ON THE BACKWARD PARABOLIC EQUATION:
A CRITICAL SURVEY OF SOME CURRENT METHODS(*)

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Consider the equation
\[ u_t - \Delta u = 0 \quad (x, t) \text{ in } D \times (0, 1) \]  
with boundary condition
\[ u = 0 \quad \text{on } \partial D \]  
where $D$ is a domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial D$. The problem is to find a solution $u(x, t)$ of (1)--(2) subject to the "terminal" condition
\[ u(x, 1) = g(x). \]  

By a solution of the problem (1)--(3), henceforth called BPP, we mean a continuous mapping
\[ t \rightarrow u(t) \]  
from $[0, 1]$ to $L_2(D)$, which is an $H_0^1 \cap H^2(D)$-valued $C^1$-map on $(0, 1)$, satisfies (1) in the strong sense, and assumes value $g$ at $t = 1$, i.e.,
\[ u(1) = g. \]  

Now, it is immediately seen that the problem does not have a solution for arbitrary $g$ in $L_2(D)$, because $u(\cdot, t)$ has certain well-known regularity properties that are not shared by arbitrary functions in $L_2(D)$. Furthermore, as will be seen later, even on the set of the $g$'s for which the problem does have a solution, the latter does not vary continuously with $g$. In other words, the problem is ill-posed. The ill-posedness of the problem is linked to the fact that heat conduction is an irreversible phenomenon in time. The problem, in such setting, is intractable numerically. Indeed, $g$ is usually a result of experimental

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measurements (hence subject to errors), and the slightest error in the measure-
ment may lead to a new $g$ for which the problem either has no solution, or else,
for which the solution departs considerably from the exact solution. We are
facing the task of regularizing the problem, i.e., finding a solution $v_\beta(t; g), \beta > 0,$
to (1)–(2), stable with respect to variations in $g,$ that satisfies
\begin{align*}
(5) \quad v_\beta(1, g) & \rightarrow g \quad \beta \rightarrow 0, \\
(6) \quad v_\beta(0, u(1)) & \rightarrow u(0) \quad \beta \rightarrow 0,
\end{align*}
for any solution $u(t)$ of (1)–(2) (we write $u(t)$ for the function $u(t)x = u(x, t)$).

The one-parameter family $v_\beta(t, g), \beta > 0,$ is called a quasi-solution of the
BPP. The second question that arises is how to pick the right $\beta.$ We shall
return to this matter. For now, we simply note that the problem has been
considered for the past three decades, with such pioneering papers as e.g. the
work of F. John in 1960 [7]. But it is during the past two decades that the area
of research has been particularly active, with the work of Lattès–Lions on the
method of quasi-reversibility (see [6], [8]), with the applications of Tikhonov’s
theory of regularization [4], with the work of K. Miller [8] on stabilized
quasi-reversibility, and with the method of integral equations (see [1a], [1b] for
instance).

From the point of view of our exposition, it seems easiest to start with the
method of integral equations. In fact, this method will form the central point of
our discussion (Section 1). Other methods will be sketched in Section 2. The
advantages and drawbacks of each method will be discussed (Section 3). The
paper concludes with an open problem together with some clues to its solution.

1. The method of integral equations

Let $S(t), t \geq 0,$ be the semigroup generated by $\Delta$ on $L_2(D)$
(dom $\Delta = H^1_0 \cap H^2(D)$). For any $v$ in $L_2(D),$ put
\begin{equation}
(7) \quad u(t) = S(t)v.
\end{equation}
Then $u(t)$ is the solution of (1)–(2) with initial value $v.$ Thus the BPP is
converted into the problem of solving the integral equation
\begin{equation}
(8) \quad Kv = g, \quad K \equiv S(1).
\end{equation}
The operator $K$ has the following properties
\begin{align*}
(9) \quad & K \text{ is one to one}, \\
(10) \quad & K \text{ is self-adjoint and strictly positive}.
\end{align*}
From (10), we infer that range$(K)$ is dense in $L_2(D),$ and since it is a proper
subspace of $L_2(D),$ we see that $K^{-1}$ is not continuous. We have thus proved
that our BPP is ill-posed. Observe that the density of range$(K)$ is of
independent interest, and is of particular relevance when we consider our BPP as a problem of control.

Consider (8) with \( g \), the experimentally measured value of \( u(1) \). Suppose

\[ |u(1) - g| \leq \varepsilon \]

where \( |\cdot| \) denotes the \( L_2 \)-norm. Suppose it is known, e.g. by a physical reasoning, that

\[ |u(0)| \leq E. \]

For \( \beta = \varepsilon/E \), define

\[ v_\beta = (\beta + K)^{-1} g \]

where we have written \( \beta \) for \( \beta I \) (for regularization of integral operators, see Tikhonov and Arsenin [11]). Put

\[ u_\beta(t, g) = S(t) v_\beta, \quad t \geq 0. \]

Then \( u_\beta(t, g) \) satisfies (1)–(2) and, furthermore

\[ |u_\beta(t, g) - u(t)| \leq 2\alpha e^t E^{1-t} \]

where

\[ \alpha \equiv (1 + \beta)^{-1}. \]

The inequality (15) was derived using the logarithmic convexity of the \( L_2 \)-norm of any solution of (1)–(2). For a computation of \( v_\beta \), we can use successive approximation. In fact, (13) can be rewritten as

\[ v_\beta = \frac{I - K}{1 + \beta} + \frac{g}{1 + \beta}, \]

Since \( \|I - K\| \leq 1 \), it is seen that the right-hand side of (17) defines a contraction on \( L_2(D) \) of coefficient \( \leq (1 + \beta)^{-1} < 1 \). Thus (17) can be solved by successive approximation, using the contraction principle. For more details, the reader is referred to [1b].

2. The method of quasireversibility and other methods

The method of quasireversibility (QR) due to Lattès–Lions [7] consists in perturbing the original equation (1) into one for which the problem is well-posed. More precisely, it consists in solving the (well-posed) problem in \( w_e(t) \):

\[ \begin{align*}
\frac{\partial w_e}{\partial t} - Aw_e - \varepsilon \Delta^2 w_e &= 0, \quad \varepsilon > 0, \\
w_e &= 0 = Aw_e, \quad \text{on } \partial D, \\
w_e(1) &= g.
\end{align*} \]
and then putting
\begin{equation}
(21) \quad v_\varepsilon(t, g) = S(t)w_\varepsilon(0).
\end{equation}
Here $S(t)$ is the semigroup defined in Section 1. Then, $v_\varepsilon(t, g)$ depends continuously on $g$ and
\begin{equation}
(22) \quad v_\varepsilon(1, g) \rightarrow g, \quad \varepsilon \rightarrow 0.
\end{equation}

The stability of the solution $v_\nu(t, g)$ for $0 < t < 1$ is not studied in [7].

Gajewski and Zaccharias [6] studied, instead of (18), the equation
\begin{equation}
(23) \quad \frac{\partial w_\varepsilon}{\partial t} - \Delta w_\varepsilon - \varepsilon(\frac{\partial w_\varepsilon}{\partial t}) = 0, \quad \varepsilon > 0
\end{equation}
with
\begin{equation}
(24) \quad w_\varepsilon = 0, \quad \text{on } \partial D,
\end{equation}
\begin{equation}
(25) \quad w_\varepsilon(1) = g,
\end{equation}
and then they put (as in Lattès–Lions)
\begin{equation}
(26) \quad v_\varepsilon(t; g) = S(t)w_\varepsilon(0).
\end{equation}

Restricting themselves to bounded domains, the authors in [6] proved that $v_\varepsilon(1; g) \rightarrow g$ for $\varepsilon \rightarrow 0$, with an estimated rate of convergence in the case where $g$ is in $H^2_0(D)$. But, again, the stability of $v_\varepsilon(t; g)$ for $0 < t < 1$ is not studied in [6]. Ewing [4] studied the same equation (23), and, for the regularized solution, he took just $w_\varepsilon(t, g)$, not bothering to go backward and forward as in Lattès–Lions. He found the error estimate
\begin{equation}
(27) \quad |w_\varepsilon(t; g) - u(t)| \leq 4(1-t)E/(t^2\log(E/\varepsilon)) + E^{1-t}\varepsilon',
\end{equation}
where $E$ and $\varepsilon$ are as in Section 1.

K. Miller [9] considered the solution $w_f(t)$ of the equation
\begin{equation}
(28) \quad \frac{d}{dt}w_f - f(A)w_f = 0,
\end{equation}
defined on a suitable Hilbert space $H$, $A$ being a symmetric positive operator on $H$ satisfying $w_f(1) = g$, and then he put
\begin{equation}
(29) \quad v_f(t; g) = S(t)w_f(0).
\end{equation}
He found necessary and sufficient conditions on the function $f$ for the inequality
\begin{equation}
(30) \quad |v_f(t; g) - u(t)| \leq 2\varepsilon'E^{1-t}.
\end{equation}
As we can see, Miller’s method is a variant of the QR method. No better stabilized QR method is known to us.

Franklin [5], using Tikhonov’s regularization method, under the condition that $u(0)$ lies in $H^s(D)$, $D = (0, 1)$, $s$ a positive integer, and with zero Neumann condition, obtained a regularized solution $u_\varepsilon(t, g)$ with the error
estimate

\[ |u_c(0, g) - u(0)| \leq c (\log E/e)^{-s/2}, \]

where \( c > 0 \) is some (unspecified) constant.

K. Miller (loc. cit.), using the method of truncated eigenfunction expansion and under the sole condition that \( u(0) \) lies in \( L_2(D) \), obtained a stabilized approximate solution with an error estimate similar to (30). For a bounded domain, under the condition that \( u(0) \) satisfies an inequality of the form

\[ \sum_{i=1}^{\infty} \lambda_n^2 |(u(0), \varphi_n)|^2 \leq E^2, \quad s \geq 0, \quad (\cdot, \cdot) = L_2 \text{-inner product} \]

where \( (\varphi_n) \) are the eigenfunctions (orthonormalized) of \(-\Delta\) in \( H_0^1 \cap H^2(D) \) and \( (\lambda_n) \) are the corresponding eigenvalues, the authors in [2] "constructed" by truncated eigenfunction expansion a stabilized approximate solution \( u_c(t; g) \) with a sharper estimate of the error. In fact, they put

\[ u_c(t; g) = \sum_{i=1}^{N(e)} (g, \varphi_n) \varphi_n \exp(\lambda_n(1 - t)) \]

where

\[ N(e) = \max \{ n : \lambda_n \leq \log((E/e) \log E/e)^{-s/2} \} \]

(if \( \lambda_i > \log((E/e) \log E/e)^{-s/2} \), \( u_c(t; g) \) is understood to be the null function). Then, the following error estimate holds (for small \( \varepsilon > 0 \) if \( s > 0 \), and for any \( \varepsilon > 0 \) if \( s = 0 \))

\[ |u_c(t; g) - u(t)| \leq (1 + 2^s)^{1/2} E^{1-s} \varepsilon ((\log E/e)^{-s(1-s)/2} \]

Note that for \( s = 0 \), the foregoing estimates reduce to that of Miller's (loc. cit.).

\[ \text{3. Discussion and an open problem} \]

We shall point to some of the advantages and drawbacks of each of the methods outlined above. The method of truncated eigenfunctions, as used in K. Miller (loc. cit.) and in Ang-Hai [2], is certainly a most valuable tool as long as the domain \( D \) is such that the eigenfunctions and eigenvalues of \(-\Delta\) on \( D \) are available (this is the case of such simple geometries as the rectangle, the disc and their \( n \)-dimensional analogues). In general, however, they are out of reach. The problem of eigenfunction and eigenvalue perturbations is usually most delicate problem.

The method of integral equations, as presented in Section 1, provides an efficient constructive tool for calculating a regularized solution, with an advantageous error estimate. It is true that for the method to work, one needs to know the Green's function; however, there are standard methods for approximating it.
Tikhonov's method of smoothing functionals, as used in Franklin [5], is a powerful general method, which can be applied to more general problems, not necessarily linear.

The original QR method of Lattès–Lions and its stabilized version due to K. Miller are of considerable theoretical interest. Because of their generality, they can probably be successfully applied to semilinear equations of the form

\[ u_t - \Delta u = h(Vu, u), \]

with a QR perturbation of type

\[ \frac{\partial u_{f}}{\partial t} - \Delta u_{f} - \varepsilon \Delta^{2} u_{f} = h(Vu_{f}, u_{f}) \]

or

\[ \frac{\partial u_{f}}{\partial t} - f(-\Delta) = h(Vu_{f}, u_{f}) \]

where \( f \) has the stabilizing effect in K. Miller's original approach.

We conclude with an open problem and a clue to its solution. We still consider equation (1)

\[ u_t - \Delta u = 0 \quad (x, t) \text{ in } D \times (0, 1) \]

with boundary condition

\[ u \partial u / \partial n = 0, \quad u \geq 0, \quad \partial u / \partial n \geq 0 \quad \text{on } D \]

where \( \partial / \partial n \) is the derivative along the outer normal at the boundary. The problem is to find a solution satisfying

\[ u(1) = g. \]

The problem was raised in Payne [10]. A rather natural way to look at it would consist in converting it into a problem involving a nonlinear field equation with zero Neumann condition. In fact, let \( v = u^2 \). Then, \( v \) satisfies

\[ v_t - \Delta v = -|Vv|^2 / 2v \]

with boundary condition

\[ \partial v / \partial n = 0 \quad \text{on } D \]

and terminal condition

\[ v(1) = g. \]

To regularize problem (40)–(42), one might try the QR method with a regularized equation given in (35) or (36). Alternatively, one could formulate (40)–(41) as an integral equation (nonlinear), and then use Tikhonov's method.
References


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