TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

ISOMORPHISMS AND AUTOMORPHISMS OF INTEGRAL GROUP RINGS

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This is an extended version of two lectures I gave at the Banach Center in February 1988. I would like to take this opportunity to thank the Banach Center and my colleagues there for the hospitality I have encountered.

I. Introduction

When Frobenius, Burnside and Schur started representation theory of finite groups at the end of the last century, they were considering "representations" of an abstract finite group G in terms of matrices, and they hoped to get information on G by manipulating with these matrices. The prime example of a successful application of this idea is Burnside's theorem (1911), which states that a group of order $p^{\alpha}q^{\beta}$, p and q different primes, is solvable. (A purely group-theoretical proof was given only in 1972 by Bender [Be].)

More precisely, a *representation of G of degree n* over a commutative ring R – in early times one had $R = \mathbf{C}$ or $\mathbf{Q}$ and later a finite field $\mathbf{F}$ of characteristic p dividing $|G|$ – is a homomorphism

$$\varphi\colon G \to \mathrm{GL}(n, R). \tag{1.1}$$

Since the group $\mathrm{GL}(n, R)$ embeds into the R-algebra $M_n(R)$ of $(n \times n)$-matrices over R, one had in a natural way an R-algebra associated with these matrices, namely Λ_{φ}, the R-algebra generated in $M_n(R)$ by the matrices $\{\varphi(g)\colon g \in G\}$. Studying the various representations is essentially equivalent to studying the

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This paper is in final form and no version of it will be submitted for publication elsewhere.

various R-algebras Λ_φ. However, there is a universal R-algebra, the *group ring of G over R*:

$$RG = \{\sum_{g \in G} r_g g\},$$

which has as R-basis the group elements; the addition is componentwise, and the multiplication is induced from the group multiplication. Now RG maps via the representation φ *onto* any of the R-algebras Λ_φ, and thus essentially contains all information which can be obtained from the R-representations.

Thus the original question of Burnside, Frobenius and Schur could be phrased as follows:

(1.2) Which properties of the abstract finite group G can be recovered from the ring structure of $\mathbf{C}G$, or more generally of RG?

Since $\mathbf{Z}$ is the universal commutative ring, we have a natural homomorphism $\mathbf{Z}G \to RG$ from the integral group ring $\mathbf{Z}G$ to any of the group algebras RG. Thus if one does not want to restrict attention just to one ring R or one particular field, it is natural to *ask the question* (1.2) *for the integral group ring* $\mathbf{Z}G$.

This brings us to the theme of the present lectures: May it be possible that for some class of rings R, the group ring RG determines the group G? More precisely,

(1.3) Does $RG \simeq RH$ imply $G \simeq H$?

This would imply—in the spirit of Burnside, Frobenius and Schur—that all properties of G can be recovered from its R-representations. In the early times of representation theory, this question was not raised, since for complex representations—only those were considered at the beginning—lots of nonisomorphic groups have isomorphic complex group algebras. (By Maschke's theorem, two abelian groups have isomorphic complex group algebras if and only if they have the same order.)

Moreover in 1971 E. Dade [Da] gave an example of two nonisomorphic groups G and H of order p^3q^6 which have isomorphic group rings over *every* field. Hence by studying ordinary and modular representations alone, one cannot distinguish G and H. However, the reader should note that G and H are not p-groups, and from p-modular representation theory one should only expect results on the "p-part" of G from the group ring over a field of characteristic p.

So, up to now two problems remain open:

(1.4) Does $\mathbf{Z}G \simeq \mathbf{Z}H$ imply $G \simeq H$?

(1.5) Does $\mathbf{F}_pG \simeq \mathbf{F}_pH$, G and H p-groups, $\mathbf{F}_p$ the field with p elements, imply $G \simeq H$?

Here I shall concentrate on (1.4), the *isomorphism problem*.

II. The isomorphism problem and units

With the group ring RG we associate the *augmentation map*

$$\mathscr{E}_G\colon RG \to R, \quad \sum r_g g \mapsto \sum r_g,$$

which has as kernel the *augmentation ideal* $I_R(G)$, freely generated over R by the elements $\{g-1\}_{g\in G\setminus\{1\}}$. A given isomorphism of R-algebras $\alpha\colon RG \to RH$ can easily be modified to commute with the augmentation: replace $\alpha(g)$ by $\alpha(g)\mathscr{E}_H(\alpha(g))^{-1}$ (note that $\mathscr{E}_H(\alpha(g))$ is invertible).

(2.1) We assume henceforth that *automorphisms between group rings are always augmented.*

G. Higman in his 1939 thesis [Hi] first considered the isomorphism problem in connection with his study of units in group rings. There he proved the following marvellous result:

(2.2) THEOREM (Higman). *Let G and H be finite abelian groups, and let $\alpha\colon \mathbf{Z}G \to \mathbf{Z}H$ be an augmented automorphism. Then α is induced from a group homomorphism, i.e. $\alpha(g)\in H$ for every $g\in G$.*

Before I sketch a proof of this, let me draw attention to the connection with the units $U(RG)$ of RG. Let us denote by $V(RG)$ the units of augmentation one, i.e.

$$V(RG) = U(RG)\cap(1+I_R(G)).$$

Then it is easily seen that $U(RG) = V(RG)\cdot U(R)$, where $U(R)$ denotes the units in R.

Let us look more closely at $U(\mathbf{Z}G)$: Since $\mathbf{Q}G$ is semisimple we have

$$\mathbf{Q}G \simeq \prod_{i=1}^{s} M_{n_i}(D_i),$$

where the D_i are finite-dimensional skew fields over $\mathbf{Q}$, and hence

(2.3) $$U(\mathbf{Z}G) \subset \prod_{i=1}^{s} \mathrm{GL}(n_i, D_i)$$

is a *commensurable arithmetic group.*

In addition to RG being an R-algebra, it is also an R-Hopf algebra, the Hopf algebra structure being induced from the anti-involution

(2.4) $$*_G\colon RG \to RG, \quad \sum_{g\in G} r_g g \mapsto \sum_{g\in G} r_g g^{-1}.$$

A characterization of those Hopf algebras which arise this way from group rings was given by Takahashi [Ta].

Higman actually proves a much stronger result:

(2.5) THEOREM (Higman). *If $u \in V(\mathbf{Z}A)$ is a unit of finite order in the integral group ring of the finite abelian group A, then $u \in G$, i.e. the only units of finite order in $V(\mathbf{Z}A)$ are the group elements.*

Proof. More generally, let $u \in V(\mathbf{Z}G)$, for an arbitrary finite group G, be a central unit of order n. (We shall here only deal with the case where *n is odd*.) Then u^*, the image of u under $*_G$, is also a central unit of order n, and thus $v = uu^*$ is a central unit in $\mathbf{Z}G$ with $v^n = 1$; moreover, v is fixed under $*_G$. It is easily seen that on the centre Z of $\mathbf{Q}G$, $Z = \prod_{i=1}^{s} K_i$, where the K_i are algebraic number fields, $*_G$ induces the complex conjugation. Thus v lies in a product of real fields. Since the order of v is odd, we conclude $v = 1$. Thus

$$1 = \Big(\sum_{g \in G} z_g g\Big)\Big(\sum_{g \in G} z_g g^{-1}\Big),$$

where $u = \sum_{g \in G} z_g g$. We now consider the coefficient of 1 in the above product and conclude

$$1 = \sum z_g^2.$$

Since $z_g \in \mathbf{Z}$, we have $u = \pm g_0$ for some $g_0 \in G$, and thus $u = g_0$, u being augmented. (The proof for n even is similar.)

In this connection we point out that this argument has heavily used the fact that in $\mathbf{Z}$ a sum of squares is one if and only if there is only one nonzero summand, equal to ± 1.

The isomorphism problem $\mathbf{Z}G \cong \mathbf{Z}H$ via α as augmented algebras appears in a new light if we assume that α commutes with $*$, i.e.

$$\text{(2.6)} \qquad \text{if} \quad \alpha(g) = \sum_{h \in H} z_h h, \quad \text{then} \quad \alpha(g^{-1}) = \sum_{h \in H} z_h h^{-1}.$$

In this case we have

(2.7) PROPOSITION [Ba]. *Let α: $\mathbf{Z}G \to \mathbf{Z}H$ be an augmented isomorphism commuting with $*$. Then α is induced from a group isomorphism from G to H.*

Proof. Let $\alpha(g) = \sum z_h h$; then

$$1 = \alpha(g)\cdot\alpha(g^{-1}) = \Big(\sum_{h \in H} z_h h\Big)\cdot\Big(\sum_{h \in H} z_h h^{-1}\Big),$$

and the same argument as above shows $\alpha(g) = h$ for some $h \in H$.

A consequence of Higman's result is the following:

(2.8) COROLLARY. *Let α: $\mathbf{Z}A \to \mathbf{Z}B$ be an augmented isomorphism with A and B abelian. Then α commutes with $*$.*

After these rather special observations we turn to the general situation.

Given an augmented isomorphism α: $RG \to RH$, the elements $\{\alpha(g)$: $g \in G\}$ form a finite subgroup in $V(RH)$ consisting of R-linearly independent elements. Conversely, given any finite subgroup V in $V(RH)$ of order $|H|$ whose elements are linearly independent over R, we have an augmented isomorphism

$$\alpha\colon RV \to RH.$$

S. D. Berman has observed that the linear independence is often automatic.

(2.9) PROPOSITION [Ber]. *Let R be an integral domain of characteristic zero such that no rational prime divisor of $|G|$ is a unit in R. Let $V \leqslant V(RG)$ be a finite subgroup. Then the elements in V are R-linearly independent in RG.*

This result allows to *phrase the isomorphism problem differently*:

(2.10) Let V be a finite subgroup in $V(\mathbf{Z}G)$ with $|V| = |G|$; is then $V \simeq G$? (Note that in view of (2.9) this is equivalent to the isomorphism problem (1.3) for $\mathbf{Z}$.)

One might go even *one step further than the isomorphism problem*, and ask: Let $V \leqslant V(\mathbf{Z}G)$ and assume $|V| = |G|$. How is V embedded in $V(\mathbf{Z}G)$? Higman (2.2) gave the answer for abelian G. However, in general, one cannot expect that $V = G$, since $V(\mathbf{Z}G)$ is in general not abelian, and so we have conjugation with the units in $V(\mathbf{Z}G)$, which do not necessarily stabilize G, since G is not normal in $V(\mathbf{Z}G)$. Moreover, even for the dihedral group D of order 8, there exists a unit $a \in \mathbf{Q}D \setminus V(\mathbf{Z}D)$ such that

$$D \neq a \cdot D \cdot a^{-1} \in \mathbf{Z}D,$$

and this conjugation is not inner. So for G not abelian, the obstructions to $V = G$ are not just the inner automorphisms of $V(\mathbf{Z}G)$. In this connection, Zassenhaus [Za] made a far-reaching conjecture:

(2.11) ZASSENHAUS CONJECTURE I. *Let $V \leqslant V(\mathbf{Z}G)$ be a finite subgroup with $|V| = |G|$. Then there exists a unit $a \in \mathbf{Q}G$ with $a \cdot V \cdot a^{-1} = G$.*

Because of (2.9) this is equivalent to

(2.12) ZASSENHAUS CONJECTURE II. *Let α: $\mathbf{Z}G \to \mathbf{Z}H$ be an augmented isomorphism. Then there exists a group isomorphism ϱ: $G \to H$ such that $\alpha \cdot \varrho^{-1}$: $\mathbf{Z}G \to \mathbf{Z}G$ is a central automorphism, i.e. an automorphism leaving the centre of $\mathbf{Z}G$ elementwise fixed.*

In view of the structure of the centre Z of $\mathbf{Q}G$, $Z = \prod K_i$, this conjecture would imply that every automorphism σ of Z which stabilizes $\mathbf{Z}G$—note that σ induces an automorphism of $\mathbf{Q}G$—is induced from a group automorphism;

this is a strong statement. Prior to 1985 there were some special classes of groups for which the Zassenhaus conjecture was verified:

1) Higman's result (2.2) shows that it is true for abelian groups.
2) Ritter–Sehgal [RiSe] proved it for certain metacyclic groups.

III. Special properties of G detected by $\mathbf{Z}G$

Let us return to the original question of (1.2).

A direct extension of Higman's result was given by Glauberman and Berman.

(3.1) THEOREM. *Let* α: $\mathbf{Z}G \to \mathbf{Z}H$ *be an augmented automorphism and let*

$$K_g = \sum_{x \in G/C_G(g)} {}^g x$$

be a class sum. Then $\alpha(K_g) = K_h$ *is a class sum in* $\mathbf{Z}H$.

In a similar spirit, the lattices of normal subgroups of G and H are isomorphic:

(3.2) THEOREM (Berman, Glauberman, Sehgal, cf. [Se]). *Let* N *be a normal subgroup of* G, *and let* α: $\mathbf{Z}G \to \mathbf{Z}H$ *be an augmented automorphism. Then* $\alpha(\sum_{n \in N} n) = \sum_{m \in M} m$ *for a normal subgroup* M *in* H.

These results also hold for RG if R is a Dedekind domain of characteristic zero in which no prime divisor of $|G|$ is invertible. It would be important for our results on the conjugacy problem for defect groups to know whether the hypotheses in (3.1, 3.2) could be weakened as follows: if the rational prime p is not a unit in S, is there still a correspondence between the normal p-subgroups in G and those in H?

An extension of the above results was recently obtained by Kimmerle, Lyons and Sandling, who showed, using heavily the classification of finite simple groups:

(3.3) THEOREM [KLS]. *Let* α: $\mathbf{Z}G \to \mathbf{Z}H$ *be an augmented automorphism, and let*

$$1 = N_0 < \ldots < N_t = G$$

be a chief series of G. *Then there exists a chief series*

$$1 = M_0 < \ldots < M_\tau = H$$

of H *such that* $t = \tau$ *and the chief factors are isomorphic, even with the same indices*: $N_i/N_{i-1} \simeq M_i/M_{i-1}$, $1 \leqslant i \leqslant t$.

Note that this shows in particular that simple groups are determined by their integral group rings.

The next result of Whitcomb pushes the isomorphism problem further to metabelian groups, and it seems to give rise to a possible induction for the solvable case:

(3.4) THEOREM [Wh]. *Let* α: $\mathbf{Z}G \to \mathbf{Z}H$ *be an augmented isomorphism, and* A *an abelian normal subgroup of* G *corresponding under* α *to* B *which is abelian normal in* H (3.1, 3.2). *Then we have a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \to & I(A)G & \to & \mathbf{Z}G & \to & \mathbf{Z}G/A & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \bar\alpha} & & \\ 0 & \to & I(B)H & \to & \mathbf{Z}H & \to & \mathbf{Z}H/B & \to & 0 \end{array}$$

Assume that $\bar\alpha$ *is induced from a group isomorphism* $\bar\varrho$: $G/A \to H/B$. *Then* $G \simeq H$.

(3.5) COROLLARY. *In* (3.4) *assume that* G *is metabelian with* G/A *abelian. Then* $G \simeq H$.

In fact, by Higman's result (2.2), $\bar\alpha$ is induced from a group isomorphism.

Proof of (3.4). $I(A)G$ is the kernel of the map $\mathbf{Z}G \to \mathbf{Z}G/A$. Now Heinz Hopf has observed that

$$\gamma\colon\ I(A)G/I(A)I(G) \to A, \qquad (a-1)\cdot g + I(A)I(G) \mapsto a,$$

is an isomorphism of *left* $\mathbf{Z}G$-modules, where G acts on A via conjugation. Hence the diagram in (3.4) gives rise to the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & \mathbf{Z}G & \to & \mathbf{Z}G/A & \to & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \bar\alpha} & & \\ 0 & \to & B & \to & \mathbf{Z}H & \to & \mathbf{Z}H/B & \to & 0. \end{array}$$

We also have the commutative diagram—arising from the natural embeddings—

$$\begin{array}{ccc} G/A & \xrightarrow{\varkappa_1} & \mathbf{Z}G/A \\ \downarrow{\scriptstyle \bar\varrho} & & \downarrow{\scriptstyle \bar\alpha} \\ H/B & \xrightarrow{\varkappa_2} & \mathbf{Z}H/B. \end{array}$$

An easy cohomology argument now shows that the group extensions

$$\begin{array}{ccccccccc} 1 & \to & A & \to & G & \to & G/A & \to & 1 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \bar\varrho} & & \\ 1 & \to & B & \to & H & \to & H/B & \to & 1 \end{array}$$

are isomorphic, via the pullbacks along $\varkappa_1$ and $\varkappa_2$.

Note, however, that the proof does not give any information of how G is embedded into $V(\mathbf{Z}H)$; also, it does not give any clue for the Zassenhaus conjecture. As a matter of fact, the Zassenhaus conjecture (2.11) is not at all suited for an induction as (3.3) might suggest.

When Leonard Scott and myself started to consider these questions 5 years ago, the Zassenhaus conjecture was at first a tempting target for a counterexample. At that time we were not successful, and eventually we started to believe that it might be true for certain classes of groups; in fact, it was a guide for our work on p-groups. However, recently it has turned out that the Zassenhaus conjecture is too strong.

One might even go one step further than Zassenhaus and ask – this question was briefly raised by Berman and Rossa [BR] for complete rings:

(3.6) CONJUGACY PROBLEM. *If $V(RG) = V(SH)$, are G and H conjugate in $V(RG)$?*

The results which we have obtained so far let it appear reasonable to ask the following questions:

(3.7) Let R be a complete Dedekind domain with residue field of characteristic $p > 0$. Let B_0 be the principal block of RG. Is the "defect group" of B_0 uniquely determined up to conjugacy in B_0?

IV. Our answers

A. Progress on the isomorphism problem

The next results were obtained by Leonard Scott and myself in 1985 [RS1] and 1987 [RS3]:

(4.1) *Let G be a finite group such that there exists an exact sequence $1 \to A \to G \to N \to 1$, where A is abelian and N is nilpotent. If $\mathbf{Z}G \simeq \mathbf{Z}H$, then G is isomorphic to H.*

Again, this is a statement of an abstract isomorphism, and it does not give any information about the embeddings into $V(\mathbf{Z}G)$.

(4.2) THEOREM. *Let G be a finite solvable group, and assume that $\mathbf{Z}G \simeq \mathbf{Z}H$. Then for every prime p, $G/O_{p'}(G)$ is isomorphic to $H/O_{p'}(H)$, where $O_{p'}(X)$ is the largest normal subgroup of X of order prime to p. In particular, the Sylow p-subgroups of G and H are isomorphic.*

As we shall see below, in this case one can say more about the embedding of H into $V(\mathbf{Z}G)$.

B. Drawbacks and progress on the Zassenhaus conjecture

Leonard Scott and I have tried hard to prove the Zassenhaus conjecture (2.12) for abelian-by-nilpotent groups, however unsuccessfully. Eventually we started looking more carefully for a counterexample. We constructed a counterexample to (2.12) [RS2]:

(4.3) THEOREM. *There exists a finite metabelian group G and an augmented automorphism α of $\mathbf{Z}G$ such that $\alpha\cdot\varrho$ is not a central automorphism for any group automorphism ϱ of G.*

This shows that the Zassenhaus conjecture is false, even for metabelian groups. It was known to hold, however, for nilpotent groups of class 2 and for certain metabelian groups [RiSe].

Before we found the counterexample, we made some positive progress on the Zassenhaus conjecture [RS]:

(4.4) THEOREM. *Let G be a finite nilpotent group, and assume that $\mathbf{Z}G = \mathbf{Z}H$ as augmented algebras. Then there exists a unit a in $\mathbf{Q}G$ normalizing $V(\mathbf{Z}G)$ such that $aGa^{-1} = H$, i.e. any finite subgroup U in $V(\mathbf{Z}G)$ with $|U| = |G|$ is conjugate to G in $\mathbf{Q}G$.*

(4.5) THEOREM. *Let G be a finite group with a normal p-subgroup N such that for the centralizer of N in G we have $C_G(N) \subset N$. (Alternatively phrased, G is a p-constrained group with $O_{p'}(G) = 1$.) Assume that α is an augmented automorphism of $\mathbf{Z}G$. Then the Zassenhaus conjecture holds for G and $\alpha(G)$, i.e. G and $\alpha(G)$ are conjugate in $\mathbf{C}G$.*

We point out that (4.5) applies in particular to p-solvable groups G with $O_{p'}(G) = 1$. A comparison between (4.3) and (4.5) shows how sensitive the Zassenhaus conjecture is to the internal structure of the underlying group.

C. Progress on the conjugacy problem

As we have remarked above, $\mathbf{Z}D_8$, the group ring of the dihedral group of order 8, has another subgroup $U \simeq D_8$ in $V(\mathbf{Z}D_8)$ which is not conjugate in $V(\mathbf{Z}D_8)$ to D_8; however, U and D_8 are conjugate in $\mathbf{Q}D_8$ by a matrix with determinant 3. Now 3 becomes a unit in $\hat{\mathbf{Z}}_2$, the 2-adic integers. Moreover, we have the philosophy that for a p-group P, the natural group ring is not $\mathbf{Z}P$, but rather $\hat{\mathbf{Z}}_pP$, the p-adic group ring. This was one of the reasons why we concentrated on $\hat{\mathbf{Z}}_pP$. Another reason is the following: In order to prove that the isomorphism problem has a positive answer for p-groups P, one wants to use induction on the order of P. Neither the isomorphism problem nor the Zassenhaus conjecture are suited for induction. In fact, if the automorphism α of $\mathbf{Z}G$ is on the quotient $\mathbf{Z}G/N$ conjugation with a unit $a \in \mathbf{Q}G/N$, there is no reason to believe that a can be lifted to a unit in $\mathbf{Q}G$ normalizing $\mathbf{Z}G$. However, if one considers p-groups over $\hat{\mathbf{Z}}_p$, and if a is in $V(\hat{\mathbf{Z}}_pP/N)$, then it can be lifted to a unit in $V(\hat{\mathbf{Z}}_pP)$, and we can use induction. Following this philosophy, we were able to extend our result on p-groups [RS], and obtained in 1986 the following result [RS3]:

(4.6) THEOREM. *Let G be a finite group with normal Sylow p-subgroup, such* *that* $O_{p'}(G) = 1$. *Let* α *be an augmented automorphism of* $\hat{\mathbf{Z}}_pG$. *Then there exists a unit* $u \in V(\hat{\mathbf{Z}}_pG)$ *such that* $uGu^{-1} = \alpha(G)$.

The theorem we have proved is actually more general: Let G be a p-constrained group with $O_{p'}(G) = 1$ and let $N = O_p(G)$. If α is an augmented automorphism of $\hat{\mathbf{Z}}_pG$, stabilizing $I_{\hat{\mathbf{Z}}_p}(N)G$, then G and $\alpha(G)$ are conjugate in $V(\hat{Z}_pG)$.

This result applies in particular to p-groups, as one might have expected according to our philosophy. However, what is really surprising is that the above groups are by no means all p-groups (the symmetric group on 3 elements satisfies the hypotheses of (4.6) for $p = 3$), and that even the p'-parts of these groups can be detected p-adically. Note that for a q-group Q, $q \neq p$,

$$V(\hat{\mathbf{Z}}_pQ) = \prod \mathrm{GL}(n_i, R_i),$$

where the R_i are unramified extensions of $\hat{\mathbf{Z}}_p$. Also, for groups which satisfy the hypotheses of (4.6), the Zassenhaus conjecture is true. Again, comparing (4.6) with our counterexample to the Zassenhaus conjecture, (4.3)—recall that our counterexample is metabelian—one sees how delicate these problems are.

We have stated in (4.4) that the Zassenhaus conjecture is true for $\mathbf{Z}N$, provided N is a nilpotent group. Our result (4.6) is so strong that it allows to compute the Picard group of $\mathbf{Z}N$ semilocally [RS], and from that we have:

(4.7) THEOREM. *For nilpotent groups* N, *the conjugacy problem has in general a negative answer for both* $V(\mathbf{Z}N)$ *and* $V(\hat{\mathbf{Z}}_pN)$.

Let us pause for a moment to contemplate about further possibilities: Let G be a finite group with a normal Sylow p-subgroup and with $O_{p'}(G) = 1$. Then a consequence of (4.6) is the following: Let $U \leqslant V(\hat{\mathbf{Z}}_pG)$, with $U \simeq G$, and such that the elements of U are linearly independent over $\hat{\mathbf{Z}}_p$. Then U and G are conjugate in $V(\hat{\mathbf{Z}}_pG)$. I do not know whether the hypothesis that the elements of U are linearly independent can be dropped. Our results, and the evidence we have gathered up to now, make it reasonable to ask whether $V(\hat{\mathbf{Z}}_pG)$ has the

(4.8) SYLOW PROPERTY. Let G be a finite group with a normal Sylow p-subgroup P such that $O_{p'}(G) = 1$. Let U be a finite p-subgroup of $V(\hat{\mathbf{Z}}_pG)$. Is U conjugate in $V(\hat{\mathbf{Z}}_pG)$ to a subgroup of P?

One consequence of this would be that for groups as above, the vertices of indecomposable $\hat{\mathbf{Z}}_pG$-lattices would be unique up to conjugacy in $V(\hat{\mathbf{Z}}_pG)$.

The result (4.6) can be interpreted as a first step to prove Sylow's theorems for finite subgroups of $V(RG)$, provided G is a p-group. In an attempt to give an answer to this question we were guided by the fact that for a finite group H, the connectedness of the spectrum of the cohomology ring $H^*(H, \mathbf{F}_p)$ is equivalent to Sylow's theorems for the p-subgroups of H. We tried unsuccessfully to mimic John Carlson's proof that the variety of an indecomposable module is connected [Ca].

More precisely, let V be a profinite p-group, and denote by $H^*(V, \mathbf{F}_p)$ the continuous (even-dimensional for p odd) cohomology ring of V with coefficients in $\mathbf{F}_p$, the field with p elements. If the spectrum of $H^*(V, \mathbf{F}_p)$ is connected, we say that *the variety of V is connected*. We shall write $VC(V, \mathbf{F}_p)$ for the variety of $H^*(V, \mathbf{F}_p)$.

Though we could not reach our original goal, we were able to prove for R a complete Dedekind domain of characteristic zero with residue field of characteristic p:

(4.9) THEOREM. *The following conditions are equivalent for a p-group G:*

(i) *Every finite p-subgroup U of V(RG) is conjugate in V(RG) to a subgroup of G.*

(ii) *For every p-subgroup P of G, the natural inclusion*

$$N_G(P)/P \to N_{V(RG)}(P)/P$$

induces a continuous map

$$VC(N_G(P)/P, \mathbf{F}_p) \to VC(N_{V(RG)}(P)/P, \mathbf{F}_p),$$

which is a bijection.

(iii) *The variety of $N_{V(RG)}(P)/P$ is connected for every p-subgroup P of G.*

(In the statement of the result we have used $N_A(B)$ to denote the normalizer of B in A).

We want to point out that the statements (i) and (iii) are not true in general for profinite p-groups: in fact, we have examples of unit groups of orders where (i) is false. It would be interesting to have a group-theoretical criterion for when (i) is true for profinite p-groups.

Let us return to a discussion of (4.8) in case G is a p-group. Some years ago we found a proof of (4.8) for G a 2-group [R1]. Later on we were able to handle groups of order p^3, and we developed a sketch of the proof in the general p-group case in November 1985 [S1]. Since in this proof we had not worked out *all* the details, we only made at the Arcata meeting in 1986 the conjecture that (4.8) is true for p-groups. In October 1986 we learnt that Al Weiss from the University of Alberta [W] had a different proof of (4.8) for p-groups:

(4.10) SUBGROUP RIGIDITY THEOREM. *Let G be a finite p-group, and U a finite subgroup of $V(\hat{\mathbf{Z}}_pG)$. Then U is conjugate in $V(\hat{\mathbf{Z}}_pG)$ to a subgroup of G.*

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