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ANDRÉ-QUILLEN COHOMOLOGY FOR COMMUTATIVE COALGEBRAS

BY

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The aim of this paper is to give a cohomology theory for commutative coalgebras dual to the theory of André and Quillen for commutative algebras (see [1] and [8]).

For any commutative coalgebra C, we consider two categories: the category C-Comod of all left comodules over C and the category Coalg-C of all coalgebras over C. Fundamental elementary properties of those categories are contained in Section 1. In the category Coalg-C we distinguish a full subcategory of free C-coalgebras consisting of all coalgebras of the form $k[X]^\circ \otimes C$, where X is an arbitrary set (see Section 3).

In Section 5 we show that for any C-coalgebra D there exists a cosimplicial free C-resolution, i. e., a cosimplicial object $X^\cdot = \{X^n\}$ in Coalg-C such that all X^n are free and the cochain complex associated with $X^\cdot$ in C-Comod is acyclic. The cohomologies $H^n(C, D, M)$ and $\bar{H}^n(C, D, M)$ of D with coefficients in a D-comodule M are defined as cohomology objects of the cochain complex $\mathrm{Coder}_C(M, X^\cdot)$ and $\mathrm{Codiff}_C(M, X^\cdot)$, respectively, where $\mathrm{Coder}_C(M, \cdot)$ and $\mathrm{Codiff}_C(M, \cdot)$ are certain functors from the category $(D$, Coalg-$C)$ to the category of vector spaces. Basic properties of both these functors, generalizing certain results of Lee [6], are presented in Section 2.

One of the main results of this paper is Theorem 5.5 which shows that any sequence $B \to C \to D$ of coalgebra morphisms induces appropriate long exact sequences of cohomologies H^n and $\bar{H}^n$. To prove this theorem we show that H^n and $\bar{H}^n$ are cohomologies of a certain triple (5.3) with coefficients in Coder and Codiff, respectively.

Following André [1], we define cohomologies $A^n(C, D, M)$ and $\bar{A}^n(C, D, M)$ using objects $k[X_1, \ldots, X_n]^\circ \otimes C$ as models. By Lemma 5.7, it follows that $A^n = H^n$.

In Section 3 a functor $\mathcal{S}_C$: Comod-$C \to$ Coalg-C is constructed, which is right adjoint to the appropriate forgetting functor. This pair of functors induces another triple, cohomologies of which will be considered in Section 5

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1. Coalgebras and comodules. Throughout this paper k will denote a fixed field. Symbols $\otimes$ and Hom (without subscripts) mean that these functors are taken over k. The category of all vector spaces over k will be denoted by Mod-k.

A k-coalgebra is a vector space C together with two k-linear maps $\Delta_C: C \to C \otimes C$ and $\varepsilon_C: C \to k$ satisfying

$$(I \otimes \Delta_C)\Delta_C = (\Delta_C \otimes I)\Delta_C \quad \text{and} \quad (\varepsilon_C \otimes I)\Delta_C = I = (I \otimes \varepsilon_C)\Delta_C.$$

Δ_C is called *comultiplication*, and ε_C is called *counit*. If no confusion can arise, the subscripts C will be omitted.

A k-linear map $f: C \to D$ is a *morphism of k-coalgebras* if $\Delta_D f = (f \otimes f)\Delta_C$. The set of all coalgebra morphisms from C to D will be denoted by $\mathrm{Coalg}(C, D)$.

A k-coalgebra is *commutative* if $t\Delta = \Delta$, where

$$t: \ V \otimes W \to W \otimes V$$

is a twisting morphism defined by $t(v \otimes w) = w \otimes v$. If C and D are k-coalgebras, then $C \otimes D$ is also a k-coalgebra with morphisms $(I \otimes t \otimes I)(\Delta_C \otimes \Delta_D)$ and $\varepsilon_C \otimes \varepsilon_D$. It is easy to verify that C is commutative iff Δ_C is a morphism of k-coalgebras.

An important role in our considerations will be played by the functors

$$* : \text{Coalg-}k \to \text{Alg-}k \quad \text{and} \quad \circ : \text{Alg-}k \to \text{Coalg-}k.$$

For any k-coalgebra C, $C^* = \mathrm{Hom}(C, k)$ is an algebra over k together with structural maps $\Delta^*|_{C^* \otimes C^*}$ and ε^* (see [9], 1.1.1). If A is a k-algebra together with maps $\mu: A \otimes A \to A$ and $\eta: k \to A$, then A° is a subspace of A^* consisting of all linear maps $f: A \to k$ such that $\ker f$ contains an ideal of finite codimension. By Section 6 in [9], the natural map $A^\circ \otimes A^\circ \to (A \otimes A)^\circ$ is an isomorphism and the structural maps μ° and η° are defined as restrictions of μ^* and η^*, respectively, to A°.

$*$ is a right adjoint functor to $\circ$, that is there exists a functorial isomorphism

$$\mathrm{Coalg}(C, A^\circ) \approx \mathrm{Alg}(A, C^*)$$

for any k-coalgebra C and any k-algebra A (see [9], Section 6, p. 118).

If C is a k-coalgebra, a *left C-comodule* is a vector space M together with a linear map $\varrho: M \to C \otimes M$ such that

$$I = (\varepsilon \otimes I)\varrho \quad \text{and} \quad (\Delta \otimes I)\varrho = (I \otimes \varrho)\varrho.$$

The map ϱ is called *comultiplication*. Similarly one can define a *right C-comodule*.

A morphism $f: M \to N$ of C-comodules M and N is a linear map satisfying $(I \otimes f)\varrho = \varrho f$. The vector space of all C-comodule morphisms from M to N will be denoted by $\mathrm{Hom}_C(M, N)$.

The category of all left (right) C-comodules is denoted by C-Comod (Comod-C). It is easy to prove that this is an abelian category with arbitrary direct sums and products. The product in C-Comod we denote by $\tilde{\prod}$. If $M_a \in C$-Comod for $a \in I$, then $\tilde{\prod}_I M_a$ is a maximal rational C^*-submodule of $\prod M_a$ (see [9], p. 37).

Let M be a right C-comodule and N a left C-comodule. Following Milnor and Moore [7], we define *cotensor product* $M \square_C N$ as a kernel of the map

$$\varrho \otimes I - I \otimes \varrho : M \otimes N \to M \otimes C \otimes N.$$

If $f \in \mathrm{Hom}(M, M')$ and $g \in \mathrm{Hom}(N, N')$, then the linear map

$$f \square g : M \square_C N \to M' \square_C N'$$

is a restriction of $f \otimes g$ to $M \square_C N$. One can readily check that $\square_C$ is a left exact functor commuting with arbitrary direct sums. Now, $M \square_C C \approx M$, since the following sequence is exact:

$$0 \to M \xrightarrow{\varrho} M \otimes C \xrightarrow{\varrho \otimes I - I \otimes \Delta} M \otimes C \otimes C.$$

Since the category C-Comod has sufficiently many injective objects, one can define derived functors of the functor $\square_C$. They will be denoted by Cotor_n^C. Clearly, $\mathrm{Cotor}_0^C = \square_C$.

If C is commutative, then any left C-comodule is a right one with $t\varrho$ as the structural morphism. If M has both left and right C-comodule structures, then $M \square_C N$ is a left C-comodule with a comultiplication defined by the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \square_C N \arrow[r] \arrow[d] & M \otimes N \arrow[r] \arrow[d, "\varrho' \otimes I"'] & M \otimes C \otimes N \arrow[d, "\varrho' \otimes I \otimes I"'] \\
0 \arrow[r] & C \otimes M \square_C N \arrow[r] & C \otimes M \otimes N \arrow[r] & C \otimes M \otimes C \otimes N
\end{tikzcd}$$

Throughout this paper we shall consider only commutative coalgebras. Therefore, the word "commutative" will be omitted, and "coalgebra" and "comodule" will mean "commutative coalgebra" and "left comodule", respectively.

1.1. Definition. Let C and D be k-coalgebras. D is called a *C-coalgebra with a structural morphism* $\bar{\varepsilon}$ if $\bar{\varepsilon}: D \to C$ is a k-coalgebra morphism. A k-coalgebra morphism $f: D \to D'$ is called a *morphism of C-coalgebras* if $\bar{\varepsilon}' f = \bar{\varepsilon}$. The set of all C-coalgebra morphisms from D to D' will be denoted by Coalg-$C(D, D')$.

One can check that if M is a D-comodule and D is a C-coalgebra, then M is a C-comodule with $(\bar{\varepsilon}\otimes I)\varrho$ as the comultiplication. Clearly, $\varrho M\subset D\square_C M$ and ϱ is a C-coalgebra map. Hence $\Delta D\subset D\square_C D$ and Δ is a C-comodule morphism. Theorefore, we can consider comultiplications as the maps

$$\bar{\Delta}: D\to D\square_C D \quad \text{and} \quad \bar{\varrho}: M\to D\square_C M.$$

The category of all C-coalgebras will be denoted by Coalg-C. It is easy to check that this category is isomorphic to a category defined as follows. Objects are C-comodules D together with C-comodule morphisms $\bar{\Delta}: D\to D\square_C D$ and $\bar{\varepsilon}: D\to C$ satisfying

$$(I\square \bar{\Delta})\bar{\Delta} = (\bar{\Delta}\square I)\bar{\Delta} \quad \text{and} \quad (\bar{\varepsilon}\square I)\bar{\Delta} = I = (I\square \bar{\varepsilon})\bar{\Delta}.$$

Morphisms are C-comodule morphisms $f: D\to D'$ such that $(f\square f)\bar{\Delta} = \bar{\Delta}'f$ and $\bar{\varepsilon} = \bar{\varepsilon}'f$. If M is a D-comodule, and $\bar{\varrho}: M\to D\square_C M$ is a comultiplication, then

$$(I\square \bar{\varrho})\bar{\varrho} = (\bar{\Delta}\square I)\bar{\varrho} \quad \text{and} \quad I = (\bar{\varepsilon}\square I)\bar{\varrho}.$$

1.2. If D and D' are C-coalgebras, then $D\square_C D'$ together with morphisms $(I\square t\square I)(\bar{\Delta}\square \bar{\Delta}')$ and $\bar{\varepsilon}\square \bar{\varepsilon}'$ is also a C-coalgebra. It is a product in the category Coalg-C.

1.3. One can verify without difficulty that if D and D' are C-coalgebras, B is a D-coalgebra, M is a $D\square_C D'$-comodule, and N is a D'-comodule, then there are the following functorial isomorphisms:

$$\mathrm{Hom}_{D\square_C D'}(M, D\square_C N) \approx \mathrm{Hom}_{D'}(M, N)$$

and

$$\text{Coalg-}D(B, D\square_C D') \approx \text{Coalg-}C(B, D').$$

1.4. If M_a for $a\in I$ and N are C-comodules, then $\big(\tilde{\prod\limits_{a\in I}} M_a\big)\square_C N$ is a vector space contained in $\big(\prod\limits_{a\in I} M_a\big)\otimes N$, $\tilde{\prod\limits_{a\in I}}(M_a\square_C N)$ is a vector subspace of $\prod\limits_{a\in I}(M_a\otimes N)$ and, therefore, there exists a natural embedding

$$\Big(\tilde{\prod_{a\in I}} M_a\Big)\square_C N \hookrightarrow \tilde{\prod_{a\in I}}(M_a\square_C N).$$

1.5. For any C-coalgebra D, one can define a category $(D$, Coalg-$C)$. Its objects are morphisms of C-coalgebras $\eta: D\to B$ and its morphisms are C-coalgebra maps $f: B\to B'$ such that $f\eta = \eta'$. If M is a D-comodule, then a C-comodule $D\oplus M$ has a C-coalgebra structure whenever $\bar{\Delta}\oplus(\bar{\varrho}+t\bar{\varrho})$ is a comultiplication and $\bar{\varepsilon}\oplus 0$ is a counit. Such a C-coalgebra will be denoted by $D{*}M$. The natural inclusion $D\hookrightarrow D{*}M$ is an object of $(D$, Coalg-$C)$ (see [6]).

2. Coderivations.

2.1. Definition. Let D be a C-coalgebra and let M be a D-comodule. A morphism $f: M \to D$ of C-comodules is called a *C-coderivation* if

$$\bar{\Delta} f = (f \square I) t \bar{\varrho} + (I \square f) \bar{\varrho}.$$

A vector space of all C-coderivations from M to D will be denoted by $\mathrm{Coder}_C(M, D)$.

2.2. Lemma. *For any C-coderivation $f: M \to D$, we have $\bar{\varepsilon} f = 0$.*

In fact, since C is a C-coalgebra with the comultiplication $I = \bar{\Delta}_C: C \to C = C \square_C C$ and the counit $I: C \to C$, we have

$$\bar{\varepsilon} f = \bar{\Delta}_C \bar{\varepsilon} f = (\bar{\varepsilon} \square \bar{\varepsilon}) \bar{\Delta}_D f = (\bar{\varepsilon} \square \bar{\varepsilon})(f \square I) t \bar{\varrho} + (\bar{\varepsilon} \square \bar{\varepsilon})(I \square f) \bar{\varrho} = \bar{\varepsilon} f + \bar{\varepsilon} f,$$

whence $\bar{\varepsilon} f = 0$.

2.3. Lemma. *Let $g: M \to N$ be a D-comodule morphism.*

a. *If $f: N \to D$ is a C-coderivation, then so is fg.*

b. *If g is an epimorphism and fg a C-coderivation, then f is a C-coderivation.*

It follows from 2.3.a that $\mathrm{Coder}_C(\cdot, D)$ is a functor from D-Comod to Mod-k.

2.4. Lemma. *For any C-coalgebra D, the map*

$$d = I \square \bar{\varepsilon} - \bar{\varepsilon} \square I : D \square_C D \to D$$

is a C-coderivation.

Proof. Since $D \square_C D$ is a D-comodule with the comultiplication $\bar{\Delta} \square I$, we have

$$(d \square I)(I \square \bar{\Delta}) + (I \square d)(\bar{\Delta} \square I) = I \square I - \bar{\varepsilon} \square \bar{\Delta} + \bar{\Delta} \square \bar{\varepsilon} - I \square I = \bar{\Delta} d.$$

Let $L(C, D)$ be a cokernel of the C-comodule map $\bar{\Delta}: D \to D \square_C D$. Sometimes we will write L instead of $L(C, D)$. Let s be a D-comodule morphism such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\bar{\Delta}} & D \square_C D & \xrightarrow{p} & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\Delta}} & & \downarrow{\scriptstyle (I \square t \square I)(\bar{\Delta} \square \bar{\Delta})} & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & D \square_C D & \xrightarrow[\bar{\Delta} \square \bar{\Delta}]{} & D \square_C D \square_C D \square_C D & \xrightarrow[p \square p]{} & L \square_C L & \longrightarrow & 0
\end{array}$$

commutes. $\mathrm{Ker}\, s$ will be denoted by $J(C, D)$ (or by J), and the natural inclusion $J \hookrightarrow L$ by h.

Since $d \bar{\Delta} = 0$, by 2.3, there exists a unique C-coderivation $i: L \to D$ such that $ip = d$.

2.5. Theorem. *The C-coderivation $j = ih : J(C, D) \to D$ induces a natural equivalence*

$$j_* : \mathrm{Hom}_D(\cdot, J(C, D)) \xrightarrow{\approx} \mathrm{Coder}_C(\cdot, D)$$

*of functors from D-*Comod *to k-*Mod.

Proof. For any C-coderivation $f : M \to D$, the morphism $p(I \Box f)\bar{\varrho}$ is a D-comodule map, since there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r, "\bar{\Delta}"] \arrow[d, "\bar{\Delta}"] & D \Box_C D \arrow[r, "p"] \arrow[d, "\bar{\Delta}\Box I"] & L \arrow[r] \arrow[d, "\bar{\varrho}_L"] & 0 \\
0 \arrow[r] & D \Box_C D \arrow[r, "I\Box\bar{\Delta}"'] & D \Box_C D \Box_C D \arrow[r, "I\Box p"] & D \Box_C L \arrow[r] & 0
\end{tikzcd}$$

and

$$\bar{\varrho}_L(I \Box f)\bar{\varrho} = (I \Box p)(\bar{\Delta} \Box I)(I \Box f)\bar{\varrho} = \big(I \Box p(I \Box f)\bar{\varrho}\big)\bar{\varrho}.$$

Further,

$$\begin{aligned} sp(I \Box f)\bar{\varrho} &= (p \Box p)(I \Box t \Box I)(\bar{\Delta} \Box \bar{\Delta})(I \Box f)\bar{\varrho} \\ &= (p \Box p)(I \Box t \Box I)\big(\Delta \Box \big((I \Box f)\bar{\varrho} + (f \Box I)t\bar{\varrho}\big)\big)\bar{\varrho} \\ &= \big(p\bar{\Delta} \Box p(I \Box f)\bar{\varrho}\big)\bar{\varrho} + \big(p(I \Box f)\bar{\varrho} \Box p\bar{\Delta}\big)t\bar{\varrho} = 0 \end{aligned}$$

and

$$ip(I \Box f)\bar{\varrho} = d(I \Box f)\bar{\varrho}f.$$

Hence, j_* is an epimorphism. It is easy to check that $p(I \Box i)\varrho_L = I$. Since

$$hg = p(I \Box i)\varrho_L hg = p(I \Box i)(I \Box hg)\bar{\varrho} = p(I \Box jg)\bar{\varrho}$$

whenever $g : M \to J$ is a D-comodule morphism, j_* is a monomorphism, and the proof is complete.

The cotensor product $J(C, D) \Box_D M$ will be denoted by $\mathrm{Codiff}_C(M, D)$ whenever M is a D-comodule.

2.6. Lemma. *If D is a C-coalgebra, M a D-comodule and*

$$(D \xrightarrow{\eta} B) \in \mathrm{Ob}(D, \text{Coalg-}C),$$

then M is a B-comodule and there exists a functorial isomorphism

$$\mathrm{Coder}_C(M, B) \approx (D, \text{Coalg-}C)(D * M, B).$$

For $C = k$ this is a well-known fact (see [6]), and the proof for an arbitrary coalgebra C is analogous.

The following proposition follows from 2.6, 2.5 and 1.3.

2.7. Proposition. *If B and D are C-coalgebras, and M is a $B\square_C D$-comodule, then there exist the following functorial isomorphisms:*

$$\mathrm{Coder}_B(M, B\square_C D) \approx \mathrm{Coder}_C(M, D),$$
$$\mathrm{Coder}_C(M, B)\oplus \mathrm{Coder}_C(M, D) \approx \mathrm{Coder}_C(M, B\square_C D),$$
$$J(C, B)\square_C D\oplus J(C, D)\square_C B \approx J(C, B\square_C D),$$
$$J(B, B\square_C D) \approx B\square_C J(C, D).$$

2.8. Proposition. *Any sequence of k-coalgebra morphisms*

$$D \xrightarrow{f} B \xrightarrow{g} C$$

induces the following exact sequences:

$$0 \longrightarrow \mathrm{Coder}_B(M, D) \hookrightarrow \mathrm{Coder}_C(M, D) \xrightarrow{f^*} \mathrm{Coder}_C(M, B),$$
$$0 \longrightarrow \mathrm{Codiff}_B(M, D) \longrightarrow \mathrm{Codiff}_C(M, D) \longrightarrow \mathrm{Codiff}_C(M, B).$$

Proof. For any B-coderivation $b: M \to D$, we have $f^*b = fb = 0$ by 2.2. Assume a C-coderivation $c: M \to D$ such that $f^*c = 0$. Then

$$(f\square I)\Delta c = (f\square I)\big((c\square I)t\bar{\varrho}+(I\square c)\bar{\varrho}\big) = (I\square c)(f\square I)\varrho$$

and, therefore, c is a B-coderivation. Consequently, the first sequence is exact. The exactness of the second one follows from 2.5.

2.9. It is easy to check that, for any k-algebra A and any A°-comodule M, there exists a functorial isomorphism of vector spaces,

$$\mathrm{Coder}_k(M, A^\circ) \approx \mathrm{Der}_k(A, M^*),$$

where an A-module structure of M^* is induced in a natural way from the A°-comodule structure of M.

3. Free and symmetric coalgebras.

3.1. Definition. Let C be a k-coalgebra and X an arbitrary set. A C-coalgebra $C\{X\} = k[X]^\circ\otimes C$ is called the *free C-coalgebra* over X. Any morphism $f: X \to Y$ induces (in a natural way) a morphism $C\{f\}$: $C\{Y\} \to C\{X\}$ of C-coalgebras such that $C\{\cdot\}$: Set $\to$ Coalg-C is a contravariant functor.

The following properties of the functor $C\{\cdot\}$ can be easily verified:

3.2. $C\{\cdot\}$ is a right adjoint to the functor

$$\overline{U}: \text{Coalg-}C \xrightarrow{\ *\ } \text{Alg-}C^* \xrightarrow{\ U\ } \text{Set},$$

where U is a forgetting functor. The isomorphism

$$\text{Coalg-}C(D, C\{X\}) \approx \mathrm{Set}(X, D^*)$$

is defined in the following manner. A C-coalgebra map $f: D \to C\{X\}$ $= k[X]^\circ\otimes C$ corresponds to a set morphism $\bar{f}: X \to D^*$ such that $\bar{f}(x)d$

$= \big((I\otimes \varepsilon_C)f(d)\big)x$ for any $x\in X$ and $d\in D$. Consequently, any C-coalgebra morphism f is determined by a collection of functionals $\bar{f}(x)$, where $x\in X$.

3.3. If $i: X \hookrightarrow Y$ is an inclusion, then $p = C\{i\}: C\{Y\}\to C\{X\}$ is surjective.

3.4. If $\mathrm{s}X$ denotes the collection of all finite subsets of X, and $\overline{\prod}$ and $\overline{\varprojlim}$ are, respectively, the product and inverse limit in the category Coalg-C, then

$$C\{X\} = \overline{\prod_{x\in X}} C\big\{\{x\}\big\} = \overline{\varprojlim_{Y\in \mathrm{s}X}} C\{Y\}.$$

3.5. Proposition. *For any finite set Y and any C-coalgebra morphism $f: C\{X\}\to C\{Y\}$, there exists a finite subset X_0 of X and a C-coalgebra morphism $g: C\{X_0\}\to C\{Y\}$ such that the diagram*

$$\begin{tikzcd} C\{X\} \arrow[rr, "p"] \arrow[dr, "f"'] & & C\{X_0\} \arrow[dl, "g"] \\ & C\{Y\} & \end{tikzcd}$$

commutes.

Proof. Since Y is a finite set, by 3.2 the map f is determined by a finite set of linear maps $\bar{f}(y): C\{X\}\to k$ for $y\in Y$. It is sufficient to show that any such $\bar{f}(y)$ can be factorized by a certain $C\{X_0\}$, where X_0 is finite. Let $g: k[X]^\circ\otimes C\to k$ be a non-zero linear map. Then there are $h_1,\ldots,h_h\in k[X]^\circ$ and linearly independent $c_1,\ldots,c_h\in C$ such that $k[X]^\circ = \ker g\oplus \sum k(h_i\otimes c_i)$. We can find in $k[X]$ an ideal I_0 contained in $\bigcap \ker h_i$ and a finite subset $X_0\subset X$ such that $k[X] = I_0\oplus k[X_0]$. Let $\pi: k[X]\to k[X]$ be a k-algebra morphism defined by $\pi|_{I_0} = 0$ and $\pi|_{k[X_0]} = I$. Then the induced C-coalgebra morphism

$$r: k[X_0]^\circ\otimes C\to k[X]^\circ\otimes C$$

satisfies conditions $pr = I$ and $grp = g$. Therefore, gr is a required map from $k[X_0]^\circ\otimes C$ to k, and the proof is complete.

Let M be a $C\{X\}$-comodule. Then 2.7 and 2.8 yield

$$\begin{aligned}\mathrm{Coder}_C(M, C\{X\}) &= \mathrm{Coder}_k(M, k[X]^\circ) = \mathrm{Der}_k(k[X], M^*) = \prod_{x\in X} M^* \\ &= \prod_{x\in X}\mathrm{Hom}_C(M, C\{X\}) = \mathrm{Hom}_{C\{X\}}\Big(M, \tilde{\prod_{x\in X}} C\{X\}\Big) \\ &= \mathrm{Hom}_{C\{X\}}\Big(M, \Big(\prod_{x\in X} k\Big)\otimes C\{X\}\Big),\end{aligned}$$

where $\tilde{\prod}$ denotes the product in the category $C\{X\}$-Comod.

3.6. COROLLARY.

$$\mathrm{Coder}_C(M, C\{X\}) = \prod_{x\in X} M^* \quad \text{and} \quad J(C, C\{X\}) = \Big(\prod_{x\in X} k\Big)\otimes C\{X\}.$$

Let M be a C-comodule

$$M_n = \underbrace{M\,\square_C \cdots \square_C\, M}_{n}, \qquad M_0 = C, \qquad M_{k,l} = M_k \square_C M_l,$$

$$M_{q,r,s} = M_q \square\, M_r \square\, M_s.$$

Natural product projections of

$$N = \tilde{\prod_{n\geqslant 0}} M_n, \qquad K = \tilde{\prod_{k,l\geqslant 0}} M_{k,l}, \qquad Q = \tilde{\prod_{q,r,s\geqslant 0}} M_{q,r,s}$$

will be denoted by p_n, $p_{k,l}$, $p_{q,r,s}$, respectively. Consider C-comodule morphisms $\psi : N \to K$, $\psi_1, \psi_2 : K \to Q$ uniquely determined by the formulas

$$p_{k,l}\psi = p_{k+l}, \qquad p_{q,r,s}\psi_1 = p_{q+r,s}, \qquad p_{q,r,s}\psi_2 = p_{q,r+s}.$$

Let $\hat{T}_C M = \psi^{-1}(\mathrm{Im}\, i)$, where i is the natural embedding $N\square_C N \hookrightarrow K$ (see 1.4). For $f: M\to M'$ being a C-comodule morphism, let

$$\hat{T}_C f : \hat{T}_C M \to \hat{T}_C M'$$

be the unique map determined by the equalities $p_n \hat{T}_C f = (f\square \cdots \square f)p_n$.

3.7. LEMMA. *$\hat{T}_C$ is a covariant functor from* C-Comod *to the category of non-commutative C-coalgebras.*

Proof. Let $\hat{\Delta} : \hat{T}_C M \to N_C \square N$ and $\hat{\varepsilon} : \hat{T}_C M \to C$ be C-comodule maps such that $i\hat{\Delta} = \psi|_{\hat{T}_C M}$ and $\hat{\varepsilon} = p_0|_{\hat{T}_C M}$. It is easy to verify that

$$i_2(I\square \psi)\hat{\Delta} = \psi_2\psi = \psi_1\psi = i_1(\psi\square I)\hat{\Delta},$$

where i_1 and i_2 are the natural embeddings $K\square_C N \hookrightarrow Q$ and $N\square_C K \hookrightarrow Q$, respectively. Therefore, it follows that $(I\square\psi)\hat{\Delta}$ and $(\psi\square I)\hat{\Delta}$ are factorized by $N\square_C N\square_C N$, and so we can consider $\hat{\Delta}$ as a morphism

$$\hat{T}_C M \to \hat{T}_C M \square_C \hat{T}_C M.$$

$\hat{\Delta}$ is a comultiplication and $\hat{\varepsilon}$ is a counit, since

$$p_n(I\square p_0)\hat{\Delta} = (p_n\square p_0)\hat{\Delta} = p_{n,0}\psi = p_n = p_n(p_0\square I)\Delta$$

and

$$\bar{i}(I\square \hat{\Delta})\hat{\Delta} = \psi_2\psi = \psi_1\psi = \bar{i}(\hat{\Delta}\square I)\hat{\Delta},$$

where $\bar{i}$ is the natural embedding $N\square_C N\square_C N \hookrightarrow Q$. Now it is sufficient to show that $\mathrm{Im}\,\hat{T}_C f \subset \hat{T}_C M'$ and that $T_C f$ is a C-coalgebra morphism.

This follows from the equalities

$$\begin{aligned} p'_{k,l}\psi'\hat{T}_C f &= (f\Box\ldots\Box f)p_{k+l} = (f\Box\ldots\Box f)p_{k,l}\psi|_{\hat{T}_C M} = (f\Box\ldots\Box f)p_{k,l}i\hat{\Delta} \\ &= (f\Box\ldots\Box f)(p_k\Box p_l)\hat{\Delta} = \big((f\Box\ldots\Box f)p_k\Box(f\Box\ldots\Box f)p_l\big)\hat{\Delta} \\ &= (p'_k\hat{T}_C f\Box p'_l\hat{T}_C f)\hat{\Delta} = (p'_k\Box p'_l)\hat{T}_C f\hat{\Delta} = p'_{k,l}i'(\hat{T}_C f\Box \hat{T}_C f). \end{aligned}$$

3.8. Definition. The maximal commutative subcoalgebra $\hat{S}_C M$ of $\hat{T}_C M$ is called the *symmetric C-coalgebra* over M. The existence and the uniqueness follow from [9], Section 3, p. 63.

It is clear that we have a covariant functor

$$\hat{S}_C : C\text{-Comod} \to \text{Coalg-}C.$$

3.9. PROPOSITION. *$\hat{S}_C$ is right adjoint to the forgetting functor U:* Coalg-$C \to C$-Comod.

Proof. Let D be a C-coalgebra and M a D-comodule. We define a map

$$\Phi\colon \operatorname{Hom}_C(UD, M) \to \text{Coalg-}C(D, \hat{S}_C M)$$

as follows. If $f\colon D\to M$ is a C-comodule map, then there exists a unique C-comodule morphism

$$\bar{f}: D \to N = \widetilde{\prod_{n\geqslant 0}} M_n$$

such that

$$p_0\bar{f} = \varepsilon_D, \quad p_1\bar{f} = f, \quad p_n f = (f\Box\ldots\Box f)(\Delta\Box I\Box\ldots\Box I)\ldots\Delta.$$

Observe that $\operatorname{Im}\psi\bar{f}\subset N\Box_C N$, since

$$p_{k,l}\psi\bar{f} = p_{k+l}\bar{f} = (p_k\bar{f}\Box p_l\bar{f})\Delta = (p_k\Box p_l)(\bar{f}\Box\bar{f})\Delta = p_{k,l}i(\bar{f}\Box\bar{f})\Delta.$$

Consequently, $\operatorname{Im}\bar{f}\subset\hat{T}_C M$. Futhermore, since D is commutative, we have $\operatorname{Im}\bar{f}\subset\hat{S}_C M$ and so, using the above equalities, one can check that $\bar{f}: D\to\hat{S}_C M$ is a morphism of C-coalgebras. We put $\Phi f = \bar{f}$. Clearly, Φ is a monomorphism. It is easy to see that if $g\colon D\to S_C M$ is a C-coalgebra map, then

$$p_n g = (p_1 g_n\Box\ldots\Box p_1 g)(\Delta\Box I\Box\ldots\Box I)\ldots\Delta.$$

Thus Φ is an epimorphism, and the proof is complete.

3.10. COROLLARY. *$\hat{S}_C(V^*\otimes C) = C\{X\}$ whenever V is a vector space with basis X.*

The proof follows from the natural equivalence of functors:

$$\begin{aligned} \text{Coalg-}C\big(\cdot, \hat{S}_C(V^*\otimes C)\big) &\approx \operatorname{Hom}_C(\cdot, V^*\otimes C) \approx \operatorname{Hom}_k(\cdot, V) \approx \operatorname{Set}(X, \cdot) \\ &\approx \text{Coalg-}C(\cdot, C\{X\}). \end{aligned}$$

For $C = k$, $\hat{S}_k M$ is the cofree cocommutative coalgebra in the sense of Sweedler (see [9], Section 6, p. 129).

It follows from [4] that there exists an anti-equivalence of categories of C-coalgebras and profinite C^*-algebras given by the pair of functors

$$*\colon \text{Coalg-}C \to \text{Prof-}C^* \quad \text{and} \quad \hom(\cdot, k)\colon \text{Prof-}C^* \to \text{Coalg-}C,$$

where $\hom(A, k)$ denotes the vector space of all continuous morphisms from A to k (k has the discrete topology). Then the above construction of S_C is dual to the construction of the functor $\hat{S}_{C^*}$ from the category of pseudocompact modules over C^* to the category of profinite C^*-algebras (see [4], p. 85-86).

4. Triples and André-Appelgate cohomology. This section contains some of the results concerning triples and André-Appelgate cohomology which can be obtained by dualization of results from [2] and which are needed in our further considerations.

Throughout this section $\mathfrak{C}$ will be a fixed category.

A collection $X = \{X^n\}_{n\geqslant 0}$ of objects of $\mathfrak{C}$ together with morphisms $\varepsilon_i^n\colon X^{n-1} \to X^n$ and $\delta_i^n\colon X^{n+1} \to X^n$ $(0 \leqslant i \leqslant n)$ is a *cosimplicial object* in $\mathfrak{C}$ if, for any n,

$$\varepsilon_j \varepsilon_i = \varepsilon_i \varepsilon_j \quad \text{and} \quad \delta_j \varepsilon_i = \varepsilon_i \delta_{j-1} \quad \text{for } i < j,$$

$$\delta_j \delta_i = \delta_i \delta_{j+1} \quad \text{for } i \leqslant j,$$

$$\delta_i \varepsilon_i = I = \delta_i \varepsilon_{i+1} \quad \text{and} \quad \delta_j \varepsilon_i = \varepsilon_{i-1} \delta_j \quad \text{for } i > j+1.$$

An *augmented cosimplicial object* in $\mathfrak{C}$ is a cosimplicial object X with a morphism $\varepsilon_0\colon X^{-1} \to X^0$ such that $\varepsilon_1 \varepsilon_0 = \varepsilon_0 \varepsilon_0$.

Let $\mathfrak{T} = (T, \eta, \mu)$, where $T\colon \mathfrak{C} \to \mathfrak{C}$ is a covariant functor, and $\eta\colon \mathrm{Id} \to T$ and $\mu\colon T \circ T \to T$ are natural transformations of functors. $\mathfrak{T}$ is a *triple* in $\mathfrak{C}$ if $\mu(\eta T) = \mathrm{Id} = \mu(T\eta)$ and $\mu(T\mu) = \mu(\mu T)$. If $\mathfrak{T}$ is a triple and $\mathfrak{C}$ is an object of $\mathfrak{C}$, then $\mathfrak{T}C = \{T^{n+1}C\}$ together with $\varepsilon_i = T^{n-i}\eta T^i$ and $\delta_i = T^{n-i}\mu T^i$ is an augmented cosimplicial object.

An object C of $\mathfrak{C}$ is called $\mathfrak{T}$-*injective* if $\eta(C)\colon C \to TC$ splits.

A $\mathfrak{T}$-*resolution* of C is a cochain complex $\{X^n, d^n\}$ in $Z\mathfrak{C}$ (the additive category over $\mathfrak{C}$) such that all X^n are $\mathfrak{T}$-injective and the sequence

$$0 \to Z\mathfrak{C}(C, TD) \to Z\mathfrak{C}(X^\circ, TD) \to \ldots$$

is exact in the category of abelian groups for any $D \in \mathrm{Ob}\,\mathfrak{C}$.

A *cosimplicial* $\mathfrak{T}$-*resolution* of C is an augmented cosimplicial object X such that the associated cochain complex in $Z\mathfrak{C}$,

$$KX = \left\{X^n, \sum_{i=0}^{n+1} (-1)^i \varepsilon_i^{n+1}\right\},$$

is a $\mathfrak{T}$-resolution of C.

$\mathfrak{T}C$ is a cosimplicial $\mathfrak{T}$-resolution of C called the *standard $\mathfrak{T}$-resolution.*

Let $\mathfrak{A}$ denote an arbitrary abelian category. For any functor E: $\mathfrak{C} \to \mathfrak{A}$, $H^n(C, E)_{\mathfrak{T}}$ is defined as the n-th cohomology object of the cochain complex

$$0 \longrightarrow EX^0 \xrightarrow{d^0} EX^1 \xrightarrow{d^1} EX^2 \xrightarrow{d^2} \dots \qquad \text{with } d^n = \sum_{i=0}^{n+1} (-1)^i E\varepsilon_i^{n+1},$$

where X is a certain cosimplicial $\mathfrak{T}$-resolution of C. The object $H^n(C, E)_{\mathfrak{T}}$ is independent of the choice of X.

4.1. We have

$$H^n(C, E)_{\mathfrak{T}} = \begin{cases} 0 & \text{for } n > 0, \\ EC & \text{for } n = 0, \end{cases}$$

whenever C is $\mathfrak{T}$-injective.

4.2. If C and D have the cosimplicial $\mathfrak{T}$-resolutions X and Y, respectively, such that $X \Pi Y$ is a cosimplicial $\mathfrak{T}$-resolution of $C \Pi D$, then

$$H^n(C, E)_{\mathfrak{T}} \oplus H^n(D, E)_{\mathfrak{T}} = H^n(C \Pi D, E)_{\mathfrak{T}}$$

for any product-preserving functor E.

4.3. Let $D \in \operatorname{Ob}\mathfrak{C}$. Then $(\mathfrak{C}, D)$ is a category of morphisms $C \to D$ in $\mathfrak{C}$, and morphisms in this category are commutative triangles

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & D & \end{tikzcd}$$

If E: $\mathfrak{C} \to \mathfrak{A}$ is a functor, then (E, D): $(\mathfrak{C}, D) \to \mathfrak{A}$ is defined by

$$(E, D)(C \to D) = \ker(EC \to ED).$$

Any triple $\mathfrak{T}$ in $\mathfrak{C}$ induces (in a natural way) a triple $(\mathfrak{T}, D)$ in $(\mathfrak{C}, D)$ such that

$$(\mathfrak{T}, D)(C \to D) = TC \Pi D \to D.$$

4.4. If

$$H^n(C \Pi TB, E)_{\mathfrak{T}} = \begin{cases} H^n(C, E)_{\mathfrak{T}} & \text{for } n > 0, \\ H^0(C, E)_{\mathfrak{T}} \oplus ETB & \text{for } n = 0 \end{cases}$$

for any $C, B \in \operatorname{Ob}\mathfrak{C}$, then an arbitrary sequence of morphisms $C \to D \to B$ in $\mathfrak{C}$ induces an exact sequence

$$0 \to H^0\big(C \to D, (E, D)\big)_{(\mathfrak{T}, D)} \to \dots \to H^{n-1}\big(C \to B, (E, B)\big)_{(\mathfrak{T}, B)}$$
$$\to H^{n-1}\big(D \to B, (E, B)\big)_{(\mathfrak{T}, B)} \to H^n\big(C \to D, (E, D)\big)_{(\mathfrak{T}, D)} \to \dots,$$

whenever E preserves the product.

4.5. Let $\mathfrak{B}$ be an arbitrary category. Any pair of adjoint functors $U\colon \mathfrak{C} \to \mathfrak{B}$ and $F\colon \mathfrak{B} \to \mathfrak{C}$ induces a certain triple $\mathfrak{T}$ with $T = FU$ and appropriate η and μ (see [2]). If $\mathfrak{B}$ is an abelian category and X is an augmented cosimplicial object in $\mathfrak{C}$ such that X^n are $\mathfrak{T}$-injective for $n \geqslant 0$, and if there exists a contraction of the cochain complex

$$\Big\{UX^n, \sum_{i=0}^{n+1} (-1)^i U\varepsilon_i^{n+1}\Big\} \quad \text{in } \mathfrak{B},$$

then X is a cosimplicial $\mathfrak{T}$-resolution.

4.6. Let $\mathfrak{A}$ be an abelian category with arbitrary products. Let us distinguish a small and full subcategory $\mathfrak{M}$ of $\mathfrak{C}$. Objects of $\mathfrak{M}$ will be called *models*. For any functor $E\colon \mathfrak{M} \to \mathfrak{A}$, the *cochain complex* $\{C^n(\mathfrak{M}, E), d^n\}$ is defined as follows:

$$C^n(\mathfrak{M}, E) = \prod_{M_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} M_n} EM_n, \quad \text{where } M_i \in \mathfrak{M},$$

$$d^{n-1} = \sum_{i=0}^{n} (-1)^i \varepsilon_i^n,$$

and $\varepsilon_i^n\colon C^{n-1} \to C^n$ is determined by the formula

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle \varepsilon_i^n = \begin{cases} \langle a_1, \ldots, a_{n-1}\rangle & \text{for } i = 0, \\ \langle a_0, \ldots, a_i a_{i-1}, \ldots, a_{n-1}\rangle & \text{for } 0 < i < n, \\ E(a_{n-1})\langle a_0, \ldots, a_{n-2}\rangle & \text{for } i = n, \end{cases}$$

where $\langle a_0, \ldots, a_{n-1}\rangle\colon C^n(\mathfrak{M}, E) \to EM_n$ is a product structural map. Furthermore, the n-th cohomology object of that cochain complex will be denoted by $H^n(\mathfrak{M}, E)$.

4.7. Let $C \in \operatorname{Ob}\mathfrak{C}$. Then $E_0\colon (C, \mathfrak{M}) \to \mathfrak{A}$ is a functor defined by $E_0(C \to M) = EM$. The *André-Appelgate cohomology* of C with coefficients in E and models $\mathfrak{M}$ is a collection of objects

$$A^n(C, E) = H^n\big((C, \mathfrak{M}), E_0\big),$$

4.8. We have

$$A^n(M, E) = \begin{cases} 0 & \text{for } n > 0, \\ EM & \text{for } n = 0, \end{cases}$$

whenever M is a model.

5. Cohomology of coalgebras. A cosimplicial object in Coalg-C will be called a *cosimplicial C-coalgebra.* The chain

$$UX^{\cdot} = (0 \longrightarrow X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \ldots) \quad \text{with } d^n = \sum_{i=0}^{n} (-1)^i \varepsilon_i^{n+1}$$

is called the *associated chain complex* of $X^{\cdot}$ in C-Comod.

A cosimplicial free *C-resolution* of C-coalgebra D is an augmented cosimplicial C-coalgebra $X^{\cdot}$ such that $X^{-1} = D$, $UX^{\cdot}$ is acyclic, and X^n are free for $n \geqslant 0$.

Let us consider a triple $\mathfrak{F}_C$ in Coalg-C defined by the pair of adjoint functors

$$\text{Coalg-}C \underset{\overline{U}}{\overset{C\{\cdot\}}{\rightleftarrows}} \text{Set},$$

where $\overline{U}$ is the functor defined in 3.2. It is clear that any free C-coalgebra is $\mathfrak{F}_C$-injective.

The following lemma is a consequence of well-known facts concerning a simplicial group homology and of results from [1], Section 5:

5.1. Lemma. *$X^{\cdot}$ is a cosimplicial free C-resolution of D iff $X^{\cdot}$ is a cosimplicial $\mathfrak{F}_C$-resolution of D.*

5.2. Lemma. *Assume that* $\mathrm{Cotor}_n^C(B, D) = 0$ *for* $n > 0$. *If* $\{X^n\}$ *and* $\{Y^n\}$ *are cosimplicial free C-resolutions of B and D, respectively, then* $\{X^n \square_C Y^n\}$ *is a cosimplicial C-resolution of* $B \square_C D$.

Proof. It follows from Section 2 in [3] that there exists a chain homotopy $U(X^{\cdot} \square_C Y^{\cdot}) \simeq UX^{\cdot} \square_C UY^{\cdot}$. Any free C-coalgebra is injective as a C-comodule and, therefore, $UX^{\cdot}$ and $UY^{\cdot}$ are injective resolutions of B and D, respectively, in C-Comod. Thus $U(X^{\cdot} \square_C Y^{\cdot})$ is acyclic iff $\mathrm{Cotor}_n^C(B, D) = 0$ for $n > 0$. If X^n and Y^n are free C-coalgebras, then also $X^n \square_C Y^n$ is free.

Now let D be a C-coalgebra and M a D-comodule. We define cohomologies $H^n(C, D, M)$ and $\overline{H}^n(C, D, M)$ of D with coefficients in M as n-th cohomology objects of the cochain complex $\mathrm{Coder}_C(M, X^{\cdot})$ and $\mathrm{Codiff}_C(M, X^{\cdot})$, respectively, where $X^{\cdot}$ is an arbitrary cosimplicial free C-resolution of D (taken without $X^{-1} = D$).

5.3. It is easy to see that

$$H^n(C, D, M) = H^n(D \xrightarrow{I} D, E_C)_{(D, \mathfrak{F}_C)},$$

$$\overline{H}^n(C, D, M) = H^n(D \xrightarrow{I} D, \overline{E}_C)_{(D, \mathfrak{F}_C)},$$

where E_C and $\overline{E}_C$ are functors from $(D, \text{Coalg-}C)$ to k-Mod defined by

$$E_C(D \to B) = \mathrm{Coder}_C(M, B), \qquad \overline{E}_C(D \to B) = \mathrm{Codiff}_C(M, B)$$

and

$$(D, F_C)(D \to B) = (D \to F_C D \to F_C B).$$

Hence $\bar{H}^n(C, D, M)$ and $H^n(C, D, M)$ are independent of the choice of the resolution $X^{\cdot}$.

5.4. COROLLARY. a. $H^0(C, D, M) = \mathrm{Coder}_C(M, D)$.

b. $H^n(C, B, M) \oplus H^n(C, D, M) = H^n(C, B \square_C D, M)$, *whenever*

$$\mathrm{Cotor}_n^C(B, D) = 0 \quad \textit{for } n > 0.$$

Proof. Statement a is consequence of the exactness of the sequence

$$0 \to \mathrm{Coder}_C(M, D) \to \mathrm{Coder}_C(M, F_C D) \to \mathrm{Coder}_C(M, F_C^2 D)$$

which can be easily verified. Statement b follows immediately from 5.2, 5.3 and 4.2.

5.5. THEOREM. *An arbitrary sequence $D \to B \to C$ in* Coalg-k *induces the exact sequence*

$$0 \to \mathrm{Coder}_B(M, D) \to \mathrm{Coder}_C(M, D) \to \ldots$$
$$\to H^{n-1}(C, B, M) \to H^n(B, D, M) \to H^n(C, D, M) \to H^n(C, B, M) \to \ldots$$

for any D-comodule M.

Proof. Observe that

$$H^n\big(D \to C, (E_k, C)\big)_{(\mathfrak{F}_k, C)} = H^n(C, D, M).$$

Indeed,

$$(F_k, C)(D \to C) = (F_k D \otimes C \to C) = (F_C D \to C),$$

$$(E_k, C)(D \to C) = \ker\big(\mathrm{Coder}_k(M, D) \to \mathrm{Coder}_k(M, C)\big) = \mathrm{Coder}_C(M, D).$$

Since $\mathrm{Cotor}_n^k(B, C) = 0$ for $n > 0$, we have, by 5.2 and 4.2,

$$H^n(B, E_k)_{\mathfrak{F}_k} \oplus H^n(C, E_k)_{\mathfrak{F}_k} = H^n(B \otimes C, E_k)_{\mathfrak{F}_k}.$$

Thus the theorem follows from 4.4 and 5.4.

Of course, the functor $\bar{H}^n$ has analogous properties as H^n.

In Coalg-C we can consider other triple $\mathfrak{S}_C$ induced by the pair of adjoint functors

$$\text{Coalg-}C \underset{U}{\overset{\hat{S}_C}{\leftrightarrows}} C\text{-Comod}.$$

Using this triple, one can define cohomologies $G^n(C, D, M)$ and $\bar{G}^n(C, D, M)$ similarly as H^n and $\bar{H}^n$ in 5.3.

5.6. LEMMA. *$\{X^n \square_C Y^n\}$ is a cosimplicial $\mathfrak{S}_C$-resolution of $B \square_C D$ whenever $\{X^n\}$ and $\{Y^n\}$ are standard $\mathfrak{S}_C$-resolutions.*

Proof. For any C-coalgebra D, the standard $\mathfrak{S}_C$-resolution is in C-Comod$\{\hat{S}_C^{n+1}(s)\}$, where $s\colon \hat{S}_C D \to D$ is a C-comodule map such that $s\varepsilon_0 = I$. Hence $U\{X^n \square_C Y^n\}$ has a contraction and the lemma follows by 4.5.

For $C = k$, the standard $\mathfrak{F}_k$-resolution is also an $\mathfrak{S}_k$-resolution, since the complex $U\mathfrak{F}_k D$ has a contraction in k-Mod (see 3.10 and 4.5). Hence, in such a case, $\bar{H}^n$ and $\bar{G}^n$ coincide and so do H^n and G^n.

It follows by 5.6 that

$$G^n(C, B, M) \oplus G^n(C, D, M) = G^n(C, B \square_C D, M),$$

whenever $B, D \in$ Coalg-C. Hence 5.5 is true also for G^n.

Let us consider a category $(D, \text{Coalg-}C)$ and its full subcategory

$$\mathfrak{M} = \{D \to k[X_1 \ldots X_n]^\circ \otimes C\}_{n \geqslant 1}.$$

The André-Appelgate cohomology of $I \colon D \to D$ with coefficients in E_C or $\bar{E}_C$ and models $\mathfrak{M}$ will be denoted by $A(C, D, M)$ or $\bar{A}(C, D, M)$, respectively. It is obvious that $A^n(C, D, M)$ is the n-th cohomology object of the cochain complex $\{C^n(D), d^n\}$, where

$$C^n(D) = \prod_{D \xrightarrow{a_0} C\{X_{i_0}\} \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} C\{X_{i_n}\}} \operatorname{Coder}(M, C\{X_{i_n}\})$$

($C\{X_i\}$ are models), and d^n is defined similarly as in 4.6.

From Propositions 4.1 and 8.1 in [1] we infer that $A^n(C, D, M) = H^n(C, D, M)$ for any C-coalgebra D if the following lemma holds:

5.7. Lemma. *For any set X,*

$$A^n(C, C\{X\}, M) = \begin{cases} 0 & \text{for } n > 0, \\ \operatorname{Coder}_C(M, C\{X\}) & \text{for } n = 0. \end{cases}$$

To prove this lemma we need the following notion:

5.8. Definition (see [5]). An inverse system of k-modules $C = \{C_i\}_{i \in I}$ is *weakly flabby* if the natural homomorphism

$$\varprojlim_I C \to \varprojlim_J C$$

is surjective whenever J is a directed subset of I.

Proof of Lemma 5.7. By 4.8, the lemma follows for X finite. Now observe that any inclusion $i \colon Y \hookrightarrow X$ induces an epimorphism $T^n \colon C^n(C\{X\}) \to C^n(C\{Y\})$ such that

$$\langle a_0, \ldots, a_n \rangle T^n = \langle a_0 p, \ldots, a_n \rangle, \quad \text{where } p = C\{i\}.$$

Let B^n be a cochain complex and let $\{R^n Y\}_{Y \in 8X}$ be a collection of maps $R^n Y \colon B^n \to C^n(C\{Y\})$ satisfying

$$C^n(C\{i_1\}) R^n Y = R^n Y_1$$

whenever $i_1 \colon Y_1 \hookrightarrow Y$ is an inclusion.

We define R^n: $B^n \to C^n(C\{X\})$ by setting

$$\langle a_0', \ldots, a_n' \rangle R^n = \langle a_0, \ldots, a_n \rangle R^n Y,$$

where Y and a_0' are such that $a_0 p = a_0'$ (see 3.5). Then it follows that $T^m R^n = R^n Y$ and it is easy to see that

$$\varprojlim_{Y \in sX} C^n(C\{Y\}) = C^n(C\{X\}).$$

Consequently, $\{C^n(C\{Y\})\}_{Y \in sX}$ is a weakly flabby system, since, for any directed subset $s_1 X$ of sX, we have

$$\varprojlim_{Y \in s_1 X} C^n(C\{Y\}) = C^n(C\{Z\}) \quad \text{with } Z = \bigcup_{Y \in s_1 X} Y$$

and, therefore, the map p: $C^n(C\{X\}) \to C^n(C\{Z\})$ is an epimorphism. Furthermore, $\{\mathrm{Coder}_C(M, C\{Y\})\}_{Y \in sX}$ is also a weakly flabby system. Indeed,

$$\mathrm{Coder}_C(M, C\{X\}) = \prod_{x \in X} M^* = \varprojlim_{Y \in sX} \prod_{y \in Y} M^* = \varprojlim_{Y \in sX} \mathrm{Coder}_C(M, C\{Y\}),$$

$$\mathrm{Coder}_C(M, C\{Z\}) = \varprojlim_{Y \in s_1 X} \mathrm{Coder}_C(M, C\{Y\}),$$

and the natural map

$$\mathrm{Coder}_C(M, C\{X\}) \to \mathrm{Coder}_C(M, C\{Z\})$$

is surjective. Then we have an exact sequence of weakly flabby systems

$$0 \to \{\mathrm{Coder}_C(M, C\{Y\})\} \to \{C^0(C\{Y\})\} \to \ldots$$

By Theorems 1.8 and 1.9 of Jensen [5], the sequence

$$0 \to \mathrm{Coder}_C(M, C\{X\}) \to C^0(C\{X\}) \to C^1(C\{X\}) \to \ldots$$

is exact, and the lemma follows.

By similar considerations for $\bar{A}^n$, one can obtain

$$\bar{A}^n(C, C\{X\}, M) = \begin{cases} 0 & \text{for } n > 0, \\ \prod_{x \in X} M & \text{for } n = 0. \end{cases}$$

Since, by 3.6,

$$\bar{H}^0(C, C\{X\}, M) = \Big(\prod_{x \in X} k\Big) \otimes M,$$

we have $\bar{A} \neq \bar{H}$.

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