

## ON THE APPROXIMATION OF THE SIGNORINI PROBLEM WITH FRICTION

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### Introduction

In practice we often meet problems in which one deformable body comes in contact with another. The deformation of each body depends not only on given known forces but also on contact tractions along contact surfaces which are not known a priori. Mathematical formulation of these problems leads to a variational inequality of elliptic type, i.e., to the problem of finding a minimizer of the functional of potential energy  $J$  over a closed, convex subset  $K$  of a Hilbert space  $V$ . Approximation of problems of such a type by finite elements have been studied by Oden and Kikuchi ([11]) and by Haslinger and Hlaváček ([5]–[7]).

In all these papers no friction is assumed (contact surfaces are supposed to be lubricated). The aim of this paper is to extend some known results to contact problems with friction. The main difficulty from the practical point of view is the fact that the functional  $J$  is not differentiable in general, so that a direct application of finite elements to the classical variational formulation is not a good way to its approximation. To overcome this difficulty we use the classical duality approach, by means of which the minimization problem for the non-differentiable functional  $J$  will be replaced by a saddle-point (or mixed) formulation for a Lagrange function  $\mathcal{Q}$ , smooth in all its components.

Approximation of contact problems with friction by the finite element method, based on the mixed formulation, will be studied.

For the sake of simplicity we restrict ourselves to the plane case where an *elastic* body  $\Omega$  is supported by a rigid foundation (the so-called Signorini problem). The extension of all results to the case of a finite number of elastic bodies in contact is straightforward.

### 1. Primal and mixed formulation of the Signorini problem with friction

Let  $\Omega \subset R_2$  be a bounded domain whose Lipschitz boundary  $\partial\Omega$  is decomposed as follows:

$$\partial\Omega = \Gamma_u \cup \Gamma_P \cup \Gamma_K,$$

where  $\Gamma_u$ ,  $\Gamma_P$ ,  $\Gamma_K$  are open parts in  $\partial\Omega$ , mutually disjoint,  $\Gamma_u$  and  $\Gamma_K$  are non-empty.

By a *classical solution* of the Signorini problem with friction we denote a displacement field  $u = (u_1, u_2)$  satisfying

$$(1) \quad \tau_{ij,j}(u) + F_i = 0 \quad \text{in } \Omega, \quad i = 1, 2 \quad (1)$$

where  $\tau_{ij}(u)$  is the  $i, j$ -th component of the stress tensor  $\tau(u)$ , corresponding to the strain tensor  $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$  by means of the linear Hooke law

$$\tau_{ij}(u) = c_{ijkl} \varepsilon_{kl}(u) \quad (\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})).$$

The coefficients  $c_{ijkl}$  satisfy the usual property of symmetry

$$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e. in } \Omega$$

and the ellipticity condition

$$\exists \alpha = \text{const} > 0: \quad c_{ijkl}(x) \xi_{ij} \xi_{kl} \geq \alpha \xi_{ij} \xi_{ij} \quad \text{a.e. in } \Omega \quad \forall \xi_{ij} = \xi_{ji}.$$

Moreover, the following system of boundary conditions for  $u$  is prescribed:

$$(2) \quad u = 0 \quad \text{on } \Gamma_u;$$

$$(3) \quad \tau_{ij}(u) n_j = P_i \quad \text{on } \Gamma_P, \quad i = 1, 2;$$

$$(4) \quad \begin{cases} u_n \equiv u \cdot n \leq 0, & T_n(u) \equiv \tau_{ij}(u) n_i n_j \leq 0, \\ u_n \cdot T_n(u) = 0 & \text{on } \Gamma_K; \end{cases}$$

$$(5) \quad \begin{cases} \text{for } \forall x \in \Gamma_K \text{ such that } u_n(x) = 0: \\ |T_t(u)(x)| \leq g(x); \\ |T_t(u)(x)| < g(x) \Rightarrow u_t \equiv u \cdot t = 0; \\ |T_t(u)(x)| = g(x) \Rightarrow \exists \lambda \geq 0, u_t = -\lambda T_t(u). \end{cases}$$

$F$  and  $P$  are given body forces and surface tractions, respectively.  $n, t$  denote the unit normal and the tangential vector to  $\partial\Omega$ , respectively and  $g$  is a given non-negative function.

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(1)  $\tau_{ij,j} \stackrel{\text{def.}}{=} \sum_{j=1}^2 \frac{\partial \tau_{ij}}{\partial x_j}$  (the summation convention will be used).

The set of conditions (4) are classical *unilateral* boundary conditions describing the fact that  $\Omega$  is unilaterally supported by a rigid foundation. (5) describes the simplest model, involving friction.

In order to give the variational formulation of our problem, we introduce the space  $V = V \times V$ , where

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_u\}$$

and the closed convex subset  $K$  of  $V$ :

$$K = \{v \in V \mid v_n \leq 0 \text{ on } \Gamma_K\}.$$

Finally, let

$$J(v) = \frac{1}{2}a(v, v) - L(v) + j(v),$$

with

$$a(v, v) = \int_{\Omega} \tau_{ij}(v) \varepsilon_{ij}(v) dx,$$

$$L(v) = \int_{\Omega} F_i v_i dx + \int_{\Gamma_P} P_i v_i ds$$

and

$$j(v) = \int_{\Gamma_K} g |v_t| ds.$$

We suppose that  $F \in (L^2(\Omega))^2$ ,  $P \in (L^2(\Gamma_P))^2$ ,  $g \in L^\infty(\Gamma_K)$ ,  $g \geq 0$ .

A function  $u \in K$  such that

$$(\mathcal{P}) \quad J(u) \leq J(v) \quad \forall v \in K$$

will be called a *variational solution* of the Signorini problem with friction.

It is easy to verify that  $(\mathcal{P})$  is equivalent to

$$(\mathcal{P}') \quad \text{to find } u \in K \text{ such that}$$

$$a(u, v - u) + j(v) - j(u) \geq L(v - u) \quad \forall v \in K.$$

Integrating by parts in  $(\mathcal{P}')$ , we formally obtain (1)–(5). It is not difficult to prove

**THEOREM 1.** *There exists a unique variational solution of the Signorini problem with friction.*

As already mentioned, formulation  $(\mathcal{P})$  is not suitable for numerical approximation. The main difficulty arises from the fact that  $J$  is not differentiable. To overcome this difficulty the classical duality approach will be used.

Clearly,

$$j(v) = \sup_{\mu \in \mathcal{A}} \int_{\Gamma_K} g \mu v_i ds,$$

where

$$\mathcal{A} = \{\mu \in L^2(\Gamma_K) \mid |\mu| \leq 1 \text{ on } \text{supp } g, \mu = 0 \text{ on } \Gamma_K \setminus \text{supp } g\}.$$

( $\mathcal{P}$ ) can be written in the following equivalent form:

$$(6) \quad \inf_K J(v) = \inf_K \sup_{\mathcal{A}} \mathfrak{L}(v, \mu),$$

where  $\mathfrak{L}: V \times \mathcal{A} \rightarrow \mathbb{R}_1$  is a Lagrange function given by

$$\mathfrak{L}(v, \mu) = \frac{1}{2} a(v, v) - L(v) + \int_{\Gamma_K} g \mu v_i ds.$$

Instead of ( $\mathcal{P}$ ), we shall consider the following problem:

( $\tilde{\mathcal{P}}$ ) to find  $(w, \lambda) \in K \times \mathcal{A}$  such that

$$\mathfrak{L}(w, \mu) \leq \mathfrak{L}(w, \lambda) \leq \mathfrak{L}(v, \lambda) \quad \forall v \in K, \mu \in \mathcal{A},$$

i.e.,  $(w, \lambda)$  is a saddle-point of  $\mathfrak{L}$  on  $K \times \mathcal{A}$ .

Using the partial differentiation of  $\mathfrak{L}$  with respect to  $v, \mu$ , we obtain the equivalent form of ( $\tilde{\mathcal{P}}$ ) (see [2]):

( $\tilde{\mathcal{P}}'$ ) to find  $(w, \lambda) \in K \times \mathcal{A}$  such that

$$a(w, v - w) + (g\lambda, v_i - w_i)_{0, \Gamma_K} \geq L(v - w) \quad \forall v \in K,$$

$$(g(\mu - \lambda), w_i)_{0, \Gamma_K} \leq 0 \quad \forall \mu \in \mathcal{A}.$$

( $\tilde{\mathcal{P}}$ ) (or ( $\tilde{\mathcal{P}}'$ )) will be called the *mixed* formulation of the Signorini problem with a given friction. The relation between ( $\tilde{\mathcal{P}}$ ) and ( $\mathcal{P}$ ) is given by

THEOREM 2. *There exists a unique solution  $(w, \lambda)$  of ( $\tilde{\mathcal{P}}$ ) and we have*

$$(7) \quad w = u, \quad g\lambda = T_t(u),$$

where  $u \in K$  is the solution of ( $\mathcal{P}$ ).

*Proof.* The existence of a solution of ( $\tilde{\mathcal{P}}$ ) is a direct consequence of Korn's inequality and the boundedness of  $\mathcal{A}$  (see also [2], Ch. VI, Remark 2.1). (7) follows from ( $\tilde{\mathcal{P}}'$ ) and Green's formula.

## 2. Approximation of $(\tilde{\mathcal{P}})$

Let  $\Omega \subset R_2$  be a *polygonal* bounded domain and let  $\{\mathcal{T}_h\}$ ,  $h \rightarrow 0+$  be a *regular family* of triangulations of  $\bar{\Omega}$  which is compatible with the decomposition of  $\partial\Omega$  into  $\Gamma_u$ ,  $\Gamma_P$  and  $\Gamma_K$ . Moreover, we shall suppose that  $\Gamma_K$  is a *straight line segment*.

With every  $\mathcal{T}_h$  we associate a finite-dimensional space  $V_h = V_h \times V_h$  where

$$V_h = \{v_h \in C(\bar{\Omega}) \mid v_h|_{T_i} \text{ is linear } \forall T_i \in \mathcal{T}_h, v_h = 0 \text{ on } \Gamma_u\}$$

and a closed, convex subset  $K_h$  of  $V_h$ , given by

$$K_h = \{v_h \in V_h \mid (v_h \cdot n)(a_i) \leq 0 \quad \forall i = 1, 2, \dots, m\}.$$

$a_1, \dots, a_{m(h)}$  are nodes of  $\mathcal{T}_h$ , lying on  $\Gamma_K$ . One can easily verify that  $K_h \subset K \quad \forall h \in (0, 1)$ , i.e.,  $K_h$  is an internal approximation of  $K$ .

Let  $\{\mathcal{T}_H\}$ ,  $H \in (0, 1)$  be a partition of  $\Gamma_K$ , compatible with the boundary of  $\text{supp } g$  in  $\Gamma_K$ , the nodes of which will be denoted by  $b_1, b_2, \dots, b_{m(H)}$ ,  $H = \max \text{length } \overline{b_i b_{i+1}}$ . These nodes do not coincide with  $a_1, a_2, \dots, a_{m(h)}$ , in general. Next, we shall write  $h = H$  if and only if  $b_1 = a_1, \dots, b_{m(H)} = a_{m(h)}$ .

Let

$$L_H = \{\mu_H \in L^2(\Gamma_K) \mid \mu_H|_{b_i b_{i+1}} \text{ is constant, } i = 1, \dots, m(H)\}$$

and

$$\Lambda_H = \{\mu_H \in L_H \mid |\mu_H| \leq 1 \text{ on } \text{supp } g, \mu_H = 0 \text{ on } \Gamma_K \setminus \text{supp } g\}.$$

By the *approximation* of  $(\tilde{\mathcal{P}})$  we denote the problem of finding a saddle-point  $(w_h, \lambda_H)$  of  $\Omega$  on  $K_h \times \Lambda_H$ , i.e.,

$$(\tilde{\mathcal{P}}_{hH}) \quad \Omega(w_h, \mu_H) \leq \Omega(w_h, \lambda_H) \leq \Omega(v_h, \lambda_H) \quad \forall v_h \in K_h, \forall \mu_H \in \Lambda_H.$$

In view of the boundedness of  $\Lambda_H$  and Korn's inequality, the following existence result holds:

**THEOREM 3.** *There exists a solution  $(w_h, \lambda_H)$  of  $(\tilde{\mathcal{P}}_{hH})$  the first component of which is uniquely determined.*

The second component  $\lambda_H$  is not uniquely determined in general. Let us formulate the conditions which guarantee its uniqueness.

Let

$$\overset{\circ}{K}_h = \{v_h \in V_h \mid v_h \cdot n = 0 \text{ on } \Gamma_K\}.$$

THEOREM 4. If

$$(8) \quad (g\mu_H, v_h)_{0,\Gamma_K} = 0 \quad \forall v_h \in \overset{\circ}{K}_h \quad \text{implies} \quad \mu_H \equiv 0 \quad \text{on } \Gamma_K,$$

the second component  $\lambda_H$  is uniquely determined.

*Proof.* Let  $(w_h, \lambda_H^1), (w_h, \lambda_H^2)$  be two solutions of  $(\tilde{\mathcal{P}}_{hH})$ . The second inequality in  $(\tilde{\mathcal{P}}_{hH})$  and the fact that  $K_h$  is a convex cone with vertex at 0 lead to

$$(9) \quad a(w_h, v_h) + (g\lambda_H^i, v_h)_{0,\Gamma_K} \geq L(v_h) \quad \forall v_h \in K_h, \quad i = 1, 2.$$

Let  $v_h \in \overset{\circ}{K}_h$ . Since  $\overset{\circ}{K}_h$  is a linear set, the sign " $\geq$ " in (9) can be replaced by " $=$ ". Subtracting these two equations for  $i = 1, 2$ , we get

$$(g(\lambda_H^1 - \lambda_H^2), v_h)_{0,\Gamma_K} = 0 \quad \forall v \in \overset{\circ}{K}_h.$$

From this and (8), the uniqueness of  $\lambda_H$  follows.

*Remark 1.* Let  $g$  be a piecewise constant on  $\Gamma_K$ . Then  $g\mu_H, \mu_H \in L_H$  is piecewise constant, while  $v_h$  for  $v_h \in \overset{\circ}{K}_h$  is continuous, piecewise linear on  $\Gamma_K$ . (8) means that the set of continuous, piecewise linear functions over the partition of  $\Gamma_K$ , given by points  $a_1, a_2, \dots, a_{m(h)}$ , is "sufficiently" large, or in other words that  $h_{\partial\Omega}/H$  is "sufficiently" small.  $h_{\partial\Omega}$  is defined as max length  $\overline{a_i a_{i+1}}$ .

### 3. Convergence results

The aim of this section is to prove the convergence of  $w_h$  to  $w$  and  $\lambda_H$  to  $\lambda$  in suitable norms. To this end we shall apply the following abstract convergence result, whose proof is given by Haslinger [4].

Let  $V, L$  be two Hilbert spaces,  $K \subseteq V$ ,  $\Lambda \subseteq L$  their non-empty closed convex subsets,  $\Lambda$  bounded. Let

$$\mathfrak{L}(v, \mu) = \frac{1}{2}a(v, v) - \langle f, v \rangle + b(v, \mu) - [q, \mu]$$

be a Lagrange function defined on  $V \times L$ , where

$a: V \times V \rightarrow R_1$  is a symmetric, bounded and  $V$ -elliptic bilinear form;

$f \in V', q \in L'$ , i.e.,  $f, q$  are linear continuous functionals on  $V, L$ , respectively;

$b: V \times L \rightarrow R_1$  is a bounded bilinear form on  $V \times L$ .

Let  $\{K_h\}, \{\Lambda_H\}$ ,  $K_h \subseteq V$ ,  $\Lambda_H \subseteq L$ ,  $h, H \in (0, 1)$  be closed convex subsets, and let  $\Lambda_H$  be uniformly bounded in  $L$ , i.e., there exists a positive constant  $c$  such that

$$\|\mu_H\|_L \leq c \quad \forall \mu_H \in \Lambda_H, \quad \forall H \in (0, 1).$$

Let  $(w, \lambda)$ ,  $(w_h, \lambda_H)$  be saddle-points of  $\mathcal{Q}$  on  $K \times \Lambda$ ,  $K_h \times \Lambda_H$ , respectively.

**THEOREM 5.** *Let  $h \rightarrow 0 +$  if and only if  $H \rightarrow 0 +$ . Moreover, let us suppose that*

$$(10) \quad \forall v \in K \exists v_h \in K_h: v_h \rightarrow v \text{ in } V;$$

$$(11) \quad \forall \mu \in \Lambda \exists \mu_H \in \Lambda_H: \mu_H \rightarrow \mu \text{ in } L;$$

$$(12) \quad v_h \in K_h, v_h \rightarrow v \text{ (weakly) in } V \text{ implies } v \in K;$$

$$(13) \quad \mu_H \in \Lambda_H, \mu_H \rightarrow \mu \text{ in } L \text{ implies } \mu \in \Lambda;$$

$$(14) \quad \exists r > 0 \exists \{v_h\}, v_h \in K_h \text{ such that } \|v_h\| \leq r \forall h \in (0, 1).$$

*Let the saddle-point  $(w, \lambda)$  of  $\mathcal{Q}$  on  $K \times \Lambda$  be unique. Then*

$$w_h \rightarrow w \text{ in } V, \quad \lambda_H \rightarrow \lambda \text{ in } L.$$

A direct consequence of this theorem is

**THEOREM 6.** *Let  $h \rightarrow 0 +$  if and only if  $H \rightarrow 0 +$ . Then*

$$w_h \rightarrow w \text{ in } (H^1(\Omega))^2, \\ \lambda_H \rightarrow \lambda \text{ in } L^2(\Gamma_K).$$

*Proof.* In this theorem we set  $V = \mathbf{V}$ ,  $L = L^2(\Gamma_K)$  with  $K$  and  $\Lambda$  defined in Section 1,

$$\langle f, v \rangle = L(v) \quad \forall v \in V, \quad q = 0, \quad b(v, \mu) = (g\mu, v_t)_{0, \Gamma_K}.$$

Since  $K_h \subset K$ ,  $\Lambda_H \subset \Lambda \forall h, H \in (0, 1)$ , (12), (13) are automatically satisfied. Clearly (14) holds as well. Using the usual regularization of  $w$ , one can obtain (11). So the most difficult is the verification of (10). This density result has been proved by Hlaváček and Lovíšek ([9]), for example.

*Remark 2.* Taking into account the relation between  $(\mathcal{P})$  and  $(\tilde{\mathcal{P}})$ , we see that  $w_h$  can be taken as the approximation of the displacement field  $u$  and  $g\lambda_H$  as the approximation of  $T_t(u)$  on  $\Gamma_K$ .

*Remark 3.* The rate of convergence of  $w_h$  to  $w$  and  $\lambda_H$  to  $\lambda$  in terms of  $h$  and  $H$  can be derived provided the exact solution  $w = u$  is smooth enough. Unfortunately, it is not so in our case. The solution  $u$  is not too smooth, even if  $\Omega$  is a domain with a smooth boundary and  $g$  has a special form (see [10]).

*Remark 4.* An alternative approach of solving these problems is possible. Using additional dualization of the constraint  $v \in K$ , it is possible to obtain a problem without constraints.

The Lagrange multiplier related to this dualization plays the role of  $T_n(u)$  on  $\Gamma_K$  (see [8]). The approach described here seems to be useful if the so-called *semicoercive* case is assumed, i.e., if  $J$  is not coercive on the whole  $V$  and if some additional assumptions on  $F$ ,  $P$  and  $g$  are imposed (see [1]).

*Remark 5.* More interesting and much more difficult is the model involving friction and obeying the so-called *Coulomb law* (see [1], [10] and [3]).

#### 4. Numerical realization

$(\tilde{\mathcal{P}}_{hH})$  is defined as the problem of finding a saddle-point in finite dimension, so that the classical method of Uzawa can be used to discover  $(w_h, \lambda_H)$  (see [2]).

A specific property of our problem, however, enables us to modify this method to obtain a very economical tool of its realization. An explicit form of Uzawa's method is the following:

We choose an arbitrary  $\lambda^{(0)} \in \Lambda_H$  (we write simply  $\lambda^{(0)}$  instead of  $\lambda_H^{(0)}$ , etc.). Starting from  $\lambda^{(n)}$ , we define  $w^{(n)} \in K_h$  as the solution of

$$\Omega(w^{(n)}, \lambda^{(n)}) \leq \Omega(v, \lambda^{(n)}) \quad \forall v \in K_h.$$

Then we replace  $\lambda^{(n)}$  by  $\lambda^{(n+1)}$  as follows:

$$\lambda^{(n+1)} = P_{\Lambda_H}(\lambda^{(n)} + \varrho w^{(n)}), \quad \varrho > 0,$$

where  $P_{\Lambda_H}$  is the projection of  $L^2(\Gamma_K)$  onto the convex set  $\Lambda_H$ .

It is well known that there exist positive numbers  $\varrho_1, \varrho_2$ ,  $\varrho_1 < \varrho_2$  such that for every  $\varrho$  satisfying  $\varrho_1 < \varrho < \varrho_2$

$$w^{(n)} \rightarrow w_h, \quad n \rightarrow \infty,$$

where  $w_h$  is the first component of the saddle-point of  $\Omega$  on  $K_h \times \Lambda_H$  (see [2]). As regards the behaviour of  $\{\lambda^{(n)}\}$ , we have

**THEOREM 7.** *Let (8) be satisfied. Then*

$$\lambda^{(n)} \rightarrow \lambda_H, \quad n \rightarrow \infty,$$

where  $\lambda_H$  is the second component of the saddle-point of  $\Omega$  on  $K_h \times \Lambda_H$ .

*Proof.* Since  $\{\lambda^{(n)}\}$  is bounded, we can choose a subsequence  $\{\lambda^{(n')}\} \subset \{\lambda^{(n)}\}$  such that

$$\lambda^{(n')} \rightarrow \lambda^*, \quad n' \rightarrow \infty.$$

As  $\lambda^{(n)} \in \Lambda_H \quad \forall n$  and  $\Lambda_H$  is closed,  $\lambda^* \in \Lambda_H$ . From the definition of  $w^{(n)}$  it follows that

$$a(w^{(n)}, v) + (g\lambda^{(n)}, v)_{0, \Gamma_K} = L(v) \quad \forall v \in \overset{\circ}{K}_h.$$

The limit passage for  $n' \rightarrow \infty$  gives

$$a(w_h, v) + (g\lambda^*, v_t)_{0, \Gamma_K} = L(v) \quad \forall v \in \overset{\circ}{K}_h.$$

On the other hand,

$$a(w_h, v) + (g\lambda_H, v_t)_{0, \Gamma_K} = L(v) \quad \forall v \in \overset{\circ}{K}_h.$$

From these two equations and (8) we deduce that  $\lambda^* = \lambda_H$ . As  $\lambda_H$  is unique, the whole sequence  $\lambda^{(n)}$  tends to  $\lambda_H$ .

As already mentioned, the problem in question has some specific properties. First of all, the matrix of rigidity is the same during the whole iterative process, while only a few components of the linear term vary. Finally, the number of constrained components is small in comparison with the total number of components. These facts can be used for the modification of Uzawa's method. Let us suppose that the constrained unknowns are arranged in such a way that they are listed last. Then the idea of substructuring inequalities can be applied (see [11]), i.e., the unconstrained unknowns can be eliminated and Uzawa's method can be applied to a small matrix only, the range of which is equal to the number of constrained unknowns.

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