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Stability of stochastic differential equations driven by general semimartingales

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Introduction

This thesis summarizes the author's research on convergence of solutions of finite-dimensional stochastic differential equations (SDE's) driven by general semimartingales.

We start with the following question. Let X be a solution of the Doléans-Dade and Protter equation

$$X_t = H_t + \int_0^t f(X_{s-})\,dZ_s, \qquad t \in \mathbb{R}^+, \tag{0.1}$$

where H is a d-dimensional initial process, Z is a d-dimensional semimartingale and $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a continuous function satisfying linear growth condition. For every $n \in \mathbb{N}$, let H^n be a perturbation of H and a semimartingale Z^n be a perturbation of Z. The problem is to find conditions on the convergence of the data $\{H^n\}$ and $\{Z^n\}$ assuring the convergence of the solution X^n,

$$X^n_t = H^n_t + \int_0^t f(X^n_{s-})\,dZ^n_s, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}, \tag{0.2}$$

to the solution X of the SDE (0.1) in various topologies. In other words, we ask whether the Doléans-Dade and Protter equation is stable under perturbations of the initial and driving processes.

One cannot expect such a stability under the only assumption of the convergence of $\{H^n\}$ and $\{Z^n\}$, since such stability is not satisfied even in the deterministic case. Therefore some additional condition on the sequence of driving semimartingales is indispensable. We assume that the sequence of semimartigales $\{Z^n\}$ satisfies the condition introduced by Stricker [Str85] (we call it after [JMP89] the condition (UT)), which means that the sequence of semimartingales $\{Z^n\}$ satisfies uniformly the condition from the theorem of Bichteler, Dellacherie and Mokobodzki (see Chapter 1).

Chapters 2–4 are devoted to the solution of the above problem for SDE's of the type (0.1) and some generalizations.

In Chapter 5 we investigate stability of finite-dimensional Stratonovich SDE's driven by general semimartingales. As an easy application we obtain a generalization of the results of Wong and Zakai. Let us recall that when Z is a Wiener process and Z^n are obtained by its linear interpolation, then X^n are solutions of the SDE of the type (0.1), while X is a solution of the equation with the so-called "correction term". Therefore, the results concerning the stability of Doléans-Dade and Protter equations are not directly applicable to the important Wong–Zakai type of approximations.

In Chapters 6–10 the stability criteria are proved for finite-dimensional SDE's with reflections on a very general domain D satisfying the assumptions (A) and (B) earlier introduced by Lions and Sznitman [LiSz84].

In all three cases: the classical SDE's of the Doléans-Dade and Protter type, the Stratonovich SDE's and the SDE's with reflections on the boundary, the same approach has been applied. First using (UT) and the functional limit theorem for stochastic integrals given by [JMP89], we prove very general theorems on weak convergence of solutions. Then by applying a change of probability idea from [Sło89] we deduce statements concerning strong convergence.

This method has enabled us to address the following problems:

- existence and stability of weak solutions,
- stability of strong solutions for equations, whose coefficients need not be Lipschitz,
- existence, uniqueness and stability of solutions of Stratonovich equations with jumps,
- approximation of Wong–Zakai type,
- existence, uniqueness and stability of solutions of SDE's with reflection for very general domains.

In particular, for all the three types of equations mentioned above there has been obtained a theoretical basis for computer simulations of solutions. In some cases the rate of convergence for those simulations can be determined as well.

Kurtz and Protter [KuPr91a, KuPr91b] showed that under a condition like (UT), it was possible to arrive at similar results on the convergence of solutions of SDE's (0.1). However, the methods of proof used by Kurtz and Protter are completely different from the ones used in our work. In Proposition 1.4 we prove that in the case of weakly convergent sequences of semimartingales the condition considered in [KuPr91a] is equivalent to (UT).

Definitions and required results from the general theory of stochastic processes and SDE's can be found in [DeMe80, JaSh87] and [Pro90].

Some of the results contained in this thesis have already been published in the series of papers [Sło89, MéSł91, Sło93, Sło94, Sło95] and they are being included here for the sake of clarity and completeness. The new theorems, to be found mainly in Chapters 3, 5 and 9, dealing both with the convergence of the flows of solutions and Stratonovich equations with jumps, broaden the scope of the theory.

The most important results of the work are summarized in Chapter 0.

0. Announcement of results

Chapter 1. Condition (UT)

In the late seventies Bichteler, Dellacherie and Mokobodzki characterized a semimartingale Z as a "good integrator", i.e. as a process for which for every $q \in \mathbb{R}^+$ the family of integrals of the form

$$\int_0^q U_s \, dZ_s$$

over the set U of all uniformly bounded simple predictable processes is tight in $\mathbb{R}$ (we will say throughout that a family of random elements is tight if the family of their laws is tight). Later on the theory of "good integrators" was intensively developed (see e.g. [Bich81, Pro90]).

Stricker [Str85] introduced a counterpart to the above condition for the sequence of semimartingales $\{Z^n\}$ (we call it after [JMP89] the condition (UT)). It is satisfied if the sequence of semimartingales $\{Z^n\}$ satisfies uniformly the condition from the theorem of Bichteler, Dellacherie and Mokobodzki.

A simple characterization of (UT) is given in Theorem 1.1. Namely, for every $a > 0$ we may decompose a semimartingale Z^n into the sum of three processes

$$Z^n = J^{n,a} + M^{n,a} + B^{n,a}, \qquad n \in \mathbb{N}, \tag{0.3}$$

where

$$J_t^{n,a} = \sum_{0<s\le t} \Delta Z_s^n \mathbf{1}_{\{|\Delta Z_s^n|>a\}},$$

$M^{n,a}$ is a locally square integrable martingale with $M_0^{n,a} = 0$ and $B^{n,a}$ is a predictable process of bounded variation with $B_0^{n,a} = 0$. Theorem 1.1 asserts that $\{Z^n\}$ fulfills the condition (UT) if and only if for some $a > 0$ and for every $q \in \mathbb{R}^+$ the families of random variables $\{\mathrm{Var}(J^{n,a})_q\}$, $\{\mathrm{Var}(B^{n,a})_q\}$, $\{[M^{n,a}]_q\}$ are tight in $\mathbb{R}$. This enables one to give conditions assuring that the sequences of stochastic integrals driven by $\{Z^n\}$ as well as the sequences of some other functionals of $\{Z^n\}$ satisfy (UT).

If it is known that if $\{Z^n\}$ satisfies some additional assumptions, then the characterization of (UT) is even simpler. In Propositions 1.4–1.6 we examine sequences of semimartingales $\{Z^n\}$ satisfying (UT) such that either

- $\{Z^n\}$ is weakly convergent in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ or
- $\sup_{t\le q} |Z_t^n - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$ or
- $Z^n \to_{\mathcal{D}} Z$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, where Z is a semimartingale with continuous trajectories.

Theorem 1.10 deals with the especially important case when $\{Z^n\}$ is a sequence of solutions to stochastic differential equations. Some examples of sequences of semimartingales satisfying (UT) are also given.

All results in Chapter 1 except Theorem 1.10 and examples from Section 1.4 are taken from [MéSł91].

Chapter 2. Weak convergence of solutions

We consider the sequence $\{X^n\}$ of the solutions to the SDE (0.2), where $f:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ is a given Lipschitz continuous function. It is well known (see e.g. [Dol78, Pro77]) that in this case for every $n\in\mathbb{N}$ there exists a unique strong solution of the SDE (0.2).

The main theorem (Theorem 2.1) gives sufficient conditions under which $\{X^n\}$ is weakly convergent in the Skorokhod topology J_1. The above problem has been discussed earlier by several authors. For example Emery [Eme78] and Protter [Pro78, Pro90] have proved stability of solutions to (0.1) using a very strong topology in the space of semimartingales. In that approach a common probability basis $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ is assumed, i.e. all the processes are adapted to a common filtration. We omit this assumption. Theorem 2.1 says that if $\{Z^n\}$ is a sequence of semimartingales satisfying (UT) then the following implication holds:

$$(H^n,Z^n)\underset{\mathcal{D}}{\rightarrow}(H,Z)\text{ in }\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})\Rightarrow(X^n,H^n,Z^n)\underset{\mathcal{D}}{\rightarrow}(X,H,Z)\text{ in }\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d}),$$

where X is a unique strong solution to the equation (0.2). In Corollaries 2.2–2.6 we give modifications of Theorem 2.1 for some special classes of semimartingales like local martingales and processes with bounded variation as well as for sequences of functions $\{f^n\}$ instead of f.

The main tool in the proof of Theorem 2.1 is a new tightness criterion given in Proposition 2.7. Let us remark that this criterion has recently been used to prove tightness of solutions of stochastic difference-differential equations (see e.g. [FuKu92, Fuj92a, Fuj92b]).

Theorem 2.1 is applicable directly to many problems of mathematical finance (see e.g. [DoPr92], [FöSc93]) or to investigation of the convergence rate of numerical schemes for SDE's (see e.g. [KuPr91b]). Other applications can be found in [Yam84, Yam86]).

All the above results are taken from [Sło89].

Chapter 3. Convergence in probability

In this chapter we study stability of solutions to the SDE's (0.2) in the topology of convergence in probability in the space $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. More precisely, we start from Theorem 2.1 to deduce that if $\{Z^n\}$ is a sequence of semimartingales satisfying (UT) then the following implication is true:

$$(H^n,Z^n)\underset{\mathcal{P}}{\rightarrow}(H,Z)\text{ in }\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})\Rightarrow(X^n,H^n,Z^n)\underset{\mathcal{P}}{\rightarrow}(X,H,Z)\text{ in }\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d}),$$

where X is a unique strong solution to the equation (0.1). In fact, we give a stronger version of the above result. Namely, in Section 3.1 we study SDE's of the form

$$X_t^x=x+H_t+\int_0^t f(X_{s-}^x)\,dZ_s,\qquad t\in\mathbb{R}^+,x\in\mathbb{R}^d. \tag{0.4}$$

The flow of the above SDE (0.4) is a function $\mathbb{R}^d\ni x\mapsto X_\cdot^x\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. We investigate flows of SDE's solutions driven by general semimartingales and their convergence. Let

$\{X^{n,x}\}$ be a family of strong solutions to the SDE

$$X_t^{n,x} = x + H_t^n + \int_0^t f(X_{s-}^{n,x})\,dZ_s^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N},\ x \in \mathbb{R}. \tag{0.5}$$

Let $\delta^d(\cdot,\cdot)$ denote a distance on the space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ generating the topology J_1.

Theorem 3.1 asserts that if $\{Z^n\}$ is a sequence of semimartingales satisfying (UT) then the implications given below are true:

- $(H^n, Z^n) \underset{\mathcal{P}}{\to} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \Rightarrow \sup_{x\in K} \delta^{3d}((X^{n,x}, H^n, Z^n), (X^x, H, Z)) \underset{\mathcal{P}}{\to} 0$, for every bounded subset $K \subset \mathbb{R}^d$,
- $\sup_{t\le q} |H_t^n - H_t| \underset{\mathcal{P}}{\to} 0$ and $\sup_{t\le q} |Z_t^n - Z_t| \underset{\mathcal{P}}{\to} 0$, $\in \mathbb{R}^+ \Rightarrow \sup_{x\in K}\sup_{t\le q} |X_t^{n,x} - X_t^x| \underset{\mathcal{P}}{\to} 0$, for every bounded subset $K \subset \mathbb{R}^d$, $q \in \mathbb{R}^+$.

If we assume additionally that $f \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ then, as is well known (see e.g. [Pro90]), the flow $x \mapsto X_t^x(\omega)$ is differentiable and for $k = 1, \ldots, d$ the process $D_{k,t}^x$ defined by

$$D_{k,t}^x(\omega) = \frac{\partial}{\partial x^k}(X_t^x(\omega))$$

is a unique strong solution to the appropriate d-dimensional SDE. Using Theorem 3.1 we prove stability of solutions to that SDE and we show some results on convergence in probability of derivatives of flows (see Theorem 3.4).

Let us consider an array $\{\{t_{nk}\}\}$ of nonnegative numbers such that in the nth row the sequence $T_n = \{t_{nk}\}$ forms a partition of $\mathbb{R}^+$ such that $0 = t_{n0} < t_{n1} < \ldots$, $\lim_{k\to\infty} t_{nk} = +\infty$ and

$$\max_k (t_{nk} - t_{n,k-1}) \to 0 \qquad \text{as } n \to +\infty. \tag{0.6}$$

For the array $\{\{t_{nk}\}\}$ we define a sequence of summation rules $\varrho^n : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varrho_t^n = \max\{t_{nk} : t_{nk} \le t\}$. For every $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ the sequence $\{x^{\varrho^n}\}$ denotes the following discretizations of x:

$$x_t^{\varrho^n} = x_{\varrho_t^n} = x_{t_{nk}} \qquad \text{for } t \in [t_{nk}, t_{n,k+1}),\ k \in \mathbb{N} \cup \{0\},\ n \in \mathbb{N}.$$

In Section 3.3 we consider Euler and Euler–Peano schemes for the SDE (0.2), i.e. the sequences of solutions to discrete SDE's corresponding to the SDE (0.2)

$$\overline{X}_t^n = H_t^{\varrho^n} + \int_0^t f(\overline{X}_{s-}^n)\,dZ_s^{\varrho^n},$$

$$\widehat{X}_t^n = H_t + \int_0^t f(\widehat{X}_{s-}^{n,\varrho^n})\,dZ_s, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}.$$

Due to Theorem 3.1 it is possible to prove the convergence in probability of these schemes. Moreover, Theorem 1.10 allows us to discuss the rate of convergence.

The above results are generalizations of earlier ones proved in [Sło89].

Chapter 4. SDE's with past-dependent and non-Lipschitz coefficients

In Section 4.1 we consider convergence of solutions of SDE's with memory of the form

$$X_t^n = H_t^n + \int_0^t f^n(s, X_\cdot^n)\, dZ_s^n + \int_0^t g^n(X_\cdot^n)\, dA_s^n, \qquad t \in \mathbb{R}^+,$$

where f^n, g^n are predictable functions from $\mathbb{R}^+ \times \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ into $\mathbb{R}$ (with respect to the filtration $\mathcal{F}_t = \bigcap_{s>t} \sigma(x_u : u \le s, x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}))$, Z^n is a local martingale and A^n is a process with nondecreasing trajectories.

Theorem 4.1 says that if $f^n \to f$, $g^n \to g$ in an appropriate sense, $\{Z^n\}$ satisfies a condition slightly stronger than (UT) and $(H^n, Z^n, A^n) \to_{\mathcal{D}} (H, Z, A)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^3)$, where H, Z, A are processes with continuous trajectories then

$$X^n \underset{\mathcal{D}}{\to} X \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}),$$

provided that the SDE

$$X_t = H_t + \int_0^t f(s, X_\cdot)\, dZ_s + \int_0^t g(s, X_\cdot)\, dA_s, \qquad t \in \mathbb{R}^+,$$

possesses a unique weak solution X. This enables us to generalize earlier results of this type proved in [Yam86, Yam87] and [Zan90]. Since the time change arguments are used in the proofs, the results of Section 4.1 are essentially one-dimensional.

In Section 4.2 we consider stability of finite-dimensional SDE with memory. The assumptions on $\{Z^n\}$ have been relaxed compared to the ones in Section 4.1. We consider an arbitrary sequence of d-dimensional semimartingales $\{Z^n\}$ satisfying (UT). The main result of the section, Theorem 4.2, asserts that if $f^n \to f$ in an appropriate sense and $\{f^n(\cdot, X_\cdot^n)\}$ is approximated by some arrays of semimartingales, where X^n is a solution of a d-dimensional SDE of the form

$$X_t^n = H_t^n + \int_0^t f^n(s, X_\cdot^n)\, dZ_s^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}, \tag{0.7}$$

then

$$(H^n, Z^n) \underset{\mathcal{D}}{\to} (H, Z) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \Rightarrow\ (X^n, H^n, Z^n) \underset{\mathcal{D}}{\to} (X, H, Z) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

provided that the SDE

$$X_t = H_t + \int_0^t f(s, X_\cdot)\, dZ_s, \qquad t \in \mathbb{R}^+, \tag{0.8}$$

has a unique weak solution X. Next, in Lemma 4.4 we describe some classes of coefficients $\{f^n\}$, f satisfying the assumptions of Theorem 4.2. As a consequence of Theorem 4.2 and Lemma 4.4 we state in Corollary 4.5 conditions assuring existence of weak solutions to the SDE (0.8).

The remaining part of Section 4.2 concerns convergence in probability of solutions to the SDE's (0.7). We assume additionally that a limit solution X to the SDE (0.8) satisfies the following two conditions:

- X is adapted to the natural filtration of (H, Z),
- pathwise uniqueness for the SDE (0.8) holds.

In Theorem 4.6 we give conditions under which $X^n \to_{\mathcal{P}} X$. For example if f is continuous, satisfies linear growth condition and

$$\|f(t,x) - f(t,y)\|^2 \le \varrho(|x-y|^2), \qquad t \in \mathbb{R}^+,\ x, y \in \mathbb{R},$$

where

$$\text{(0.9)} \qquad \varrho : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is strictly increasing and concave with } \varrho(0) = 0 \text{ and } \int_{0^+} \frac{du}{\varrho(u)} = +\infty,$$

and $\|\cdot\|$ denotes the usual norm in the space of linear operators from $\mathbb{R}^d$ into $\mathbb{R}^d$, then it is possible to approximate in probability the solution of the SDE (0.8). Note that in this case the function f need not be Lipschitz continuous.

All the results in Chapter 4 are taken or can be deduced from [MéSł91]. Similar results were proved also in [KuPr91a] by using different methods.

Chapter 5. Stratonovich SDE's

In this chapter variants of Theorem 2.1 and Theorem 3.1 for finite-dimensional Stratonovich SDE's driven by general d-dimensional semimartingales are given. We begin with Example 5.1 which shows that for this purpose we cannot use Stratonovich inegrals driven by semimartingales with jumps introduced by Meyer [Mey76]. Instead of Meyer's definition, a definition proposed by Marcus [Mar78, Mar81] and in the general case by Kurtz, Pardoux and Protter [KPP92] is adopted to the equation of the following form:

$$\text{(0.10)} \qquad \begin{aligned} X_t &= X_0 + \int_0^t f(X_s) \circ dZ_s \\ &= X_0 + \int_0^t f(X_{s-})\, dZ_s + \frac{1}{2} \int_0^t f' f(X_{s-})\, d[Z]^c_s \\ &\quad + \sum_{s \le t} \{\varphi(f \Delta Z_s, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\}, \qquad t \in \mathbb{R}^+, \end{aligned}$$

(the notation "$\circ$" indicates that we do not deal here with a standard Itô type stochastic integral), where for given $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, $\varphi(g, x)$ denotes the value at time $u = 1$ of the solution to the ordinary differential equation

$$\begin{aligned} \frac{dy}{du}(u) &= g(y(u)), \\ y(0) &= x \in \mathbb{R}^d. \end{aligned}$$

It turns out that the above definition has been properly chosen for obtaining the stability results under (UT).

Let $\{Z^n\}$ be a sequence of semimartingales and $\{X^n\}$ be a sequence of strong solutions corresponding to $\{Z^n\}$, i.e.

$$
\begin{aligned}
(0.11)\qquad X^n_t &= X^n_0 + \int_0^t f(X^n_s)\circ dZ^n_s \\
&= X^n_0 + \int_0^t f(X^n_{s-})\,dZ^n_s + \frac{1}{2}\int_0^t f'f(X^n_{s-})\,d[Z^n]^c_s \\
&\quad + \sum_{s\le t}\{\varphi(f\Delta Z^n_s, X^n_{s-}) - X^n_{s-} - f(X^n_{s-})\Delta Z^n_s\},\ t\in\mathbb{R}^+,\ n\in\mathbb{N}.
\end{aligned}
$$

If $\{Z^n\}$ satisfies (UT), f belongs to $\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ and f, $f'f$ satisfy linear growth condition then Theorem 5.2 asserts that the following implication holds:

$$
(X^n_0, Z^n)\underset{\mathcal{D}}{\to}(X_0,Z) \text{ in } \mathbb{R}^d\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d) \Rightarrow X^n \underset{\mathcal{D}}{\to} X \text{ in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d),
$$

provided that the SDE (0.10) has a unique weak solution X.

It is also possible to give a version of the above result for the convergence in probability. In Theorem 5.4 we prove that if $\{Z^n\}$ satisfies (UT), the function f belongs to $\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ and f, $f'f$ satisfy a linear growth condition then the following two implications are true:

- $X^n_0 \underset{\mathcal{P}}{\to} X_0$ and $Z^n \underset{\mathcal{P}}{\to} Z$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d) \Rightarrow (X^n,Z^n)\underset{\mathcal{P}}{\to}(X,Z)$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$,
- $X^n_0 \underset{\mathcal{P}}{\to} X_0$ and $\sup_{t\le q}|Z^n_t - Z_t| \underset{\mathcal{P}}{\to} 0,\in\mathbb{R}^+ \Rightarrow \sup_{t\le q}|X^n_t - X_t|\underset{\mathcal{P}}{\to} 0,\ q\in\mathbb{R}^+$,

provided that the solution X to the SDE (0.10) is adapted to the natural filtration of (X_0,Z) and the pathwise uniqueness property for the SDE (0.10) holds (see (4.21)).

In Corollary 5.5 we approximate the solution X of the SDE (0.10) in probability by solutions of discrete SDE's.

Consider the sequence $\{T_n\}$ of partitions of $\mathbb{R}^+$ satisfying the condition (0.6). Let $\{\widehat{Z}^{\varrho^n}\}$ be a sequence of linear approximations of Z, i.e.

$$
\widehat{Z}^{\varrho^n}_t = Z_{t_{nk}} + \frac{t-t_{nk}}{t_{n,k+1}-t_{nk}}(Z_{t_{n,k+1}} - Z_{t_{nk}}),\qquad t\in[t_{nk},t_{n,k+1}[,\ n\in\mathbb{N},\ k\in\mathbb{N}\cup\{0\}.
$$

Since the trajectories of $\widehat{Z}^{\varrho^n}$ are continuous and have bounded variation, it is clear that for every $n\in\mathbb{N}$ there exists a unique strong solution to the SDE

$$
(0.12)\qquad Y^n_t = X_0 + \int_0^t f(Y^n_s)\circ d\widehat{Z}^{\varrho^n}_s = X_0 + \int_0^t f(Y^n_{s-})\,d\widehat{Z}^{\varrho^n}_s,\qquad t\in\mathbb{R}^+,
$$

provided that f is locally Lipschitz continuous and satisfies a linear growth condition. The above type of SDE's was introduced by Wong and Zakai [WoZa65] for Z being a Wiener process. Those authors have shown that Y^n tends to the solution of the Stratonovich SDE (0.10). Kurtz, Pardoux and Protter [KPP92] have recently discussed Wong–Zakai type of approximations for Stratonovich SDE's driven by general semimartingales. In this case the approximating sequence $\{Y^n\}$ is as before a sequence of continuous processes, but the limit process X need not have continuous trajectories. Therefore the approximating solutions need not converge in the Skorokhod topology J_1. Using very subtle results on Meyer–Zheng topology (see e.g. EtKu86, Ku91]) Kurtz, Pardoux and Protter have been able to prove that

$$Y_t^n \underset{\mathcal{P}}{\rightarrow} X_t,$$

except possibly for a countable set of t's.

It turns out that the above result is an easy corollary of Theorem 5.2. In fact we generalize it slightly. Theorem 5.6 says that if f satisfies the assumptions of Corollary 5.5 then

$$\sup_{t \le q,\, t \in T_n} |Y_t^n - X_t| \underset{\mathcal{P}}{\rightarrow} 0, \qquad q \in \mathbb{R}^+,$$

where X is a unique strong solution of the SDE (0.10). In particular,

$$Y_t^n \underset{\mathcal{P}}{\rightarrow} X_t,$$

provided $\Delta Z_t = 0$ or $t \in \liminf_{n \to +\infty} T_n$.

Chapter 6. Skorokhod problem, deterministic case

In Chapters 6 we discuss existence, uniqueness and stability of solutions (X, K) of d-dimensional Skorokhod problems on a domain D,

$$x_t = y_t + k_t, \qquad t \in \mathbb{R}^+, \tag{0.13}$$

where $x_t \in \overline{D} = D \cup \partial D$, $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and k is a bounded variation function with variation $\mathrm{Var}(k)$ increasing only, when $x_t \in \partial D$. The problem of existence and uniqueness of solutions to (0.13) has been discussed for the first time by Skorokhod [Sko61] for $d = 1$ and $D = \mathbb{R}^+$. Later on, many attempts have been made to generalize Skorokhod's results to larger class of domains. The papers [ChKM80, KaCh78, IkWa81, McK63, Pro80, Wat71] are devoted to the case of reflection in the half-space, i.e. $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. For the first time the reflection in a more general domain was discussed by Tanaka [Tan79], where D is a convex subset of $\mathbb{R}^d$. Then Lions and Sznitman [LiSz84] investigated domains satisfying the conditions (A) and (B) given in Chapter 6 along with the admissibility condition which means, roughly speaking, that D can be approximated in some sense by smooth domains. Finally, Saisho [Sai87] has omitted the admissibility condition.

We assume as in [Sai87] that D is a general domain satisfying the conditions (A) and (B), whereas y is any function from $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. We prove existence and uniqueness for such problems provided y has jumps bounded by some constant r_0, $r_0 \in [0, +\infty]$, depending on the region D only, i.e. $|\Delta y| < r_0$ (for example if D is convex then we can take $r_0 = \infty$). Let us note that if D is a convex domain then problem (0.13) has been recently considered by Anulova and Liptzer [AnLi89] in order to characterize the diffusion approximation for processes with reflecting boundary. We also deal with the question of convergence of solutions to equations of the type (0.13).

Chapter 7. Skorokhod problem, nondeterministic case

In Chapter 7 we consider a random case of (0.13). More precisely we assume that $Y = H + Z$ (where H is a process with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and Z is a semimartingale decomposed into the sum of a local martingale M and of a process with bounded variation V), and we investigate the property of the solution (X, K) of the Skorokhod problem

$$X_t = Y_t + K_t = H_t + Z_t + K_t = H_t + M_t + V_t + K_t, \quad t \in \mathbb{R}^+.$$

In particular, in Theorem 7.2 we discuss the conditions assuring an estimate of the type

$$\mathrm{Var}(K)_q \le C \sup_{t\le q} |Y_t|.$$

Further, Theorem 7.3 allows us to estimate $E\sup_s |X_s - \widehat{X}_s|^2$, where $(\widehat{X}, \widehat{K})$ is a solution of the Skorokhod problem corresponding to $\widehat{Y}$ ($\widehat{Y}_t = H_t + \widehat{Z}_t$) and $\widehat{Z}$ is a semimartingale decomposed into the sum of a local martingale $\widehat{M}$ and a process with bounded variation $\widehat{V}$, i.e. $\widehat{Y} = H_t + \widehat{M}_t + \widehat{V}_t$. In Theorem 7.3 we prove that under some assumptions on Y and $\widehat{Y}$,

$$E \sup_{t\le q} |X_t - \widehat{X}_t|^2 \le CE\{[M - \widehat{M}]_q + \mathrm{Var}(V - \widehat{V})_q^2\}.$$

These two estimates play a key role in Chapter 8.

We also discuss asymptotical behaviour of sequences of solutions to Skorokhod problems. Let $\{Y^n\}$ be a sequence of processes of the form $Y^n = H^n + Z^n$ and let $\{(X^n, K^n)\}$ be a sequence of solutions to the Skorokhod problems associated with $\{Y^n\}$. Propositions 7.6 and 7.8 say that if $|\Delta Y^n| \le c < r_0$, $n \in \mathbb{N}$, and the conditions (A), (B) are fulfilled then

- $\{H^n\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $\{Z^n\}$ satisfies (UT) $\Rightarrow$ $\{\mathrm{Var}(K^n)_q\}$ is tight in $\mathbb{R}$,
- $\{H^n\}$, $\{Z^n\}$ satisfy (UT) $\Rightarrow$ $\{X^n\}$ satisfies (UT), $\{\mathrm{Var}(K^n)_q\}$ is tight in $\mathbb{R}$,
- $(H^n, Z^n) \underset{\mathcal{D}}{\to} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ $\Rightarrow$ $(H^n, Z^n, X^n) \underset{\mathcal{D}}{\to} (H, Z, X)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$.

All the results are taken from [Sło93].

Chapter 8. SDE's with reflecting boundary

In Chapter 8 we investigate a d-dimensional SDE on a domain D with reflecting boundary condition

$$X_t = H_t + \int_0^t f(X_{s-})\, dZ_s + K_t, \quad t \in \mathbb{R}^+, \tag{0.14}$$

where Z is a semimartingale, X is a reflecting process on $\overline{D} = D \cup \partial D$ and K is a bounded variation process with variation $\mathrm{Var}(K)$ increasing only when $X_t \in \partial D$.

Let $\{Z^n\}$ be a sequence of semimartingales satisfying (UT). We consider a sequence $\{X^n\}$ of solutions to SDE's of the form (0.14), i.e.

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + K_t^n, \quad t \in \mathbb{R}^+,\ n \in \mathbb{N}. \tag{0.15}$$

In Theorem 8.1, our main result in this chapter, we give some sufficient conditions under which $\{X^n\}$ converges in law to the solution X of the SDE (0.14). As a consequence in Corollary 8.2, we obtain existence of weak solution to the SDE (0.14) provided that f is continuous and bounded, i.e. $\|f\| \le L < +\infty$ and $|\Delta Y| + L|\Delta Z| < r_0$. If additionally f is Lipschitz continuous, then using discrete approximations of the solution to the SDE (0.14), which are constructed in a manner analogous to Euler's formula, we prove Theorem 8.3 on the existence and uniqueness of strong solution to the SDE (0.14). Then we consider the convergence in law and in probability of strong solutions of the SDE (0.15).

Let us note that for $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$ the problem of existence and uniqueness of solutions of SDE's like (0.14) has been examined by Chaleyat-Maurel, El Karoui and Marchal [ChKM80], and Protter [Pro80]. Unfortunately their approach cannot be extended to domains satisfying only the conditions (A) and (B).

All these results were originally published in [Sło93].

Chapter 9. Flows of SDE's with reflecting boundary

In Chapter 9 we discuss the SDE's with reflecting boundary of the form

$$X_t^x = x + H_t + \int_0^t f(X_{s-}^x)\, dZ_s + K_t^x, \qquad t \in \mathbb{R}^+,\ x \in \overline{D}, \tag{0.16}$$

where $H_0 = 0$. The flow of (0.16) is the function $\overline{D} \ni x \mapsto X_\cdot^x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$.

Theorem 9.3 says that if f is Lipschitz continuous and bounded, i.e. there exists a constant $L > 0$ such that $\|f(x) - f(y)\| \le L|x-y|$, $\|f(x)\| \le L$ for all $x, y \in \overline{D}$ and $|\Delta H| + L|\Delta Z| < r_0$, then under the conditions (A) and (B) there exists a function $X(t,\omega,x)$ on $\mathbb{R}^+ \times \Omega \times \mathbb{R}^d$ such that:

- for each $x \in \overline{D}$ the process $X_t^x(\omega) = X(t,\omega,x)$ is a solution of (0.16),
- for almost all $\omega \in \Omega$, the flow $x \mapsto X_\cdot^x(\omega)$ from $\overline{D}$ into $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is continuous in the topology of uniform convergence on compact sets.

We also give a version of Theorem 3.1 Namely, let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, and let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes, $H_0^n = 0$. Assume $\{X^{n,x}\}$ is a family of solutions to the SDE

$$X_t^{n,x} = x + H_t^n + \int_0^t f(X_{s-}^{n,x})\, dZ_s^n + K_t^{n,x}, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N},\ x \in \overline{D}. \tag{0.17}$$

Theorem 9.4 asserts that under (A), (B) and the assumptions on f from Theorem 9.3 the following implications are true:

- $(H^n, Z^n) \underset{\mathcal{P}}{\to} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ and $|\Delta H| + L|\Delta Z| < r_0$

  $$\Rightarrow \sup_{x \in K \cap \bar{D}} \delta^{4d}((X^{n,x}, K^{n,x}, H^n, Z^n), (X^x, K^x, H, Z)) \underset{\mathcal{P}}{\to} 0,$$

  for every bounded subset $K \subset \mathbb{R}^d$,
- $\sup_{t \le q} |H_t^n - H_t| \underset{\mathcal{P}}{\to} 0$, $\sup_{t \le q} |Z_t^n - Z_t| \underset{\mathcal{P}}{\to} 0$, $q \in \mathbb{R}^+$ and $|\Delta H| + L|\Delta Z| < r_0$

  $$\Rightarrow \sup_{x \in K \cap \bar{D}} \sup_{t \le q} |X_t^{n,x} - X_t^x| \underset{\mathcal{P}}{\to} 0 \text{ and } \sup_{x \in K \cap \bar{D}} \sup_{t \le q} |K_t^{n,x} - K_t^x| \underset{\mathcal{P}}{\to} 0,$$

  for every bounded subset $K \subset \mathbb{R}^d$, $q \in \mathbb{R}^+$.

Chapter 10. Numerical schemes for SDE's with reflecting boundary

In Chapter 10 we investigate L^p convergence as well as almost sure convergence of time-discretization schemes for d-dimensional stochastic differential equation on a domain D with reflecting boundary condition. Given a function $f : \overline{D} = D \cup \partial D \to \mathbb{R}^d \otimes \mathbb{R}^d$, $f(x) = \{f_{ij}(x)\}_{i,j=1,\ldots,d}$ we consider the SDE

$$X_t = X_0 + \int_0^t f(X_s)\,dW_s + \int_0^t g(X_s)\,ds + K_t, \qquad t \in \mathbb{R}^+, \tag{0.18}$$

where W_t is a d-dimensional Wiener process, X_t is a reflecting process on $\overline{D}$ and K_t is a bounded variation process with variation $\mathrm{Var}(K)_t$ increasing only when $X_t \in \partial D$. In the paper [Sai87] it is proven under the conditions (A), (B) that if f, g are Lipschitz continuous and bounded on $\overline{D}$ then there exists a unique strong solution to the SDE (0.18) (this fact can be also deduced from the general Theorem 8.3).

Let us consider an array $\{\{t_{nk}\}\}$ of nonnegative numbers such that in each nth row the sequence $T_n = \{t_{nk}\}$ forms a partition on $\mathbb{R}^+$ such that

$$\max_k (t_{nk} - t_{n,k-1}) \le 1/n, \qquad n \in \mathbb{N}.$$

In this chapter we assume that D is either a convex set or a general domain satisfying the conditions (A) and (B) and we consider Euler and Euler–Peano schemes for the SDE (0.18). More precisely, we investigate the approximations $\{\overline{X}^n\}$ and $\{\widehat{X}^n\}$, which are the solutions to the appropriate SDE's with reflecting boundary conditions

$$\overline{X}^n_t = X_0 + \int_0^t f(\overline{X}^n_{s-})\,dW^{\varrho^n}_s + \int_0^t g(\overline{X}^n_{s-})\,d\varrho^n_s + \overline{K}^n_t, \qquad t \in \mathbb{R}^+, \tag{0.19}$$

and

$$\widehat{X}^n_t = X_0 + \int_0^t f(\widehat{X}^{n,\varrho^n}_{s-})\,dW_s + \int_0^t g(\widehat{X}^{n,\varrho^n}_{s-})\,ds + \widehat{K}^n_t, \qquad t \in \mathbb{R}^+. \tag{0.20}$$

Note that if $D = \mathbb{R}^d$ then $\overline{X}^n_t = \widehat{X}^{n,\varrho^n}_t$ and (0.19) is a classical Euler scheme considered for the first time by Maruyama [Mar55] (see also Section 3.3). In the case $D \ne \mathbb{R}^d$ the equality $\overline{X}^n_t = \widehat{X}^{n,\varrho^n}_t$ need not be satisfied. In this case the rate of mean-square convergence in the above schemes has been earlier examined only if $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$ by Chitashvili and Lazrieva [ChLa81], Kinkladze [Ki83] (the scheme (0.19)) and Lépingle [Lép93] (the scheme (0.20)). We give the rate of L^p as well as almost sure convergence for the schemes (0.19), (0.20) for a broader class of domains than the half-space.

Let D be a convex domain in $\mathbb{R}^d$. For the Euler scheme we show that there exists a sequence $\{\tau_n\}$ of stopping times with $\tau_n \to_{\mathcal{P}} +\infty$ such that

$$E \sup_{t \le q} |\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \qquad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}, \tag{0.21}$$

and

$$n^{1/4-\varepsilon} \sup_{t \le q} |\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t| \to 0 \qquad \mathcal{P}\text{-a.s.}, \varepsilon > 0,\ q \in \mathbb{R}^+. \tag{0.22}$$

In this case for the Euler–Peano scheme we obtain

$$E\sup_{t\le q}|\widehat{X}^n_t - X_t|^{2p} = \mathcal{O}(1/n^p), \qquad q\in\mathbb{R}^+,\ p\in\mathbb{N}, \tag{0.23}$$

and

$$n^{1/2-\varepsilon}\sup_{t\le q}|\widehat{X}^n_t - X_t| \to 0 \qquad \mathcal{P}\text{-a.s.}, \varepsilon>0,\ q\in\mathbb{R}^+. \tag{0.24}$$

For a large class of convex domains we are able to strengthen (0.21) and (0.22). If D is a convex subset of $\mathbb{R}^d$ satisfying the condition (β) which is automatically fulfilled for D bounded or $d<3$, then we prove that

$$E\sup_{t\le q}|\overline{X}^n_t - X_t|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \qquad \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N},$$

and

$$n^{1/4-\varepsilon}\sup_{t\le q}|\overline{X}^n_t - X_t| \to 0 \qquad \mathcal{P}\text{-a.s.}, \varepsilon>0,\ q\in\mathbb{R}^+.$$

If D is a convex polyhedron, i.e. $D=\bigcap_{i=1}^N D_i$, where D_i is a closed half-space, we can prove even more, namely that

$$E\sup_{t\le q}|\overline{X}^n_t - X_t|^{2p} = \mathcal{O}(1/n^{p-\varepsilon}), \qquad \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}, \tag{0.25}$$

and

$$n^{1/2-\varepsilon}\sup_{t\le q}|\overline{X}^n_t - X_t| \to 0 \qquad \mathcal{P}\text{-a.s.}, \varepsilon>0,\ q\in\mathbb{R}^+. \tag{0.26}$$

Let us mention that in the case of the half-space some results similar to (0.25) have been obtained in [ChLa81, Ki83], while (0.23) was announced in [Lép93].

Now let D be a general domain satisfying the conditions (A) and (B). For the Euler scheme we show that (0.21) and (0.22) are also true. In the case of the Euler–Peano scheme we prove that there exists an array $\{\{\tau^k_n\}\}$ of stopping times such that

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathcal{P}(\tau^k_n\le q)=0$$

and

$$E\sup_{t\le q}|\widehat{X}^{n,\tau^k_n}_t - X^{\tau^k_n}_t|^{2p} = \mathcal{O}(1/n^p), \qquad k\in\mathbb{N},\ q\in\mathbb{R}^+,\ p\in\mathbb{N}.$$

We can also find a sequence $\{\tau_n\}$ of stopping times with $\tau_n\to_{\mathcal{P}}+\infty$ such that

$$n^{1/2-\varepsilon}\sup_{t\le q}|\widehat{X}^{n,\tau_n}_t - X^{\tau_n}_t| \to 0 \qquad \mathcal{P}\text{-a.s.}, \varepsilon>0,\ q\in\mathbb{R}^+.$$

In Section 10.4 we prove convergence of numerical schemes for slightly more general sequences $\{\{t_{nk}\}\}$ of partitions satisfying the condition

$$\max_k(t_{nk}-t_{n,k-1})\le 1/n^\alpha, \qquad n\in\mathbb{N},\ \alpha>0. \tag{0.27}$$

The above results are taken from [Sło94, Sło95].

1. Condition (UT)

1.1. Main criterion. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space and let $(\mathcal{F}_t)$ be a filtration on $(\Omega,\mathcal{F},\mathcal{P})$ satisfying the usual conditions. A process Z is called an $(\mathcal{F}_t)$ *adapted semimartingale* if $Z = M + V$, where M is an $(\mathcal{F}_t)$ adapted local martingale and V is an $(\mathcal{F}_t)$ adapted process with locally bounded variation. The famous theorem of Bichteler, Dellacherie and Mokobodzki [DeMe80, Theorem 2.5] provides an equivalent definition: Z is an $(\mathcal{F}_t)$ adapted semimartingale if and only if the following condition is satisfied:

(T) for every $q \in \mathbb{R}^+$ the family of random variables

$$\Big\{\int_0^q U_s\,dZ_s : U \in \mathbf{U}_q\Big\} \quad \text{is tight in } \mathbb{R},$$

where $\mathbf{U}_q$ is the class of predictable processes of the form

$$U_s = U_0 + \sum_{i=0}^{k} U_i \mathbf{1}_{\{t_i<s\le t_{i+1}\}}$$

such that $0 = t_0 < t_1 < \ldots < t_k = q$ and every U_i is $\mathcal{F}_{t_i}$ measurable, $|U_i| \le 1$ for every $i \in \mathbb{N}\cup\{0\}$, $k \in \mathbb{N}$.

Now, let $\{Z^n\}$ be a sequence of semimartingales defined possibly on different probability spaces $(\Omega^n,\mathcal{F}^n,\mathcal{P}^n)$, $n \in \mathbb{N}$, and adapted to different filtrations $(\mathcal{F}^n_t)$, $n \in \mathbb{N}$. We will assume that the sequence $\{Z^n\}$ fulfills the above condition (T) uniformly with respect to $n \in \mathbb{N}$. More precisely, we will investigate the sequences of semimartingales satisfying the condition (UT) introduced by Stricker [Str85]:

(UT) for every $q \in \mathbb{R}^+$ the family of random variables

$$\Big\{\int_0^q U^n_s\,dZ^n_s : n \in \mathbb{N}, U^n \in \mathbf{U}^n_q\Big\} \quad \text{is tight in } \mathbb{R},$$

where $\mathbf{U}^n_q$ is the class of predictable processes of the form

$$U^n_s = U^n_0 + \sum_{i=0}^{k} U^n_i \mathbf{1}_{\{t_i<s\le t_{i+1}\}}$$

such that $0 = t_0 < t_1 < \ldots < t_k = q$ and every U^n_i is $\mathcal{F}^n_{t_i}$ measurable, $|U^n_i| \le 1$ for every $i \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, $k \in \mathbb{N}$.

The condition (UT) appeared very fruitful in the theory of limit theorems for stochastic integrals. Under this condition Jakubowski, Mémin and Pages [JMP89] proved a functional limit theorem for stochastic integrals. The main results of Stricker [Str85] and Jakubowski, Mémin and Pages [JMP89] under the condition (UT) are described in Appendix B.

We start with a simple characterization of (UT). For every $a > 0$, the process Z^n, $Z^n_0 = 0$ can be decomposed into the sum of three processes

$$Z^n = J^{n,a} + M^{n,a} + B^{n,a}, \quad n \in \mathbb{N}, \tag{1.1}$$

where $J_t^{n,a} = \sum_{0<s\le t} \Delta Z_s^n \mathbf{1}_{\{|\Delta Z_s^n|>a\}}$, $M^{n,a}$ is a locally square integrable martingale, $M_0^{n,a} = 0$ and $B^{n,a}$ is a predictable process with bounded variation, $B_0^{n,a} = 0$.

THEOREM 1.1. *The following four conditions are equivalent*:

(i) $\{Z^n\}$ *satisfies the condition* (UT),

(ii) *there exists* $a > 0$ *such that for every* $q \in \mathbb{R}^+$ *the families of random variables* $\{\mathrm{Var}(J^{n,a})_q\}$, $\{\mathrm{Var}(B^{n,a})_q\}$, $\{[M^{n,a}]_q\}$ *are tight in* $\mathbb{R}$,

(iii) *for every* $q \in \mathbb{R}^+$ *the family of random variables*

$$\Big\{\int_0^q U_s^n\,dZ_s^n : n \in \mathbb{N}, U^n \in \mathbf{U}^n\Big\} \quad \text{is tight in } \mathbb{R},$$

where $\mathbf{U}^n$ *denotes the class of predictable processes* U^n *such that* $\sup_{t\le q}|U_t^n| \le 1$, $n \in \mathbb{N}$,

(iv) *for every* $q \in \mathbb{R}^+$ *and* $\varepsilon > 0$ *there exists* $\alpha > 0$ *such that for every* $n \in \mathbb{N}$ *and every predictable processes* U^n,

$$\mathcal{P}[\sup_{t\le q}|U_t^n| > \alpha] < \alpha \Rightarrow \mathcal{P}\Big[\sup_{t\le q}\Big|\int_0^t U_s^n\,dZ_s^n\Big| > \varepsilon\Big] < \varepsilon.$$

Proof. It is clear that (iii)⇔(iv) and (iii)⇒(i). We show (i)⇒(ii) and (ii)⇒(iii).

(ii)⇒(iii). Since $\{\mathrm{Var}(Z^n - M^{n,a})_q\}$ is tight in $\mathbb{R}$, it is obvious that

$$\Big\{\sup_{t\le q}\Big|\int_0^t U_s^n\,d(Z^n - M^{n,a})_s\Big|\Big\} \quad \text{is tight in } \mathbb{R}.$$

On the other hand, for every $\varepsilon > 0$ there exists a constant $K > 0$ such that $\mathcal{P}[[M^{n,a}]_q > K] < \varepsilon, n \in \mathbb{N}$. Define $\tau^n = \inf\{s > 0 : [M^{n,a}]_s > K\} \wedge q$. Then $\mathcal{P}[\tau^n < q] < \varepsilon$ and from the sequence of inequalities

$$\begin{aligned}\mathcal{P}\Big[\sup_{t\le q}\Big|\int_0^t U_s^n\,dM_s^{n,a}\Big| > K\Big] &\le \mathcal{P}\Big[\sup_{t\le \tau^n}\Big|\int_0^t U_s^n\,dM_s^{n,a}\Big| > K\Big] + \varepsilon \\ &\le K^{-2}E\Big[\sup_{t\le \tau^n}\Big|\int_0^t U_s^n\,dM_s^{n,a}\Big|^2\Big] + \varepsilon \\ &\le 4K^{-2}E[[M^{n,a}]_{\tau^n}] + \varepsilon \le 4K^{-2}(K + 4a^2) + \varepsilon,\end{aligned}$$

we deduce that $\{\sup_{t\le q}|\int_0^t U_s^n\,dM_s^{n,a}|\}$ is tight in $\mathbb{R}$.

(i)⇒(ii). Due to Theorem (B.1), $\{[Z^n]_q\}$ is tight in $\mathbb{R}$. Hence for every $a > 0$,

$$\Big\{\mathrm{Var}(J^{n,a})_q = \sum_{0<s\le q}|\Delta Z_s^n|\mathbf{1}_{\{|\Delta Z_s^n|>a\}}\Big\}$$

is also tight in $\mathbb{R}$. Moreover, it is obvious that $\{B^{n,a} + M^{n,a}\}$ satisfies (UT). Using once more (B.1) for every $\varepsilon > 0$ there exists $K > 0$ such that $\mathcal{P}[[B^{n,a} + M^{n,a}]_q > K] < \varepsilon$, $n \in \mathbb{N}$. Define $\tau^n = \inf\{s > 0 : [B^{n,a} + M^{n,a}]_s > K\} \wedge q$. Then $\mathcal{P}[\tau^n < q] < \varepsilon$, $n \in \mathbb{N}$ and by [DeMe80, Chap. 7, Theorem 55] we get

$$\begin{aligned}\mathcal{P}[[M^{n,a}]_q > K] &\le K^{-2}E[M^{n,a}]_{\tau^n}] + \varepsilon \le K^{-2}E[[B^{n,a} + M^{n,a}]_{\tau^n}] + \varepsilon \\ &\le K^{-2}(K + a^2) + \varepsilon,\end{aligned}$$

and thus $\{[M^{n,a}]\}$ is tight in $\mathbb{R}$. As a consequence also $\{B^{n,a}\}$ satisfies (UT). Since $B^{n,a}$ is predictable, we can approximate $\mathrm{Var}(B^{n,a})$ by the sums of the form $\sum U^n_{t_i}(B^{n,a}_{t_{i+1}} - B^{n,a}_{t_i})$, where $U^n_{t_i} = \pm 1$. Therefore (UT) for $\{B^{n,a}\}$ implies that $\{\mathrm{Var}(B^{n,a})_q\}$ is tight in $\mathbb{R}$. ∎

The following two corollaries are easy consequences of Theorem 1.1.

COROLLARY 1.2. *If* $\{Z^n\}$ *satisfies* (UT), *and* $\{U^n\}$ *is a sequence of predictable processes such that* $\{\sup_{t\le q}|U^n_t|\}$ *is tight in* $\mathbb{R}$, *then the sequence of stochastic integrals* $\{\int_0^{\cdot} U^n_s\,dZ^n_s\}$ *also satisfies* (UT).

Proof. For every $\varepsilon > 0$ there exists a constant K and a stopping time τ^n such that $\mathcal{P}[\tau^n < q] < \varepsilon/2$ and $\sup_{t\le q}|U^{n,\tau^n}_t| \le K, n \in \mathbb{N}$. Now, the proof is immediate by Theorem 1.1(iii). ∎

COROLLARY 1.3. *Assume* $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}, F \in \mathcal{C}^{1,2}$. *If* $\{Z^n\}$ *satisfies* (UT) *then* $\{(F(t, Z^n_t))\}$ *also satisfies* (UT).

Proof. The proof follows by Itô's formula, Theorem 1.1 and Corollary 1.2. ∎

1.2. Convergence of semimartingales and (UT). It is possible to give a slightly simpler characterization of (UT) in the case when the sequence $\{Z^n\}$ of semimartingales is weakly convergent.

PROPOSITION 1.4. *Assume* $Z^n \to_{\mathcal{D}} Z$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$.

(i) $\{Z^n\}$ *satisfies* (UT) *if and only if there exists* $a > 0$ *such that for every* $q \in \mathbb{R}^+$ $\{\mathrm{Var}(B^{n,a})_q\}$ *is tight in* $\mathbb{R}$.

(ii) *If* $\{Z^n\}$ *is a sequence of local martingales such that for every* $q \in \mathbb{R}^+$,

$$\sup_{n\in\mathbb{N}} E[\sup_{t\le q}|\Delta Z^n_s|] < +\infty,$$

then $\{Z^n\}$ *satisfies* (UT).

(iii) $\{Z^n\}$ *satisfies* (UT) *if and only if there exists* $a > 0$ *and a family of stopping times* $\{\tau^n_\alpha\}$, $\alpha > 0$, $n \in \mathbb{N}$, *such that* $\mathcal{P}[\tau^n_\alpha \ge \alpha] \ge \alpha^{-1}$ *and for every* $q \in \mathbb{R}^+$,

$$\sup_{n\in\mathbb{N}} E\{[M^{n,a}]_{q\wedge\tau^n_\alpha} + \mathrm{Var}(B^{n,a})_{q\wedge\tau^n_\alpha}\} < \infty.$$

Proof. (i) See [JMP89, Lemma 3.1].

(ii) See [Jac79b, p. 342].

(iii) Assume that the condition described in (iii) is satisfied. Then in particular $\{\mathrm{Var}(B^{n,a})_q\}$ is tight in $\mathbb{R}$ and due to (i), $\{Z^n\}$ fulfills (UT). Conversely, assume that $\{Z^n\}$ satisfies (UT). Define $\tau^n_c = \inf\{s > 0 : [M^{n,a}]_s \vee \mathrm{Var}(B^{n,a})_s \ge c\}$, $n \in \mathbb{N}$, $c > 0$. By Theorem 1.1(ii) there exists c_α such that $\mathcal{P}[\tau^n_{c_\alpha} \ge \alpha] \ge \alpha^{-1}$ and (iii) is satisfied. ∎

In the sequel the following assumption will sometimes be considered:

$$\sup_{t\le q}|Z^n_t - Z_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+. \tag{1.2}$$

If we know that the sequence of semimartingales $\{Z^n\}$ is adapted to the same filtration $(\mathcal{F}_t)$ and $\{Z^n\}$ fulfills (UT) it is possible to describe (1.2) more precisely.

PROPOSITION 1.5. *Let* $\{Z^n\}$ *fulfil* (UT) *and be adapted to the same filtration* $(\mathcal{F}_t)$. *Then the condition* (1.2) *is satisfied if and only if for every* $a > 0$,

$$\sup_{t\le q}\{|B_t^{n,a} - B_t^a| + |M_t^{n,a} - M_t^a| + |J_t^{n,a} - J_t^a|\} \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

where $Z^n = J^{n,a} + M^{n,a} + B^{n,a}$, $Z = J^a + M^a + B^a$ (see (1.1)).

Proof. Due to Bichteler, Dellacherie and Mokobodzki's theorem Z is an $(\mathcal{F}_t)$ adapted semimartingale and as a consequence the sequence $\{Z^n - Z\}$ satisfies (UT). By [Jac79b, p. 342], $[Z^n - Z]_q \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$. Since

$$\sup_{t\le q}|J_t^{n,a} - J_t^a| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

we also have the convergence $[B^{n,a} + M^{n,a} - B^a - M^a]_q \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$. By the Lenglart–Rebolledo inequality for every ε, $\eta > 0$,

$$\begin{aligned}\mathcal{P}[[M^{n,a} - M^a]_q > \varepsilon] \le \varepsilon^{-1}E[[M^{n,a} - M^a + B^{n,a} - B^a]_q \wedge \eta + 4a^2]\\ + \mathcal{P}[[M^{n,a} - M^a + B^{n,a} - B^a]_q > \eta].\end{aligned}$$

Hence we deduce that $[M^{n,a} - M^a]_q \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, which is equivalent to the convergence

$$\sup_{t\le q}|M_t^{n,a} - M_t^a| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+.$$

Since

$$\sup_{t\le q}|B_t^{n,a} - B_t^a| \le \sup_{t\le q}|Z_t^n - J_t^{n,a} - Z_t + J_t^a| + \sup_{t\le q}|M_t^{n,a} - M_t^a|, \qquad q \in \mathbb{R}^+,$$

the proof is complete. ∎

Let $\{Z^n\}$ be a sequence of semimartingales. Until the end of Section 1.2 we assume that $Z^n \to_{\mathcal{D}} Z$, where Z is a process with continuous trajectories. Let us consider the canonical decomposition of Z^n of the form (1.1) for fixed $a > 0$. For simplicity we will write briefly $Z^n = J^n + M^n + B^n$, $n \in \mathbb{N}$. We will also write $\nu^n[dt, dx]$ for the dual, predictable projection of the measure of jumps of Z^n, $n \in \mathbb{N}$. A sequence of processes $\{Y^n\}$ is *C-tight* in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ if it is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and each of its limit points has continuous trajectories.

PROPOSITION 1.6. *Let* $\{Z^n\}$ *be a sequence of semimartingales satisfying the condition* (UT). *If* $Z^n \to_{\mathcal{D}} Z$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, *where* Z *is a semimartingale possessing continuous trajectories*, *then*

$$\{(Z^n, M^n, [M^n], B^n)\} \quad \textit{is C-tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}),$$

and every limit point is of the form $(Z, M', [M'], B')$, *where* M' *is a local martingale adapted to the natural filtration of* (Z, M') *such that* $[M'] = [Z]$ *and* B' *is a process with locally bounded variation.*

Proof. Since Z has continuous trajectories it is clear that $\sup_{t\le q}|J_t^n| \to_{\mathcal{P}} 0$ and $\sup_{t\le q}|\Delta Z_t^n| \to_{\mathcal{P}} 0$. On the other hand, for every $\varepsilon > 0$,

$$\sup_{t\le q}|\Delta B_t^n| \le \varepsilon + \sum_{t\le q}\nu^n[\{t\}\times\{x : |x| > \varepsilon\}] \le \varepsilon + a\nu^n[[0,q]\times\{x : |x| > \varepsilon\}]$$

Therefore, in view of the convergence $\nu^n[[0,q]\times\{x:|x|>\varepsilon\}]\to_{\mathcal P}0$ and [JaSh87, Lemma 4.22] we have $\sup_{t\le q}|\Delta B^n_t|\to_{\mathcal P}0$ and $\sup_{t\le q}|\Delta M^n_t|\to_{\mathcal P}0$.

Now, we show that

$$\{[M^n]\}\quad\text{is C-tight.}\tag{1.3}$$

Due to the convergence $\sup_{t\le q}|\Delta M^n_t|^2\to_{\mathcal P}0$ it is sufficient to check that $\{[M^n]\}$ is tight in $\mathbb D(\mathbb R^+,\mathbb R^d)$. We use the well known Aldous criterion [Ald78a, Theorem 1]. By Theorem 1.1, $\{[M^n]\}$ is tight in $\mathbb R$. Let $\{\delta_n\}$ and $\{\sigma^n\}$ be two sequences of constants and of stopping times, respectively, such that $\delta_n\downarrow 0$ and $\sigma^n\le q, n\in\mathbb N$. By the Lenglart–Rebolledo inequality for every $\varepsilon,\eta>0$ we have

$$\begin{aligned}\mathcal P[[M^n]_{(\sigma^n+\delta_n)\wedge\tau^n}-[M^n]_{\sigma^n\wedge\tau^n}>\varepsilon]&\\ \le \varepsilon^{-1}E[[M^n+B^n]_{(\sigma^n+\delta_n)\wedge\tau^n}-[M^n+B^n]_{(\sigma^n+\delta_n)\wedge\tau^n}\wedge\eta+a^2]&\\ +\mathcal P[[M^n+B^n]_{(\sigma^n+\delta_n)\wedge\tau^n}-[M^n+B^n]_{(\sigma^n+\delta_n)\wedge\tau^n}>\eta],&\end{aligned}$$

where $\tau^n=\inf\{t:[M^n+B^n]_t>K\}\wedge(q+1)$ and K is such that $\mathcal P[\tau^n<q+1]\le\varepsilon$. Since $\{[M^n+B^n]\}$ is C-tight and the constants $\varepsilon,\eta>0$ are arbitrarily small we deduce from the above inequality that

$$[M^n]_{\sigma^n+\delta_n}-[M^n]_{\sigma^n}\underset{\mathcal P}{\to}0,$$

and the proof of (1.3) is finished.

Due to (1.3), $\{M^n\}$ is C-tight in $\mathbb D(\mathbb R^+,\mathbb R^d)$. Hence the sequence $\{B^n\}$ is also C-tight in $\mathbb D(\mathbb R^+,\mathbb R^d)$. In view of (UT) and Theorem 1.1, every limit point of it is a process with locally bounded variation. ■

Remark 1.7. If $\{Z^n\}$ is a sequence of local martingales satisfying (UT) and $Z^n\to_{\mathcal D}Z$ then Z need not be a local martingale, as the following example shows (see [JaSh87, p. 454]). Let $Z^n_t=\sum_{k=1}^{[nt]}U^n_k$, where U^n_k are independent identically distributed random variables such that

$$\mathcal P[U^n_k=n]=\frac{1}{n^2},\quad \mathcal P\Big[U^n_k=-\frac{1}{n(1-n^{-2})}\Big]=1-\frac{1}{n^2},\quad n\ge 2.$$

Then $Z^n_t\to_{\mathcal P}-t$ and for $a=1$, $B^n_t=\sum_{k=1}^{[nt]}E[U^n_k\mathbf 1_{\{U^n_k=n\}}]$. Therefore $B^n_t\to_{\mathcal P}-t$, $\mathrm{Var}(B^n)_t\to_{\mathcal P}t$, $t\in\mathbb R^+$, and due to Theorem 1.1, $\{Z^n\}$ satisfies (UT).

Corollary 1.8. *If $\{Z^n\}$ is a sequence of semimartingales satisfying* (UT) *and $\{B^n\}$ is C-tight then the following two conditions are equivalent*:

(i) *$\{Z^n\}$ is C-tight*,
(ii) *$\{[Z^n]\}$ is C-tight.*

Proof. (i)⇒(ii). This implication is a consequence of [Jac79b].

(ii)⇒(i). In this case $\sup_{t\le q}|J^n_t|\to_{\mathcal P}0$, $q\in\mathbb R^+$, and as a consequence $\{[M^n+B^n]\}$ is C-tight. By the arguments from the proof of (1.3), $\{[M^n]\}$ is C-tight, too. Hence C-tightness of $\{M^n\}$ follows. ■

Proposition 1.9. *Assume that $\{Z^n\}$ is a sequence of semimartingales such that $\{\mathrm{Var}(B^n)_q\}$ is C-tight and $Z^n\to_{\mathcal D}Z$, where Z is a process with continuous trajectories. Let $\{U^n\}$ be a sequence of uniformly, locally bounded and predictable processes. Then* *the sequence of stochastic integrals* $\{\int_0^{\cdot} U_s^n\,dZ_s^n\}$ *is C-tight. If additionally* $\mathrm{Var}(B^n)_t \to_{\mathcal{P}} 0, t \in \mathbb{R}^+$, *then every limit point of* $\{\int_0^{\cdot} U_s^n\,dZ_s^n\}$ *is a continuous, local martingale.*

P r o o f. Since $\{Z^n\}$ satisfies (UT), due to Proposition 1.5,

$$\{(M^n, B^n, [M^n])\} \quad \text{is C-tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1}).$$

Moreover, for every $\varepsilon > 0$ there exists a constant K and a stopping time τ^n such that $\mathcal{P}[\tau^n < q] < \varepsilon/2$ and $\sup_{t\le q}|U_t^{n,\tau^n}| \le K$, $n \in \mathbb{N}$. Therefore the following implications are true:

$$\{[M^n]\} \text{ is C-tight} \Rightarrow \Big\{\int_0^{\cdot} U_s^n\,d[M^n]_s^{\tau^n}\Big\} \text{ is C-tight} \Rightarrow \Big\{\int_0^{\cdot} U_s^n\,dM_s^{n,\tau^n}\Big\} \text{ is C-tight},$$

and

$$\{\mathrm{Var}(B^n)_\cdot\} \text{ is C-tight} \Rightarrow \Big\{\int_0^{\cdot} U_s^n\,dB_s^{n,\tau^n}\Big\} \text{ is C-tight}.$$

Since $\varepsilon > 0$ is arbitrarily small the sequences $\{\int_0^{\cdot} U_s^n\,dM_s^n\}$, $\{\int_0^{\cdot} U_s^n\,dB_s^n\}$ are C-tight, and the proof of the first assertion follows.

If $\mathrm{Var}(B^n)_t \to_{\mathcal{P}} 0$, $t \in \mathbb{R}^+$ then $\sup_{t\le q}|\int_0^t U_s^n\,dB_s^n| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$. On the other hand, we deduce from Proposition 1.6 that every limit point of the sequence $\{\int_0^{\cdot} U_s^n\,dM_s^n\}$ has to be a local martingale and the proof of the second assertion is finished. ∎

1.3. Solutions of SDE's and (UT). In this section we consider SDE's of the form

$$X_t^n = H_t^n + \int_0^t f(s, X_\cdot^n)\,dZ_s^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}, \tag{1.4}$$

where $\{Z^n\}$ is a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, $Z_0^n = 0$, $\{H^n\}$ is a sequence of $(\mathcal{F}_t^n)$ adapted initial processes, $f : \mathbb{R}^+ \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ is predictable and such that

$$\|f(t,x)\| \le L(1 + \sup_{s<t}|x_s|), \qquad x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d). \tag{1.5}$$

THEOREM 1.10. *Let* $\{Z^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted semimartingales, and let* $\{H^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted processes. Assume that* f *satisfies* (1.5) *and* $\{X^n\}$ *is a sequence of strong solutions to the SDE* (1.4).

(i) *If* $\{\sup_{t\le q}|H_t^n|\}$ *is tight in* $\mathbb{R}$ *and* $\{Z^n\}$ *satisfies* (UT) *then* $\{\sup_{t\le q}|X_t^n|\}$ *is tight in* $\mathbb{R}$.

(ii) *If* $\{H^n\}$ *and* $\{Z^n\}$ *satisfy* (UT) *then* $\{X^n\}$ *satisfies* (UT).

P r o o f. (i) Define

$$\tau_k^n = \inf\{t : \sup_{s\le t}|H_t^n| \vee \mathrm{Var}(J^{n,a})_t \vee [M^{n,a}]_t \vee \langle M^{n,a}\rangle_t \vee \mathrm{Var}(B^{n,a})_t > k\} \wedge k,$$

$n, k \in \mathbb{N}$. It is clear from Theorem 1.1 that

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathcal{P}[\tau_k^n < q] = 0, \qquad q \in \mathbb{R}^+. \tag{1.6}$$

On the other hand, for every stopping time σ^n,

$$\begin{aligned}\sup_{t<\sigma^n\wedge\tau_k^n}|X^n|^2 &\le 2\Big\{\sup_{t<\sigma^n\wedge\tau_k^n}|H_t^n|^2+\sup_{t<\sigma^n\wedge\tau_k^n}\Big|\int_0^t f(s,X_\cdot^n))\,dZ_s^n\Big|^2\Big\}\\ &\le 2k^2+6C(k,d)\Big\{\sup_{t<\sigma^n\wedge\tau_k^n}\int_0^t\|f(s,X_\cdot^n)\|^2\,d\operatorname{Var}(J^{n,a})_s\\ &\quad+\sup_{t<\sigma^n\wedge\tau_k^n}\int_0^t\|f(s,X_\cdot^n)\|^2\,d\operatorname{Var}(B^{n,a})_s\Big\}\\ &\quad+6\sup_{t<\sigma^n\wedge\tau_k^n}\Big|\int_0^t f(s,X_\cdot^n)\,dM_s^{n,a}\Big|^2.\end{aligned}$$

By the Doob type inequality proved in [MéPe77],

$$E\sup_{t<\sigma^n\wedge\tau_k^n}\Big|\int_0^t(f(s,X_\cdot^n)\,dM_s^{n,a}\Big|^2\le 4dE\int_0^{(\sigma^n\wedge\tau_k^n)-}\|f(s,X_\cdot^n)\|^2\,d([M^{n,a}]_s+\langle M^{n,a}\rangle_s),$$

and we have

$$\begin{aligned}&E\sup_{t<\sigma^n\wedge\tau_k^n}|X_t^n|^2\\ &\quad\le 2k^2+6C(k,d)L^2\Big\{2k+2E\sup_{t<\sigma^n\wedge\tau_k^n}\int_0^t\sup_{u\le s}|X_{u-}^n|^2\,d(\operatorname{Var}(J^{n,a})+\operatorname{Var}(B^{n,a}))_s\Big\}\\ &\qquad+24dL^2\Big\{2k+E\sup_{t<\sigma^n\wedge\tau_k^n}\int_0^t\sup_{u\le s}|X_{u-}^n|^2\,d([M^{n,a}]+\langle M^{n,a}\rangle)_s\Big\}.\end{aligned}$$

Therefore for every stopping time σ^n,

$$\begin{aligned}&E\sup_{t<\sigma^n}|X^{n,\tau_k^n-}|^2\\ &\quad\le 2k^2+C(k,d,L)E\int_0^{\sigma^n-}\sup_{u\le s}|X_{u-}^{n,\tau_k^n-}|^2\,d(\operatorname{Var}(J^{n,a})+\operatorname{Var}(B^{n,a})+[M^{n,a}]+\langle M^{n,a}\rangle)_s\end{aligned}$$

In view of Lemma C1,

$$E\sup_{t<\tau_k^n}|X_t^n|^2\le 2k^2\exp\{4kC(k,d,L)\}.$$

This and (1.6) finish the proof of (i).

(ii) By (i) and (1.5), $\{\sup_{t\le q}\|f(t,X_\cdot^n)\|\}$ is tight in $\mathbb{R}$, $q\in\mathbb{R}^+$. In view of Corollary 1.2, $\{\int_0^\cdot f(s,X_\cdot^n)\,dZ_s^n\}$ satisfies (UT) and $\{X^n\}$ is a sum of two sequences of processes satisfying (UT). ■

1.4. Some examples. It is clear that a sequence $\{Z^n=(Z^{n1},\ldots,Z^{nd})\}_{n\in\mathbb{N}}$ of d-dimensional semimartingales satisfies (UT) if and only if (UT) holds for every sequence $\{Z^{ni}\}$, $i=1,\ldots,d$. Therefore, without loss of generality, we may and will consider only one-dimensional processes.

Example 1.11. Let $\{U_k\}$ be a sequence of independent identically distributed random variables such that $EU_k = 0$, $EU_k^2 = 1$, $k \in \mathbb{N}$. Define

$$Z_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} U_k, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}.$$

By Donsker's invariance principle, $Z^n \to_{\mathcal{D}} W$, where W is a standard Wiener process. If we denote $\mathcal{F}_n = \sigma(U_1, \ldots, U_k)$ and $\mathcal{F}_t^n = \mathcal{F}_{[nt]}$ for $t \in \mathbb{R}^+, n \in \mathbb{N}$ then $\{Z^n\}$ is a sequence of $(\mathcal{F}_t^n)$ adapted local martingales such that

$$E \sup_{t \le q} |\Delta Z_t^n| \le \bigg(E\bigg(\frac{1}{n} \max_{k \le [nt]} U_k^2\bigg)\bigg)^{1/2} \le \bigg(E\bigg(\frac{1}{n} \sum_{k=1}^{[nt]} U_k^2\bigg)\bigg)^{1/2} \le \sqrt{t}.$$

By Proposition 1.4(ii), $\{Z^n\}$ satisfies (UT).

The next example shows that if we disturb the sequence $\{Z^n\}$ slightly then (UT) need not be satisfied.

Example 1.12. Let $\{U_k\}$ be defined as above. Define

$$\widehat{Z}_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} U_k + (nt - [nt]) \frac{U_{[nt]+1}}{\sqrt{n}}, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}.$$

Then $\{\widehat{Z}^n\}$ is a sequence of continuous processes with bounded variation. Therefore the predictable and bounded variation part in the decomposition (1.1) of $\widehat{Z}^n$ is exactly equal to $\widehat{Z}^n$, i.e. $\widehat{Z}^n = B^{n,a}$. Since $\sup_{t \le q} |\widehat{Z}_t^n - Z_t^n| \to_{\mathcal{P}} 0$, it is clear that $B^{n,a} \to_{\mathcal{D}} W$. Hence $\operatorname{Var}(B^{n,a})_q \to_{\mathcal{P}} +\infty$ and by Theorem 1.1, (UT) is not satisfied for the sequence $\{\widehat{Z}^n\}$.

Similar examples can be constructed by using a sequence of discretizations of a given semimartingale.

Example 1.13. Let Z be an $(\mathcal{F}_t)$ adapted semimartingale. Consider the sequence $\{T_n\}$ of partitions of $\mathbb{R}^+$ satisfying (0.6). Define

$$\begin{aligned} Z_t^{\varrho^n} &= Z_{t_{nk}}, \\ \mathcal{F}_t^{\varrho^n} &= \mathcal{F}_{t_{nk}} \qquad \text{for } t \in [t_{nk}, t_{n,k+1}),\ k \in \mathbb{N} \cup \{0\},\ n \in \mathbb{N}. \end{aligned}$$

Then $\{Z^{\varrho^n}\}$ is a sequence of $(\mathcal{F}_t^{\varrho^n})$ adapted processes with bounded variation. Due to the theorem by Bichteler, Dellacherie and Mokobodzki $\{Z^{\varrho^n}\}$ satisfies (UT).

Example 1.14. Let Z be an $(\mathcal{F}_t)$ adapted semimartingale and let $\{\widehat{Z}^{\varrho^n}\}$ be a sequence of linear approximations of Z, i.e.

$$\widehat{Z}_t^{\varrho^n} = Z_{t_{nk}} + \frac{t - t_{nk}}{t_{n,k+1} - t_{nk}} (Z_{t_{n,k+1}} - Z_{t_{nk}}), \qquad t \in [t_{nk}, t_{n,k+1}[,\ n \in \mathbb{N},\ k \in \mathbb{N} \cup \{0\}.$$

Then $\{\widehat{Z}^{\varrho^n}\}$ is a sequence of processes with bounded variation and with continuous trajectories adapted to the same filtration $(\mathcal{F}_t)$. In this case, similarly to Example 1.12, we have $B^{n,a} = \widehat{Z}^{\varrho^n}$. Due to Theorem 1.1, $\{\widehat{Z}^{\varrho^n}\}$ satisfies (UT) if and only if Z is a process with bounded variation.

2. Weak convergence of solutions

2.1. Main results. Let Z be an $(\mathcal{F}_t)$ adapted semimartingale, $Z_0 = 0$, and H an $(\mathcal{F}_t)$ adapted process. Assume that $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a Lipschitz continuous function with a constant L, $L > 0$, i.e.

$$\|f(x) - f(y)\| \le L|x - y|, \qquad x, y \in \mathbb{R}^d$$

(in the sequel we write for simplicity $f \in \mathbb{L}(L)$). It is well known by theorems of [Dol78, Pro77] that there exists a unique strong solution of the SDE

$$X_t = H_t + \int_0^t f(X_{s-})\, dZ_s, \qquad t \in \mathbb{R}^+. \tag{2.1}$$

Now, let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, $Z_0^n = 0$, and let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes. Then for every $n \in \mathbb{N}$ there exists a unique strong solution X^n corresponding to Z^n and H^n, i.e.

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\, dZ_s^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}. \tag{2.2}$$

The purpose of this chapter is to give some sufficient conditions under which $\{X^n\}$ converges in law to X.

Theorem 2.1. *Let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, and let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes. Assume that $\{X^n\}$ is the sequence of strong solutions to the SDE* (2.2) *and $\{Z^n\}$ satisfies* (UT). *If*

$$(H^n, Z^n) \underset{\mathcal{D}}{\to} (H, Z) \qquad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$$

then

$$(X^n, H^n, Z^n) \underset{\mathcal{D}}{\to} (X, H, Z) \qquad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where X is the unique strong solution to the SDE (2.1).

We defer the proof of Theorem 2.1 to Section 2.3.

Our method allows obtaining a stability theorem for strong solutions of SDE's which are slight modifications of the SDE's (2.1), (2.2).

Let $\{X^n\}$ be the sequence of unique strong solutions of SDE's

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\, dM_s^n + \int_0^t g(X_{s-}^n)\, dB_s^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N},$$

where $f \in \mathbb{L}(L)$, $g : \mathbb{R}^d \to \mathbb{R}^d$ is also a Lipschitz continuous function and $\{M^n\}, \{B^n\}$ are sequences of local martingales and of processes with locally bounded variations, respectively ($M_0^n = B_0^n = 0$).

Corollary 2.2. *Assume that* $\sup_n E \sup_{t \le q} |\Delta M_t^n| < +\infty$, $q \in \mathbb{R}^+$, *and for every* $q \in \mathbb{R}^+$, $\{\mathrm{Var}(B^n)_q\}$ *is tight in* $\mathbb{R}$. *If* $(H^n, M^n, B^n) \to_{\mathcal{D}} (H, M, B)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$ *then*

$$(X^n, H^n, M^n, B^n) \underset{\mathcal{D}}{\to} (X, H, M, B) \qquad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

where X is the unique strong solution of the SDE

$$X_t = H_t + \int_0^t f(X_{s-})\,dM_s + \int_0^t g(X_{s-})\,dB_s, \qquad t \in \mathbb{R}^+.$$

Another type of SDE which can be examined by our method is

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\,dZ_s^n + \int_0^t g(X_{s-}^n)\,dv[Z^n]_s, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N},$$

where f, g are as above and $v[Z^n] = ([Z^{n1}], \ldots, [Z^{nd}])$, $n \in \mathbb{N}$.

COROLLARY 2.3. *Assume that* $\{Z^n\}$, $\{H^n\}$ *satisfy the assumptions of Theorem* 2.1. *Then*

$$(X^n, H^n, Z^n) \underset{\mathcal{D}}{\rightarrow} (X, H, Z) \qquad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where X is the unique strong solution of the SDE

$$X_t = H_t + \int_0^t f(X_{s-})\,dZ_s + \int_0^t g(X_{s-})\,dv[Z]_s, \qquad t \in \mathbb{R}^+.$$

It is also possible to generalize slightly the above results by using a sequence of functions $\{f^n\}$, $f^n : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, instead of f. We restrict our attention to the sequences of functions $\{f^n\}$ satisfying the following condition (L):

DEFINITION 2.4. We say that a sequence of functions $f^n : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, satisfies the condition (L) if and only if the three conditions below are fulfilled:

(i) $f^n(t,\cdot) \in \mathbb{L}(L)$, $t \in \mathbb{R}^+$, $n \in \mathbb{N}$,

(ii) for every $w \in \mathbb{R}^d$, $f^n(\cdot, w)$ is left continuous and admits right-hand limits,

(iii) for every sequence $\{(x_n, z_n)\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, $(x, z) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ such that $(x_n, z_n) \to (x, z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ we have $(y_n, z_n) \to (y, z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(d+1)})$, where $y_n(s) = f^n(s+, x_n(s))$, $s \in \mathbb{R}^+$, $n \in \mathbb{N}$.

Assume that $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $f(\cdot, w)$ is continuous for every $w \in \mathbb{R}^d$. Then in view of Lemma A.1 every sequence $\{f^n\}$ of functions satisfying the conditions (i) and (ii) and such that $f^n(s_n, w_n) \to f(s, w)$ as $n \uparrow +\infty$, and $(s_n, w_n) \to (s, w)$ for all $(s, w) \in \mathbb{R}^+ \times \mathbb{R}^d$ fulfills the condition (L).

COROLLARY 2.5. *Assume that* $\{f^n\}$ *satisfies the condition* (L). *If* $\{X^n\}$ *is a sequence of strong solutions to the SDE* (2.1) *with* f^n *instead of* f, *then the conclusion of Theorem* 2.1 *is also true.*

Let $\mathbb{L}^{\mathrm{loc}}$ denote the set of functions f which are locally Lipschitz continuous, i.e. for each f there exists a sequence of constants $\{L_k\}$ and domains $\{D_k\}$ such that $D_k \uparrow \mathbb{R}^d$ and

$$\|f(x) - f(y)\| \le L_k|x - y|, \qquad x, y \in D_k,\ k \in \mathbb{N}.$$

We consider the functions satisfying the linear growth condition, i.e.

$$\|f(t,x)\| \le L(1 + |x|), \qquad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d. \tag{2.3}$$

Corollary 2.6. *Assume that the sequence $\{f^n\}$ of functions satisfies the conditions* (ii), (iii) *from Definition* 2.4. *Additionally, suppose that for every $n \in \mathbb{N}$, $f^n \in \mathbb{L}^{\mathrm{loc}}$ and f^n fulfills* (2.3). *If $\{X^n\}$ is a sequence of strong solutions of the SDE* (2.1) *with f^n instead of f, then the conclusion of Theorem* 2.1 *is true.*

2.2. Tightness criterion. The following tightness criterion will be our main tool in the proof of Theorem 2.1.

Proposition 2.7. (i) *Let $\{X^n\}$ be a sequence of processes. Assume that there exist three families of positive constants, $\{\delta^i\}$, $\{\{\varrho^i_j\}\}$, $\{q_j\}$, where $\delta^i \to 0$, $\varrho^i_j \to 0$ as $i \uparrow +\infty$, $q_j \to +\infty$ as $j \uparrow +\infty$ and for fixed $i \in \mathbb{N}$ there exists a sequence $\{\{\sigma^i_{nk}\}\}$ of partitions of $\mathbb{R}^+$ by stopping times, $0 = \sigma^i_{n0} < \sigma^i_{n1} < \ldots$, $\lim_{k\to+\infty} \sigma^i_{nk} = +\infty$, such that*

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\max_{k\in D^i_{nj}} (\sigma^i_{n,k+1} - \sigma^i_{nk}) > \delta^i] = 0, \tag{2.4}$$

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\min_{k\in D^i_{nj}} (\sigma^i_{n,k+1} - \sigma^i_{nk}) \le \varrho^i_j] = 0, \qquad j \in \mathbb{N}, \tag{2.5}$$

where we set $D^i_{nj} = [k : \sigma^i_{n,k+1} \le q_j]$, $i, j, n \in \mathbb{N}$. If, additionally,

$$\{\sup_{t\le q_j} |X^n_t|\} \quad \textit{is tight in } \mathbb{R},\ j \in \mathbb{N}, \tag{2.6}$$

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le\delta^i} |X^n_t| \ge \varepsilon] = 0, \qquad \varepsilon > 0, \tag{2.7}$$

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\max_{k\in D^i_{nj}} \sup_{\sigma^i_{nk}\le t<\sigma_{n,k+1}} |X^n_t - X^n_{\sigma^i_{nk}}| \ge \varepsilon] = 0, \qquad \varepsilon > 0,\ j \in \mathbb{N}, \tag{2.8}$$

then $\{X^n\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$.

(ii) *Conversely, if a sequence $\{X^n\}$ is weakly convergent in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, then there exist families $\{\delta^i\}$, $\{\{\varrho^i_j\}\}$, $\{q_j\}$ of constants and arrays $\{\{\sigma^i_{nk}\}\}$ of stopping times such that the condition* (2.4)–(2.8) *are fulfilled.*

Proof. The moduli ω, ω', ω'' considered in [Bil68] are essential for the proof (the precise definitions are given also in Index of Symbols).

(i) Suppose that the conditions (2.4)–(2.8) are fulfilled. Then

$$\omega''_{X^n}(\varrho^i_j, q_j - \delta^i) \le 2w_{X^n}(i, q_j)$$

on the set $[\min_{k\in D^i_{nj}} (\sigma^i_{n,k+1} - \sigma^i_{nk}) > \varrho^i_j]$, where

$$w_{X^n}(i, q_j) = \max_{k\in D^i_{nj}} \sup_{\sigma^i_{nk}\le t<\sigma_{n,k+1}} |X^n_t - X^n_{\sigma^i_{nk}}|, \qquad i, j, n \in \mathbb{N}.$$

Hence for $\delta^i \le 1$ we have

$$\mathcal{P}[\omega''_{X^n}(\varrho^i_j, q_j - 1) \ge \varepsilon] \le \mathcal{P}[w_{X^n}(i, q_j) \ge \varepsilon/2] + \mathcal{P}[\min_{k\in D^i_{nj}} (\sigma^i_{n,k+1} - \sigma^i_{nk}) \le \varrho^i_j].$$

Now, (2.5) and (2.8) imply that for every $\varepsilon > 0$ and $j \in \mathbb{N}$,

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\omega''_{X^n}(\varrho^i_j, q_j - 1) \ge \varepsilon] = 0. \tag{2.9}$$

Therefore the assumptions of Theorem 15.3 in [Bil68] are satisfied. As a consequence $\{X^n\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$.

(ii) Now, assume that $X^n \to_{\mathcal{D}} X$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. We construct the arrays $\{\{\sigma^i_{nk}\}\}$ of stopping times as in Słomiński [Sło86]. Let $\{\delta^i\}$ be a sequence of constants $\delta^i \downarrow 0$ such that $\mathcal{P}[|\Delta X_t| = \delta^i, t \in \mathbb{R}^+] = 0$, $i \in \mathbb{N}$. We define, for fixed $i \in \mathbb{N}$,

$$\sigma^i_{n0} = 0, \quad \sigma^i_{n,k+1} = \min(\sigma^i_{nk} + \delta^i_k, \inf\{t > \sigma^i_{nk} : |\Delta X^n_t| > \delta^i\}), \tag{2.10}$$

where $\{\delta^i_k\}$ is a sequence of constants such that

$$\delta^i/2 \le \delta^i_k \le \delta^i \quad \text{and} \quad \mathcal{P}[\Delta X_{\sigma^i_k+\delta^i_k} = 0] = 1, \quad n \in \mathbb{N},\ k \in \mathbb{N}\cup\{0\}. \tag{2.11}$$

By simple calculations

$$w_{X^n}(i,q) \le \delta^i + 2\omega'_{X^n}(\delta^i, q). \tag{2.12}$$

Let $\{q_j\}$, $q_j \to +\infty$, be a sequence of continuity points of X, i.e. $\{q_j\} \subset \operatorname{Cont} X = \{q : \mathcal{P}[\Delta X_q = 0]\}$. Since

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\omega'_{X^n}(\delta^i, q_j) \ge \varepsilon] = 0, \quad \varepsilon > 0,\ j \in \mathbb{N}, \tag{2.13}$$

by (2.12) the condition (2.8) is satisfied. Using [Bil68, Theorem 15.3] once more we obtain the conditions (2.6) and (2.7). On the other hand, it is easy to see from the definitions of $\{\{\sigma^i_{nk}\}\}$ that for every $i \in \mathbb{N}$,

$$(X^n, \sigma^i_{n0}, X^n_{\sigma^i_{n0}}, \sigma^i_{n1}, X^n_{\sigma^i_{n1}}, \ldots) \underset{\mathcal{D}}{\to} (X, \sigma^i_0, X_{\sigma^i_0}, \sigma^i_1, X_{\sigma^i_1}, \ldots) \quad \text{in } (\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)\times\mathbb{R})^\infty. \tag{2.14}$$

Hence for fixed $j \in \mathbb{N}$,

$$\limsup_{n\to+\infty} \mathcal{P}[\min_{k\in D^i_{nj}} (\sigma^i_{n,k+1} - \sigma^i_{nk}) \le \varrho^i_j] \le \mathcal{P}[\min_{k\in D^i_j} (\sigma^i_{k+1} - \sigma^i_k) \le \varrho^i_j].$$

Since $0 = \sigma^i_0 < \sigma^i_1 < \ldots$ and $\max\{k : k \in D^i_j\} < +\infty$ a.s., we can choose a sequence $\{\varrho^i_j\}$ with $\varrho^i_j \to 0$ as $i \uparrow +\infty$ such that the condition (2.5) is fulfilled. Now the proof is complete. ∎

From the above very general criterion we can easily deduce the following corollary.

COROLLARY 2.8. *Let $\{X^n\}$, $\{Z^n\}$ be two sequences of $(\mathcal{F}^n_t)$ adapted processes. Assume that $(X^n, Z^n) \to_{\mathcal{D}} (X, Z)$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$. Then for every $i \in \mathbb{N}$ there exists a sequence $\{X^{ni}\}$ of $(\mathcal{F}^n_t)$ adapted processes such that*

(i) *for every $i \in \mathbb{N}$, $q \in \mathbb{R}^+$,*

$$\{\operatorname{Var}(X^{ni})_q\} \quad \text{is tight in } \mathbb{R},$$

(ii) *for every $i \in \mathbb{N}$,*

$$(X^n, X^{ni}, Z^n) \underset{\mathcal{D}}{\to} (X, X^i, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d}),$$

(iii) *for every $q \in \mathbb{R}^+$ and $\varepsilon > 0$,*

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q} |X^n_t - X^{ni}_t| \ge \varepsilon] = 0,$$

(iv) *if $\{U^n\}$ is a sequence of predictable processes such that $\{\sup_{t\le q} |U^n_t|\}$ is tight in $\mathbb{R}$ then*

$$\Big\{\Big(\int_0^{\cdot} U^n_s\, dX^{ni}_s, X^n, X^{ni}, Z^n\Big)\Big\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d+1}),\ i \in \mathbb{N}.$$

P r o o f. Define $X_t^{ni} = X^n_{\sigma^i_{nk}}$ for $\sigma^i_{nk} \le t < \sigma^i_{n,k+1}$, $t \in \mathbb{R}^+$, $n \in \mathbb{N}$, $i \in \mathbb{N}$, where $\{\{\sigma^i_{nk}\}\}$ are the same as in (2.10). By (2.14) and additionally Proposition 2.7(i) in the case (iv), the results follow. ■

R e m a r k 2.9. Proposition 2.7 has also been used as a main tool in the proof of tightness of solutions for stochastic difference-differential equations (see e.g. [FuKu92, Fuj92a, Fuj92b]).

2.3. Proofs. To prove Theorem 2.1 we use a standard technique (see for example Stroock and Varadhan [StWa79] and Yamada [Yam84]. The proof is performed in several steps.

P r o o f o f T h e o r e m 2.1.

1. *We replace the original processes X^n and H^n by stopped ones*. For fixed $N \in \mathbb{R}^+$ we define a sequence $\{\tau^n_N\}$ of stopping times by

$$\tau^n_N = \begin{cases} \inf\{t : |X^n_t| + |H^n_t| \ge N \text{ or } |X^n_{t-}| + |H^n_{t-}| \ge N\} \\ \text{or } +\infty \quad \text{if } \{\ldots\} = \emptyset,\ n \in \mathbb{N}. \end{cases}$$

If for simplicity we set $X^{n,N} = X^{\tau^n_N}$, $H^{n,N} = H^{\tau^n_N}$, $Z^{n,N} = Z^{\tau^n_N}$, $n \in \mathbb{N}$, then $X^{n,N}$ is a strong solution of the SDE

$$X^{n,N}_t = H^{n,N}_t + \int_0^t f(X^{n,N}_{s-}) \mathbf{1}_{\{|X^{n,N}_{s-}| + |H^{n,N}_{s-}| < N\}}\, dZ^n_s, \qquad t \in \mathbb{R}^+, \tag{2.15}$$

or equivalently,

$$X^{n,N}_t = H^{n,N}_t + \int_0^t f(X^{n,N}_{s-})\, dZ^{n,N}_s, \qquad t \in \mathbb{R}^+.$$

2. *We show the tightness of $\{(X^{n,N}, H^n, Z^n)\}$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$*. Since we consider the pairs of processes $\{(H^n, Z^n)\}$, our definition of stopping times $\{\{\sigma^i_{nk}\}\}$ must be different from (2.10). Let us set

$$\sigma^i_{n0} = 0, \quad \sigma^i_{n,k+1} = \min(\sigma^i_{nk} + \delta^i_k, \inf\{t > \sigma^i_{nk} : |\Delta H^n_t| + |\Delta Z^n_t| > \delta^i\}), \quad i, k \in \mathbb{N},$$

where $\{\delta^i\}$, $\{\delta^i_k\}$ are two sequences of constants such that $\delta^i \downarrow 0$, $\delta^i/2 \le \delta^i_k \le \delta^i$ and $\mathcal{P}[|\Delta H_t| + |\Delta Z_t| = \delta^i, t \in \mathbb{R}^+] = 0$, $\mathcal{P}[|\Delta H_{\sigma^i_k + \delta^i_k}| + |\Delta Z_{\sigma^i_k + \delta^i_k}| = 0] = 1$, $i \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$.

By Lemma A.1 and by the continuous mapping theorem (see e.g. [Bil68, Theorem 5.1])

$$|Z^n| + |H^n| \underset{\mathcal{D}}{\rightarrow} |Z| + |H|$$

and as a consequence of Proposition 2.7(ii), the conditions (2.4)–(2.8) are satisfied. Now, for every $i \in \mathbb{N}$, we define new sequences $\{H^{n,i}\}$, $\{Z^{n,i}\}$ of processes by

$$H^{n,i}_t = H^n_{\sigma^i_{nk}} \quad \text{for } \sigma^i_{nk} \le t < \sigma^i_{n,k+1},$$
$$Z^{n,i}_t = H^n_{\sigma^i_{nk}} \quad \text{for } \sigma^i_{nk} \le t < \sigma^i_{n,k+1},$$

for all $i \in \mathbb{N}, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Then using the continuous mapping theorem once more, we obtain the convergence $(H^n, H^{n,i}) \to_{\mathcal{D}} (H, H^i)$ and $(Z^n, Z^{n,i}) \to_{\mathcal{D}} (Z, Z^i)$.

Moreover, Lemma A.2 implies that

$$\sup_{t\le q}|H^n_t - H^{n,i}_t| \underset{\mathcal{D}}{\to} \sup_{t\le q}|H_t - H^i_t| \quad \text{in } \mathbb{R},\ q\in \operatorname{Cont} H.$$

Hence for every $\varepsilon > 0$ and every $q \in \operatorname{Cont} H$,

$$\limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q}|H^n_t - H^{n,i}_t| \ge \varepsilon] \le \mathcal{P}[\sup_{t\le q}|H_t - H^i_t| \ge \varepsilon].$$

Finally, by (2.12),

$$\sup_{t\le q}|H_t - H^i_t| \le w_H(i,q) \le \delta^i + 2\omega'_H(\delta^i, q)$$

and

$$\lim_{i\to+\infty}\limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q}|H^n_t - H^{n,i}_t| \ge \varepsilon] = 0, \quad \varepsilon > 0,\ q\in\mathbb{R}^+. \tag{2.16}$$

Observe that arguing in the same way we obtain

$$\lim_{i\to+\infty}\limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q}|Z^n_t - Z^{n,i}_t| \ge \varepsilon] = 0, \quad \varepsilon > 0,\ q\in\mathbb{R}^+. \tag{2.17}$$

Now our aim is to show a similar property for the sequence of processes $\{X^{n,N}\}$:

$$\lim_{i\to+\infty}\limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q}|X^{n,N}_t - X^{n,N,i}_t| \ge \varepsilon] = 0, \quad \varepsilon > 0,\ q\in\mathbb{R}^+, \tag{2.18}$$

where by definition $X^{n,N,i}_t = X^{n,N}_{\sigma^i_{nk}}$ for $\sigma^i_{nk} \le t < \sigma^i_{n,k+1}$, $n,i\in\mathbb{N}$, $k\in\mathbb{N}\cup\{0\}$.

First let us observe that if $t\in[\sigma^i_{nk}, \sigma^i_{n,k+1}[$ then by (2.15) we have

$$\begin{aligned}|X^{n,N}_t - X^{n,N,i}_t|^2 &\le 3|H^{n,N}_t - H^{n,N,i}_t|^2 \\ &\quad + 3\Big|\int_{(\sigma^i_{nk},t]} (f(X^{n,N}_{s-}) - f(X^{n,N,i}_{s-}))\mathbf{1}_{\{|X^{n,N}_{s-}|+|H^{n,N}_{s-}|<N\}}\, dZ^n_s\Big|^2 \\ &\quad + 3\Big|\int_{(\sigma^i_{nk},t]} f(X^{n,N,i}_{s-})\mathbf{1}_{\{|X^{n,N}_{s-}|+|H^{n,N}_{s-}|<N\}}\, dZ^n_s\Big|^2.\end{aligned}$$

Assume $\delta^i \le a$. If we apply the two equalities

$$\int_{(\sigma^i_{nk},t]} f(X^{n,N,i}_{s-})\mathbf{1}_{\{|X^{n,N}_{s-}|+|H^{n,N}_{s-}|<N\}}\, dZ^n_s = f(X^{n,N}_{\sigma^i_{nk}})(Z^{n,N}_t - Z^{n,N}_{\sigma^i_{nk}})$$

and

$$Z^n_t - Z^n_{\sigma^i_{nk}} = Z^{n,a}_t - Z^{n,a}_{\sigma^i_{nk}},$$

which are true on the interval $[\sigma^i_{NK}, \sigma^i_{n,k+1}[$, we obtain

$$\begin{aligned}|X^{n,N}_t - X^{n,N,i}_t|^2 &\le 3|H^{n,N}_t - H^{n,N,i}_t|^2 \\ &\quad + 3\Big|\int_{(\sigma^i_{nk},t]} (f(X^{n,N}_{s-}) - f(X^{n,N,i}_{s-}))\mathbf{1}_{\{|X^{n,N}_{s-}|+|H^{n,N}_{s-}|<N\}}\, dZ^n_s\Big|^2 \\ &\quad + 3|f(X^{n,N}_{\sigma^i_{nk}})\mathbf{1}_{\{|X^{n,N}_{\sigma^i_{nk}}|<N\}}(Z^n_t - Z^{n,i}_t)|^2.\end{aligned}$$

Hence for every stopping time σ^n,

$$\sup_{t<\sigma^n}|X_t^{n,N}-X_t^{n,N,i}|^2 \le 3\sup_{t<\sigma^n}|H_t^{n,N}-H_t^{n,N,i}|^2 + 12\sup_{t<\sigma^n}\Big|\int_0^t (f(X_{s-}^{n,N})-f(X_{s-}^{n,N,i}))\mathbf{1}_{\{|X_{s-}^{n,N}|+|H_{s-}^{n,N}|<N\}}\,dZ_s^{n,a}\Big|^2 + 3(LN+\|f(0)\|)^2\sup_{t<\sigma^n}(Z_t^n-Z_t^{n,i})^2. \tag{2.19}$$

Let us fix $M\in\mathbb{R}^+$ and define for every $n\in\mathbb{N}$,

$$\gamma_M^n=\inf\{t:\max([M^{n,a}]_t,\langle M^{n,a}\rangle_t,\mathrm{Var}(B^{n,a})_t,|H_t^{n,N}-H_t^{n,N,i}|,|Z_t^n-Z_t^{n,i}|)>M\}.$$

Since the jumps of the processes considered in the above definition are uniformly bounded, it is easy to observe that there exists $K>0$ such that these processes stopped at γ_M^n are bounded by K. Let us fix $q\in\mathbb{R}^+$. By Theorem 1.1, $\{\mathrm{Var}(B^{n,a})_q\}$ and $\{[M^{n,a}]\}$ are tight in $\mathbb{R}$. By the Lenglart type inequality proved in [Reb80] for every $D,G>0$,

$$\mathcal{P}[\langle M^{n,a}\rangle_q\ge G]\le G^{-1}E([M^{n,a}]_q\wedge(D+a^2))+\mathcal{P}[[M^{n,a}]_q\ge D].$$

Therefore, the sequence $\{\langle M^{n,a}\rangle_q\}$ is also tight in $\mathbb{R}$. Combining the above properties with (2.16) and (2.17) we obtain

$$\lim_{M\to+\infty}\limsup_{n\to+\infty}\mathcal{P}[\gamma_M^n\le q]=0,\qquad q\in\mathbb{R}^+. \tag{2.20}$$

Below we discuss only the processes stopped at the stopping time $\gamma_M^n\wedge q$. For simplicity we write $X^{n,N}$, $H^{n,N}$, $Z^{n,N}$ instead of $X^{n,N,\gamma_M^n\wedge q}$, $H^{n,N,\gamma_M^n\wedge q}$, $Z^{n,N,\gamma_M^n\wedge q}$. Applying (2.19) we get

$$\begin{aligned}E\sup_{t<\sigma^n}|X_t^{n,N}-X_t^{n,N,i}|^2 &\le 3E\sup_{t<\sigma^n}|H_t^{n,N}-H_t^{n,N,i}|^2\\ &\quad+24E\sup_{t<\sigma^n}\Big|\int_0^t (f(X_{s-}^{n,N})-f(X_{s-}^{n,N,i}))\mathbf{1}_{\{|X_{s-}^{n,N}|+|H_{s-}^{n,N}|<N\}}\,dB_s^{n,a}\Big|^2\\ &\quad+24E\sup_{t<\sigma^n}\Big|\int_0^t (f(X_{s-}^{n,N})-f(X_{s-}^{n,N,i}))\mathbf{1}_{\{|X_{s-}^{n,N}|+|H_{s-}^{n,N}|<N\}}\,dM_s^{n,a}\Big|^2\\ &\quad+3(LN+\|f(0)\|)^2\sup_{t<\sigma^n}|Z_t^n-Z_t^{n,i}|^2.\end{aligned}$$

By the Doob type inequality proved by Métivier and Pellaumail [MéPe77],

$$\begin{aligned}&E\sup_{t<\sigma^n}\Big|\int_0^t (f(X_{s-}^{n,N})-f(X_{s-}^{n,N,i}))\mathbf{1}_{\{|X_{s-}^{n,N}|+|H_{s-}^{n,N}|<N\}}\,dM_s^{n,a}\Big|^2\\ &\quad\le 4dE\int_0^{\sigma^n-}\|f(X_{s-}^{n,N})-f(X_{s-}^{n,N,i})\|^2\mathbf{1}_{\{|X_{s-}^{n,N}|+|H_{s-}^{n,N}|<N\}}\,d([M^{n,a}]_s+\langle M^{n,a}\rangle_s).\end{aligned}$$

Since $f \in \mathbb{L}(L)$, we have

$$
\begin{aligned}
&E \sup_{t<\sigma^n} |X_t^{n,N} - X_t^{n,N,i}|^2 \\
&\qquad \le 24L^2 E \operatorname{Var}(B^{n,a})_{\sigma^n-} \int_0^{\sigma^n-} \sup_{u\le s} |X_{s-}^{n,N} - X_{s-}^{n,N,i}|^2 \, d\operatorname{Var}(B^{n,a})_s \\
&\qquad\quad + 96L^2 dE \int_0^{\sigma^n-} \sup_{u\le s} |X_{s-}^{n,N} - X_{s-}^{n,N,i}|^2 \, d([M^{n,a}]_s + \langle M^{n,a}\rangle_s) + \varepsilon_n^i,
\end{aligned}
$$

where

$$
\varepsilon_n^i = 3E \sup_{t<\sigma^n} |H_t^{n,N} - H_t^{n,N,i}|^2 + 3(LN + \|f(0)\|)^2 \sup_{t<\sigma^n} |Z_t^n - Z_t^{n,i}|^2.
$$

If we put

$$
\begin{aligned}
Y_t^1 &= \sup_{s\le t} |X_s^{n,N} - X_s^{n,N,i}|^2, \qquad a_1 = 24L^2(M+2)K + 96L^2\, d(K+K), \\
Y_t^2 &= 24L^2(M+2)\operatorname{Var}(B^{n,a})_t + 96L^2\, d([M^{n,a}]_t + \langle M^{n,a}\rangle_t), \qquad a_2 = \varepsilon_n^i,
\end{aligned}
$$

then by Lemma C.1(ii) we have

$$
E \sup_{t<\gamma_M^n\wedge q} |X_s^{n,N} - X_s^{n,N,i}|^2 \le \varepsilon_n^i e^{a_1}, \qquad q \in \mathbb{R}^+,\ M \in \mathbb{R}^+.
$$

Since $\lim_{i\to+\infty} \limsup_{n\to+\infty} \varepsilon_n^i = 0$, it follows from (2.20) that (2.18) is satisfied.

Finally, we show that the conditions (2.6) and (2.7) in Proposition 2.7 are satisfied. It is clear by (2.15) that

$$
\begin{aligned}
\sup_{t\le q} |X_t^{n,N}| &\le N + \sup_{t\le q} |\Delta X_t^{n,N}| \\
&\le N + \sup_{t\le q} |\Delta H_t^{n,N}| + (LN + \|f(0)\|) \sup_{t\le q} |\Delta Z_t^n|
\end{aligned}
$$

and that the expressions on the right-hand side of the above inequality are tight in $\mathbb{R}$. Quite similar arguments imply (2.7).

The tightness of $\{(X^{n,N}, H^n, Z^n)\}$ follows now from a suitably adjusted version of Proposition 2.7 with $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$ instead of $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$.

3. *We prove that every subsequence $\{n'\} \subset \{n\}$ contains a subsequence $\{n''\}$ such that*

$$
(X^{n''}, H^{n''}, Z^{n''}) \underset{\mathcal{D}}{\to} (X, H, Z) \qquad \textit{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).
$$

Let $\{N_i\}$ be a sequence of positive constants, $N_i \uparrow +\infty$. Let us fix $i \in \mathbb{N}$.

Since $\{(X^{n,N_1}, H^n, Z^n)\}$ is tight, from every subsequence $\{n'\} \subset \{n\}$ we can choose a further subsequence $\{n_1'\}$ such that

$$
(X^{n_1',N_1}, H^{n_1'}, Z^{n_1'}) \underset{\mathcal{D}}{\to} (\overline{Y}, \overline{H}, \overline{Z}) \qquad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),
$$

where $\mathcal{L}(\overline{H},\overline{Z}) = \mathcal{L}(H,Z)$, where $(\overline{Y},\overline{H},\overline{Z})$ are defined on some probability space $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathcal{P}})$. In view of Lemma 12 of [StWa79] there exists M with $N_1 - 1 < M \le N_1$ such that $\inf\{t : |\overline{H}_t| + |\overline{Y}_t| > M \text{ or } |\overline{H}_{t-}| + |\overline{Y}_{t-}| > M\} = \inf\{t : |\overline{H}_t| + |\overline{Y}_t| \ge M$ or $|\overline{H}_{t-}| + |\overline{Y}_{t-}| \ge M\}$ almost surely. Then by the continuous mapping theorem and by Corollary A.6 (Appendix),

$$(X^{n'_1,M}, H^{n'_1,M}, U^{n'_1}, Z^{n'_1}) \underset{\mathcal{D}}{\to} (\overline{Y}^M, \overline{H}^M, \overline{U}, \overline{Z}) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(d+3)}),$$

where $U^{n'_1}$, $\overline{U}$ are defined as follows:

$$U_s^{n'_1} = f(X_s^{n'_1,M})\mathbf{1}_{\{|X_s^{n'_1,M}|+|H_s^{n'_1,M}|<M\}}, \qquad \overline{U}_s = f(\overline{Y}_s^M)\mathbf{1}_{\{|\bar{Y}_s^M|+|\bar{H}_s^M|<M\}}.$$

Due to the functional limit theorem for stochastic integrals proved by Jakubowski, Mémin and Pages (see Theorem B.2) we have

$$\Big(X^{n'_1,M}, H^{n'_1,M}, \int_0^{\cdot} U_{s-}^{n'_1}\, dZ_s^{n'_1}\Big) \underset{\mathcal{D}}{\to} \Big(\overline{Y}^M, \overline{H}^M, \int_0^{\cdot} \overline{U}_{s-}\, d\overline{Z}_s\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

Hence Lemma A.2 implies the convergence

$$\sup_{t\le q}\Big|X_t^{n'_1,M} - H_t^{n'_1,M} - \int_0^t U_{s-}^{n'_1}\, dZ_s^{n'_1}\Big| \underset{\mathcal{D}}{\to} \sup_{t\le q}\Big|\overline{Y}_t^M - \overline{H}_t^M - \int_0^t \overline{U}_{s-}\, d\overline{Z}_s\Big|$$

for every $q \in \operatorname{Cont} \overline{Y}^M \cap \operatorname{Cont} \overline{H}^M \cap \operatorname{Cont} \overline{Z}$. As a consequence,

$$\overline{Y}_t^M = \overline{H}_t^M + \int_0^t f(\overline{Y}_{s-}^M)\, d\overline{Z}_s^M, \qquad t \in \mathbb{R}^+.$$

On the other hand, on $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathcal{P}})$ there exists a strong solution $\overline{X}$ such that

$$\overline{X}_t = \overline{H}_t + \int_0^t f(\overline{X}_{s-})\, d\overline{Z}_s, \qquad t \in \mathbb{R}^+.$$

By simple calculations $\overline{Y}_t = \overline{X}_t$ for $t \le \overline{\tau}_M$. However, since $\mathcal{L}(\overline{H},\overline{Z}) = \mathcal{L}(H,Z)$ it follows by standard arguments that

$$\mathcal{L}(\overline{X},\overline{H},\overline{Z}) = \mathcal{L}(X,H,Z), \qquad \mathcal{L}(\overline{X}^M,\overline{H},\overline{Z}) = \mathcal{L}(X^M,H,Z).$$

Therefore

$$(X^{n'_1,M}, H^{n'_1}, Z^{n'_1}) \underset{\mathcal{D}}{\to} (X^M, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

Again, since $\{(X^{n'_1,N_2}, H^{n'_1}, Z^{n'_1})\}$ is tight, there exists a subsequence $\{n'_2\} \subset \{n'_1\}$ such that

$$(X^{n'_2,N_2}, H^{n'_2}, Z^{n'_2}) \underset{\mathcal{D}}{\to} (Y', H', Z') \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where $\mathcal{L}(H',Z') = \mathcal{L}(H,Z)$ and by the same arguments there exists M' with $M' > M$, $N_2 - 1 < M' \le N_2$ for which

$$(X^{n'_2,M'}, H^{n'_2}, Z^{n'_2}) \underset{\mathcal{D}}{\to} (X^{M'}, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

This procedure may be continued. Next, by the diagonal procedure we can choose a subsequence $\{n''\}$ such that there exists a sequence $\{M_k\}$ of constants with $M_k \uparrow +\infty$

and

$$(X^{n'',M_k}, H^{n''}, Z^{n''}) \underset{\mathcal{D}}{\to} (X^{M_k}, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),\ k \in \mathbb{N}.$$

Finally, we recall that $\tau^n_{M_k} = \inf\{t : |X^n_t| + |H^n_t| \geq M_k \text{ or } |X^n_{t-}| + |H^n_{t-}| \geq M_k\}$. In order to finish the proof it is sufficient to show that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\tau^n_{M_k} \leq q] = 0, \qquad q \in \mathbb{R}^+. \tag{2.21}$$

Fix $q \in \operatorname{Cont} X \cap \operatorname{Cont} H$. Since

$$\mathcal{P}[\tau^n_{M_k} \leq q] = \mathcal{P}[\sup_{t\leq q}(|X^n_t| + |H^n_t|) \geq M_k] = \mathcal{P}[\sup_{t\leq q}(|X^{n,M_{k+1}}_t| + |H^{n,M_{k+1}}_t|) \geq M_k],$$

it follows from the convergence $|X^{n,M_{k+1}}| + |H^{n,M_{k+1}}| \to_{\mathcal{D}} |X^{M_{k+1}}| + |H^{M_{k+1}}|$ that

$$\limsup_{n\to+\infty} \mathcal{P}[\tau^n_{M_k} \leq q] \leq \mathcal{P}[\sup_{t\leq q}(|X^{M_{k+1}}_t| + |H^{M_{k+1}}_t|) \geq M_k] = \mathcal{P}[\sup_{t\leq q}(|H_t| + |X_t| \geq M_k].$$

Hence (2.21) holds and $\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\sup_{t\leq q} |X^n_t - X^{n,M_k}_t| \geq \varepsilon] = 0$ for all $\varepsilon > 0$, $q \in \mathbb{R}^+$. The proof is complete. ∎

Proof of Corollary 2.2. Note that the sequence $\{M^n\}$ satisfies the condition (UT) by Proposition 1.4. Since $\{\operatorname{Var}(B^n)_q\}$ is tight in $\mathbb{R}$, $\{B^n\}$ also satisfies (UT). Define

$$\sigma^i_{n0} = 0, \qquad \sigma^i_{n,k+1} = \min(\sigma^i_{nk} + \delta^i_k, \inf\{t > \sigma^i_{nk} : |\Delta H^n_t| + |\Delta B^n_t| + |\Delta M^n_t| > \delta^i)$$

for $n, k, i \in \mathbb{N}$, where $\{\delta^i\}$, $\{\delta^i_k\}$ are two sequences of constants such that $\delta^i \downarrow 0$, $\delta^i/2 \leq \delta^i_k \leq \delta^i$ and $\mathcal{P}[|\Delta H_t| + |\Delta B_t| + |\Delta M_t| = \delta^i,\ t \in \mathbb{R}^+] = 0$, $\mathcal{P}[|\Delta H_{\sigma^i_k+\delta^i_k}| + |\Delta B_{\sigma^i_k+\delta^i_k}| + |\Delta M_{\sigma^i_k+\delta^i_k}| = 0] = 1$, $i \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$. Finally, we use the arguments similar to those in the proof of Theorem 2.1. ∎

Proof of Corollary 2.3. Due to Theorem 1.8 in [Jac79b] the condition (UT) implies $(H^n, Z^n, v[Z^n]) \to_{\mathcal{D}} (H, Z, v[Z])$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$. We complete the proof arguing as in Corollary 2.2. ∎

Proof of Corollary 2.5. The proof is also very similar to the proof of Theorem 2.1. Some additional arguments are necessary in Step 2 only. Let us return to the estimate (2.19). We have additionally to prove that

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}\Big[\max_{k\in D^i_{nj}} \sup_{\sigma^i_{nk}<t<\sigma^i_{n,k+1}} \Big| \int_{(\sigma^i_{nk},t]} f^n(s, X^{n,N,i}_{s-})\, dZ^{n,N}_s \Big| \geq \varepsilon\Big] = 0,$$

for every $q \in \mathbb{R}^+$, $\varepsilon > 0$, but this easily follows from the condition (L). ∎

Proof of Corollary 2.6. By Theorem 1.10, $\{\sup_{t\leq q} |X^n_t|\}$ is tight in $\mathbb{R}$, $q \in \mathbb{R}^+$. Therefore we can apply Corollary 2.5. ∎

3. Convergence in probability

3.1. Flows and strong convergence of solutions. In this section we study SDE's of the form

$$X^x_t = x + H_t + \int_0^t f(X^x_{s-})\, dZ_s, \qquad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \tag{3.1}$$

where Z is an $(\mathcal{F}_t)$ adapted semimartingale, $Z_0 = 0$, and H is an $(\mathcal{F}_t)$ adapted process. The flow of the above SDE (3.1) is a function $\mathbb{R}^d \ni x \mapsto X^x_\cdot \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Ikeda, Watanabe [IkWa81] and Bally [Bal89] considered the convergence in probability of flows of solutions to diffusion equations. In the present section we investigate flows of solutions to SDE's driven by general semimartingales and their convergence. With the help of Theorem 2.1 we obtain the following theorem on the uniform convergence of flows.

THEOREM 3.1. *Let $\{Z^n\}$ be a sequence of $(\mathcal{F}^n_t)$ adapted semimartingales satisfying* (UT), $Z^n_0 = 0$, *and let $\{H^n\}$ be a sequence of $(\mathcal{F}^n_t)$ adapted processes. Assume $f \in \mathbb{L}(L)$ and that $\{X^{n,x}\}$ is a sequence of strong solutions of the SDE*

$$X^{n,x}_t = x + H^n_t + \int_0^t f(X^{n,x}_{s-})\, dZ^n_s, \qquad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d. \tag{3.2}$$

(i) *If $(H^n, Z^n) \to_{\mathcal{P}} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ then*

$$\sup_{x\in K} \delta^{3d}((X^{n,x}, H^n, Z^n), (X^x, H, Z)) \underset{\mathcal{P}}{\to} 0,$$

for every bounded subset $K \subset \mathbb{R}^d$.

(ii) *If $\sup_{t\le q} |H^n_t - H_t| \to_{\mathcal{P}} 0$ and $\sup_{t\le q} |Z^n_t - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, then*

$$\sup_{x\in K} \sup_{t\le q} |X^{n,x}_t - X^x_t| \underset{\mathcal{P}}{\to} 0 \quad \text{for every bounded subset } K \subset \mathbb{R}^d,\ q \in \mathbb{R}^+.$$

Proof. (i) In the first step we show that

$$(X^{n,x}, H^n, Z^n) \underset{\mathcal{P}}{\to} (X^x, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),\ x \in \mathbb{R}^d. \tag{3.3}$$

Let us assume that $\mathcal{Q}$ is another probability measure on $(\Omega, \mathcal{F})$, absolutely continuous with respect to the measure $\mathcal{P}$, i.e. $\mathcal{Q} \ll \mathcal{P}$. It is well known that if Z is a semimartingale on $(\Omega, \mathcal{F}, \mathcal{P})$ then also Z is a semimartingale on $(\Omega, \mathcal{F}, \mathcal{Q})$ (see e.g. [DeMe80]). Here we can obtain more:

LEMMA 3.2. *Let $\{Z^n\}$ be a sequence of semimartingales on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the condition* (UT). *If $\mathcal{Q} \ll \mathcal{P}$ then $\{Z^n\}$ is a sequence of semimartingales on $(\Omega, \mathcal{F}, \mathcal{Q})$ such that the condition* (UT) *holds, too.*

Proof. Let $\{\{U^n_\alpha\}\}$ be a family of processes as in (UT). Assume that $\{\{U^n_\alpha\}\}$ is tight on $(\Omega, \mathcal{F}, \mathcal{P})$. Then for every $\varepsilon > 0$ there exists $N > 0$ such that $\sup_{n,\alpha} \mathcal{P}[|U^n_\alpha| > N] < \varepsilon$. Since on $(\Omega, \mathcal{F}, \mathcal{Q})$ the discrete processes in the condition (UT) are exactly the same (with respect to the probability $\mathcal{Q}$), we have

$$\sup_{n,\alpha} \mathcal{Q}[|U^n_\alpha| > N] = \sup_{n,\alpha} \int_{[|U^n_\alpha| > N]} \left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right) d\mathcal{P}.$$

The result follows because $(d\mathcal{Q}/d\mathcal{P})$ is integrable with respect to $d\mathcal{P}$. ∎

Now, let us fix $B \in \mathcal{F}$, $\mathcal{P}(B) > 0$. Define $\mathcal{Q}_B = \mathcal{P}(A|B)$ for every $A \in \mathcal{F}$. Obviously $\mathcal{Q}_B \ll \mathcal{P}$ and $(d\mathcal{Q}/d\mathcal{P}) = \mathbf{1}_B/\mathcal{P}(B)$.

Let $\{H^n\}$, $\{Z^n\}$ be two sequences of processes satisfying the assumptions of Theorem 3.1(i). By Lemma 3.2, $\{Z^n\}$ is a sequence of semimartingales on $(\Omega, \mathcal{F}, \mathcal{Q}_B)$ for which (UT) is fulfilled. Moreover, the stochastic integral

$$\int f(X^{n,x}_{s-})\, dZ^n_s$$

calculated with respect to $\mathcal{P}$ is for $\mathcal{Q}_B$-almost all $\omega \in \Omega$ equal to the integral calculated with respect to $\mathcal{Q}_B$. Thus

$$(H^n, Z^n) \underset{\mathcal{D}(\mathcal{Q}_B)}{\longrightarrow} (H, Z) \quad \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

and

$$X^{n,x}_t = x + H^n_t + \int_0^t f(X^n_{s-})\, dZ^n_s, \qquad t \in \mathbb{R}^+,\ \mathcal{Q}_B\text{-a.e.}$$

As a consequence, by Theorem 2.1,

$$(X^{n,x}, H^n, Z^n) \underset{\mathcal{D}(\mathcal{Q}_B)}{\longrightarrow} (X^x, H, Z) \quad \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

and hence for all bounded and continuous mappings $\Phi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \to \mathbb{R}$,

$$\lim_{n\to+\infty} \int_\Omega \Phi(X^{n,x}, H^n, Z^n)\, d\mathcal{Q}_B = \int_\Omega \Phi(X^x, H, Z)\, d\mathcal{Q}_B, \tag{3.4}$$

or equivalently

$$\lim_{n\to+\infty} \int_B \Phi(X^{n,x}, H^n, Z^n)\, d\mathcal{P} = \int_B \Phi(X^x, H, Z)\, d\mathcal{P}. \tag{3.5}$$

Since (3.5) holds for all $B \in \mathcal{F}$ with $\mathcal{P}(B) > 0$ and all bounded continuous mappings $\Phi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \to \mathbb{R}$, the proof of (3.3) is complete.

In the second step we will prove the following lemma:

LEMMA 3.3. *Assume that* $\{Z^n\}$ *satisfies* (UT), $f \in \mathbb{L}(L)$ *and that for every* $x \in \mathbb{R}^d$, $\{X^{n,x}\}$ *is a sequence of strong solutions to the SDE* (3.2). *Then there exist arrays* $\{\{\tau^n_k\}\}$ *of stopping times and* $\{\{C^p_k\}\}$ *of constants such that*

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\tau^n_k < q] = 0, \qquad q \in \mathbb{R}^+,$$

and

$$\limsup_{n\to+\infty} E \sup_{t<\tau^n_k} |X^{n,x}_t - X^{n,y}_t|^{2p} \le C^p_k |x-y|^{2p}, \qquad x, y \in \mathbb{R}^d,\ p \in \mathbb{N}.$$

Proof. Define, for fixed $a \in \mathbb{R}^+$, $\tau^n_k = \inf\{t : \mathrm{Var}(J^{n,a})_t \vee [M^{n,a}]_t \vee \langle M^{n,a}\rangle_t \vee \mathrm{Var}(B^{n,a})_t > k\} \wedge k$, $n, k \in \mathbb{N}$. It is clear from Theorem 1.1 that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\tau^n_k < q] = 0, \qquad q \in \mathbb{R}^+.$$

On the other hand, for every stopping time σ^n,

$$
\begin{aligned}
\sup_{t<\sigma^n\wedge\tau^n_k} |X^{n,x}_t - X^{n,y}_t|^{2p}
&\le 2^{p-1}\Big\{|x-y|^{2p} + \sup_{t<\sigma^n\wedge\tau^n_k}\Big|\int_0^t (f(X^{n,x}_{s-}) - f(X^{n,y}_{s-}))\,dZ^n_s\Big|^{2p}\Big\}\\
&\le 2^{p-1}|x-y|^{2p}\\
&\quad + 6^{p-1}\Big\{k^{2p-1}\sup_{t<\sigma^n\wedge\tau^n_k}\int_0^t \|f(X^{n,x}_{s-}) - f(X^{n,y}_{s-})\|^{2p}\,d\,\mathrm{Var}(J^{n,a})_s\\
&\quad + k^{2p-1}\sup_{t<\sigma^n\wedge\tau^n_k}\int_0^t \|f(X^{n,x}_{s-}) - f(X^{n,y}_{s-})\|^{2p}\,d\,\mathrm{Var}(B^{n,a})_s\\
&\quad + \sup_{t<\sigma^n\wedge\tau^n_k}\Big|\int_0^t (f(X^{n,x}_{s-}) - f(X^{n,y}_{s-}))\,dM^{n,a}_s\Big|^{2p}\Big\}.
\end{aligned}
$$

By the Doob type inequality proved by Pratelli [Pra83] there exists a constant C_p such that

$$
\begin{aligned}
&E\sup_{t<\sigma^n\wedge\tau^n_k} |X^{n,x}_t - X^{n,y}_t|^{2p}\\
&\quad\le 2^{p-1}|x-y|^{2p} + 6^{p-1}k^{2p-1}L^{2p}E\sup_{t<\sigma^n\wedge\tau^n_k}\int_0^t |X^{n,x}_{s-} - X^{n,y}_{s-}|^{2p}\,d(\mathrm{Var}(J^{n,a}) + \mathrm{Var}(B^{n,a}))_s\\
&\qquad + 6^{p-1}L^{2p}C(p,d)E\sup_{t<\sigma^n\wedge\tau^n_k}\int_0^t |X^{n,x}_{s-} - X^{n,y}_{s-}|^{2p}\,d([M^{n,a}] + \langle M^{n,a}\rangle)_s.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&E\sup_{t<\sigma^n} |X^{n,x,\tau^n_k-}_t - X^{n,y,\tau^n_k-}_t|^{2p}\\
&\quad\le 2^{p-1}|x-y|^{2p} + C(p,d,k,L)\\
&\qquad\times E\int_0^{\sigma^n-}\sup_{u\le s}|X^{n,x,\tau^n_k-}_{u-} - X^{n,y,\tau^n_k-}_{u-}|^{2p}\,d(\mathrm{Var}(J^{n,a}) + \mathrm{Var}(B^{n,a}) + [M^{n,a}] + \langle M^{n,a}\rangle)_s
\end{aligned}
$$

and due to Lemma C.1,

$$
E\sup_{t<\tau^n_k} |X^{n,x}_t - X^{n,y}_t|^{2p} \le 2^{p-1}|x-y|^{2p}C'(p,d,k,L).
$$

The proof of Lemma 3.3 is complete. ∎

It is obvious that similarly we can define a sequence $\{\tau_k\}$ of stopping times such that $\lim_{k\to+\infty}\mathcal{P}[\tau_k<q]=0$, $q\in\mathbb{R}^+$, and

$$
E\sup_{t<\tau_k}|X^x_t - X^y_t|^{2p} \le C^p_k|x-y|^{2p}, \qquad x,y\in\mathbb{R}^d,\ p\in\mathbb{N}.
$$

Now we are able to finish the proof of Theorem 3.1(i). Let $S(k)$ denote the set of points of K with coordinates of the form $j2^{-k}$, $j\in\mathbb{Z}$. Assume that $\{\tau_k\}$ and $\{\{\tau^n_k\}\}$ are exactly as in Lemma 3.3. Then by (3.3),

$$\limsup_{n\to+\infty} \mathcal{P}[\sup_{x\in K}\sup_{t\le q} \delta^3((X^{n,x},H^n,Z^n),(X^x,H,Z)) > \varepsilon]$$

$$\le \limsup_{n\to+\infty} \mathcal{P}[\sup_{x\in S(k)}\sup_{t\le q} \delta^3((X^{n,x},H^n,Z^n),(X^x,H,Z)) > \varepsilon/3]$$

$$+ \limsup_{n\to+\infty} \mathcal{P}[\sup_{|x-y|\le 2^{-k}}\sup_{t\le q} |X_t^{n,x} - X_t^{n,x}| > \varepsilon/3]$$

$$+ \mathcal{P}[\sup_{|x-y|\le 2^{-k}}\sup_{t\le q} |X_t^{x} - X_t^{x}| > \varepsilon/3]$$

$$\le \limsup_{n\to+\infty} \mathcal{P}[\sup_{|x-y|\le 2^{-k}}\sup_{t<\tau_k^n} |X_t^{n,x} - X_t^{n,x}| > \varepsilon/3]$$

$$+ \mathcal{P}[\sup_{|x-y|\le 2^{-k}}\sup_{t<\tau_k} |X_t^{x} - X_t^{x}| > \varepsilon/3] + \limsup_{n\to+\infty} \mathcal{P}[\tau_k^n < q] + \mathcal{P}[\tau_k < q].$$

In order to estimate the right hand side of the above inequality we use Lemma 3.3 and the arguments from the proof of Proposition 2.2 of [IkWa81] (see also [Bal89, Proposition 5.2]).

(ii) It is clear that $(H^n, Z^n) \to_{\mathcal{P}} (H,Z)$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$. Therefore the condition (3.5) is satisfied. Now, let us note that

$$\Delta X_t^x = \Delta H_t + f(X_{t-})\Delta Z_t, \tag{3.6}$$

and if $\Delta X_t^x \ne 0$ then $\Delta H_t \ne 0$ or $\Delta Z \ne 0$. Due to Corollary A.4 we obtain

$$\sup_{t\le q} |X_t^{n,x} - X_t^x| \underset{\mathcal{P}}{\to} 0, \qquad q\in\mathbb{R}^+,\ x\in\mathbb{R}.$$

Finally, we use Lemma 3.3 and repeat the arguments from the proof of (i). ∎

3.2. Convergence of derivatives of flows. In this section we assume additionally that $f\in\mathcal{C}_b^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$. It is well known that in this case the flow $x\mapsto X_t^x(\omega)$ is differentiable (see e.g. [Pro90]). Given the function $f:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$, $f(x)=\{f_{ik}\}_{i,k=1,\dots,d}$, we rewrite the SDE (3.1) in the form

$$X_t^i = x^i + H_t^i + \sum_{j=1}^d \int_0^t f_{ij}(X_{s-})\,dZ_s^j, \qquad i=1,\dots,d,\ t\in\mathbb{R}^+,$$

where $H_t=(H_t^1,\dots,H_t^d)$ is an $(\mathcal{F}_t)$ adapted initial process, $Z_t=(Z_t^1,\dots,Z_t^d)$ is an $(\mathcal{F}_t)$ adapted semimartingale, $Z_0=0$ and $x=(x^1,\dots,x^d)\in\mathbb{R}^d$.

If we define, for $k=1,\dots,d$,

$$D_{k,t}^x(\omega) = \frac{\partial}{\partial x^k}(X_t^x(\omega)), \qquad x\in\mathbb{R}^d,$$

then the process $D_{k,t}^x$ is a unique strong solution to the following SDE:

$$D_{k,t}^{x,i} = \delta_k^i + \sum_{j=1}^d\sum_{l=1}^d \int_0^t \frac{\partial f_{ij}}{\partial x^l}(X_{s-}^x) D_{k,s-}^{x,l}\,dZ_s^j, \qquad i=1,\dots,d,\ t\in\mathbb{R}^+,$$

where $\delta_k^i=1$ if $i=k$ and 0 otherwise. In [IkWa81, page 429] convergence of derivatives of flows of Itô SDE's has been considered. We give some parallel results for SDE's with respect to general semimartingales.

THEOREM 3.4. *Let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales satisfying* (UT), *and let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes. Assume $f \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ and let $\{X^{n,x}\}$ be a sequence of strong solutions of the SDE* (3.2).

(i) *If $(H^n, Z^n) \to_{\mathcal{D}} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ then*

$$(D_1^{n,x}, \ldots, D_d^{n,x}, X^{n,x}, H^n, Z^n) \underset{\mathcal{D}}{\to} (D_1^x, \ldots, D_d^x, X^x, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(d+3)}).$$

(ii) *If $(H^n, Z^n) \to_{\mathcal{P}} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ then*

$$\sup_{x \in K} \delta^{d(d+3)}((D_1^{n,x}, \ldots, D_d^{n,x}, X^{n,x}, H^n, Z^n), (D_1^x, \ldots, D_d^x, X^x, H, Z)) \underset{\mathcal{P}}{\to} 0,$$

for every bounded subset $K \subset \mathbb{R}^d$.

(iii) *If $\sup_{t \le q} |H_t^n - H_t| \to_{\mathcal{P}} 0$ and $\sup_{t \le q} |Z_t^n - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, then*

$$\sup_{x \in K} \sup_{t \le q} |D_{k,t}^{n,x} - D_{k,t}^x| \underset{\mathcal{P}}{\to} 0,$$

for every bounded subset $K \subset \mathbb{R}^d, q \in \mathbb{R}^+, k = 1, \ldots, d$.

Proof. (i) It is clear that $\{D_k^{n,x}\}$ are the solutions to

$$\begin{aligned} D_{k,t}^{n,x,i} &= \delta_k^i + \sum_{j=1}^d \sum_{l=1}^d \int_0^t \frac{\partial f_{ij}}{\partial x^l}(X_{s-}^{n,x}) D_{k,s-}^{n,x,l}\, dZ_s^{n,j} \\ &= \delta_k^i + \sum_{l=1}^d \int_0^t D_{k,s-}^{n,x,l}\, d\widehat{Z}_s^{n,l}, \quad i = 1, \ldots, d, \ t \in \mathbb{R}^+, \end{aligned}$$

Since $f \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$, the sequence $\{\widehat{Z}^n\}$ satisfies (UT) and $\widehat{Z}^n \to_{\mathcal{D}} \widehat{Z}$, where

$$\widehat{Z}_t^l = \sum_{j=1}^d \int_0^t \frac{\partial f_{ij}}{\partial x^l}(X_{s-}^x)\, dZ_s^j, \quad l = 1, \ldots, d.$$

Therefore now it is enough to use Theorem 2.1. Analogously we deduce (ii) and (iii) from Theorem 3.1(i), (ii). ∎

3.3. Euler schemes for SDE's driven by general semimartingales. Let X be a strong solution to the SDE of the form (2.1), i.e.

$$X_t = H_t + \int_0^t f(X_{s-})\, dZ_s, \quad t \in \mathbb{R}^+,$$

where $f \in \mathbb{L}(L)$, Z is an $(\mathcal{F}_t)$ adapted semimartingale, $Z_0 = 0$ and H is an $(\mathcal{F}_t)$ adapted process only. It is well known that if Z is a Wiener process then the solution X can be uniformly approximated by solutions of discrete SDE's (see e.g. [Mar55, Ku71, PaTa85, KlPl92]). In this section we discuss such an approximation without any restrictions on the semimartingale Z.

Let us consider the sequence $\{T_n\}$ of partitions of $\mathbb{R}^+$ satisfying the condition (0.6). In this section we investigate Euler and Euler–Peano schemes for SDE (2.1). In the classical

Euler scheme the approximations $\{\overline{X}^n\}$ are solutions to the SDE's

$$\overline{X}^n_t = H^{\varrho^n}_t + \int_0^t f(\overline{X}^n_{s-})\,dZ^{\varrho^n}_s, \quad t \in \mathbb{R}^+,\ n \in \mathbb{N}. \tag{3.7}$$

Since for every $n \in \mathbb{N}$ both processes Z^{ϱ^n} and H^{ϱ^n} have locally bounded variation, they are semimartingales and there is no problem with the existence and uniqueness of a solution to (3.7). Moreover, the special form of H^{ϱ^n} and Z^{ϱ^n} implies that

$$\overline{X}^n_t = \begin{cases} H_0 & \text{if } t \in [0, t_{n1}[, \\ \overline{X}^n_{t_{n,k-1}} + f(\overline{X}^n_{t_{n,k-1}})(Z_{t_{nk}} - Z_{t_{n,k-1}}) & \text{if } t \in [t_{nk}, t_{n,k+1}[, \end{cases} \tag{3.8}$$

for all $k \in \mathbb{N} \cup \{0\}$.

THEOREM 3.5. *Assume $f \in \mathbb{L}(L)$. Let X be a strong solution of the SDE* (2.1) *and let $\{\overline{X}^n\}$ be a sequence of solutions to discrete SDE's* (3.7). *Then*

(i) $(\overline{X}^n, H^{\varrho^n}, Z^{\varrho^n}) \to_{\mathcal{P}} (X, H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$,

(ii) $\sup_{t\le q} |\overline{X}^n_t - X^{\varrho^n}_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$.

Proof. As is observed in [Sło87], $H^{\varrho^n} \to H$ and $Z^{\varrho^n} \to Z$ almost surely in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Moreover, in view of Lemma A.1, $(H^{\varrho^n}, Z^{\varrho^n}) \to (H, Z)$ almost surely in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. The conclusion (i) is true since due to the theorem of Bichteler, Dellacherie and Mokobodzki the sequence $\{Z^{\varrho^n}\}$ of discrete semimartingales satisfies (UT) and Theorem 3.1(i) can be used. Lemma A.1 implies $(\overline{X}^n, X^{\varrho^n}) \to_{\mathcal{P}} (X, X)$, and by Lemma A.2 the conclusion (ii) follows. ■

Sometimes the Euler–Peano scheme is more convenient. It concerns the following sequence of approximations:

$$\widehat{X}^n_t = H_t + \int_0^t f(\widehat{X}^{n,\varrho^n}_{s-})\,dZ_s, \quad t \in \mathbb{R}^+. \tag{3.9}$$

It is clear that $\widehat{X}^{n,\varrho^n} = \overline{X}^n$, $n \in \mathbb{N}$, and that $\widehat{X}^n$ are adapted to the same filtration $(\mathcal{F}_t)$, $n \in \mathbb{N}$.

THEOREM 3.6. *Assume $f \in \mathbb{L}(L)$. Let X be a strong solution of the SDE* (2.1) *and let $\{\widehat{X}^n\}$ be a sequence of solutions of SDE's* (3.9). *Let $\{\alpha_n\}$ be a sequence of positive constants diverging to infinity. If the sequence $\{\alpha_n \int_0^{\cdot} (Z^i_{s-} - Z^{\varrho^n,i}_{s-})\,dZ^j_s\}_{n\in\mathbb{N}}$ of $(\mathcal{F}_t)$ adapted processes satisfies* (UT) *for $i, j = 1, \ldots, d$, then also $\{\alpha_n(\widehat{X}^n - X)\}_{n\in\mathbb{N}}$ satisfies* (UT).

Proof. Define $Y^n_t = \alpha_n(\widehat{X}^n_t - X_t)$, $t \in \mathbb{R}^+, n \in \mathbb{N}$. For simplicity, assume $d = 1$. By the Lipschitz continuity of f,

$$\begin{aligned} Y^n_t &= \int_0^t \alpha_n(f(\widehat{X}^n_{s-}) - f(X_{s-}))\,dZ_s + \int_0^t \alpha_n(f(\widehat{X}^{n,\varrho^n}_{s-}) - f(\widehat{X}^n_{s-}))\,dZ_s \\ &= \int_0^t Y^n_{s-} U^n_s\,dZ_s + \int_0^t f(\widehat{X}^{n,\varrho^n}_{s-}) V^n_{s-} \alpha_n (Z_{s-} - Z^{\varrho^n}_{s-})\,dZ_s, \quad t \in \mathbb{R}^+, \end{aligned}$$

where U^n, V^n are predictable processes such that $\{\sup_{t\le q} |U^n_t|\}_{n\in\mathbb{N}}$ and $\{\sup_{t\le q} |V^n_t|\}_{n\in\mathbb{N}}$ are tight in $\mathbb{R}$. Since $|f(x)| \le L(1 + |x|)$, $x \in \mathbb{R}$, also $\{\sup_{t\le q} |f(\widehat{X}^{n,\varrho^n}_{t-})|\}_{n\in\mathbb{N}}$ is tight in $\mathbb{R}$. Therefore, by Corollary 1.2,

$$\Big\{\int_0^t f(\widehat{X}^{n,\varrho^n}_{s-})V^n_{s-}\alpha_n(Z_{s-}-Z^{\varrho^n}_{s-})\,dZ_s\Big\}_{n\in\mathbb{N}}\quad\text{satisfies (UT)}.$$

Finally, it is enough to use Theorem 1.10(ii). ∎

R e m a r k 3.7. Assuming additionally that the sequences $\{\alpha_n\int_0^{\cdot}(Z^i_{s-}-Z^{\varrho^n,i}_{s-})\,dZ^j_s\}$ are weakly convergent and $f\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ it is possible to characterize the limit points of $\{\alpha_n(\widehat{X}^n-X)\}$ (see [KuPr91b]).

COROLLARY 3.8. *Under the assumptions of Theorem* 3.6 *we have the following convergences*:

(i) *for every* $q\in\mathbb{R}^+$, $\varepsilon>0$,

$$\alpha_n^{1-\varepsilon}\sup_{t\le q}|\widehat{X}^n_t-X_t|\underset{\mathcal{P}}{\to}0,$$

(ii) *for every* $q\in\mathbb{R}^+$, $\varepsilon>0$,

$$\alpha_n^{2-\varepsilon}[\widehat{X}^n_t-X_t]_q\underset{\mathcal{P}}{\to}0,$$

(iii) *for every* $q\in\mathbb{R}^+$, $\varepsilon>0$,

$$\alpha_n^{2-\varepsilon}[\overline{X}^n_t-X^{\varrho^n}_t]\underset{\mathcal{P}}{\to}0.$$

P r o o f. According to Theorem B.1 the sequences $\{\sup_{t\le q}|Y^n_t|\}$ and $\{[Y^n]\}$ are tight in $\mathbb{R}$. Since $\alpha_n\to+\infty$, the proof of (i) and (ii) is complete. In order to prove (iii) it is sufficient to observe that the condition (UT) being satisfied for $\{Y^n\}$ implies the same condition for $\{Y^{n,\varrho^n}\}$. ∎

COROLLARY 3.9. *Assume* $f\in\mathbb{L}$ (L). *Let* W *be a standard Wiener process and*

$$\max_k(t_{nk}-t_{n,k-1})\le 1/n,\qquad n\in\mathbb{N}.$$

Then the sequences $\{n^{1/2}(\widehat{X}^n-X)\}$, $\{n^{1/2}(\overline{X}^n-X^{\varrho^n})\}$ *satisfy* (UT), *and the conditions* (i)–(iii) *from Corollary* 3.8 *are satisfied with* $\alpha_n=n^{1/2}$.

P r o o f. Define

$$M^n_t=n^{1/2}\int_0^t(W^i_s-W^{\varrho^n,i}_{s-})\,dW^j_s,\qquad t\in\mathbb{R}^+,\ n\in\mathbb{N}.$$

Then $\{M^n\}$ is a sequence of continuous martingales such that

$$[M^{n,a}]_t=[M^n]_t=n\int_0^t(W^i_s-W^{\varrho^n,i}_{s-})^2\,ds,\qquad t\in\mathbb{R}^+,\ n\in\mathbb{N}.$$

Since $E[M^n]_t\le t$, $\{[M^n]_t\}$ is tight in $\mathbb{R}$ and $\{M^n\}$ satisfies (UT) in view of Theorem 1.1. Finally, by Theorem 3.6 and Corollary 3.8 the result follows. ∎

4. Stability of SDE's with past-dependent and non-Lipschitz coefficients

4.1. Convergence to the solution of SDE driven by a continuous semimartingale. Let f, g be two predictable mappings from $\mathbb{R}^+ \times \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ into $\mathbb{R}$ such that

$$\sup_{s\le t}(|f(s,x)| + |g(s,x)|) \le K(1 + \sup_{s<t}|x_s|) \qquad x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}),\ t \in \mathbb{R}^+. \tag{4.1}$$

In this section we discuss stability of SDE's with memory when a limit process X has the form

$$X_t = H_t + \int_0^t f(s, X_.)\, dZ_s + \int_0^t g(s, X_.)\, dA_s, \qquad t \in \mathbb{R}^+, \tag{4.2}$$

where Z is an $(\mathcal{F}_t)$ adapted continuous semimartingale, H, A are two $(\mathcal{F}_t)$ adapted processes with continuous trajectories and additionally the trajectories of A are nondecreasing, $Z_0 = A_0 = 0$.

We say that the SDE (4.2) has a *weak solution* if there exists a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$ with a filtration $(\widehat{\mathcal{F}}_t)$ satisfying the usual conditions and $(\widehat{\mathcal{F}}_t)$ adapted processes $\widehat{X}, \widehat{H}, \widehat{Z}, \widehat{A}$ such that $\mathcal{L}(\widehat{H}, \widehat{Z}, \widehat{A}) = \mathcal{L}(H, Z, A)$ and (4.2) holds for $\widehat{X}, \widehat{H}, \widehat{Z}, \widehat{A}$ in place of H, X, Z, A. If any two weak solutions $\widehat{X}$ and $\widehat{X}'$ of SDE (4.2), possibly defined on different probability spaces, are such that $\mathcal{L}(\widehat{X}) = \mathcal{L}(\widehat{X}')$ we say that the *weak uniqueness* holds for (4.2).

Assume that f^n, g^n are predictable functions from $\mathbb{R}^+ \times \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ into $\mathbb{R}$ satisfying the condition (4.1) and such that

$$f^n(s_n, x_n) \to f(s,x), \qquad g^n(s_n, x_n) \to g(s,x), \tag{4.3}$$

as $n \uparrow +\infty$, and $(s_n, x_n) \to (s,x)$ in $\mathbb{R} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ for all $(s,x) \in \mathbb{R}^+ \times \mathbb{D}(\mathbb{R}^+, \mathbb{R})$. For each $n \in \mathbb{N}$ denote by X^n the solution of (4.2) with the coefficients f^n, g^n, i.e.

$$X^n_t = H^n_t + \int_0^t f^n(s, X^n_.)\, dZ^n_s + \int_0^t g^n(X^n_.)\, dA^n_s, \qquad t \in \mathbb{R}^+, \tag{4.4}$$

where Z^n is a local martingale and A^n a process with nondecreasing trajectories, $Z^n_0 = A^n_0 = 0$.

THEOREM 4.1. *Let $\{X^n\}$ be the solutions of the SDE* (4.4), *where $\{f^n\}, \{g^n\}$ satisfy the conditions* (4.1), (4.3). *If $\int_{\{|x|>a\}} |x|\nu^n[[0,t] \times dx] \to_{\mathcal{P}} 0$, $t \in \mathbb{R}^+$, and*

$$(H^n, Z^n, A^n) \underset{\mathcal{D}}{\to} (H, Z, A) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^3)$$

then

(i) *$\{(X^n, H^n, Z^n, A^n)\}$ is C-tight in $\mathbb{D}(\mathbb{R}^4)$ and every limit point is a weak solution of the SDE* (4.2),

(ii) *if additionally the SDE* (4.2) *has a unique weak solution X then*

$$X^n \underset{\mathcal{D}}{\to} X \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}).$$

Proof. (i) Since $\{Z^n\}$ and $\{A^n\}$ satisfy (UT) we can use the results from Sections 1.2 and 1.3. Define $B_t^n = \int_0^t g^n(s, X_\cdot^n)\, dA_s^n$, $L_t^n = \int_0^t f^n(s, X_\cdot^n)\, dZ_s^n$, for every $t \in \mathbb{R}^+$, $n \in \mathbb{N}$. Due to Theorem 1.10 the sequences

$$\{\sup_{t\le q} |X_t^n|\},\ \{\sup_{t\le q} |f^n(t, X_\cdot^n)|\},\ \{\sup_{t\le q} |g^n(t, X_\cdot^n)|\} \quad \text{are tight in } \mathbb{R}.$$

By Proposition 1.9 the sequence $\{L^n\}$ is C-tight and each of its limit points is a local martingale. Moreover, we can deduce that

$$\{(H^n, B^n, L^n, [L^n], [L^n, Z^n], X^n, Z^n, A^n)\} \quad \text{is C-tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^8)$$

Without loss of generality we assume that

$$(H^n, B^n, L^n, [L^n], [L^n, Z^n], X^n, Z^n, A^n) \underset{\mathcal{D}}{\to} (H', B', L', [L'], [L', Z'], X', Z', A') \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^8).$$

The proof of (i) will be finished if we show that X' is a weak solution to (4.2), i.e.

$$B' = \int_0^{\cdot} g(s, X_\cdot')\, dA_s' \quad \text{and} \quad L' = \int_0^{\cdot} f(s, X_\cdot')\, dZ_s'. \tag{4.5}$$

By the Skorokhod representation theorem we may and will assume that

$$(H^n, B^n, L^n, [L^n], [L^n, Z^n], X^n, Z^n, A^n) \to (H', B', L', [L'], [L', Z'], X', Z', A') \quad \mathcal{P}\text{-a.s. in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^8).$$

Define $g^n(s) = g^n(s, X_\cdot^n)$, $g(s) = g(s, X_\cdot')$, $s \in \mathbb{R}^+$. Then by the arguments from the paper by Yamada [Yam86] we have

$$\int_0^t g^n(s, X_\cdot^n)\, dA_s^n = \int_0^{A_t^n} g^n(C_s^n)\, ds,$$

where $C_t^n = \inf\{s : A_s^n \ge t\}$, and

$$\int_0^t g(s, X_\cdot')\, dA_s = \int_0^{A_t} g(C_s)\, ds,$$

where $C_t = \inf\{s : A_s \ge t\}$. Next, from the convergence $\sup_{t\le q} |A_t^n - A_t| \to 0$ $\mathcal{P}$-a.s. we deduce that

$$\sup_{t\le q} \Big| \int_0^{A_t^n} g^n(C_s^n)\, ds - \int_0^{A_t} g(C_s)\, ds \Big| \to 0 \quad \mathcal{P}\text{-a.s.}$$

Exactly the same way we can prove that

$$[L^n]_t = \int_0^t (f^n(s, X_\cdot^n))^2\, d[Z^n]_s \to \int_0^t (f(s, X_\cdot'))^2\, d[Z']_s \quad \mathcal{P}\text{-a.s.}$$

$$[L^n, Z^n]_t = \int_0^t f^n(s, X_\cdot^n)\, d[Z^n]_s \to \int_0^t f(s, X_\cdot')\, d[Z']_s \quad \mathcal{P}\text{-a.s.},$$

uniformly with respect to t on every compact subset of $\mathbb{R}^+$. On the other hand, $[L^n] \to [L']$ and $[L^n, Z^n] \to [L', Z']$. Therefore

$$[L']_t = \int_0^t (f(s, X'_\cdot))^2 \, d[Z']_s, \qquad [L', Z']_t = \int_0^t f(s, X'_\cdot) \, d[Z']_s,$$

which implies that $L'_t = \int_0^t f(s, X'_\cdot) \, dZ_s$ and completes the proof of (4.5).

(ii) This is an obvious consequence of (i). ■

4.2. General case. In the present section we consider convergence of solutions of d-dimensional SDE's with memory when a limit process X has the form

$$X_t = H_t + \int_0^t f(s, X_\cdot) \, dZ_s, \qquad t \in \mathbb{R}^+, \tag{4.6}$$

where Z is an $(\mathcal{F}_t)$ adapted semimartingale, $Z_0 = 0$, H is an $(\mathcal{F}_t)$ adapted process and $f : \mathbb{R}^+ \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfies (1.5). Let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, $Z_0^n = 0$, and $\{H^n\}$ a sequence of $(\mathcal{F}_t^n)$ adapted processes. Similarly to Section 4.1 we consider the solutions X^n corresponding to Z^n and H^n, i.e.

$$X_t^n = H_t^n + \int_0^t f^n(s, X_\cdot^n) \, dZ_s^n, \qquad t \in \mathbb{R}^+, \ n \in \mathbb{N}, \tag{4.7}$$

Here the assumptions on $\{Z^n\}$ have been relaxed compared with the ones in Section 4.1. We assume only that $\{Z^n\}$ satisfies (UT). This generalization is possible because the class of the functions $\{f^n\}$ has been thinned down. Now we demand that $\{f^n(\cdot, X_\cdot^n)\}$ is approximated by some arrays of semimartingales and $\{f^n\}$ satisfies:

(4.8) $f^n(\cdot, x)$ is left continuous and admits right-hand limits, $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $n \in \mathbb{N}$, and for every sequence $\{(x_n, z_n)\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ and $(x, z) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ such that $(x_n, z_n) \to (x, z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ we have $(y_n, z_n) \to (y, z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(d+1)})$, where $y_n(s) = f^n(s+, x_n)$.

THEOREM 4.2. *Let $\{X^n\}$ be a sequence of solutions of the SDE* (4.7). *Assume that for $\{f^n\}$, the conditions* (4.1), (4.8) *are true and there exists a family of semimartingales $\{\{Y^{ni}\}\}$ such that for every $i \in \mathbb{N}$, $\{Y^{ni}\}$ fulfills (UT),*

$$\lim_{k \to +\infty} \limsup_{n \to +\infty} \mathcal{P}[\sup_{t \le q} \|Y_{t-}^{ni} - f^n(s, X_\cdot^n)\| \ge \varepsilon] = 0, \qquad \varepsilon > 0, \ q \in \mathbb{R}^+. \tag{4.9}$$

If $\{Z^n\}$ is a sequence of semimartingales satisfying (UT) *and*

$$(H^n, Z^n) \underset{\mathcal{D}}{\to} (H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$$

then

(i) *$\{(X^n, H^n, Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$ and every limit point is a weak solution of the SDE* (4.6),

(ii) *if additionally* (4.6) *has a unique weak solution X, then*

$$X^n \underset{\mathcal{D}}{\to} X \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).$$

P r o o f. We start with a very general tightness criterion for a sequence of stochastic integrals, which is a consequence of Corollary 2.8.

LEMMA 4.3. *Let $\{Y^n\}, \{Z^n\}$ be two sequences of semimartingales satisfying the condition* (UT). *If $\{H^n\}$ is a sequence of processes such that $\{(H^n, Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ then also*

$$\Big\{\Big(H^n, \int_0^{\cdot} Y^n_{s-}\, dZ^n_s, Z^n\Big)\Big\} \quad \textit{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1}).$$

P r o o f. Without loss of generality we assume that

$$(H^n, Z^n) \underset{\mathcal{D}}{\rightarrow} (H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Let $\{\{Z^{ni}\}\}$ be a family of processes defined in Corollary 2.8, i.e.

$$\{(H^n, \int_0^{\cdot} Y^n_{s-}\, dZ^{ni}_s, Z^{ni}, Z^n)\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}), \tag{4.10}$$

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q} |Z^n_t - Z^{ni}_t| \ge \varepsilon] = 0, \qquad \varepsilon > 0,\ q \in \mathbb{R}^+. \tag{4.11}$$

$$\{Z^n - Z^{ni}\} \quad \text{satisfies (UT)},\ i \in \mathbb{N}. \tag{4.12}$$

Integration by parts leads to the formula

$$\int_0^t Y^n_{s-}\, d(Z^n - Z^{ni})_s = Y^n_t(Z^n_t - Z^{ni}_t) + \int_0^t (Z^n_{s-} - Z^{ni}_{s-})\, dY^n_s + [Z^n - Z^{ni}, Y^n]_t. \tag{4.13}$$

It is clear by Theorem 1.1 that

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}\Big[\sup_{t\le q} \Big|\int_0^t (Z^n_{s-} - Z^{ni}_{s-})\, dY^n_s\Big| \ge \varepsilon\Big] = 0, \qquad \varepsilon > 0,\ q \in \mathbb{R}^+.$$

By the Kunita–Watanabe inequality,

$$\begin{aligned}\sup_{t\le q} |[Z^n - Z^{ni}, Y^n]_t &\le [Z^n - Z^{ni}]_q [Y^n]_t \\ &= \Big\{|Z^n - Z^{ni}|^2_q - 2\int_0^q (Z^n_{s-} - Z^{ni}_{s-})\, d(Z^n - Z^{ni})_s\Big\}[Y^n]_q.\end{aligned}$$

From Theorem B.1, $\{[Y^n]_q\}$ is tight in $\mathbb{R}$. Therefore using once more Theorem 1.1 we obtain

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}[\sup_{t\le q} |[Z^n - Z^{ni}, Y^n]_t| \ge \varepsilon] = 0, \qquad \varepsilon > 0,\ q \in \mathbb{R}^+. \tag{4.14}$$

The condition (UT) for $\{Y^n\}$ implies also that $\{\sup_{t\le q} |Y^n_t|\}$ is tight in $\mathbb{R}$ (see Theorem B.1). As a consequence it follows by (4.11) and (4.13) that

$$\lim_{i\to+\infty} \limsup_{n\to+\infty} \mathcal{P}\Big[\sup_{t\le q} \Big|\int_0^t Y^n_{s-}\, d(Z^n - Z^{ni})_s\Big| \ge \varepsilon\Big] = 0, \qquad \varepsilon > 0,\ q \in \mathbb{R}^+. \tag{4.15}$$

Combining (4.15) with (4.10) we complete the proof of the lemma. ∎

(i) According to the above lemma $\{(H^n, \int_0^{\cdot} Y^{ni}_{s-}\, dZ^n_s, Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$. Therefore by (4.9) also $\{(X^n, H^n, \int_0^{\cdot} f^n(s, X^n_{\cdot})\, dZ^n_s, Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$. Assume that

$$\Big(X^n, H^n, \int_0^{\cdot} f^n(s, X^n_{\cdot})\, dZ^n_s, Z^n\Big) \underset{\mathcal{D}}{\to} (X', H', Y', Z') \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).$$

Then it is clear by Theorem B.2 that $Y'_t = \int_0^t f(s, X'_{\cdot})\, dZ'_s$ and

$$X'_t = H'_t + \int_0^t f(s, X'_{\cdot})\, dZ'_s, \quad t \in \mathbb{R}^+,$$

where $\mathcal{L}(H', Z') = \mathcal{L}(H, Z)$, i.e. X' is a weak solution to the SDE (4.6).

(ii) easily follows from (i). ■

LEMMA 4.4. *Assume that* $g : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ *is continuous and there exists a constant* $K > 0$ *such that*

$$\|g(t, y)\| \le K(1 + |y|), \quad y \in \mathbb{R}^d,\ t \in \mathbb{R}^+.$$

If f^n *and* f *have one of the following forms*:

(i) $f^n(s, x) = f(s, x) = g(s, x_{s-})$, $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $s \in \mathbb{R}^+$, $n \in \mathbb{N}$,

(ii) $f^n(s, x) = f(s, x) = \int_0^s g(t, x_{t-})\, dt$, $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $s \in \mathbb{R}^+$, $n \in \mathbb{N}$,

(iii) $f^n(s, x) = f(s, x) = \sup_{t \le s} g(t, x_{t-})$, $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $s \in \mathbb{R}^+$, $n \in \mathbb{N}$.

Then the assumptions (4.1), (4.8) *and* (4.9) *of Theorem* 4.2 *hold.*

Proof. (i) It is clear that here the assumptions (4.1) and (4.8) are satisfied. We will restrict our attention only to the proof of (4.9). It is well known that for continuous $g \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ we can construct a sequence $\{g^i\}$ of functions such that for every $i \in \mathbb{N}$, $g^i \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ and

$$\sup_{x \in K} \sup_{t \le q} \|g^i(t, x) - g(t, x)\| \to 0, \tag{4.16}$$

for all compact subsets K of $\mathbb{R}^d$ and $q \in \mathbb{R}^+$.

Since $H^n \to_{\mathcal{D}} H$, by Corollary 2.8 there exists a family $\{\{H^{ni}\}\}$ of processes such that

$$\lim_{i \to +\infty} \limsup_{n \to +\infty} \mathcal{P}[\sup_{t \le q} |H^n_t - H^{ni}_t| \ge \varepsilon] = 0, \quad \varepsilon > 0,\ q \in \mathbb{R}^+. \tag{4.17}$$

$$\{H^{ni}\} \quad \text{satisfies (UT)},\ i \in \mathbb{N}. \tag{4.18}$$

Define for every $n, k \in \mathbb{N}$,

$$X^{ni}_t = H^{ni} + \int_0^{\cdot} g(s, X^n_{s-})\, dZ^n_s, \quad Y^{ni}_t = g^i(t, X^{ni}_t).$$

We know by Theorem 1.10 that $\{\sup_{t \le q} |X^n_t|\}$ is tight in $\mathbb{R}$. Therefore by (4.18) and Corollary 1.2, $\{X^{ni}\}$ is a sum of two sequences of processes satisfying (UT). Thus for $\{X^{ni}\}$ the property (UT) holds, too. Moreover, since $g^i \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$, by Corollary 1.3 for every $i \in \mathbb{N}$,

$$\{g^i(X^{ni})\} \text{ satisfies the condition (UT).} \tag{4.19}$$

Also the second property in (4.9) easily follows due to (4.16) and (4.17).

(ii), (iii) We use exactly the same arguments as in (i). ■

COROLLARY 4.5. *Assume that f is of the form considered in Corollary* 4.4(i)–(iii). *Then there exists a weak solution to the SDE* (4.6).

Proof. Let $\{T_n\}$ be a sequence of partitions of $\mathbb{R}^+$ satisfying (0.6). Put $Z^n = Z^{\varrho^n}$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exists a strong solution of (4.7). Since $\{Z^{\varrho^n}\}$ satisfies (UT) and $(H^{\varrho^n}, Z^{\varrho^n}) \to (H, Z)$ $\mathcal{P}$-a.s., Theorem 4.2 and Corollary 4.4 complete the proof. ∎

In the sequel we consider convergence in probability of solutions of SDE's. We assume that the limit solution X of the SDE (4.6) additionally satisfies the following two conditions:

(4.20) X is adapted to the natural filtration of (H, Z),

(4.21) pathwise uniqueness holds for (4.6).

i.e. for any two weak solutions $\widehat{X}$, $\widehat{X}'$ of the SDE (4.6) corresponding to processes $(\widehat{H}, \widehat{Z})$, $(\widehat{H}', \widehat{Z}')$, and defined on a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$ with filtration $(\widehat{\mathcal{F}}_t)$ the following implication holds:

$$\mathcal{P}[(\widehat{H}_t, \widehat{Z}_t) = (\widehat{H}'_t, \widehat{Z}'_t); t \in \mathbb{R}^+] = 1 \Rightarrow \mathcal{P}[\widehat{X}_t = \widehat{X}'_t; t \in \mathbb{R}^+] = 1.$$

THEOREM 4.6. *Let $\{Z^n\}$ be a sequence of $(\mathcal{F}^n_t)$ adapted semimartingales satisfying the condition* (UT) *and $\{H^n\}$ a sequence of $(\mathcal{F}^n_t)$ adapted processes. Assume that $\{X^n\}$ is a sequence of strong solutions of the SDE* (4.7), *where $\{f^n\}$ fulfills the conditions* (1.5), (4.8) *and* (4.9) *and additionally* (4.20) *and* (4.21).

(i) *If $(H^n, Z^n) \to_{\mathcal{P}} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ then*

$$(X^n, H^n, Z^n) \underset{\mathcal{P}}{\to} (X, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

(ii) *If $\sup_{t\le q} |H^n_t - H_t| \to_{\mathcal{P}} 0, \sup_{t\le q} |Z^n_t - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$ then*

$$\sup_{t\le q} |X^n_t - X_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

where X is a unique strong solution of the SDE (4.6).

Proof. (i) We follow the method used in the proof of Theorem 3.1. Since we assume the convergence in probability, all the processes are defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We should show that

$$\lim_{n\to\infty} \int_B \Phi(X^n, H^n, Z^n)\, d\mathcal{P} = \int_B \Phi(X, H, Z)\, d\mathcal{P}, \tag{4.22}$$

for all bounded and continuous mappings $\Phi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \to \mathbb{R}$, and all $B \in \mathcal{F}^{(H,Z)}$. By a standard approximation of measurable functions we can replace (4.22) by

$$\lim_{n\to\infty} \int_\Omega \Psi(H, Z)\Phi(X^n, H^n, Z^n)\, d\mathcal{P} = \int_\Omega \Psi(H, Z)\Phi(X, H, Z)\, d\mathcal{P}, \tag{4.23}$$

for all bounded and continuous mappings $\Psi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \to \mathbb{R}$ and $\Phi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \to \mathbb{R}$. On the other hand, (4.21) implies weak uniqueness for (4.6) and due to Theorem 4.2,

$$\lim_{n\to\infty} \int_\Omega \Psi(H^n, Z^n)\Phi(X^n, H^n, Z^n)\, d\mathcal{P} = \int_\Omega \Psi(H, Z)\Phi(X, H, Z)\, d\mathcal{P}. \tag{4.24}$$

By the continuity of Ψ, $\Psi(H^n, Z^n) \to_{\mathcal{P}} \Psi(H, Z)$, and it follows by the boundedness of Ψ and Φ that (4.24) implies (4.23).

(ii) We deduce the result from (i) using exactly the same arguments as in the proof of Theorem 3.1(ii). ■

Corollary 4.7. *Assume that f is of the form considered in Corollary* 4.4 *and the conditions* (4.20) *and* (4.21) *hold. Let X be a strong solution of the SDE* (4.6) *and let $\{T_n\}$ be a sequence of partitions of* $\mathbb{R}^+$ *satisfying* (0.6). *If $\{\overline{X}^n\}$ is a sequence of solutions to discrete SDE's* (4.7) *with $Z^n = Z^{\varrho^n}$, $H^n = H^{\varrho^n}$, $n \in \mathbb{N}$, then*

(i) $(\overline{X}^n, H^{\varrho^n}, Z^{\varrho^n}) \to_{\mathcal{P}} (X, H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$,

(ii) $\sup_{t\le q} |\overline{X}^n_t - X^{\varrho^n}_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$.

Proof. We repeat the arguments from the proof of Theorem 3.5. ■

The classical example of non-Lipschitz coefficient f which however assures pathwise uniquness, was given by Yamada and Watanabe [YaWa71] in the case of diffusion equations (a larger class of coefficients was considered in [Tani92]). It was proved by Tudor [Tud84] that this example works also for SDE's driven by general semimartingales. By using Lemma C.3 we can give the following version of his result.

Example 4.8. Assume that f is continuous, satisfies (1.5) and

$$\|f(t,x) - f(t,y)\|^2 \le \varrho(|x-y|^2), \qquad t \in \mathbb{R}^+,\ x, y \in \mathbb{R}^d,$$

where $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing and concave with $\varrho(0) = 0$ and

$$\int_{0+} \frac{du}{\varrho(u)} = +\infty.$$

Then the assumptions (4.8) and (4.9) are satisfied, as are (4.20) and (4.21). ■

5. Stability of Stratonovich SDE's

5.1. Weak and strong convergence of solutions. In the present section we give versions of Theorem 2.1 and Theorem 3.1 for Stratonovich SDE's driven by general semimartingales. Let U, Z be two $(\mathcal{F}_t)$ adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that Z is a semimartingale, $Z_0 = 0$ and their mutual quadratic variation process $[U, Z]$ exists. Meyer [Mey76] has defined the Stratonovich integral for general semimartingales as follows:

$$(S)\int_0^t U_{s-}\,dZ_s = \int_0^t U_{s-}\,dZ_s + \frac{1}{2}[U, Z]^c_t, \qquad t \in \mathbb{R}^+.$$

Using this definition we can consider a Stratonovich SDE of the form

$$\begin{aligned} X_t &= X_0 + (S)\int_0^t f(X_{s-})\,dZ_s,\\ &= X_0 + \int_0^t f(X_{s-})\,dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-})\,d[Z]^c_s, \qquad t \in \mathbb{R}^+, \end{aligned}$$

provided that $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ or in an equivalent form, for every $i = 1, \ldots, d$,

$$X_t^i = X_0^i + \sum_{j=1}^{d} \int_0^t f_{ij}(X_{s-})\, dZ_s^j + \frac{1}{2} \sum_{j=1}^{d} \sum_{l=1}^{d} \sum_{m=1}^{d} \int_0^t \frac{\partial f_{ij}}{\partial x_l} f_{lm}(X_{s-})\, d[Z^j, Z^m]_s^c.$$

Obviously if f, $f'f$ are Lipschitz continuous (we will say throughout that f, $f'f$ have some property if the coefficients f_{ij}, $\partial f_{ij}/\partial x_l f_{lm}$ possess this property for $i, j, l, m = 1, \ldots d$), then there exists a unique strong solution to the above SDE. Let us consider a sequence $\{Z^n\}$ of semimartingales and the corresponding sequence $\{X^n\}$ of solutions, i.e.

$$X_t^n = X_0^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + \frac{1}{2} \int_0^t f'f(X_{s-}^n)\, d[Z^n]_s^c, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}.$$

A natural question arises whether under (UT) it is possible to obtain stability results similar to the ones for Doléans-Dade and Protter equations. The answer is negative as the following simple counterexample shows.

Example 5.1. Let $\{T_n\}$ be a sequence of partitions of $\mathbb{R}^+$ satisfying (0.6). Assume that for every $n \in \mathbb{N}$, Z^n is of the form $Z_t^n = Z_t^{\varrho^n}$. Then $[Z^{\varrho^n}]^c = 0$ and

$$X_t^n = (S) \int_0^t f(X_{s-}^n)\, dZ_s^{\varrho^n} = \int_0^t f(X_{s-}^n)\, dZ_s^n, \qquad t \in \mathbb{R}^+.$$

By Theorem 3.1, $X^n \to_{\mathcal{P}} Y$, where Y is a unique strong solution to the following equation:

$$Y_t = \int_0^t f(Y_{s-})\, dZ_s, \qquad t \in \mathbb{R}^+,$$

and of course $X \neq Y$, provided that $[Z]^c \neq 0$.

One may ask whether it is possible to define a Stratonovich SDE for which stability theorem would be true. Marcus [Mar78, Mar81] has proposed the following form of the equation:

$$\begin{aligned} X_t = X_0 + \int_0^t f(X_s) \circ dZ_s &= X_0 + \int_0^t f(X_{s-})\, dZ_s + \frac{1}{2} \int_0^t f'f(X_{s-})\, d[Z]_s^c \\ &\quad + \sum_{s \le t} \{\varphi(f \varDelta Z_s, X_{s-}) - X_{s-} - f(X_{s-}) \varDelta Z_s\}, \end{aligned} \tag{5.1}$$

where for given $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, $\varphi(g, x)$ denotes the value at time $u = 1$ of the solution to the following ordinary differential equation:

$$\frac{dy}{du}(u) = g(y(u)), \qquad y(0) = x \in \mathbb{R}^d.$$

(The notation "$\circ$" is used to indicate that we do not deal with a standard Itô type stochastic integral). More precisely, for $i = 1, \ldots, d$,

$$X_t^i = X_0^i + \sum_{j=1}^{d} \int_0^t f_{ij}(X_{s-})\, dZ_s^j + \frac{1}{2} \sum_{j=1}^{d} \sum_{l=1}^{d} \sum_{m=1}^{d} \int_0^t \frac{\partial f_{ij}}{\partial x_l} f_{lm}(X_{s-})\, d[Z^j, Z^m]_s^c$$

$$+\sum_{s\le t}\{\varphi^i(f\Delta Z_s, X_{s-}) - X^i_{s-} - \sum_{j=1}^{d} f_{ij}(X_{s-})\Delta Z^j_s\}, \qquad t\in\mathbb{R}^+.$$

Let us note that the definition of Stratonovich SDE given by Marcus coincides with the definition given by Meyer if Z has continuous trajectories. In his paper [Mar81], Marcus has restricted himself to the case of semimartingales with a finite number of jumps on every interval $[0,t]$. Recently, Kurtz, Pardoux and Protter [KPP92] have proved that Marcus' definition is correct for general semimartingales. They have shown that there exists a unique strong solution to the SDE (5.1), provided that f, $f'f$ are Lipschitz continuous. It turns out that the above definition is a suitable one for obtaining stability results, under (UT).

Let $\{Z^n\}$ be a sequence of semimartingales, $Z^n_0 = 0$ and let $\{X^n\}$ be a family of strong solutions corresponding to $\{Z^n\}$, i.e.

$$\begin{aligned}(5.2)\qquad X^n_t &= X^n_0 + \int_0^t f(X^n_s)\circ dZ^n_s\\ &= X^n_0 + \int_0^t f(X^n_{s-})\,dZ^n_s + \frac{1}{2}\int_0^t f'f(X^n_{s-})\,d[Z^n]^c_s\\ &\quad+\sum_{s\le t}\{\varphi(f\Delta Z^n_s, X^n_{s-}) - X^n_{s-} - f(X^n_{s-})\Delta Z^n_s\}, \qquad t\in\mathbb{R}^+,\end{aligned}$$

or in an equivalent form, for $i = 1,\dots,d$,

$$\begin{aligned}X^{ni}_t &= X^{ni}_0 + \sum_{j=1}^{d}\int_0^t f_{ij}(X^n_{s-})\,dZ^{nj}_s + \frac{1}{2}\sum_{j=1}^{d}\sum_{l=1}^{d}\sum_{m=1}^{d}\int_0^t \frac{\partial f_{ij}}{\partial x_l} f_{lm}(X^n_{s-})\,d[Z^{nj},Z^{nm}]^c_s\\ &\quad+\sum_{s\le t}\{\varphi^i(f\Delta Z^n_s, X^n_{s-}) - X^{ni}_{s-} - \sum_{j=1}^{d} f_{ij}(X^n_{s-})\Delta Z^{nj}_s\}, \qquad t\in\mathbb{R}^+,\ n\in\mathbb{N}.\end{aligned}$$

In the sequel we assume that the function f belongs to $\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ and

$$(5.3)\qquad f, f'f \quad \text{possess the linear growth property.}$$

We say that the SDE (5.1) has a *weak solution* if there exists a probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathcal{P}})$ with filtration $(\widehat{\mathcal{F}}_t)$ satisfying the usual conditions and $(\widehat{\mathcal{F}}_t)$ adapted processes $\widehat{X}$, $\widehat{Z}$ such that $\mathcal{L}(\widehat{X}_0,\widehat{Z}) = \mathcal{L}(X_0,Z)$ and (5.1) holds for processes $\widehat{X}$, $\widehat{Z}$ instead of X, Z. If any two weak solutions $\widehat{X}$ and $\widehat{X}'$ of SDE (5.1), possibly defined on two different probability spaces, are such that $\mathcal{L}(\widehat{X}) = \mathcal{L}(\widehat{X}')$, we say that the *weak uniqueness* for the SDE (5.1) holds.

Now, we are ready to formulate our main theorem.

THEOREM 5.2. *Let $\{Z^n\}$ be a sequence of $(\mathcal{F}^n)$ adapted semimartingales satisfying* (UT) *and such that $(X^n_0, Z^n)\to_{\mathcal{D}}(X_0,Z)$ in $\mathbb{R}^d\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. Assume that $\{X^n\}$ is a sequence of strong solutions to the SDE* (5.2). *If $f\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ and satisfies* (5.3) *then*

(i) *$\{(X^n,Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$ and every limit point of $\{X^n\}$ is a weak solution of the SDE* (5.1),

(ii) *if additionally the SDE* (5.1) *has a unique weak solution* X *then*

$$X^n \underset{\mathcal{D}}{\to} X \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).$$

Proof. Let $\{c_k\}$ be a sequence of constants, $c_k \uparrow +\infty$ such that $\mathcal{P}[|\Delta Z_t| = c_k; t \in \mathbb{R}^+] = 0$, $k \in \mathbb{N}$. Define

$$\tau_k^n = \inf\{t > 0 : |\Delta Z_t^n| \geq c_k\} \quad \text{and} \quad \tau_k = \inf\{t > 0 : |\Delta Z_t| \geq c_k\}, \qquad n, k \in \mathbb{N}.$$

By Proposition 2.7 in [JaSh87],

$$(\tau_k^n, Z^{n,\tau_k^n -}) \underset{\mathcal{D}}{\to} (\tau_k, Z^{\tau_k -}) \quad \text{in } \mathbb{R} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).$$

and $\lim_{k\to+\infty} \lim_{n\to+\infty} \mathcal{P}[\tau_k^n \leq q] = 0$, $q \in \mathbb{R}^+$. Since for every $k \in \mathbb{N}$, $\{Z^{n,\tau_k^n -}\}$ satisfies (UT), we may and will assume that there exists a constant $c > 0$ such that

$$|\Delta Z^n| \leq c, \qquad n \in \mathbb{N}.$$

Similarly to [KPP92] we define

$$h^n(s,x) = \frac{\varphi(\Delta Z_s^n f, x) - x - f(x)\Delta Z_s^n}{|\Delta Z_s^n|^2}, \qquad (s,x) \in \mathbb{R}^+ \times \mathbb{R}^d,\ n \in \mathbb{N}.$$

It has been observed in [KPP92] that the equation (5.2) can be rewritten in the form

$$X_t^n = X_0^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + \frac{1}{2}\int_0^t f'f(X_{s-}^n)\, d[Z^n]_s^c + \int_0^t h^n(s, X_{s-}^n)\, d[Z^n]_s^d.$$

Unfortunately, in the present case h^n need not be Lipschitz continuous. However, by (5.3) and by Gronwall's lemma,

$$\begin{aligned} |h^n(s, X_{s-}^n)| &\leq \sup_{\theta\in[0,1]} |f'f(\varphi(f\Delta Z_s, X_{s-}^n, \theta))| \\ &\leq C(1 + \sup_{\theta\in[0,1]} |\varphi(f\Delta Z_s, X_{s-}^n, \theta)|) \\ &\leq C(1 + |X_{s-}^n|e^c), \qquad s \in \mathbb{R}^+,\ n \in \mathbb{N}. \end{aligned}$$

Therefore by the arguments from the proof of Theorem 1.10 we deduce that

$$\{\sup_{t\leq q} |X_t^n|\} \quad \text{is tight in } \mathbb{R}. \tag{5.4}$$

On the other hand, we know due to [Jac79b] that

$$(Z^n, [[Z^n]]) \underset{\mathcal{D}}{\to} (Z, [[Z]]) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(d+1)}).$$

Hence

$$\{(Z^n, [[Z^n]]^c, [Z^n]^d, [[Z^n]])\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(1+2d)+1}). \tag{5.5}$$

Moreover, by (5.4),

$$\Big\{\Big(Z^n, \int_0^{\cdot} f'f(X_{s-}^n)\, d[Z^n]_s^c, \int_0^{\cdot} h^n(s,\cdot, X_{s-}^n)\, d[Z^n]_s^d\Big)\Big\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}). \tag{5.6}$$

Hence we can rewrite SDE (5.2) in the form

$$X_t^n = X_0^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + H_t^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}, \tag{5.7}$$

where $\{(Z^n, H^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. By a simple extension of Theorem 2.1,

$$\{(X^n, Z^n, [[Z^n]])\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)}).$$

In view of (5.3) and (5.4) without loss of generality we may assume that f, $f'f$, h^n are bounded by some constant $C > 0$ and

$$(X^n, Z^n, [[Z^n]]) \underset{\mathcal{D}}{\rightarrow} (\widehat{X}, \widehat{Z}, [[\widehat{Z}]]) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)}). \tag{5.8}$$

The proof will be finished if we show that $\widehat{X}$ is the solution to the following SDE:

$$\begin{aligned}\widehat{X}_t = \widehat{X}_0 + \int_0^t f(\widehat{X}_{s-})\, d\widehat{Z}_s + \frac{1}{2}\int_0^t f'f(\widehat{X}_{s-})\, d[\widehat{Z}]^c_s \\ + \sum_{s\le t}\{\varphi(f\Delta\widehat{Z}_s, \widehat{X}_{s-}) - \widehat{X}_{s-} - f(\widehat{X}_{s-})\Delta\widehat{Z}_s\}.\end{aligned}$$

Let $\{\varepsilon_k\}$ be a sequence of constants with $\varepsilon_k \downarrow 0$ such that $\mathcal{P}[|\Delta Z_t| = \varepsilon_k; t \in \mathbb{R}^+] = 0$, $k \in \mathbb{N}$. Define $J^{n,\varepsilon_k}_t = \sum_{0<s\le t} \Delta Z^n_s \mathbf{1}_{\{|\Delta Z^n_s| > \varepsilon_k\}}$, $\widehat{J}^{\varepsilon_k}_t = \sum_{0<s\le t} \Delta Z_s \mathbf{1}_{\{|\Delta \widehat{Z}_s| > \varepsilon_k\}}$, $Z^{n,\varepsilon_k} = Z^n_t - J^{n,\varepsilon_k}_t$, and $\widehat{Z}^{\varepsilon_k} = \widehat{Z}_t - \widehat{J}^{\varepsilon_k}_t$, $k, n \in \mathbb{N}$.

Since

$$(X^n, Z^n, [[Z^{n,\varepsilon_k}]], [J^{n,\varepsilon_k}]) \underset{\mathcal{D}}{\rightarrow} (\widehat{X}, \widehat{Z}, [[\widehat{Z}^{\varepsilon_k}]], [\widehat{J}^{\varepsilon_k}]) \quad \text{in} \quad \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)+1}) \tag{5.9}$$

and

$$\sum_{s\le t} |\Delta\widehat{Z}_s|^2 \mathbf{1}_{\{|\Delta Z^n_s| > \varepsilon_k\}} \to [\widehat{Z}]^d_t \quad \mathcal{P}\text{-a.s., } t \in \mathbb{R}^+,$$

we can find a sufficiently slowly increasing sequence $k_n \uparrow +\infty$ such that

$$(X^n, Z^n, [[Z^{n,\varepsilon_{k_n}}]], [J^{n,\varepsilon_{k_n}}]) \underset{\mathcal{D}}{\rightarrow} (\widehat{X}, \widehat{Z}, [[\widehat{Z}]]^c, [\widehat{Z}]^d) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)+1}). \tag{5.10}$$

For the sake of simplicity define

$$\phi(y, x) = \varphi(fy, x) - x - f(x)y, \qquad x, y \in \mathbb{R}^d.$$

Then due to the condition $\mathcal{P}[|\Delta\widehat{Z}_t| = \varepsilon_k; t \in \mathbb{R}^+] = 0, k \in \mathbb{N}$, and (5.10) we have the convergence

$$\begin{aligned}&\Big(X^n, Z^n, [[Z^{n,\varepsilon_{k_n}}]], [J^{n,\varepsilon_{k_n}}], \sum_{s\le\cdot} \phi(\Delta Z^n_s, X^n_{s-})\mathbf{1}_{\{|\Delta Z^n_s| > \varepsilon_k\}}\Big) \\ &\underset{\mathcal{D}}{\rightarrow} \Big(\widehat{X}, \widehat{Z}, [[\widehat{Z}]]^c, [\widehat{Z}]^d, \sum_{s\le t} \phi(\Delta\widehat{Z}_s, \widehat{X}_{s-})\mathbf{1}_{\{|\Delta\widehat{Z}_s| > \varepsilon_k\}}\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(3+d)+1}).\end{aligned} \tag{5.11}$$

By the boundedness of $f'f$,

$$\sum_{s\le t} |\phi(\Delta Z^n_s, X^n_{s-})| \mathbf{1}_{\{\varepsilon_k \ge |\Delta Z^n_s| > \varepsilon_{k_n}\}} \le C \sum_{s\le t} |\Delta Z^n_s|^2 \mathbf{1}_{\{\varepsilon_k \ge |\Delta Z^n_s| > \varepsilon_{k_n}\}}.$$

Since $\sum_{s\le t} |\Delta\widehat{Z}_s|^2 \mathbf{1}_{\{\varepsilon_k \ge |\Delta\widehat{Z}_s|\}} \downarrow 0$ $\mathcal{P}$-a.s., $t \in \mathbb{R}^+$, and

$$\sum_{s\le\cdot} |\Delta Z^n_s|^2 \mathbf{1}_{\{\varepsilon_k \ge |\Delta Z^n_s|\}} \underset{\mathcal{D}}{\rightarrow} \sum_{s\le\cdot} |\Delta\widehat{Z}_s|^2 \mathbf{1}_{\{\varepsilon_k \ge |\Delta\widehat{Z}_s|\}} \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}),$$

we deduce from (5.11) that

$$\Big(X^n, Z^n, [[Z^{n,\varepsilon_{k_n}}]], \sum_{s\le\cdot} \phi(\Delta Z^n_s, X^n_{s-})\mathbf{1}_{\{|\Delta Z^n_s|>\varepsilon_{k_n}\}}\Big) \underset{\mathcal{D}}{\to} \Big(\widehat{X}, \widehat{Z}, [[\widehat{Z}]]^c, \sum_{s\le\cdot} \phi(\Delta\widehat{Z}_s, \widehat{X}^x_{s-})\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(3+d)}). \tag{5.12}$$

In view of Theorem B.2,

$$\Big(X^n, \int_0^\cdot f(X^n_{s-})\,dZ^n_s, \int_0^\cdot f'f(X^n_{s-})\,d[Z^{n,\varepsilon_{k_n}}]_s, \sum_{s\le\cdot} \phi(\Delta Z^n_s, X^n_{s-})\mathbf{1}_{\{|\Delta Z^n_s|>\varepsilon_{k_n}\}}\Big)$$
$$\underset{\mathcal{D}}{\to} \Big(\widehat{X}, \int_0^\cdot f(\widehat{X}_{s-})\,d\widehat{Z}_s, \int_0^\cdot f'f(\widehat{X}_{s-})\,d[\widehat{Z}]^c, \sum_{s\le\cdot} \phi(\Delta\widehat{Z}_s, \widehat{X}_{s-})\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).$$

Therefore it remains to check that

$$I^n_\cdot = \frac{1}{2}\int_0^\cdot f'f(X^n_{s-})\,d[Z^n]^c_s + \sum_{s\le\cdot} \phi(\Delta Z^n_s, X^n_{s-})\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}} - \frac{1}{2}\int_0^\cdot f'f(X^n_{s-})\,d[Z^{n,\varepsilon_n}]_s \underset{\mathcal{P}}{\to} 0.$$

For fixed $n \in \mathbb{N}$ we have

$$\begin{aligned}
\sup_{s\le t}|I^n_s| &\le \sum_{s\le t}\Big|\phi(\Delta Z^n_s, X^n_{s-}) - \frac{1}{2}f'f(X^n_{s-})\Delta Z^n_s\Delta Z^{n\star}_s\Big|\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}} \\
&\le \sum_{s\le t}\sup_{\theta\in[0,1]}|(f'f(\varphi(f\Delta Z^n_s, X^n_{s-},\theta)) - f'f(X^n_{s-}))\Delta Z^n_s\Delta Z^{n\star}_s|\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}} \\
&\le \sup_{s\le t}\sup_{\theta\in[0,1]}|f'f(\varphi(f\Delta Z^n_s, X^n_{s-},\theta)) - f'f(X^n_{s-})|\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}}\sum_{s\le t}|\Delta Z^n_s|^2,
\end{aligned}$$

and by (5.3),

$$\begin{aligned}
&\sup_{s\le t}\sup_{\theta\in[0,1]}|\varphi(f\Delta Z^n_s, X^n_{s-},\theta) - X^n_{s-}|\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}} \\
&\qquad \le \varepsilon_{k_n}C\sup_{s\le t}\sup_{\theta\in[0,1]}\{1+|\varphi(f\Delta Z^n_s, X^n_{s-},\theta)|\} \le \varepsilon_{k_n}C\{1+\sup_{s\le t}|X^n_{s-}|e^c\}.
\end{aligned}$$

Thus the continuity of $f'f$ implies the convergence $\sup_{t\le q}|I^n_t| \to_{\mathcal{P}} 0$ and the proof of (i) is complete.

(ii) Follows immediately from (i). ■

Corollary 5.3. *Let Z be an $(\mathcal{F}_t)$ adapted semimartingale. If $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^d)$ and satisfies* (5.3), *then there exists a weak solution of the SDE* (5.1).

Proof. Let $\{T_n\}$ be a sequence of partitions of $\mathbb{R}^+$ satisfying (0.6). We put $Z^n = Z^{\varrho^n}$. For every $n \in \mathbb{N}$ there exists a unique strong solution to the SDE (5.1). Since $Z^{\varrho^n} \to Z$ a.s. in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $\{Z^{\varrho^n}\}$ satisfies (UT), the result is an obvious consequence of Theorem 5.2(i). ■

It is also possible to give a version of Theorem 4.6.

THEOREM 5.4. *Let* $\{Z^n\}$ *be a sequence of* $(\mathcal{F}^n_t)$ *adapted semimartingales satisfying* (UT) *and* $\{X^n\}$ *the sequence of strong solutions of the SDE* (5.2). *Assume that* $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ *and satisfies* (5.3). *If a solution* X *of the SDE* (5.2) *is adapted to the natural filtration of* (X_0, Z) *and the pathwise uniqueness property for* (5.2) *holds* (see (4.21)) *then the following two implications are true*:

(i) *if* $X^n_0 \to_{\mathcal{P}} X_0$ *and* $Z^n \to_{\mathcal{P}} Z$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *then*

$$(X^n, Z^n) \underset{\mathcal{P}}{\to} (X, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

(ii) *if* $X^n_0 \to_{\mathcal{P}} X_0$ *and* $\sup_{t\le q}|Z^n_t - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, *then*

$$\sup_{t\le q}|X^n_t - X_t| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+.$$

Proof. (i) We use Theorem 5.2 and follow the proof of Theorem 4.6(i).

(ii) Let us note that

$$\Delta X_t = \varphi(\Delta Z_t f, X_{t-}) - X_{t-} \tag{5.13}$$

and if $\Delta X_t \neq 0$ then $\Delta Z \neq 0$. By Corollary A.3,

$$\sup_{t\le q}|X^n_t - X_t| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+.$$

and we repeat the arguments from the proof of (i). ■

COROLLARY 5.5. *Let* Z *be an* $(\mathcal{F}_t)$ *adapted semimartingale. Assume* $f\in\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^d)$ *satisfies* (5.3) *and a solution* X *to the SDE* (5.2) *is adapted to the natural filtration of* (X_0, Z) *and the pathwise uniqueness property for* (5.2) *holds. Then*

(i) *there exists a unique strong solution of the SDE* (5.2),

(ii) *if* $\{T_n\}$ *is a sequence of partitions of* $\mathbb{R}^+$ *satisfying* (0.6) *and* $\{X^n\}$ *denotes a sequence of solutions of* (5.3) *corresponding to* $Z^n = Z^{\varrho^n}$, $X^n_0 = X_0$ *then*

$$\sup_{t\le q}|X^n_t - X^{\varrho^n}_t| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+.$$

Proof. Follows from Corollary 5.3 and Theorem 5.4(i). ■

Note that there exist examples of functions such that $f'f$ need not be Lipschitz continuous and satisfy the hypothesis of Corollary 5.5, for example, if $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^d)$, satisfies (5.3) and

$$|f'f(x) - f'f(y)|^2 \le \varrho(|x-y|^2), \quad x, y \in \mathbb{R},$$

where $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing and concave with $\varrho(0) = 0$ and

$$\int_{0^+} \frac{du}{\varrho(u)} = +\infty.$$

5.2. Approximations of Wong–Zakai type. Let us consider a sequence $\{T_n\}$ of partitions of $\mathbb{R}^+$ satisfying the condition (0.6). Let $\{\widehat{Z}^{\varrho^n}\}$ be a sequence of linear approximations of Z, i.e.

$$\widehat{Z}^{\varrho^n}_t = Z_{t_{nk}} + \frac{t - t_{nk}}{t_{n,k+1} - t_{nk}}(Z_{t_{n,k+1}} - Z_{t_{nk}}), \quad t \in [t_{nk}, t_{n,k+1}[,\ n \in \mathbb{N},\ k \in \mathbb{N}\cup\{0\}.$$

It is clear that $\widehat{Z}_t^{\varrho^n} \to Z_t$ $\mathcal{P}$-a.s. provided $\Delta Z_t = 0$. If f is locally Lipschitz continuous and satisfies (2.3) then it is also obvious that for every $n \in \mathbb{N}$ there exists a unique strong solution to

$$(5.14)\qquad Y_t^n = X_0 + (S)\int_0^t f(Y_s^n)\, d\widehat{Z}_s^{\varrho^n} = X_0 + \int_0^t f(Y_{s-}^n)\, d\widehat{Z}_s^{\varrho^n}, \quad t \in \mathbb{R}^+,\ n \in \mathbb{N}.$$

The above type of SDE's was introduced by Wong and Zakai [WoZa65] in the case when Z was a Wiener process W. They showed that $\{Y^n\}$ tends to the solution of a Stratonovich SDE (5.1). Their result has been generalized in many papers (see e.g. [Mar81, Pro85, Mac87]), where the authors have considered convergence of approximating solutions in the Skorokhod topology J_1. Recently, Kurtz, Pardoux and Protter [KPP92] have discussed Wong–Zakai type approximations for Stratonovich SDE's driven by general semimartingales. In this case the approximating sequence $\{Y^n\}$ is as before a sequence of continuous processes, but the limit process X need not have continuous trajectories. Therefore the approximating solutions need not converge in the Skorokhod topology J_1. With the use of very subtle results on Meyer–Zheng topology Kurtz, Pardoux and Protter have proved that

$$Y_t^n \underset{\mathcal{P}}{\to} X_t,$$

except possibly for a countable set of t's.

It turns out that the above result is an easy corollary of our Theorem 5.2.

THEOREM 5.6. *Let $\{Y^n\}$ be a family of solutions of SDE* (5.14). *Suppose that $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ and satisfies* (5.3). *If a solution X of the SDE* (5.2) *is adapted to the natural filtration of (X_0, Z) and the pathwise uniqueness property for* (5.2) *holds*, *then*

(i)

$$\sup_{t\le q, t\in T_n} |Y_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+,$$

where X is a unique strong solution of the SDE (5.2). *In particular*,

$$Y_t^n \underset{\mathcal{P}}{\to} X_t,$$

provided $\Delta Z_t = 0$ or $t \in \liminf_{n\to+\infty} T_n$,

(ii) *if Z is a semimartingale with continuous trajectories then*

$$\sup_{t\le q} |Y_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+.$$

Proof. (i) Assume that $\{X^n\}$ is a family of strong solutions corresponding to $\{Z^{\varrho^n}\}$, i.e.

$$(5.15)\qquad \begin{aligned} X_t^n &= X_0 + \int_0^t f(X_{s-}^n)\, dZ_s^{\varrho^n} + \frac{1}{2}\int_0^t f'f(X_{s-}^n)\, d[Z^{\varrho^n}]_s^c \\ &\quad + \sum_{s\le t} \{\varphi(f\Delta Z_s^{\varrho^n}, X_{s-}^n) - X_{s-}^n - f(X_{s-}^n)\Delta Z_s^{\varrho^n}\} \\ &= X_0 + \sum_{s\le t}\{\varphi(f\Delta Z_s^{\varrho^n}, X_{s-}^n) - X_{s-}^n\}, \quad t \in \mathbb{R}^+, n \in \mathbb{N}. \end{aligned}$$

By Corollary 5.5,

$$\sup_{t\le q}|X_t^n - X_t^{\varrho_n}| \underset{\mathcal{P}}{\to} 0, \qquad q\in\mathbb{R}^+.$$

Since $Y_t^n = X_t^n$ for $t\in T_n$, the proof of (i) is complete.

(ii) In this case the solution X has continuous trajectories as well. Therefore

$$\sup_{t\le q}|X_t - X_t^{\varrho_n}| \to 0 \quad \mathcal{P}\text{-a.s., } q\in\mathbb{R}^+.$$

Similarly by the continuity of trajectories of Z we deduce $\sup_{t\le q}|\Delta Z_t^{\varrho^n}| \to 0$ $\mathcal{P}$-a.s., which implies that

$$\sup_{t\le q}|Y_t^n - Y_t^{n,\varrho_n}| \underset{\mathcal{P}}{\to} 0, \qquad q\in\mathbb{R}^+.$$

Now (ii) is an easy consequence of (i). ∎

5.3. Convergence of Stratonovich stochastic integrals. At the very beginning we recall a definition of Stratonovich integral proposed in [KPP92]. Let Z be an $(\mathcal{F}_t)$ adapted semimartingale. Assume that $f\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ and that X is a strong solution to the Stratonovich SDE (5.1).

Definition 5.7. Let $g\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$. We define the *Stratonovich integral* of $g(X)$ with respect to Z as follows:

$$\begin{aligned}\int_0^t g(X_s)\circ dZ_s &= \int_0^t g(X_{s-})\,dZ_s + \frac{1}{2}\int_0^t g'f(X_{s-})\,d[Z]_s^c\\ &\quad+\sum_{s\le t}\Big(\int_0^1\{g(\varphi(f\Delta Z_s, X_{s-},u)) - g(X_{s-})\}\,du\,\Delta Z_s\Big), \qquad t\in\mathbb{R}^+,\end{aligned}$$

where we used the notation

$$\int_0^t g'f(X_{s-})\,d[Z]_s^c = \sum_{j=1}^d\sum_{l=1}^d\sum_{m=1}^d\int_0^t \frac{\partial g_j}{\partial x_l} f_{lm}(X_{s-})\,d[Z^j,Z^m]_s^c.$$

Now, let $\{Z^n\}$ be a sequence of semimartingales and $\{X^n\}$ denote the sequence of strong solutions of the SDE (5.2), corresponding to $\{Z^n\}$.

Theorem 5.8. *Assume that* $f\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$, $g\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$ *and* f *satisfies* (5.3). *Let* $\{Z^n\}$ *be a sequence of semimartingales satisfying* (UT).

(i) *If* $(X^n,Z^n)\to_{\mathcal{D}}(X,Z)$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$ *then* X *is a solution of the Stratonovich SDE driven by a semimartingale* Z *and*

$$\Big(X^n,\int_0^{\cdot} g(X_s^n)\circ dZ_s^n, Z^n\Big) \underset{\mathcal{D}}{\to} \Big(X,\int_0^{\cdot} g(X_s)\circ dZ_s, Z\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d+1}).$$

(ii) *If* $(X_0^n,Z^n)\to_{\mathcal{D}}(X_0,Z)$ *in* $\mathbb{R}^d\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ *and* f *satisfies additionally the assumptions from Corollary* 5.5 *then*

$$\Big(X^n,\int_0^{\cdot} g(X_s^n)\circ dZ_s^n, Z^n\Big) \underset{\mathcal{D}}{\to} \Big(X,\int_0^{\cdot} g(X_s)\circ dZ_s, Z\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d+1}),$$

where X *is a unique strong solution of Stratonovich SDE driven by* Z.

Proof. (i) Due to Theorem 5.2(i), X is a solution of the Stratonovich SDE driven by the semimartingale Z. Let $\{\{Z^{n,\varepsilon_k}\}, \{\{J^{n,\varepsilon_k}\}\}$ be two families of processes defined in the proof of Theorem 5.2. Since the map

$$\mathbb{R}^{2d} \ni (y,x) \mapsto \varphi(fy,x,\cdot) \in \mathcal{C}[0,1]$$

is continuous in the topology of uniform convergence, we deduce from (5.10) that

$$\begin{aligned}\Big(X^n, \int_0^{\cdot} g(X^n_{s-})\,dZ^n_s, \int_0^{\cdot} g'f(X^n_{s-}\Big)\,d[Z^{n,\varepsilon_{k_n}}]_s,\\ \sum_{s\le\cdot}\Big(\int_0^1\{g(\varphi(f\Delta Z^n_s, X^n_{s-},u)) - g(X^n_{s-})\}\,du\Big)\Delta Z^n_s \mathbf{1}_{\{|\Delta Z^n_s|>\varepsilon_{k_n}\}}\Big)\\ \underset{\mathcal{D}}{\to} \Big(X, \int_0^{\cdot} g(X_{s-})\,dZ_s, \int_0^{\cdot} g'f(X_{s-})\,d[Z]^c,\\ \sum_{s\le\cdot}\int_0^1\{g(\varphi(f\Delta Z_s, X_{s-},u)) - g(X_{s-})\}\,du\,\Delta Z_s\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{d+3}).\end{aligned}$$

Define for every $n\in\mathbb{N}$ and $t\in\mathbb{R}^+$,

$$I^n_t = \sum_{s\le t}\Big\{\phi(\Delta Z^n_s, X^n_{s-}) - \frac{1}{2}g'f(X^n_{s-})\Delta Z^n_s \Delta Z^{n\star}_s\Big\}\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}}.$$

The proof will be finished if we show that $\sup_{t\le q}|I^n_t| \to_{\mathcal{P}} 0$.

Since the map

$$\mathbb{R} \ni v \mapsto \int_0^v g(\varphi(f\Delta Z^n_s, X^n_{s-},u))\,du\,\Delta Z^n_s \in \mathbb{R}$$

belongs to $\mathcal{C}^2$ it follows from Taylor's theorem that there exists $\theta\in(0,1)$ such that

$$\int_0^1 g(\varphi(f\Delta Z^n_s, X^n_{s-},u))\,du\,\Delta Z^n_s = g(X^n_{s-})\Delta Z^n_s + \tfrac{1}{2}g'f(\varphi(f\Delta Z^n_s, X^n_{s-},\theta))\Delta Z^n_s \Delta Z^{n\star}_s.$$

Therefore

$$\begin{aligned}\sup_{s\le t}|I^n_s| &\le \sum_{s\le t}\sup_{\theta\in[0,1]}|(g'f(\varphi(f\Delta Z^n_s, X^n_{s-},\theta)) - g'f(X^n_{s-}))\Delta Z^n_s\Delta Z^{n\star}_s|\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}}\\ &\le \sup_{s\le t}\sup_{\theta\in[0,1]}|g'f(\varphi(f\Delta Z^n_s, X^n_{s-},\theta)) - g'f(X^n_{s-})|\mathbf{1}_{\{|\Delta Z^n_s|\le\varepsilon_{k_n}\}}\sum_{s\le t}|\Delta Z^n_s|^2.\end{aligned}$$

The continuity of $g'f$ completes the proof.

(ii) In view of Theorem 5.2, $(X^n, Z^n) \to_{\mathcal{D}} (X,Z)$. Next we apply (i). ■

By the arguments from the proof of Theorem 5.4 we can deduce from Theorem 5.8 the appropriate the results on the convergence in probability. In particular, we can approximate Stratonovich integral using the processes considered in Section 5.2.

COROLLARY 5.9. *Let $\{Y^n\}$ be a sequence of solutions of the SDE* (5.14), *where f satisfies the assumptions of Corollary* 5.5 *and $g\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$. Then the following convergences hold*:

(i)

$$\sup_{t\le q, t\in T_n}\Big|\int_0^t g(Y^n_{s-})\,d\widehat{Z}^{\varrho^n}_s - \int_0^t g(X_{s-})\circ dZ_s\Big| \underset{\mathcal{P}}{\rightarrow} 0, \qquad q\in\mathbb{R}^+,$$

where X *is a unique strong solution of the SDE* (5.2), *and as a consequence*,

$$\int_0^t g(Y^n_{s-})\,d\widehat{Z}^{\varrho^n}_s \underset{\mathcal{P}}{\rightarrow} \int_0^t g(X_{s-})\circ dZ_s,$$

provided $\Delta Z_t = 0$ *or* $t\in\liminf_{n\to+\infty} T_n$.

(ii) *If* Z *is a semimartingale with continuous trajectories then*

$$\sup_{t\le q}\Big|\int_0^t g(Y^n_{s-})\,d\widehat{Z}^{\varrho^n}_s - \int_0^t g(X_{s-})\circ dZ_s\Big| \underset{\mathcal{P}}{\rightarrow} 0, \qquad q\in\mathbb{R}^+. \ \blacksquare$$

6. Skorokhod problem, deterministic case

6.1. Existence and basic properties of solutions. Let D be a domain in $\mathbb{R}^d$. Define the set $\mathcal{N}_x$ of inward normal unit vectors at $x\in\partial D$ by

$$\mathcal{N}_x = \bigcup_{r>0}\mathcal{N}_{x,r}, \qquad \mathcal{N}_{x,r} = \{\mathbf{n}\in\mathbb{R}^d : |\mathbf{n}|=1, B(x-r\mathbf{n},r)\cap D=\emptyset\},$$

where

$$B(z,r)=\{y\in\mathbb{R}^d : |y-z|<r\}, \qquad z\in\mathbb{R}^d,\ r>0.$$

Following Lions and Sznitman [LiSz84] and Saisho [Sai87] we introduce two assumptions.

(A) There exists a constant $r_0>0$ such that

$$\mathcal{N}_x = \mathcal{N}_{x,r_0}\neq\emptyset \quad \text{for every } x\in\partial D.$$

(B) There exist constants $\delta>0$, $\beta\ge 1$ such that for every $x\in\partial D$ there exists a unit vector $\mathbf{l}_x$ with the following property:

$$\langle \mathbf{l}_x,\mathbf{n}\rangle \ge \frac{1}{\beta} \quad \text{for every } \mathbf{n}\in \bigcup_{y\in B(x,\delta)\cap\partial D}\mathcal{N}_y$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$.

Remark 6.1 ([LiSz84, Sai87]). (i) $\mathbf{n}\in\mathcal{N}_{x,r}$ if and only if $\langle y-x,\mathbf{n}\rangle + \frac{1}{2r}|y-x|^2\ge 0$ for every $y\in\overline{D}$,

(ii) if $\operatorname{dist}(x,\overline{D})<r_0$, $x\notin\overline{D}$ then there exists a unique $[x]_\partial\in\overline{D}$ such that $|x-[x]_\partial| = \operatorname{dist}(x,\overline{D})$ and moreover $([x]_\partial - x)/|[x]_\partial - x|\in\mathcal{N}_{[x]_\partial}$,

(iii) if D is a convex domain in $\mathbb{R}^d$ then $r_0=+\infty$.

The Skorokhod deterministic problem is stated in the following manner.

Definition 6.2. Let $y\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ and $y_0\in\overline{D}$. We will say that a pair $(x,k)\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$ is a solution of the Skorokhod problem associated with y if

(i) $x_t = y_t + k_t$, $t\in\mathbb{R}^+$,

(ii) $x_t \in \overline{D}$, $t \in \mathbb{R}^+$,

(iii) k is a function with bounded variation on each finite interval such that $k_0 = 0$ and
$$k_t = \int_0^t \mathbf{n}_s \, d\,\mathrm{Var}(k)_s, \qquad \mathrm{Var}(k)_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} \, d\,\mathrm{Var}(k)_s,$$
where $\mathbf{n}_s \in \mathcal{N}_{x_s}$ if $x_s \in \partial D$,

(iv) $|\Delta x_t| \le |\Delta y_t|$, $t \in \mathbb{R}^+$.

Remark 6.3. Notice that we have added the condition (iv) to the classical definition of the Skorokhod problem. However, it is known that if either D is convex or the condition (A) holds, then for every pair (x, k) satisfying (i)–(iii) we have also (iv) (see e.g. [Sai87, Tan79]). On the other hand, if D is not connected then under (i)–(iii) alone, the solution of the Skorokhod problem associated with fixed discontinuous y is in general not unique. Moreover, the set of solutions need not be relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ nor locally bounded, as the following example shows.

Example 6.4. Put $D = (-\infty, 0) \cup \bigcup_{n=1}^{\infty}(4n-2, 4n)$ and $y_t = t - 9/8$ for $t < 1$ and $y_t = 1/8$ for $t \ge 1$. Then the conditions (A), (B) are satisfied with $r_0 = 1$ and one can check that the function $x_t = y_t$ for $t < 1$ and $x_t = 0$ for $t \ge 1$ solves the problem associated with y. Moreover, for each $n \in \mathbb{N}$ the function x^n defined as $x^n_t = y_t$ for $t < 1$ and $x^n_t = 4n - 2$ for $t \ge 1$ is a solution of the problem, too.

We start with the following simple lemma:

Lemma 6.5. *Assume the condition* (A). *If* $y_0 \in \overline{D}$ *and* $|\Delta y| < r_0$ *then the Skorokhod problem has at most one solution.*

Proof. We will use the inequality proved by Saisho (see [Sai87, 2.6]). Let y, $\widehat{y} \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and let (x, k), $(\widehat{x}, \widehat{k})$ be the solutions associated with y and $\widehat{y}$, respectively. Then
$$|x_t - \widehat{x}_t|^2 \le |y_t - \widehat{y}_t|^2 + \frac{1}{r_0}\int_0^t |x_s - \widehat{x}_s|^2 \, d(\mathrm{Var}(k)_s + \mathrm{Var}(\widehat{k})_s) + 2\int_0^t (y_t - y_s - \widehat{y}_t + \widehat{y}_s)\, d(k_s - \widehat{k}_s). \tag{6.1}$$
First assume $|\Delta y| \le r_0/4$. Suppose that $(x, k), (\widehat{x}, \widehat{k})$ are two solutions of the Skorokhod problem associated with $y = \widehat{y}$. By (6.1),
$$|x_t - \widehat{x}_t|^2 \le \frac{1}{r_0}\int_0^{t-} |x_s - \widehat{x}_s|^2 \, d(\mathrm{Var}(k)_s + \mathrm{Var}(\widehat{k})_s) + \frac{1}{r_0}|x_t - \widehat{x}_t|^2(|\Delta k_t| + |\Delta \widehat{k}_t|).$$
Since $|\Delta x_t|, |\Delta \widehat{x}_t| \le |\Delta y_t|$ and $|\Delta y_t| \le r_0/4$, it is clear that $|\Delta k_t| + |\Delta \widehat{k}_t| \le r_0/2$ and as a consequence
$$|x_t - \widehat{x}_t|^2 \le \frac{2}{r_0}\int_0^{t-} |x_t - \widehat{x}_t|^2 \, d(\mathrm{Var}(k)_s + \mathrm{Var}(\widehat{k})_s).$$
Therefore by Gronwall's lemma (see e.g. Lemma C.2) $x = \widehat{x}$.

Now, let $|\Delta y| < r_0$. Define

$$s_1 = \inf\{t > 0 : |\Delta y_t| > r_0/4\}, \quad s_m = \inf\{t > s_{m-1} : |\Delta y_t| > r_0/4\}, \quad m \geq 2.$$

Obviously, on each finite interval there exists only finitely many jumps greater than $r_0/4$. By the arguments used previously $x_t = \widehat{x}_t$ for $t \in [0, s_1)$ and in particular $x_{s_1-} = \widehat{x}_{s_1-}$. Therefore using the notation from Remark 6.1(ii) and setting $[x]_\partial = x$ for every $x \in \overline{D}$ we can write

$$x_{s_1} = \widehat{x}_{s_1} = [x_{s_1-} + \Delta y_{s_1}]_\partial.$$

Hence $x_t = \widehat{x}_t$ for $t \in [0, s_1]$. Analogously we obtain the equality $x = \widehat{x}$ on every interval $[s_{m-1}, s_m]$. The proof is finished. ∎

PROPOSITION 6.6. *Let D satisfy the conditions* (A) *and* (B). *Assume that* (x, k) *is a solution of the Skorokhod problem associated with* $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $|\Delta y| \leq c$ *for some constant* $c < r_0$. *If* $\sup_t |y_t| \leq a$ *then there exists a constant* C *depending on* a *and* c (*and also on* r_0, δ, β) *such that*

$$\mathrm{Var}(k)_{t_m} - \mathrm{Var}(k)_{t_{m-1}} \leq C\omega_y[t_{m-1}, t_m]$$

provided that $t_m < +\infty$, *where* $t_0 = \inf\{t : x_t \in \partial D\}$, $t_m^\delta = \inf\{t > t_{m-1} : |x_t - x_{t_{m-1}}| \geq \delta\}$, $t_m = \inf\{t \geq t_m^\delta : x_t \in \partial D\}$, $m \in \mathbb{N}$.

P r o o f. If $t \in [t_{m-1}, t_m^\delta[$ and $\mathbf{l} = \mathbf{l}_{x_{t_{m-1}}}$ then by (B) we have

$$\begin{aligned}\langle \mathbf{l}, x_t - x_{t_{m-1}}\rangle &= \langle \mathbf{l}, y_t - y_{t_{m-1}}\rangle + \langle \mathbf{l}, k_t - k_{t_{m-1}}\rangle \\ &\geq \langle \mathbf{l}, y_t - y_{t_{m-1}}\rangle + \frac{1}{\beta}(\mathrm{Var}(k)_t - \mathrm{Var}(k)_{t_{m-1}}).\end{aligned}$$

Therefore,

$$\mathrm{Var}(k)_{t_m^\delta -} - \mathrm{Var}(k)_{t_{m-1}} \leq \beta\{\sup_{t_{m-1} \leq t < t_m^\delta} |x_t - x_{t_{m-1}}| + \omega_y[t_{m-1}, t_m^\delta[\}. \tag{6.2}$$

On the other hand, by the inequality 2.7 in [Sai87] for $t \in [t_{m-1}, t_m^\delta[$

$$\begin{aligned}|x_t - x_{t_{m-1}}|^2 &\leq |y_t - y_{t_{m-1}}|^2 + \frac{1}{r_0}\int_{t_{m-1}}^{t} |x_s - x_{t_{m-1}}|^2 \, d\,\mathrm{Var}(k)_s + 2\int_{t_{m-1}}^{t} (y_t - y_s)\, dk_s \\ &\leq \omega_y^2[t_{m-1}, t_m^\delta[+ \frac{1}{r_0}\int_{t_{m-1}}^{t-} |x_s - x_{t_{m-1}}|^2 \, d\,\mathrm{Var}(k)_s \\ &\quad + \frac{1}{r_0}|x_t - x_{t_{m-1}}|^2 |\Delta k_t| + 2\omega_y[t_{m-1}, t_m^\delta[(\mathrm{Var}(k)_{t_m^\delta -} - \mathrm{Var}(k)_{t_{m-1}}).\end{aligned}$$

Hence for $t \in [t_{m-1}, t_m^\delta[$,

$$\begin{aligned}|x_t - x_{t_{m-1}}|^2 \leq \frac{r_0}{r_0 - c}\Big\{&\omega_y^2[t_{m-1}, t_m^\delta[+ \frac{1}{r_0}\int_{t_{m-1}}^{t-} |x_s - x_{t_{m-1}}|^2 \, d\,\mathrm{Var}(k)_s \\ &+ 2\omega_y[t_{m-1}, t_m^\delta[(\mathrm{Var}(k)_{t_m^\delta -} - \mathrm{Var}(k)_{t_{m-1}})\Big\}\end{aligned}$$

and by Gronwall's lemma,

$$\sup_{t_{m-1}\le t<t_m^\delta}|x_t - x_{t_{m-1}}|^2 \le \frac{r_0}{r_0-c}\{\omega_y^2[t_{m-1},t_m^\delta[+ 2\omega_y[t_{m-1},t_m^\delta[(\mathrm{Var}(k)_{t_m^\delta-} - \mathrm{Var}(k)_{t_{m-1}})\} \times \exp\left\{\frac{1}{r_0-c}(\mathrm{Var}(k)_{t_m^\delta} - \mathrm{Var}(k)_{t_{m-1}})\right\}.$$

Therefore using (6.2) and putting,

$$w = \sup_{t_{m-1}\le t<t_m^\delta}|x_t - x_{t_{m-1}}|, \qquad b = \omega_y[t_{m-1},t_m^\delta[,$$

we have

$$w^2 \le \frac{r_0}{r_0-c}\{b^2 + 2b\beta(w+b)\}\exp\left\{\frac{1}{r_0-c}\beta(w+b)\right\}.$$

By easy calculations $w \le \delta$ and $b \le 2a$. Hence it is clear that there exists a constant C_1 such that

$$w^2 \le C_1(b^2 + 2bw).$$

Since $b, w \ge 0$, there exists also a constant C_2 for which $w \le C_2 b$. Thus, using (6.2) once more we get

$$\mathrm{Var}(k)_{t_m^\delta-} - \mathrm{Var}(k)_{t_{m-1}} \le \beta(C_2+1)\omega_y[t_{m-1},t_m^\delta[.$$

Thus

$$\begin{aligned}\mathrm{Var}(k)_{t_m^\delta} - \mathrm{Var}(k)_{t_{m-1}} &\le \beta(C_2+1)\omega_y[t_{m-1},t_m^\delta[+ |\Delta k_{t_m^\delta}| \\ &\le \beta(C_2+1)\omega_y[t_{m-1},t_m^\delta[+ |\Delta y_{t_m^\delta}| \\ &\le \{\beta(C_2+1)+1\}\omega_y[t_{m-1},t_m^\delta].\end{aligned}$$

Therefore if $t_m = t_m^\delta$ the proof is complete. Assume now $t_m^\delta < t_{m-1}$. Since k is constant in the interval $]t_m^\delta, t_{m-1}[$,

$$\begin{aligned}\mathrm{Var}(k)_{t_m} - \mathrm{Var}(k)_{t_{m-1}} &\le \{\beta(C_2+1)+1\}\omega_y[t_{m-1},t_m^\delta] + |\Delta k_{t_m}| \\ &\le \{\beta(C_2+1)+2\}\omega_y[t_{m-1},t_m]\end{aligned}$$

and it is enough to put $C = \beta(C_2+1)+2$. ■

6.2. Convergence of solutions

COROLLARY 6.7. *Under the assumptions of Proposition* 6.6 *there exists a constant C depending on a and c such that*

$$\delta \le |x_{t_m^\delta} - x_{t_{m-1}}| \le C\omega_y[t_{m-1},t_m^\delta],$$

provided that $t_m^\delta < +\infty$, $m \in \mathbb{N}$.

Proof. This follows easily from the definition of t_m^δ and the estimate $w \le C_2 b$ from the proof of Proposition 6.6. ■

COROLLARY 6.8. *Assume the conditions* (A) *and* (B). *Let $\{y^n\} \subset \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $|\Delta y^n| \le c < r_0$, for every $n \in \mathbb{N}$. Let $\{(x^n,k^n)\}$ be a sequence of solutions of the Skorokhod* *problem associated with the sequence* $\{y^n\}$. *If* $\{y^n\}$ *is relatively compact in* $\mathbb{D}([0,q],\mathbb{R}^d)$ *then*

(i) *there exists a constant* C *depending on* q, c, $\sup_n \sup_{t\le q}|y^n_t|$ *and on the modulus of continuity* ω'_{y^n} *on* $[0,q]$ *such that*

$$\mathrm{Var}(k^n)_t - \mathrm{Var}(k^n)_s \le C\omega_{y^n}[s,t], \qquad n\in\mathbb{N},\ 0\le s<t\le q,$$

(ii) $\mathrm{Var}(k^n)_q \le 2C\sup_{t\le q}|y^n_t|$,

(iii) $\{(x^n,y^n,k^n,\mathrm{Var}(k^n))\}$ *is relatively compact in* $\mathbb{D}([0,q],\mathbb{R}^{4d})$.

Proof. (i) By the version of the Arzelà–Ascoli theorem in $\mathbb{D}([0,q],\mathbb{R}^d)$ [Bil68, Theorem 14.3], the sequence $\{y^n\}$ is relatively compact in $\mathbb{D}([0,q],\mathbb{R}^d)$ if and only if the following two conditions are satisfied:

$$\sup_n \sup_{t\le q}|y^n_t| < +\infty, \tag{6.3}$$

$$\lim_{h\downarrow 0}\sup_n \omega'_{y^n}(h,q) = 0. \tag{6.4}$$

It is obvious that we can rewrite (6.4) in the form

$$\forall_{\varepsilon>0}\exists_{\gamma>0,(s_k)} \quad \omega_{y^n}[s^n_{k-1},s^n_k[\, <\varepsilon, \quad k=1,\ldots,r_n,\ n\in\mathbb{N}, \tag{6.5}$$

where $0=s^n_0<\ldots<s^n_{r_n}=q$, $s^n_k-s^n_{k-1}>\gamma$, $k=1,\ldots,r_n$. Similarly to t_m, t^δ_m we define the times t^n_m, $t^{n,\delta}_m$ for x^n, $n,m\in\mathbb{N}$. By Corollary 6.7 there exists a constant C depending on q, c and $a=\sup_n\sup_{t\le q}|y^n_t|$ (and also on r_0, δ, β) such that

$$\delta \le |x^n_{t^{n,\delta}_m} - x^n_{\tau^n_{m-1}}| \le C\omega_{y^n}[\tau^n_{m-1},t^{n,\delta}_m], \tag{6.6}$$

provided that $t^{n,\delta}_m\le q$. Hence for such m and $\varepsilon=\delta/C$,

$$0<\varepsilon\le\omega_{y^n}[\tau^n_{m-1},t^{n,\delta}_m]\le\omega_{y^n}[\tau^n_{m-1},t^n_m].$$

As a consequence every interval $[s^n_{k-1},s^n_k[$ in (6.5) contains at most one point t^n_m for every $n\in\mathbb{N}$. Therefore by Proposition 6.6,

$$\mathrm{Var}(k^n)_t-\mathrm{Var}(k^n)_s\le\Big(\frac{q}{\gamma}+1\Big)C\omega_{y^n}[s,t], \qquad s,t\le q,$$

which yields (i).

(ii) is a trivial consequence of (i) and of the estimate

$$\omega_{y^n}[0,q]\le 2\sup_{t\le q}|y^n_t|.$$

(iii) By (i) and (ii) we can estimate $\sup_{t\le q}|x^n_t|$, $\sup_{t\le q}|k^n_t|$, $\mathrm{Var}(k^n)_q$ by $\sup_{t\le q}|y^n_t|$, and the moduli of continuity $\omega_{x^n}[s,t]$, $\omega_{k^n}[s,t]$, $\omega_{\mathrm{Var}(k^n)}[s,t]$ by $\omega_{y^n}[s,t]$, $s,t\le q$. Thus, by simple calculations we deduce that the conditions (6.3), (6.5) are satisfied for $\{(x^n,y^n,k^n,\mathrm{Var}(k^n))\}\subset\mathbb{D}([0,q],\mathbb{R}^{4d})$ instead of $\{y^n\}\subset\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. ∎

THEOREM 6.9. *Suppose that a domain* D *satisfies the conditions* (A) *and* (B). *Let* $\{y^n\}\subset\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $y^n_0\in\overline{D}$ *and let* $\{(x^n,k^n)\}$ *be a sequence of solutions to the Skorokhod problem associated with* $\{y^n\}$. *If* $y^n\to y$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ *and* $|\Delta y|<r_0$ *then*

$$(x^n,k^n,y^n)\to(x,k,y) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d}),$$

where (x,k) *is a solution of the Skorokhod problem corresponding to* y.

P r o o f. Let $\{c_k\}$ be a sequence of constants, $c_k \uparrow r_0$ such that $|\Delta y_t| \neq c_k$ for all $t \in \mathbb{R}^+$, $k \in \mathbb{N}$. Define

$$s_k = \inf\{t > 0 : |\Delta y_t| \geq c_k\} \quad \text{and} \quad s_k^n = \inf\{t > 0 : |\Delta y_t^n| \geq c_k\}, \quad n, k \in \mathbb{N}.$$

By Proposition 2.7 of [JaSh87], $s_k^n \to s_k$ and $y^{n,s_k^n-} \to y^{s_k-}$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Let us fix $q \in \mathbb{R}^+$ such that $\Delta y_q = 0$. Then $y^{n,s_k^n-} \to y^{s_k-}$ in $\mathbb{D}([0,q], \mathbb{R}^d)$. Since $|\Delta y^{n,s_k^n-}| \leq c_k$, by Corollary 6.8(iii),

$$\{(x^{n,s_k^n-}, k^{n,s_k^n-}, \operatorname{Var}(k^{n,s_k^n-}), y^{n,s_k^n-})\}$$

is relatively compact in $\mathbb{D}([0,q], \mathbb{R}^{3d+1})$. On the other hand,

$$\lim_{k\to+\infty} \lim_{n\to+\infty} s_k^n = +\infty.$$

Therefore letting $q \uparrow +\infty$ we deduce that

$$\{(x^n, k^n, \operatorname{Var}(k^n), y^n)\} \quad \text{is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}). \tag{6.7}$$

Now, assume that there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$(x^{n'}, k^{n'}, \operatorname{Var}(k^{n'}), y^{n'}) \to (x, k, b, y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}). \tag{6.8}$$

Since $|\Delta y| < r_0$, according to Lemma 6.5 we only have to check that (x, k) is a solution of the Skorokhod problem associated with y. Therefore all we have to show is the following:

$$\operatorname{Var}(k)_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} \, d\operatorname{Var}(k)_s, \quad t \in \mathbb{R}^+, \tag{6.9}$$

$$k_t = \int_0^t \mathbf{n}_s \, d\operatorname{Var}(k)_s, \quad \text{where } \mathbf{n}_s \in \mathcal{N}_{x_s} \quad \text{if } x_s \in \partial D,\ t \in \mathbb{R}^+. \tag{6.10}$$

By Remark 6.1 for every continuous z with values in $\overline{D}$ and every $n' \in \mathbb{N}$,

$$\int_0^t (z_s - x_s^{n'}) \, dk_s^{n'} + \frac{1}{2r_0} \int_0^t |z_s - x_s^{n'}|^2 \, d\operatorname{Var}(k^{n'})_s \geq 0. \tag{6.11}$$

Since z is continuous, by (6.8),

$$(z - x^{n'}, |z - x^{n'}|^2, k^{n'}, \operatorname{Var}(k^{n'})) \to (z - x, |z - x|^2, k, b) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d+2}),$$

and it follows by Corollary 6.8(ii) and Corollary B.3 that (6.11) implies

$$\int_0^t (z_s - x_s) \, dk_s + \frac{1}{2r_0} \int_0^t |z_s - x_s|^2 \, db_s \geq 0. \tag{6.12}$$

The last part of the proof follows the proof of [LiSz84, Theorem 1.1] and we give it below for completeness only. By the definition of b, $k_t - k_s \leq \operatorname{Var}(k)_t - \operatorname{Var}(k)_s \leq b_t - b_s$ for $s \leq t$ and hence there exists a bounded measurable function h_s with $|h_s| \leq 1$ such that $dk_s = h_s \, db_s$, $\operatorname{Var}(k)_s = |h_s| \, db_s$. Therefore by (6.12),

$$\langle z_s - x_s, h_s \rangle + \frac{1}{2r_0} |z_s - x_s|^2 \geq 0 \quad db_s\text{-a.e.} \tag{6.13}$$

Let $D^\circ = \overline{D}\backslash\partial D$ and let $\{\phi_m\}$ be a sequence of continuous functions $\phi_m : \mathbb{R}^d \to \mathbb{R}$ with $\phi_m \uparrow \mathbf{1}_{D^\circ}$, $\phi_m = 0$ on $\mathbb{R}^d\backslash D^\circ$. Since by Definition 6.2,

$$\forall_m \quad \int_0^t \phi_m(X_s^{n'})\, d\operatorname{Var}(k^{n'})_s = 0,$$

it follows by Lemma B.3 that $\int_0^t \mathbf{1}_{\{x_s\in D^\circ\}}\, db_s = 0$, which yields $x_s \in \partial D\, d\operatorname{Var}(k)_s$-a.e. and (6.9). On the other hand, by (6.13),

$$\langle z_s - x_s, h_s\rangle + \frac{1}{2r_0}|z_s - x_s|^2 \ge 0 \quad dk_s\text{-a.e.} \tag{6.14}$$

In order to finish the proof it is sufficient to deduce from (6.9) that

$$\mathbf{n}_s = h_s/|h_s| \quad d\operatorname{Var}(k)_s\text{-a.e.}$$

Indeed, we have

$$k_t = \int_0^t h_s\, db_s = \int_0^t \frac{h_s}{|h_s|}|h_s|\, db_s = \int_0^t \mathbf{n}_s\, d\operatorname{Var}(k)_s,$$

which gives (6.10). The equality $\mathbf{n}_s = h_s/|h_s|$ is trivial by Remark 6.1 if $|h_s| = 1$. Assume that $0 < |h_s| < 1$ and define $z = x_s - r_0h_s, C = |z - [z]_\partial|$, where $[z]_\partial$ is uniquely determined by Remark 6.1. Then $|z - x_s| = r_0|h_s|$ and by (6.14),

$$\begin{aligned}
0 &\le \left\langle [z]_\partial - x_s, \frac{x_s - z}{r_0|h_s|}\right\rangle + \frac{1}{2r_0|h_s|}|[z]_\partial - x_s|^2\\
&= -r_0|h_s| + \left\langle [z]_\partial - z, \frac{x_s - z}{r_0|h_s|}\right\rangle + \frac{1}{2r_0|h_s|}|[z]_\partial - x_s|^2\\
&= -r_0|h_s| + \frac{1}{2r_0|h_s|}(|[z]_\partial - z|^2 + |z - x_s|^2)\\
&= -r_0|h_s| + \frac{1}{2r_0|h_s|}(C^2 + r_0^2|h_s|^2).
\end{aligned}$$

Hence $C \ge r_0|h_s|$ and again using Remark 6.1, $x_s = [z]_\partial$. Since $[z]_\partial - z/|[z]_\partial - z| = h_s/|h_s|$, the proof is finished. ■

It is proved in Saisho [Sai87] that if y is continuous then the solution of the Skorokhod problem corresponding to y can be approximated by the solutions of discrete Skorokhod problems. Now, we discuss such a problem for $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Let us consider a sequence $\{T_n\}$ of partitions of $\mathbb{R}^+$ satisfying the condition (0.6). It is easy to see that

$$y^{\varrho^n} \to y \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d). \tag{6.15}$$

Now, let us fix $n \in \mathbb{N}$. It is observed in [Sai87] that if $|\Delta y| < r_0$ then for sufficiently large $n \in \mathbb{N}$ the solution (x^n, k^n) of the Skorokhod problem associated with a discretization $y^{\varrho^n} \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ has the form

$$x_t^n = \begin{cases} y_0 & \text{if } t \in [0, t_{n1}),\\ [x^n_{t_{n,k-1}} + y_{t_{nk}} - y_{t_{n,k-1}}]_\partial & \text{if } t \in [t_{nk}, t_{n,k+1}),\ k \in \mathbb{N}, \end{cases} \tag{6.16}$$

and

$$
(6.17) \qquad k_t^n = \begin{cases} 0 & \text{if } t \in [0, t_{n1}), \\ k^n_{t_{n,k-1}} + x_t^n - x^n_{t_{n,k-1}} & \\ -y_{t_{nk}} + y_{t_{n,k-1}} & \text{if } t \in [t_{nk}, t_{n,k+1}),\ k \in \mathbb{N}. \end{cases}
$$

COROLLARY 6.10. *Assume the conditions* (A) *and* (B). *Let* $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $|\Delta y| < r_0$ *and* $y_0 \in \overline{D}$. *Then there exists a unique solution* (x, k) *of the Skorokhod problem associated with* y. *Moreover*, *if* $\{(x^n, k^n)\}$ *is a sequence of solutions to the Skorokhod problem corresponding to the sequence of discretizations* $\{y^{\varrho^n}\}$ *then*

(i) $(x^n, k^n, y^{\varrho^n}) \to (x, k, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$,

(ii) $\sup_{t \le q} |x_t^n - x_t^{\varrho^n}| \to 0$ *and* $\sup_{t \le q} |k_t^n - k_t^{\varrho^n}| \to 0$, $q \in \mathbb{R}^+$.

Proof. The uniqueness was proved in Lemma 6.5. The existence part and property (i) follow immediately from Theorem 6.9 and (6.15). To see (ii) first observe that by (i) and, for example, by Lemma A.1,

$$(x^n, x^{\varrho^n}) \to (x, x) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Hence $x^n - x^{\varrho^n} \to 0$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, which leads to (ii). ∎

COROLLARY 6.11. *Assume the conditions* (A) *and* (B). *Let* $\{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and let* $\{(x^n, k^n)\}$ *be a sequence of solutions to the Skorokhod problem corresponding to the sequence* $\{y^n\}$. *If* $\sup_{t \le q} |y_t^n - y_t| \to 0, q \in \mathbb{R}^+$, *where* $|\Delta y| < r_0$ *then* $\sup_{t \le q} |x_t^n - x_t| \to 0$ *and* $\sup_{t \le q} |k_t^n - k_t| \to 0, q \in \mathbb{R}^+$, *where* (x, k) *is a solution of the Skorokhod problem associated with* y.

Proof. Since the uniform convergence on compact subsets of $\mathbb{R}^+$ is stronger than the convergence in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, it follows by Theorem 6.9 that in particular

$$(x^n, y^n) \to (x, y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

On the other hand, if $|\Delta x_t| > 0$, then by the nature of the Skorokhod problem $|\Delta y_t| > 0$. Using Lemma A.3 we complete the proof. ∎

Now, let μ be a probability measure on $\mathbb{R}^+$ which is equivalent to the Lebesgue measure. For $x, y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ write

$$d_\mu(x, y) = \inf\{\varepsilon > 0 : \mu\{t \in \mathbb{R}^+ : |x_t - y_t| \ge \varepsilon\} \le \varepsilon\}.$$

So d_μ metrizes the topology of convergence in measure, which is weaker than the usual topology J_1 on $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. For $y^n, y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ we will write $y^n \to_\mu y$ if and only if $d_\mu(y^n, y) \to 0$. This kind of convergence turns out to be very useful in many problems connected with convergence of semimartingales and SDE's (see e.g. [Ald89, MeZh84, Mey89, Str85]). There arises the problem whether it is possible to obtain some stability results for the sequence of solutions to the Skorokhod problems with the use of the notion of the convergence in measure. The answer is negative. It is not true even in the classical case $d = 1, D = \mathbb{R}^+$, EXAMPLE 6.12. Define

$$y_t^n = \begin{cases} -nt & \text{if } t \in [0, n^{-1}), \\ nt - 2 & \text{if } t \in [n^{-1}, 2n^{-1}), \\ 0 & \text{if } t \in [2n^{-1}, +\infty). \end{cases}$$

Then it is well known that

$$k_t^n = \sup_{s \le t} \max(0, -y_s^n) = \begin{cases} nt & \text{if } t \in [0, n^{-1}), \\ 1 & \text{otherwise.} \end{cases}$$

In this case $y^n \to_\mu y = 0$ but $k^n \to_\mu 1$ and $x^n \to_\mu 1$. On the other hand, it is obvious that the solution of the Skorokhod problem associated with $y = 0$ is of the form $(0, 0)$. ■

7. Skorokhod problem, nondeterministic case

7.1. Main estimates. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the usual conditions.

DEFINITION 7.1. Let Y be an $(\mathcal{F}_t)$ adapted process and $Y_0 \in \overline{D}$. We say that a pair (X, K) of $(\mathcal{F}_t)$ adapted processes solves the Skorokhod problem associated with Y if and only if for every $\omega \in \Omega, (X(\omega), K(\omega))$ is a solution of the Skorokhod problem corresponding to $Y(\omega)$.

Let us note that as a consequence of Corollary 6.10 for every process Y such that $Y_0 \in \overline{D}$, $|\Delta Y| < r_0$, there exists a unique solution of the Skorokhod problem associated with Y.

In this section we will give some estimates for the solution (X, K) assuming Y is of the form

$$Y_t = H_t + Z_t = H_t + M_t + V_t, \quad t \in \mathbb{R}^+, \tag{7.1}$$

where H is an $(\mathcal{F}_t)$ adapted process, Z is an $(\mathcal{F}_t)$ adapted semimartingale, $Z_0 = 0$, decomposed into the sum of a local martingale M and of a process with bounded variation $V, M_0 = V_0 = 0$.

THEOREM 7.2. *Assume the conditions* (A) *and* (B). *Let* $Y_0 \in \overline{D}$, $|\Delta Y| \le c < r_0$ *and let* Y *be as in* (7.1), *where* $\sup_t |Z_t|$, $\sup_t |H_t|$ *are bounded by some constant* a, *and* M, V *are a square integrable martingale and a process with square integrable variation, respectively* (*i.e.* $E[M]_\infty$, $E\operatorname{Var}(V)^2_\infty < +\infty$). *Then for every* $(\mathcal{F}_t)$ *stopping time* σ *there exists a sequence* $\{\sigma_j\}$ *of* $(\mathcal{F}_t)$ *stopping times and a sequence* $\{C_j\}$ *of constants such that*

$$\operatorname{Var}(K)_{\sigma_j \wedge \sigma} \le C_j \sup_{t \le \sigma_j \wedge \sigma} |Y_t| \quad \textit{on the set } [\sigma < +\infty],$$

where $\mathcal{P}[\sigma_j \le \sigma, \sigma < +\infty] \to 0$ *and for every* $j \in \mathbb{N}$ *the constant* C_j *depends only on* $\sigma, a, c, E[M]_\infty, E\operatorname{Var}(V)^2_\infty$ *and on the modulus of continuity* ω'_H.

Proof. Without loss of generality we assume $\mathcal{P}[\sigma < +\infty] = 1$. Define

$$\gamma_0^i = 0, \quad \gamma_{k+1}^i = \min(\gamma_k^i + b_k^i, \inf\{t > \gamma_k^i : |\Delta H_t| > b^i\}), \tag{7.2}$$

where $\{b^i\}, \{\{b^i_k\}\}$ are two families of constants such that $b^i \downarrow 0$, $b^i/2 \le b^i_k \le b^i$ and $\mathcal{P}[|\Delta H_t| = b^i, t \in \mathbb{R}^+] = 0$, $\mathcal{P}[\Delta H_{\gamma^i_k + b^i_k} = 0] = 1$. With this notation we introduce a sequence of discretizations of the process H. We set

$$H^i_t = H_{\gamma^i_k} \quad \text{if } t \in [\gamma^i_k, \gamma^i_{k+1}),\ k \in \mathbb{N} \cup \{0\}, \\ = \sum_{\gamma^i_k \le t} (H_{\gamma^i_k} - H_{\gamma^i_{k+1}}), \qquad t \in \mathbb{R}^+. \tag{7.3}$$

Let q_j be such that $\mathcal{P}[q_j \le \sigma] \le 1/j$ and $\mathcal{P}[\Delta H_{q_j \wedge \sigma} = 0] = 1$. By simple calculations (see e.g. [Sło89]), $|\Delta(H - H^i)| \le b^i$ and

$$\sup_{t \le q_j \wedge \sigma} |(H - H^i)_t| \le b^i + 2\omega'_H(b^i, q_j \wedge \sigma).$$

Obviously if $i \uparrow +\infty$ then $\sup_{t \le q_j \wedge \sigma} |(H - H^i)_t| \to 0$ $\mathcal{P}$-a.e. Analogously to Proposition 1, we define $\tau_0 = \inf\{t : X_t \in \partial D\}$, $\tau^\delta_m = \inf\{t > \tau_{m-1} : |X_t - X_{\tau_{m-1}}| \ge \delta\}$, $\tau_m = \inf\{t \ge \tau^\delta_m : X_t \in \partial D\}$. By Corollary 6.7, there exists a constant C depending on a such that

$$\delta \le |X_{\tau^\delta_m} - X_{\tau_{m-1}}| \le C\omega_Y[\tau_{m-1}, \tau^\delta_m], \qquad \text{provided that } \tau^\delta_m \le \sigma. \tag{7.4}$$

For $\varepsilon = \delta/C$ set $\xi_i = \inf\{t : |(H - H^i)_t| \ge \varepsilon/8 \text{ or } |(H - H^i)_{t-}| \ge \varepsilon/8\}$, $i \in \mathbb{N}$. It is easy to see that

$$\sup_{t \le \xi_i \wedge q_j \wedge \sigma} |(H - H^i)_t| \le \varepsilon/8 + b^i. \tag{7.5}$$

Therefore we can choose $i = i(j)$ so large that $b^{i(j)} \le \varepsilon/8$ and

$$\mathcal{P}[\xi_{i(j)} \le q_j \wedge \sigma] = \mathcal{P}[\sup_{t \le q_j \wedge \sigma} |(H - H^i)_t| \ge \varepsilon/8] \le 1/j.$$

Also we can choose h_j so small that $\mathcal{P}[\omega'_H(h_j, q_j) \ge b^{i(j)}] \le 1/j$. Now, observe that if $\omega'_H(h_j, q_j) < b^{i(j)}$ then every interval of length less than or equal to h_j contains at most one jump of H greater than $b^{i(j)}$. Hence on the set $\omega'_H(h_j, q_j) < b^{i(j)}$, $\gamma^{i(j)}_k > q_j$ for $k \ge k(j)$, where

$$k(j) = \left[\frac{q_j}{h_j \wedge 2^{-1} b^{i(j)}}\right] + 1,$$

and γ^i_k is defined by (7.2). On the other hand,

$$\mathcal{P}[\gamma^{i(j)}_{k(j)} \le \sigma] \le \mathcal{P}[q_j \le \sigma] + \mathcal{P}[\omega'_H(h_j, q_j) \ge b^{i(j)}] \le 2/j$$

and $\mathcal{P}[q_j \wedge \xi_{i(j)} \wedge \gamma^{i(j)}_{k(j)} \le \sigma] \le 3/j$.

Put $\alpha_j = q_j \wedge \xi_{i(j)} \wedge \gamma^{i(j)}_{k(j)}$. By (7.4) and (7.5),

$$0 < \varepsilon/2 \le \omega_M[\tau_{m-1}, \tau^\delta_m] + \omega_{H^{i(j)}+V}[\tau_{m-1}, \tau^\delta_m] \quad \text{if } \tau^\delta_m \le \alpha_j \wedge \sigma.$$

In the sequel we consider only processes stopped at $\alpha_j \wedge \sigma$. For simplicity we write M, $H^{i(j)}$, V for the stopped processes $M^{\alpha_j \wedge \sigma}$, $H^{i(j), \alpha_j \wedge \sigma}$, $V^{\alpha_j \wedge \sigma}$. We have for every $m \in \mathbb{N}$,

$$\begin{aligned} \mathbf{1}_{\{\tau^\delta_m \le \alpha_j \wedge \sigma\}} \varepsilon^2/2 &\le \omega^2_M[\tau_{m-1}, \tau^\delta_m] + \omega^2_{H^{i(j)}+V}[\tau_{m-1}, \tau^\delta_m] \\ &\le 2 \sup_{\tau_{m-1} \le s \le \tau^\delta_m} |M_s - M_{\tau_{m-1}}|^2 + 2(\mathrm{Var}(H^{i(j)})_{\tau^\delta_m} - \mathrm{Var}(H^{i(j)})_{\tau_{m-1}})^2 \\ &\quad + 2(\mathrm{Var}(V)_{\tau^\delta_m} - \mathrm{Var}(V)_{\tau_{m-1}})^2. \end{aligned}$$

Define $L^m_u = M_{\tau_{m-1}+u} - M_{\tau_{m-1}}$, $m \in \mathbb{N}$. Then L^m is an $(\mathcal{F}_{\tau_{m-1}+u})$ adapted square integrable martingale and $\eta_m = \tau^\delta_m - \tau_{m-1}$ is an $(\mathcal{F}_{\tau_{m-1}+u})$ stopping time. Hence, summing up to n and integrating the above inequalities we get

$$\begin{aligned}\mathcal{P}[\tau^\delta_n \le \alpha_j \wedge \sigma] n\varepsilon^2/2 &\le \sum_{m=1}^n \{2E \sup_{u \le \eta_m} |L^m_u|^2 + 2(\mathrm{Var}(H^{i(j)})_{\tau^\delta_m} - \mathrm{Var}(H^{i(j)})_{\tau_{m-1}})^2 \\ &\quad + 2(\mathrm{Var}(V)_{\tau^\delta_m} - \mathrm{Var}(V)_{\tau_{m-1}})^2\} \\ &\le \sum_{m=1}^n 8E|L^m_{\eta_m}|^2 + 2E\,\mathrm{Var}(H^{i(j)})^2_\infty + 2E\,\mathrm{Var}(V)^2_\infty \\ &\le \sum_{m=1}^n 8E([M]_{\tau^\delta_m} - [M]_{\tau_{m-1}}) + 2(2ak(j))^2 + 2E\,\mathrm{Var}(V)^2_\infty \\ &\le 8E[M]_\infty + 8a^2k^2(j) + 2E\,\mathrm{Var}(V)^2_\infty.\end{aligned}$$

Since the right hand side of the last inequality is finite, for sufficiently large $n = n(j)$,

$$\mathcal{P}[\tau^\delta_{n(j)} \le \alpha_j \wedge \sigma] \le 1/j \quad \text{and} \quad \mathcal{P}[\tau^\delta_{n(j)} \wedge \alpha_j \le \sigma] \le 4/j.$$

Finally, if we define $\sigma_j = \tau^\delta_{n(j)} \wedge \alpha_j$, by Proposition 6.6 we conclude that for a constant C in (7.4), we have

$$\mathrm{Var}(K)_{\sigma_j \wedge \sigma} \le (n(j)+1)2C \sup_{t \le \sigma_j \wedge \sigma} |Y_t| = C_j \sup_{t \le \sigma_j \wedge \sigma} |Y_t|,$$

which is our claim. ∎

Assume now that there is given another $(\mathcal{F}_t)$ adapted process $\widehat{Y}$ such that $\widehat{Y}_0 \in \overline{D}$, $|\Delta \widehat{Y}| < r_0$ and $\widehat{Y}$ admits the decomposition

$$\widehat{Y}_t = H_t + \widehat{Z}_t = H_t + \widehat{M}_t + \widehat{V}_t, \qquad t \in \mathbb{R}^+, \tag{7.6}$$

where $\widehat{Z}$ is an $(\mathcal{F}_t)$ adapted semimartingale, $\widehat{Z}_0 = 0$ and $\widehat{M}$ is an $(\mathcal{F}_t)$ adapted local martingale, $\widehat{M}_0 = 0$ and $\widehat{V}$ is an $(\mathcal{F}_t)$ adapted process with bounded variation, $\widehat{V}_0 = 0$. Assume that $(\widehat{X}, \widehat{K})$ is a solution of the Skorokhod problem corresponding to $\widehat{Y}$. In Theorem 7.3 below we estimate $E \sup_s |X_s - \widehat{X}_s|^2$, which is the crucial step in proofs of uniqueness in Section 8.2.

THEOREM 7.3. *Assume the conditions* (A) *and* (B). *Let* $Y_0, \widehat{Y}_0 \in \overline{D}$, $|\Delta Y|$, $|\Delta \widehat{Y}| \le r_0/4$ *and let processes* Y, $\widehat{Y}$ *satisfy* (7.1) *and* (7.6), *respectively, where* M, $\widehat{M}$ *are square integrable martingales and* V, $\widehat{V}$ *are processes with square integrable variation. If* $r_0 < +\infty$ *we assume additionally that there exists a constant* a *such that* $\mathrm{Var}(K)_\infty$, $\mathrm{Var}(\widehat{K})_\infty \le a$. *Then there exists a constant* C *depending on* a (*and also on* r_0, β, δ) *such that for every* $(\mathcal{F}_t)$ *stopping time* σ,

$$E \sup_{t \le \sigma} |X_t - \widehat{X}_t|^2 \le CE\{[M - \widehat{M}]_\sigma + \mathrm{Var}(V - \widehat{V})^2_\sigma\}.$$

P r o o f. For simplicity we only consider the processes stopped at σ. First we discuss the more complicated case $r_0 < +\infty$. By (6.1) for every $t \in \mathbb{R}^+$,

$$|X_t - \widehat{X}_t|^2 \le 2\{|Z_t - \widehat{Z}_t|^2 + \frac{1}{r_0}\int_0^{t-} |X_s - \widehat{X}_s|^2\, d(\mathrm{Var}(K)_s + \mathrm{Var}(\widehat{K})_s) + 2\int_0^t (Z_t - Z_s - \widehat{Z}_t + \widehat{Z}_s)\, d(K_s - \widehat{K}_s)\}.$$

Since $K_0 = \widehat{K}_0 = 0$, by the integration by parts formula we obtain,

$$\begin{aligned}2\int_0^t (Z_t - Z_s - \widehat{Z}_t + \widehat{Z}_s)\, d(K_s - \widehat{K}_s) &= 2\int_0^t (K_{s-} - \widehat{K}_{s-})\, d(Z_s - \widehat{Z}_s)\\ &= 2\int_0^t (X_{s-} - \widehat{X}_{s-})\, d(Z_s - \widehat{Z}_s) - 2\int_0^t (Z_{s-} - \widehat{Z}_{s-})\, d(Z_s - \widehat{Z}_s)\\ &= 2\int_0^t (X_{s-} - \widehat{X}_{s-})\, d(Z_s - \widehat{Z}_s) + [Z - \widehat{Z}]_t - |Z_t - \widehat{Z}_t|^2.\end{aligned}$$

Hence for every stopping time τ,

$$E\sup_{t\le\tau}|X_t - \widehat{X}_t|^2 \le 2\bigg\{\frac{1}{r_0}E\int_0^{\tau-} |X_s - \widehat{X}_s|^2\, d(\mathrm{Var}(K)_s + \mathrm{Var}(\widehat{K})_s) + 2E\sup_{t\le\tau}\int_0^t (X_{s-} - \widehat{X}_{s-})\, d(Z_s - \widehat{Z}_s) + E[Z - \widehat{Z}]_\tau\bigg\}.$$

By simple calculations based on Burkholder–Davies–Gundy's and Schwarz's inequalities

$$\begin{aligned}E\sup_{t\le\tau}\Big|\int_0^t (X_{s-} - \widehat{X}_{s-})\, d(M_s - \widehat{M}_s)\Big| &\le C_0 E\Big(\int_0^\tau |X_{s-} - \widehat{X}_{s-}|^2\, d[M - \widehat{M}]_s\Big)^{1/2}\\ &\le C_0(E\sup_{s\le\tau}|X_{s-} - \widehat{X}_{s-}|)^{1/2}(E[M - \widehat{M}]_\tau)^{1/2},\end{aligned}$$

and

$$\begin{aligned}E\sup_{t\le\tau}\Big|\int_0^t (X_{s-} - \widehat{X}_{s-})\, d(V_s - \widehat{V}_s)\Big| &\le E\sup_{s\le\tau}|X_{s-} - \widehat{X}_{s-}|\,\mathrm{Var}(V - \widehat{V})_\tau\\ &\le (E\sup_{s\le\tau}|X_{s-} - \widehat{X}_{s-}|^2)^{1/2}(E\,\mathrm{Var}(V - \widehat{V})_\tau^2)^{1/2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}E[Z - \widehat{Z}]_\tau &\le 2E[M - \widehat{M}]_\tau + 2E[V - \widehat{V}]_\tau\\ &\le 2E[M - \widehat{M}]_\tau + 2E\,\mathrm{Var}(V - \widehat{V})_\tau.\end{aligned}$$

If we set

$$x = (E\sup_{t\le\tau}|X_t - \widehat{X}|^2)^{1/2},$$

$$b_1 = (E[M - \widehat{M}]_\tau + E\,\mathrm{Var}(V - \widehat{V})^2_\tau)^{1/2},$$

$$b_2 = \frac{1}{r_0} E \int_0^{\tau-} |X_s - \widehat{X}_s|^2\, d(\mathrm{Var}(K)_s + \mathrm{Var}(\widehat{K})_s),$$

then by the above calculations we deduce that there exists a constant C_1 such that

$$x^2 \le C_1(b_2 + 2b_1 x + b_1^2). \tag{7.7}$$

Since $x, b_1 > 0$ it is clear that $x^2 \le C_2(b_2 + b_1^2)$ for some constant C_2. If we set in Lemma C.1(i), $Y_t^1 = \sup_{s\le t} |X_s - \widehat{X}_s|^2, Y_t^2 = C_2(\mathrm{Var}(K)_t + \mathrm{Var}(\widehat{K})_t) < C_2 2a + 1$ and $a_2 = C_2 E\{[M - \widehat{M}]_\sigma + \mathrm{Var}(V - \widehat{V})^2_\sigma\}$ then the proof in the case $r_0 < +\infty$ is complete.

Finally, let us note that in the case $r_0 = +\infty$ instead of (7.7) we obtain a simpler inequality, namely $x^2 \le \widehat{C}_1(2b_1 x + b_1^2)$. Hence there exists a constant $\widehat{C}_2$ such that $x^2 \le \widehat{C}_2 b_1^2$ and we get the desired result putting $\tau = \sigma$. ∎

Since $X = Y + K$ it is easy to obtain from Theorem 7.3 the following result.

Corollary 7.4. *Under the assumptions of Theorem* 7.3, *there exists a constant* C *such that for every stopping time* σ,

$$E \sup_{t\le\sigma} |K_t - \widehat{K}_t|^2 \le CE\{[M - \widehat{M}]_\sigma + \mathrm{Var}(V - \widehat{V})^2_\sigma\}. \ ∎$$

The estimates proved in Theorem 7.3 and Corollary 7.4 are very similar to those obtained previously by Chaleyat-Maurel, El Karoui and Marchal [ChKM80, Proposition 8] in the case $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. Moreover, using their method based on [MéPe77, Theorem 1] we can deduce from Corollary 7.4 estimates on the interval $[0, \sigma[$ in place of $[0, \sigma]$.

Corollary 7.5. *Under the assumptions of Theorem* 7.3, *there exists a constant* C *such that for every stopping time* σ,

$$E \sup_{t<\sigma} |K_t - \widehat{K}_t|^2 \le CE\{[M - \widehat{M}]_{\sigma-} + \langle M - \widehat{M}\rangle_{\sigma-} + \mathrm{Var}(V - \widehat{V})^2_{\sigma-}\}. \ ∎$$

7.2. Tightness and convergence of solutions to Skorokhod problems. Now, we will discuss applications of Theorems 6.9, 7.2 and 7.3 for sequences of solutions to the Skorokhod problems. Let $\{Y^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes and let $\{(X^n, K^n)\}$ be a sequence of solutions to Skorokhod problems associated with $\{Y^n\}$. We will assume that Y^n is of the form

$$Y^n = H^n + Z^n, \tag{7.8}$$

where $\{Z^n\}$ is a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, $Z_0^n = 0$, satisfying the condition (UT). Let for every $n \in \mathbb{N}$, Z^n be decomposed into the sum of three processes,

$$Z^n = J^n + M^n + B^n, \tag{7.9}$$

where $J_t^n = \sum_{0<s\le t} \Delta Z_s^n \mathbf{1}_{\{|\Delta Z_s^n|>1\}}$, M^n is a locally square integrable martingale, $M_0^n = 0$ and B^n is a predictable process with bounded variation, $B_0^n = 0$. Then in view of Theorem 1.1, $\{Z^n\}$ satisfies (UT) if and only if for every $q \in \mathbb{R}^+$ the families of random variables $\{\mathrm{Var}(J^n)_q\}$, $\{\mathrm{Var}(B^n)_q\}$, $\{[M^n]_q\}$ are tight in $\mathbb{R}$. The following proposition plays the key role in the proof of existence of solutions to the SDE's.

Proposition 7.6. *Assume the conditions* (A) *and* (B). *Let* $\{Y^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted processes*, $|\Delta Y^n| \le c < r_0$, *and let* $\{(X^n, K^n)\}$ *be a sequence of solutions to the Skorokhod problem associated with* $\{Y^n\}$. *Assume that every* Y^n *is of the form* (7.8), *where* $\{H^n\}$ *is a tight* (*in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$) *sequence of* $(\mathcal{F}_t^n)$ *adapted processes and* $\{Z^n\}$ *is a sequence of* $(\mathcal{F}_t^n)$ *adapted semimartingales satisfying* (UT). *Then*

(i) *for every* $q \in \mathbb{R}^+$ *the family of random variables* $\{\mathrm{Var}(K^n)_q\}$ *is tight in* $\mathbb{R}$,

(ii) *if* $H^n = Y_0^n$ *then* $\{X^n\}$ *satisfies* (UT).

Proof. (i) Define $\tau^{n,a} = \inf\{t : |H_t^n| \vee |Z_t^n| \ge a\}$, $a \in \mathbb{R}^+$, $n \in \mathbb{N}$. Since by Theorem B.1, $\{\sup_{t\le q} |Z_t^n|\}$, $q \in \mathbb{R}^+$, is tight in $\mathbb{R}$, and by the tightness of $\{H^n\}$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $\{\sup_{t\le q} |H_t^n|\}$, $q \in \mathbb{R}^+$, is tight in $\mathbb{R}$, it is clear that

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathcal{P}(\tau^{n,a} \le q) = 0, \qquad q \in \mathbb{R}^+. \tag{7.10}$$

Further, by easy calculations

$$\omega'_{H^{n,\tau^{n,a}-}}(h, q) \le \omega'_{H^n}(h, q) \quad \text{and} \quad \sup_{t\le q} |H_t^{n,\tau^{n,a}-}| \le \sup_{t\le q} |H_t^n|, \quad q \in \mathbb{R}^+,$$

and hence the sequence $\{H^{n,\tau^{n,a}-}\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, too. On the other hand, by the definition of (UT) also the sequence $\{Z^{n,\tau^{n,a}}\}$ satisfies (UT). But

$$Z_t^{n,\tau^{n,a}-} = Z_t^{n,\tau^{n,a}} - \Delta Z_{\tau^{n,a}}^n \mathbf{1}_{\{t\ge\tau^{n,a}\}} = Z_t^{n,\tau^{n,a}} + V_t^n,$$

where $\mathrm{Var}(V^n)_q \le 2\sup_{t\le q} |Z_t^n|$, $q \in \mathbb{R}^+$. Thus using once more Theorem B.1 and the definition of (UT), it is clear that $\{Z^{n,\tau^{n,a}-}\}$ satisfies (UT), too. Therefore by (7.10), without loss of generality, we can assume $Z^n = Z^{n,\tau^{n,a}-}$, $H^n = H^{n,\tau^{n,a}-}$ and $\sup_t |Z_t^n| \le a$, $\sup_t |H_t^n| \le a$ for some constant $a \in \mathbb{R}^+$. Analogously to (7.9) we decompose Z^n as a sum of J^n, M^n, and B^n. We define $\gamma^{n,b} = \inf\{t : \mathrm{Var}(J^n)_t \vee [M^n]_t \vee \mathrm{Var}(B^n)_t \ge b\}$. By Theorem 1.1(ii),

$$\lim_{b\to\infty} \limsup_{n\to\infty} \mathcal{P}(\gamma^{n,b} \le q) = 0, \qquad q \in \mathbb{R}^+,$$

and as before we can assume $H^n = H^{n,\gamma^{n,b}}, J^n = J^{n,\gamma^{n,b}}, M^n = M^{n,\gamma^{n,b}}, B^n = B^{n,\gamma^{n,b}}$ for some constant b. Now we can write $Z^n = M^n + V^n, V^n = J^n + B^n$, where $[M^n]_\infty \le b + 4, \mathrm{Var}(V^n)_\infty^2 \le 2\,\mathrm{Var}(J^n)_\infty^2 + 2\,\mathrm{Var}(B^n)_\infty^2 \le 2(b^2 + 4a^2) + 2(b^2 + 1)$. It is enough to use Theorem 7.2. Indeed, setting $\sigma = q$ we have

$$\mathrm{Var}(K^n)_{\sigma_j^n\wedge q} \le C_j^n \sup_{t\le\sigma_j^n\wedge q} |Y_t^n| \le C_j^n a, \qquad j, n \in \mathbb{N}. \tag{7.11}$$

By the definition of σ_j^n, C_j^n, we have

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathcal{P}(\sigma_j^n \le q) = 0 \quad \text{and} \quad \limsup_{n\to\infty} C_j^n < +\infty$$

for every $j \in \mathbb{N}$, which together with (7.11) gives (i).

(ii) The sequence $\{K^n\}$ of processes satisfies (UT) by (i). Since $\{Y_0^n\}$ is bounded in probability and $\{Z^n\}$ satisfies (UT), the sequence $\{X^n = Y_0^n + Z^n + K^n\}$ satisfies (UT), too. ∎

Assume for the moment that a sequence $\{Y^n\}$ of processes is of the form considered in (ii), i.e. $\{Y^n\}$ satisfies (UT). From the theory of convergence in Meyer and Zheng's sense ([MeZh84, Str85]) we conclude by (ii) that there exist processes X, K, Y, a subsequence $\{n'\} \subset \{n\}$ and a set A of full Lebesgue measure such that the finite-dimensional distributions of $(X_t^{n'}, K_t^{n'}, Y_t^{n'})_{t\in A}$ converge to those of $(X_t, K_t, Y_t)_{t\in A}$. Since the convergence in Meyer and Zheng's sense is given by the topology of convergence in measure, Example 6.12 in Section 6.2 shows that (X, K) need not be a solution associated with Y. Therefore the notion of convergence in Meyer and Zheng's sense is not appropriate in stability theorems for solutions of the Skorokhod problem. This kind of convergence seems to be too weak.

Now let us assume additionally that all the semimartingales $Y^n = Y_0^n + Z^n$ are adapted to the same filtration $(\mathcal{F}_t)$. If Y^n tends to Y in the space of semimartingales $\mathcal{H}^2$ (see e.g. [DeMe80]) then by Theorem 7.3, $E\sup_t |X_t^n - X_t|^2 \to 0$ and $E\sup_t |K_t^n - K_t|^2 \to 0$, where (X, K) is a solution corresponding to Y, i.e. the convergence in the space $\mathcal{S}^2$ holds. The following example from [Sha88] shows that this kind of convergence cannot be strenghtened to convergence in $\mathcal{H}^2$.

Example 7.7. Let $d = 1$, $D = \mathbb{R}^+$ and let M be a continuous square integrable martingale. Then from the definition of local time we have

$$M_t^+ = \int_0^t \mathbf{1}_{\{M_s>0\}}\, dM_s + \tfrac{1}{2}L_t^0(M),$$

$$(M_t - a_n)^+ = \int_0^t \mathbf{1}_{\{M_s>a_n\}}\, dM_s + \tfrac{1}{2}L_t^{a_n}(M),$$

where $a_n > 0, a_n \downarrow 0$. It is clear that $((M_\cdot - a_n)^+, \frac{1}{2}L_\cdot^{a_n}(M))$ is a solution of the Skorokhod problem associated with $\int_0^\cdot \mathbf{1}_{\{M_s>a_n\}}\, dM_s$, $((M_\cdot^+, \frac{1}{2}L_\cdot^0(M))$ is a solution of the Skorokhod problem associated with $\int_0^\cdot \mathbf{1}_{\{M_s>0\}}\, dM_s$ and

$$\Big\|\int_0^\cdot \mathbf{1}_{\{0<M_s\le a_n\}}\, dM_s\Big\|_{\mathcal{H}^2} = \Big(E\int_0^\infty \mathbf{1}_{\{0<M_s\le a_n\}}\, d[M]_s\Big)^{1/2} \to 0.$$

On the other hand,

$$|L_\cdot^0(M) - L_\cdot^{a_n}(M)|_q \to 2L_q^0(M) \qquad \mathcal{P}\text{-a.e.}$$

and hence the convergence $\|(M - a_n)^+ - M^+\|_{\mathcal{H}^2} \to 0$ does not hold. ∎

What we can do is to give convergence results in law and in probability in the Skorokhod topology, which is stronger than Meyer and Zheng's topology.

Proposition 7.8. *Assume the conditions* (A) *and* (B). *Let* $\{H^n\}, \{Z^n\}, \{Y^n\}$ *be three sequences of processes,* $Y_0^n \in \overline{D}$ *and let* $\{(X^n, K^n)\}$ *be a sequence of solutions of the Skorokhod problem associated with* $\{Y^n\}$. *If*

$$(H^n, Z^n, Y^n) \underset{\mathcal{D}}{\to} (H, Z, Y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$$

(*resp.* $(H^n, Z^n, Y^n) \to_{\mathcal{P}} (H, Z, Y)$) *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$), *and* $|\Delta Y| < r_0$ *then*

$$(H^n, Z^n, X^n, K^n, Y^n) \underset{\mathcal{D}}{\to} (H, Z, X, K, Y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, R^{5d})$$

(resp. $(H^n, Z^n, X^n, K^n, Y^n) \to_{\mathcal{P}} (H, Z, X, K, Y))$ *in* $\mathbb{D}(\mathbb{R}^+, R^{5d}))$, *where* (X, K) *is a solution of the Skorokhod problem associated with the process* Y*.*

Proof. The statement easily follows from Theorem 6.9 and the Skorokhod representation theorem. ∎

By Corollaries 6.10 and 6.11 we have the next

COROLLARY 7.9. *Assume the conditions* (A) *and* (B). *Let* (X, K) *be a solution of the Skorokhod problem corresponding to a process* $Y, Y_0 \in \overline{D}, |\Delta Y| < r_0$.

(i) *Let* $\{(X^n, K^n)\}$ *be a sequence of solutions to the Skorokhod problem associated with a sequence of processes* $\{Y^n\}$*. If* $\sup_{t\le q} |Y^n_t - Y_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, *then*

$$\sup_{t\le q} |X^n_t - X_t| \underset{\mathcal{P}}{\to} 0 \quad \text{and} \quad \sup_{t\le q} |K^n_t - K_t| \underset{\mathcal{P}}{\to} 0 \quad \text{for every } q \in \mathbb{R}^+.$$

(ii) *Let* $\{T_n\}$ *be a sequence of partitions of* $\mathbb{R}^+$ *satisfying the condition* (0.6) *and for every* $n \in \mathbb{N}$, *let* Y^n *be of the form* Y^{ϱ^n}*. Then*

$$\sup_{t\le q} |X^n_t - X^{\varrho^n}_t| \underset{\mathcal{P}}{\to} 0 \quad \text{and} \quad \sup_{t\le q} |K^n_t - K^{\varrho^n}_t| \underset{\mathcal{P}}{\to} 0 \quad \text{for every } q \in \mathbb{R}^+. \ \blacksquare$$

8. SDE's with reflecting boundary

8.1. Existence and stability of weak solutions. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the usual conditions. Given a function $f : \overline{D} \to \mathbb{R}^d \otimes \mathbb{R}^d$, $f(x) = \{f_{ik}\}_{i,k=1,\ldots,d}$ we consider the following SDE:

$$X^i_t = H^i_t + \sum_{k=1}^d \int_0^t f_{ik}(X_{s-})\, dZ^i_s + K^i_t, \qquad i = 1, \ldots, d,\ t \in \mathbb{R}^+, \tag{8.1}$$

where $H_t = (H^1_t, \ldots, H^d_t)$ is an $(\mathcal{F}_t)$ adapted process and $Z_t = (Z^1_t, \ldots, Z^d_t)$ is an $(\mathcal{F}_t)$ adapted semimartingale, $H_0 \in \overline{D}, Z_0 = 0$. For simplicity of notation we write briefly

$$X_t = H_t + \int_0^t f(X_{s-})\, dZ_s + K_t, \qquad t \in \mathbb{R}^+.$$

We say that the SDE (8.1) has a *strong solution* if there exists a pair (X, K) of $(\mathcal{F}_t)$ adapted processes satisfying the conditions

(8.2) X is $\overline{D}$-valued,

(8.3) K is a process with locally bounded variation such that $K_0 = 0$ and

$$K_t = \int_0^t \mathbf{n}_s \, d\,\mathrm{Var}(K)_s, \qquad \mathrm{Var}(K)_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}}\, d\,\mathrm{Var}(K)_s,$$

$t \in \mathbb{R}^+$, where $\mathbf{n}_s \in \mathcal{N}_{X_s}$ if $X_s \in \partial D$,

(8.4) $|\Delta X_t| \le |\Delta H_t + f(X_{t-})\Delta Z_t|, \quad t \in \mathbb{R}^+.$

If any two $(\mathcal{F}_t)$ adapted solutions (X, K), (X', K') on $(\Omega, \mathcal{F}, \mathcal{P})$ of the SDE (8.1) satisfy $\mathcal{P}[(X_t, K_t) = (X'_t, K'_t), t \in \mathbb{R}^+] = 1$ then we say that *strong uniqueness* holds for (8.1).

We say that the SDE (8.1) has a *weak solution* if there exists a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$ with filtration $(\widehat{\mathcal{F}}_t)$ satisfying the usual conditions and $(\widehat{\mathcal{F}}_t)$ adapted processes $\widehat{X}, \widehat{K}, \widehat{H}, \widehat{Z}$ such that $\mathcal{L}(\widehat{H}, \widehat{Z}) = \mathcal{L}(H, Z)$ and (8.1)–(8.4) hold for the processes $\widehat{X}$, $\widehat{K}$, $\widehat{H}$, $\widehat{Z}$ taken instead of X, K, H, Z. If any two weak solutions $(\widehat{X}, \widehat{K})$ and $(\widehat{X}', \widehat{K}')$ of SDE (8.1), possibly defined on different probability spaces, are such that $\mathcal{L}(\widehat{X}, \widehat{K}) = \mathcal{L}(\widehat{X}', \widehat{K}')$ we say that *weak uniqueness* holds for (8.1).

Assume $\{Z^n\}$ is a sequence of $(\mathcal{F}^n)$ adapted semimartingales and let $\{H^n\}$ be a sequence of $(\mathcal{F}^n)$ adapted processes, $H_0^n \in \overline{D}, Z_0^n = 0$. We consider a sequence $\{X^n\}$ of solutions of SDE's of the form (8.1), i.e.

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + K_t^n, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N}. \tag{8.5}$$

Now, we are ready to formulate our main theorem.

THEOREM 8.1. *Assume the conditions* (A) *and* (B). *Let $\{Z^n\}$ be a sequence of $(\mathcal{F}^n)$ adapted semimartingales satisfying* (UT) *and let $\{H^n\}$ be a sequence of $(\mathcal{F}^n)$ adapted processes. Let $\{(X^n, K^n)\}$ be a sequence of strong solutions to the SDE* (8.5). *If f is continuous, $\|f(x)\| \le L < +\infty$, $x \in \overline{D}$ and*

$$(H^n, Z^n) \underset{\mathcal{D}}{\rightarrow} (H, X) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

where $|\Delta H| + L|\Delta Z| < r_0$, then

(i) *$\{(X^n, K^n, H^n, Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$ and every limit point of $\{(X^n, K^n)\}$ is a weak solution of the SDE* (8.1),

(ii) *if additionally the SDE* (8.1) *has a unique weak solution (X, K), then*

$$(X^n, K^n) \underset{\mathcal{D}}{\rightarrow} (X, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Proof. (i) First let us note that the joint weak convergence of (H^n, Z^n) to (H, Z) and the continuous mapping theorem imply

$$(H^n, Z^n, |H^n_{\cdot}| + L|Z^n_{\cdot}|) \underset{\mathcal{D}}{\rightarrow} (H, Z, |H_{\cdot}| + L|Z_{\cdot}|) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \tag{8.6}$$

Let $\{c_k\}$ be a sequence of constants with $c_k \uparrow r_0$ such that $\mathcal{P}[|\Delta H_t| + L|\Delta Z_t| = c_k, t \in \mathbb{R}^+] = 0$, $k \in \mathbb{N}$. Define

$$\sigma_k = \inf\{t > 0 : |\Delta H_t| + L|\Delta Z_t| \ge c_k\}, \quad \sigma_k^n = \inf\{t > 0 : |\Delta H_t^n| + L|\Delta Z_t^n| \ge c_k\},$$

$k, n \in \mathbb{N}$. By (8.6) and [JaSh87, Proposition 3.15] for every $k \in \mathbb{N}$,

$$(H^{n,\sigma_k^n -}, Z^{n,\sigma_k^n -}, \sigma_k^n) \underset{\mathcal{D}}{\rightarrow} (H^{\sigma_k -}, Z^{\sigma_k -}, \sigma_k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \times \mathbb{R}.$$

Since $|\Delta H| + L|\Delta Z| < r_0$, $\sigma_k \uparrow +\infty$ $\mathcal{P}$-a.e. and as a consequence

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathcal{P}[\sigma_k^n \le q] = 0.$$

Therefore without loss of generality we may assume that there exists a constant $c < r_0$ such that

$$|\Delta H^n| + L|\Delta Z^n| \le c, \qquad n \in \mathbb{N}.$$

Due to Corollary 1.2 the sequence of stochastic integrals $\{\int_0^\cdot f(X^n_{s-})\,dZ^n_s\}$ satisfies (UT), hence by Proposition 7.6 and Corollary 7.9, the sequences of random variables

$$\{\mathrm{Var}(K^n)_q\}, \{\sup_{t\le q}|X^n_t|\} \quad \text{are tight in } \mathbb{R}, q\in\mathbb{R}^+.$$

Now, assume that $\{b^i\}$, $\{b^i_k\}$ are exactly the same as in (7.2). Put

$$\gamma^{ni}_0=0,\qquad \gamma^{ni}_{k+1}=\min(\gamma^{ni}_k+b^i_k,\inf\{t>\gamma^{ni}_k : |\Delta H^n_t|>b^i\}),$$

and

$$\begin{aligned} H^{ni}_t &= H^n_{\gamma^{ni}_k} \qquad \text{if } t\in[\gamma^{ni}_k,\gamma^{ni}_{k+1}[\\ &= \sum_{\gamma^{ni}_k\le t}(H^n_{\gamma^{ni}_{k+1}}-H^n_{\gamma^{ni}_k}),\qquad t\in\mathbb{R}^+. \end{aligned}$$

As observed in the proof of Theorem 2.1, for every $i\in\mathbb{N}$,

$$(H^n_{\gamma^{ni}_0},\gamma^{ni}_0,H^n_{\gamma^{ni}_1},\gamma^{ni}_1,\ldots)\underset{\mathcal{D}}{\to}(H_{\gamma^i_0},\gamma^i_0,H_{\gamma^i_1},\gamma^i_1,\ldots)\quad \text{in } \mathbb{R}^\infty.$$

As a consequence, for every $i\in\mathbb{N}$,

(8.7) $\quad H^{ni}\underset{\mathcal{D}}{\to}H^i \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d),$

(8.8) $\quad \mathrm{Var}(H^{ni})_q\underset{\mathcal{D}}{\to}\mathrm{Var}(H^i)_q \quad$ in $\mathbb{R}$, provided that $\mathcal{P}[|\Delta H^i_q|=0]=1,\ q\in\mathbb{R}^+$.

On the other hand, it is well known that for continuous $f:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$, we can construct a sequence $\{f^i\}$ of functions such that for every $i\in\mathbb{N}, f^i\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ and

$$\sup_{x\in K}\|f^i(x)-f(x)\|\to 0, \tag{8.9}$$

for all compact subsets K of $\mathbb{R}^d$. For $n,i\in\mathbb{N}$, define

$$X^{ni}=H^{ni}+\int_0^\cdot f(X^n_{s-})\,dZ^n_s+K^n_s,$$

$$Y^{ni}=H^n+\int_0^\cdot f^i(X^{ni}_{s-})\,dZ^n_s,$$

$$Y^n=H^n+\int_0^\cdot f(X^n_{s-})\,dZ^n_s.$$

By simple calculations based on (2.12),

$$\lim_{i\to\infty}\limsup_{n\to\infty}\mathcal{P}[\sup_{t\le q}|H^{ni}_t-H^n_t|\ge\varepsilon]=0,\qquad \varepsilon>0,\ q\in\mathbb{R}^+.$$

Therefore due to (8.9) and Theorem 1.1 we have

$$\lim_{i\to\infty}\limsup_{n\to\infty}\mathcal{P}[\sup_{t\le q}|Y^{ni}_t-Y^n_t|\ge\varepsilon]=0,\qquad \varepsilon>0,\ q\in\mathbb{R}^+. \tag{8.10}$$

On the other hand, by (8.8), $\{\mathrm{Var}(H^{ni})_q\}$ is tight in $\mathbb{R}$. Hence $\{X^{ni}\}$ is a sum of three sequences of processes satisfying (UT). As a consequence $\{X^{ni}\}$ satisfies (UT), too. Moreover, since $f^i\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$, by Corollary 1.3 for every $i\in\mathbb{N}$,

$$\{f^i(X^{ni})\}\quad \text{satisfies the condition (UT).} \tag{8.11}$$

Therefore due to Lemma 4.3 for every $i \in \mathbb{N}$,

$$\Big\{\Big(H^n, Z^n, \int_0^{\cdot} f^i(X^{ni}_{s-})\,dZ^n_s\Big)\Big\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

Hence by (8.10) also $\{(H^n, Z^n, Y^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$.

Assume that there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$(H^{n'}, Z^{n'}, Y^{n'}) \underset{\mathcal{D}}{\rightarrow} (\widehat{H}, \widehat{Z}, \widehat{Y}) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where $\mathcal{L}(\widehat{H}, \widehat{Z}) = \mathcal{L}(H, Z)$. Then by Proposition 7.8,

$$(H^{n'}, Z^{n'}, X^{n'}, K^{n'}, Y^{n'}) \underset{\mathcal{D}}{\rightarrow} (\widehat{H}, \widehat{Z}, \widehat{X}, \widehat{K}, \widehat{Y}) \quad \text{in } \mathbb{D}(\mathbb{R}^+, R^{5d}),$$

where $(\widehat{X}, \widehat{K})$ is the solution of the Skorokhod problem corresponding to $\widehat{Y}$. Moreover, by using Theorem B.2,

$$\Big(H^{n'}, X^{n'}, K^{n'}, \int_0^{\cdot} f(X^{n'}_{s-})\,dZ^n_s\Big) \underset{\mathcal{D}}{\rightarrow} \Big(\widehat{H}, \widehat{X}, \widehat{K}, \int_0^{\cdot} f(\widehat{X}_{s-})\,d\widehat{Z}_s\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).$$

Finally, by the continuous mapping theorem, we conclude that $(\widehat{X}, \widehat{K})$ is a solution of the SDE (8.1).

(ii) follows immediately by (i). ∎

Corollary 8.2. *Assume the conditions* (A)–(B). *Let f be continuous and bounded*, $\|f(x)\| \le L < +\infty, x \in \overline{D}$. *If $|\Delta H| + L|\Delta Z| < r_0$ and $H_0 \in \overline{D}$ then there exists a weak solution of the SDE* (8.1).

Proof. Let $\{T_n\}$ be a sequence of partitions of $\mathbb{R}^+$ satisfying the condition (0.6). By simple calculations we can prove that for sufficiently large n there exists a unique strong solution of the following discrete stochastic differential equation:

$$X^n_t = H^{\varrho^n}_t + \int_0^t f(X^n_{s-})\,dZ^{\varrho^n}_s + K^n_t. \tag{8.12}$$

Moreover, the special form of H^{ϱ^n} and Z^{ϱ^n} implies that

$$X^n_t = \begin{cases} H_0 & \text{if } t \in [0, t_{n1}[, \\ [X^n_{t_{n,k-1}} + H_{t_{nk}} - H_{t_{n,k-1}} & \\ \quad + f(X^n_{t_{n,k-1}})(Z_{t_{nk}} - Z_{t_{n,k-1}})]_\partial & \text{if } t \in [t_{nk}, t_{n,k+1}[,\ k \in \mathbb{N}, \end{cases} \tag{8.13}$$

and

$$K^n_t = \begin{cases} 0 & \text{if } t \in [0, t_{n1}[, \\ K^n_{t_{n,k-1}} + X^n_t - X^n_{t_{n,k-1}} & \\ -\{H_{t_{nk}} - H_{t_{n,k-1}} + & \\ \quad + f(X^n_{t_{n,k-1}})(Z_{t_{nk}} - Z_{t_{n,k-1}})\} & \text{if } t \in [t_{nk}, t_{n,k+1}[,\ k \in \mathbb{N}. \end{cases} \tag{8.14}$$

By Lemma A.1,

$$(H^{\varrho^n}, Z^{\varrho^n}) \to (H, Z) \quad \mathcal{P}\text{-a.e.} \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Since according to Bichteler, Dellacherie and Mokobodzki's theorem the sequence $\{Z^{\varrho^n}\}$ satisfies the condition (UT), the conclusion follows easily by Theorem 8.1(i). ∎

8.2. Existence, uniqueness and stability of strong solutions

THEOREM 8.3. *Assume the conditions* (A) *and* (B). *Let f be Lipschitz continuous and bounded, i.e. there exists a constant $L > 0$ such that $\|f(x)-f(y)\| \le L|x-y|$, $\|f(x)\| \le L$ for all $x, y \in \overline{D}$. If $|\Delta H|+L|\Delta Z| < r_0$, $H_0 \in \overline{D}$, then there exists a unique strong solution of the SDE* (8.1).

Proof. By Corollary 8.2 there exists a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$, a filtration $(\widehat{\mathcal{F}}_t)$ and $(\widehat{\mathcal{F}}_t)$ adapted processes $\widehat{H}, \widehat{Z}, \widehat{X}$ defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$ such that $\mathcal{L}(\widehat{H}, \widehat{Z}) = \mathcal{L}(H, Z)$ and (8.1)–(8.4) hold. Assuming the Lipschitz continuity of f, we will show a little more, namely that there exists a measurable map $F : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \to \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ (depending only on f) such that

$$(\widehat{X}, \widehat{K}) = F(\widehat{H}, \widehat{Z}), \tag{8.15}$$

which shows the pathwise uniqueness for (8.1) on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$. Moreover, if we return to our basic probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the filtration $(\mathcal{F}_t)$ and with $(\mathcal{F}_t)$ adapted processes H, Z, then defining $(X, K) = F(H, Z)$ we obtain a strong solution on the space $(\Omega, \mathcal{F}, \mathcal{P})$. This also means that the solution of (8.1) on $(\Omega, \mathcal{F}, \mathcal{P})$ is unique in the strong sense. Without loss of generality we may assume that the solution (X, K) of (8.1) is defined on the basic probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and satisfies (8.1)–(8.4). Let $\{(X^n, K^n)\}$ be a sequence of solutions to the discrete SDE (8.12) defined by (8.13) and (8.14). We will show that

$$(X^n, K^n) \underset{\mathcal{P}}{\to} (X, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}), \tag{8.16}$$

which leads to (8.15) for (X, K) and (H, Z) in place of $(\widehat{X}, \widehat{K})$ and $(\widehat{H}, \widehat{Z})$.

The proof of (8.16) is long, so it is divided into two steps. But first let us define $Y = \int_0^{\cdot} f(X_{s-})\, dZ_s$ and let $\{(\overline{X}^n, \overline{K}^n)\}$ be a sequence of solutions to the Skorokhod problem associated with the sequence $\{Y^{\varrho^n}\}$ (we recall that for sufficiently large n, $(\overline{X}^n, \overline{K}^n)$ is defined by (6.16) and (6.17)). By Corollary 7.9(ii) for every $q \in \mathbb{R}^+$,

$$\sup_{t \le q} |\overline{X}^n_t - X^{\varrho^n}_t| \to 0 \quad \mathcal{P}\text{-a.e.}, \tag{8.17}$$

and

$$\sup_{t \le q} |\overline{K}^n_t - K^{\varrho^n}_t| \to 0 \quad \mathcal{P}\text{-a.e.} \tag{8.18}$$

Step 1. *We assume additionally that* $\sup_t |H_t|$, $\sup_t |Z_t|$ *are bounded by the constant* $\frac{r_0}{8(L+1)} \wedge 1$. By our assumption Z is a special semimartingale. Hence Z is uniquely decomposed into the sum

$$Z = M + V,$$

where M is a locally square integrable martingale, $M_0 = 0$, and V is a process with bounded variation, $V_0 = 0$. Define $\tau^a = \inf\{t > 0 : |M_t| \vee \mathrm{Var}(V)_t \ge a\}$. Obviously $\tau^a \uparrow +\infty$ and we can assume that the processes H, M, V are stopped at τ^a for some fixed $a \in \mathbb{R}^+$.

For n, $i \in \mathbb{N}$ let us define

$$\tau_n^i = \inf\Big\{t > 0 : \Big|\int_0^t f(X_{s-}^n)\,dZ_s^{\varrho^n}\Big| \vee \Big|\int_0^{\varrho_n(t)} f(X_{s-})\,dZ_s\Big| \vee [M^{\varrho^n}]_t \vee \langle M^{\varrho^n}\rangle_t \vee \mathrm{Var}(V^{\varrho^n})_t \geq i\Big\}.$$

Then it is clear that for every $q \in \mathbb{R}^+$,

$$\lim_{i\to\infty}\limsup_{n\to\infty}\mathcal{P}[\tau_n^i \leq q] = 0, \tag{8.19}$$

and the processes $\int_0^\cdot f(X_{s-}^n)\,dZ_s^{\varrho^n}$, $\int_0^{\varrho_n(\cdot)} f(X_{s-})\,dZ_s$, $[M^{\varrho^n}]$, $\langle M^{\varrho^n}\rangle$, $\mathrm{Var}(V^{\varrho^n})$. stopped at τ_n^i are uniformly bounded in n by some constant K_1. In view of (8.19) we can restrict our attention to the processes stopped at τ_n^i. By Theorem 7.2 for fixed $q \in \mathbb{R}^+$, there exists a sequence $\{\sigma_n^j\}$ of $(\mathcal{F}_t^{\varrho^n})$ stopping times and a sequence $\{C_n^j\}$ of constants such that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\mathcal{P}[\sigma_n^j \leq q] = 0, \tag{8.20}$$

$$\lim_{j\to\infty}\limsup_{n\to\infty} C_n^j < +\infty, \tag{8.21}$$

and $\mathrm{Var}(K^n)_{q\wedge\sigma_n^j}, \mathrm{Var}(\overline{K}^n)_{q\wedge\sigma_n^j} \leq C_n^j(K_1 + \frac{r_0}{8(L+1)})$. In the sequel due to (8.20) and (8.21), we can and will assume that every $(\mathcal{F}_t^{\varrho^n})$ adapted process is stopped at $q\wedge\sigma_n^j\wedge\tau_n^i$ and

$$\mathrm{Var}(K^n)_{q\wedge\sigma_n^j\wedge\tau_n^i}, \mathrm{Var}(\overline{K}^n)_{q\wedge\sigma_n^j\wedge\tau_n^i} \leq K_2.$$

Now, suppose that γ_n is an $(\mathcal{F}_t^{\varrho^n})$ stopping time. Then

$$\begin{aligned}E\sup_{t<\gamma_n}|\overline{X}_t^n - X_t^n|^2 &\leq 3E\sup_{t<\gamma_n}\Big|\int_0^t f(\overline{X}_{s-}^n) - f(X_{s-}^n)\,dZ_s^{\varrho^n}\Big|^2\\ &\quad + 3E\sup_{t<\gamma_n}|\overline{K}_t^n - K_t^n|^2 + 3\varepsilon_n = 3I_1^n + 3I_2^n + 3\varepsilon_n,\end{aligned}$$

where $\varepsilon_n = E\sup_{t<\gamma_n}|\int_0^{\varrho_n(t)} f(X_{s-})\,dZ_s - \int_0^t f(\overline{X}_{s-}^n)\,dZ_s^{\varrho^n}|^2$. We recall that by the Doob type inequality proved in [MéPe80],

$$\begin{aligned}I_1^n &\leq C(d)\Big\{E\int_0^{\gamma_n-}\|f(\overline{X}_{s-}^n) - f(X_{s-}^n)\|^2\,d([M^{\varrho^n}]_s + \langle M^{\varrho^n}\rangle_s)\\ &\quad + E\,\mathrm{Var}(V^{\varrho^n})_{\gamma_n-}\int_0^{\gamma_n-}\|f(\overline{X}_{s-}^n) - f(X_{s-}^n)\|^2\,d\,\mathrm{Var}(V^{\varrho^n})_s\Big\}\\ &\leq C(d,L,K_1)E\int_0^{\gamma_n-}\sup_{u\leq s}|\overline{X}_{u-}^n - X_{u-}^n|^2\,d([M^{\varrho^n}]_s + \langle M^{\varrho^n}\rangle_s + \mathrm{Var}(V^{\varrho^n})_s)\end{aligned}$$

On the other hand, by Corollary 7.5,

$$I_2^n \leq C(d)\Big\{E\int_0^{\gamma_n-}\|f(\overline{X}_{s-}^n) - f(X_{s-}^n)\|^2\,d([M^{\varrho^n}]_s + \langle M^{\varrho^n}\rangle_s)$$

$$+\ E\operatorname{Var}(V^{\varrho^n})_{\gamma_n-}\int_0^{\gamma_n-}\|f(\overline{X}^n_{s-})-f(X^n_{s-})\|^2\,d\operatorname{Var}(V^{\varrho^n})_s$$

$$+\ E\Big[\int_0^{\varrho_n(\cdot)} f(X_{s-})\,dM_s-\int_0^{\cdot} f(\overline{X}^n_{s-})\,dM^{\varrho^n}_s\Big]_{\gamma_n-}$$

$$+\ E\Big|\int_0^{\varrho_n(\cdot)} f(X_{s-})\,dV_s-\int_0^{\cdot} f(\overline{X}^n_{s-})\,dV^{\varrho^n}_s\Big|^2_{\gamma_n-}\Big\}$$

$$\le C(d,L,K_1)E\int_0^{\gamma_n-}\sup_{u\le s}|\overline{X}^n_{u-}-X^n_{u-}|^2\,d([M^{\varrho^n}]_s+\langle M^{\varrho^n}\rangle_s+\operatorname{Var}(V^{\varrho^n})_s)$$

$$+\ 2C(d)(\varepsilon^1_n+\varepsilon^2_n).$$

Therefore for every $(\mathcal{F}^{\varrho^n}_t)$ stopping time γ_n,

$$E\sup_{t<\gamma_n}|\overline{X}^n_t-X^n_t|^2\le C(d,L,K_1)$$

$$\times\, E\int_0^{\gamma_n-}\sup_{u\le s}|\overline{X}^n_{u-}-X^n_{u-}|^2\,d([M^{\varrho^n}]_s+\langle M^{\varrho^n}\rangle_s+\operatorname{Var}(V^{\varrho^n})_s)$$

$$+\,3\varepsilon_n+C(d)\varepsilon^1_n+C(d)\varepsilon^2_n.$$

If we define $\varepsilon(n)=3\varepsilon_n+C(d)\varepsilon^1_n+C(d)\varepsilon^2_n$, then due to Lemma C.1(ii),

$$E\sup_{t<q\wedge\sigma^j_n\wedge\tau^i_n}|\overline{X}^n_t-X^n_t|^2\le\varepsilon(n)\exp\{C(d,L,K_1)3K_1\}.$$

Finally, let us observe that by simple calculations based on (8.17) and by the arguments from the proof of [DeMe80, Theorem 15, VIII], $\varepsilon(n)\to 0$. Hence

$$E\sup_{t<q\wedge\sigma^j_n\wedge\tau^i_n}|\overline{X}^n_t-X^n_t|\to 0,\qquad j,\ i\in\mathbb{N},\ q\in\mathbb{R}^+.$$

Since

$$E\sup_{t<q\wedge\sigma^j_n\wedge\tau^i_n}|\overline{K}^n_t-K^n_t|^2$$

$$\le C_1(d,L,K_1)E\int_0^{q\wedge\sigma^j_i\wedge\tau^i_n}\sup_{u\le s}|\overline{X}^n_{u-}-X^n_{u-}|^2\,d([M^{\varrho^n}]_s+\langle M^{\varrho^n}\rangle_s+\operatorname{Var}(V^{\varrho^n})_s)$$

$$+\,C_1(d)(\varepsilon^1_n+\varepsilon^2_n),$$

it follows by (8.17)–(8.20) that

$$\sup_{t\le q}|X^n_t-X^{\varrho^n}_t|\underset{\mathcal{P}}{\to}0,\qquad q\in\mathbb{R}^+,$$

and

$$\sup_{t\le q}|K^n_t-K^{\varrho^n}_t|\underset{\mathcal{P}}{\to}0,\qquad q\in\mathbb{R}^+,$$

which give (8.16).

S t e p 2. *We show how to omit the assumption* $\sup_t |H_t|, \sup_t |Z_t| < \frac{r_0}{8(L+1)} \wedge 1$. Define

$$\sigma_0 = 0, \quad \sigma_{k+1} = \inf\{t > \sigma_k : |H_t - H_{\sigma_k} + X_{\sigma_k}| \vee |Z_t - Z_{\sigma_k}| \geq \frac{r_0}{8(L+1)} \wedge 1\}, \quad k \in \mathbb{N}.$$

By the arguments used in Step 1,

$$\sup_{t\leq q} |X^{n,\varrho_n^\star(\sigma_1)-} - X^{\sigma_1-}_{\varrho_n(t)}| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+, \tag{8.22}$$

and as a consequence

$$X^n_{\varrho_n^\star(\sigma_1)-} \underset{\mathcal{P}}{\to} X_{\sigma_1-}, \tag{8.23}$$

where $\varrho_n^\star(t) = \inf\{t_{nk} : t_{nk} \geq t\}$ (observe that in contrast to $\varrho_n(\sigma_1)$, $\varrho_n^\star(\sigma_1)$ is an $(\mathcal{F}_t^{\varrho^n})$ stopping time!). By simple calculations,

$$X_t^{n,\varrho_n^\star(\sigma_1)} - X^{\sigma_1}_{\varrho_n(t)} = X_t^{n,\varrho_n^\star(\sigma_1)-} - X^{\sigma_1-}_{\varrho_n(t)} + (\Delta X^n_{\varrho_n^\star(\sigma_1)} - \Delta X_{\sigma_1})1_{\{t\geq\varrho_n^\star(\sigma_1)\}}. \tag{8.24}$$

On the other hand,

$$\Delta X^n_{\varrho_n^\star(\sigma_1)} = [X^n_{\varrho_n^\star(\sigma_1)-} + \Delta X^{\varrho^n}_{\varrho_n^\star(\sigma_1)} + f(X^n_{\varrho_n^\star(\sigma_1)-})\Delta Z^{\varrho^n}_{\varrho_n^\star(\sigma_1)}]_\partial - X^n_{\varrho_n^\star(\sigma_1)-}$$

and

$$\Delta X_{\sigma_1} = [X_{\sigma_1-} + \Delta H_{\sigma_1-} + f(X_{\sigma_1-})\Delta Z_{\sigma_1}]_\partial - X_{\sigma_1-}.$$

Therefore by (8.23) and by the convergence $\Delta H^{\varrho^n}_{\varrho_n^\star(\sigma_1)} \to \Delta H_{\sigma_1}, \Delta Z^{\varrho^n}_{\varrho_n^\star(\sigma_1)} \to \Delta Z_{\sigma_1}$ $\mathcal{P}$-a.e. we get $\Delta X^{\varrho^n}_{\varrho_n^\star(\sigma_1)} \to \Delta X_{\sigma_1}$ $\mathcal{P}$-a.e. Due to (8.24),

$$\sup_{t\leq q} |X_t^{n,\varrho_n^\star(\sigma)-} - X^{\sigma_1}_{\varrho_n(t)}| \underset{\mathcal{P}}{\to} 0, \quad q \in \mathbb{R}^+.$$

If we replace the processes H, Z by $H_{\sigma_1+\cdot} - H_{\sigma_1} + X_{\sigma_1}$ and $Z_{\sigma_1+\cdot} - Z_{\sigma_1}$, respectively, we can deduce the convergence X^n to X on the interval $[\sigma_1, \sigma_2[$, etc. Since $\sigma_k \uparrow +\infty$, the proof is complete. ■

The proof of Theorem 8.3 implies immediately the following:

COROLLARY 8.4. *Assume the conditions* (A) *and* (B). *Let* $|\Delta H| + L|\Delta Z| < r_0$ *and let* (X, K) *be a unique strong solution of the SDE* (8.1), *where* f *is Lipschitz continuous and bounded*, *i.e.* $\|f(x)\| \leq L, \|f(x) - f(y)\| \leq L|x-y|$ *for* $x, y \in \overline{D}$. *If* $\{(X^n, K^n)\}$ *is a sequence of solutions to the discrete SDE* (8.12) *defined by* (8.13) *and* (8.14) *then*

(i) $\sup_{t\leq q} |X_t^n - X_t^{\varrho^n}| \to_{\mathcal{P}} 0$ *and* $\sup_{t\leq q} |K_t^n - K_t^{\varrho^n}| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$,

(ii) $(X^n, K^n, H^{\varrho^n}, Z^{\varrho^n}) \to_{\mathcal{P}} (X, K, H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$. ■

COROLLARY 8.5. *Assume the conditions* (A) *and* (B). *Let* $\{Z^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted semimartingales satisfying the condition* (UT) *and let* $\{H^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted processes. Let* $\{(X^n, K^n)\}$ *be a sequence of strong solutions to the SDE* (8.5), *where* f *is Lipschitz continuous and bounded*, *i.e.* $\|f(x)\| \leq L$, $\|f(x) - f(y)\| \leq L|x-y|$ *for* $x, y \in \overline{D}$. *Then*

(i) *if* $(H^n, Z^n) \to_{\mathcal{D}} (H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$(X^n, K^n, H^n, Z^n) \underset{\mathcal{D}}{\to} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

(ii) *if* $(H^n, Z^n) \to_{\mathcal{P}} (H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$(X^n, K^n, H^n, Z^n) \underset{\mathcal{P}}{\to} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

(iii) *if* $\sup_{t\le q}|H^n_t - H_t| \to_{\mathcal{P}} 0$, $\sup_{t\le q}|Z^n_t - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$ *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$\sup_{t\le q}|X^n_t - X_t| \underset{\mathcal{P}}{\to} 0 \quad \text{and} \quad \sup_{t\le q}|K^n_t - K_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

where (X, K) *is a unique strong solution of the SDE* (8.1).

Proof. (i) Due to Corollary 8.4(ii), the SDE (8.1) has a unique weak solution. Thus by Theorem 8.1(i),

$$(X^n, K^n) \underset{\mathcal{D}}{\to} (X, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Moreover, in this case the joint law of (X, K, H, Z) is uniquely determined. Therefore we can strengthen the conclusion of Theorem 8.1(ii) to the convergence

$$(X^n, K^n, H^n, Z^n) \underset{\mathcal{D}}{\to} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).$$

(ii) We use the method of the proof of Theorem 3.1. Since we assume the convergence in probability, all processes are defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $B \in \mathcal{F}, \mathcal{P}(B) > 0$. Define $\mathcal{Q}_B(A) = \mathcal{P}(A|B)$ for every $A \in \mathcal{F}$. By Lemma 3.2, $\{Z^n\}$ is a sequence of semimartingales on $(\Omega, \mathcal{F}, \mathcal{Q}_B)$ for which the condition (UT) is satisfied. Then

$$(H^n, Z^n) \underset{\mathcal{D}(\mathcal{Q}_B)}{\longrightarrow} (H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

and

$$X^n_t = H^n_t + \int_0^t f(X^n_{s-})\,dZ^n_s + K^n_t, \qquad t \in \mathbb{R}^+,\ \mathcal{Q}_B\text{-a.e.}$$

As a consequence, by (i),

$$(X^n, K^n, H^n, Z^n) \underset{\mathcal{D}(\mathcal{Q}_B)}{\longrightarrow} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

Hence for all bounded and continuous mappings $\Phi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}) \to \mathbb{R}$,

$$\lim_{n\to\infty}\int_\Omega \Phi(X^n, K^n, H^n, Z^n)\,d\mathcal{Q}_B = \int_\Omega \Phi(X, K, H, Z)\,d\mathcal{Q}_B. \tag{8.25}$$

Since (8.25) holds for all $B \in \mathcal{F}$ with $\mathcal{P}(B) > 0$ and all bounded continuous mappings $\Phi : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}) \to \mathbb{R}$, we have

$$(X^n, K^n, H^n, Z^n) \underset{\mathcal{P}}{\to} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \tag{8.26}$$

(iii) It is clear by the part (ii) that the condition (8.26) is satisfied. Now, let us note that

$$\Delta X_t = \Delta H_t + f(X_{t-})\Delta Z_t + \Delta K_t, \tag{8.27}$$

and if $\Delta K_t \ne 0$ then $\Delta H_t \ne 0$ or $\Delta Z \ne 0$. Using Corollary A.4 we obtain

$$\sup_{t\le q}|K^n_t - K_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+.$$

Finally, let us observe that by (8.26) we have

$$(X^n, K^n, |H_\cdot^n| + L|Z_\cdot^n|) \underset{\mathcal{P}}{\to} (X, K, |H_\cdot| + L|Z_\cdot|) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

Since it is clear that if $\Delta X_t \neq 0$ then $|\Delta H_t| + L|\Delta Z_t| \neq 0$ or $\Delta K_t \neq 0$, using once more Corollary A.4, we get

$$\sup_{t\le q} |X_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

and the proof is complete. ∎

9. Flows of SDE's with reflecting boundary

9.1. L^p-estimates for solutions of Skorokhod problems starting from different points. Let (X^x, K^x) be a solution to the Skorokhod problem associated with a process Y^x of the form

$$Y_t^x = x + H_t + M_t + V_t, \qquad t \in \mathbb{R}^+,\ x \in \overline{D}, \tag{9.1}$$

where M is an $(\mathcal{F}_t)$ adapted local martingale, V is an $(\mathcal{F}_t)$ process with bounded variation and H is an $(\mathcal{F}_t)$ process such that $H_0 = M_0 = V_0 = 0$. Assume also that there is given another $(\mathcal{F}_t)$ adapted process $\widehat{Y}^y$ which admits the decomposition

$$\widehat{Y}_t^y = y + H_t + \widehat{M}_t + \widehat{V}_t, \qquad t \in \mathbb{R}^+,\ y \in \overline{D}, \tag{9.2}$$

where $\widehat{M}$ is an $(\mathcal{F}_t)$ adapted local martingale $\widehat{M}_0 = 0$ and $\widehat{V}$ is an $(\mathcal{F}_t)$ adapted process with bounded variation, $\widehat{V}_0 = 0$. Let $(\widehat{X}^y, \widehat{K}^y)$ be a solution of the Skorokhod problem corresponding to $\widehat{Y}^y$.

THEOREM 9.1. *Assume D satisfies* (A) *and* (B). *Let $x, y \in \overline{D}$ and let processes Y^x, $\widehat{Y}^y$ fulfill* (9.1) *and* (9.2), *respectively, where $M, \widehat{M}$ are square integrable martingales and $V, \widehat{V}$ are processes with square integrable variation. If $r_0 < +\infty$ assume additionally that $|\Delta Y^x|$, $|\Delta \widehat{Y}^y| \le r_0/4$ and there exists a constant a such that* $\mathrm{Var}(K^x)_\infty, \mathrm{Var}(\widehat{K}^y)_\infty \le a$. *Then for every $p \in \mathbb{N}$ there exists a constant C_p depending on a and r_0 such that for every $(\mathcal{F}_t)$ stopping time σ,*

(i) $E\sup_{t\le\sigma} |X_t^x - \widehat{X}_t^y|^{2p} \le C_p\{|x-y|^{2p} + E\{[M-\widehat{M}]_\sigma^p + \mathrm{Var}(V-\widehat{V})_\sigma^{2p}\}\}$

(ii) $E\sup_{t\le\sigma} |K_t^x - \widehat{K}_t^y|^{2p} \le C_p\{|x-y|^{2p} + E\{[M-\widehat{M}]_\sigma^p + \mathrm{Var}(V-\widehat{V})_\sigma^{2p}\}\}$

(iii) $E\sup_{t<\sigma} |K_t^x - \widehat{K}_t^y|^{2p} \le C_p\{|x-y|^{2p} + E\{\langle M-\widehat{M}\rangle_{\sigma-}^p + [M-\widehat{M}]_{\sigma-}^p + \mathrm{Var}(V-\widehat{V})_{\sigma-}^{2p}\}\}$.

Proof. (i) By [Sai87, Lemma 2] for every $t \in \mathbb{R}^+$,

$$|X_t^x - \widehat{X}_t^y|^2 \le \Big\{|Y_t^x - \widehat{Y}_t^y|^2 + \frac{1}{r_0}\int_0^t |X_s^x - \widehat{X}_s^y|^2\, d(\mathrm{Var}(K^x)_s + \mathrm{Var}(\widehat{K}^y)_s) + 2\int_0^t (Y_t^x - Y_s^x - \widehat{Y}_t^y + \widehat{Y}_s^y)\, d(K_s^x - \widehat{K}_s^y)\Big\}. \tag{9.3}$$

Since $|\varDelta K^x|$, $|\varDelta \widehat{K}^y| \le r_0/4$, it follws that

$$|X^x_t-\widehat{X}^y_t|^2 \le 2\Big\{|Y^x_t-\widehat{Y}^y_t|^2+\frac{1}{r_0}\int_0^{t-}|X^x_s-\widehat{X}^y_s|^2\,d(\mathrm{Var}(K^x)_s+\mathrm{Var}(\widehat{K}^y)_s) + 2\int_0^t (Y^x_t-Y^x_s-\widehat{Y}^y_t+\widehat{Y}^y_s)\,d(K^x_s-\widehat{K}^y_s)\Big\}.$$

On the other hand, by the integration by parts formula,

$$\begin{aligned}&2\int_0^t (Y^x_t-Y^x_s-\widehat{Y}^y_t+\widehat{Y}^y_s)\,d(K^x_s-\widehat{K}^y_s)\\ &= 2\int_0^t (X^x_{s-}-\widehat{X}^y_{s-})\,d(M_s-\widehat{M}_s)+2\int_0^t (X^x_{s-}-\widehat{X}^y_{s-})\,d(V_s-\widehat{V}_s)+[Y^x-\widehat{Y}^y]_t-|Y^x_t-\widehat{Y}^y_t|^2.\end{aligned}$$

Hence for every stopping time τ and $p\in\mathbb{N}$,

$$\begin{aligned}E\sup_{t\le\tau}|X^x_t-\widehat{X}^y_t|^{2p} \le 2^p4^{p-1}\Big\{&E\Big(\frac{1}{r_0}\int_0^{\tau-}|X^x_s-\widehat{X}^y_s|^2\,d(\mathrm{Var}(K^x)_s+\mathrm{Var}(\widehat{K}^y)_s)\Big)^p\\ &+2^pE\sup_{t\le\tau}\Big|\int_0^t (X^x_{s-}-\widehat{X}^y_{s-})\,d(M_s-\widehat{M}_s)\Big|^p\\ &+2^pE\sup_{t\le\tau}\Big|\int_0^t (X^x_{s-}-\widehat{X}^y_{s-})\,d(V_s-\widehat{V}_s)\Big|^p+E[Y^x-\widehat{Y}^y]^p_\tau\Big\}.\end{aligned}$$

Next by the Burkholder–Davis–Gundy and Schwarz inequalities

$$\begin{aligned}&E\sup_{t\le\tau}\Big|\int_0^t (X^x_{s-}-\widehat{X}^y_{s-})\,d(M_s-\widehat{M}_s)\Big|^p\\ &\qquad\le c(p)E\Big(\int_0^\tau |X^x_{s-}-\widehat{X}^y_{s-}|^2\,d[M-\widehat{M}]_s\Big)^{p/2}\\ &\qquad\le c(p)(E\sup_{s\le\tau}|X^x_{s-}-\widehat{X}^y_{s-}|^{2p})^{1/2}(E[M-\widehat{M}]^p_\tau)^{1/2},\end{aligned}$$

and

$$\begin{aligned}E\sup_{t\le\tau}\Big|\int_0^t (X^x_{s-}-\widehat{X}^y_{s-})\,d(V_s-\widehat{V}_s)\Big|^p &\le E\sup_{s\le\tau}|X^x_{s-}-\widehat{X}^y_{s-}|^p\,\mathrm{Var}(V-\widehat{V})^p_\tau\\ &\le (E\sup_{s\le\tau}|X^x_{s-}-\widehat{X}^y_{s-}|^{2p})^{1/2}(E\,\mathrm{Var}(V-\widehat{V})^{2p}_\tau)^{1/2}.\end{aligned}$$

By simple calculations

$$\begin{aligned}E[Y^x-\widehat{Y}^y]^p_\tau &\le 3^p\{|x-y|^{2p}+E[M-\widehat{M}]^p_\tau+E[V-\widehat{V}]^p_\tau\}\\ &\le 3^p\{|x-y|^{2p}+E[M-\widehat{M}]^p_\tau+E\,\mathrm{Var}(V-\widehat{V})^{2p}_\tau\}.\end{aligned}$$

If $r_0 < +\infty$, then there exists a constant $c(p,a,r_0)$ such that,

$$\Big(\frac{1}{r_0}\int_0^{\tau-} |X_s^x - \widehat{X}_s^y|^2\, d(\mathrm{Var}(K^x)_s + \mathrm{Var}(\widehat{K}^y)_s)\Big)^p \le c(p,a,r_0)\int_0^{\tau-} |X_s^x - \widehat{X}_s^y|^{2p}\, d(\mathrm{Var}(K^x)_s + \mathrm{Var}(\widehat{K}^y)_s).$$

If we define

$$u = (E\sup_{t\le\tau} |X_t^x - \widehat{X}^y|^{2p})^{1/2},$$
$$b_1 = (|x-y|^{2p} + E[M-\widehat{M}]_\tau^p + E\,\mathrm{Var}(V-\widehat{V})_\tau^{2p})^{1/2},$$
$$b_2 = \frac{1}{r_0}E\int_0^{\tau-} |X_s^x - \widehat{X}_s^y|^{2p}\, d(\mathrm{Var}(K^x)_s + \mathrm{Var}(\widehat{K}^y)_s),$$

then from calculations we deduce that there exists a constant C_1 such that

$$u^2 \le C_1(b_2 + 2b_1u + b_1^2).$$

Since $0 \le b_1, u < +\infty$ it is clear that $u^2 \le C_2(b_2 + b_1^2)$ for some constant C_2. To finish the proof it is sufficient to use Lemma C.2 of the Appendix. If we set in Lemma C.2, $Y_t^1 = \sup_{s\le t\wedge\sigma} |X_s^x - \widehat{X}_s^y|^{2p}, Y_t^2 = C_2(\mathrm{Var}(K^x)_{t\wedge\sigma} + \mathrm{Var}(\widehat{K}^y)_{t\wedge\sigma}) < C_2 2a + 1$ then the proof of (i) is complete.

(ii) follows from (i) and the Burkholder–Davis–Gundy inequality.

(iii) can be deduced from (ii) with the use of the version of Metivier–Pellaumail inequality proved in Pratelli [Pra83] and the arguments from the paper of Chaleyat-Maurel, El Karoui and Marchal [ChKM80]. ∎

COROLLARY 9.2. *Assume D is a convex subset in $\mathbb{R}^d$. Let Y_0, $\widehat{Y}_0^y \in \overline{D}$ and let processes $Y^x, \widehat{Y}$ fulfil* (9.1) *and* (9.2), *respectively, where M, $\widehat{M}$ are square integrable martingales and V, $\widehat{V}$ are processes with square integrable variation. Then the estimates* (i), (ii) *and* (iii) *are true.*

Proof. It is clear that in this case the condition (A) is satisfied with $r_0 = +\infty$. Let $\tau_k = \inf\{t \in \mathbb{R}^+ : |X_t^x| \text{ or } |\widehat{X}_t^y| \ge k\}$, $k \in \mathbb{N}$. Then of course $X_t^{x,\tau_k-}, \widehat{X}_t^{y,\tau_k-} \in \overline{D}_k$, where $D_k = D \cap \{x \in \mathbb{R}^d : |x| < k\}$. Next, due to Theorem 9.1(iii), for every $k \in \mathbb{N}$ (the condition (B) is satisfied for every bounded and convex set) we have

$$E\sup_{t<\tau_k\wedge\sigma} |K_t^x - \widehat{K}_t^y|^{2p} \le C_p|x-y|^{2p} + E\{\langle M-\widehat{M}\rangle_{(\tau_k\wedge\sigma)-}^p + [M-\widehat{M}]_{(\tau_k\wedge\sigma)-}^p + \mathrm{Var}(V-\widehat{V})_{(\tau_k\wedge\sigma)-}^{2p}\}.$$

Since C_p does not depend on δ and β and $\tau_k \uparrow +\infty$, letting $k \uparrow +\infty$ we get (iii) for every convex set in $\mathbb{R}^d$. Similarly,

$$E\sup_{t\le\sigma} |K_t^x - \widehat{K}_t^y|^{2p} = \lim_{k\to+\infty} E\sup_{t<\tau_k} |K_t^{x,\sigma} - \widehat{K}_t^{y,\sigma}|^{2p}$$

$$\le C_p\{|x-y|^{2p} + \lim_{k\to+\infty} E\{\langle M^\sigma - \widehat{M}^\sigma\rangle^p_{\tau_k-} + [M^\sigma - \widehat{M}^\sigma]^p_{\tau_k-}$$
$$+ \mathrm{Var}(V^\sigma - \widehat{V}^\sigma)^{2p}_{\tau_k-}\}\}$$
$$= C_p\{|x-y|^{2p} + E\{\langle M - \widehat{M}\rangle^p_\sigma + [M - \widehat{M}]^p_\sigma + \mathrm{Var}(V - \widehat{V})^{2p}_\sigma\}\}$$
$$\le (C_p + p^p)\{|x-y|^{2p} + E\{[M - \widehat{M}]^p_\sigma + \mathrm{Var}(V - \widehat{V})^{2p}_\sigma\}\}$$

and we obtain (ii). Finally, the estimate (i) is an obvious consequence of (ii) and of the Burkholder–Davis–Gundy inequality. ∎

9.2. Flows of solutions. In this section we discuss the SDE's with reflecting boundary of the form

$$X^x_t = x + H_t + \int_0^t f(X^x_{s-})\,dZ_s + K^x_t, \qquad t\in\mathbb{R}^+,\ x\in\overline{D}, \tag{9.4}$$

where Z is an $(\mathcal{F}_t)$ adapted semimartingale and H an $(\mathcal{F}_t)$ adapted process, $Z_0 = H_0 = 0$. The flow of SDE (9.4) is a function $\overline{D}\ni x\mapsto X^x_\cdot \in \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$.

THEOREM 9.3. *Assume the conditions* (A) *and* (B) *are satisfied. Let f be Lipschitz continuous and bounded*, *i.e. there exists a constant $L>0$ such that $\|f(x)-f(y)\|\le L|x-y|$*, *$\|f(x)\|\le L$ for all $x,y\in\overline{D}$. If $|\Delta H| + L|\Delta Z| < r_0, H_0\in\overline{D}$ then there exists a function $X(t,\omega,x)$ on $\mathbb{R}^+\times\Omega\times\mathbb{R}^d$ such that*

(i) *for each $x\in\overline{D}$ the process $X^x_t(\omega) = X(t,\omega,x)$ is a solution of* (9.4);

(ii) *for almost all $\omega\in\Omega$, the flow $x\mapsto X^x_\cdot(\omega)$ from $\overline{D}$ into $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ is continuous in the topology of uniform convergence on compact sets.*

P r o o f. By Theorem 8.3 we know that for every $x\in\overline{D}$ there exists a unique strong solution to the SDE (9.3).

Assume we are able to prove that there exists a sequence $\{\tau_k\}$ of stopping times such that $\lim_{k\to+\infty}\mathcal{P}[\tau_k\le q] = 0$, $q\in\mathbb{R}^+$, and

$$E\sup_{t<\tau_k}|X^x_t - X^y_t|^{2p} \le C^p_k|x-y|^{2p}, \qquad x,y\in\mathbb{R},\ p\in\mathbb{N}. \tag{9.5}$$

Then Kolmogorov's lemma (see e.g. [Pro90, Theorem 53, Chap. IV]) implies the result.

It remains to show that $\{\tau_k\}$ in (9.5) exists. First, let us note that by the arguments from the proof of Theorem 7.2 and Proposition 7.6 for every compact set $K\subset\mathbb{R}^d$ there can be found a sequence $\{\sigma_k\}$ of stopping times such that $\lim_{k\to+\infty}\mathcal{P}[\sigma_k\le q] = 0$, $q\in\mathbb{R}^+$, and

$$\sup_{x\in K}\sup_{t<\sigma_k}\mathrm{Var}(K^x)_t \le k. \tag{9.6}$$

Define for fixed $a\in\mathbb{R}^+$,

$$\tau_k = \sigma_k\wedge\inf\{t;\mathrm{Var}(J^a)_t\vee[M^a]_t\vee\langle M^a\rangle_t\vee\mathrm{Var}(B^a)_t\ge k\}$$
$$\wedge\inf\{t:|\Delta H_t| + L|\Delta Z_t|\ge r_0/4\}, \qquad k\in\mathbb{N}.$$

The inequality (9.5) can be shown with the use of Theorem 9.1 and the methods from the proof of Lemma 3.3. ∎

Let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, and let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes, $Z_0^n = H_0^n = 0$. Assume $\{X^{n,x}\}$ is a family of solutions to the following SDE:

$$X_t^{n,x} = x + H_t^n + \int_0^t f(X_{s-}^{n,x})\,dZ_s^n + K_t^{n,x}, \qquad t \in \mathbb{R}^+,\ n \in \mathbb{N},\ x \in \overline{D}. \tag{9.7}$$

THEOREM 9.4. *Assume the conditions* (A) *and* (B). *Let* $\{Z^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted semimartingales satisfying the condition* (UT) *and let* $\{H^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted processes. Let* $\{(X^{n,x}, K^{n,x})\}$ *be a family of strong solutions to the SDE* (9.7), *where* f *is Lipschitz continuous and bounded*, *i.e.* $\|f(x)\| \le L$, $\|f(x) - f(y)\| \le L|x-y|$ *for* $x, y \in \overline{D}$. *Then*

(i) *if* $(H^n, Z^n) \to_{\mathcal{P}} (H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^2)$ *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$\sup_{x \in K \cap \bar{D}} \delta^{4d}((X^{n,x}, K^{n,x}, H^n, Z^n), (X^x, K^x, H, Z)) \underset{\mathcal{P}}{\to} 0,$$

for every bounded subset $K \subset \mathbb{R}^d$,

(ii) *if* $\sup_{t \le q} |H_t^n - H_t| \to_{\mathcal{P}} 0$, $\sup_{t \le q} |Z_t^n - Z_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$\sup_{x \in K \cap \bar{D}} \sup_{t \le q} |X_t^{n,x} - X_t^x| \underset{\mathcal{P}}{\to} 0 \quad \textit{and} \quad \sup_{x \in K \cap \bar{D}} \sup_{t \le q} |K_t^{n,x} - K_t^x| \underset{\mathcal{P}}{\to} 0,$$

for every bounded subset $K \subset \mathbb{R}^d$, $q \in \mathbb{R}^+$, *where* (X^x, K^x) *is a family of strong solutions to the SDE* (9.4).

Proof. (i) By Corollary 8.4,

$$(X^{n,x}, K^{n,x}, H^n, Z^n) \underset{\mathcal{P}}{\to} (X^x, K^x, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),\ x \in \overline{D}. \tag{9.8}$$

On the other hand, by the arguments from the proof of Theorem 9.4, there exist arrays $\{\{\tau_k^n\}\}$ of stopping times and $\{\{C_k^p\}\}$ of constants such that

$$\lim_{k \to +\infty} \limsup_{n \to +\infty} \mathcal{P}[\tau_k^n < q] = 0, \qquad q \in \mathbb{R}^+,$$

and

$$\limsup_{n \to +\infty} E \sup_{t < \tau_k^n} |X_t^{n,x} - X_t^{n,y}|^{2p} \le C_k^p |x-y|^{2p}, \qquad x, y \in \mathbb{R},\ p \in \mathbb{N}. \tag{9.9}$$

Finally, we repeat the arguments from the proof of Theorem 3.1.

(ii) In view of (9.9) we deduce (ii) from (i). ∎

10. Numerical schemes for SDE's with reflecting boundary

10.1. Euler schemes for SDE's with reflecting boundary. In this chapter we investigate L^p convergence as well as almost sure convergence of time-discretization schemes for d-dimensional stochastic differential equation (SDE) on a domain D with reflecting boundary condition. Given a function $f : \overline{D} = D \cup \partial D \to \mathbb{R}^d \otimes \mathbb{R}^d$, $f(x) = \{f_{ij}(x)\}_{i,j=1,\ldots,d}$

we consider the following SDE:

$$X_t^i = X_0^i + \sum_{j=1}^{d} \int_0^t f_{ij}(X_s)\, dW_s^j + \int_0^t g_i(X_s)\, ds + K_t^i, \tag{10.1}$$

$i = 1, \ldots, d$, $t \in \mathbb{R}^+$, where $W_t = (W_t^1, \ldots, W_t^d)$ is a d-dimensional Wiener process, $W_0 = 0$, $X_t = (X_t^1, \ldots, X_t^d)$ is a reflecting process on $\overline{D}$ and $K_t = (K_t^1, \ldots, K_t^d)$ is a bounded variation process with variation $\mathrm{Var}(K)_t$ increasing only when $X_t \in \partial D$ (the precise definition was given in Section 8.1). In [Tan79] and [Sai87] it is proven that if f, g are Lipschitz continuous and bounded on $\overline{D}$, i.e. there exists a constant $L > 0$ such that for every $x, y \in \overline{D}$,

$$\|f(x) - f(y)\| + |g(x) - g(y)| \le L|x - y|, \qquad \|f(x)\|, |g(x)| \le L, \tag{10.2}$$

where $\|\cdot\|$ denotes the usual norm in the space of linear operators from $\mathbb{R}^d$ into $\mathbb{R}^d$, and the conditions (A), (B) from Section 6.1 are satisfied, then there exists a unique strong solution to the SDE (10.1).

Let us consider the sequence $\{T_n\}$ of partitions of $\mathbb{R}^+$ with the property

$$\max_k (t_{nk} - t_{n,k-1}) \le 1/n, \qquad n \in \mathbb{N}. \tag{10.3}$$

In the present chapter we assume that D is either a convex set or a general domain satisfying the conditions (A) and (B), and we consider Euler and Euler–Peano schemes for the SDE (1). More precisely, we investigate the approximations $\{\overline{X}^n\}$ and $\{\widehat{X}^n\}$ which are the solutions to the appropriate SDE's with reflecting boundary conditions

$$\overline{X}_t^n = X_0 + \int_0^t f(\overline{X}_{s-}^n)\, dW_s^{\varrho^n} + \int_0^t g(\overline{X}_{s-}^n)\, d\varrho_s^n + \overline{K}_t^n, \qquad t \in \mathbb{R}^+, \tag{10.4}$$

and

$$\widehat{X}_t^n = X_0 + \int_0^t f(\widehat{X}_{s-}^{n,\varrho^n})\, dW_s + \int_0^t g(\widehat{X}_{s-}^{n,\varrho^n})\, ds + \widehat{K}_t^n, \qquad t \in \mathbb{R}^+. \tag{10.5}$$

It has been observed in Chapter 8 and [Sai87, pp. 473–474] that

$$\sup_{t \le q} |\overline{X}_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

and

$$\sup_{t \le q} |\widehat{X}_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,$$

respectively. Note that if $D = \mathbb{R}^d$ then $\overline{X}_t^n = \widehat{X}_t^{n,\varrho^n}$ and (10.4) is a classical Euler scheme first considered in Maruyama [Mar55] (see also Section 3.3). In the case $D \neq \mathbb{R}^d$ the equality $\overline{X}_t^n = \widehat{X}_t^{n,\varrho^n}$ need not be satisfied. In this case the rate of mean-square convergence in the above schemes was considered before only if $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$ by Chitashvili and Lazrieva [ChLa81], Kinkladze [Ki83] (the scheme (10.4)) and Lépingle [Lép93] (the scheme (10.5)). Let us observe that in this case we can write down the explicit formulas for the solutions $\overline{X}^n$ and $\widehat{X}^n$.

EXAMPLE 10.1. Assume $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. Then the solutions $\overline{X}^n$ and $\widehat{X}^n$ have the following forms:

$$\overline{X}_t^{n,i} = \begin{cases} \overline{Y}_t^{n,1} - 0 \wedge \inf_{s\le t} \overline{Y}_t^{n,1} & \text{if } i=1, \\ \overline{Y}_t^{n,i} & \text{if } i=2,\dots,d, \end{cases}$$

where

$$\overline{Y}_t^{n,i} = \begin{cases} X_0^i & \text{if } t\in[0,t_{n1}[, \\ \overline{Y}_{t_{n,k-1}}^{n,i} + \sum_{j=1}^d f_{ij}(\overline{Y}_{t_{n,k-1}}^n)(W_{t_{nk}}^j - W_{t_{n,k-1}}^j) & \\ \qquad +g_i(\overline{Y}_{t_{n,k-1}}^n)(t_{nk}-t_{n,k-1})\} & \text{if } t\in[t_{nk},t_{n,k+1}[, \end{cases}$$

and

$$\widehat{X}_t^{n,i} = \begin{cases} \widehat{Y}_t^{n,1} - 0 \wedge \inf_{s\le t} \widehat{Y}_t^{n,1} & \text{if } i=1, \\ \widehat{Y}_t^{n,i} & \text{if } i=2,\dots,d, \end{cases}$$

where

$$\widehat{Y}_t^{n,i} = \begin{cases} X_0^i & \text{if } t=0, \\ \widehat{Y}_{t_{nk}}^{n,i} + \sum_{j=1}^d f_{ij}(\widehat{Y}_{t_{nk}}^n)(W_t^j - W_{t_{nk}}^j) & \\ \qquad +g_i(\widehat{Y}_{t_{nk}}^n)(t-t_{nk})\} & \text{if } t\in]t_{nk},t_{n,k+1}]. \ \blacksquare \end{cases}$$

In this chapter we give the rate of L^p as well as of almost sure convergence for the schemes (10.4), (10.5) for domains more general than the half-space. Apart from the conditions (A) and (B), we will consider the following condition (β) introduced by Tanaka [Tan79]:

(β) There exist constants $\varepsilon > 0$ and $\delta > 0$ such that for every $x \in \partial D$ we can find $x_0 \in D$ such that $B(x_0,\varepsilon) \subset D$ and $|x - x_0| \le \delta$.

Remark 10.2 ([Tan79]). If D is a convex domain in $\mathbb{R}^d$ with nonempty interior then $r_0 = +\infty$ and the assumptions (β), (A) and (B) are satisfied for $d = 1, 2$. For $d > 2$ there exists a sequence $\{D_k\}$ of bounded convex sets satisfying the conditions (β), (A) and (B) such that $D_k \uparrow D$. In this case (β)$\Rightarrow$(B) and we can put $D_k = D \cap \{x \in \mathbb{R}^d : |x| < k\}$, $k \in \mathbb{N}$.

Now, let W be an $(\mathcal{F}_t)$ adapted Wiener process. We will say that the SDE (10.1) has a *strong solution* if there exists a pair (X,K) of $(\mathcal{F}_t)$ adapted processes such that (X,K) is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^{\cdot} f(X_s)\,dW_s + \int_0^{\cdot} g(X_s)\,ds.$$

If any two $(\mathcal{F}_t)$ adapted solutions (X,K), (X',K') on $(\Omega,\mathcal{F},\mathcal{P})$ of the SDE (10.1) satisfy $\mathcal{P}[(X,K)=(X',K') : t\in\mathbb{R}^+]=1$ then we say that *strong uniqueness* holds for the SDE (10.1). Similarly we will say that the SDE (10.5) has a *strong solution* if there exists a pair $(\widehat{X}^n,\widehat{K}^n)$ of $(\mathcal{F}_t)$ adapted processes such that $(\widehat{X}^n,\widehat{K}^n)$ is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^{\cdot} f(\widehat{X}_{s-}^{n,\varrho^n})\,dW_s + \int_0^{\cdot} g(\widehat{X}_{s-}^{n,\varrho^n})\,ds.$$

Let $(\mathcal{F}_t^{\varrho^n})$ be a discretization of $(\mathcal{F}_t)$, i.e. $\mathcal{F}_t^{\varrho^n} = \mathcal{F}_{t_{nk}}$, $t \in [t_{nk}, t_{n,k+1}[$. We will say that the SDE (10.4) has a *strong solution* if there exists a pair $(\overline{X}^n, \overline{K}^n)$ of $(\mathcal{F}_t^{\varrho^n})$ adapted processes such that $(\overline{X}^n, \overline{K}^n)$ is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^{\cdot} f(\overline{X}_{s-}^n)\, dW_s^{\varrho^n} + \int_0^{\cdot} g(\overline{X}_{s-}^n)\, d\varrho_s^n.$$

It is easy to prove that the pair $(\overline{X}^n, \overline{K}^n)$ defined by the recurrent formula

$$\overline{X}_t^n = \begin{cases} X_0 & \text{if } t \in [0, t_{n1}[, \\ [\overline{X}_{t_{n,k-1}}^n + f(\overline{X}_{t_{n,k-1}}^n)(W_{t_{nk}} - W_{t_{n,k-1}}) & \\ \qquad + g(\overline{X}_{t_{n,k-1}}^n)(t_{nk} - t_{n,k-1})]_\partial & \text{if } t \in [t_{nk}, t_{n,k+1}[, \end{cases}$$

and

$$\overline{K}_t^n = \begin{cases} 0 & \text{if } t \in [0, t_{n1}[, \\ \overline{K}_{t_{n,k-1}}^n + \overline{X}_t^n - \overline{X}_{t_{n,k-1}}^n & \\ -\{f(\overline{X}_{t_{n,k-1}}^n)(W_{t_{nk}} - W_{t_{n,k-1}}) + g(\overline{X}_{t_{n,k-1}}^n)(t_{nk} - t_{n,k-1})\} & \text{if } t \in [t_{nk}, t_{n,k+1}[\end{cases}$$

is a strong solution to (10.4) on the interval $[0, \gamma_n[$, where $\gamma_n = \inf\{t : |\Delta W_t^{\varrho^n}| + 1/n \geq r_0/L\}$. If D is a convex domain then due to [Tan79] strong uniqueness for the SDE (10.4) holds, too.

Now we use recurrence to construct $(\widehat{X}^n, \widehat{K}^n)$. We put $\widehat{X}_0 = X_0$ and then for $t \in [0, t_{n1}]$, $(\widehat{X}_t^n, \widehat{K}_t^n)$ is a solution to the Skorokhod problem associated with

$$X_0 + f(X_0)W_t + g(X_0)t$$

(we know that under our assumptions on the domain D the solution really exists and is unique). If we have defined $(\widehat{X}_t^n, \widehat{K}_t^n)$ for $t \in [0, t_{nk}]$ then for $t \in [t_{nk}, t_{n,k+1}]$, $(\widehat{X}_t^n, \widehat{K}_t^n)$ is a solution to the Skorokhod problem associated with

$$\widehat{X}_{t_{nk}}^n + f(\widehat{X}_{t_{nk}}^n)(W_t - W_{t_{nk}}) + g(\widehat{X}_{t_{nk}}^n)(t - t_{nk}).$$

By construction, strong uniqueness for (10.5) holds for every D which is either convex or satisfies both (A) and (B).

10.2. Convex domains. In this section we assume that D is a convex domain in $\mathbb{R}^d$. We would like to stress that in this case, due to Corollary 9.2,

$$E \sup_{t \leq q} |X_t - X_0|^{2p} \leq C_p d^p L^{2p} (q^p + q^{2p}).$$

Let (X^n, K^n) be a solution of the Skorokhod problem associated with

$$X_0 + \int_0^{\varrho^n_{\cdot}} f(X_s)\, dW_s + \int_0^{\varrho^n_{\cdot}} g(X_s)\, ds.$$

Using once more Corollary 9.2 and the Burkholder–Davis–Gundy inequality gives

$$E \sup_{t \leq q} |X_t^n - X_0|^{2p} \leq C_p E\Big\{\Big[\int_0^{\varrho^n_{\cdot}} f(X_s)\, dW_s\Big]_q^p + \Big|\int_0^{\varrho^n_{\cdot}} g(X_s)\, ds\Big|_q^p\Big\}$$

$$\le C(p,d)E\Big\{\sup_{t\le\varrho^n_q}\Big|\int_0^t f(X_s)\,dW_s\Big|^{2p}+\Big|\int_0^{\cdot} g(X_s)\,ds\Big|^p_{\varrho^n_q}\Big\}$$
$$= C(p,d,L,q)<+\infty.$$

Notice that Corollary 9.2 implies also integrability of

$$\sup_{t\le q}|\overline{X}^n_t-X_0|^{2p}\quad\text{and}\quad\sup_{t\le q}|\widehat{X}^n_t-X_0|^{2p}.$$

PROPOSITION 10.3. *Assume D is a convex domain in $\mathbb{R}^d$. Then for every sequence $\{\tau_n\}$ of stopping times*

$$E\sup_{t<q}|X^{n,\tau_n-}_t-\overline{X}^{n,\tau_n-}_t|^{2p}=\mathcal{O}(E\sup_{t<q}|X^{\tau_n}_t-X^{n,\tau_n-}_t|^{2p}),\qquad q\in\mathbb{R}^+,\ p\in\mathbb{N}.$$

P r o o f. For every $q\in\mathbb{R}^d$,

$$E\sup_{t<q}|X^{n,\tau_n-}_t-\overline{X}^{n,\tau_n-}_t|^{2p}$$
$$\le 3^{p-1}\Big\{E\sup_{t<q\wedge\tau_n}\Big|\int_0^t(f(X^n_{s-})-f(\overline{X}^n_{s-}))\,dW^{\varrho^n}_s$$
$$+\int_0^t(g(X^n_{s-})-g(\overline{X}^n_{s-}))\,d\varrho^n_s\Big|^{2p}+E\sup_{t<q\wedge\tau_n}|K^n_t-\overline{K}^n_t|^{2p}+\varepsilon^n_1\Big\}$$
$$=3^{p-1}\{I^n_1+I^n_2+\varepsilon^n_1\},$$

where

$$\varepsilon^n_1=E\sup_{t<q\wedge\tau_n}\Big|\int_0^{\varrho^n_t}(f(X_s)-f(X^n_{s-}))\,dW_s+\int_0^{\varrho^n_t}(g(X_s)-g(X^n_{s-}))\,ds\Big|^{2p}.$$

By the Burkholder–Davis–Gundy inequality,

$$\varepsilon^n_1\le E\sup_{t<q\wedge\tau_n}\Big|\int_0^t(f(X_s)-f(X^n_{s-}))\,dW_s+\int_0^t(g(X_s)-g(X^n_{s-}))\,ds\Big|^{2p}$$
$$\le E\sup_{t\le q}\Big|\int_0^t(f(X^{\tau_n}_s)-f(X^{n,\tau_n-}_{s-}))\,dW_s+\int_0^t(g(X^{\tau_n}_s)-g(X^{n,\tau_n-}_{s-}))\,ds\Big|^{2p}$$
$$\le C(p,d,L,q)E\sup_{t\le q}|X^{\tau_n}_t-X^{n,\tau_n-}_{t-}|^{2p}=C(p,d,L,q)E\sup_{t<q}|X^{\tau_n}_t-X^{n,\tau_n-}_t|^{2p}$$

and

$$I^n_1\le E\Big\{\sup_{t<q\wedge\tau_n}\Big|\int_0^t(f(X^n_{s-})-f(\overline{X}^n_{s-}))\,dW_s+\int_0^t(g(X^n_{s-})-g(\overline{X}^n_{s-}))\,ds\Big|^{2p}\Big\}$$
$$\le C(p,d,L,q)E\Big\{\Big(\int_0^q\sup_{u\le s}|X^{n,\tau_n-}_{u-}-\overline{X}^{n,\tau_n-}_{u-}|^2\,ds\Big)^p+\Big(\int_0^q\sup_{u\le s}|X^{n,\tau_n-}_{u-}-\overline{X}^{n,\tau_n-}_{u-}|\,ds\Big)^{2p}\Big\}$$
$$\le C(p,d,L,q)\int_0^q E\sup_{u<s}|X^{n,\tau_n-}_u-\overline{X}^{n,\tau_n-}_u|^{2p}\,ds.$$

For every τ_n define

$$\sigma_n = \min\{t_{nk} : t_{nk} \geq \tau_n\}.$$

Then by simple calculations σ_n is an $(\mathcal{F}^{\varrho^n})$ stopping time such that for every $(\mathcal{F}^{\varrho^n})$ adapted step process Y (i.e. of the form $Y_t = Y_{t_{nk}}, t \in [t_{nk}, t_{n,k+1}[$) we have $Y_t^{\tau_n -} = Y_t^{\sigma_n -}$ and $\sup_{t<q\wedge\tau_n} |Y_t| = \sup_{t<q\wedge\sigma_n} |Y_t|$. Therefore by using Corollary 9.2 and the Burkholder–Davis–Gundy inequality we get

$$\begin{aligned}
I_2^n &= \sup_{t<q\wedge\sigma_n} |K_t^n - \overline{K}_t^n|^{2p} \\
&\leq C(p)E\Big\{\Big[\int_0^{\cdot}(f(X_{s-}^n) - f(\overline{X}_{s-}^n))\,dW_s^{\varrho^n}\Big]_{(q\wedge\sigma_n)-}^p \\
&\quad + \Big\langle \int_0^{\cdot}(f(X_{s-}^n) - f(\overline{X}_{s-}^n))\,dW_s^{\varrho^n}\Big\rangle_{(q\wedge\sigma_n)-}^p \\
&\quad + \Big|\int_0^{\cdot}(g(X_{s-}^n) - g(\overline{X}_{s-}^n))\,d\varrho_s^n\Big|_{(q\wedge\sigma_n)-}^{2p} + \varepsilon_2^n\Big\} \\
&\leq C(p,d,L,q)E\Big\{\Big[\int_0^{\cdot}(f(X_{s-}^{n,\tau_n-}) - f(\overline{X}_{s-}^{n,\tau_n-}))\,dW_s^{\varrho^n}\Big]_q^p \\
&\quad + \int_0^q E\sup_{u\leq s}|X_{u-}^{n,\tau_n-} - \overline{X}_{u-}^{n,\tau_n-}|^{2p}\,d\varrho_s^n + \varepsilon_2^n\Big\} \\
&\leq C(p,d,L,q)E\Big\{\sup_{t\leq q}\Big|\int_0^t(f(X_{s-}^{n,\tau_n-}) - f(\overline{X}_{s-}^{n,\tau_n-}))\,dW_s\Big|_q^{2p} \\
&\quad + \int_0^q E\sup_{u\leq s}|X_{u-}^{n,\tau_n-} - \overline{X}_{u-}^{n,\tau_n-}|^{2p}\,ds + \varepsilon_2^n\Big\} \\
&\leq C(p,d,L,q)\Big\{\int_0^q E\sup_{u<s}|X_u^{n,\tau_n-} - \overline{X}_u^{n,\tau_n-}|^{2p}\,ds + \varepsilon_2^n\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_2^n &= E\Big\{\Big[\int_0^{\varrho^n_\cdot} f(X_s)\,dW_s - \int_0^{\cdot} f(X_{s-}^n)\,dW_s^{\varrho^n}\Big]_{(q\wedge\sigma_n)-}^p \\
&\quad + \Big\langle \int_0^{\varrho^n_\cdot} f(X_s)\,dW_s - \int_0^{\cdot} f(X_{s-}^n)\,dW_s^{\varrho^n}\Big\rangle_{(q\wedge\sigma_n)-}^p \\
&\quad + E\Big|\int_0^{\varrho^n_\cdot} g(X_s)\,ds - \int_0^{\cdot} g(X_{s-}^n)\,d\varrho_s^n\Big|_{(q\wedge\sigma_n)-}^{2p}\Big\} \\
&\leq C(p,d,L,q)E\sup_{t<q}|X_t^{\tau_n} - X_t^{n,\tau_n-}|^{2p}.
\end{aligned}$$

From the above estimates, it is clear that there exist two constants C_1 and C_2 such that for every $q \in \mathbb{R}^d$,

$$E \sup_{t<q} |X_t^{n,\tau_n-} - \overline{X}_t^{n,\tau_n-}|^{2p} \le C_1 E \sup_{t<q} |X_t^{\tau_n} - X_t^{n,\tau_n-}|^{2p} + C_2 \int_0^q E \sup_{u<s} |X_u^{n,\tau_n-} - \overline{X}_u^{n,\tau_n-}|^{2p}\, ds.$$

To finish the proof we use Lemma C.2 of the Appendix. If we set in Lemma C.2, $h_q = E\sup_{t<q} |X_t^{n,\tau_n-} - \overline{X}_t^{n,\tau_n-}|^{2p}$, $k_q = C_2 q$ we obtain

$$E \sup_{t<q} |X_t^{n,\tau_n-} - \overline{X}_t^{n,\tau_n-}|^{2p} \le C_1 E \sup_{t<q} |X_t^{\tau_n} - X_t^{n,\tau_n-}|^{2p} \exp\{C_2 q\}$$

and the proof is complete. ∎

Theorem 10.4. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.3).

(i) *If D is a convex polyhedron then*

$$E \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} = \mathcal{O}(1/n^{p-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}.$$

(ii) *If D is a convex domain satisfying the condition* (β) *then*

$$E \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}.$$

(iii) *If D is any convex domain then there exists a sequence $\{\tau_n\}$ of stopping times such that $\tau_n \to_{\mathcal{P}} +\infty$ and*

$$E \sup_{t\le q} |\overline{X}_t^{n,\tau_n-} - X_t^{\tau_n}|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}.$$

Proof. (i) Due to Proposition 10.3 (with $\tau_n = +\infty, n \in \mathbb{N}$) it is sufficient to prove that

$$E \sup_{t\le q} |X_t^n - X_t|^{2p} = \mathcal{O}(1/n^{p-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+.$$

Next, by [DuIs91, Theorem 2.2] there exists a constant $c(p) > 0$ such that

$$\sup_{t\le q} |X_t^n - X_t|^{2p} \le c(p) \sup_{t\le q} \Big| \int_{\varrho_t^n}^t f(X_s)\, dW_s + \int_{\varrho_t^n}^t g(X_s)\, ds \Big|^{2p} \tag{10.6}$$

and therefore in order to prove (i) it is enough to use the following simple lemma:

Lemma 10.5. *Let H, G be two predictable processes with values in $\mathbb{R}^d \otimes \mathbb{R}^d$ and $\mathbb{R}^d$, respectively, such that $\sup_{t\le q} \|H_t\|, \sup_{t\le q} |G_t| \le L < +\infty$ for some constant $L > 0$ and let Y be a process with continuous trajectories of the form $Y_t = \int_0^t H_s\, dW_s + \int_0^t G_s\, ds$, $t \in \mathbb{R}^+$. Then*

$$E \sup_{t\le q} |Y_t - Y_t^{\varrho^n}|^{2p} \le E\omega_Y(1/n, q)^{2p} = \mathcal{O}(1/n^{p-\varepsilon}), \quad q \in \mathbb{R}^+,\ p \in \mathbb{N},\ \varepsilon > 0.$$

Proof. Without loss of generality we may and will assume that $d = 1$, $g = 0$. Then $\int_0^t H_s\, dW_s = B \circ A_t$, for some standard (possibly stopped) Wiener process B and A defined by $A_t = \int_0^t H_s^2\, ds$. Hence $A_t - A_s \le L^2(t-s)$, $A_q \le L^2 q$ and

$$\omega_{B\circ A}(\delta, q) \le \omega_B(\delta L^2, L^2 q).$$

Since

$$\mathcal{P}(\omega_B(\delta L^2, L^2 q)^{2p} \geq y) \leq \frac{16L\sqrt{\delta}((q+2\delta)/\delta)}{\sqrt{2\pi y^{1/p}}} \exp\Big(-\frac{y^{1/p}}{8\delta L^2}\Big)$$

(see e.g. [Ute81]), we can write

$$\begin{aligned} n^{p-\varepsilon} E\omega_B(L^2/n, L^2 q)^{2p} &\leq 1 + n^{p-\varepsilon} \int_{n^{-p+\varepsilon}}^{\infty} \mathcal{P}\Big(\omega_B\Big(\frac{L^2}{n}, L^2 q\Big)^{2p} > y\Big) dy \\ &\leq 1 + n^{-\varepsilon} \int_{n^{-\varepsilon/p}}^{\infty} \frac{16L(qn+2)}{\sqrt{2\pi y}} \exp\Big(-\frac{y}{8L^2}\Big) y^{p-1}\, dy \\ &\leq 1 + C(p,L,q,\pi) n \int_{n^{-\varepsilon/p}}^{\infty} \exp\Big(-\frac{y}{8L^2}\Big) y^{p-1}\, dy \\ &\leq 1 + C(p,L,q,\pi) n \exp\Big(-\frac{n^{\varepsilon}}{8L^2}\Big) = \mathcal{O}(1). \ \blacksquare \end{aligned}$$

(ii) For a general convex domain we have the following inequality [Tan79, Lemma 2.2]:

$$\begin{aligned} \sup_{t\leq q} |X_t^n - X_t|^{2p} \leq C(p)\Big\{ &\sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds\Big|^{2p} \\ &+ \sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds\Big|^{p} (\mathrm{Var}(K^n)_q^p + \mathrm{Var}(K)_q^p)\Big\} \end{aligned}$$

instead of (10.6). Therefore by Schwarz's inequality we have

$$\begin{aligned} E\sup_{t\leq q} |X_t^n - X_t|^{2p} \leq C(p)\Big\{ &E\sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds\Big|^{2p} \\ &+ \Big(E\sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds\Big|^{2p}\Big)^{1/2} \\ &\times (E\,\mathrm{Var}(K^n)_q^{2p} + E\,\mathrm{Var}(K)_q^{2p})^{1/2}\Big\}. \end{aligned}$$

To finish the proof in view of Proposition 10.3 (as before with $\tau_n = +\infty, n \in \mathbb{N}$) and Lemma 10.5, it is enough to show

$$E\,\mathrm{Var}(K)_q^{2p} < +\infty \tag{10.7}$$

and

$$\sup_n E\,\mathrm{Var}(K^n)_q^{2p} < +\infty. \tag{10.8}$$

Now, let Y be a process with continuous trajectories and let (X,K) be a solution of the Skorokhod problem associated with Y. Due to [Tan79, 2.11(b)] there exist constants $c,h>0$ such that if n is so large that $\omega_{Y(\omega)}(1/n,q) < h$ then

$$\mathrm{Var}(K(\omega))_q \leq (nq+1)c\sup_{t\leq q} |Y_t(\omega)|, \qquad \omega \in \Omega. \tag{10.9}$$

Define

$$N = \inf\{n \in \mathbb{N} : \omega_Y(1/n, q) < h/2\} - 1.$$

Then if $N \geq 1$ we have

$$h/2 \leq \omega_Y(1/N, q) < h \tag{10.10}$$

and

$$\mathrm{Var}(K)_q^{2p} \leq c(p)(Nq+1)^{2p} \sup_{t\leq q} |Y_t|^{2p}. \tag{10.11}$$

By Lemma 10.5 and (10.10),

$$\begin{aligned} EN^{2p} &\leq (h/2)^{-8p} EN^{2p}\omega_Y(1/N, q)^{8p} \\ &= (h/2)^{-8p} \sum_{n=1}^{\infty} En^{2p}\omega_Y(1/n, q)^{8p} 1_{\{N=n\}} \\ &\leq (h/2)^{-8p} \sum_{n=1}^{\infty} n^{-2p+\varepsilon} n^{4p-\varepsilon} E\omega_Y(1/n, q)^{8p} < +\infty \end{aligned}$$

for every $p \in \mathbb{N}$. In view of Schwarz's inequality, (10.7) is immediate from (10.11). On the other hand, it is known that (10.9) is true also if $\omega'_{Y(\omega)}(1/n, q) < h$ and Y is any process with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ (see e.g. [AnLi89]). Since by simple calculations

$$\omega'_{Y^{\varrho^n}}(h, q) \leq \omega_Y(h, q),$$

it is clear that

$$\mathrm{Var}(K^n)_q^{2p} \leq c(p)(Nq+1)^{2p} \sup_{t\leq q} |Y_t^{\varrho^n}|^{2p}, \qquad n \in \mathbb{N},$$

and by the arguments used previously the property (10.8) easily follows.

(iii) Define $\tau_n^k = \inf\{t \in \mathbb{R}^+ : |X_t| + |X_t^n| + |\overline{X}_t^n| > k\}$, $n, k \in \mathbb{N}$. Then of course

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathcal{P}(\tau_n^k \leq q) = 0 \tag{10.12}$$

and for fixed $k \in \mathbb{N}$, $X_t^{\tau_n^k}, X_t^{n,\tau_n^k-}, \overline{X}_t^{n,\tau_n^k-} \in \overline{D}_k$. Since D_k satisfies the condition (β) (see Remark 10.2(ii)), from the proof of (ii) we deduce

$$E\sup_{t\leq q} |X_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \qquad \varepsilon > 0,\ q \in \mathbb{R}^+,\ k \in \mathbb{N},\ p \in \mathbb{N}.$$

Hence, due to Proposition 10.3, also

$$E\sup_{t\leq q} |\overline{X}_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \qquad \varepsilon > 0,\ q \in \mathbb{R}^+,\ k \in \mathbb{N},\ p \in \mathbb{N}.$$

Since $\varepsilon > 0$ can be chosen as small as desired, we have in fact the convergence

$$\sup_{p'\leq p} n^{p'/2-\varepsilon} E\sup_{t\leq q} |\overline{X}_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p'} \to 0 \tag{10.13}$$

for every $\varepsilon > 0$, $k \in \mathbb{N}$, $q \in \mathbb{R}^+$, $p \in \mathbb{N}$. Set

$$a_{nk}(\varepsilon^{-1}, q, p) = \sup_{p'\leq p} n^{p'/2-\varepsilon} E\sup_{t\leq q} |\overline{X}_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p'} + \mathcal{P}(\tau_n^k \leq q)$$

for every $\varepsilon > 0$, $k \in \mathbb{N}$, $q \in \mathbb{R}^+$, $p \in \mathbb{N}$. Now we use the following lemma:

Lemma 10.6. *Let* $S = \prod_{j=1}^{l} S_j$, *where* S_j *are* $\mathbb{N}$ *or* $\mathbb{R}^+$. *Assume that for every* $\alpha \in S$, $\{\{a_{nk}(\alpha)\}\}$ *is an array of nonnegative numbers such that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} a_{nk}(\alpha) = 0, \qquad \alpha \in S, \tag{10.14}$$

$$\alpha \le \alpha' \Rightarrow a_{nk}(\alpha) \le a_{nk}(\alpha'). \tag{10.15}$$

Then there exists a sequence $\{k_n\}$ *with* $k_n \uparrow +\infty$ *sufficiently slowly for which*

$$a_{nk_n}(\alpha) \to 0, \qquad \alpha \in S.$$

Proof. For every $m \in \mathbb{N}$ there exists a number $c(m) > c(m-1)$ ($c(0) = 1$) such that for all $k \ge c(m)$,

$$\limsup_{n\to+\infty} a_{nk}((m, m, \ldots, m)) \le 1/m,$$

where $(m, m, \ldots, m) \in S$. Set $\alpha_k = (m, m, \ldots, m)$ if $k \in [c(m), c(m+1)[$, $m \in \mathbb{N}$ Then

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} a_{nk}(\alpha_k) = 0$$

and $\alpha_k \uparrow +\infty$. Similarly, for every $m \in \mathbb{N}$ there exists a number $b(m) > b(m-1)$ ($b(0) = 1$) such that for $n \ge b(m)$,

$$a_{nm}(\alpha_m) \le 2 \limsup_{n\to+\infty} a_{nm}(\alpha_m).$$

Finally, if we set $k_n = m$ for $n \in [b(m), b(m+1)[$ then $a_{n,k_n}(\alpha_{k_n}) \to 0$ and the desired result follows by (10.15). ∎

By (10.12), (10.13) and Lemma 10.6 we can choose sequence $\{k_n\}$ with $k_n \uparrow +\infty$ sufficiently slowly such that

$$a_{nk_n}(\varepsilon^{-1}, q, p) \to 0 \quad \text{and} \quad \tau_n^{k_n} \underset{\mathcal{P}}{\to} +\infty.$$

Thus, setting $\tau_n = \tau_n^{k_n}$, $n \in \mathbb{N}$, we complete the proof. ∎

Remark 10.7. As observed in [DuIs91, Prop. 4.1], the estimate (10.6) is true only for a convex polyhedron with nonempty interior. Therefore it seems to be impossible to obtain the rate of convergence $\mathcal{O}(1/n^{p-\varepsilon})$ for other convex domains in $\mathbb{R}^d$.

Corollary 10.8. *Assume* f, g *satisfy* (10.2) *and* $\{T_n\}$ *satisfies* (10.3).

(i) *If* D *is a convex polyhedron then*

$$n^{1/2-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t| \to 0 \quad \mathcal{P}\text{-a.s.}, \ \varepsilon > 0, \ q \in \mathbb{R}^+.$$

(ii) *If* D *is a convex domain satisfying the condition* (β) *then*

$$n^{1/4-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t| \to 0 \quad \mathcal{P}\text{-a.s.}, \ \varepsilon > 0, \ q \in \mathbb{R}^+.$$

(iii) *If* D *is any convex domain then there exists a sequence* $\{\tau_n\}$ *of stopping times such that* $\tau_n \to_{\mathcal{P}} +\infty$ *and*

$$n^{1/4-\varepsilon} \sup_{t\le q} |\overline{X}_t^{n,\tau_n-} - X_t^{\tau_n}| \to 0 \quad \mathcal{P}\text{-a.s.}, \ \varepsilon > 0, \ q \in \mathbb{R}^+.$$

P r o o f. (i) Due to the Borel–Cantelli lemma it is sufficient to find a sequence $\{\alpha_n\}$ with $\alpha_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \mathcal{P}(n^{1/2-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t| > \alpha_n) < +\infty.$$

Let p be so large that $2p\varepsilon > 1$. Set $\varepsilon' = (2p\varepsilon - 1)/4$, $\alpha_n = n^{-\varepsilon'/2p}$. Then by Chebyshev's inequality and Theorem 10.4(i) used for ε' in place of ε,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathcal{P}(n^{1/2-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t| > \alpha_n) &\le \sum_{n=1}^{\infty} \frac{1}{n^{-\varepsilon'}} E n^{p-2p\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} \\ &\le C(p,d,L,q) \sum_{n=1}^{\infty} \frac{1}{n^{-\varepsilon'}} \frac{n^{p-2p\varepsilon}}{n^{p-\varepsilon'}} \\ &= C(p,d,L,q) \sum_{n=1}^{\infty} n^{-2p\varepsilon+2\varepsilon'}.\end{aligned}$$

Since $-2p\varepsilon + 2\varepsilon' < -1$, the proof is finished.

(ii), (iii). It is enough to use Theorem 10.4(ii), (iii), respectively and to follow the proof of (i). ∎

Corollary 10.9. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.3). *If D is a convex domain then*

(i) $E \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} \to 0$, $q \in \mathbb{R}^+$, $p \in \mathbb{N}$,

(ii) $n^{1/4-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t| \to_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, $\varepsilon > 0$.

P r o o f. In view of Corollary 10.8(iii),

$$n^{p/2-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} \underset{\mathcal{P}}{\to} 0, \qquad q \in \mathbb{R}^+,\ p \in \mathbb{N},\ \varepsilon > 0,$$

and we get (ii). Then to prove (i) it is enough to observe that by Corollary 9.2,

$$\sup_n E \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} < +\infty, \qquad p \in \mathbb{N}. ∎$$

Theorem 10.10. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.3). *If D is a convex domain then*

$$E \sup_{t\le q} |\widehat{X}_t^n - X_t|^{2p} = \mathcal{O}(1/n^p), \qquad q \in \mathbb{R}^+.$$

P r o o f. For every $t \in \mathbb{R}^+$,

$$\begin{aligned}E|\widehat{X}_t^n - X_t|^{2p} &\le 3^{p-1}\Big\{E\Big|\int_0^t (f(\widehat{X}^n)_s - f(X_s))\,dW_s + \int_0^t (g(\widehat{X}_s^n) - g(X_s))\,ds\Big|^{2p} \\ &\quad + E|\widehat{K}_t^n - K_t|^{2p} \\ &\quad + E\Big|\int_0^t (f(\widehat{X}_{s-}^{n,\varrho^n}) - f(\widehat{X}_s^n))\,dW_s + \int_0^t (g(\widehat{X}_{s-}^{n,\varrho^n}) - g(\widehat{X}_s^n))\,ds\Big|^{2p}\Big\} \\ &= 3^{p-1}\{I_1^n + I_2^n + \varepsilon_n\}.\end{aligned}$$

It is clear that

$$I_1^n \le C(p,d,L,q)\int_0^t E|\widehat{X}_s^n - X_s|^{2p}\,ds$$

and

$$\varepsilon_n \le C(p,d,L,q)\int_0^t E|\widehat{X}_{s-}^{n,\varrho^n} - \widehat{X}_s^n|^{2p}\,ds.$$

Let $s \in\,]t_{nk}, t_{n,k+1}]$, $s \le q$. Then of course $\widehat{X}_{s-}^{n,\varrho^n} = \widehat{X}_{t_{nk}}^n$. By Corollary 9.2 for $\mathcal{F}_{t_{nk}+\cdot}$ adapted processes we have

$$\begin{aligned} E|\widehat{X}_{t_{nk}}^n - \widehat{X}_s^n|^{2p} &\le C_p E\Big\{\Big[\int_{t_{nk}}^{\cdot} f(\widehat{X}_{u-}^{n,\varrho^n})\,dW_u\Big]_s^p + \Big|\int_{t_{nk}}^{\cdot} g(\widehat{X}_{u-}^{n,\varrho^n})\,ds\Big|_s^{2p}\Big\} \\ &\le C(p,d,L,q)/n^p. \end{aligned}$$

Hence for every $s \le q$,

$$E|\widehat{X}_{s-}^{n,\varrho^n} - \widehat{X}_s^n|^{2p} \le C(p,d,L,q)/n^p. \tag{10.16}$$

On the other hand, by Corollary 9.2 once more,

$$\begin{aligned} I_2^n &\le E\sup_{t\le q}|\widehat{K}_t^n - K_t|^{2p} \\ &\le C_p\Big\{E\Big[\int_0^{\cdot}(f(\widehat{X}_{s-}^{n,\varrho^n}) - f(X_s))\,dW_s\Big]_q^p + E\Big|\int_0^{\cdot}(g(\widehat{X}_{s-}^{n,\varrho^n}) - g(X_s))\,ds\Big|_q^{2p}\Big\} \\ &\le C(p,d,L,q)\Big\{\int_0^q E|\widehat{X}_s^n - X_s|^{2p}\,ds + \int_0^q E|\widehat{X}_{s-}^{n,\varrho^n} - \widehat{X}_s^n|^{2p}\,ds\Big\}. \end{aligned}$$

Putting the above estimates together we see that for every $t \le q$ there exist constants $C_1, C_2 > 0$ such that

$$E|\widehat{X}_t^n - X_t|^{2p} \le C_1/n^p + C_2\int_0^t E|\widehat{X}_s^n - X_s|^{2p}\,ds,$$

hence by Lemma C.2 of Appendix C,

$$\sup_{t\le q} E|\widehat{X}_t^n - X_t|^{2p} \le \frac{C_1}{n^p}\exp\{C_2 q\}, \tag{10.17}$$

which, when combined with (10.16), yields

$$\sup_{t\le q} E|\widehat{X}_{t-}^{n,\varrho^n} - X_t|^{2p} = \mathcal{O}(1/n^p), \qquad q \in \mathbb{R}^+. \tag{10.18}$$

Finally, by the Burkholder–Davis–Gundy inequality and (10.17),

$$\begin{aligned} E\sup_{t\le q}|\widehat{X}_t^n - X_t|^{2p} \le 3^{p-1}\Big\{&E\sup_{t\le q}\Big|\int_0^t(f(\widehat{X}_{s-}^{n,\varrho^n}) - f(X_s))\,dW_s\Big|^{2p} \\ &+ E\sup_{t\le q}\Big|\int_0^t(g(\widehat{X}_{s-}^{n,\varrho^n}) - g(X_s))\,ds\Big|^{2p} + E\sup_{t\le q}|\widehat{K}_t^n - K_t|^{2p}\Big\} \end{aligned}$$

$$\leq C(p,d,L,q)\Big\{\int_0^q E|\widehat{X}^{n,\varrho^n}_{s-} - X_s|^{2p}\,ds + E\sup_{t\leq q}|\widehat{K}^n_t - K_t|^{2p}\Big\}$$

$$\leq C(p,d,L,q)\Big\{\int_0^q E|\widehat{X}^{n,\varrho^n}_{s-} - X_s|^{2p}\,ds + \int_0^q E|\widehat{X}^n_s - X_s|^{2p}\,ds\Big\} + \mathcal{O}(1/n^p),$$

and thus, in view of (10.17) and (10.18) the result follows. ∎

By using the arguments from the proof of Corollary 10.8 we can easily deduce from Theorem 10.10 the following rate of almost sure convergence.

COROLLARY 10.11. *Assume f, g satisfy* (10.2) *and* $\{T_n\}$ *satisfies* (10.3). *If D is a convex domain in $\mathbb{R}^d$ then*

$$n^{1/2-\varepsilon}\sup_{t\leq q}|\widehat{X}^n_t - X_t| \to 0 \quad \mathcal{P}\text{-a.s.}, \ \varepsilon > 0,\ q\in\mathbb{R}^+. \ ∎$$

10.3. General domains. In this section we assume that D satisfies the conditions (A) and (B). In that case, if $r_0 < +\infty$, the processes $|X_t - X_0|^{2p}$, $|X^n_t - X_0|^{2p}$, $|\overline{X}^n_t - X_0|^{2p}$, and $|\widehat{X}^n_t - X_0|^{2p}$ are not necessarily integrable. Therefore the results are weaker than their analogues in Section 10.2.

THEOREM 10.12. *Assume f, g satisfy* (10.2) *and* $\{T_n\}$ *satisfies* (10.3). *If the conditions* (A) *and* (B) *are satisfied then there exists a sequence $\{\tau_n\}$ of stopping times with $\tau_n \to_{\mathcal{P}} \infty$ such that*

$$E\sup_{t\leq q}|\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \qquad \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}.$$

Proof. Define

$$\tau^k_n = \inf\Big\{t : |K|_t + |K^n|_t + |\overline{K}^n|_t > k \text{ or } |\Delta W^{\varrho^n}_t| + \frac{1}{n} \geq \frac{r_0}{4L}\Big\}, \qquad k,n\in\mathbb{N}.$$

Due to [Sai87, Proposition 3.1],

$$\lim_{k\to\infty}\limsup_{n\to\infty}\mathcal{P}(\tau^k_n \leq q) = 0, \qquad q\in\mathbb{R}^+. \tag{10.19}$$

By using the arguments from the proof of Proposition 10.3 we show that

$$E\sup_{t<q}|X^{n,\tau^k_n-}_t - \overline{X}^{n,\tau^k_n-}_t|^{2p} = \mathcal{O}(E\sup_{t<q}|X^{\tau^k_n}_t - X^{n,\tau^k_n-}_t|^{2p}). \tag{10.20}$$

On the other hand, since $|\Delta K^{n,\tau^k_n-}| \leq r_0/4$, it follows by [Sai87, Lemma 2.3(i)] that

$$\sup_{t\leq q}|X^{n,\tau^k_n-}_t - X^{\tau^k_n}_t|^{2p} \leq C(p)\Big\{\sup_{t\leq q}\Big|\int_{\varrho^n_t}^t f(X_s)\,dW_s + \int_{\varrho^n_t}^t g(X_s)\,ds\Big|^{2p}$$

$$+\sup_{t\leq q}\Big|\int_{\varrho^n_t}^t f(X_s)\,dW_s + \int_{\varrho^n_t}^t g(X_s)\,ds\Big|^{p}(V(K^{n,\tau^k_n-})^p_q + \mathrm{Var}(K^{\tau^k_n})^p_q)$$

$$+\Big(\frac{1}{r_0}\int_0^{q-}|X^{n,\tau^k_n-}_s - X^{\tau^k_n}_s|^2\,(dV(K^{n,\tau^k_n-})_s + \mathrm{Var}(K^{\tau^k_n})_s)\Big)^p\Big\}.$$

By the definition of τ_n^k, $\mathrm{Var}(K^{n,\tau_n^k-}) \le k$ and $\mathrm{Var}(K^{\tau_n^k}) \le k$. Therefore by using Lemma C.2 we have

$$\sup_{t\le q} |X_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p} \le C(p,k)\Big\{ \sup_{t\le q} \Big| \int_{\varrho_t^n}^t f(X_s)\,dW_s + \int_{\varrho_t^n}^t g(X_s)\,ds \Big|^{2p} + \sup_{t\le q} \Big| \int_{\varrho_t^n}^t f(X_s)\,dW_s + \int_{\varrho_t^n}^t g(X_s)\,ds \Big|^{p} \Big\}. \tag{10.21}$$

Because of Lemma 10.5 and (10.21),

$$E \sup_{t\le q} |X_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \quad \varepsilon > 0,\ q\in\mathbb{R}^+,\ k\in\mathbb{N},\ p\in\mathbb{N}.$$

Next, in view of (10.20)

$$E \sup_{t\le q} |\overline{X}_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p} = \mathcal{O}(1/n^{p/2-\varepsilon}), \quad \varepsilon > 0,\ q\in\mathbb{R}^+,\ k\in\mathbb{N},\ p\in\mathbb{N}.$$

Hence

$$\sup_{p'\le p} n^{p'/2-\varepsilon} E \sup_{t\le q} |\overline{X}_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p'} \to 0 \tag{10.22}$$

for every $\varepsilon > 0$, $k\in\mathbb{N}$, $q\in\mathbb{R}^+$, $p\in\mathbb{N}$. Finally, the result follows from (10.19), (10.22) and Lemma 10.6. ∎

From the above theorem we can deduce easily

COROLLARY 10.13. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.3). *If the conditions* (A) *and* (B) *are satisfied then*

(i) *there exists a sequence $\{\tau_n\}$ of stopping times with $\tau_n \to_{\mathcal{P}} +\infty$ such that*

$$n^{1/4-\varepsilon} \sup_{t\le q} |\overline{X}_t^{n,\tau_n-} - X_t^{\tau_n}| \to 0 \quad \mathcal{P}\text{-a.s.},\ k\in\mathbb{N},\ \varepsilon>0,\ q\in\mathbb{R}^+,$$

(ii) $n^{1/4-\varepsilon} \sup_{t\le q} |\overline{X}_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \quad k\in\mathbb{N},\ \varepsilon>0,\ q\in\mathbb{R}^+.$ ∎

THEOREM 10.14. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.3). *If the conditions* (A) *and* (B) *are satisfied then there exists an array $\{\{\tau_n^k\}\}$ of stopping times such that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathcal{P}(\tau_n^k \le q) = 0, \quad q\in\mathbb{R}^+,$$

and

$$E \sup_{t\le q} |\widehat{X}_t^{n,\tau_n^k-} - X_t^{\tau_n^k}|^{2p} = \mathcal{O}(1/n^p), \quad q\in\mathbb{R}^+,\ k\in\mathbb{N}.$$

Proof. It is sufficient to put $\tau_n^k = \inf\{t : \mathrm{Var}(\widehat{K}^n)_t + \mathrm{Var}(K)_t > k\}$ and use the arguments from the proof of Theorem 10.12. ∎

Theorem 10.14 and Lemma 10.6 immediately lead to the following

COROLLARY 10.15. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.3). *If the conditions* (A) *and* (B) *are satisfied then* (*i*) *there exists a sequence* $\{\tau_n\}$ *of stopping times with* $\tau_n \to_{\mathcal{P}} +\infty$ *such that*

$$n^{1/2-\varepsilon} \sup_{t\le q} |\widehat{X}_t^{n,\tau_n-} - X_t^{\tau_n}| \to 0 \qquad \mathcal{P}\text{-a.s.},\ k\in\mathbb{N},\ \varepsilon>0,\ q\in\mathbb{R}^+,$$

(ii) $n^{1/2-\varepsilon} \sup_{t\le q} |\widehat{X}_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \qquad k\in\mathbb{N},\ \varepsilon>0,\ q\in\mathbb{R}^+.$ ■

10.4. Some generalizations. In the present section we discuss more general sequences of partitions on $\mathbb{R}^+$. Let us consider an array $\{\{t_{nk}\}\}$ of nonnegative numbers such that in the nth row the sequence $T_n = \{t_{nk}\}$ forms a partition of $\mathbb{R}^+$ with $0 = t_{n0} < t_{n1} < \ldots$, $\lim_{k\to\infty} t_{nk} = +\infty$ and

$$\max(t_{nk} - t_{n,k-1}) \le 1/n^\alpha, \qquad n\in\mathbb{N},\ \alpha>0. \tag{10.23}$$

In the sequel we restrict our attention to the Euler scheme for the SDE (10.1). However, by using more general sequences of partitions of $\mathbb{R}^+$ satisfying (10.23), one can obtain analogues of our results for the Euler–Peano scheme, too.

THEOREM 10.16. *Assume* f, g *satisfy* (10.2) *and* $\{T_n\}$ *satisfies* (10.23).

(i) *If* D *is a convex polyhedron then*

$$\sum_{n=1}^\infty n^{p(\alpha-\varepsilon)} E \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} < +\infty, \qquad \varepsilon>0,\ q\in\mathbb{R}^+, p>1/\varepsilon.$$

(ii) *If* D *is a convex domain satisfying the condition* (β) *then*

$$\sum_{n=1}^\infty n^{p(\alpha-\varepsilon)/2} E \sup_{t\le q} |\overline{X}_t^n - X_t|^{2p} < +\infty, \qquad \varepsilon>0, q\in\mathbb{R}^+, p>2/\varepsilon.$$

(iii) *If* D *is any convex domain then there exists a sequence* $\{\tau_n\}$ *of stopping times such that* $\tau_n \to_{\mathcal{P}} +\infty$ *and*

$$\sum_{n=1}^\infty n^{p(\alpha-\varepsilon)/2} E \sup_{t\le q} |\overline{X}_t^{n,\tau_n-} - X_t^{\tau_n}|^{2p} < +\infty, \qquad \varepsilon>0,\ q\in\mathbb{R}^+,\ p>2/\varepsilon.$$

Proof. (i) Due to Proposition 10.3 (with $\tau_n = +\infty, n\in\mathbb{N}$) it is sufficient to prove that

$$\sum_{n=1}^\infty n^{p(\alpha-\varepsilon)} E \sup_{t\le q} |X_t^n - X_t|^{2p} < +\infty, \qquad q\in\mathbb{R}^+.$$

By (10.6) in order to prove (i) it is enough to show that for $p > 1/\varepsilon$,

$$\sum_{n=1}^\infty n^{p(\alpha-\varepsilon)} E \sup_{t\le q} \Big| \int_{\varrho_t^n}^t f(X_s)\, dW_s + \int_{\varrho_t^n}^t g(X_s)\, ds \Big|^{2p} < +\infty. \tag{10.24}$$

Without loss of generality we may and will assume that $d=1$, $g=0$. Then $\int_0^t f(X_s)\, dW_s = B\circ A_t$ for some standard (possibly stopped) Wiener process B and A defined by $A_t = \int_0^t f(X_s)^2\, ds$. Hence $A_t - A_s \le L^2(t-s)$, $A_q \le L^2 q$ and

$$\sup_{t\le q} \Big| \int_{\varrho_t^n}^t f(X_s)\, dW_s \Big| \le \omega_{B\circ A}(n^{-\alpha}, q) \le \omega_B(n^{-\alpha}L^2, L^2 q).$$

Let p be so large that $p\varepsilon > 1$. Set $\delta = (p\varepsilon - 1)/2$. Since

$$\mathcal{P}(\omega_B(n^{-\alpha}L^2, L^2 q)^{2p} \geq y) \leq \frac{16L\sqrt{n^{-\alpha}}(\frac{q+2n^{-\alpha}}{n^{-\alpha}})}{\sqrt{2\pi y^{1/p}}} \exp\Big(-\frac{y^{1/p}}{8n^{-\alpha}L^2}\Big)$$

(see e.g. [Ute81]) we can write

$$\begin{aligned}\sum_{n=1}^{\infty} & n^{p(\alpha-\varepsilon)} E\omega_B(n^{-\alpha}L^2, L^2 q)^{2p} \\ & \leq \sum_{n=1}^{\infty} n^{-p\varepsilon+\delta} + \sum_{n=1}^{\infty} n^{p(\alpha-\varepsilon)} \int_{n^{-p\alpha+\delta}}^{\infty} \mathcal{P}(\omega_B(n^{-\alpha}L^2, L^2 q)^{2p} > y)\, dy \\ & \leq \sum_{n=1}^{\infty} n^{-p\varepsilon+\delta} + C(p,L,q) \sum_{n=1}^{\infty} n^{\alpha-p\varepsilon} \int_{n^{\delta/p}}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\Big(-\frac{y}{8L^2}\Big) y^{p-1}\, dy \\ & \leq \sum_{n=1}^{\infty} n^{-p\varepsilon+\delta} + C(p,L,q,\pi) \sum_{n=1}^{\infty} P(n) \exp(-n^{\delta/p}) \leq +\infty,\end{aligned}$$

where $P(n)$ is some polynomial in $n \in \mathbb{N}$.

(ii) For a general convex domain we have the following inequality [Tan79, Lemma 2.2]:

$$\begin{aligned}\sup_{t\leq q} |X_t^n - X_t|^{2p} \leq C(p)\Big\{ & \sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds \Big|^{2p} \\ & + \sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds \Big|^{p} (\mathrm{Var}(K^n)_q^p + \mathrm{Var}(K)_q^p)\Big\}\end{aligned}$$

instead of (10.6). Therefore by Schwarz's inequality we have

$$\begin{aligned}E\sup_{t\leq q} |X_t^n - X_t|^{2p} \leq C(p)\Big\{ & E\sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds \Big|^{2p} \\ & + \Big(E\sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds \Big|^{2p}\Big)^{1/2} \\ & \times (E\,\mathrm{Var}(K^n)_q^{2p} + E\,\mathrm{Var}(K)_q^{2p})^{1/2}\Big\}.\end{aligned}$$

On the other hand, from the proof of (10.24) it is clear that for $p > 2/\varepsilon$,

$$\sum_{n=1}^{\infty} n^{p(\alpha-\varepsilon)/2} \Big(E\sup_{t\leq q} \Big| \int_{\varrho_t^n}^{t} f(X_s)\, dW_s + \int_{\varrho_t^n}^{t} g(X_s)\, ds \Big|^{2p}\Big)^{1/2} < +\infty. \tag{10.25}$$

Therefore to finish the proof in view of Proposition 10.3 (as before with $\tau_n = +\infty, n \in \mathbb{N}$) it is enough to observe that

$$E\,\mathrm{Var}(K)_q^{2p} < +\infty, \qquad \sup_n E\,\mathrm{Var}(K^n)_q^{2p} < +\infty. \tag{10.26}$$

(iii) It is sufficient to use (10.25) and to follow the proof of Theorem 10.4(iii). ∎

Theorem 10.16 immediately leads to the following

COROLLARY 10.17. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.23).

(i) *If D is a convex polyhedron then*

$$\begin{cases} n^{(\alpha-\varepsilon)/2}\sup_{t\le q}|\overline{X}^n_t - X_t| \to 0 \quad \mathcal{P}\text{-}a.s., & \varepsilon>0,\ q\in\mathbb{R}^+,\\ n^{p(\alpha-\varepsilon)}E\sup_{t\le q}|\overline{X}^n_t - X_t|^{2p}\to 0, & \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}.\end{cases}$$

(ii) *If D is a convex domain satisfying the condition (β) then*

$$\begin{cases} n^{(\alpha-\varepsilon)/4}\sup_{t\le q}|\overline{X}^n_t - X_t| \to 0 \quad \mathcal{P}\text{-}a.s., & \varepsilon>0,\ q\in\mathbb{R}^+,\\ n^{p(\alpha-\varepsilon)/2}E\sup_{t\le q}|\overline{X}^n_t - X_t|^{2p}\to 0, & \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}.\end{cases}$$

(iii) *If D is any convex domain then there exists a sequence $\{\tau_n\}$ of stopping times such that $\tau_n \to_{\mathcal{P}} +\infty$ and*

$$\begin{cases} n^{(\alpha-\varepsilon)/4}\sup_{t\le q}|\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t| \to 0 \quad \mathcal{P}\text{-}a.s., & \varepsilon>0,\ q\in\mathbb{R}^+,\\ n^{p(\alpha-\varepsilon)/2}E\sup_{t\le q}|\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t|^{2p}\to 0, & \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}. \ \blacksquare\end{cases}$$

Now we assume that D satisfies the conditions (A) and (B). In this case, if $r_0 < +\infty$, the processes $|X_t - X_0|^{2p}$, $|X^n_t - X_0|^{2p}$ and $|\overline{X}^n_t - X_0|^{2p}$ are not necessarily integrable.

THEOREM 10.18. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.23). *If the conditions* (A) *and* (B) *are fulfilled then there exists a sequence $\{\tau_n\}$ of stopping times, $\tau_n \to_{\mathcal{P}} \infty$, such that*

$$\sum_{n=1}^{\infty} n^{p(\alpha-\varepsilon)/2}E\sup_{t\le q}|\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t|^{2p} < +\infty, \qquad \varepsilon>0,\ q\in\mathbb{R}^+,\ p>2/\varepsilon.$$

P r o o f. We follow the proof of Theorem 10.12. $\blacksquare$

From the above theorem we can deduce

COROLLARY 10.19. *Assume f, g satisfy* (10.2) *and $\{T_n\}$ satisfies* (10.23). *If the conditions* (A) *and* (B) *are fulfilled then there exists a sequence $\{\tau_n\}$ of stopping times, $\tau_n \to_{\mathcal{P}} +\infty$, such that*

$$\begin{cases} n^{(\alpha-\varepsilon)/4}\sup_{t\le q}|\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t| \to 0 \quad \mathcal{P}\text{-}a.s., & \varepsilon>0,\ q\in\mathbb{R}^+,\\ n^{p(\alpha-\varepsilon)/2}E\sup_{t\le q}|\overline{X}^{n,\tau_n-}_t - X^{\tau_n}_t|^{2p}\to 0, & \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}. \ \blacksquare\end{cases}$$

Appendix A. Convergence in the Skorokhod topology J_1

In this section we consider in detail the basic properties of the convergence of elements of $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ in the Skorokhod topology J_1. It is well known that there exists a distance δ^d on $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ under which $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ is a Polish space (e.g. [Bil68, JaSh87]). We say that $x^n \to x$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ if and only if $\delta^d(x^n,x)\to 0$. The equivalences given below can be easily obtained from [Ald78b, Proposition 29.2] (see also [JaSh87, Chapter VI.2]).

LEMMA A.1. *Assume that $x^n \to x$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ and $y^n\to y$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. Then the following four conditions are equivalent*:

(i) $(x^n,y^n)\to(x,y)$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$,

(ii) *for every* $t \in \mathbb{R}^+$ *there exists a sequence* $\{t_n\}$, $t_n \to t$ *such that* $x^n_{t_n} \to x_t$, $x^n_{t_n-} \to x_{t-}$, $y^n_{t_n} \to y_t$ *and* $y^n_{t_n-} \to y_{t-}$,

(iii) $x^n + y^n \to x + y$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$,

(iv) $|x^n| + |y^n| \to |x| + |y|$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R})$. ■

Let us denote by x^t the element x stopped at $t \in \mathbb{R}^+$, i.e. $x^t_s = x_{t\wedge s}$, $s \in \mathbb{R}^+$. By Lemma A.1 we can obtain the following result.

LEMMA A.2. *Suppose that* $(x^n, y^n) \to (x, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. *Then*:

(i) *for every continuity point* $q \in \mathbb{R}^+$ *of* x *and* y *and for every sequence* $\{q_n\}$ *with* $q_n \to q$, $(x^{n,q_n}, y^n) \to (x^q, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and* $\sup_{t\le q_n} |x^n_t - y^n_t| \to \sup_{t\le q} |x_t - y_t|$,

(ii) *for every discontinuity point* $q \in \mathbb{R}^+$ *of* x *or* y *and for all sequences* $\{u_n\}$, $\{w_n\}$ *with* $u_n \to q$, $w_n \to q$, $u_n < q_n \le w_n$, $n \in \mathbb{N}$, *where* $\{q_n\}$, $q_n \to q$ *is defined in Lemma* A1(ii), *we have the convergences* $(x^{n,w_n}, y^n) \to (x^q, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, $\sup_{t\le w_n} |x^n_t - y^n_t| \to \sup_{t\le q} |x_t - y_t|$, *and* $\sup_{t\le u_n} |x^n_t - y^n_t| \to \sup_{t\le q} |x_{t-} - y_{t-}|$. ■

LEMMA A.3. *Suppose that* $(x^n, y^n) \to (x, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and* $\sup_{t\le q} |y^n_t - y_t| \to 0, q \in \mathbb{R}^+$. *If the implication*

$$\Delta x_t \ne 0 \Rightarrow \Delta y \ne 0$$

is true, *then*

$$\sup_{t\le q} |x^n_t - x_t| \to 0, \quad q \in \mathbb{R}^+.$$

P r o o f. Observe that it is enough to show that

$$x^n_{v_n} - x_{v_n} \to 0 \tag{A.1}$$

for all sequences $\{v_n\}$ with $v_n \to t$. First assume that t is a continuity point of x. Then by the nature of Skorokhod convergence $x^n_{v_n} \to x_t$. Since in this case $x_{v_n} \to x_t$, the property (A.1) is obvious. Now, suppose that $\Delta x_t \ne 0$. Then by the hypothesis $\Delta y_t \ne 0$, too. Let $\{v_{n'}\}$ be a subsequence of $\{v_n\}$ satisfying $v_{n'} \ge t_{n'}$ and let $\{v_{n''}\}$ be another subsequence for which $v_{n''} < t_{n''}$, where $\{t_n\}$ is defined in Lemma A.1(ii). Then $y^{n'}_{v_{n'}} \to y_t$ and $y^{n''}_{v_{n''}} \to y_{t-}$. Since we have in fact the uniform convergence of y_n to y, it follows that $y^n_{v_n} - y_{v_n} \to 0$. Combining the above properties we get $y_{v_{n'}} \to y_t$, $y_{v_{n'}} \to y_{t-}$. Hence for almost all n' and n'',

$$v_{n'} \ge t \quad \text{and} \quad v_{n''} < t. \tag{A.2}$$

On the other hand, by (A.2), $x_{v_{n'}} \to x_t$ and $x_{v_{n''}} \to x_{t-}$. Now, arguing exactly in the same way as for the sequence $\{y_n\}$, we obtain $x^{n'}_{v_{n'}} \to x_t$ and $x^{n''}_{v_{n''}} \to x_{t-}$. Therefore the conclusion (A.2) follows. ■

COROLLARY A.4. *Suppose that* $(x^n, y^n, z^n) \to (x, y, z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$ *and that* $\sup_{t\le q} |y^n_t - y_t| \to 0, q \in \mathbb{R}^+$ *and* $\sup_{t\le q} |z^n_t - z_t| \to 0$, $q \in \mathbb{R}^+$. *If the implication*

$$\Delta x_t \ne 0 \Rightarrow (\Delta y \ne 0 \text{ or } \Delta z_t \ne 0)$$

is true, *then*

$$\sup_{t\le q} |x^n_t - x_t| \to 0, \quad q \in \mathbb{R}^+. \ ■$$

LEMMA A.5. *Suppose that* $(x^n, y^n) \to (x, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. *If we denote* $t_N^n = \inf\{t : |x_t^n| \geq N$ *or* $|x_{t-}^n| \geq N\}$, $t_N = \inf\{t : |x_t| \geq N$ *or* $|x_{t-}| \geq N\}$ *and* $t_{N+}^n = \inf\{t : |x_t^n| > N$ *or* $|x_{t-}^n| > N\}$, $t_{N+} = \inf\{t : |x_t| > N$ *or* $|x_{t-}| > N\}$, *for some* $N > 0$, $n \in \mathbb{N}$, *then under the assumptions* $t_N = t_{N+}$, $|x_{t_N-}| \neq N$ *we have*:

(i) $(x^{n,t_N^n}, y^n) \to (x^{t_N}, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$,
(ii) $(x^{n,t_{N+}^n}, y^n) \to (x^{t_N}, y)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$.

Proof. We start with the proof of (i). For simplicity we write $x^{n,N}$ instead of x^{n,t_N^n}. By Lemma A.2 it is sufficient to show that

(A.3) $t_N^n \to t_N$,
(A.4) if t_N is a discontinuity point of x and $\{t_n\}$ is defined in Lemma A.2(ii), then for almost all n, $t_N^n \geq t_n$.

Let us fix $\varepsilon > 0$ and let s_1, s_2 be two continuity points of x for which $s_1 < t_N < s_2$ and $s_2 - s_1 < \varepsilon$. By hypothesis and by Lemma A.2,

$$\sup_{t \leq s_1} |x_t^n| \to \sup_{t \leq s_1} |x_t| = N_1 < N \leq \sup_{t \leq t_N} |x_t| = N_0$$

and

$$\sup_{t \leq s_1} |x_t^n| \to \sup_{t \leq s_2} |x_t| = N_2 > N_0 \geq N.$$

Assume that $t_N^{n'} < s_1$ for countably many n'. Then we have the following contradiction:

$$N \leq \sup_{t \leq t_N^{n'}} |x_t^{n'}| \leq \sup_{t \leq s_1} |x_t^{n'}| \to N_1 < N.$$

As a consequence $s_1 \leq t_N^n$ for almost all $n \in \mathbb{N}$. On the other hand, suppose that $t_N^{n''} > s_2$ for countably many n''. Then

$$N \geq \sup_{t \leq t_N^{n''}} |x_{t-}^{n''}| \geq \sup_{t \leq s_2} |x_t^{n''}| \to N_2 > N.$$

Therefore, $s_1 \leq t_N^n \leq s_2$ for almost all $n \in \mathbb{N}$. Since we can take $\varepsilon > 0$ arbitrarily small, the conclusion (A.3) follows. Now, assume that t_N is a discontinuity point of x and for countably many n', $t_N^{n'} < t^{n'}$. By Lemma A.2 it is clear that

$$N \leq \sup_{t \leq t_N^{n'}} |x_t^{n'}| \to \sup_{t \leq t_N} |x_{t-}| < N.$$

Hence (A.4) is satisfied. It is easy to see that for $\{t_{N+}^n\}$ the proof is exactly the same as for $\{t_N^n\}$. ■

COROLLARY A.6. *Suppose that* $(x^n, y^n, z^n) \to (x, y, z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$. *Define* $t_N^n = \inf\{t : |x_t^n| + |y_t^n| \geq N$ *or* $|x_{t-}^n| + |y_{t-}^n| \geq N\}$, *and* $t_{N+}^n = \inf\{t : |x_t^n| + |y_t^n| > N$ *or* $|x_{t-}^n| + |y_t^n| > N\}$, $k_t^n = \mathbf{1}_{\{|x_t^n| + |y_t^n| < N\}}$ *and* t_N, t_{N+}, k_t *for* x *and* y, *respectively. Then under the assumptions* $t_N = t_{N+}$, $|x_{t_N-}| \neq N$ *we have*

$$(x^{n,N}, k^{n,N}, y^{n,N}, z^n) \to (x^N, k^N, y^N, z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}),$$

and

$$(x^{n,N+}, k^{n,N}, y^{n,N+}, z^n) \to (x^N, k^N, y^N, z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}),$$

where the elements x^n, y^n, k^n *are stopped at* t_N^n *and* t_{N+}^n, *respectively.* ■

Appendix B. (UT) and convergence of stochastic integrals

In this chapter we present earlier results obtained under (UT) on the convergence of semimartingales and stochastic integrals proved by Stricker and Jakubowski, Mémin and Pages.

THEOREM B.1 (see e.g. [Str85]). *Assume that* $\{Z^n\}$ *is a sequence of semimartingales satisfying* (UT).

(i) *For every* $q \in \mathbb{R}^+$ *the families* $\{\sup_{t\le q}|Z_t^n|\}$, $\{[Z^n]_q\}$ *of random variables are tight in* $\mathbb{R}$.

(ii) *If additionally* $Z^n \to_{\mathcal{D}} Z$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ *then* Z *is also a semimartingale adapted to its natural filtration.* ∎

THEOREM B.2 (see e.g. [JMP89, Théorème 2.6]). *Assume that* $\{Z^n\}$ *is a sequence of semimartingales satisfying* (UT). *If* $\{Y^n\}$, $\{U^n\}$ *are two sequences of* $(\mathcal{F}_t^n)$ *adapted processes such that*

$$(Y^n, U^n, Z^n) \underset{\mathcal{D}}{\to} (Y, U, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d}),$$

then Z *is a semimartingale adapted to the natural filtration of* (Y,U,Z) *and*

$$\Big(Y^n, U^n, Z^n, \int_0^{\cdot} U^n_{s-}\,dZ^n_s\Big) \underset{\mathcal{D}}{\to} \Big(Y, U, Z, \int_0^{\cdot} U_{s-}\,dZ_s\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d+1}). \ \blacksquare$$

COROLLARY B.3 (see e.g. [JMP89, Proposition 2.9]). *Let* $\{f^n\}$, $\{g^n\} \subset \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ *and for every* $q\in\mathbb{R}^+, \sup_n \mathrm{Var}(g^n)_q < +\infty$. *If* $(f^n,g^n)\to(f,g)$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$ *then*

$$\Big(f^n, g^n, \int_0^{\cdot} f^n_s\,dg^n_s\Big) \to \Big(f, g, \int_0^{\cdot} f_s\,dg_s\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d+1}). \ \blacksquare$$

Appendix C. Gronwall's lemma

LEMMA C.1. *Let* Y^1, Y^2 *be two increasing processes with trajectories in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R})$ *such that* $EY^1_\infty < +\infty$, $Y^2_0 = 0$, $Y^2_\infty \le a_1$ *for some constant* a_1. *If one of the following two conditions is satisfied*:

(i) *for every stopping time* τ,

$$EY^1_\tau \le a_2 + E\int_0^{\tau-} Y^1_s\,dY^2_s,$$

(ii) *for every stopping time* τ,

$$EY^1_{\tau-} \le a_2 + E\int_0^{\tau-} Y^1_{s-}\,dY^2_s,$$

for some constant $a_2>0$, *then* $EY^1_\infty \le a_2 e^{a_1}$.

Proof. (i) We provide the proof of part (i) for the sake of completeness only. We define $\tau_t = \inf\{s : Y_s^2 \ge t\}$. Then by [DeMe80, VI, Theorem 55] we have

$$\begin{aligned}EY^1_{\tau_t} &\le E\int_0^{\tau_t-} Y_s^1\,dY_s^2 + a_2 = E\int_0^{\infty} Y_s^1 \mathbf{1}_{\{s<\tau_t\}}\,dY_s^2 + a_2\\ &\le E\int_0^{\infty} Y^1_{\tau_u}\mathbf{1}_{\{\tau_u<\tau_t,\tau_u<+\infty\}}\,du + a_2 \le \int_0^t EY^1_{\tau_u}\,du + a_2\end{aligned}$$

and it is sufficient to use the classical Gronwall lemma for $f(t) = EY^1_{\tau_t}$. Hence $EY^1_{\tau_t} \le a_2e^t$ and assuming $t \uparrow a_1$ we obtain our claim.

(ii) See [Mac87, Lemma 2]. ■

From the proof of part (i) we deduce that the result is true for every increasing and adapted process Y^1 such that $EY^1_\infty < +\infty$ (i.e. Y^1 need not have trajectories in $\mathbb{D}(\mathbb{R}^+,\mathbb{R})$).

LEMMA C.2 ([Sai87, Lemma 2.2]). *Let $k \in \mathbb{D}(\mathbb{R}^+,\mathbb{R})$ be a nondecreasing function with $k_0 = 0$, $k_q \le a_1$ and let h be a nonnegative Borel measurable function on $\mathbb{R}^+$. If*

$$h_q \le a_2 + \int_0^{q-} h_s\,dk_s, \qquad q \in \mathbb{R}^+,$$

for some constant $a_2 > 0$ then $h_q \le a_2e^{a_1}$.

Proof. Define $Y_t^1 = \sup_{s\le t} h_s$, $Y_t^2 = k_t$, $t \in \mathbb{R}^+$. By Lemma C.1 the conclusion of Lemma C.2 follows. ■

By using the methods from the proof of Lemma C.1 we can easily obtain the following:

LEMMA C.3. *Let Y^1, Y^2 be two increasing processes with trajectories in $\mathbb{D}(\mathbb{R}^+,\mathbb{R})$ such that $EY^1_\infty < +\infty$, $Y^2_0 = 0$, $Y^2_\infty \le a_1$ for some constant a_1. Assume that $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing and concave function such that $\varrho(0) = 0$ and*

$$\int_{0^+} \frac{du}{\varrho(u)} = +\infty. \tag{C.1}$$

Define the function $\overline{\varrho} : (0,+\infty)\times\mathbb{R}^+ \to \mathbb{R}^+$ as the solution of

$$\int_x^{\overline{\varrho}(x,y)} \frac{du}{\varrho(u)} = y, \qquad (x,y) \in (0,+\infty)\times\mathbb{R}^+.$$

If one of the following two conditions is satisfied:

(i) *for every stopping time τ,*

$$EY^1_\tau \le a_2 + E\int_0^{\tau-} \varrho(Y_s^1)\,dY_s^2,$$

(ii) *for every stopping time τ,*

$$EY^1_{\tau-} \le a_2 + E\int_0^{\tau-} \varrho(Y^1_{s-})\,dY_s^2,$$

for some constant $a_2 > 0$, then $EY^1_\infty \le \overline{\varrho}(a_2,a_1)$. ■

R e m a r k C.4. It is clear by (C.1) and by the definition of $\overline{\varrho}$ that

$$\lim_{x\to 0}\overline{\varrho}(x,y)\to 0,\qquad y\in(0,+\infty).$$

Note that if $\varrho(u)=u$ then $\overline{\varrho}(x,y)=xe^y$, i.e. Lemma C.1 can be deduced from Lemma C.3.

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