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ON THE BACKWARD PARABOLIC EQUATION: A CRITICAL SURVEY OF SOME CURRENT METHODS(*)

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Consider the equation

(1) $$u_t - \Delta u = 0 \qquad (x,t) \text{ in } D \times (0,1)$$

with boundary condition

(2) $$u = 0 \qquad \text{on } \partial D$$

where D is a domain in $\boldsymbol{R}^n$ with a sufficiently smooth boundary ∂D. The problem is to find a solution $u(x,t)$ of (1)–(2) subject to the "terminal" condition

(3) $$u(x,1) = g(x).$$

By a solution of the problem (1)–(3), henceforth called BPP, we mean a continuous mapping

$$t \to u(t)$$

from $[0,1]$ to $L_2(D)$, which is an $H_0^1 \cap H^2(D)$-valued C^1-map on $(0,1)$, satisfies (1) in the strong sense, and assumes value g at $t = 1$, i.e.,

(4) $$u(1) = g.$$

Now, it is immediately seen that the problem does not have a solution for arbitrary g in $L_2(D)$, because $u(\cdot, l)$ has certain well-known regularity properties that are not shared by arbitrary functions in $L_2(D)$. Furthermore, as will be seen later, even on the set of the g's for which the problem does have a solution, the latter does not vary continuously with g. In other words, the problem is ill-posed. The ill-posedness of the problem is linked to the fact that heat conduction is an irreversible phenomenon in time. The problem, in such setting, is intractable numerically. Indeed; g is usually a result of experimental

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measurements (hence subject to errors), and the slightest error in the measurement may lead to a new g for which the problem either has no solution, or else, for which the solution departs considerably from the exact solution. We are facing the task of *regularizing* the problem, i.e., finding a solution $v_\beta(t;g)$, $\beta > 0$, to (1)–(2), stable with respect to variations in g, that satisfies

$$v_\beta(1,g) \to g \qquad \beta \to 0, \tag{5}$$

$$v_\beta(0,u(1)) \to u(0) \qquad \beta \to 0, \tag{6}$$

for any solution $u(t)$ of (1)–(2) (we write $u(t)$ for the function $u(t)x = u(x,t)$).

The one-parameter family $v_\beta(t,g)$, $\beta > 0$, is called a *quasi-solution* of the BPP. The second question that arises is how to pick the right β. We shall return to this matter. For now, we simply note that the problem has been considered for the past three decades, with such pioneering papers as e.g. the work of F. John in 1960 [7]. But it is during the past two decades that the area of research has been particularly active, with the work of Lattès–Lions on the method of quasi-reversibility (see [6], [8]), with the applications of Tikhonov's theory of regularization [4], with the work of K. Miller [8] on stabilized quasi-reversibility, and with the method of integral equations (see [1a], [1b] for instance).

From the point of view of our exposition, it seems easiest to start with the method of integral equations. In fact, this method will form the central point of our discussion (Section 1). Other methods will be sketched in Section 2. The advantages and drawbacks of each method will be discussed (Section 3). The paper concludes with an open problem together with some clues to its solution.

1. The method of integral equations

Let $S(t)$, $t \geqslant 0$, be the semigroup generated by Δ on $L_2(D)$ ($\operatorname{dom}\Delta = H_0^1 \cap H^2(D)$). For any v in $L_2(D)$, put

$$u(t) = S(t)v. \tag{7}$$

Then $u(t)$ is the solution of (1)–(2) with initial value v. Thus the BPP is converted into the problem of solving the integral equation

$$Kv = g, \qquad K \equiv S(1). \tag{8}$$

The operator K has the following properties

(9) K is one to one,

(10) K is self-adjoint and strictly positive.

From (10), we infer that range(K) is dense in $L_2(D)$, and since it is a proper subspace of $L_2(D)$, we see that K^{-1} is not continuous. We have thus proved that our BPP is ill-posed. Observe that the density of range(K) is of independent interest, and is of particular relevance when we consider our BPP as a problem of control.

Consider (8) with g, the experimentally measured value of $u(1)$. Suppose

$$|u(1)-g| \leqslant \varepsilon \tag{11}$$

where $|\cdot|$ denotes the L_2-norm. Suppose it is known, e.g. by a physical reasoning, that

$$|u(0)| \leqslant E. \tag{12}$$

For $\beta = \varepsilon/E$, define

$$v_\beta = (\beta + K)^{-1} g \tag{13}$$

where we have written β for βI (for regularization of integral operators, see Tikhonov and Arsenin [11]). Put

$$u_\beta(t, g) = S(t) v_\beta, \qquad t \geqslant 0. \tag{14}$$

Then $u_\beta(t, g)$ satisfies (1)–(2) and, furthermore

$$|u_\beta(t, g) - u(t)| \leqslant 2\alpha^t \varepsilon^t E^{1-t} \tag{15}$$

where

$$\alpha \equiv (1+\beta)^{-1}. \tag{16}$$

The inequality (15) was derived using the logarithmic convexity of the L_2-norm of any solution of (1)–(2). For a computation of v_β, we can use successive approximation. In fact, (13) can be rewritten as

$$v_\beta = \frac{I-K}{1+\beta} + \frac{g}{1+\beta}. \tag{17}$$

Since $\|I-K\| \leqslant 1$, it is seen that the right-hand side of (17) defines a contraction on $L_2(D)$ of coefficient $\leqslant (1+\beta)^{-1} < 1$. Thus (17) can be solved by successive approximation, using the contraction principle. For more details, the reader is referred to [1b].

2. The method of quasireversibilty and other methods

The method of quasireversibility (QR) due to Lattès–Lions [7] consists in perturbing the original equation (1) into one for which the problem is well-posed. More precisely, it consists in solving the (well-posed) problem in $w_\varepsilon(t)$:

$$\partial w_\varepsilon/\partial t - \Delta w_\varepsilon - \varepsilon \Delta^2 w_\omega = 0, \qquad \varepsilon > 0, \tag{18}$$

$$w_\varepsilon = 0 = \Delta w_\varepsilon, \qquad \text{on } \partial D, \tag{19}$$

$$w_\varepsilon(1) = g, \tag{20}$$

and then putting

(21) $$v_\varepsilon(t,g) = S(t)w_\varepsilon(0).$$

Here $S(t)$ is the semigroup defined in Section 1. Then, $v_\varepsilon(t,g)$ depends continuously on g and

(22) $$v_\varepsilon(1,g)\to g, \quad \varepsilon\to 0.$$

The stability of the solution $v_\beta(t,g)$ for $0<t<1$ is not studied in [7].

Gajewski and Zaccharias [6] studied, instead of (18), the equation

(23) $$\partial w_\varepsilon/\partial t - \Delta w_\varepsilon - \varepsilon(\partial w_\varepsilon/\partial t) = 0, \quad \varepsilon > 0$$

with

(24) $$w_\varepsilon = 0, \quad \text{on } \partial D,$$

(25) $$w_\varepsilon(1) = g,$$

and then they put (as in Lattès–Lions)

(26) $$v_\varepsilon(t;g) = S(t)w_\varepsilon(0).$$

Restricting themselves to bounded domains, the authors in [6] proved that $v_\varepsilon(1;g)\to g$ for $\varepsilon\to 0$, with an estimated rate of convergence in the case where g is in $H_0^1(D)$. But, again, the stability of $v_\varepsilon(t;g)$ for $0<t<1$ is not studied in [6]. Ewing [4] studied the same equation (23), and, for the regularized solution, he took just $w_\varepsilon(t,g)$, not bothering to go backward and forward as in Lattès–Lions. He found the error estimate

(27) $$|w_\varepsilon(t;g)-u(t)| \leqslant 4(1-t)E/\big(t^2\log(E/\varepsilon)\big)+E^{1-t}\varepsilon^t,$$

where E and ε are as in Section 1.

K. Miller [9] considered the solution $w_f(t)$ of the equation

(28) $$\frac{d}{dt}w_f - f(A)w_f = 0,$$

defined on a suitable Hilbert space H, A being a symmetric positive operator on H satisfying $w_f(1)=g$, and then he put

(29) $$v_f(t;g) = S(t)w_f(0).$$

He found necessary and sufficient conditions on the function f for the inequality

(30) $$|v_f(t;g)-u(t)| \leqslant 2\varepsilon^t E^{1-t}.$$

As we can see, Miller's method is a variant of the QR method. No better stabilized QR method is known to us.

Franklin [5], using Tikhonov's regularization method, under the condition that $u(0)$ lies in $H^s(D)$, $D=(0,1)$, s a positive integer, and with zero Neumann condition, obtained a regularized solution $u_\varepsilon(t,g)$ with the error estimate

$$|u_\varepsilon(0,g)-u(0)| \leqslant c(\log E/\varepsilon)^{-s/2}, \tag{31}$$

where $c > 0$ is some (unspecified) constant.

K. Miller (*loc. cit.*), using the method of truncated eigenfunction expansion and under the sole condition that $u(0)$ lies in $L_2(D)$, obtained a stabilized approximate solution with an error estimate similar to (30). For a bounded domain, under the condition that $u(0)$ satisfies an inequality of the form

$$\sum_{1}^{\infty} \lambda_n^s |(u(0), \varphi_n)|^2 \leqslant E^2, \quad s \geqslant 0, \quad (\cdot,\cdot) = L_2\text{-inner product} \tag{32}$$

where (φ_n) are the eigenfunctions (orthonormalized) of $-\Delta$ in $H_0^1 \cap H^2(D)$ and (λ_n) are the corresponding eigenvalues, the authors in [2] "constructed" by truncated eigenfunction expansion a stabilized approximate solution $u_\varepsilon(t,g)$ with a sharper estimate of the error. In fact, they put

$$u_\varepsilon(t;g) = \sum_{1}^{N(\varepsilon)} (g, \varphi_n)\varphi_n \exp(\lambda_n(1-t)) \tag{33 a}$$

where

$$N(\varepsilon) = \max\{n : \lambda_n \leqslant \log((E/\varepsilon)\log E/\varepsilon)^{-s/2}\}$$

(if $\lambda_1 > \log((E/\varepsilon)\log E/\varepsilon)^{-s/2}$, $u_\varepsilon(t;g)$ is understood to be the null function). Then, the following error estimate holds (for small $\varepsilon > 0$ if $s > 0$, and for any $\varepsilon > 0$ if $s = 0$):

$$|u_\varepsilon(t,g) - u(t)| \leqslant (1+2^s)^{1/2} E^{1-t} \varepsilon^t (\log E/\varepsilon)^{-s(1-t)/2} \tag{33 b}$$

Note that for $s = 0$, the foregoing estimates reduce to that of Miller's (*loc. cit.*).

3. Discussion and an open problem

We shall point to some of the advantages and drawbacks of each of the methods outlined above. The method of truncated eigenfunctions, as used in K. Miller (*loc. cit.*) and in Ang–Hai [2], is certainly a most valuable tool as long as the domain D is such that the eigenfunctions and eigenvalues of $-\Delta$ on D are available (this is the case of such simple geometries as the rectangle, the disc and their n-dimensional analogues). In general, however, they are out of reach. The problem of eigenfunction and eigenvalue perturbations is usually a most delicate problem.

The method of integral equations, as presented in Section 1, provides an efficient constructive tool for calculating a regularized solution, with an advantageous error estimate. It is true that for the method to work, one need to know the Green's function; however, there are standard methods for approximating it.

Tikhonov's method of smoothing functionals, as used in Franklin [5], is a powerful general method, which can be applied to more general problems, not necessarily linear.

The original QR method of Lattès–Lions and its stabilized version due to K. Miller are of considerable theoretical interest. Because of their generality, they can probably be successfully applied to semilinear equations of the form

$$u_t - \Delta u = h(\nabla u, u), \tag{34}$$

with a QR perturbation of type

$$\partial u_\varepsilon/\partial t - \Delta u_\varepsilon - \varepsilon\Delta^2 u_\varepsilon = h(\nabla u_\varepsilon, u_\varepsilon) \tag{35}$$

or

$$\partial u_f/\partial t - f(-\Delta) = h(\nabla u_f, u_f) \tag{36}$$

where f has the stabilizing effect in K. Miller's original approach.

We conclude with an open problem and a clue to its solution. We still consider equation (1)

$$u_t - \Delta u = 0 \quad (x,t) \text{ in } D\times(0,1) \tag{37}$$

with boundary condition

$$u\partial u/\partial n = 0, \quad u \geqslant 0, \quad \partial u/\partial n \geqslant 0 \quad \text{on } \partial D \tag{38}$$

where $\partial/\partial n$ is the derivative along the outer normal at the boundary. The problem is to find a solution satisfying

$$u(1) = g. \tag{39}$$

The problem was raised in Payne [10]. A rather natural way to look at it would consist in converting it into a problem involving a nonlinear field equation with zero Neumann condition. In fact, let $v = u^2$. Then, v satisfies

$$v_t - \Delta v = -|\nabla v|^2/2v \tag{40}$$

with boundary condition

$$\partial v/\partial n = 0 \quad \text{on } \partial D \tag{41}$$

and terminal condition

$$v(1) = g. \tag{42}$$

To regularize problem (40)–(42), one might try the QR method with a regularized equation given in (35) or (36). Alternatively, one could formulate (40)–(41) as an integral equation (nonlinear), and then use Tikhonov's method.

References

[1a] Dang Dinh Ang, *The inverse time problem for some diffusion equations*, Math. Geo. Workshop, IGU, Venice, Italy 1984.

[1b] — *Stabilized approximate solutions to the inverse time problem for a parabolic equation*, J. Math. Anal. Appl. 111 (1) (1985), 148–155.

[2] Dang Dinh Ang and Dang Dinh Hai, *On the backward parabolic equation*, (to appear).

[3] D. Colton and J. Wimp, *The construction of solutions to the heat equation backward in time*, Math. Methods Appl. Sci. 1 (1979), 32–39.

[4] R. E. Ewing, *The approximation of certain parabolic equations backward in time by Sobolev equations*, SIAM J. Math. Anal. 6 (1975), 283–294.

[5] J. Franklin, *On Tikhonov's method for ill-posed problems*, Math. Comp. 28 (1974), 889–907.

[6] H. Gajewski and K. Zaccharias, *Zur Regularisierung einer Klass nichtkorrekter Probleme bei Evolutiongleichungen*, J. Math. Anal. Appl. 38 (1972), 784–789.

[7] F. John, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math. 13 (1960), 551–585.

[8] R. Lattès and J. L. Lions, *Méthode de quasireversibilité et applications*, Dunod, Paris 1967.

[9] K. Miller, *Stabilized quasireversibility and other nearly-best possible methods for nonwell posed problems*, Lecture Notes in Math. 316, 161–176.

[10] L. E. Payne, *Improperly posed problems in partial differential equations*, Regional Conf. SIAM 1975.

[11] A. N. Tikhonov and V. Y. Arsenin, *Solutions of ill posed problems*, Winston, Washington 1977.

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