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On invariants of certain proper isomorphism classes of (μ, Δ, γ)-systems

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Dedicated to Prof. K. Kuratowski
at the fourty years anniversary of His scientific work

On invariants of certain proper isomorphism classes of (μ, Δ, γ)-systems

§ 1. Let X be a topological space. By $\pi_r(X)$ we mean a homotopy group of X. If $r \geqslant 2$, then $\pi_r(X)$ is a direct sum of cyclic groups of prime power order. We write $\pi_r(X) = \sum_{q(r,X)} Z_{q(r,X)}$, where $Z_{q(r,X)}$ denotes a cyclic group of order $q(r, X)$. Evidently $\{q(r, X)\}$ is a set of integers, among which some may be duplicated. If $a_1, \ldots, a_n$ (n may be finite or infinite) are distinct integers among the set $\{q(r, X)\}$, and if a_i occurs t_i times in $\{q(r, X)\}$, then both a_i and t_i must be invariants of the homotopy type of X. This claims general measure to produce numerical homotopy invariants besides Betti numbers and torsions (1). In [3] and [4], we have introduced relative block invariants, characteristic coefficients and characteristic polynomials. The first two kinds are integer invariants but the third kind are irreducible polynomials with coefficients in the group of integers reduced mod 2. They have been proved [5] to be intimately related with homotopy groups of polyhedra.

By a (μ, Δ, γ)-system we mean an algebraic structure introduced in [7] and [2] which is an abstract description of relations between cohomology groups produced by Steenrod homomorphisms. Here we take two Steenrod homomorphisms:

$$\gamma_j^m : H^m(2) \to H^{m+j}(2), \qquad \gamma_k^n : H^n(2) \to H^{n+k}(2),$$

their related cohomology groups H^r $(r = m, m+1, m+j, m+j+1, n, n+1, n+k, n+k+1)$, $H^s(2)$ $(s = m, m+j, n, n+k)$ and their related homomorphisms (2)

$$\begin{aligned} &\mu : H^s \to H^s(2), \\ &\Delta : H^s(2) \to H^{s+1} \end{aligned} \qquad s = m, m+j, n, n+k,$$

(1) Most primitive invariant of this type is the secondary torsion in [7]. Afterwards we have extended it to block invariants in [2].

(2) The homomorphism μ is natural. Let ${}_2H^{n+1}$ be the subgroup of H^{n+1} consisting of those elements x satisfying $2x = 0$. According to J. H. G. Whitehead we have defined $H^n(2)$ to be $\mu H^n + \Delta_2^* H^{n+1}$. Now Δ is such a homomorphism that $\Delta^{-1}(0) = \mu H^n$ and $\Delta\Delta^* = 1$.

to construct special (μ, Δ, γ)-systems. In [2] it is remarked that block invariants together with Betti numbers and torsions are not enough to characterize isomorphism classes of these special (μ, Δ, γ)-systems, if among the sets of groups $(H^m, H^{m+1}, H^{m+j}, H^{m+j+1})$ and $(H^n, H^{n+1}, H^{n+k}, H^{n+k+1})$ there are common groups. In the simplest case there is one common group between the sets $(H^m, H^{m+1}, H^{m+j}, H^{m+j+1})$ and $(H^n, H^{n+1}, H^{n+k}, H^{n+k+1})$, but in the most complicated case there exist three common groups. The isomorphism class of (μ, Δ, γ)-systems of the first case has been studied in [3]. But in the latter case we have $j = k = 2$ and $m = n+1$, which system plays an important role in the homotopy type of A_n^3-polyhedron $(n \geqslant 3)$(3) and is called A_n^3-system. In [5] we have proved the following

THEOREM. *The complete set of invariants of proper isomorphism class of A_n^3-systems are Betti numbers, torsions, block invariants, relative block invariants, characteristic coefficients and characteristic polynomials.*

This Theorem was first announced in [3]. But the proof in [4] needs to be clarified as it has been hastened by the application of this Theorem to classify homotopy types of A_n^3-polyhedra (4). The purpose of this paper is to produce new definitions of relative block invariants, characteristic coefficients and characteristic polynomials, and consequently to give a new proof of the above Theorem.

§ 2. Let $s(r, 0) = 0$ and let $s(r, i)$ $(i = 1, \ldots, n_r+1)$ be an increasing sequence of integers. Let H^{r+n} be an abelian group with generators $u^r(1), \ldots, u^r(s(r, n_r+1)), \ldots$ such that (5) $u^r(s(r, i)+j)$, $j = 1, \ldots, s(r, i+1)-s(r, i)$, are of order $2^{m(r,i+1)}$, where $m(r, i) < m(r, i+1)$ and $m(r, n_r+1) = \infty$. We often write $2\varrho^r_{s(r,i)+j}$ as the order of $u^r(s(r, i)+j)$.

An automorphism of H^{r+n} may be given by

$$g_{r+n}u^r(i) = \sum_{j=1}^{s(r,n_r)} a^r_{i,j}u^r(j), \qquad i = 1, \ldots, s(r, n_r),$$

$$(1) \quad g_{r+n}u^r(a) = \sum_{b=s(r,n_r+1)}^{s(r,n_r+1)} \beta^r_{a,b}u^r(b) + \sum_{j=1}^{s(r,n_r)} a^r_{a,j}u^r(j) + \ldots, \qquad a = s(r, n_r)+1, \ldots, s(r, n_r+1),$$

(3) See [8]. But the main theorem in [8] needs to be corrected, see S. C. Chang, *On a theorem of Shiraiwa*, not published.

(4) For example see S. C. Chang, *Normal forms and homotopy groups of polyhedra* (Chinese), to appear in Acta Math. Sinica.

(5) Generators of odd prime power orders are not mentioned here.

where $\alpha^r_{\cdot\cdot}$ and $\beta^r_{\cdot\cdot}$ are integers, the square matrix $[\beta^r_{a,b}]$ being unimodular, the determinant $|\alpha^r_{i,j}|$ odd, and

$$\varrho^r_i \alpha^r_{i,j} \equiv 0 (\mathrm{mod}\, \varrho^r_j). \tag{2}$$

By ${}_2H^{n+1+r}$ we mean the subgroup of H^{n+1+r} such that an element x of H^{n+1+r} belongs to ${}_2H^{n+1+r}$ if, and only if, $2x = 0$. The generators of ${}_2H^{n+1+r}$ are evidently $\varrho^{1+r}_1 u^{1+r}(1), \ldots, \varrho^{1+r}_{n_{1+r}} u^{1+r}\big(s(1+r, n_{1+r})\big)$. We put

$$\sigma^{1+r}_s = \varrho^{1+r}_{s(1+r, n_{1+r}+1)-s+1}, \tag{3}$$

$$v^{1+r}_k = u^{1+r}\big(s(1+r, n_{1+r}) - k + 1\big). \tag{4}$$

Substituting $1+r$ for r in (1) and (2) we see that

$$g_{r+1+n}\, \sigma^{1+r}_k v^{1+r}_k = \sum_{s=1}^{s(1+r, n_{1+r})} \frac{\sigma^{1+r}_k}{\sigma^{1+r}_s} \alpha^{1+r}_{s(1+r, n_{1+r})-k+1, s(1+r, n_{1+r})-s+1} (\sigma^{1+r}_s v^{1+r}_s) \tag{5}$$

$$k = 1, \ldots, s(1+r, n_{1+r}),$$

where

$$\frac{\sigma^{1+r}_k}{\sigma^{1+r}_s} \alpha^{1+r}_{s(1+r, n_{1+r})-k+1, s(1+r, n_{1+r})-s+1}$$

are integers and the determinant

$$\frac{\sigma^{1+r}_k}{\sigma^{1+r}_s} \alpha^{1+r}_{s(1+r, n_{1+r})-k+1, s(1+r, n_{1+r})-s+1}$$

is odd.

Denote the generators of $H^{n+r}(2)$ by

$$x^r_i = \mu u^r(i), \quad i = 1, \ldots, s(r, n_r+1),$$

$$y^r_j = \Delta^* \sigma^{r+1}_j v^{r+1}(j), \quad j = 1, \ldots, s(1+r, n_{r+1}).$$

Then g_{r+n} and g_{r+1+n} induce an automorphism $\Phi^r: H^{n+r}(2) \to H^{n+r}(2)$ as follows:

$$\left\{\begin{aligned}
\Phi^r x^r_i &= \sum_{j=1}^{s(r, n_r)} \alpha^r_{i,j} x^r_j, \quad i = 1, \ldots, s(r, n_r); \\
\Phi^r x^r_a &= \sum_{j=1}^{s(r, n_r)} \alpha^r_{a,j} x^r_j + \sum_{b=s(r, n_r)+1}^{s(r, n_r+1)} \beta^r_{a,b} x^r_b, \quad a = s(r, n_r)+1, \ldots, s(r, n_r+1); \\
\Phi^r y^r_k &= \sum_{j=1}^{s(r, n_r+1)} \delta_{kj} x^r_j + \sum_{s=1}^{s(1+r, n_{1+r})} \frac{\sigma^{1+r}_k}{\sigma^{1+r}_s} \alpha^{1+r}_{s(1+r, n_{1+r})-k+1, s(1+r, n_{1+r})-s+1} y^r_s, \\
&\qquad k = 1, \ldots, s(1+r, n_{1+r});
\end{aligned}\right. \tag{6}$$

in which $\delta_{k,j}$ are arbitrary integers, and each of the integers, $\alpha^r_{\dots}$, $\beta^r_{\dots}$, δ_k and

$$\frac{\sigma_k^{1+r}}{\sigma_s^{1+r}}\, a^{1+r}_{s(1+r,n_{1+r})-k+1,\, s(1+r,\, n_{1+r})-s+1},$$

actually means the reduced mod2 class determined by the integer. Evidently $\Phi^r\mu = \mu g^r$ and $\Delta\Phi^r = g^{r+1}\Delta$. The automorphism Φ^r may be represented by a matrix $[\Phi^r]$ in which we first arrange the rows (or columns) corresponding to $x^r_1, \dots, x^r_{s(r,n_r+1)}$ according to their subscript indices and then arrange those rows (or columns) corresponding to $y^r_1, \dots, y^r_{s(1+r,n_{1+r})}$ also according to their subscript indices. Thus we actually obtain an ordering of the generators of $H^{n+r}(2)$ and call it as *R-ordering*. Now $[\Phi^r]$ has $s(r, n_r+1)+s(1+r, n_{1+r})$ rows and columns. We divide $[\Phi^r]$ into n_r+1+n_{1+r} row strips and column strips such that the i-th row strip (or column strip) contains $s(r, i)-s(r, i-1)$ rows (or columns) if $i \leqslant n_r+1$ and $s(1+r, n_{1+r}-i+n_r+2)-s(1+r, n_{1+r}-i+n_r+1)$ rows (or columns) if $n_r+1 < i \leqslant 1+n_r+n_{1+r}$. The intersection of the i-th row strip and the j-th column strip is a matrix $M^r_{i,j}$. Then we may write

$$[\Phi^r] = [M^r_{i,j}].$$

By (2) and the definition of σ_k^{1+r} we see that $M^r_{i,j} \equiv 0 \pmod 2$, if $i > j$

If all the elements of a matrix lying above the principal diagona are trivial, we call it a *lower matrix*. And we call it *upper matrix*, if al the elements lying below the principal diagonal are trivial. The matrix $[M^r_{i,j}]$ is a lower matrix.

Let us arrange the rows and columns of the matrix $[\Phi^r]$ in another way as follows: Allow the rows (columns) corresponding to y^r_j's to be arranged before those rows (columns) corresponding to x^r_i's and the row (column) corresponding to y^r_i's before that to y^r_j if $i > j$ and the row (column) corresponding to x^r_l before that to x^r_i if $l > i$. This actually gives a new ordering of the generators. We call it *C-ordering*. We divide the matrix so arranged also to $n_{r+1}+n_r+1$ row strips and column strips such that the i-th strip contains $s(1+r, i)-s(1+r, i-1)$ rows or columns if $i \leqslant n_{r+1}$ and contains $s(r, n_r+2-i+n_{r+1})-s(r, n_r+1-i+n_{r+1})$ rows or columns if $i > n_{r+1}$. Let $L^r_{i,j}$ denote the matrix which is the intersection of the i-th row strip and the j-th column strip. Then the matrix $[L^r_{i,j}]$ is now an upper matrix.

We have chosen definite ordering of the basis of $H^{n+r}(2)$ so that Φ^0 and Φ^2 are represented by lower matrices $[M^0_{i,j}]$ and $[M^1_{i,j}]$ while Φ^1 and Φ^2, by upper matrices $[L^2_{a,b}]$ and $[L^3_{a',b'}]$ respectively.

The automorphism Φ^r is induced by g_{r+n} and g_{r+1+n}. But (2) and (6) tell us that among the coefficients $\alpha^r_{i,j}$, only those will produce influence upon Φ^r, for which $\varrho^r_j \leqslant \varrho^r_i$; furthermore, the last equation of (6) shows that

$a_{i,j}^{1+r}$ will produce influence upon Φ^r if, and only if, $\varrho_i^{1+r} \leqslant \varrho_j^{1+r}$. We remark that g_{n+r} $(r = 1, 2, 3)$ produces effects upon Φ^{r-1} and Φ^r.

Let θ be a square matrix of n rows whose element $\theta_{i,j}$ is 1 if, and only if, $i+j = n+1$ and $\theta_{i,j}$ is zero if $i+j \neq n+1$. In case A is a square matrix of n rows, then we denote $\theta A \theta$ by $\dot{A}$.

Automorphisms Φ^r of $H^{n+r}(2)$ are called (μ, Δ)-*automorphisms* if, and only if, $\Delta\Phi^r = g_{n+r+1}\Delta$ and $\mu g_{n+r} = \Phi^r \mu$.

LEMMA 1. *The automorphisms* $\Phi^r: H^{n+r}(2) \to H^{n+r}(2)$ $(r = 0, 1, 2, 3)$ *are* (μ, Δ)-*automorphisms if, and only if, the following conditions are satisfied*:

(i) *The matrices* $[M_{i,j}^0]$ *and* $[M_{i',j'}^1]$ *are lower matrices if the rows and columns of these matrices are arranged according to R-ordering of their corresponding generators. The matrices* $[L_{h,k}^2]$ *and* $[L_{h',k'}^3]$ *are upper matrices if the rows and columns of these matrices are arranged according to C-ordering of their corresponding generators.*

(ii) *We have*

$$\dot{M}_{i,i}^1 = M_{n_0+2+n_1-i,n_0+2+n_1-i}^0, \quad i = 1, \ldots, n_1; \tag{7}$$

$$M_{j,j}^1 = L_{n_3+j-n_1,n_3-n_1+j}^2, \quad j = n_1+2, \ldots, n_1+n_2+1; \tag{8}$$

$$\dot{L}_{i,i}^3 = L_{n_3+2-i+n_4,n_3+n_4+2-i}^2, \quad i = n_4+2, \ldots, n_4+n_3+1. \tag{9}$$

This lemma is proved in [4]. Here we give an outline of the proof. Since $[M_{i,j}^0]$ is influenced by $a_{l,k}^{n+1}$ if $\varrho_l^{n+1} \leqslant \varrho_k^{n+1}$, while $[M_{h,k}^1]$ by $a_{l,k}^{n+1}$ if $\varrho_l^{n+1} \geqslant \varrho_k^{n+1}$, $a_{l,k}^{n+1}$ will produce effect upon both $[M_{i,j}^0]$ and $[M_{h,k}^1]^{l,k}$ only if $\varrho_l^{n+1} = \varrho_k^{n+1}$. Those particular $a_{l,k}^{n+1}$ determining $M_{i,i}^1$ and $M_{n_0+2+n_1-i,n_0+2+n_1-i}$ $(i = 1, \ldots, n_1)$. Due to orderings of the rows and columns previously given to the matrices representing Φ^0 and Φ^1 we reach (7). Similar arguments give (8) and (9). The sufficiency is a construction of automorphisms g_{n+r} $(r = 0, 1, 2, 3, 4)$ of H^{n+r} so that they induce Φ^i $(i = 0, \ldots, 3)$ as (μ, Δ)-automorphisms. This is an extensive argument of that used in [3] and follows as natural consequence of some Lemmas in [6].

Our purpose is to determine the complete and independent system of invariants of proper isomorphism class of A_n^3-systems. By Lemma 1 in [4] what we need is to determine the system of invariants of proper automorphism class of A_n^3-systems. The procedure has been devised by Lemma 2 in [4]. We write down the basis of $H^n(2)$ according to R-ordering as x_i $(i = 1, \ldots, s(0, n_0+1)+s(1, n_1))$ and that of $H^{n+2}(2)$ according to C-ordering as y_l $(l = 1, \ldots, s(2, n_2+1)+s(3, n_3))$. The homomorphism γ_n^2 becomes

$$\gamma_n^2 x_i = \sum c_{i,l} y_l, \quad i = 1, \ldots, s(0, n_0+1)+s(1, n_1),$$

which is represented by a matrix $[c_{ij}]$. If $[c_{ij}] \sim [c'_{ij_2}]$, i. e. there exist (μ, Δ)-automorphisms $[M^0_{i,j}]$ and $[L^2_{k,l}]$ such that

$$[M^0_{i,j}][c_{j,l}] = [c'_{i,l}][L^2_{l,k}],$$

the normal form of the homomorphism γ^2_n is determined by block invariants, $\tau_{i,l}$ $(i = 1, \ldots, n_0+n_1+1;\ l = 1, \ldots, n_2+n_3+1)$. We define

$$\tau_{i,n_2+n_3+2} = \begin{cases} s(0, i) - s(0, i-1) - \sum\limits_{l=1}^{n_2+n_3+1} \tau_{i,l} & (i = 1, \ldots, n_0+1), \\ s(1, n_1+n_0+2-i) - s(1, n_1+n_0+1-i) - \sum\limits_{l=1}^{n_2+n_3+1} \tau_{i,l} & \\ & (i = n_0+2, \ldots, n_0+n_1+1); \end{cases}$$

$$\tau_{n_0+n_1+1,l} = \begin{cases} s(3, j) - s(3, j-1) - \sum\limits_{i=1}^{n_0+n_1+1} \tau_{i,l} & (l = 1, \ldots, n_3), \\ s(2, n_2+n_3+2-l) - s(2, n_2+n_3+1-l) - \sum\limits_{i=1}^{n_0+n_1+1} \tau_{i,l} & \\ & (l = n_3+1, \ldots, n_3+n_2+1). \end{cases}$$

The normal form of γ^2_n may be constructed as follows: We take a matrix $[M_{\alpha,\beta}]$ of $(n_0+n_1+1)(n_2+n_3+2)$ rows and $(n_0+n_1+2)(n_2+n_3+1)$ columns. By $\tau_{i,j}$ we also mean a unit square matrix of $\tau_{i,j}$ rows. $[M_{\alpha,\beta}]$ is a normal form if: (i) $M_{\alpha,\beta}$ denotes matrices, (ii) $M_{\alpha,\beta}$ and $M_{\alpha,\beta'}$ are matrices of the same number of rows, (iii) $M_{\alpha,\beta}$ and $M_{\alpha',\beta}$ are matrices of the same number of columns, (iv) $M_{i(n_2+n_3+2),\beta}$ consists of τ_{i,n_2+n_3+2} rows, (v) $M_{\alpha,j(n_0+n_1-2)}$ consists of $\tau_{n_0+n_1+2,j}$ columns and (vi) all the matrices $M_{\alpha,\beta}$ are zero matrices except

$$M_{(i-1)(n_2+n_3+2)+j,(j-1)(n_0+n_1+2)+i} = \tau_{i,j},$$
$$i = 1, \ldots, n_0+n_1+1; \qquad j = 1, \ldots, n_2+n_3+1.$$

Let M be a matrix of r rows and s columns. Finite sequences of positive integers $\{a_i\}$ $(i = 1, \ldots, n)$ and $\{b_j\}$ $(j = 1, \ldots, m)$ satisfying $\sum\limits_1^m a_i = r$, $\sum\limits_1^m b_j = s$ are utilized to divide M into n row strips M_i $(i = 1, \ldots, n)$ and m column strips M^j $(j = 1, \ldots, m)$ such that M_l contains the

$$\Big(\sum_{i=1}^{l-1} a_i + p\Big)\text{-th} \qquad (p = 1, \ldots, a_l),$$

rows of M and M^k contains the

$$\Big(\sum_{j=1}^{k-1} b_j+q\Big)\text{-th} \qquad (q=1,\ldots,b_k)$$

columns. The intersection between M_l and M^k is a matrix $M_{l,k}$. Then the matrix $[M_{l,k}]$ is called the *subdivision of M according to $\{a_i\}$ and $\{b_j\}$*. If M is a square matrix and if $\{a_i\}=\{b_i\}$ then $[M_{l,k}]$ is called the *subdivision of M according to $\{a_i\}$*.

We arrange $\tau_{i,j}$ $(i=1,\ldots,n_0+n_1+1;\ j=1,\ldots,n_2+n_3+2)$ as a sequence φ such that $\tau_{i,j}<\tau_{i',j'}$ if $i<i'$ and $\tau_{ij}<\tau_{ij'}$, if $j<j'$. The subdivision of $(\Phi^0)(=[M^0_{ij}])$ according to φ actually subdivide $M^0_{i,j}$ into a matrix(6) $[M^0_{i,j;u,v}]$, $u,v=1,\ldots,n_2+n_3+2$. We also may arrange $\tau_{i,j}$ $(j=1,\ldots,n_2+n_2+1,\ i=1,\ldots,n_0+n_1+2)$ as a sequence ψ such that $\tau_{i,j}<\tau_{i'j'}$, if $j<j'$ and $\tau_{ij}<\tau_{i'j}$ if $i<i'$. By ψ the subdivision of $[\Phi^2]$ $(=[L^2_{h,k}])$ induces a subdivision of $L^2_{h,k}$ into a matrix $[L^2_{h,k;r,s}]$ $(r,s=1,\ldots,n_0+n_1+2)$.

If M is a normal form of γ^2_n mentioned above, then we have

LEMMA 2. *If $[M^0_{i,j}]M=M[L^2_{h,k}]$ then*

(i) $[M^0_{i,i;u,v}]$ $(i=1,\ldots,n_0+n_1+1)$ *are unimodular upper matrices,*

(ii) $[L^2_{l,l;r,s}]$ $(l=1,\ldots,n_2+n_3+1)$ *are unimodular lower matrices,*

(iii) $M^0_{i,i;u,u}=L^2_{u,u;i,i}$, $\quad i=1,\ldots,n_0+n_1+1,\ u=1,\ldots,n_2+n_3+1.$

Conversely, if the matrices $[M^0_{i,i;u,v}]$ $(i=1,\ldots,n_0+n_1+1)$ and $[L^2_{l,l;r,s}]$ $(l=1,\ldots,n_2+n_3+1)$, satisfying (i), (ii) *and* (iii) *are given, then there are lower matrix $[M^0_{i,j}]$ and upper matrix $[L^2_{h,k}]$ such that $M^0_{i,i}$ and $L^2_{l,l}$ coincide with the given matrices $[M^0_{i,i;u,v}]$ and $[L^2_{l,l;r,s}]$ respectively and that $[M^0_{i,j}]\,M=M[L^2_{h,k}]$.*

Lemma 2 is implied in Lemmas 4 and 5 in [4].

Here I mention again the Lemma 2 in [3], which tells us that what we need is to find the complete system of invariants of γ_{n+1} under those (μ,Δ)-automorphisms (of the A^3_n-system) which keep the normal form, M, of γ^2_n invariant. In other words, we need to find the complete and independent set of common properties of γ^2_{n+1} and γ'^2_{n+1} if

$$[M'_{a,b}][\gamma^2_{n+1}]=[\gamma'^2_{n+1}][L^3_{h,k}], \tag{10}$$

where the lower matrix $[M'_{a,b}]$ and the upper matrix $[L^3_{h,k}]$ must be restricted by (7), (8), (9) and (i), (ii), (iii), of Lemma 2. Now (7) and (i) tell us that the subdivision of $M'_{i,i}$ $(i=1,2,\ldots,n_1)$ according to the sequence $\tau_{i,1},\tau_{i,2},\ldots,\tau_{i,n_2+n_3+2}$ is a unimodular upper matrix, i. e. the

(6) Note that in the matrix $[M^0_{i,j;u,v}]$ the indices i and j are fixed while u and v indicate the number of rows or columns respectively. It is similar in $[L^2_{h,k;r,s}]$.

subdivision of $M^1_{i,i}$ according to the sequence $\tau_{i,n_2+n_3+2}, \tau_{i,n_2+n_3+1}, \dots, \tau_{i,1}$ is a unimodular lower matrix. Similarly (8) and (ii) reveal that the subdivision of $M^1_{j,j}$ $(j = n_1+2, \dots, n_1+n_2+1)$ according to the sequence $\tau_{1,n_3-n_1+j}, \tau_{2,n_3-n_1+j}, \dots, \tau_{n_0+n_1+2,n_3-n_1+j}$ becomes a unimodular lower matrix. Meanwhile (9) and (ii) give us that $L^3_{i,i}$ $(i = n_4+2, \dots, n_4+n_3+1)$ is subdivided to be a unimodular upper matrix according to the sequence $\tau_{n_0+n_1+2,n_3+n_4+2-i}, \tau_{n_0+n_1+1,n_3+n_4+2-i}, \dots, \tau_{1,n_3+n_4+2-i}$. In short, the subdivision of the matrix $[M^1_{a,b}]$ is a lower matrix written as $[\mathfrak{M}_{i,j}]$ $(i, j = 1, \dots, n_1(n_2+n_3+2)+1+n_2(n_0+n_1+2))$ according to the sequence of integers

$$
\tag{11}
\left\{
\begin{array}{l}
\tau_{n_0+n_1+1,n_2+n_3+2}, \tau_{n_0+n_1+1,n_2+n_3+1}, \dots, \tau_{n_0+n_1+1,n_3+2}, \tau_{n_0+n_1+1,n_3+1}, \\
\qquad\qquad\qquad \tau_{n_0+n_1+1,n_3}, \dots, \tau_{n_0+n_1+1,2}, \tau_{n_0+n_1+1,1}; , \dots; \\
\tau_{n_0+2,n_2+n_3+2}, \tau_{n_0+2,n_2+n_3+1}, \dots, \tau_{n_0+2,n_3+2}, \tau_{n_0+2,n_3+1}, \tau_{n_0+2,n_3}, \dots, \\
\qquad\qquad\qquad\qquad\qquad\qquad \tau_{n_0+2,2}, \tau_{n_0+2,1}; \\
s(1, n_1+1)-s(1, n_1); \\
\tau_{1,n_3+2}, \tau_{2,n_3+2}, \dots, \tau_{n_0+1,n_3+2}, \tau_{n_0+2,n_3+2}, \dots, \tau_{n_0+n_1+1,n_3+2}, \\
\qquad\qquad\qquad\qquad\qquad\qquad \tau_{n_0+n_1+2,n_3+2}; \dots; \\
\tau_{1,n_3+n_2+1}, \tau_{2,n_3+n_2+1}, \dots, \tau_{n_0+1,n_3+n_2+1}, \tau_{n_0+2,n_3+n_2+1}, \dots, \\
\qquad\qquad\qquad\qquad \tau_{n_0+n_1+1,n_3+n_2+1}, \tau_{n_0+n_1+2,n_3+n_2+1}.
\end{array}
\right.
$$

Similarly the subdivision of $[L^3_{h,k}]$ is an upper matrix, written as $[\mathfrak{L}_{h,k}]$ $(h, k = 1, \dots, n_4+1+n_3(n_0+n_1+2))$, according to the sequence of integers

$$
\tag{12}
\left\{
\begin{array}{l}
s(4, 1), s(4,2)-s(4, 1), \dots, s(4, n_4)-s(4, n_4-1); \\
\qquad\qquad\qquad\qquad\qquad\qquad s(3, n_3+1)-(3, n_3); \\
\tau_{n_0+n_1+2,n_3}, \tau_{n_0+n_1+1,n_3}, \dots, \tau_{n_0+2,n_3}, \tau_{n_0+1,n_3}, \dots, \tau_{1,n_3}; \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\tau_{n_0+n_1+2,1}, \tau_{n_0+n_1+1,1}, \dots, \tau_{n_0+2,1}, \tau_{n_0+1,1}, \dots, \tau_{1,1}.
\end{array}
\right.
$$

Besides these, writing for brevity $M_{i;u}$ and $L_{h;k}$ for $M_{i,i;u,u}$ and $L_{h,h;k}$, respectively, we have the following relations:

$$
\begin{aligned}
\dot{M}^1_{i;u} &= M^0_{n_2+n_1+2-i;n_2+n_3+3-u}, \quad i = 1, \dots, n_1; u = 1, \dots, n_3+n_2+2; && \text{(by (7))} \\
&= L^2_{n_2+n_3+3-u;n_2+n_1+2-i}, \quad i = 1, \dots, n_1; u = 2, \dots, n_3+n_2+2; && \text{(by (iii))} \\
&= \dot{L}^3_{n_4-n_2-1+u;i}, \quad i = 1, \dots, n_1; u = n_2+3, \dots, n_3+n_2+2; && \text{(by (9))}
\end{aligned}
$$

and

$$
\dot{M}^1_{i;u} = M^1_{n_2+n_1+3-u;n_2+n_1+2-i}, \quad u = 2, \dots, n_2+1; i = 1; \dots, n_1. \qquad \text{(by (8))}
$$

In new notations they are

$$(13)\qquad \mathfrak{M}_{(i-1)(n_2+n_3+2)+u,*} = \mathfrak{L}_{(u-u_2-3)(n_0+n_1+2)+n_4+1+i,*},$$
$$i = 1, \ldots, n_1;\ u = n_2+3, \ldots, n_3+n_2+2;$$

$$(14)\qquad \mathfrak{M}_{(i-1)(n_2+n_3+2)+u,*} = \mathfrak{M}_{(n_2+n_1+2-u)(n_2+n_3+2)+(n_2+n_1+2-i),*},$$
$$i = 1, \ldots, n_1;\ u = 2, \ldots, n_2+1,$$

where $\mathfrak{M}_{r,*}$ (or $L_{r,*}$) means $\mathfrak{M}_{r,r}$ (or $L_{r,r}$).

Our algebraic problem is to classify those matrices such that $[\gamma'^2_{n+1}] \sim [\gamma^2_{n+1}]$ if, and only if, there exist unimodular matrices M and L satisfying

$$M[\gamma^2_{n+1}] = [\gamma'^2_{n+1}]L,$$

and having the following properties: The subdivision $[\mathfrak{M}_{i,j}]$ of M according to the sequence (11) is a lower matrix, the subdivision $[L_{h,k}]$ of L according to the sequence (12) is an upper matrix; and, furthemore, (13) and (14) must hold. Before going on to prove the theorem stated in § 1 we need to introduce several useful lemmas.

§ 3. Let $[\gamma]$ be a matrix of $\sum_{i=1}^{m+1} s_i$ rows and $\sum_{j=1}^{n} r_j$ columns. Hereafter elements of matrices are always integers reduced mod 2. Let A be a unimodular square matrix of $\sum_{i=1}^{m+1} s_i$ rows, of which the subdivision, $[A_{i,j}]$, with respect to the sequence $s_1, \ldots, s_{m+1}$ is a lower matrix. By B we mean a unimodular square matrix of $\sum_{j=1}^{n} r_j$ rows, whose subdivision $[B_{h,k}]$ with respect to the sequence $r_1, \ldots, r_n$ is an upper matrix.

Sometimes there exist suitable A and B so that the matrix $A[\gamma]B^{-1}$ becomes

$$(15)\qquad [\gamma'] = \begin{array}{c} \left[\begin{array}{cccccccccccc}
\tau_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau_{1,2} & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \tau_{1,n} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \tau_{m,1} & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tau_{m,2} & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & \tau_{m,n} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & \cdot & 0 & 0 & c_n
\end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} s_1 \\ \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} s_m \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} s_{m+1} \end{array} \\
\underbrace{\qquad\qquad}_{n_1}\ \underbrace{\qquad\qquad}_{n_2}\ \underbrace{\qquad\qquad}_{n_n}
\end{array}$$

where $\tau_{i,j}$ means a unit matrix of $\tau_{i,j}$ rows. I remark that once "1" appears in certain column or row of $\tau_{i,j}$ in $[\gamma']$, then "1" would never appear in the same column or row in $[\gamma']$. For convenience the matrix $[\gamma']$ is written as $\begin{bmatrix} N \\ C \end{bmatrix}$, where N denotes the first $\sum_1^m s_i$ rows. By Q we mean the set of integers

$$\left\{\sum_{i=1}^{j-1} r_i + k \quad (j = 1, \ldots, n;\ k = 1, \ldots, \sum_{l=1}^{m} \tau_{l,j})\right\}.$$

In B we delete those columns and rows whose indices of columns and rows belong to Q; finally a square upper matrix B' remains. In the matrix C we remove those trivial columns whose indices of columns belong to Q and finally we obtain a matrix $\bar{C}$. Then we have

LEMMA 3. *If*

$$A\begin{bmatrix} N \\ C \end{bmatrix} = \begin{bmatrix} N \\ C' \end{bmatrix} B, \tag{16}$$

then B' is unimodular and $A_{m+1,m+1}\bar{C} = \bar{C}'B'$. Conversely, if $A_{m+1,m+1}$ is a unimodular matrix and if B' is a unimodular matrix whose subdivision with respect to

$$r_1 - \sum_{i=1}^{m} \tau_{i1}, \ldots, r_n - \sum_{i=1}^{m} \tau_{im}$$

is an upper matrix, and if $A_{m+1,m+1}\bar{C} = \bar{C}'B'$, then we may extend $A_{m+1,m+1}$ and B' to A and B such that (16) *holds.*

Proof. Let the matrix A be written as

$$\begin{bmatrix} A_1 & 0 \\ A_{m+1} & A_{m+1,m+1} \end{bmatrix},$$

where A_1 is a unimodular lower matrix. By (16) we have $A_1 N = NB$. By Lemma 2 the subdivision of $B_{h,k}$ according to the sequences

$$\left\{\tau_{1,h}, \ldots, \tau_{m,h}, r_h - \sum_{i=1}^{m} \tau_{i,h}\right\} \quad \text{and} \quad \left\{\tau_{1,k}, \ldots, \tau_{m,k}, r_k - \sum_{i=1}^{m} \tau_{i,k}\right\}$$

becomes a lower matrix $[B_{h,k;u,v}]$ (h, k are fixed). Hence B' must be unimodular since B is unimodular too. By simple calculation we may have $A_{m+1,m+1}\bar{C} = \bar{C}'B'$. Conversely, if B' is the given unimodular matrix,

then we may take $B'_{h,k}$ as $B_{h,k;m+1,m+1}$ and take

$$B_{h,k;u,v} = 0 \quad \text{if} \quad h \neq k,\ u+v < 2m+2,$$

$$B_{h,h;u,v} = \begin{cases} 0 & \text{if} \quad u \neq v, \\ \tau_{u,h} & \text{if} \quad u = v < m+1. \end{cases}$$

Take A_1 as the unit matrix. We have $A_1 N = N = NB$. To determine A_{m+1} it requires to inspect

$$A_{m+1} N + A_{m+1,m+1} C = C' B. \tag{17}$$

If we examine both sides of (17) about those columns whose indices of columns are not belonging to Q, we obtain $A_{m+1,m+1}\bar{C} = \bar{C}' B'$ which has been given to be true. Since N is normal, in each column of N whose index of column belongs to Q we find a unique 1, and in each row of N there is, at most, one 1 only; we may therefore easily find A_{m+1} to satisfy (17) by comparing those columns of both sides of (17) when their indices of columns belong to Q. This concludes the Lemma.

Suppose $[\gamma]$ and $[\gamma']$ are matrices of $\sum_{i=1}^{m+1} s_i$ rows and $\sum_{i=1}^{n} r_i$ columns. If their subdivisions with respect to $s_1, \ldots, s_{m+1}$ and $r_1, \ldots, r_n$ are $[\gamma_{i,j}]$ and $[\gamma'_{i,j}]$ and if rank of $[\gamma_{1,1}, \ldots, \gamma_{1,k}] =$ rank of $[\gamma'_{1,1}, \ldots, \gamma'_{1,k}]$, $k = 1, \ldots, m' \leqslant n$, then the block invariants $\tau_{1,k}$ $(k = 1, \ldots, m')$ of $[\gamma]$ and $[\gamma']$ are the same. Now $[\gamma]$ may be reduced to the following subnormal form ([7]):

$$[\bar{\gamma}] = \begin{array}{c} \left[\begin{array}{cccccccccc} \tau_{1,1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \tau_{1,m} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & \tau_{1,m'} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \gamma_{m'+1} & \ldots & \gamma_n \\ \underbrace{0 \quad c_1}_{r_1} & & \underbrace{0 \quad c_2}_{r_2} & & \ldots & \underbrace{0 \quad c_{m'}}_{r_{m'}} & & \underbrace{c_{m'+1}}_{r_{m'+1}} & \ldots & c_n \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} s_1 \\ \sum_{i=2}^{m+1} s_i \end{array} \end{array} \tag{18}$$

while $[\gamma']$ may be reduced to a similar subnormal form $[\bar{\gamma}']$, obtained from (18) by replacing $\gamma_{m'+1}, \ldots, \gamma_m$ and $c_1, c_2, \ldots, c_m$ with $\gamma'_{m'+1}, \ldots, \gamma'_m$ and $c'_1, \ldots, c'_m$ respectively.

LEMMA 4. *If $A[\bar{\gamma}] = [\bar{\gamma}']B$, then the subdivision $[A_{1,1;u,v}]$ of $A_{1,1}$, with respect to the sequence $\tau_{1,1}, \ldots, \tau_{1,m'}, s_1 - \sum_{j=1}^{m'} \tau_{1,j}$, is an upper matrix,*

([7]) There exist A and B such that $A[\gamma] = [\bar{\gamma}]B$.

and

$$(19)\quad \begin{bmatrix} A_{1,1;m'+1,m'+1} & 0 & 0 & 0 \\ * & A_{2,2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & A_{m+1,2} & \cdots & A_{m+1,m+1} \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 & \gamma_{m'+1} & \cdots & \gamma_n \\ c_1 & c_2 & \dots & c_{m'} & c_{m'+1} & \dots & c_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 & \gamma'_{m'+1} & \cdots & \gamma'_m \\ c'_1 & c'_2 & \dots & c'_{m'} & c'_{m'+1} & \dots & c'_n \end{bmatrix} B',$$

where B' is the unimodular square matrix obtained from B by striking out those columns and rows whose indices of columns or rows belong to the set

$$\bar{Q}\Big\{\sum_{v=1}^{j-1} r_v + i,\ i = 1, \dots, \tau_{1,j};\ j = 1, \dots, m'\Big\}.$$

Conversely, if (19) *is true, where the matrices*

$$\begin{bmatrix} A_{1,1;m'+1,m'+1} & & & 0 \\ * & A_{2,2} & & \\ \vdots & & \ddots & \\ * & A_{m+1,2} & \cdots & A_{m+1,m+1} \end{bmatrix},$$

and B' are unimodular matrices, the subdivision of B', accordingly to the sequence $r_1 - \tau_{1,1}, \dots, r_{m'} - \tau_{1,m'}, r_{m'+1}, \dots, r_n$ being an upper matrix, then we may extend (19) *to $A[\bar{y}] = [\bar{y}']B$.*

Proof. Since the subdivision of A with respect to the sequence $s_1, \dots, s_{m+1}$ is a unimodular lower matrix $[A_{ij}]$ $(i, j = 1, \dots, m+1)$ and since the subdivision of B with respect to $r_1, \dots, r_n$ is a unimodular upper matrix $[B_{r,s}]$ $(r, s = 1, \dots, n)$, the elements of the first s_1 rows and first $\sum_{i=1}^{m'} r_i$ columns of both sides of $A[\bar{y}] = [\bar{y}']B$, after comparison, give the following equality:

(20)

$$A_{1,1}\begin{bmatrix} \tau_{1,1} & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & \tau_{1,2} & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \tau_{1,m'} & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 \end{bmatrix} = \begin{bmatrix} \tau_{1,1} & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & \tau_{1,2} & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \tau_{1,m'} & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 \end{bmatrix} \begin{bmatrix} B_{1,1} & & & \\ 0 & B_{2,2} & & * \\ & & \ddots & \\ 0 & & & B_{m',m'} \end{bmatrix}$$

which tells us that the subdivision of $A_{1,1}$ with respect to the sequence $\tau_{1,1}, \ldots, \tau_{1,m'}, s_1 - \sum_{i=1}^{m'} \tau_{1,i}$ is a unimodular upper matrix $[A_{1,1}; u, v]$ $(u, v = 1, \ldots, m'+1)$, and the subdivision of $B_{i,j}$ $(i, j = 1, \ldots, m')$ with respect to the sequences $\{\tau_{1,i}, r_i - \tau_{1,i}\}$ and $\{\tau_{1,j}, r_j - \tau_{1,j}\}$ is a lower matrix (8) $[B_{i,j;h,k}]$ $(h, k = 1, 2)$. Now in the matrix

$$\begin{bmatrix} B_{1,1} & & * \\ & \ddots & \\ 0 & & B_{m',m'} \end{bmatrix}$$

strike out rows and columns whose indices of rows or columns belong to $\bar{Q}$ and reach a new matrix which is unimodular since $B_{i,j;1,2} = 0$ $(i, j = 1, \ldots, m')$. Consequently B' is unimodular. We subdivide the matrix $B_{h,m'+i}$ $(h = 1, \ldots, m';\ i = 1, \ldots, n-m)$ by the sequences $\{\tau_{1,h}, r_h - \tau_{1,h}\}$ and $\{r_{m'+i}\}$, the last sequence consisting of one term only, into the matrix

$$\begin{bmatrix} B_{h,m'+i;1} \\ B_{h,m'+i;2} \end{bmatrix}.$$

Comparing the elements of the first $\sum_{j=1}^{m'} \tau_{1,j}$ rows and the last $\sum_{u=m'+1}^{n} r_u$ columns in the both sides of $A[\bar{y}] = [\bar{y}']B$, we have

$$(21) \quad A_{1,1;r,m'+1} \cdot \gamma_{m'+i} = \tau_{1,r} \cdot B_{r,m'+i;1}, \quad (i = 1, \ldots, n-m';\ \ r = 1, \ldots, m').$$

Comparing the elements of the last $\sum_{i=1}^{m+1} s_i - \sum_{j=1}^{m'} \tau_{1,j}$ rows we have two cases. In the first case we consider only those elements whose indices of columns do not belong to $\bar{Q}$. Then we obtain (19). Denote the matrix $[A_{i,1}]$ $(i = 2, \ldots, m+1)$ by $A_{*,1}$ and subdivide $A_{*,1}$ into $\{A_{*,1;j}\}$, $j = 1, \ldots, m'+1$, by the sequences $\{\sum_{i=2}^{m'+1} s_i\}$ and $\{\tau_{1,1}, \ldots, \tau_{1,m'}, s_1 - \sum_{i=1}^{m'} \tau_{1,i}\}$, the sequence $\{\sum_{i}^{m'+1} s_i\}$ consisting of one term only. Then we consider the contrary case when the indices of columns belong to $\bar{Q}$ and obtain

$$(22) \qquad A_{*,1;j} \cdot \tau_{1,j} = \sum_{l=1}^{j} c_l' B_{l,j;2,1} \quad (j = 1, \ldots, m').$$

The converse may be proved by constructing A and B so that $A[\bar{y}] = [\bar{y}']B$ if (19) is given to be true. It is easy to extend $A_{1,1;m+1,m+1}$ to a uni-

(8) Further relations actually exist between $A_{1,1;u,u}$ and $B_{i,i;v,v}$. See Lemma 2 above.

modular upper matrix $[A_{1,1;i,j}]$, $i, j = 1, \ldots, m'+1$. Put $B_{i,j;1,2} = 0$, $B_{i,j;1,1} = A_{1,1;i,j}$, $i, j = 1, \ldots, m'$, then (20) evidently holds.

We determine $B_{r,m'+k;1}$ $(k = 1, \ldots, n-m'; r = 1, \ldots, m')$ by (21) and $A_{k,1;j}$ by (22). Besides these $B_{i,j;1,1}(i,j=1,\ldots,m')$, $B_{r,m'+k;2}$ $(k=1,\ldots,n-m')$, $B_{u,v}$ $(u, v = m'+1, \ldots, n)$, $A_{1,1;m'+1,m'+1}$ and $A_{k,l}$ $(h, l = 2, \ldots, m+1)$ are given in (19) except $B_{i,j;2,1}$ $(i, j = 1, \ldots, m')$ which may be arbitrary. Hence we have determined the upper matrix $[B_{h,k}]$ $(h, k = 1, \ldots, n)$ and the lower matrix $[A_{a,b}]$ $(a, b = 1, \ldots, m+1)$ where $A_{1,1}$ is subdivided in order to be an upper matrix $[A_{1,1;u,v}]$ $(u, v = 1, \ldots, m'+1)$. It is evident that A and B are unimodular. Since (19), (20), (21) and (22) all hold, therefore $A[\bar{\gamma}] = [\bar{\gamma}']B$. This concludes the lemma.

§ 4. Let γ, γ', A and B be matrices as in § 3. If $A_{1,1} = B_{i,i}$ and if $\tau_{1,1} = \tau_{1,2} = \ldots = \tau_{1,i-1} = 0$, then in the equation $A[\bar{\gamma}] = [\bar{\gamma}']B$ we first pay attention to

$$A_{1,1}\gamma_{1,i} = \gamma'_{1,i}B_{i,i} = \gamma'_{1,i}A_{1,1} \tag{23}$$

and wish to reduce the matrix $\gamma_{1,i}$ and $\gamma'_{1,i}$ to their normal form. Now $A_{1,1}$ is a unimodular matrix. It is known that the normal form of the square matrix, $\gamma_{1,i}$, under symmetric transformations is a diagonal matrix (see [8], p. 84).

$$[P_1, P_2, \ldots, P_N, 0, \ldots, 0] \tag{24}$$

where P_i $(i = 1, \ldots, n)$ are square matrices determined by characteristic coefficients and characteristic polynomials. In detail, if

$$|\lambda I - \gamma_{1,i}| = f_1^{p_1}(\lambda) f_2^{p_2}(\lambda) \ldots f_k^{p_k}(\lambda),$$

$f_i(\lambda)$ being irreducible polynomials with integers reduced mod 2 as coefficients, then $f_i(\lambda)$ are called *characteristic polynomials* of $[\gamma_{1,i}]$. Among the invariant factors of $[\lambda I - \gamma_{1,i}]$ if $f_i^{p_{ij}}(\lambda)$ $(j = 1, \ldots, n_i)$ happen as factors, then $\sum_{j=1}^{n_i} p_{ij} = p_i$. If $f_k(\lambda) = \lambda$, we arrange $p_{k,j}$ in descending order (9) that

$$p_{k,1} = \ldots = p_{k,j_1} > p_{k,j_1+1} = \ldots = p_{k,j_2} > \ldots > p_{k,j_w+1} = \ldots = p_{k,n_k} = 1,$$

where $n_k - j_w$ may be trivial.

Let the degree of $f_i(\lambda)$ be t_i, then

$$s_1 = r_i = \sum_{i=1}^{k-1} \sum_{j=1}^{n_i} p_{i,j} t_i + \sum_{j=1}^{j_w} p_{k,j} + n_k - j_w.$$

(9) $p_{k,l}$ may all be trivial.

We give an ordering of $p_{i,j}$ $(i = 1, \ldots, k-1;\ j = 1, \ldots, n_i$ and $i = k;$ $j = 1, \ldots, j_w)$ such that $p_{i,j} \prec p_{i+1,j}$ and $p_{i,j} \prec p_{i,j+1}$ and write the ordered sequence $\{p_{i,j}\}$ as $\{P_l'\}$. If $p_{i,j} = P_l'$, then construct the matrix

$$P_l = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ & & & \cdot & & 0 \\ & & 0 & & \ddots & 1 \\ a_{m(l)} & a_{m(l)-1} & & & \ldots & a_1 \end{bmatrix}, \tag{25}$$

where $m(l)$ denotes $p_{i,j} \cdot t_i$ and $a_i (i = 1, \ldots, m(l))$ are coefficients in $f_i^{p_{i,j}}(\lambda) = \lambda^{m(l)} + a_1 \lambda^{m(l)-1} + \ldots + a_{m(l)}$. Finally, we completely determine the diagonal matrix (24), since $N = \sum_{i=1}^{k-1} n_i + j_w$.

Suppose we have actually reduced $[\gamma]$ to semi-normal form such that $\gamma_{1,i}$ has the normal form (24). If A is a unimodular matrix, what follows is to characterize those A such that

$$A[P_1, \ldots, P_N, 0, \ldots, 0]A^{-1} = [P_1, \ldots, P_N, 0, \ldots, 0].$$

Let

$$P = [P_1, \ldots, P_M], \qquad M = \sum_{i=1}^{k-1} n_i, \qquad Q = [P_{M+1}, \ldots, P_N]$$

and the subdivision of A with respect to the sequence

$$\Big\{\sum_{i=1}^{k-1} \sum_{j=1}^{n_i} p_{i,j} k_i, \sum_{j=1}^{n_k} p_{k,j}, s_1 - \sum_{i=1}^{k-1} \sum_{j=1}^{n_i} p_{i,j} t_i - \sum_{j=1}^{n_k} p_{k,i}\Big\}$$

be

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}.$$

Since $A[P, Q, 0] = [P, Q, 0]A$, we have

$$PA_{1,3} = A_{3,1}P = A_{3,2}Q = QA_{2,3} = 0,$$

$$A_{1,2}Q = PA_{1,2}, \qquad A_{2,1}P = QA_{2,1}, \qquad QA_{2,2} = A_{2,2}Q.$$

Now P is unimodular, hence $A_{1,3} = A_{3,1} = 0$. From $QA_{2,3} = 0$ we know that all the elements of $A_{2,3}$ are trivial except those of the $\sum_{j=1}^{l-1} p_{k,j} + 1$-th $(l = 1, \ldots, j_w)$ rows. Similarly from $A_{3,2}Q = 0$ all the elements

of $A_{3,2}$ are trivial except those of the $\sum_{j=1}^{l} p_{k,j}$-th $(l = 1, \ldots, j_w)$ columns. We know that $A_{1,2} = 0$, by comparing elements column by column of both sides of $A_{1,2}Q = PA_{1,2}$. Similarly $A_{2,1} = 0$.

Furthermore we need to investigate the properties of the matrix $A_{2,2}$. By I_i we mean the unit square matrix of $(p_{k,i}-1)$ $(i = 1, \ldots, j_w)$ rows. Then the square matrix Q may be written as

$$\begin{bmatrix} 0 & I_1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & I_2 & 0 & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & 0 & & 0 & I_{j_w} \\ & & & & & 0 & 0 \end{bmatrix}.$$

Let $A_{2,2} = [a_{i,j}]$, $i, j = 1, \ldots, \sum_{i=1}^{j_w} p_{k,i}$. Through the relation $QA_{2,2} = A_{2,2}\, Q$ we have

$$(26) \quad \begin{cases} a_{i+1,j} = a_{i,j-1}, \\ i = \sum_{l=1}^{r-1} p_{k,l}+1, \ldots, \sum_{l=1}^{r} p_{k,l}-1, \quad j = \sum_{l=1}^{s-1} p_{k,l}+2, \ldots, \sum_{l=1}^{s} p_{k,l}; \end{cases}$$

$$(27) \quad \begin{cases} a_{i,j} = 0, \\ i = \sum_{l=1}^{r} p_{k,l}, \quad j = \sum_{l=1}^{s-1} p_{k,l}+1, \ldots, \sum_{l=1}^{s} p_{k,l}-1; \end{cases}$$

$$(28) \quad \begin{cases} a_{i,j} = 0, \\ i = \sum_{l=1}^{r-1} p_{k,l}+2, \ldots, \sum_{l=1}^{r} p_{k,l}, \quad j = \sum_{l=1}^{s-1} p_{k,l}+1; \quad r, s = 1, \ldots, j_w. \end{cases}$$

Under these circumstances the matrix $A_{2,2}$ has its particular structure. Let us subdivide $A_{2,2}$ according to the sequence $p_{k,1}, \ldots, p_{k,w}$ so that $A_{2,2}$ becomes $[A_{2,2;r,s}]$ $(r, s = 1, \ldots, k)$. The elements of $A_{2,2;r,s}$ are equal by (24) if they lie on one and the same line parallel to the principal diagonal. Those elements of the matrix $A_{2,2;r,s}$ must be zero, if they are of the last row but not of the last column by (25). At the same time (26) tells that all the elements of the first column of $A_{2,2;r,s}$ are zero except the element of the first row (which is also possible to be zero). If $p_{k,r} = p_{k,s}$ then $A_{2,2;r,s}$ is a square upper

matrix. When $p_{k,r} > p_{k,s}$, then writing $\sum_{l=1}^{r-1} p_{k,l} = u_r$ we have

$$A_{2,2;r,s} = \begin{array}{c} \left[\begin{array}{cccc} a_{\mu_r+1,\mu_s+1} & a_{\mu_r+1,\mu_s+2} \cdots & a_{\mu_r+1,\mu_{s+1}-1} & a_{\mu_r+1,\mu_s+1} \\ 0 & a_{\mu_r+1,\mu_s+1} \cdots & \cdots & a_{\mu_r+1,\mu_{s+1}-1} \\ & 0 & \ddots & \vdots \\ & & & a_{\mu_r+1,\mu_s+1} \\ 0 & & & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\} p_{k,s} \\ \left.\vphantom{a}\right\} p_{k,r}-p_{k,s} \end{array} \\ \underbrace{\qquad\qquad\qquad\qquad}_{p_{k,s}} \end{array}$$

If $p_{k,r} < p_{k,s}$, then

$$A_{2,2;r,s} = \left[\begin{array}{cccc} \overbrace{0 \ldots 0}^{p_{k,s}-p_{k,r}} & a_{\mu_r+1,\mu_s+1} \cdots & a_{\mu_r+2,\mu_s+1} & a_{\mu_r+1,\mu_s+1} \\ & \ddots & & a_{\mu_r+2,\mu_s+1} \\ 0 & & \ddots & \vdots \\ & & & a_{\mu_r+1,\mu_s+1} \end{array}\right] \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} p_{k,r}$$

Let Q^* be the set of integers consisting of μ_{r+1} $(r = 0, 1, \ldots, j_w-1)$ and $n_k-\varrho$ $(\varrho = 0, 1, \ldots, j_w-1)$. In the matrix

$$\begin{bmatrix} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{bmatrix} \tag{29}$$

deleting all the columns and rows whose indices of columns or rows do not belong to Q^* we obtain a unimodular matrix, Θ, whose subdivision, $[\theta_{\alpha,\beta}]$, $\alpha, \beta, = 1, \ldots, w+1$ with respect to $\{j_1, j_2-j_1, \ldots, j_w-j_{w-1}, n_k-j_w\}$ is an upper matrix. Let Q'^* be the set of integers consisting of μ_r $(r = 1, \ldots, j_w)$ and $n_k-j_w+1, \ldots, n_k$. In the matrix (29), deleting all the columns and rows whose indices of columns or rows do not belong to Q'^*, we obtain a unimodular matrix Θ', whose subdivision $[\theta'_{\alpha,\beta}]$, with espect to $\{j_1, j_2-j_1, \ldots, j_w-j_{w-1}, n_k-j_w\}$, is a lower matrix. Applying (26) to square matrices $A_{2,2;r,r}$ we know

$$\theta'_{\alpha,\alpha} = \theta_{\alpha,\alpha}, \qquad \alpha = 1, \ldots, w. \tag{30}$$

If θ and θ' are given satisfying (30), than we may determine the matrix (29) by virtue of (26), (27) and (28).

To summarize the above results we have

LEMMA 5. *Let* $[P_1, \ldots, P_N, 0, \ldots, 0]$ *be the diagonal matrix* (24), *where* $P_1, \ldots, P_{j_w}(j_w < N)$ *are proper while* $P_{j_w+1}, \ldots, P_N$ *are degenerate. If* A *is a unimodular square matrix such that*

(31) $$A[P_1, \ldots, P_N, 0, \ldots, 0] = [P_1, \ldots, P_N, 0, \ldots, 0]A,$$

then A must be of the following form:

(32) $$A = \begin{bmatrix} A_{1,1} & 0 \\ 0 & B \end{bmatrix}.$$

where $A_{1,1}$ is a square matrix of $\sum_{i=1}^{k-1}\sum_{j=1}^{n_i} p_{i,j}t_j$ rows and B is a square matrix satisfying (26), (27) *and* (28). *After deleting from B those columns and rows whose indices of columns and rows do not belong to the set Q^* (Q'^*), the remaining part is a unimodular upper (lower) matrix Θ (Θ') if Θ (Θ') is subdivided with respect to the sequence $\{j_1, j_2-j_1, \ldots, j_w-j_{w-1}, n_k-j_w\}$. Between Θ and Θ' the condition* (30) *holds. Conversely, if Θ and Θ' are given unimodular, upper and lower matrices satisfying* (30), *then Θ and Θ' may be extended to B and hence to A such that* (31) *holds.*

We return to the beginning of this section to study common invariants of the matrices $[0, \ldots, 0, \gamma(1, u), \ldots, \gamma(1, n)]$ and $A_{(1,1)}[0, \ldots, 0, \gamma(1, u), \ldots, \gamma(1, n)]B^{-1} = [0, \ldots, 0, \gamma'(1, u), \ldots, \gamma'(1, n)]$ which are of s_1 rows and $\sum_{j=1}^{n} r_j$ columns. Since $B(u, u) = A(1, 1)$ we may without loss of generality assume that $\gamma(1, u)$ and $\gamma'(1, u)$ have been reduced to the same normal form (24) which is written as $N(1, u)$. Let

(33) $$A(1, 1)[\overbrace{0, \ldots, 0}^{u-1}, N(1, u), \gamma(1, u+1), \ldots, \gamma(1, n)]$$
$$= [\overbrace{0, \ldots, 0}^{u-1}, N(1, u), \gamma'(1, u+1), \ldots, \gamma'(1, n)]B.$$

Let H denote the set of row-indices of those rows where 1 happens in the normal form of $N(1, u)$.

Since $A_{1,1}N(1, u) = N(1, u)B(u, u) = N(1, u)A_{1,1}$ the matrix $A_{1,1}$ has its particular property according to Lemma 5. In $A_{1,1}$ we delete those rows and columns whose indices of rows and columns belong to H and obtain a unimodular lower matrix θ^1 which is subdivided according to the sequence $\{j_1, j_2-j_1, \ldots, j_w-j_{w-1}, n_k-j_w\}$. This sequence, I repeat, is determined by the normal form, $N(1, u)$. By $\bar{\gamma}(1, i)$ and $\bar{\gamma}'(1, i)$ $(i = u+1, \ldots, n)$ we mean matrices obtained from $\gamma(1, i)$ and $\gamma'(1, i)$ by deleting rows whose indices of rows belong to H. The equation (33) is equivalent to

$$\theta'[\overbrace{0, \ldots, 0}^{u}, \bar{\gamma}(1, u+1), \ldots, \bar{\gamma}(1, n)] = [\overbrace{0, \ldots, 0}^{u}, \bar{\gamma}'(1, u+1), \ldots, \bar{\gamma}'(1, n)]B,$$

or

$$(34)\qquad \theta'[\bar{\gamma}(1,u+1),\ldots,\bar{\gamma}(1,n)] = [\bar{\gamma}'(1,u+1),\ldots,\bar{\gamma}'(1,n)]\begin{bmatrix} B_{u+1,u+1}\cdots B_{u+1,u} \\ \ddots \quad \vdots \\ 0 \qquad B_{n,n}\end{bmatrix}.$$

Classical argument gives us a complete set of invariants which should have the matrices $[\bar{\gamma}(1,u+1),\ldots\bar{\gamma}(1,n)]$ and $[\bar{\gamma}'(1,u+1),\ldots,\bar{\gamma}'(1,n)]$ so that ϑ' and $B_{i,j}$ $(i,j=u+1,\ldots,n)$ may exist to satisfy (34). These invariants are relative block invariants, $\sigma_{i,j}$ $(i=1,\ldots,w+1;\ j=u+1,\ldots,n)$. Both $[\bar{\gamma}(1,u+1),\ldots,\bar{\gamma}(1,n)]$ and $[\bar{\gamma}(1,u+1),\ldots,\bar{\gamma}'(1,n)]$ may be reduced to the same normal form as follows:

$$(35)\qquad \begin{array}{c} \begin{array}{cccccccccccc} \overbrace{\qquad\qquad}^{r_{u+1}} & & & \overbrace{\qquad\qquad}^{r_{u+2}} & & & & & & & \end{array} \\ \left[\begin{array}{cccccccccccc} \sigma_{1,u+1} & 0\ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \sigma_{1,u+2} & \ldots & 0 & & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & \sigma_{1,n} & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0\ldots & \sigma_{w+1,n+1} & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0\ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & \sigma_{w+1,n} & 0 \\ 0 & 0\ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} j_1 \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} n_k - j_w \end{array}$$

where $\sigma_{i,j}$ means a unit matrix of $\sigma_{i,j}$ rows.

We start from $A[\gamma]=[\gamma']B$ again. If $\gamma(1,i)=0$, $i=1,\ldots,u-1$, and if $A_{1,1}=B(u,u)$, then $\gamma(1,u)$ and $\gamma'(1,u)$ are assumed to have been reduced to the same normal form (23). Meantime we assume that elements of $\gamma(1,i)$ and $\gamma'(1,i)$ $(i=u+1,\ldots,n)$ are trivial if these elements lie in those rows whose row-indices belong to H. Furthermore, we assume that $[\bar{\gamma}(1,u+1),\ldots,\bar{\gamma}(1,n)]$ and $[\bar{\gamma}'(1,u+1),\ldots,\gamma'(1,n)]$ are reduced to the same normal form (35)([10]). Our purpose at this moment is to study the properties of A, B if they satisfy $A[\gamma]=[\gamma']B$ where $[\gamma]$ and $[\gamma']$ have the assumed properties. Afterwards, we are to determine

([10]) If $\gamma(1,i)=0$ $(i=1,\ldots,u-1)$ and $A_{1,1}=B_{u,u}$ then the assumptions made here will not lose generality.

the complete set of invariants existing between those particular matrices, $[\gamma]$ and $[\gamma']$, if there exist A and B such that $A[\gamma] = [\gamma']B$. Since $A_{1,1}[\gamma_{1,u}] = [\gamma'_{1,u}]B(u, u)$ (i. e. $A_{1,1}\ N(1, u) = N(1, u)A_{1,1}$) the matrix $A_{1,1}$ has the properties of B in Lemma 5. Hence we have the matrices θ and θ'. Applying Lemma 2 to (34)(11) we have the following result: If we subdivide θ' according to the sequence of integers

$$\sigma_{1,u+1}, \ldots, \sigma_{1,n}; j_1 - \sum_{i=u+1}^{n} \sigma_{1,i}; \sigma_{2,u+1}, \ldots, \sigma_{2,n},$$

$$j_2 - j_1 - \sum_{i=u+1}^{n} \sigma_{2,i}; \ldots; \sigma_{m+1,u+1}, \ldots, \sigma_{m+1,n}, n_k - j_k - \sum_{i=u+1}^{n} \sigma_{m,i}$$

then the subdivision of $\theta'_{\alpha,\beta}$ $(\alpha, \beta = 1, \ldots, k)$ is an upper matrix $[\vartheta'_{\alpha,\beta;i,j}]$ $(i, j = 1, \ldots, n+1-u)$. Due to (30) the matrice $[\theta_{\alpha,\alpha;i,j}]$ $(\alpha = 1, \ldots, k;$ $i, j = 1, \ldots, n+1-u)$ becomes upper matrice. Furthermore, we subdivide $[B_{l,l}]$ $(l = u+1, \ldots, n)$ accordingly to the sequence

$$\sigma_{1,u+1}, \sigma_{2,u+1}, \ldots, \sigma_{m+1,u+1}, r_{u+1} -$$

$$- \sum_{i=1}^{m+1} \sigma_{i,u+1}; \ldots; \sigma_{1,n}, \sigma_{2,n}, \ldots, \sigma_{m+1,n}, r_n - \sum_{i=1}^{m+1} \sigma_{i,n};$$

then the induced subdivision $[B_{l,k;a,b}](a, b = 1, \ldots, m+2)$ of $B_{l,k}$ is a lower matrix. Besides these we have certain relations corresponding to (iii) of Lemma 2, which we omit here. If 1 occurs in the first normalized s_1 rows of $[\gamma]$ and $[\gamma']$, we denote the set of numbers of columns where 1 occurs by H. In the next $\sum_{i=2}^{m+1} s_i$ rows of $[\gamma]$ and $[\gamma']$ we may assume that 1 will not occur again in those columns whose column indices belong to H. After striking out all the columns and rows from B, if the indices of those columns and rows belong to H, we have a matrix B' whose subdivision with respect to

$$r_1, \ldots, r_{u-1}, n_k, r_{u+1} - \sum_{i=1}^{m+1} \sigma_{i,u+1}, \quad r_{u+2} - \sum_{i=1}^{m+1} \sigma_{i,u+2}, \ldots, r_n - \sum_{i=1}^{m+1} \sigma_{i,n}$$

is an upper matrix, whose diagonal elements are $B_{1,1}, \ldots, B_{u-1,u-1}, \theta,$ $B_{u+1,u+1;m+2,m+2}, \ldots, B_{n,n;m+2,m+2}$. Due to (30) the subdivisions of θ and θ' have the same diagonal elements, if they are subdivided with respect to $j_1, j_2 - j_1, \ldots, j_w - j_{w-1}, n_k - j_w$. But from (34) and Lemma 2 the subdivi-

(11) It is assumed that $[\bar{\gamma}(1, u+1), \ldots, \bar{\gamma}(1, n)]$ and $[\bar{\gamma}'(1, n+1), \ldots, \gamma'(1, n)]$ have been reduced to (35).

sion of $\theta'_{a,a}$ $(a = 1, \ldots, k)$ with respect to $\sigma_{a,1}, \ldots, a_{a,n-u}, \sigma_{a,n-u+1}$, is an upper matrix because we have assumed that $[\bar{\gamma}_{(1,u+1)}, \ldots, \bar{\gamma}(1, n)]$ and $[\bar{\gamma}'_{(1,u+1)}, \ldots, \bar{\gamma}'_{(1,n)}]$ have been reduced to the same normal form. Finally the subdivision of B' with respect to $r_1, \ldots, r_{u-1}; \sigma_{1,1}, \ldots, \sigma_{1,n-u}, \sigma_{1,n-u+1};$ $\sigma_{2,1}, \ldots, \sigma_{2,n-u+1}; \ldots; \sigma_{w+1,1}, \ldots, \sigma_{w+1,n-w+1}; r_{u+1} - \sum_{i=1}^{m+1} \sigma_{i,u+1}, \ldots, r_n - \sum_{i=1}^{m+1} \sigma_{i,n}$ is an upper matrix.

In both sides of $A[\gamma] = [\gamma']B$, the first s_1 rows have been now naturally equal. Comparing the next $\sum_{i=2}^{m+1} s_i$ rows we obtain relations similar to (17). These relations split into two sets, one of them obtained by comparing elements of those columns with indices of columns in H. This actually tells how to determine $A_{i,1} (i = 2, \ldots, m+1)$. The other set is obtained by comparing elements of those columns whose indices of columns do not belong to H. This is

$$(36) \qquad \begin{bmatrix} A_{2,2} & & & 0 \\ & A_{3,3} & & \\ & & \ddots & \\ * & & & A_{m+1,m+1} \end{bmatrix} [\bar{\gamma}] = [\bar{\gamma}'] B^1,$$

where $[\bar{\gamma}]$ and $[\bar{\gamma}']$ are matrices obtained from $[\gamma]$ and $[\gamma']$ respectively by removing first s_1 rows and all the columns whose indices of columns belong to H. If (36) holds then we may extend $[A_{i,j}]$ $(i, j = 2, \ldots, m+1)$ and B^1 to A and B so that $A[\gamma] = [\gamma']B$. Hence we have

LEMMA 6. *If A is a unimodular square matrix of $\sum_{i=1}^{m+1} s_i$ rows, the subdivision $[A_{i,j}]$ $(i, j = 1, \ldots, m+1)$ of A with respect to $s_i, \ldots, s_{m+1}$ is a lower matrix. If B is a unimodular square matrix of $\sum_{j=1}^{n} r_j$ rows, the subdivision $[B_{h,k}]$ $(h, k = 1, \ldots, n)$ of B with respect to $r_i, \ldots, r_n$ is an upper matrix. Let $[\gamma]$ and $[\gamma']$ be matrices of $\sum_{1}^{m+1} s_i$ rows and $\sum_{j=1}^{n} r_j$ columns. By $[\gamma_{i,h}]$ and $[\gamma'_{i,h}]$ we mean the subdivisions of $[\gamma]$ and $[\gamma']$ respectively according to $s_i, \ldots, s_{m+1}$ and $r_1, \ldots, r_n$. If $\gamma_{1,l} = \gamma'_{1,l} = 0$ $(l = 1, \ldots, u-1)$ and if $A_{1,1} = B_{u,u}$ then the necessary and sufficient conditions that there exist such matrices A and B satisfying $A[\gamma] = [\gamma']B$ are as follows:*

(i) *The matrices $\gamma_{1,u}$ and $\gamma'_{1,u}$ possess the same characteristic polynomials and characteristic coefficients.*

(ii) *The matrices $[\bar{\gamma}_{1,u+1}, \ldots, \bar{\gamma}_{1,n}]$ and $[\bar{\gamma}'_{1,u+1}, \ldots, \bar{\gamma}'_{1,n}]$ possess the same relative block invariants $\sigma_{i,j}$ $(i = 1, \ldots, w+1; j = u+1, \ldots, n)$.*

(iii) (36) *must be true.*

Lemma 6 reduces $A\gamma = \gamma' B$ to (36) and the invariants mentioned in (i) and (ii) of the Lemma.

§ 5. Before dealing with the most general case we treat a particular one to clarify our method. Suppose $n_0 = 0$, $n_1 = n_2 = n_3 = 1$, $n_4 = 0$, $\tau_{2,4} = \tau_{2,2} = s(1,2) - s(1,1) = \tau_{3,3} = \tau_{1,3} = \tau_{3,1} = s(3,2) - s(3,1) = 0$. Then the sequence (11) becomes $\tau_{2,3}, \tau_{2,1}, \tau_{2,3}$ while the sequence (12) becomes $s(3,2) - s(3,1), \tau_{2,1}, \tau_{1,1}$. The automorphisms A and B are now given by

$$(37) \qquad \begin{array}{c} \\ \tau_{2,3} \\ \tau_{2,1} \\ \tau_{2,3} \end{array} \begin{array}{c} \begin{array}{ccc} \tau_{2,3} & \tau_{2,1} & \tau_{2,3} \end{array} \\ \begin{bmatrix} A_1 & 0 & 0 \\ * & A_2 & 0 \\ * & * & A_1 \end{bmatrix} \end{array}$$

and

$$(38) \qquad \begin{array}{c} \\ s(3,2)-s(3,1) \\ \tau_{2,1} \\ \tau_{1,1} \end{array} \begin{array}{c} \begin{array}{ccc} s(3,2)-s(3,1) & \tau_{2,1} & \tau_{1,1} \end{array} \\ \begin{bmatrix} B_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & B_3 \end{bmatrix} \end{array}$$

respectively. If $[\gamma]$ and $[\gamma']$ are matrices of $\tau_{2,3} + \tau_{2,1} + \tau_{2,3}$ rows and $s(3,2) - s(3,1) + \tau_{2,1} + \tau_{1,1}$ columns, and if $A[\gamma] = [\gamma']B$, then we define $[\gamma] \sim [\gamma']$. Our purpose here is to determine the complete and independent set of invariants of this sort of equivalence class.

For clearness we write

$$\gamma^2 = \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} \\ \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} \end{bmatrix}, \qquad \gamma'^2 = \begin{bmatrix} \gamma'_{1,1} & \gamma'_{1,2} & \gamma'_{1,3} \\ \gamma'_{2,1} & \gamma'_{2,2} & \gamma'_{2,3} \\ \gamma'_{3,1} & \gamma'_{3,2} & \gamma'_{3,3} \end{bmatrix}.$$

The first $\tau_{2,3}$ rows of $A[\gamma] = [\gamma']B$ give us

$$(39) \qquad A_1[\gamma_{1,1}\gamma_{1,2}\gamma_{1,3}] = [\gamma'_{1,1}\gamma'_{1,2}\gamma'_{1,3}] \begin{bmatrix} B_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & B_3 \end{bmatrix}.$$

By means of the method used in defining block invariants we know that $[\gamma_{1,1}\ \gamma_{1,1}\ \gamma_{1,2}]$ and $[\gamma'_{1,1}\ \gamma'_{1,2}\ \gamma'_{1,3}]$ must have common relative block invariants $\theta'_{1,1}, \theta'_{1,2}, \theta'_{1,3}$. The matrix $[\gamma]$ may be reduced to the following semi-normal form:

$$(40)\qquad [\tilde{\gamma}] = \begin{array}{c} \left[\begin{array}{cccccc} \theta_{1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_{1,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta_{1,3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{2,1}^{(1)} & 0 & \gamma_{2,2}^{(1)} & 0 & \gamma_{2,3}^{(1)} \\ 0 & \gamma_{3,1}^{(1)} & 0 & \gamma_{3,2}^{(1)} & 0 & \gamma_{3,3}^{(1)} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} \tau_{2,3} \\ \tau_{2,1} \\ \tau_{2,3} \end{array} \\ \underbrace{\qquad\qquad}_{s(3,2)-s(3,1)} \quad \underbrace{\qquad\qquad}_{\tau_{2,1}} \quad \underbrace{\qquad\qquad}_{\tau_{1,1}} \end{array}$$

Similarly $[\gamma']$ is reduced to a matrix $[\tilde{\gamma}']$, like (40) with $\tilde{\gamma}_{i,j}^{(1)}$ $(i = 2, 3;$ $j = 1, 2, 3)$ in place of $\gamma_{i,j}^{(1)}$. By Lemma 1 we know that the subdivision of A_1 with respect to $\theta_{1,1}, \theta_{1,2}, \theta_{1,3}, \tau_{2,3}-\theta_{1,1}-\theta_{1,2}-\theta_{1,3}$ is an upper matrix $[A_{1;u,v}]$ $(u, v = 1, \ldots, 4)$ if $A[\tilde{\gamma}] = [\tilde{\gamma}']B$. Meanwhile the subdivision of the matrix B according to the sequence $\theta_{1,1}, s(3,2)-s(3,1)-$ $-\theta_{1,1}, \theta_{1,2}, \tau_{2,1}-\theta_{1,2}, \theta_{1,3}, \tau_{1,1}-\theta_{1,3}$ induces subdivisions of $B_1, A_2, B_3, \ldots$ which become lower matrices. Especially A_2 is subdivided to $[A_{2;x,y}]$ $(x, y = 1, 2)$, which is a lower matrix. By Lemma 3 we know $A\tilde{\gamma} = \tilde{\gamma}'B$ if, and only if [12],

$$(41_1)\qquad \begin{array}{c} \theta_{1,2} \\ \\ \tau_{2,1}-\theta_{1,2} \end{array} \begin{bmatrix} A_{2;1,1} & 0 & 0 \\ A_{2;2,1} & A_{2;2,2} & 0 \\ * & * & \dot{A}_1 \end{bmatrix} \begin{bmatrix} \gamma_{2,1}^{(1)} & \gamma_{2,2}^{(1)} & \gamma_{2,3}^{(1)} \\ \gamma_{3,1}^{(1)} & \gamma_{3,2}^{(1)} & \gamma_{3,3}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{\gamma}_{2,1}^{(1)} & \tilde{\gamma}_{2,2}^{(1)} & \tilde{\gamma}_{2,3}^{(1)} \\ \tilde{\gamma}_{3,1}^{(1)} & \tilde{\gamma}_{3,2}^{(1)} & \tilde{\gamma}_{3,3}^{(1)} \end{bmatrix} \begin{bmatrix} B_{1;2,2} & * & * \\ 0 & A_{2;2,2} & * \\ 0 & 0 & B_{3;2,2} \end{bmatrix},$$

$\dot{A}_1$ being now a lower matrix. By (iii) of Lemma 2 we know that

$$(42_1)\qquad A_{1;2,2} = A_{2;1,1}.$$

To (41) we repeat the above argument μ times and find the invariants $\theta_{i,j}$ $(i = 1, \ldots, \mu; j = 1, 2, 3)$. Suppose $\theta_{\nu,2} \neq 0$ if $\nu < \mu$ but $\theta_{\mu,2} = 0$. The matrix $[\gamma]$ is reduced to the following semi-normal form

---

[12] The number of rows of $\gamma_{2,1}^{(1)}$ (or $\tilde{\gamma}_{2,1}'$) is equal to that of

$$\begin{bmatrix} A_{2;1,1} & 0 \\ A_{2;2,1} & A_{2;2,2} \end{bmatrix}.$$

$$
(43)\quad \left[\begin{array}{cccccccccccccc}
\theta_{1,1} & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & \theta_{1,2} & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & \theta_{1,3} & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & \theta_{2,1} & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & 0 & \theta_{2,2} & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & \theta_{2,3} & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots \theta_{\mu,1} & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots \theta_{\mu,3} & 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 & 0 \\
0 & 0 & \dots 0 & \gamma_{2,1}^{(\mu)} & 0 & 0 & \dots \gamma_{2,2}^{(\mu)} & 0 & 0 & 0 & \dots 0 & \gamma_{2,3}^{(\mu)} \\
0 & 0 & \dots 0 & \gamma_{3,1}^{(\mu)} & 0 & 0 & \dots \gamma_{3,2}^{(\mu)} & 0 & 0 & 0 & \dots 0 & \gamma_{3,3}^{(\mu)}
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l}\tau_{1,3}\\ \\ \\ \end{array}\right\} \\
\left.\begin{array}{l}\theta_{1,2}\\ \\ \\ \end{array}\right\} \\
\\
\left.\begin{array}{l}\theta_{\mu-1,2}\\ \\ \\ \end{array}\right\} \\
\tau_{3,1}
\end{array}
\Bigg\}\tau_{2,1}
$$

while $[\gamma']$ is reduced to the same semi-normal form but replacing $\gamma_{h,k}^{(\mu)}$ by $\tilde{\gamma}_{h,k}^{(\mu)}$ $(h = 2, 3;\ k = 1, 2, 3)$. Then (41) is replaced by

$$
(41\mu)\quad \begin{bmatrix} A_{2;2,2}^{(\mu-1)} & 0 \\ * & A_1 \end{bmatrix} \begin{bmatrix} \gamma_{2,1}^{(\mu)} & \gamma_{2,2}^{(\mu)} & \gamma_{2,3}^{(\mu)} \\ \gamma_{3,1}^{(\mu)} & \gamma_{3,2}^{(\mu)} & \gamma_{3,3}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \tilde{\gamma}_{2,1}^{(\mu)} & \tilde{\gamma}_{2,2}^{(\mu)} & \tilde{\gamma}_{2,3}^{(\mu)} \\ \tilde{\gamma}_{3,1}^{(\mu)} & \tilde{\gamma}_{3,2}^{(\mu)} & \tilde{\gamma}_{3,3}^{(\mu)} \end{bmatrix} \begin{bmatrix} B_{1;2,2}^{(\mu-1)} & * & * \\ 0 & A_{2;2,2}^{(\mu-1)} & * \\ 0 & 0 & B_{3;2,2}^{(\mu-1)} \end{bmatrix},
$$

where A_1, by induction, becomes an upper matrix if it is subdivided to $[A_{1;i,j}]$ $(i, j = 1, \dots, 3\mu)$ according to the sequence of integers $\theta_{1,1}, \theta_{2,1}, \dots, \theta_{\mu,1}, \theta_{\mu,3}, \theta_{\mu-1,2} - \sum_{i=1}^{3} \theta_{\mu,i}, \theta_{\mu-1,3}, \theta_{\mu-2,2} - \sum_{i=1}^{3} \theta_{\mu-1,i}, \dots, \theta_{2,3}, \theta_{1,2} - \sum_{i=1}^{3} \theta_{2,i}, \theta_{1,3}, \tau_{1,3} - \sum_{i=1}^{3} \theta_{1,i}$.

The equation (41μ) is the necessary and sufficient condition for $[\gamma] \sim [\gamma']$ if $\theta_{\mu,2} = 0$ but $\theta_{\nu,2} \neq 0$ as $\nu < \mu$.

The equation (41μ) gives

$$
A_{2;2,2}^{(\mu-1)} \gamma_{2,1}^{(\mu)} = \tilde{\gamma}_{2,1}^{(\mu)} B_{1;2,2}^{(\mu-1)}
$$

together with further relations which we mention afterwards. Let $\theta_{\mu+1,1}$ be the rank of $\gamma_{2,1}^{(\mu)}$ or $\tilde{\gamma}_{2,1}^{(\mu)}$. Without loss of generality we assume that

$$
(43)\qquad \gamma_{2,1}^{(\mu)} = \tilde{\gamma}_{2,1}^{(\mu)} = \begin{bmatrix} \theta_{\mu+1,1} & 0 \\ 0 & 0 \end{bmatrix},
$$

that the elements of $\gamma_{2,2}^{(\mu)}, \gamma_{2,3}^{(\gamma)}, \tilde{\gamma}_{2,2}^{(\mu)}, \tilde{\gamma}_{2,3}^{(\mu)}$ are trivial if they lie in the first $\theta_{\mu+1,1}$ rows, and that elements of $\tilde{\gamma}_{3,1}^{(\mu)}$ and $\gamma_{3,1}^{(\mu)}$ are trivial if they lie in the first $\theta_{\mu+1,1}$ columns. Since $A_{2,2}^{(\mu-1)}$ and $B_{1;2,2}^{(\mu-1)}$ satisfy $A_{2,2}^{(\mu-1)} \gamma_{2,1}^{(\mu)} = \tilde{\gamma}_{2,1}^{(\mu)} B_{1;2,2}^{(\mu-1)}$, where $\gamma_{2,1}^{(\mu)}$ and $\tilde{\gamma}_{2,1}^{(\mu)}$ take the same normal form (43), therefore the subdivision of $A_{2,2}^{(\mu-1)}$ according to $\theta_{\mu+1,1}, \tau_{i,1} - \sum_{i=1}^{\mu} \theta_{i,2} - \theta_{\mu+1,1}$ is an upper matrix.

Now applying Lemma 5 to (41μ) we write down the necessary and sufficient condition so that (41μ) is true as

$$(44)\qquad \begin{bmatrix} A_{2,2}^{(\mu)} & 0 \\ * & \dot{A}_1 \end{bmatrix} \begin{bmatrix} 0 & \gamma_{2,2}^{(\mu+1)} & \gamma_{2,3}^{(\mu+1)} & \gamma_{2,4}^{(\mu+1)} \\ \gamma_{3,1}^{(\mu+1)} & \gamma_{3,2}^{(\mu+1)} & \gamma_{3,3}^{(\mu+1)} & \gamma_{3,4}^{(\mu+1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \tilde{\gamma}_{2,2}^{(\mu+1)} & \tilde{\gamma}_{2,3}^{(\mu+1)} & \tilde{\gamma}_{2,4}^{(\mu+1)} \\ \tilde{\gamma}_{3,1}^{(\mu+1)} & \tilde{\gamma}_{3,2}^{(\mu+1)} & \tilde{\gamma}_{3,3}^{(\mu+1)} & \tilde{\gamma}_{3,4}^{(\mu+1)} \end{bmatrix} \begin{bmatrix} B_1 & * & * & * \\ & B_2 & * & * \\ 0 & & A_{2,2}^{(\mu)} & * \\ & & & B_3 \end{bmatrix},$$

where the sum of the numbers of columns of $\gamma_{2,2}^{(\mu+1)}$ and $\gamma_{2,3}^{(\mu+1)}$ is the number of columns of $\gamma_{2,2}^{(\mu)}$. In this case we consider $A_{2,2}^{(\mu)}\gamma_{2,2}^{(\mu+1)} = \tilde{\gamma}_{2,2}^{(\mu+1)}B_2$ and find an invariant $\theta_{\mu+2,2}$. Repeat ν times and find the invariants $\theta_{\mu+1,1}, \theta_{\mu+2,2}, \ldots$ $\ldots, \theta_{\mu+\nu+1,2}$, where $\theta_{\mu+\nu+1,2}$ is the first trivial invariant. The matrix $[\gamma]$ may be reduced to the semi-normal form (45), p. 26, where $\delta_i = \theta_{\mu+1+i,2}$ $(i = 1, \ldots, \nu-1)$. The semi-normal form of $[\gamma']$ may be obtained from (45) with $\tilde{\gamma}_{i,j}^{(\mu+\nu-1)}$ $(i = 2, 3;\ j = 1, \ldots, \nu+3)$ replacing $\gamma_{i,j}^{(\mu+\nu-1)}$ in (45). In order that $A[\gamma] = [\gamma']B$, $[\gamma]$ and $[\gamma']$ taking the semi-normal forms (45), the necessary and sufficient condition is the existence of unimodular matrices

$$\begin{bmatrix} B_{2,2}^{(\mu+\nu-1)} & 0 \\ * & \dot{A} \end{bmatrix}$$

and

$$\begin{bmatrix} B_1^{(\mu+\nu-1)} & & & & * \\ & \ddots & & & \\ & & B_{\nu+1}^{(\mu+\nu-1)} & & \\ & 0 & & B_{2,2}^{(\mu+\nu-1)} & \\ & & & & B_{\nu+2}^{(\mu+\nu-1)} \end{bmatrix}$$

such that

$$(46)\qquad \begin{bmatrix} B_{2,2}^{(\mu+\nu-1)} & 0 \\ * & \dot{A} \end{bmatrix} \begin{bmatrix} 0 & \ldots & 0 & \gamma_{2,\nu+2}^{(\mu+\nu)} & \gamma_{2,\nu+3}^{(\mu+\nu)} \\ \gamma_{3,1}^{(\mu+\nu)} & \ldots & \gamma_{3,\nu+1}^{(\mu+\nu)} & \gamma_{3,\nu+2}^{(\mu+\nu)} & \gamma_{3,\nu+3}^{(\mu+\nu)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \ldots & 0 & \tilde{\gamma}_{2,\nu+2}^{(\mu+\nu)} & \tilde{\gamma}_{2,\nu+3}^{(\mu+\nu)} \\ \tilde{\gamma}_{3,1}^{(\mu+\nu)} & \ldots & \tilde{\gamma}_{3,\nu+1}^{(\mu+\nu)} & \gamma_{3,\nu+2}^{(\mu+\nu)} & \gamma_{3,\nu+3}^{(\mu+\nu)} \end{bmatrix} \begin{bmatrix} B_1^{(\mu+\nu-1)} & & & & \\ & \ddots & & & * \\ & & B_{\nu+1}^{(\mu+\nu-1)} & & \\ & 0 & & B_{2,2}^{(\mu+\nu-1)} & \\ & & & & B_{\nu+2}^{(\mu+\nu-1)} \end{bmatrix}$$

$$
(45)\quad
\left[\begin{array}{cccccc|c|ccc|c|ccc|c|ccc|c}
\theta_{1,1} & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & & \theta_{1,2} & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & & \vdots & \dots & & 0 & \theta_{1,3} & 0 & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & & 0 \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\hline
0 & 0 & \dots & \theta_{\mu,1} & 0 & 0 & & 0 & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & \theta_{\mu,3} & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & \theta_{\mu+1,1} & 0 & & 0 & 0 & & \vdots & \dots & & 0 & 0 & 0 & \dots & 0 & 0 \\
 & 0 & & & & & & 0 & \delta_1 & 0 & \vdots & & & 0 & & & & & \\
\hline
 & 0 & & & & 0 & & 0 & 0 & 0 & \vdots & \delta_{\nu-1} & 0 & 0 & 0 & & 0 & & \\
\hline
 & 0 & & & & 0 & & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \gamma_{2,\nu+2}^{(\mu+\nu-1)} & 0 & & & \gamma_{2,\nu+3}^{(\mu+\nu-1)} \\
\hline
 & 0 & & & & \gamma_1^{(\mu+\nu-1)} & & 0 & & \gamma & \vdots & 0 & \gamma & \gamma & \gamma_{3,\nu+2}^{(\mu+\nu-1)} & 0 & & & \gamma_{3,\nu+3}^{(\mu+\nu-1)} \\
 & & & & & \underbrace{\qquad\qquad}_{s(3,2)-s(3,1)-\sum_{i=1}^{\mu+1}\theta_{i,1}} & & & & \underbrace{\quad}_{\theta_{\mu+1}} & & \vdots & \underbrace{\quad}_{\theta_{\mu+\nu-1,2}} & & \underbrace{\quad}_{\theta_{\mu+\nu,2}} & & & & &
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}\tau_{1,3} \\
\\
\left.\begin{array}{l}\\ \\ \\ \\ \\ \\ \end{array}\right\}\tau_{2,1}-\sum_{i=1}^{\mu}\theta_{i,2}- \\
\qquad -\theta_{\mu+1,1}-\sum_{i=\mu+2}^{\mu+\nu}\theta_{i,2}
\end{array}
$$

which is obtained by successive application of Lemma 5. Obviously (46) is reduced to (44) if $\nu = 1$ and $\theta_{\mu+2,2} = 0$. To (46) Lemma 6 is used. Both $\gamma_{2,\nu+2}^{(\mu+\nu)}$ and $\tilde{\gamma}_{2,\nu+2}^{(\mu+\nu)}$ are easily reduced to one and the same normal form, (24), dependent upon characteristic coefficients and characteristic polynomials. The matrix $[\lambda I - \tilde{\gamma}_{2,\nu+1}^{(\mu+\nu)}]$ has invariant factors, of which powers of λ may occur. Arrange these powers in descending way as follows:

$$h_1 = \ldots = h_{j_1} > h_{j_1+1} = \ldots = h_{j_2} > \ldots > h_{j_w+1} = \ldots = h_{j_{w+1}} = 1,$$

then by Lemma 5 a square matrix θ' is introduced, of which the subdivision according to $j_1, j_2 - j_1, \ldots, j_{w+1} - j_w$ is a lower matrix.

Let $\bar{\gamma}_{2,\nu+3}^{(\mu+\nu)}$ and $\bar{\gamma}_{2,\nu+3}^{'(\mu+\nu)}$ be the matrices obtained from $\gamma_{2,\nu+3}^{(\mu+\nu)}$ and $\tilde{\gamma}_{2,\nu+3}^{(\mu+\nu)}$ by removing those rows where 1 has occurred in the normal form of $\gamma_{2,\nu+2}^{(\mu+\nu)}$ (or $\tilde{\gamma}_{2,\nu+2}^{(\mu+\nu)}$). By (45) we have

$$\theta' \bar{\gamma}_{2,\nu+3}^{(\mu+\nu)} = \bar{\gamma}_{2,\nu+3}^{'(\mu+\nu)} B_{\nu+2}^{(\mu+\nu-1)}, \tag{46}$$

where θ' and $B_{\nu+2}^{(\mu+\nu-1)}$ are unimodular square matrices. By the method used to define block invariants we may have the invariants $\theta_{\mu+\nu+j,3}$ $(j = 2, \ldots, w+2)$, which determine the common normal form of $\bar{\gamma}_{2,\nu+3}^{(\mu+\nu)}$ and $\bar{\gamma}_{2,\nu+3}^{'(\mu+\nu)}$.

Finally we write down the equation (36) for this case. It is

$$\dot{A}\,[\gamma_{3,1}^{(\mu+\nu)} \ldots \gamma_{3,\nu+1}^{(\mu+\nu)} \gamma_{3,\nu+2}^{'(\mu+\nu)} \gamma_{3,\nu+3}^{'(\mu+\nu)}] \tag{47}$$

$$= [\tilde{\gamma}_{3,1}^{(\mu+\nu)} \ldots \tilde{\gamma}_{3,\nu+1}^{(\mu+\nu)} \tilde{\gamma}_{3,\nu+2}^{'(\mu+\nu)} \tilde{\gamma}_{3,\nu+3}^{'(\mu+\nu)}] \begin{bmatrix} B_1^{(\mu+\nu-1)} & & & & \\ & \ddots & & & * \\ & & B_{\nu+1}^{(\mu+\nu-1)} & & \\ & 0 & & \theta & \\ & & & & B_{\nu+2}^{'(\mu+\nu+1)} \end{bmatrix}$$

where $\gamma_{3,\nu+2}^{'(\mu+\nu)}$, $\tilde{\gamma}_{3,\nu+2}^{'(\mu+\nu)}$ are obtained from $\gamma_{3,\nu+2}^{(\mu+\nu)}$, $\gamma_{3,\nu+2}^{(\mu+\nu)}$ respectively, by striking out those columns where 1 happens in the normal form of $\gamma_{2,\nu+3}^{(\mu+\nu-1)}$ (or $\gamma_{2,\nu+3}^{'(\mu+\nu-1)}$) and $\gamma_{3,\nu+3}^{'(\mu+\nu)}$, $\tilde{\gamma}_{3,\nu+3}^{'(\mu+\nu)}$ are obtained from $\gamma_{3,\nu+3}^{(\mu+\nu)}$, $\tilde{\gamma}_{3,\nu+3}^{(\mu+\nu)}$ respectively, by striking out those columns where 1 occurs in the normal form of $\bar{\gamma}_{2,\nu+3}^{(\mu+\nu)}$ (or $\bar{\gamma}_{2,\nu+3}^{'(\mu+\nu)}$) and θ, $B_{\nu+2}^{'(\mu+\nu-1)}$ are obtained from $B_{2,2}^{(\mu+\nu-1)}$, $B_{\nu+2}^{(\mu+\nu-1)}$ by the method described in Lemma 6. In (46) we have assumed that $\bar{\gamma}_{2,\nu+3}^{(\mu+\nu)}$ and $\bar{\gamma}_{2,\nu+3}^{'(\mu+\nu)}$ have been reduced to the same normal form (35) with $n = u+1$, $\sigma_{i,\mu+1} = \theta_{\mu+\nu+i+1,3}$ $(i = 1, \ldots, w+1)$. Hence $\theta'_{a,a}$ is an upper matrix if it is subdivided according to $\theta_{\mu+\nu+a+1,3}, ja - j_{a-1} - \theta_{\mu+\nu+a+2,3}$ $(j_0 = 0)$. By (30) we know that θ is an upper matrix, if it is subdivided according to $\theta_{\mu+\nu+2,3}, j_1 - \theta_{\mu+\nu+2,3}$, $\theta_{\mu+\nu+3,3}, j_2 - j_1 - \theta_{\mu+\nu+3,3}; \ldots; \theta_{\mu+\nu+w+2,3}, j_{w+1} - j_w - \theta_{\mu+\nu+w+2,3}$, since $\dot{A}$ has been subdivided to a lower matrix according to a sequence of 3μ terms.

The complete set of invariants, which exist between the matrices

$$[\gamma_{3,1}^{(\mu+\nu-1)}, \dots \gamma_{3,\nu+1}^{(\mu+\nu-1)}, \gamma'^{(\mu+\nu-1)}_{3,\nu+2}, \gamma'^{(\mu+\nu-1)}_{3,\nu+3}]$$

$$\text{and} \quad [\tilde{\gamma}_{3,1}^{(\mu+\nu-1)}, \dots \tilde{\gamma}_{3,\nu+1}^{(\mu+\nu-1)}, \quad \gamma'^{(\mu+\nu+1)}_{3,\nu+2}, \tilde{\gamma}'^{(\mu+\nu-1)}_{3,\nu+3}]$$

may be determined easily. They are $\theta_{\mu+v+w+2+i,j}$ $(i = 1, \dots, 3\mu;\ j = 1, \dots, v+2w+4)$. The normal form of $[\gamma_{3,1}^{(\mu+\nu-1)}, \dots, \gamma'^{(\mu+\nu-1)}_{3,\nu+2}, \gamma'^{(\mu+\nu-1)}_{3,\nu+3}]$ may also be found easily. Consequently we reach the normal form of $[\gamma]$. We call all the invariants $\theta_{\mu,\nu}$ as relative block invariants. In this particular case our theorem has actually been proved.

§ 6. Return to consider the most general case where A is a unimodular lower matrix $[A_{i,a}]$, if A is subdivided according to (11), B is a unimodular upper matrix $[B_{j,h}]$, if B is subdivided according to (12) and relations (13), (14) must hold. For convenience we write the sequence (11) as $\{a_i\}(i = 1, \dots, M)$ and (12) as $\{b_j\}(j = 1, \dots, N)$. Let $e_1, \dots, e_\varrho; f_1, \dots, f_\sigma; g_1, \dots, g_\varrho$, be three increasing sequences of integers, any two sequences of them having no common integer, such that $g_1 > \mathrm{Max}(e_\varrho, f_\sigma)$ and $g_\varrho \leqslant M$. The conditions (13) and (14) are written as [13]

$$\dot{A}_{e_r,e_r} = A_{g_r,g_r} \qquad (r = 1, \dots, \varrho) \tag{48}$$

and

$$A_{f_s,f_s} = B_{l_s,l_s} \qquad (s = 1, \dots, \sigma) \tag{49}$$

respectively. Our problem is to find the complete set of invariants of equivalence classes of matrices $[\gamma]$ where $[\gamma]\sim[\gamma']$ means existence of A and B with the prescribed properties such that $A[\gamma] = [\gamma']B$. In case $\sigma = 0$. The first $\sum_{i=1}^{g_1-1} a_i$ rows of both sides of $A[\gamma] = [\gamma']B$ give us

$$A_1[\gamma_1] = [\gamma_1']B, \tag{50}$$

where $[\gamma_1]$ and $[\gamma_1']$ denote the first $\sum_{r=1}^{g_1-1} a_i$ rows of $[\gamma]$ and $[\gamma']$ respectively, A_1 being the lower matrix obtained from A by removing last $\sum_{i=g_1}^{M} a_i$ rows and columns. No doubt we may reduce $[\gamma_1]$ and $[\gamma_1']$ to the same normal form, N, determined by relative block invariants. By Lemma 2 we know what A_1 and B should be if $A_1 N = NB$. If A_{e_r,e_r} are subdivided to be upper matrices, then A_{g_r,g_r} $(r = 1, \dots, \varrho)$ are subdivided to be lower matrices

[13] $\sigma = n_1 n_3$, $\varrho = n_1 n_2$.

by (48). Assume that the first $\sum_{i=1}^{\varrho_1-1} a_i$ rows of $[\gamma]$ and $[\gamma']$ have been reduced to the same normal form N. Let us denote $[\gamma]$ and $[\gamma']$ by $\begin{bmatrix} N \\ C \end{bmatrix}$ and $\begin{bmatrix} N \\ C' \end{bmatrix}$ respectively. By Lemma 3 $A[\gamma] = [\gamma']B$ is reduced to $A_2\bar{C} = \bar{C}'B'$, A^2 and B' being the matrices obtained from A and B according to Lemma 3. Obviously A_2 is a lower matrix while B' an upper one within the consideration of Lemma 2 and (48). Now $\varrho = \sigma = 0$. Therefore we may find a complete set of common invariants of $\bar{C}$ and $\bar{C}'$ and reduce them to one and the same normal form determined by their common invariants.

In case $\sigma > 0$ we have succeded in § 5 to reduce σ from 1 to zero in the special example. If we may reduce σ by 1, then the induction will assure us the required proof. Suppose $e_1, \ldots, e_{\varrho'} < f_1 < e_{\varrho'+1}$. We compare the first $\sum_{i=1}^{f_1-1} a_i$ rows of both sides of $A\gamma = \gamma' B$ and have

$$(51) \qquad \begin{bmatrix} A_{1,1} & & \\ & \ddots & 0 \\ * & & A_{f_1-1,f_1-1} \end{bmatrix} \begin{bmatrix} \gamma_{1,1} & \cdots & \gamma_{1,N} \\ \vdots & & \vdots \\ \gamma_{f_1-1,1} & \cdots & \gamma_{f_1-1,N} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma'_{1,1} & \cdots & \gamma'_{1,N} \\ \vdots & & \vdots \\ \gamma'_{f_1-1,1} & \cdots & \gamma_{f_1-1,N} \end{bmatrix} \begin{bmatrix} B_{1,1} & \cdots & B_{1,N} \\ & \ddots & \vdots \\ 0 & & B_{N,N} \end{bmatrix}$$

if $[\gamma_{i,j}]$ and $[\gamma'_{i,j}]$ denote the subdivisions of $[\gamma]$ and $[\gamma']$ according to the sequences $\{a_i\}$ and $\{b_j\}$. It is obvious that the matrices $[\gamma_{i,j}]$ and $[\gamma'_{i,j}]$ $(i = 1, \ldots, f_1-1;\ j = 1, \ldots, N)$ have common relative block invariants $\theta^{(1)}_{i,j}$ $(i = 1, \ldots, f_1-1;\ j' = 1, \ldots, N)$, which completely decide the common normal form, $\tilde{N}$, of them. If

$$\begin{bmatrix} A_{1,1} & & \\ & \ddots & 0 \\ * & & A_{f_1-1,f_1-1} \end{bmatrix} \tilde{N} = \tilde{N} \begin{bmatrix} B_{1,1} & \cdots & B_{1,N} \\ & \ddots & \vdots \\ 0 & & B_{N,N} \end{bmatrix},$$

by Lemma 2 the subdivision $[A_{i,j;u,v}]$ $(u, v = 1, \ldots, N+1)$ of $A_{i,j}$ $(i, j = 1, \ldots, f_1-1)$ with respect to the sequences $\{\theta^{(1)}_{i,1}, \ldots, \theta^{(1)}_{i,N}, a_i - \sum_{j=1}^{N} \theta^{(1)}_{i,j}\}$ and $\{\theta^{(1)}_{j,1}, \ldots, \theta^{(1)}_{j,N}, a_j - \sum_{k=1}^{N} \theta^{(1)}_{j,k}\}$ is an upper matrix, while the subdivision, $[B_{h,k;r,s}]$ $(r, s = 1, \ldots, f_1)$ of $B_{h,k}$ $(h, k = 1, \ldots, N)$ with respect to the sequences $\{\theta^{(1)}_{1,h}, \ldots, \theta^{(1)}_{f_1-1,h}, b_h - \sum_{i=1}^{f_1-1} \theta^{(1)}_{i,h}\}$ and $\{\theta^{(1)}_{1,k}, \ldots, \theta^{(1)}_{f_1-1,k}, b_k - \sum_{i=1}^{f_1-1} \theta^{(1)}_{i,k}\}$

is a lower matrix. Particularly

$$A_{i,i;u,u} = B_{u,u;i,i}, \quad i = 1, \ldots, f_1 - 1;\ u = 1, \ldots, N.$$

If the matrices $[\gamma]$ and $[\gamma']$ have been reduced to $\begin{bmatrix} \tilde{N} \\ C \end{bmatrix}$ and $\begin{bmatrix} \tilde{N} \\ C' \end{bmatrix}$ respectively, Lemma 3 tells us that $A[\gamma] = [\gamma']B$ if, and only if,

$$(52) \qquad \begin{bmatrix} A_{f_1 f_1} & 0 & \cdots & 0 \\ & A_{f_1-1,f_1+1} & 0 & \vdots \\ * & & \ddots & 0 \\ & & & A_{M,M} \end{bmatrix} [\bar{C}] = [\bar{C}'] \begin{bmatrix} B_{1,1,f_1,f_1} & \cdots & B_{1,N;f_1,f_1} \\ & \ddots & \vdots \\ 0 & & B_{N_1 N;f_1,f_1} \end{bmatrix}$$

where A_{f_s,f_s} $(s = 1, \ldots, \sigma)$ should be replaced by the lower matrix $B_{l_s l_s;i,j}$ $(i, j = 1, \ldots, f_1)$ according to (49) and A_{g_r,g_r} $(r = 1, \ldots, \sigma')$ should be replaced by the lower matrix $\dot{A}_{e_r,e_r}$ because of (48) and $e_{\varrho'} < f_0 < e_{\varrho'+1}$. Consequently the matrix

$$\begin{bmatrix} A_{f_1,f_1} & & 0 \\ & \ddots & \\ * & & A_{M,M} \end{bmatrix}$$

is a lower matrix if A_{f_s,f_s} $(s = 1, \ldots, \sigma)$ and A_{g_r,g_r} $(r = 1, \ldots, \varrho')$ are suitably subdivided. Suppose it has been done and write the subdivided matrix as $[A^{(1)}_{i,h}]$. Then due to (48) there are σ elements in the principal diagonal of $[A^{(1)}_{i,h}]$ which will also appear in the principal diagonal of $[B_{h,k;f_1,f_1}]$ $(h, k = 1, \ldots, N)$. The problem to classify matrices $\{[\gamma]\}$ under (52) is the same thing as to classify the matrices $\{[\gamma]\}$ under the relation $A[\gamma] = [\gamma']B$ where A and B should satisfy (48), (49) etc. given in the begining of this section. But in (52) the matrix A_{f_1,f_1} is subdivided according to the sequence

$$\vartheta^{(1)}_{1,l_1}, \ldots, \theta^{(1)}_{f_1-1,l_1}, b_{l_1} - \sum_{i=1}^{f_1-1} \vartheta^{(1)}_{i,l_1}.$$

Hence what we are to overcome is the problem mentioned at the beginning of the section with a further restriction that

$$(53) \qquad \text{number of rows of } A_{i,i} = \theta^{(1)}_{i,l_1},\ i = 1, \ldots, f_1 - 1.$$

Repeat this process T times, reach invariants, $\theta^{(t)}_{i,j}$ $(t = 1, \ldots, T)$ and hence get semi-normalized form of equivalence classes of $[\gamma]$'s. Since $\theta^{t}_{i,j}$ are positive integers and

$$b_{l_1} - \sum_{t=1}^{T} \sum_{i=1}^{f_1-1} \vartheta^{(t)}_{i,l_1} \geqslant 0$$

$\theta_{i,l_1}^{(T)}$ would be zero for all i if T is sufficiently large. Therefore we may assume besides (48), (49) and (53) that $f_1 = 1$. If it is so, we know that

$$(54)\qquad A_{1,1}[\gamma_{1,1}, \ldots, \gamma_{1,l_1-1}] = [\gamma'_{1,1}, \ldots, \gamma'_{1,l_1-1}]\begin{bmatrix} B_{1,1} & \ldots & B_{1,l_1-1} \\ & \ddots & \vdots \\ 0 & & B_{l_1-1,l_1-1} \end{bmatrix},$$

where $[\gamma_{i,j}]$ (or $[\gamma'_{i,j}]$) denotes the subdivision of $[\gamma]$ (or $[\gamma']$) according to the sequences $\{a_i\}$ and $\{l_j\}$. By the classical argument we determine invariants $\vartheta_{1,j}^{(T+1)}$ $(j = 1, \ldots, l_1-1)$ and reduce both $[\gamma_{1,1}, \ldots, \gamma_{1,l_1-1}]$ and $[\gamma'_{1,l}, \ldots, \gamma'_{1,l_1-1}]$ to their normal form. After this reduction, Lemma 2 shows that $A_{1,1}$ becomes an upper matrix if $A_{1,1}$ is subdivided according to

$$\theta_{1,1}^{(T+1)}, \ldots, \theta_{1,l_1-1}^{(T+1)}, a_1 - \sum_{j=1}^{l_1-1} \theta_{1,j}^{(T+1)}.$$

Lemma 4 tells that $A\gamma = \gamma' B$ is equivalent to

$$(55)\qquad \begin{bmatrix} A_{1,1;s,s} & 0 & \ldots & 0 \\ * & A_{2,2} & & 0 \\ \vdots & & \ddots & \\ * & A_{M,2} & & A_{M,M} \end{bmatrix}\begin{bmatrix} 0 & \ldots & 0 & \tilde{\gamma}_{1,l_1} & \ldots & \gamma_{1,N} \\ C_1 & \ldots & C_{l_1-1} & C_{l_1} & \ldots & C_N \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \ldots & 0 & \tilde{\gamma}'_{1,l_1} & \ldots & \tilde{\gamma}'_{1,N} \\ C'_1 & \ldots & C'_{l_1-1} & C'_{l_1} & \ldots & C_N \end{bmatrix} B',$$

where B' is obtained from B by removing certain columns and rows. Assume that (54) remains to be true if $[\gamma_{1,1}, \ldots, \gamma_{1,l_1-1}]$ and $[\gamma'_{1,1}, \ldots, \gamma'_{1,l_1-1}]$ are both reduced to common normal form. Since $A_{1,1} = B_{l_1,l_1}$ and since $A_{1,1}$ is required to be an upper matrix if it is subdivided according to $\theta_{1,1}^{(T+1)}, \ldots, \theta_{1,l_1-1}^{(T+1)}, a_1 - \sum_{j=1}^{l_1-1} \theta_{1,j}^{(T+1)}$, therefore the subdivision of B' according to the sequence

$$\Big\{b_1 - \theta_{1,1}^{(T+1)}, \ldots, b_{l_1-1} - \theta_{1,l_1-1}^{(T+1)}, \theta_{1,1}^{(T+1)}, \ldots, \theta_{1,l_1-1}^{(T+1)}, a_1 - \sum_{j=1}^{l_1-1} \theta_{1,j}^{(T+1)}, b_{l_1+1}, \ldots, b_N\Big\}$$

is an upper matrix. If we subdivide $\begin{bmatrix} \tilde{\gamma}_{1,l_1} \\ C_{l_1} \end{bmatrix}$ and $\begin{bmatrix} \tilde{\gamma}'_{1,l_1} \\ C'_{l_1} \end{bmatrix}$ according to the two sequences ([14])

$$\Big\{\sum_{1}^{M} a_i - \sum_{j=1}^{l_1-1} \vartheta_{1,j}^{(T+1)}\Big\} \quad \text{and} \quad \Big\{\vartheta_{1,1}^{(T+1)}, \ldots, \vartheta_{1,l_1-1}^{(T+1)}, a_1 - \sum_{j=1}^{l_1-1} \vartheta_{1,j}^{(T+1)}\Big\}$$

([14]) The first sequence contains one term only.

into

$$\begin{bmatrix} \tilde{\gamma}_{1,l_1;1} & \tilde{\gamma}_{1,l_1;2} & \cdots & \tilde{\gamma}_{1,l_1;l_1} \\ c_{l_1;1} & c_{l_1;2} & \cdots & c_{l_1;l_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\gamma}'_{1,e_1;1} & \tilde{\gamma}'_{1,l_1;2} & \cdots & \tilde{\gamma}_{1,b_1;l_1} \\ c'_{l_1;1} & c_{l_1;2} & \cdots & c'_{l_1;l_1} \end{bmatrix}$$

respectively, then (55) may be written as

$$A\begin{bmatrix} 0\ldots 0 & \tilde{\gamma}_{1,l_1;1}\cdots\tilde{\gamma}_{1,l_1;l_1} & \tilde{\gamma}_{1,l_1+1}\cdots\tilde{\gamma}_{1,N} \\ c_1\ldots c_{l_1-1} & c_{l_1;1}\cdots\ c_{l_1;l_1} & c_{l_1+1}\ \cdots\ c_N \end{bmatrix}$$

$$= \begin{bmatrix} 0\ldots 0 & \tilde{\gamma}'_{1,l_1;1}\cdots\tilde{\gamma}'_{1,l_1;l_1} & \tilde{\gamma}'_{1,l_1+1}\cdots\tilde{\gamma}'_{1,N} \\ c'_1\ldots c'_{l_1-1} & c_{l'_1;1}\ \cdots c_{l'_1;l_1} & c'_{l_1+1}\ldots c'_N \end{bmatrix} B',$$

where A is a unimodular lower matrix, B' a unimodular upper matrix. Moreover, the diagonal elements (matrices) satisfy relations like (48), (49) with $f_1 = 1$. The number σ in (48) is not increased at all. We repeat the argument to obtain invariants $\theta_{1,j}^{(T+2)}$ $(j = 1, \ldots, l_1-1)$ and normalize $[\tilde{\gamma}_{1,l_1;1}, \ldots, \tilde{\gamma}_{1,l_1;l_1-1}]$ and $[\tilde{\gamma}'_{1,l_1;1}, \ldots, \tilde{\gamma}'_{1,l_1;l_1-1}]$. If the repetition goes on T' times and we have found the invariants $\theta_{1,j}^{(T+q)}$ $(j = 1, \ldots, l_1-1,$ $q = 1, 2, \ldots, T')$. If T' is sufficiently large, $\theta_{1,j}^{(T+T')}$ may be trivial for all j since $\theta_{1,j}^{(T+q)}$ are non-negative and

$$a_1 - \sum_{h=1}^{T'}\sum_{j=1}^{l_1-1} \theta_{1,j}^{(T+h)} \geqslant 0.$$

Repeated application of Lemma 4 will reduce the problem to that mentioned in the beginning of this section with additional conditions that $f_1 = 1$ and $\gamma_{1,1} = \ldots = \gamma_{1,l_1-1} = 0$. Then Lemma 6 is applicable. We reduce $A[\gamma] = [\gamma']B$ to

$$(36^*) \qquad \begin{bmatrix} A_{2,2} & & \\ & \ddots & 0 \\ * & & A_{MM} \end{bmatrix} [\bar{\gamma}] = [\bar{\gamma}']B',$$

where B' is obtained from B by removing certain columns and rows. No doubt B' is an upper matrix. Without loss of generality we assume that the first a_1 rows of $[\gamma]$ and $[\gamma']$ have been reduced to the same normal form and

$$A_{1,1}[0, \ldots, 0, \gamma_{1,l_1}, \ldots, \gamma_{1,N}] = [0, \ldots, 0, \gamma_{1,l}, \ldots, \gamma_{1,N}]B.$$

Consequently through the proof of Lemma 6 we know that B_{l_s,l_s} $(s = 2, \ldots, \sigma)$ become lower matrices if they are suitably subdivided. By means of (48) we know that A_{f_s,f_s} $(s = 2, \ldots, \sigma)$ become lower matrices. Therefore the matrix $[A_{ij}]$ $(i, j = 2, \ldots, M)$ is still a lower matrix

if $A_{j_s j_s}$ $(s = 2, \ldots, \sigma)$ are replaced by their subdivision. After this replacement the matrix $[A_{i,j}]$ $(i, j = 2, \ldots, M)$ becomes a lower matrix. The number of common diagonal elements of A' and B is now $\sigma - 1$. This completes the proof.

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