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JACOBIAN MATRIX AND p-BASIS

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In this paper, p is a prime number and a ring is always a commutative ring with identity. Let R be a ring of characteristic p and R^p denote the subring $\{x^p | x \in R\}$. Let R' be a subring of R. A subset of Γ of R is said to be p-independent over R' if, for any subset $\{b_1, \ldots, b_n\}$ of Γ, the set of monomials $b_1^{e_1} \ldots b_n^{e_n}$ $(0 \leqslant e_i \leqslant p-1)$ is linearly independent over $R^p[R']$. Γ is called a *p-basis* of R over R' if it is p-independent over R' and $R^p = R^p[R', \Gamma]$.

The aim of this paper is to check up on a relation between the existence of p-basis and the Jacobian condition for various Frobenius sandwiches of rings. We omit almost all the proof and the reader may refer to [7].

Let R be an artinian local ring of char p and let M be the maximal ideal. Let $\{x_1, \ldots, x_n\}$ be a minimal system of generators for M and k be a coefficient field of R. Then, we may set $R = k[x_1, \ldots, x_n] = k[[X_1, \ldots, X_n]]/I$, where $k[[X_1, \ldots, X_n]]$ is a power series ring in n indeterminates over k, I an ideal of $k[[X_1, \ldots, X_n]]$ and $x_i = X_i \bmod I$.

Let R be a ring, J an ideal of R and let D be a set of derivations of R. Then, J is called a D-ideal if $d(J) \subset J$ for all d in D.

THEOREM 1 (Cor. 1 of [6]). *Let R be an artinian local ring of char p, k a coefficient field of R and $\{x_1, \ldots, x_n\}$ be a minimal system of generators for the maximal ideal M of R. Then, the following three conditions are equivalent:*

(1) *$\{x_1, \ldots, x_n\}$ is a p-basis of R over k.*

(2) *There exist n k-derivations $d_1, \ldots, d_n \in \mathrm{Der}_k(R)$ such that $d_i x_j = \delta_{ij}$ and $d_i^p = 0$ $(i, j = 1, \ldots, n)$.*

(3) *There exists a D-ideal I of $k[[X_1, \ldots, X_n]]$ such that $R = k[x_1, \ldots, x_n] = k[[X_1, \ldots, X_n]]/I$ and $D = \{\partial/\partial X_1, \ldots, \partial/\partial X_n\}$.*

P. Nousiainen proved the following interesting and useful result:

This paper is in final form and no version of it will be submitted for publication elsewhere.

THEOREM 2 (6.2 Theorem of [10]). *Let k be a field of char p. Let $R = k[X_1, \ldots, X_n]$ be the polynomial ring in n indeterminates over k and $f_1, \ldots, f_n \in R$. Then, the following conditions are equivalent:*

(1) *There exist n k-derivations $d_1, \ldots, d_n \in \mathrm{Der}_k(R)$ such that $d_i f_j = \delta_{ij}$ $(i, j = 1, \ldots, n)$.*

(2) *There exist n k-derivations $d_1, \ldots, d_n \in \mathrm{Der}_k(R)$ such that $\det(d_i f_j) \in k^*$.*

(3) *The Jacobian matrix $(\partial f_j/\partial X_i)$ is invertible.*

(4) $k[X_1, \ldots, X_n] = k[X_1^p, \ldots, X_n^p, f_1, \ldots, f_n]$.

(5) *$\{f_1, \ldots, f_n\}$ is a p-basis of $k[X_1, \ldots, X_n]$ over $k[X_1^p, \ldots, X_n^p]$.*

P. Nousiainen proved this result by a p-Jacobian matrix in his thesis. We may prove this result by a little bit different way (Theorem 4.2 of [7]). Anyway, this result is just our assertion itself. That is, the invertibility of Jacobian matrix $(\partial f_j/\partial X_i)$ is equivalent to that $\{f_1, \ldots, f_n\}$ is a p-basis of $k[X_1, \ldots, X_n]$ over $k[X_1^p, \ldots, X_n^p]$.

In the case of formal power series rings, we have the following:

PROPOSITION 3. *Let k be a field of characteristic p and $k[[X]] = k[[X_1, \ldots, X_n]]$ be the formal power series ring in n indeterminates over k. Let $\{f_1, \ldots, f_n\}$ be a set of n power series without constant terms in $k[[X]]$. Then, the following conditions are equivalent:*

(1) *There exist n k-derivations $d_1, \ldots, d_n \in \mathrm{Der}_k(k[[X]])$ such that $d_i f_j = \delta_{ij}$ $(i, j = 1, \ldots, n)$.*

(2) *There exist n k-derivations $d_1, \ldots, d_n \in \mathrm{Der}_k(k[[X]])$ such that $\det(d_i f_j) \notin M$.*

(3) *The Jacobian matrix $(\partial f_j/\partial X_i)$ is invertible.*

(4) $k[[X_1, \ldots, X_n]] = k[[f_1, \ldots, f_n]]$.

(5) $k[[X_1, \ldots, X_n]] = k[[X_1^p, \ldots, X_n^p]][f_1, \ldots, f_n]$.

(6) *$\{f_1, \ldots, f_n\}$ is a p-basis of $k[[x_1, \ldots, x_n]]$ over $k[[x_1^p, \ldots, x_n^p]]$.*

(7) *$\{f_1, \ldots, f_n\}$ is a regular system of parameters of the regular local ring $k[[X_1, \ldots, X_n]]$.*

This proposition is straightforward from Proposition 5 of § 4, Chap. III, [1]. The proof of Proposition 3 is very simple, but it has many interesting applications.

Let (R, M, k) be a local ring of characteristic p. Then R^p is a local ring with the maximal ideal $M^p = \{x^p | x \in M\}$. Since $M \cap R^p = M^p$, the natural map $R^p/M^p \to R/M = k$ is surjective and its image is equal to $(R/M)^p = k^p$. In view of the above injection, the residue field R^p/M^p of R^p can be identified with the subfield k^p of k. R' denotes an intermediate local ring between R and R^p, M' the maximal ideal and k' the residue field. It is clear that R dominates R', and so we may identify the residue field k' of R' with the subfield of k and we assume that

$k \supset k' \supset k^p$. For any subset A of R, we denote by $\bar{A}$ the set of residue classes of the elements of A modulo M. When we say "$\bar{A}$ is a p-basis", we tacitly assume that A maps injectively to $\bar{A}$.

PROPOSITION 4. *Let (R, M, k) be a local ring of char p. Let A be a subset of R such that $\bar{A}$ is a p-basis of k over k'. If R' is regular, $R'[A]$ is a regular local ring with the maximal ideal $M'R'[A]$.*

Let (R, M, k) be a local ring of characteristic p and k_0 be a subfield of R. k_0 is said to be a quasicoefficient field of R if the residue field $k = R/M$ is 0-etale over the image of k_0 in k. A local ring of chap p has a quasicoefficient field k_0 and the M-adic completion R^* of R has a coefficient field K containing k_0 (cf. Theorem 28.3 of [9]). Then, every derivation of R (into itself) over k_0 is uniquely extended to a derivation of R^* (into itself) over K. Therefore, we can identify $\mathrm{Der}_{k_0}(R)$ with an R-submodule of $\mathrm{Der}_K(R^*)$ and $\mathrm{Der}_{k_0}(R^*) = \mathrm{Der}_K(R^*)$.

PROPOSITION 5. *Let R and R' be semilocal rings of characteristic p such that $R \supset R' \supset R^p$ and R is a finitely generated R'-module. For the subset Γ of R, the following conditions are equivalent:*

(1) *Γ is a p-basis of R over R'.*
(2) *Γ is a p-basis of R^* over $(R')^*$.*

THEOREM 6. *Let (R, M, k) be an n-dimensional regular local ring of char p and let k_0 be a quasicoefficient field of R. Suppose that R is a finitely generated $R^p[k_0]$-module. Then, for the elements $f_1, \ldots, f_n$ of M, the following conditions are equivalent:*

(1) *There exist n k_0-derivations $d_1, \ldots, d_n$ of R into itself such that $d_i f_j = \delta_{ij}$ for $i, j = 1, \ldots, n$.*
(2) *There exist n k_0-derivations $d_1, \ldots, d_n$ of R into itself such that* $\det(d_i f_j) \notin M$.
(3) *$\{f_1, \ldots, f_n\}$ is a p-basis of R over k_0.*
(4) *$\{f_1, \ldots, f_n\}$ is a regular system of parameters of R.*

COROLLARY 7. *Let R be a regular local ring of char p and k_0 be a quasicoefficient field of R. Then, if R is a finitely generated $R^p[k_0]$-module, R has a p-basis over k_0 and R has also a p-basis over R^p. More precisely, if $\{f_1, \ldots, f_n\}$ is a regular system of parameters of R and A is a p-basis of k_0 over $(k_0)^p$, then $\{f_1, \ldots, f_n\}$ is a p-basis of R over k_0 and $A \cup \{f_1, \ldots, f_n\}$ is a p-basis of R over R^p.*

LEMMA 8 (Lemma 1 of [5]). *Let (R, M, k) be a local ring of char p and let (R', M', k') be an intermediate local ring between R and R^p. Assume that R has a p-basis of Γ over R'. Let A be an arbitrary subset of Γ such that A is a p-basis of k over k'. Then $\Gamma - A$ is a finite set and $|\Gamma - A| = \mathrm{rank}_k M/(M'R + M^2)$.*

Corollary 9. *Let R be a regular local ring of char p and k_0 be a quasicoefficient field of R. Then, the following conditions are equivalent*:

(1) *R is a finitely generated $R^p[k_0]$-module.*

(2) *R has a p-basis over k_0.*

Finally, the methods of proving Theorem 1, Theorem 2 and Proposition 3 are different from each other.

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