition. For a compact Kähler manifold we have the following isomorphisms for complex cohomologies:

$$H^r(M, \mathbb{C}) \simeq \bigoplus_{p+q=r} H_d^{p,q} \simeq \bigoplus H_{\overline{\partial}}^{p,q} \simeq \bigoplus H^q(M, \Omega^p)$$

and

$$H_d^{p,q} = \overline{H_d^{q,p}}.$$

As a special case of this decomposition we have

$$H^{p,0} \simeq H^0(M, \Omega^p) \qquad \text{(the space of holomorphic } p\text{-forms)}.$$

In fact, we have the following

PROPOSITION 9. *The holomorphic p-forms on a compact Kähler manifold are the harmonic $(p,0)$-forms for any Kähler metric.*

Proof. We have to prove equality of the spaces in question, and not only isomorphism. Since $\Delta_d = 2\Delta_{\overline{\partial}}$, we have $\mathbf{H}_d^{p,0} = \mathbf{H}_{\overline{\partial}}^{p,0}$. In the Hodge decomposition we have, in general,

$$Z_{\overline{\partial}}^{p,q} = \mathbf{H}_{\overline{\partial}}^{p,q} \oplus B_{\overline{\partial}}^{p,q}$$

with

$$B_{\overline{\partial}}^{p,q} = \overline{\partial} A^{p,q-1}.$$

Hence, for $q = 0$, we have $Z_{\overline{\partial}}^{p,0} = H_{\overline{\partial}}^{p,0}$, and $Z_{\overline{\partial}}^{p,0}$ is precisely the space of holomorphic p-forms.

The positive numbers $h^{p,q} = \dim_{\mathbb{C}} H_{\overline{\partial}}^{p,q}$ are called the *Hodge numbers.*

On a Kähler manifold they satisfy various conditions.

Proposition 10. *On a compact Kähler manifold,*

$$b_r = \sum_{p+q=r} h^{p,q}, \quad h^{p,q} = h^{n-p,n-q}, \quad h^{q,p} = h^{p,q}.$$

Corollary 5. *The odd Betti numbers of a compact Kähler manifold are even.*

Proof. $b_{2s+1} = \sum_{p=0}^{2s+1} h^{p,2s+1-p} = 2\sum_{p=0}^{s} h^{p,2s+1-p}$.

We now pause to consider a few examples. Since Betti numbers are independent of the metric, we can compute them by picking a simple metric and counting the number of harmonic forms with respect to this metric.

Example 3. Consider S^6 with the standard metric of constant curvature. It is not difficult to show that with respect to this metric the only harmonic forms are the constant functions and multiples of the volume form. Thus, $b_0 = b_6 = 1$, $b_i = 0$ for $i = 1, 2, \ldots, 5$, and the Euler number $\chi = 2$.

Example 4. Consider $S^3 \times S^3$ with the metric induced by the standard metric on S^3. Since the volume form on S^3 is harmonic, it follows that $\chi = 0$.

Example 5. Consider $S^2 \times S^2 \times S^2$. Now we have $b_0 = b_6 = 1$, $b_2 = b_4 = 3$, $b_1 = b_3 = b_5 = 0$, and $\chi = 8$.

Example 6. Consider T^6. All the constant forms are harmonic. Hence $b_0 = b_6 = 1$, $b_1 = b_5 = 6$, $b_2 = b_4 = 15$, $b_3 = 20$, and $\chi = 0$.

We also know that S^6 is not a complex manifold; $S^3 \times S^3$ is a complex manifold but does not admit any Kähler metric; $S^2 \times S^2 \times S^2$ admits a Kähler metric but not a Ricci flat one; while T^6 admits a Ricci flat Kähler metric.

10. Laplace–Beltrami operator as the generator of a canonical diffusion

With the help of the operator L_θ, given by (30) and generating E_θ-harmonic functions, we are going to define the normalized operator L^θ which is the Laplace–Beltrami operator on each leaf M of the foliation $\mathcal{L}_{k+p-1}$: On the one hand, it appears to be equal to the "tangential part" $\Delta_{\mathcal{L}}$, $\mathcal{L} = \mathcal{L}_{k+p-1}$, of the usual Laplacian Δ on $\mathbb{C}^n = \mathbb{R}^{2n}$ (Proposition 1); on the other hand, under suitable hypotheses, it appears to be the generator of a canonical diffusion whose sample paths remain to diffuse on each leaf associated with the admissible positive current θ (Theorem 6).

Precisely, let $(X_1, \ldots, X_{2n-q})$, $q := 2(k+p-1)$, be a family of orthonormal vector fields of (34) with respect to the canonical Kähler metric of $\mathbb{C}^n$, where the ideal $\mathcal{F}$ is given by (35). Then we can add complementary orthonormal vector fields $X_{2n-q+1}, \ldots, X_{2n}$ such that

$$\operatorname{span}\{X_1, \ldots, X_{2n}\} = TD_0,$$

where D_0 is a sufficiently small neighbourhood of an arbitrarily chosen, fixed point z_0 of the domain D in question, $D \subset \mathbb{C}^n$. Then the adjoint vector fields X_j^* are naturally defined by

$$\int_{D_0} (X_j^* f) g \, dV = \int_{D_0} f X_j g \, dV \qquad \text{for } f, g \in C_0^1(D_0),$$

where dV is, as before, the canonical Kähler form of $\mathbb{C}^n$. Hence,

$$X_j^* = -X_j^* - \sum_{k=1}^{2n} (\partial/\partial x_k) a_j^k \qquad \text{for } X_j = \sum_{k=1}^{2n} a_j^k(x)(\partial/\partial x_k)$$

and, consequently, we get

Lemma 11. $\sum_{j=1}^{2n} X_j^* X_j = -\Delta$, *where* Δ *is the usual Laplacian on* $\mathbb{R}^{2n}$.

Now, for any foliation $\mathcal{L} = \mathcal{L}_{k+p-1}$ of D corresponding to the annihilator (35), we set

$$\Delta_{\mathcal{L}} = - \sum_{j=1}^{2n-q} X_j^* X_j \tag{49}$$

and define $(X_j|M)^*$, the adjoint vector field on a leaf $M \in \mathcal{L}$, by

$$\int_U [(X_j|M)^* a] b \, dv_M = \int_U a(X_j b) \, dv_M \qquad \text{for } a, b \in C_0^1(U),$$

with U being an open neighbourhood of z_0 in M, where dv_M is the induced volume form. We have

Lemma 12. $(X_j|M)^* = X_j^*|M$ *for* $j = 1, \ldots, 2n - q$.

Proof. The lemma is a direct consequence of the classical theorem of Frobenius.

Next, we have

Lemma 13. *The operator* $\Delta_{\mathcal{L}}$ *has a natural restriction* $\Delta_M := \Delta_{\mathcal{L}}|M$ *to each leaf* $M \in \mathcal{L}$, *with the property* $\Delta_M = \partial_M \overline{\partial}_M$.

Proof. Firstly, X_j and X_j^* are tangent to M. Secondly, by Lemma 12 the sum $\sum X_j^* X_j$ defines normalized self-adjoint operators on M and this suffices to conclude the proof.

Now suppose we are given functions u_j, $j = 1, \ldots, p$, and a nonnegative form Ω which satisfy the hypotheses of Theorem 3. Let θ be the positive current of type $(n-1, n-1)$, defined by (27), where $(dd^c u)_r^{p-1}$ is given by (8) and the integer r is supposed to be independent of $z \in D$. Then we define the normalized operator L^θ by

$$(L^\theta \phi) dV = dd^c \phi \wedge h\theta - d^c \phi \wedge d(h\theta), \qquad \text{where } h^{-1} dV = dd^c |z|^2 \wedge \theta; \tag{50}$$

here $\phi \in C_0^2(D)$ and dV is the canonical Kähler form of $\mathbb{C}^n$. For each leaf M of $\mathcal{L}$, where $\mathcal{L} = \mathcal{L}_{k+p-1}$ is any foliation determined in Theorem 3, $L^\theta|M$ is called the *Laplace–Beltrami operator* on M. The name is motivated by

PROPOSITION 11. *Under the hypotheses of Theorem* 3, *if the form Ω is nonnegative and the integer r does not depend on $z \in D$, we have $L^\theta = \Delta_{\mathcal{L}}$, where $\mathcal{L} = \mathcal{L}_{k+p-1}$ is any foliation determined in that theorem and $\Delta_{\mathcal{L}}$ is given by* (49).

Proof. First, by definition, L^θ and $\Delta_{\mathcal{L}}$ are self-adjoint operators on $L^2(D, dV)$. Furthermore, at every point of D, they have, as is easily seen, the same second order differential term, and hence $L^\theta = \Delta_{\mathcal{L}}$.

By Lemma 13 and Proposition 11, from Theorems 4 and 5 we directly obtain

MAIN THEOREM 6. *Consider an arbitrary domain D, arbitrary functions* (11), *and an arbitrary nonnegative form Ω, satisfying all the hypotheses of Theorem* 3. *In addition, suppose that the integer $r = r(z)$ in* (13) *is independent of $z \in D$. Let, further, θ be any positive current of type $(n-1, n-1)$ of the form* (27). *Suppose that*

$$\theta \wedge (dd^c)_r^{p-1}[u_1, \ldots, u_p](z) = 0, \quad \theta \wedge (dd^c|z|^2) \neq 0, \quad z \in D.$$

Then there exists a unique diffusion X_t^θ determined by a family of transition functions (32) *with the property* (33). *The diffusion X_t^θ is the canonical diffusion with generator L^θ defined by* (50) *which is the Laplace–Beltrami operator on each leaf M of any foliation $\mathcal{L} = \mathcal{L}_{k+p-1}$ determined in Theorem* 3. *Moreover*, X_t^θ *is uniquely determined by $\mathcal{L}$ and the sample paths of X^θ remain to diffuse on M, i.e. they remain invariant on M.*

11. Laplace–Beltrami operator in the case of the sphere and the hyperboloid

It seems interesting to consider the Laplace–Beltrami operator in a more general context of Clifford analysis [23, 7]. We concentrate on the cases of the sphere and hyperboloid.

Let $\gamma_1, \ldots, \gamma_{2n+1}$ be generators of the Clifford algebra $C^{(2n+1,0)}$. We can assume that

(i) $\gamma_j^+ = \gamma_j$,
(ii) $\gamma_j \in U(2^n)$ for each $j = 1, \ldots, 2n+1$.

Let us define matrices $S_1, \ldots, S_{2n+1}$ as follows:

$$S_j = i\gamma_j, \quad j = 1, \ldots, 2n+1.$$

From the fact that S_j is antihermitian for each $j = 1, \dots, 2n+1$ it follows that (S_j) is a hermitian pre-Hurwitz system [22].

Next we introduce the Clifford variable Z so that

$$Z = y^0 I_{2^n} + \sum_{j=1}^{2n+1} y^j s_j, \tag{51}$$

where y^α, $\alpha = 0, \dots, 2n+1$, are arbitrary real constants. We can easily check the identity

$$Z^+ Z = \sum_{\alpha=0}^{2n+1} (y^\alpha)^2 I_{2^n}.$$

Let $\mathbf{u}$ and $\mathbf{v}$ be complex vectors in $\mathbb{C}^{2^n}$, satisfying

$$\mathbf{u} = Z\mathbf{v}. \tag{52}$$

Then we immediately get the equation

$$\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = \langle \mathbf{y}, \mathbf{y} \rangle \langle\langle \mathbf{v}, \mathbf{v} \rangle\rangle,$$

where $\mathbf{y}$ denotes the real vector in $\mathbb{R}^{2n+2}$ defined by $\mathbf{y}^T = (y^0, \dots, y^{2n+1})$, and $\langle\langle\, , \,\rangle\rangle$ (resp. $\langle\, , \,\rangle$) denotes the hermitian (resp. usual) scalar product in $\mathbb{C}^{2^n}$ (resp. $\mathbb{R}^{2n+2}$). This means that the system (s_j) is associated with the Hermitian pre-Hurwitz pair $((\mathbb{C}^{2^n}, I_{2^n}), (\mathbb{R}^{2n+2}, I_{2n+2}))$; cf. [22].

Let us introduce the *Laplace–Beltrami operator on the sphere* S^{2n+2} relative to the conformal metric

$$g_{uv} := 4\delta_{\mu\nu}/(1 + \langle \mathbf{y}, \mathbf{y} \rangle), \qquad \mu, \nu = 0, \dots, 2n+2,$$

by the formula

$$\Delta_{S^{2n+2}} := 4(1 + \langle \mathbf{y}, \mathbf{y} \rangle)^2 (\partial/\partial y^\mu) \left[\frac{\delta^{\mu\nu}}{(1 + \langle \mathbf{y}, \mathbf{y} \rangle)^{2n}} \partial/\partial y^\nu \right]. \tag{53}$$

Then we get the following

Proposition 12. *The operator* $P(Z)$ *defined by*

$$P(Z) := \frac{1}{1 + \langle \mathbf{y}, \mathbf{y} \rangle} \begin{bmatrix} I_{2^n} & Z^+ \\ Z & \langle \mathbf{y}, \mathbf{y} \rangle I_{2^n} \end{bmatrix}$$

satisfies the conditions

$$P(Z)^2 = P(Z), \qquad P(Z)^+ = P(Z) \tag{54}$$

and

$$[P(Z), \Delta_{S^{2n+2}} P(Z)] = 0. \tag{55}$$

Proof can be carried out by direct computation.

The operator $P(Z)$, which appears to be a projector due to (54), parametrizes a point of the Grassmannian manifold $U(2^{n+1})/U(2^n) \times U(2^n)$. Furthermore, the equation (55) can be derived from the variation of the action $S(P)$ given by

$$S(P) := \frac{1}{2} \int |g|^{1/2} \mathrm{Tr}((\partial P/\partial y^\mu)(\partial P/\partial y^\nu) g^{\mu\nu})\, dy^0 \dots dy^{2n+1}.$$

This indicates that P defines a harmonic mapping between S^{2n+2} and the Grassmannian manifold $U(2^{n+1})/U(2^n) \times U(2^n)$. For $n = 0$,

$$P : Z \mapsto P(Z)$$

defines a harmonic mapping between S^2 and $U(2^{n+1})/U(2^n) \times U(2^n) \simeq CP^1$. For $n = 1$ it defines a harmonic mapping between S^4 and HP^1. The latter space has been introduced in [33], where it has been used to construct the Yang–Mills instanton solution in the de Sitter space.

Thus we have related Proposition 12, which was first pointed out by Fujii [9], to the framework of Hermitian pre-Hurwitz pairs. The latter enables us now to generalize the results of Fujii to the noncompact cases.

Suppose that n is a nonzero integer and $\gamma_1, \dots, \gamma_{2n}$ are generators of the Clifford algebra $C^{(2m,0)}$ having the properties

(i) $\gamma_j^+ = \gamma_j$,
(ii) $\gamma_j \in U(2^n)$ for each $j = 1, \dots, 2n$.

Then we can choose γ_{2n+1} so that

$$\gamma_{2n+1} = \begin{bmatrix} I_{2^{n-1}} & 0_{2^{n-1}} \\ 0_{2^{n-1}} & -I_{2^{n-1}} \end{bmatrix}. \tag{56}$$

Thus $\gamma_1, \dots, \gamma_{2n+1}$ generate the Clifford algebra $C^{(2n+1,0)}$.

Define the matrices $S_1, \dots, S_{2n+1}$ by

$$S_j = \gamma_j, \quad j = 1, \dots, 2n; \qquad S_{2n+1} = i\gamma_{2n+1}.$$

Set $\widetilde{\kappa} := \gamma_{2n+1}$. Then we get the equations

$$S_j^+ = -\widetilde{\kappa} S_j \widetilde{\kappa}, \qquad j = 1, \dots, 2n+1.$$

This shows that (S_j) is an hermitian pre-Hurwitz system, that is, the set of all generators of an hermitian pre-Hurwitz pair.

Now we introduce the Clifford variable Z so that (51) holds. Using the notation $\overline{Z} = \widetilde{\kappa} Z^+ \widetilde{\kappa}$, it can easily be checked that

$$\overline{Z} Z = \Big[(y^0)^2 + (y^{2n+2})^2 - \sum_{j=1}^{2n} (y^j)^2 \Big] I_{2^n}$$

identically. In the sequel we shall use the notation

$$(y^0)^2 + (y^{2n+1})^2 - \sum_{j=1}^{2n} (y^j)^2 = n_{\mu\nu} y^\mu y^\nu = \langle \mathbf{y}, \mathbf{y} \rangle_\eta = \mathbf{y}^T \eta \mathbf{y}.$$

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{C}^{2n}$ defined by (52). Then we immediately check that

$$\langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle_{\tilde{\kappa}} = \langle \mathbf{y}, \mathbf{y}\rangle_\eta \langle\langle \mathbf{v}, \mathbf{v}\rangle\rangle_{\tilde{\kappa}}.$$

Here $\langle\langle\ ,\ \rangle\rangle_{\tilde{\kappa}}$ denotes the pseudo-hermitian scalar product defined by

$$\langle\langle \mathbf{w}_1, \mathbf{w}_2\rangle\rangle_{\tilde{\kappa}} = \mathbf{w}_1^+ \widetilde{\kappa} \mathbf{w}_2$$

for arbitrary vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in $\mathbb{C}^{2n}$. This shows that the system (S_j) is related to the hermitian pre-Hurwitz pair $((\mathbb{C}^{2n}, \widetilde{\kappa}), (\mathbb{R}^{2n+2}, \eta))$.

Let us now introduce the *Laplace–Beltrami operator* $\Box_{H^{2n+2}(3,2n)}$ *on the hyperboloid*

$$H^{2n+2}(3, 2n) \simeq SO(3, 2n)/SO(2, 2n)$$

of equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - \sum_{j=4}^{2n+3} (x^j)^2 = 1.$$

Relative to the conformal metric

$$g_{\mu\nu} := 4\eta_{\mu\nu}/(1 + \langle \mathbf{y}, \mathbf{y}\rangle_\eta), \qquad \mu, \nu = 1, \ldots, 2n + 1,$$

this operator is given by

$$\Box_{H^{2n+2}(3,2n)} := 4(1 + \langle \mathbf{y}, \mathbf{y}\rangle_\eta)^2 \frac{\partial}{\partial y^\mu}\left[\frac{\eta^{\mu\nu}}{(1 + \langle \mathbf{y}, \mathbf{y}\rangle_\eta)^{2n}} \frac{\partial}{\partial y^\mu}\right]. \tag{57}$$

Then we arrive at

PROPOSITION 13. *The operator* $\widehat{P}(Z)$ *defined by*

$$\widehat{P}(Z) := \frac{1}{1 + \langle \mathbf{y}, \mathbf{y}\rangle_\eta} \begin{bmatrix} I_{2^n} & \overline{Z} \\ Z & \langle \mathbf{y}, \mathbf{y}\rangle_\eta I_{2^n} \end{bmatrix}$$

satisfies

$$\widehat{P}(Z)^2 = \widehat{P}(Z), \qquad \widehat{P}(Z)^\sharp = \widehat{P}(Z), \tag{58}$$

where $\widehat{P}(Z) := \overline{\widehat{P}(Z)^T}$, *and*

$$[\widehat{P}(Z),\ \Box_{H^{2n+2}(3,2n)} \widehat{P}(Z)] = 0. \tag{59}$$

P r o o f follows by straightforward computation.

The operator $\widehat{P}(Z)$ parametrizes a point of the Grassmannian manifold

$$U(2^n, 2^n)/U(2^{n-1}, 2^{n-1}) \times U(2^{n-1}, 2^{n-1}). \tag{60}$$

Then from the equation (59) we can immediately see that $P : Z \mapsto P(Z)$ defines a harmonic mapping between the hyperboloid $H^{2n+2}(3, 2n)$ and the manifold (60).

For $n = 1$ we get the mapping $\widehat{P} : H^4(3, 2) \to U(2, 2)/U(1, 1) \times U(1, 1)$. The unitary group $U(2, 2)$ is precisely the isometry group of the twistor space (which is $\mathbb{C}^4$ endowed with the hermitian form of signature $(+, +, -, -)$).

12. Complex Hessian involving convex functions

We now apply Theorems 3–6 to the study of the complex Hessians of convex functions.

PROPOSITION 14. *Consider p convex functions $v_j : \widetilde{D} \to \mathbb{R}$, $j = 1, \ldots, p$, $\widetilde{D} \subset \mathbb{R}^n$, $n \geq p$, and the mapping $\pi : D \to \widetilde{D}$, $D \subset \mathbb{C}^n$, defined by*

$$\pi(z_1, \ldots, z_n) := (\log|z_1|, \ldots, \log|z_n|), \qquad (z_1, \ldots, z_n) \in D. \tag{61}$$

Then

$$\prod_{j=1}^{p} dd^c \pi * v_j = 0$$

if and only if, for each $\{m_1, \ldots, m_p\} \subset \{1, 2, \ldots, n\}$, $m_1 < \ldots < m_p$, we have

$$\sum_{\sigma,\tau} \operatorname{sgn}\sigma \operatorname{sgn}\tau \frac{\partial^2 v_1}{\partial x_{\sigma(m_1)} \partial x_{\tau(m_1)}} \cdot \ldots \cdot \frac{\partial^2 v_p}{\partial x_{\sigma(m_p)} \partial x_{\tau(m_p)}} = 0, \tag{62}$$

where the summation is over all permutations σ, τ of $(m_1, \ldots, m_p)$.

Proof. Since

$$\begin{aligned} dd^c v_j &= 2i\partial\overline{\partial} v_j = 2i\partial\Big(\sum_{k,l} v_{j|k} x_{k|\overline{k}l} d\overline{z}_l\Big) \\ &= i\partial\Big(\sum_{l} v_{j|l} \overline{z}_l^{-1}\Big) = \frac{1}{2} i \sum_{k,l} v_{j|kl} z_k^{-1} dz_k \wedge \overline{z}_l^{-1} d\overline{z}^l, \end{aligned}$$

where $x_k = \log|z_k|$, $k = 1, \ldots, n$, we get, with the notation $\sum_{k_1,\ldots}^{j_1,\ldots} := \sum_{j_1,\ldots,k_1,\ldots}$,

$$\begin{aligned} \prod_{j=1}^{p} dd^c v_j &= (\tfrac{1}{2} i)^p \sum_{k_1 < \ldots < k_p}^{j_1 < \ldots < j_p} v_{1|j_1 k_1} \cdot \ldots \cdot v_{p|j_p k_p} \cdot z_{j_1}^{-1} dz_{j_1} \wedge \overline{z}_{k_1}^{-1} d\overline{z}_{k_p} \\ &= (\tfrac{1}{2})^p \sum_{k_1 < \ldots < k_p}^{j_1 < \ldots < j_p} v_{1|j_1 k_1} \cdot \ldots \cdot v_{p|j_p k_p} dx_{j_1} \wedge d\theta_{k_1} \wedge \ldots \wedge dx_{j_p} \wedge d\theta_{k_p}, \end{aligned}$$

where $\theta_k = \arg z_k$, $k = 1, \ldots, n$. Since dx_j and dx_k, $j, k = 1, \ldots, n$, are linearly independent, we conclude that

$$\prod_{j=1}^{p} dd^c v_j = (\tfrac{1}{2})^p \sum_{j_1 < \ldots < j_p} dv_{1|j_1} \wedge \ldots \wedge dv_{p|j_p} \wedge d\theta_{j_1} \wedge \ldots \wedge d\theta_{j_p}. \tag{63}$$

Thus the above form vanishes if and only if the condition (62) holds.

PROPOSITION 15. *Suppose that D is an arbitrary domain in $\mathbb{C}^n$, $n \geq 2$. Consider p convex functions*

$$v_j : \widetilde{D} \to \mathbb{R}, \qquad j = 1, \ldots, p; \;\; \widetilde{D} \subset \mathbb{R}^n, \;\; 2 \leq p \leq n, \tag{64}$$

and the mapping $\pi : \widetilde{D} \to D$ defined by (61). Suppose that

(i) *For each* $\{m_1,\ldots,m_p\}\subset\{1,2,\ldots,n\}$, $m_1<\ldots<m_p$, *we have*

$$\sum_\sigma \operatorname{sgn}\sigma dv_{1|\sigma(m_1)}\wedge\ldots\wedge dv_{p|\sigma(m_p)}=0,$$

where the summation is over all permutations of $(m_1,\ldots,m_p)$.

(ii) *For every* $x\in\pi[D]$ *there exist a number* $r\in\{1,2,\ldots,p\}$ *and a sequence* $\{n_1,\ldots,n_{p-1}\}\subset\{1,2,\ldots,n\}$, $n_1<\ldots<n_{p-1}$, *such that*

$$\sum_\tau \operatorname{sgn}\tau dv_{1|\tau(n_1)}\wedge\ldots\wedge dv_{r-1|\tau(n_{r-1})}\wedge dv_{r+1|\tau(n_r)}\wedge\ldots\wedge dv_{p|\tau(n_{p-1})}\neq 0,$$

where the summation is over all permutations τ *of* $(n_1,\ldots,n_{p-1})$.

Then there exists a foliation $\mathcal{L}_{p-1}$ *of* D *by complex submanifolds of* D *of codimension* $p-1$ *with the property that for every leaf* $M\in\mathcal{L}_{p-1}$ *the functions* $\pi^*v_k|M$ *are pluriharmonic, but* $(\partial/\partial z_j)(\pi^*v_k)|M$ *are holomorphic on* M *for each* $j=1,\ldots,n$ *and* $k=1,\ldots,p$.

P r o o f. Proposition 15 follows directly from Theorem 2 and Proposition 14.

R e m a r k 11. In Proposition 15, $\pi[\mathcal{L}_{p-1}]$ can be taken as a real foliation of $\pi[D]$.

By Theorem 3 and the formula (63), Proposition 15 can be generalized as follows.

THEOREM 7. *Suppose that* D *is an arbitrary domain in* $\mathbb{C}^n$, $n\geq 2$. *Consider* p *convex functions* (44) *and the mapping* $\pi:D\to\widetilde{D}$ *defined by* (61). *Let further* Ω *be a holomorphically decomposable* (k,k)*-form*, $k<n$, *of class* $C^3(D)$ *satisfying in* D *the conditions*:

(a) *For each* $\{m_1,\ldots,m_p\}\subset\{1,2,\ldots,n\}$, $m_1<\ldots<m_p$, *we have*

$$\Omega\wedge\sum_\sigma \operatorname{sgn}\sigma dv_{1|\sigma(m_1)}\wedge\ldots\wedge dv_{p|\sigma(m_p)}=0,\qquad 2\leq k+p\leq n,$$

and

$$d\Omega\in\sum_\sigma \operatorname{ideal}[\Omega, dv_{1|\sigma(m_1)}\wedge d\arg z_{\sigma(m_1)},\ldots,dv_{p|\sigma(m_p)}\wedge d\arg z_{\sigma(m_p)}]\qquad \text{if } k=1,$$

$$d\Omega=0\qquad \text{if } k\neq 1,$$

where the summations are over all permutations σ *of* $(m_1,\ldots,m_p)$.

(b) *For every* $x\in\pi[D]$ *there exist a number* $r\in\{1,2,\ldots,p\}$ *and a sequence* $\{n_1,\ldots,n_{p-1}\}\subset\{1,2,\ldots,n\}$, $n_1<\ldots<n_{p-1}$, *such that*

$$\Omega\wedge\sum_\tau dv_{1|\tau(n_1)}\wedge\ldots\wedge dv_{r-1|\tau(n_{p-1})}\wedge dv_{r+1|\tau(n_r)}\wedge\ldots\wedge dv_{p|\tau(n_{p-1})}\neq 0,\tag{65}$$

where the summation is over all permutations τ *of* $(n_1,\ldots,n_{p-1})$.

Then there exists a foliation $\mathcal{L}_{k+p-1}$ by complex submanifolds of D of codimension $k+p-1$ with the property that for every leaf $M \in \mathcal{L}_{k+p-1}$ the functions $\pi^* v_l|M$ are pluriharmonic, but $(\partial/\partial z_j)(\pi^* v_l)|M$ are holomorphic on M for each $j = 1, \ldots, n$ and $l = 1, \ldots, p$.

Proof. Theorem 7 follows directly from Theorem 3 and Propositions 14 and 15.

Remark 12. In Theorem 7, $\pi[\mathcal{L}_{k+p-1}]$ can be taken as a real foliation of $\pi[D]$.

Now, Theorems 4 and 7 yield

COROLLARY 6. *Consider an arbitrary domain D, arbitrary functions* (64) *and* (61), *and an arbitrary nonnegative form Ω satisfying all the hypotheses of Theorem* 7. *In addition, suppose that the integer $r = r(x)$ and the sequence $(n_1, \ldots, n_{p-1})$ appearing in the condition* (b) *of that theorem are independent of $x \in \pi[D]$. Then to any positive current θ of type $(n-1, n-1)$, defined, for $z \in D$, by*

$$\theta := (\tfrac{1}{2})^p (dd^c|z|^2)^{n-k-p} \wedge \Omega \wedge \sum_{j_1<\ldots<j_p} dv_{1|j_1} \wedge d\arg z_{j_1} \wedge \ldots \wedge dv_{p|j_p} \wedge d\arg z_{j_p}, \tag{66}$$

there corresponds a unique diffusion X_t^θ determined by a family of transition functions (32) *with the property* (33).

COROLLARY 7. *Under the hypotheses of Corollary* 6, *to any foliation $\mathcal{L}_{k+p-1}$ described in Theorem* 7 *there corresponds a unique diffusion X_t^θ, that is, a Markov process with continuous sample paths, determined by a family of transition functions* (32) *with the property* (33).

Similarly, Theorems 5 and 7 and Corollary 6 yield

COROLLARY 8. *Consider any foliation $\mathcal{L}_{k+p-1}$ determined in Theorem* 7 *and let M be its arbitrary leaf. Then the sample paths of the diffusion X_t^θ, determined uniquely in Corollary* 6 *by that foliation, remain invariant on M.*

COROLLARY 9. *Under the hypotheses of Theorem* 7, *if the form Ω is nonnegative and the integer r as well as the sequence $(n_1, \ldots, n_{p-1})$ do not depend on $x \in \pi[D]$, then $L^\theta = \Delta_{\mathcal{L}}$, where $\mathcal{L} = \mathcal{L}_{k+p-1}$ is the foliation determined in that theorem and $\Delta_{\mathcal{L}}$ is given by* (49).

COROLLARY 10. *Consider an arbitrary domain D, arbitrary functions* (64), *and an arbitrary nonnegative form Ω satisfying all the hypotheses of Theorem* 7. *In addition, suppose that the integer $r = r(x)$ and the sequence $(n_1, \ldots, n_{p-1})$ in* (65) *do not depend on $x \in \pi[D]$. Let, further, θ be any positive current of type*

$(n-1, n-1)$ *of the form* (66). *Suppose that*

$$\theta \wedge \sum_{\tau} \operatorname{sgn} \tau dv_{1|\tau(n_1)} \wedge \ldots \wedge dv_{r-1|\tau(n_{r-1})}$$
$$\wedge dv_{r+1|\tau(n_r)} \wedge \ldots \wedge dv_{p|\tau(n_{p-1})}(z) \neq 0, \qquad z \in D.$$

Then there exists a unique diffusion X_t^θ *determined by a family of transition functions* (32) *with the property* (33). *The diffusion* X_t^θ *is the canonical diffusion with the generator* L^θ *defined by* (30) *which is the Laplace–Beltrami operator oneach leaf* M *of any foliation* $\mathcal{L} = \mathcal{L}_{k+p-1}$ *determined in Theorem* 7. *Moreover,* X_t^θ *is uniquely determined by* $\mathcal{L}$ *and the sample paths of* X_t^θ *remain invariant on* M.

13. Some examples of applications

Example 7. Consider the function

$$u(z_1, z_2, z_3) = \log(|z_1|^2 + |z_2|^2) + |z_3|^2$$

in $D = \{(z_1, z_2, z_3) \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 \neq 0\}$.

With the notation $R^2 := |z_1|^2 + |z_2|^2$ we get

$$\begin{aligned} dd^c u &= 2i\partial\overline{\partial} u = 2i\partial\overline{\partial} \log R^2 + 2i\partial\overline{\partial}|z_3|^2 \\ &= 2i\partial[r^{-2}(z_1 d\overline{z}_1 + z_2 d\overline{z}_2)] + 2i dz_3 \wedge d\overline{z}_3 \\ &= 2i[R^{-2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + R^2 dz_3 \wedge d\overline{z}_3) \\ &\quad - R^{-4}(\overline{z}_1 dz_1 + \overline{z}_2 dz_2) \wedge (z_1 d\overline{z}_1 + z_2 d\overline{z}_2)], \end{aligned}$$

i.e.,

$$\begin{aligned} dd^c u = 2iR^{-4}(&|z_2|^2 dz_1 \wedge d\overline{z}_1 - z_2\overline{z}_1 dz_1 \wedge d\overline{z}_2 \\ &- z_1\overline{z}_2 dz_2 \wedge d\overline{z}_2 + |z_1|^2 dz_2 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3). \end{aligned} \tag{67}$$

Hence

$$(dd^c u)^3 = -8iR^{-12} \begin{vmatrix} R^2 - |z_1|^2 & -z_2\overline{z}_1 \\ -z_2\overline{z}_2 & R^2 - |z_1|^2 \end{vmatrix} dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 = 0$$

in D. On the other hand, by (67), in D, we have

$$\begin{aligned} (dd^c u)^2 = &-4R^{-8}(|z_2|^2 dz_1 \wedge d\overline{z}_1 - z_2\overline{z}_1 dz_1 \wedge d\overline{z}_2 \\ &- z_1\overline{z}_2 dz_2 \wedge d\overline{z}_1 + |z_1|^2 dz_2 \wedge d\overline{z}_2) dz_3 \wedge d\overline{z}_3 \neq 0. \end{aligned}$$

Thus, in order to find the corresponding foliation $\mathcal{L}$ we have to determine the annihilator of dd^c. Let X be as in (23) with $n = 3$ and $X \rfloor dd^c u = 0$. The latter relation is equivalent to

$$C_1[(R^2|z_1|^2)d\overline{z}_1 - z_2\overline{z}_1 d\overline{z}_2] + C_2[-z_1\overline{z}_2 d\overline{z}_1 + (R^2 - |z_2|^2)d\overline{z}_2] + C_3 R^4 d\overline{z}_3 = 0,$$

and hence

$$C_1|z_2|^2 - C_2 z_1\overline{z}_2 = 0, \quad -C_1 z_2\overline{z}_1 + C_2|z_1|^2 = 0, \quad C_3 = 0,$$

so $C_2 = C_1 z_2/z_1$, $C_3 = 0$, and

$$X = C_1\left(\frac{\partial}{\partial z_1} + \frac{z_2}{z_1}\frac{\partial}{\partial z_2}\right) \quad \text{in } D \text{ for } z_1 \neq 0,$$

while $C_1 = C_2 z_1/z_2$, $C_3 = 0$, and

$$X = C_2\left(\frac{z_1}{z_2}\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) \quad \text{in } D \text{ for } z_2 \neq 0.$$

In the first case C_1 is an arbitrary complex constant and we get leaves $M_{C_2/C_1,C_3}$ of the foliation $\mathcal{L}$, satisfying the equations

$$(d/dt)z_1 = 1, \quad (d/dt)z_2 = z_2/z_1, \quad (d/dt)z_3 = 0,$$

i.e.,

$$\begin{aligned}(68)\quad & M_{C_2/C_1,C_3}\\ &= \{(z_1,z_2,z_3) \in D : z_1 = t + C_0,\ z_2 = (C_2/C_1)(t+C_0),\ z_3 = C_3\}.\end{aligned}$$

In the second case C_2 is an arbitrary complex constant and we get, correspondingly,

$$(d/dt)z_1 = z_1/z_2, \quad (d/dt)z_2 = 1, \quad (d/dt)z_3 = 0,$$

so the leaves are

$$\begin{aligned}(69)\quad & M'_{C_1/C_2,C_3}\\ &= \{(z_1,z_2,z_3) \in D : z_1 = (C_1/C_2)(t+C_0),\ z_2 = t + C_0,\ z_3 = C_3\}.\end{aligned}$$

The foliation $\mathcal{L}$ coincides with the foliation $\mathcal{L}_2$ provided by Theorem 7.

Next, for

$$\theta = (dd^c|z|^2) \wedge dd^c u \tag{70}$$

we have $d\theta = 0$, so the condition (30) for the E_θ-harmonicity of a form $\psi \in C_0^2(D)$ reduces to

$$(L_\theta \psi) dV = dd^c\psi \wedge \theta = 0. \tag{71}$$

Explicitly,

$$dd^c|z|^2 = 2i(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3),$$

so, by (67), we get

$$\begin{aligned}\theta = &-8R^{-4}[R^2 dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + (R^4 + |z_2|^2) dz_1 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3\\ &- z_2\overline{z}_1 dz_1 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_4 - z_1\overline{z}_2 dz_2 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3\\ &+ (R^4 + |z_1|^2) dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3].\end{aligned}$$

Therefore the condition (71) for the E_θ-harmonicity of ψ reads

$$\begin{aligned}(L_\theta\psi)dv = 8iR^{-1}[&(R^4 + |z_1|^2)\psi_{|1\overline{1}} + z_1\overline{z}_2\psi_{|1\overline{2}}\\ &+ z_2\overline{z}_1\psi_{|2\overline{1}} + (R^4 + |z_2|^2)\psi_{|2\overline{2}}\\ &+ R^2\psi_{|3\overline{3}}] dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 = 0,\end{aligned}$$

where, as before, $\psi_{|j} = \partial\psi/\partial z_j$ and $\psi_{|\bar{j}} = \partial\psi/\partial \overline{z}_j$. Consequently, the associated operator L_θ is given by

$$L_\theta = \frac{2}{R^4}\left[(R^4+|z_1|^2)\frac{\partial^2}{\partial z_1\partial\overline{z}_1} + z_1\overline{z}_2\frac{\partial^2}{\partial z_1\partial\overline{z}_2} + z_2\overline{z}_1\frac{\partial^2}{\partial z_2\partial\overline{z}_1} + (R^4+|z_2|^2)\frac{\partial^2}{\partial z_2\partial\overline{z}_2} + R^2\frac{\partial^2}{\partial z_3\partial\overline{z}_3}\right].$$

The process associated with L_θ, given by the expression in square brackets, is the solution to the following system of stochastic differential equations:

$$dz_{1t} = z_{1t}db_t, \quad dz_{2t} = z_{2t}db_t, \qquad dz_{3t} = 0, \tag{72}$$

where b_t is the stochastic complex Brownian motion. The system (72) can be solved effectively:

$$z_{1t} = C_1' \exp b_t, \quad z_{2t} = C_2' \exp b_t, \quad z_{3t} = C_3',$$

where C_1', C_2' and C_3' are arbitrary complex constants. This means that the sample paths of the unique diffusion $X_t^\theta = (z_{1t}, z_{2t}, z_{3t})$, described in Corollary 10, remain to diffuse on the leaf (68) of the foliation $\mathcal{L}$ of D for $z_1 \neq 0$ and on the leaf (69) of $\mathcal{L}$ for $z_2 \neq 0$.

Example 8. Consider the function

$$u(z_1,z_2,z_3) = [(\log|z_1|)^2 + (\log|z_2|)^2 + (\log|z_3|)^2]/\log|z_3|$$

in $D = \{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_3|>1,\ z_1z_2\neq 0\}$.

With the notation

$$v(x_1,x_2,x_3) := u(z_1,z_2,z_3), \qquad x_j = \log|z_j|,\ \ j=1,2,3,$$

we get

$$v(x_1,x_2,x_3) = (x_1^2+x_2^2)x_3^{-1} + x_3, \qquad \det[v_{|jk}] = 64|z_1z_2z_3|^2\det[u_{|j\bar{k}}],$$

and

$$[v_{|jk}] = \begin{bmatrix} 2x_3^{-1} & 0 & -2x_1x_3^{-2} \\ 0 & 2x_3^{-1} & -2x_2x_3^{-2} \\ -2x_1x_3^{-2} & -2x_2x_3^{-2} & 2(x_1^2+x_2^2)x_3^{-3} \end{bmatrix}. \tag{73}$$

Hence

$$\det[v_{|jk}] = \begin{vmatrix} 2x_3^{-1} & 0 & -2x_1x_3^{-2} \\ 0 & 2x_3^{-1} & -2x_2x_3^{-2} \\ 0 & -2x_2x_3^{-1} & 2x_2^2x_3^{-3} \end{vmatrix} = \begin{vmatrix} 2x_3^{-1} & 0 & -2x_1x_3^{-2} \\ 0 & 2x_3^{-1} & -2x_2x_3^{-2} \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{in } D,$$

so $(dd^c u)^4 = 0$ in D. On the other hand, $(dd^c u)^2 \neq 0$ in D. In fact, $\partial u = \sum_{j,k} v_{|j} x_{j|k} dz_k$ and

$$dd^c u = -2i \sum_{j,k,m} \Big(\sum_l v_{|jl} x_{l|\overline{m}} x_{j|k} + v_{|j} x_{j|k\overline{m}} \Big) dz_k \wedge d\overline{z}_m,$$
$$x_{l|\overline{m}} = \tfrac{1}{2} \overline{z}_m^{-1} \delta_{lm}, \qquad x_{j|k} = \tfrac{1}{2} z_k^{-1} \delta_{jk}, \qquad x_{j|k\overline{m}} = 0,$$

where δ_{jk} is the Kronecker symbol, and

$$v_{|1} = 2 \log |z_1| / \log |z_3|, \qquad v_{|2} = 2 \log |z_2| / \log |z_3|,$$
$$v_{|3} = 1 - [(\log |z_1|)^2 + (\log |z_2|)^2] / (\log |z_3|)^2.$$

Hence

$$\sum_{j,l} v_{|jl} x_{l|\overline{m}} x_{j|k} = \tfrac{1}{4} v_{|km} (z_k z_{\overline{m}})^{-1}$$

and

$$\begin{aligned}\sum_{j,k,m} v_{|j} x_{j|k\overline{m}} dz_k \wedge d\overline{z}_m = \sum_{k,m} & [2(\log |z_1| / \log |z_3|)(-\tfrac{1}{2} \delta_{1k} \delta_{km}) \\ & + 2(\log |z_2| / \log |z_3|)(-\tfrac{1}{2} \delta_{2k} \delta_{km}) \\ & + \{1 - [(\log |z_1|)^2 + (\log |z_2|)^2] / (\log |z_3|)^2\} \\ & \cdot (-\tfrac{1}{2} \delta_{3k} \delta_{km})] \overline{z}_k^{-2} dz_k \wedge d\overline{z}_m = 0.\end{aligned}$$

Consequently, by (73),

$$(dd^c u)^2 = \frac{1}{4} \Big(\sum_{j,k} v_{|jk} z_j^{-1} dz_j \wedge \overline{z}_k^{-1} d\overline{z}_k \Big)^2 \neq 0 \qquad \text{in } D.$$

Thus, in order to find the corresponding foliation $\mathcal{L}$ we have to determine the annihilator of $dd^c u$. Let X be as in (23) with $n = 3$ and $X \rfloor dd^c u = 0$. The latter relation is equivalent to

$$\sum_{j,k} C_j v_{|jk} (z_j \overline{z}_k)^{-1} d\overline{z}_k = 0$$

and hence, as $\overline{z}_k \neq 0$ in D,

$$\sum_j C_j v_{|jk} z_j^{-1} = 0 \qquad \text{for } k = 1, 2, 3,$$

so, with the notation $A_j = C_j / z_j$ for $j = 1, 2, 3$, we get

$$2x_3^{-1} A_1 - 2x_1 x_3^{-2} A_3 = 0, \qquad 2x_3^{-1} A_2 - 2x_2 x_3^{-2} A_3 = 0,$$
$$-2x_1 x_3^{-2} A_1 - 2x_2 x_3^{-2} A_2 + 2(x_1^2 + x_2^2) x_3^{-3} = 0.$$

This yields

$$A_2 = (x_2 / x_3) A_3 = (x_2 / x_1) A_1.$$

Thus

$$C_2 = C_1 \frac{z_2 x_2}{z_1 x_1} = C_1 \frac{z_2 \log|z_2|}{z_1 \log|z_1|}, \qquad C_3 = C_1 \frac{z_3 x_3}{z_1 x_1} = C_1 \frac{z_3 \log|z_3|}{z_1 \log|z_1|}$$

and

$$X = C_1 \left(\frac{\partial}{\partial z_1} + \frac{z_2 \log|z_2|}{z_1 \log|z_1|} \frac{\partial}{\partial z_2} + \frac{z_3 \log|z_3|}{z_1 \log|z_1|} \frac{\partial}{\partial z_3} \right),$$

where C_1 is an arbitrary complex constant.

Let $\widetilde{X} = z_1 \log|z_1| X$. Then

$$\widetilde{X} = C_1[z_1 \log|z_1|(\partial/\partial z_1) + z_2 \log|z_2|(\partial/\partial z_2) + z_3 \log|z_3|(\partial/\partial z_3)],$$

so for $k = 1, 2, 3$ we have $(d/dt)z_k = z_k \log|z_k|$, i.e.,

$$(1/\log|z_k|)d(\log z_k) = dt.$$

Further, let $t = r + is$ and $\log z_k = \alpha_k + i\beta_k$. Then we get $d\alpha_k = \alpha_k ds$, $d\beta_k = \beta_k ds$, which gives

$$\alpha_k = e^{r+c_k} \quad \text{and} \quad \beta_k = \int e^{r+c_k} ds + c_k'$$

with real constants c_k and c_k'. Thus

$$\log z_k = e^{r+c_k} + i\Big(\int e^{r+c_k} ds + c_k' \Big) = e^{c_k}\Big(e^r + i \int e^r ds \Big) + ic_k'.$$

Hence

$$e^{-c_1}(\log z_1 - ic_1') = e^{-c_2}(\log z_2 - ic_2') = e^{-c_3}(\log z_3 - ic_3'),$$

so, with $z_1 = \zeta$, we obtain

$$\log z_2 = ic_2' + e^{c_2 - c_1}(\log\zeta - ic_1'), \qquad \log z_3 = ic_3' + e^{c_3 - c_1}(\log\zeta - ic_1').$$

Therefore, we finally get

$$z_2 = e^{i\alpha} z_1^{\gamma}, \qquad z_3 = e^{i\beta} z_1^{\delta},$$

where α, β, γ, and δ are positive real constants. The equations describe leaves of the foliation $\mathcal{L}$ which coincides with the foliation $\mathcal{L}_2$ provided by Theorem 7.

Next, for θ we have $d\theta = 0$, so the condition (30) for the E_θ-harmonicity of a form $\psi \in C_0^2(D)$ reduces to (71). Explicitly,

$$\theta = -4(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3) \wedge \frac{1}{4} \sum_{j,k} v_{|jk} z_j^{-1} dz_j \wedge d\overline{z}_k^{-1} d\overline{z}_k,$$

and hence

$$\begin{aligned} \theta = & -\frac{2}{x_3} \Bigg[\left(\frac{1}{|z_1|^2} + \frac{1}{|z_2|^2} \right) dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \\ & + \left(\frac{1}{|z_1|^2} + \frac{x_1^2 + x_2^2}{x_3^2} \cdot \frac{1}{|z_3|^2} \right) dz_1 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3 \\ & + \left(\frac{x_1^2 + x_2^2}{x_3^2} \cdot \frac{1}{|z_3|^2} + \frac{1}{|z_2|^2} \right) dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 \Bigg]. \end{aligned}$$

Therefore the condition (71) for the E_θ-harmonicity of ψ reads

$$(L_\theta\psi)dV = \frac{4i}{x_3}\bigg[\bigg(\frac{1}{|z_2|^2}+\frac{x_1^2+x_2^2}{x_3^2}\cdot\frac{1}{|z_3|^2}\bigg)\psi_{|1\overline{1}}+2\frac{x_1}{x_3^2}\cdot\frac{1}{z_3\overline{z}_1}\psi_{|1\overline{3}} + \bigg(\frac{1}{|z_1|^2}+\frac{x_1^2+x_2^2}{x_3^2}\cdot\frac{1}{|z_3|^2}\bigg)\psi_{|2\overline{2}}+2\frac{x_2}{x_3^2}\cdot\frac{1}{z_3\overline{z}_2}\psi_{|2\overline{3}}+2\frac{x_1}{x_3^2}\cdot\frac{1}{z_1\overline{z}_3}\psi_{|3\overline{1}} + 2\frac{x_2}{z_2\overline{z}_3}\psi_{|3\overline{2}}+\bigg(\frac{1}{|z_1|^2}+\frac{1}{|z_2|^2}\bigg)\psi_{|3\overline{3}}\bigg]dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2\wedge dz_3\wedge d\overline{z}_3.$$

Consequently, the associated operator L_θ is given by

$$L_\theta = -\frac{1}{\log|z_3|}\bigg\{\bigg[\frac{1}{|z_2|^2}+\frac{(\log|z_1|)^2+(\log|z_2|)^2}{(\log|z_3|)^2}\cdot\frac{1}{|z_3|^2}\bigg]\frac{\partial^2}{\partial z_1\partial\overline{z}_1} + 2\frac{\log|z_1|}{(\log|z_3|)^2}\cdot\frac{1}{z_3\overline{z}_1}\cdot\frac{\partial^2}{\partial z_1\partial\overline{z}_3} + \bigg[\frac{1}{|z_1|^2}+\frac{(\log|z_1|)^2+(\log|z_2|)^2}{(\log|z_3|)^2}\cdot\frac{1}{|z_3|^2}\bigg]\frac{\partial^2}{\partial z_2\partial\overline{z}_2} + 2\frac{\log|z_2|}{(\log|z_3|)^2}\cdot\frac{1}{z_3\overline{z}_2}\cdot\frac{\partial^2}{\partial z_2\partial\overline{z}_3}+2\frac{\log|z_1|}{(\log|z_3|)^2}\cdot\frac{1}{z_1\overline{z}_3}\cdot\frac{\partial^2}{\partial z_3\partial\overline{z}_1} + 2\frac{\log|z_2|}{(\log|z_3|)^2}\cdot\frac{1}{z_2\overline{z}_3}\cdot\frac{\partial^2}{\partial z_3\partial\overline{z}_2}+\bigg(\frac{1}{|z_1|^2}q+\frac{1}{|z_2|^2}\bigg)\frac{\partial^2}{\partial z_3\partial\overline{z}_3}\bigg\}.$$

14. Hypersurfaces in $\mathbb{C}^3$ depending on two holomorphic functions

Finally, we shall investigate a particular case of Theorem 7 and Corollary 10, depending on two holomorphic functions in $\mathbb{C}^3$. We have

PROPOSITION 16. *Suppose that D is an arbitrary domain in $\mathbb{C}^3$ and $u : D \to \mathbb{R}$ is a function of the form*

$$u = |f|^2 + |g|^2, \tag{74}$$

where $dd^c|f|^2 \wedge dd^c|g|^2 \neq 0$, f and g being holomorphic.

(i) *Then the foliation $\mathcal{L}$ with leaves having, for a point $z_0 \in D$, the form*

$$M^{z_0}_{C_0^1,C_0^2} = \{z \in D : f(z) = C_0^1,\ g(z) = C_0^2\}_{\text{con}}, \qquad C_0^1 = f(z_0),\quad C_0^2 = g(z_0), \tag{75}$$

where $\{\ \}_{\text{con}}$ denotes the convex component of the set $\{\ \}$ containing z_0, has the property that for every leaf $M \in \mathcal{L}$ the function $u|M$ is pluriharmonic, but $(\partial/\partial z_j)u|M$ are holomorphic on M for each $j = 1, 2, 3$.

(ii) *Let, further, θ denote the positive current* (70). *Then there exists a unique diffusion X_t^θ determined by a family of transition functions* (32) *with the property* (33). *The diffusion X_t^θ is the canonical diffusion with generator L^θ defined by* (40) *which is the Laplace–Beltrami operator on each leaf M of $\mathcal{L}$. Moreover, X_t^θ is uniquely determined by $\mathcal{L}$ and the sample paths of X_t^θ remain to diffuse on M, i.e. they remain invariant on M.*

Proof. Clearly, $u_{|j\overline{k}} = f_{|j}\overline{f_{|k}} + g_{|j}\overline{g_{|k}}$ and $\det[u_{|j\overline{k}}] = 0$ in D, so $(dd^c u)^3 = 0$ in D. By (74) and $dd^c = 2i\partial\overline{\partial}$ we have

$$dd^c u = dd^c|f|^2 + dd^c|g|^2 = 2i(\partial f \wedge \overline{\partial f} + \partial g \wedge \overline{\partial g})$$

and

$$(dd^c u)^2 = -8(\partial f \wedge \overline{\partial f}) \wedge (\partial g \wedge \overline{\partial g}) \neq 0 \quad \text{in } D. \tag{76}$$

Thus, in order to find the corresponding foliation $\mathcal{L}$ we have to determine the annihilator of $dd^c u$. Let X be as in (23) with $n = 3$ and $X \rfloor dd^c u = 0$. The latter relation is equivalent to

$$\begin{aligned} &\overline{\partial f}[\partial f(X)] + \overline{\partial g}[\partial g(X)] = 0, \qquad \text{i.e.}\\ &\overline{f_{|j}}[\partial f(X)] + \overline{g_{|j}}[\partial g(X)] = 0, \qquad j = 1, 2, 3. \end{aligned} \tag{77}$$

Hence

$$C^1(\overline{f_{|j}} f_{|1} + \overline{g_{|j}} g_{|1}) + C^2(\overline{f_{|j}} f_{|2} + \overline{g_{|j}} g_{|2}) + C^3(\overline{f_{|j}} f_{|2} + \overline{g_{|j}} g_{|2}) = 0.$$

The determinant of this system is equal to $\det[u_{|j\overline{k}}] = 0$. Since, by (74), ∂f and ∂g do not vanish in D, it follows by (77) that $|\partial f(X)|^2 + |\partial g(X)|^2 = 0$, so $\partial f(X) = 0$ and $\partial g(X) = 0$. This means that

$$C^1 f_{|1} + C^2 f_{|2} + C^3 f_{|3} = 0, \qquad C^1 g_{|1} + C^2 g_{|2} + C^3 g_{|3} = 0.$$

Since, by (74), $|\mathrm{grad}_{\mathbb{C}} f|^2 \neq 0$ and $|\mathrm{grad}_{\mathbb{C}} g|^2 \neq 0$ in D, we see that

$$X \perp \mathrm{span}\{\mathrm{grad}_{\mathbb{C}} f, \mathrm{grad}_{\mathbb{C}} g\}.$$

Moreover, by (74), for every $z \in D$, $\mathrm{grad}_{\mathbb{C}} f$ and $\mathrm{grad}_{\mathbb{C}} g$ are linearly independent:

$$\mathrm{grad}_{\mathbb{C}} f = \sum_{j=1}^{3} f_{|j}(\partial/\partial\overline{z}_j).$$

Therefore the required foliation $\mathcal{L}$ has the form (75) for every point $z_0 \in D$, which completes the proof of (i).

The assertion (ii) is a consequence of Corollary 10. We can prove it directly. By applying the Dynkin formula (cf., e.g., [18], p. 232) to $|f - f(z_0)|^2(X_t^\theta)$ with $X_t^\theta = z_0$, we get

$$E_{z_0}[|f(X_{t\zeta}^\theta) - f(z_0)|^2] = E_{z_0}\Big\{ \int_0^t [dd^c|f(z) - f(z_0)|^2 \wedge (dd^c u)^2(X_s^\theta)/dV]ds \Big\} = 0$$

with ζ being the first hitting time ([10], p. 91) of ∂D_0, where $D_0 \subset D$, $D_0 \neq D$, is a sufficiently small neighbourhood of z_0, by virtue of the Dynkin formula, applied again to $f^2(X_t^\theta)$ and $f(X_t^\theta)$. Yet,

$$dd^c|f - f(z_0)|^2 \wedge (dd^c u)^2 = dd^c|f - f(z_0)| \wedge dd^c|f|^2 \wedge dd^c|g|^2 = 0 \quad \text{in } D,$$

so we also have $E_{z_0}[|g(X_{t\zeta}^\theta) - g(z_0)|^2] = 0$, and this suffices to conclude the proof of (ii).

Since the foliation $\mathcal{L}$ and the canonical diffusion X_t^θ are constructed in Proposition 16 effectively, we arrive at some properties of holomorphic functions on hypersurfaces, eliminating the incovenient notions of foliations and canonical diffusions. (The same concerns Examples 7 and 8 in Section 13.) The corresponding equivalent reformulation of Proposition 16 reads:

THEOREM 8. *Suppose that D is an arbitrary domain in $\mathbb{C}^3$ and $u : D \to \mathbb{R}$ is a function of the form* (74).

(i) *Then u is pluriharmonic on every hypersurface of D of the form* (75), *where $\{\ \}_{\mathrm{con}}$ denotes the convex component of the set $\{\ \}$ containing z_0, $z_0 \in D$, but $(\partial/\partial z_j)u$ are holomorphic on every such hypersurface for each $j = 1, 2, 3$.*

(ii) *Let, further, θ denote the positive current* (70). *Consider any family of transition functions p_t^θ of the form* (32) *with the property*

$$I_t^\theta(z) := \int_D p_t(z,w)\phi(w)dV(w) = T_t\phi(z) \qquad \text{for } z, w \in D, t > 0,\ \ \phi \in L^2(D, dV), \tag{78}$$

where $\{T_t : t > 0\}$ is the equicontinuous semigroup of class (C_0) with the infinitesimal generator $L_\theta - \lambda I$ such that $T_0 = I$, $T_t = \exp(tL_\theta)$ for $t > 0$, L_θ is given by (30), *ψ in* (30) *ranges over $C_0^2(D)$, dV is the canonical Kähler form on $\mathbb{C}^3$, I is the identity operator, and λ is an arbitrary complex number with $|\arg\lambda| \le \pi - \varepsilon$, ε being fixed and positive. Then all the integrals* (78) *are independent of t, i.e.,*

$$I_t^\theta(z) = I_0^\theta(z) \quad \text{or, equivalently,} \quad I_t\phi(z) = \phi(z) \quad \text{for } z \in M^{z_0}_{C_0^1, C_0^2}$$

on every hypersurface $M^{z_0}_{C_0^1, C_0^2}$ of the form (75) *whenever ϕ is holomorphic on that hypersurface.*

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