A NATURAL EQUIVALENCE RELATION ON SINGULARITIES

T.-C. KUO

Department of Pure Mathematics, University of Sydney,
Sydney, N.S.W., Australia

1. Motivation

The classification problem is always fundamental in every branch of mathematics. For singularities, one would like to classify germs of real and complex analytic functions in $n$ variables.

Let $\mathcal{A}_n$ denote the set of all real analytic germs $g: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. Each $g$ can be represented by its Taylor expansion, which has no constant term.

When should two elements of $\mathcal{A}_n$ be declared equivalent? Each equivalence class should be as large as possible, making the classification simpler; each class should also be as small as possible, so that equivalent germs are "very much similar". Therefore the task is to search for a nice and natural, God-given, equivalence relation in $\mathcal{A}_n$, an ideal compromise between these contradictory demands. This is called the Equisingularity Problem.

Consider, as an illustrative example, the Whitney family

$$W_t(x, y) = xy(x - y)(x - ty), \quad (x, y, t) \in \mathbb{R}^3.$$ 

Let us restrict the parameter $t$ to the interval $(1, \infty)$, so that $W_t$ is a non-degenerate form for each $t$; in particular, $W_0 = 0$ consists of four distinct lines.

Intuitively, $W_t$ and $W_t', t \neq t'$, are very much similar; yet, there does not exist a local $C^1$-diffeomorphism $h$ such that $W_t \circ h = W_t'$. (This can be proved using a simple Linear Algebra argument on $dh$.)

This phenomenon had cast serious doubt on the existence of an ideal equivalence relation on $\mathcal{A}_n$.

Let us not be discouraged. There is, at least, a God-given way to construct a vector field, $\vec{v}$, which generates a one-parameter family of homeomorphisms trivializing the Whitney family. Consider any point $P(x, y, t)$ off the $t$-axis. Let $\mathcal{L}_p$ denote the level surface of $W_t$ through $P$, and
\( \frac{\partial}{\partial t} \) denote the unit vector in the \( t \)-direction. Take the orthogonal projection of \( \frac{\partial}{\partial t} \) to the tangent plane of \( \mathcal{L}_P \) at \( P \), and then adjust its length so that the \( t \)-component equals 1. The resulting vector is \( \vec{v}(P) \). An easy calculation leads to

\[
\vec{v}(x, y, t) = - \frac{\partial W}{\partial \tau} \frac{\partial W}{\partial x} \frac{\partial}{\partial x} - \frac{\partial W}{\partial \tau} \frac{\partial W}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial \tau}
\]

where \( W \) stands for \( W_t(x, y) \). Along the \( t \)-axis, define

\[
\vec{v}(0, 0, t) = \frac{\partial}{\partial t}.
\]

The flow of \( \vec{v} \) trivializes \( W_t(x, y) \) topologically (cf. [K_1], [K_2], [K_3]), but, of course, not diffeomorphically.

However, one should not be satisfied with a more topological trivialization. Since \( \vec{v} \) is God-given, it must offer something stronger.

A closer examination of the components of \( \vec{v} \) reveals their resemblance to the familiar example in Calculus:

\[
f(x, y) = \frac{P_\gamma(x, y)}{x^6 + y^6}, \quad f(0, 0) = 0, \quad O(P_\gamma) \geq 7,
\]

which is continuous but not \( C^1 \).

Now, let \( \beta: (\mathbb{M}, C) \rightarrow (\mathbb{R}^2, 0) \) be the blowing-up of \( \mathbb{R}^2 \) at 0, where \( \mathbb{M} \) is the Möbius hand, \( C \) its centre circle. Two charts are needed to cover \( \mathbb{M} \). In one chart, \( \beta \) is expressed as \( \beta(X, Y) = (XY, Y) \).

Hence

\[
(f \circ \beta)(X, Y) = \frac{Y Q(X, Y)}{X^6 + 1}
\]
is analytic. The situation is similar in the other chart. We have thus made an important observation: \( f \circ \beta \) is analytic on \( \mathcal{M} \)!

Returning to the Whitney family, one finds that \( d(\beta \times \text{id})^{-1}(\tilde{v}) \) is an analytic vector field on \( \mathcal{M} \times \mathbb{R} \), tangent to \( C \times \mathbb{R} \). Hence the topological trivialization generated by \( \tilde{v} \) lifts to an analytic isomorphism, leaving \( C \times \mathbb{R} \) invariant. A detailed calculation is carried out in \([K_3]\). This result leads naturally to the notion of blow-analytic equivalence of singularities defined in the following section.

2. Blow-analytic equisingularities

The notion of blowing-up can be slightly generalized. For instance, one ought to consider a succession of them. A proper, surjective holomorphic map \( \sigma^* : \tilde{X}^* \to X^* \) of complex spaces is called a modification if \( \sigma^* \) is a biholomorphism outside \( \sigma^*^{-1}(N) \), \( N \) a thin subset of \( X^* \) \([W]\). By a modification of real spaces we shall mean a (proper surjective) real analytic map \( \sigma : \tilde{X} \to X \) whose complexification \( \sigma^* \) is a modification.

Given \( g_1, g_2 \in \mathscr{A}_n \), we say they are blow-analytically equivalent if

(i) there is a local homeomorphism \( \phi \), \( g_2 \circ \phi = g_1 \);

(ii) there exist two (real) modifications \( \mu_1, \mu_2 \), and an analytic isomorphism \( \Phi \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\{ \mathcal{M}_1, P_1^{-1}(0) \} & \xrightarrow{\mu_1} & \{ \mathbb{R}^n, 0 \} \\
\phi & \downarrow & \phi \\
\{ \mathcal{M}_2, P_2^{-1}(0) \} & \xrightarrow{\mu_2} & \{ \mathbb{R}^n, 0 \}
\end{array}
\]

(Thus, \( \phi \) is a "collapsed" isomorphism.)

A succession of blowing-ups is of course a modification. The converse is almost true: Chow's lemma asserts that if \( \mu : \mathcal{M} \to X \) is a modification, then there exists a modification \( \mu' : \mathcal{M}' \to \mathcal{M} \) such that \( \mu \circ \mu' \) is equivalent to a succession of blowing-ups \([H]\).

We are now ready to generalize what we have proved for the Whitney family. Consider a parametrized family of functions

\[ F(x, t) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \]
where \( F \) is analytic in \((x, t), F(0, t) \equiv 0\). For fixed \( t \), write \( F_t(x) \equiv F(x, t): (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)\).

**Theorem.** Suppose for each \( t \), \( F_t \) admits \( 0 \in \mathbb{R}^n \) as an isolated singularity. Then there exists a finite filtration of the parameter space \( \mathbb{R}^k \)

\[ \mathbb{R}^k = P^{(0)} \supset P^{(1)} \supset \ldots \supset P^{(i)} \supset P^{(i+1)} = \emptyset \]

by subanalytic subsets \( P^{(i)} \) with the following properties:

(i) \( \dim P^{(i)} > \dim P^{(i+1)}, P^{(i)} - P^{(i+1)} \) are smooth;

(ii) for \( t, t' \) in a same connected component of \( P^{(i)} - P^{(i+1)}, F_t \) and \( F_{t'} \) are blow-analytically equivalent.

The proof is given in [K₄].

**Conjecture.** The hypothesis that \( 0 \) be an isolated singularity is superfluous.

3. **Algebraic geometry**

Consider a real variety \( V_f = f^{-1}(0) \) defined by an analytic function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \). Given a point \( a \in V_f \), let \( T_a(f) \) denote the Taylor expansion of \( f \) centered at \( a \). Thus \( T_a(f) \in \mathcal{A}_n \).

The blow-analytic equivalence relation in \( \mathcal{A}_n \) induces an equivalence relation \( \sim_f \) on \( V_f \) as follows. Define \( a \sim_f a' \) if and only if \( T_a(f) \) and \( T_{a'}(f) \) are blow-analytically equivalent.

**Weak Conjecture.** \( V_f \) admits a (locally finite) stratification, of which each stratum is subanalytic and is contained in a single equivalence class of \( \sim_f \).

This conjecture is closely related to the conjecture in Section 2.

**Strong Conjecture.** Each equivalence class of \( \sim_f \) is an analytic manifold; these manifolds form a stratification of \( V_f \) which satisfies the \((W)\)-regularity condition \([V]\).

More details can be found in [K₄].

In an attempt to prove the conjecture of the last section, we have come across a problem on desingularization of a holomorphic map, which is formulated as the following conjecture.

Let \( \sigma: \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) be a given proper surjective holomorphic map of complex manifolds.

**Conjecture.** There exists blowing-ups \( \beta_i: \tilde{\mathcal{M}}_i \rightarrow \mathcal{M}_i, i = 1, 2 \), with possibly singular centers, whose exceptional divisors are smooth and forming normal crossing families, and a holomorphic map \( \tilde{\sigma}: \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}_2 \) such that \( \beta_2 \circ \tilde{\sigma} = \sigma \circ \beta_1 \), and \( \tilde{\sigma} \) maps each canonical stratum of \( \tilde{\mathcal{M}}_1 \) submersively.
onto a canonical stratum of $\tilde{\mathcal{M}}_2$. (The canonical stratification of $\tilde{\mathcal{M}}_1$ is provided by the normal crossing family of exceptional divisors.)

Notice that when $\mathcal{M}_2 = C$, this reduces to Hironaka's desingularization theorem.

4. Complex singularities

We may call two complex germs $g_1, g_2 \in O_n$ blow-analytically equivalent if

(i) there is a local homeomorphism $\phi$ of $(\mathbb{C}^n, 0)$ such that $g_2 \circ \phi = g_1$;
(ii) there exist real modifications $\mu_1, \mu_2$ of $\mathbb{C}^n$ (as real spaces), and a real analytic isomorphism $\Phi$ such that $\phi \circ \mu_1 = \mu_2 \circ \Phi$.

Using this definition, the theorem of Section 2 remains true for complex singularities. The proof is the same.

However, if one requires $\Phi$ to be a biholomorphism, then the problem of moduli cannot be avoided, there would be no locally finite classification in $O_n$.

References


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