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INDECOMPOSABLE REPRESENTATIONS OF ORDERS

K. W. ROGGENKAMP

Mathematisches Institut B, Universität Stuttgart
Stuttgart, Federal Republic of Germany

This is a survey of recent developments concerning orders of finite lattice type and their Auslander–Reiten quivers. In Section 1, special types of orders over complete valuation rings are described: maximal orders, hereditary orders, subhereditary orders, Bäckström orders, generalized Bäckström orders, group rings, Gorenstein orders, Schurian orders and orders of global dimension 2. In Section 2 general properties of almost split sequences and Auslander–Reiten quivers of lattices over orders and of socle projective modules over artinian algebras are described. Section 3 deals with Auslander–Reiten quivers of special types of orders: Firstly, the connection between the Auslander–Reiten quiver of an order and the Auslander–Reiten quiver of a certain subcategory of the finitely generated modules over an artinian algebra is derived. This is of particular interest for subhereditary orders, and becomes even more transparent for generalized Bäckström orders. We then report on Gorenstein orders, simple curve singularities and on group rings. Finally, the Auslander–Reiten quivers of Schurian orders are discussed; these are intimately related with the Auslander–Reiten quivers of infinite partially ordered sets of finite width.

This is an extended version of a series of lectures I gave in the Banach Center in Warsaw in April 88. I would like to take this opportunity to thank the Banach Center and my colleagues there for their hospitality.

§ 1. Special types of orders

Let R be a complete principal ideal ring with maximal ideal $\pi \cdot R$ and residue field $\mathfrak{k}$. (Examples are $R = \hat{\mathbf{Z}}_p$, the p-adic integers, and $R = \mathfrak{k}[\![t]\!]$, the ring of formal power series over $\mathfrak{k}$.) We denote by K the field od fractions of R and let A be a separable finite-dimensional K-algebra.

This paper is in final form and no version of it will be submitted for publication elsewhere.

(1.1) DEFINITION. A unital subring Λ of A is called an *R-order* in A if

(i) Λ is a finitely generated R-module,
(ii) $K \cdot \Lambda = A$, i.e. Λ contains a K-basis of A.

The aim of the local representation theory of the R-order Λ is the study of *Λ-lattices*, i.e. left Λ-modules which are finitely generated over R, and which are R-torsion-free. We denote the category of left Λ-lattices by ${}_\Lambda\mathfrak{M}^0$. Then ${}_\Lambda\mathfrak{M}^0$ is an additive category which has kernels, but not cokernels. ${}_\Lambda\mathfrak{M}^0$ is a *Krull–Schmidt* category, i.e. every $M \in {}_\Lambda\mathfrak{M}^0$ has a unique – up to isomorphism – decomposition into indecomposable Λ-lattices. Hence

(1.2) ${}_\Lambda\mathfrak{M}^0$ is well understood once one understands the indecomposable Λ-lattices and the homomorphisms between them.

Before we come to this problem we shall list some *special types of orders*.

A) Maximal orders

Since A is a separable, it has the structure

$$A = \prod_{i=1}^{t} (D_i)_{n_i}, \tag{1.3}$$

where D_i is a finite-dimensional skew field over K with centre L_i, a separable extension of K. An R-order is called *maximal* provided it is not contained in a proper overorder. All maximal orders in A are isomorphic, i.e. they are all conjugate by elements in $A \backslash \{0\}$. A typical maximal order has the structure

$$\Gamma = \prod_{i=1}^{t} (\Omega_i)_{n_i}, \tag{1.4}$$

where Ω_i is the *unique* maximal R-order in D_i. Note that Γ is then Morita equivalent to the product of the Ω_i's. The indecomposable Γ-lattices are projective and are precisely the modules

$$Q_i = (\Omega_i)_{n_i \times 1}. \tag{1.5}$$

Since A is separable every R-order Λ is contained in at least one maximal order.

B) Hereditary orders

An R-order Γ is said to be *hereditary* provided that every left Γ-lattice is projective. The structure of hereditary orders can – up to isomorphism – be described as follows ([Ha], [Bru], [Ja5]):

$$\Gamma = \begin{bmatrix} (\Omega_1)_{n_1} & & & \Omega_1 \\ & (\Omega_1)_{n_2} & & \\ & & \ddots & \\ \Pi_1\Omega_1 & & & (\Omega_1)_{n_s} \end{bmatrix} \times \begin{bmatrix} (\Omega_2)_{m_1} & & & \Omega_2 \\ & (\Omega_2)_{m_2} & & \\ & & \ddots & \\ \Pi_2\Omega_2 & & & (\Omega_2)_{m_t} \end{bmatrix} \times \ldots, \tag{1.6}$$

where Ω_i is a maximal order in D_i with $\Pi_i\Omega_i = \operatorname{rad}\Omega_i$. Γ is Morita equivalent to the corresponding ring, where all $n_i, m_i, \dots$ are equal to 1.

The following property of hereditary orders is essential in the sequel ([AG], [Ja5]):

(1.7) *Let Γ be an R-order in A. Then the following are equivalent:*

(i) $\operatorname{rad}\Gamma$ *is projective.*
(ii) $\Lambda_l = \Lambda_l(\operatorname{rad}\Gamma) = \{x \in A\colon\ x \cdot \operatorname{rad}\Gamma \subset \operatorname{rad}\Gamma\} = \Gamma$.
(iii) $\Lambda_l \cap \Lambda_r = \Gamma$, *where Λ_r is defined similarly.*

The next result shows that hereditary orders are of more importance to arbitrary orders than are maximal ones:

(1.8) *Let Λ be an R-order in A. Then*

$$\Delta_1(\Lambda) = \Lambda_r(\operatorname{rad}\Lambda) \cap \Lambda_l(\operatorname{rad}\Lambda) = \{x \in A\colon\ x \cdot \operatorname{rad}\Lambda + \operatorname{rad}\Lambda \cdot x \subset \Lambda\}$$

is equal to Λ if and only if Λ is hereditary.

Defining inductively

$$\Delta_r(\Lambda) = \Delta_1(\Delta_{r-1}(\Lambda)),$$

there must exist an $n_0 = n_0(\Lambda)$ such that $\Delta_{n_0}(\Lambda)$ is hereditary, since ascending chains of orders in A must terminate. This uniquely determined order will be called the *hereditary order associated to Λ*. It will in general not be maximal. For the sake of simplicity we shall denote it by $\Gamma(\Lambda)$.

C) Construction of general orders

Given an R-order Λ in A, we pick a hereditary order Γ in A containing Λ. Since Λ is of finite index in Γ, there exists an $n \in \mathbf{N}$ such that

$$(\operatorname{rad}\Gamma)^n \subset \Lambda.$$

Let $\mathfrak{B} = \Gamma/(\operatorname{rad}\Gamma)^n$ — note that this artinian algebra is serial and hence is very well understood in view of (1.6) — and $\mathfrak{A} = \Lambda/(\operatorname{rad}\Gamma)^n$. Then $\mathfrak{A}$ is a unital subalgebra of $\mathfrak{B}$ and Λ is the pullback of

$$\begin{array}{ccc} \Lambda & \to & \Gamma \\ \downarrow & & \downarrow \\ \mathfrak{A} & \to & \mathfrak{B} \end{array} . \tag{1.9}$$

Conversely, given $\mathfrak{B} = \Gamma/(\operatorname{rad}\Gamma)^n$ and any unital subalgebra $\mathfrak{A}$ of $\mathfrak{B}$, the pullback Λ in (1.9) is an R-order in A. With the pair (Λ, Γ) we associate the artinian algebra

$$\mathfrak{C} = \begin{bmatrix} \mathfrak{B} & {}_{\mathfrak{B}}\mathfrak{B}_{\mathfrak{A}} \\ 0 & \mathfrak{A} \end{bmatrix} \tag{1.10}$$

where ${}_{\mathfrak{B}}\mathfrak{B}_{\mathfrak{A}}$ is viewed as a $(\mathfrak{B}, \mathfrak{A})$-bimodule.

D) Subhereditary orders

Let Λ be given, and assume that we can find a hereditary order Γ such that

$$\text{rad}\,\Gamma \subset \Lambda. \tag{1.11}$$

Then Λ is said to be *subhereditary*. In this case $\mathfrak{B}$ is semisimple, and as $\mathfrak{A}$ above every unital subalgebra can occur. Γ is then not necessarily unique, but we still call it the *associated hereditary order*. In this case the algebra

$$\mathfrak{C} = \begin{bmatrix} \mathfrak{B} & {}_{\mathfrak{B}}\mathfrak{B}_{\mathfrak{A}} \\ 0 & \mathfrak{A} \end{bmatrix}$$

has a projective socle and no simple ring direct factor. Such socle projective algebras were also studied by Simson [Si1–4] and Nishida [Ni1–3]. It can be shown that every artinian $\mathfrak{k}$-algebra which has a projective socle and no simple ring direct factor arises—up to Morita equivalence—in this way [R2, Lemma 2].

E) Bäckström orders

These are very special types of subhereditary orders, where instead of $\text{rad}\,\Gamma \subset \Lambda$—note that this is always equivalent to $\text{rad}\,\Gamma \subset \text{rad}\,\Lambda$—we require

$$\text{rad}\,\Gamma = \text{rad}\,\Lambda$$

for some hereditary overorder Γ of Λ. In this case $\mathfrak{C}$ (1.10) is hereditary with $\text{rad}^2\mathfrak{C} = 0$. These orders were first considered by Bäckström [Bä]. These Bäckström orders can also be described internally as follows:

(1.12) *An R-order Λ in A is a Bäckström order if and only if there exists a hereditary order $\Gamma \supset \Lambda$ such that for every indecomposable projective Λ-lattice P, $\text{rad}_\Lambda P$ is a Γ-module.*

(1.3) EXAMPLE.

$$\Lambda = \left\{ \begin{bmatrix} \alpha & R & R & R & R \\ \pi & \alpha' & R & R & R \\ \pi & \pi & \alpha'' & \pi & R \\ \pi & \pi & \pi & \beta & R \\ \pi & \pi & \pi & \pi & \beta' \end{bmatrix} : \alpha \equiv \alpha' \equiv \alpha'' \bmod \pi,\ \beta \equiv \beta' \bmod \pi \right\}$$

is a Bäckström order with hereditary order

$$\Gamma = \begin{bmatrix} R & R & R & R & R \\ \pi & R & R & R & R \\ \pi & \pi & R & R & R \\ \pi & \pi & R & R & R \\ \pi & \pi & \pi & \pi & R \end{bmatrix}.$$

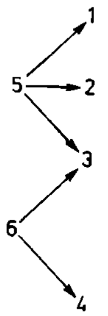

Fig. 1

In this example $\mathfrak{C}$ is the tensor algebra of the graph of Fig. 1.

F) Generalized Bäckström orders

A subhereditary order Λ is called a *generalized Bäckström order* provided the hereditary order Γ can be chosen such that $\mathfrak{C}$ in (1.10) is hereditary. The internal description of generalized Bäckström orders is as follows: There exists a hereditary overorder $\Gamma \supset \Lambda$ such that for every indecomposable Λ-lattice P,

$$\text{(1.14)} \qquad \operatorname{rad}_\Lambda P = X \oplus Q,$$

where X is a projective Λ-lattice and Q is a Γ-lattice.

(1.15) *The oriented quiver of* Λ – in case $R/\operatorname{rad} R$ is algebraically closed. Let $\mathfrak{P}_1$ be the set of isomorphism classes of indecomposable projective Λ-lattices P and $\mathfrak{P}_2$ the isomorphism classes of indecomposable Γ-lattices X. The *quiver of* Λ *with respect to* Γ – note that Γ need not be unique – is defined as follows: The *vertices* are in bijection with the elements in $\mathfrak{P} = \mathfrak{P}_1 \cup \mathfrak{P}_2$ – note that an indecomposable Γ-lattice X can very well be a projective Λ-lattice; in this case X occurs both in $\mathfrak{P}_1$ and in $\mathfrak{P}_2$, i.e. it is counted twice. To define the arrows, let P be an indecomposable projective Λ-lattice. We write

$$\text{(1.16)} \qquad \operatorname{rad}_\Lambda P \simeq \bigoplus_{i=1}^{n} P_i^{(n_i)} \oplus \bigoplus_{j=1}^{m} X_j^{(m_j)},$$

where P_i, $1 \leqslant i \leqslant n$, are indecomposable projective nonisomorphic Λ-lattices which are *not* Γ-lattices, X_j, $1 \leqslant j \leqslant m$, are indecomposable nonisomorphic Γ-lattices, and $X^{(s)}$ denotes the direct sum of s copies of X. We then draw n_i *arrows* from (P_i) to (P), where (M) denotes the isomorphism class of M, and m_j arrows from (X_j) to (P).

(1.17) Example. Let

$$\Lambda = \begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi & \pi & R & R \\ \pi & \pi & \pi & R \end{bmatrix},$$

where $R-R$ means that the corresponding elements are congruent modulo π. We put

$$\Gamma = \begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi & \pi & R & R \\ \pi & \pi & \pi & R \end{bmatrix}.$$

The projective indecomposable Λ-lattices are

$$P_1 = \begin{bmatrix} R \\ \pi \\ \pi \\ \pi \end{bmatrix}, \quad P_2 = \begin{bmatrix} R \\ R \\ \pi \\ \pi \end{bmatrix}, \quad P_3 = \begin{bmatrix} R & R \\ R & R \\ R & R \\ \pi & R \end{bmatrix};$$

the indecomposable Γ-lattices are

$$X_1 = \begin{bmatrix} R \\ \pi \\ \pi \\ \pi \end{bmatrix}, \quad X_2 = \begin{bmatrix} R \\ R \\ \pi \\ \pi \end{bmatrix}, \quad X_3 = \begin{bmatrix} R \\ R \\ R \\ \pi \end{bmatrix}, \quad X_4 = \begin{bmatrix} R \\ R \\ R \\ R \end{bmatrix}.$$

Note that $P_1 = X_1$ and $P_2 = X_2$. The set

$$\mathfrak{P} = \mathfrak{P}_1 \dot{\cup} \mathfrak{P}_2 = \{P_1, P_2, P_3\} \dot{\cup} \{X_1, X_2, X_3, X_4\}$$

has 7 points. Then $\operatorname{rad}_\Lambda P_1 = X_4$, $\operatorname{rad}_\Lambda P_2 = X_1$ (though $X_1 = P_1$), $\operatorname{rad}_\Lambda P_3 = X_2 \oplus X_3$ (though $X_2 = P_2$). Hence the quiver of Λ with respect to Γ is as shown in Fig. 2, where small Latin letters represent the isomorphism classes of

$$\begin{tikzcd} x_1 \arrow[r] & p_2 \\ x_2 \arrow[dr] & \\ & p_3 \\ x_3 \arrow[ur] & \\ x_4 \arrow[r] & p_1 \end{tikzcd}$$

Fig. 2

the capital Latin letters. Note that the quiver of Λ with respect to Γ is not connected though Λ is indecomposable as a ring.

G) Group rings of finite groups

Let G be a finite group. Then

$$RG = \{\sum_{g\in G} r_g g \colon r_g \in R\}$$

is the free R-module with basis $\{g\}_{g\in G}$ and multiplication induced from the multiplication in G. By Maschke's theorem KG is separable and thus RG is an R-order in KG, the group ring. RG has an additional property: it is a symmetric *Gorenstein order* (cf. [H]).

(1.18) We shall list some group ring explicitly [R8]: Let M_{11} be the Mathieu group and B the principal block at $p = 11$. Then B is equal to

$$(R)_{1\times 1} \; - \; \begin{bmatrix} (R)_{11\times 11} & (11R)_{11\times 10} \\ (R)_{10\times 11} & (R)_{10\times 10} \end{bmatrix} \; - \; \begin{bmatrix} (R)_{9\times 9} & (11R)_{9\times 10} & (11R)_{9\times 16} & (11R)_{9\times 10} \\ (R)_{10\times 9} & (R)_{10\times 10} & (11R)_{10\times 16} & (11R)_{10\times 10} \\ (R)_{16\times 9} & (R)_{16\times 10} & (R)_{16\times 16} & (11R)_{16\times 10} \\ (R)_{10\times 9} & (R)_{10\times 10} & (R)_{10\times 16} & (R)_{10\times 10} \end{bmatrix}$$

$$\begin{array}{ccc} (R)_{10\times 10} & (S)_{16\times 16} & (R)_{10\times 10} \end{array}$$

where $S = \mathrm{Fix}_{C_5}(R \sqrt[11]{1})$, C_5 is the cyclic group of order 5, and the congruences are given as pullbacks

$$\begin{array}{ccc} R - R & \to & R \\ \downarrow & & \downarrow \\ R & \to & \mathbf{F}_{11} \end{array} \quad \text{and} \quad \begin{array}{ccc} R - S & \to & S \\ \downarrow & & \downarrow \\ R & \to & \mathbf{F}_{11} \end{array}.$$

The congruences here are taken modulo 11. The Brauer tree is depicted in Fig. 3.

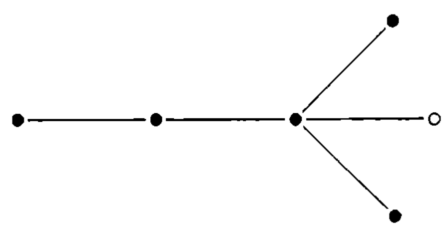

Fig. 3

H) Gorenstein orders

An R-order Λ is called a *Gorenstein order* provided

(1.19) $$\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$$

is a projective Λ-lattice, both on the left and on the right. In case $\Lambda \simeq \Lambda^*$ both as left and as right modules – note that this does not mean as bimodules – Λ is called a *symmetric Gorenstein order*. The importance of Gorenstein orders is best illustrated in terms of lattices. For any order Λ, ${}_\Lambda\mathfrak{M}^0$ has enough projective objects. These are precisely the projective left Λ-modules. The injective Λ-modules are never Λ-lattices. But still the category ${}_\Lambda\mathfrak{M}^0$ has enough injectives; in fact, let $\tilde{P}$ be a projective right Λ-lattice. Then $I = \mathrm{Hom}_R(\tilde{P}, R)$ is an injective object in ${}_\Lambda\mathfrak{M}^0$. Since an indecomposable projective Λ-lattice P has a unique maximal submodule in KP, it follows from the above construction that an indecomposable injective Λ-lattice I has a *unique minimal overmodule* $I^- \supset I$ in KI.

(1.20) *An order is Gorenstein provided every injective Λ-lattice is also projective and conversely.*

I) Schurian orders

Let $A = (K)_n$ be a simple split K-algebra. An R-order $\Lambda \subset \Gamma = (R)_n$ is said to be *Schurian* if Λ contains a complete set of orthogonal primitive idempotents $\{e_i\colon 1 \leqslant i \leqslant n\}$. For simplicity we shall assume that $e_i = e_{ii}$ are the usual matrix idempotents. W.l.o.g. we may assume

$$\Lambda = (\pi^{n_{ij}} \cdot R)_n, \qquad n_{ij} \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},$$

with $n_{ii} = 0$ and $n_{ik} + n_{kj} \geqslant n_{ij}$ for all $i, j, k = 1, \dots, n$.

The *partially ordered set* $P(\Lambda)$ of Λ is defined as follows [ZK]: $P(\Lambda)$ has as vertex set $\{1, \dots, n\} \times \mathbf{Z}$, with vertices (i, α), $1 \leqslant i \leqslant n$, $\alpha \in \mathbf{Z}$, and the partial order $\leqslant$ is generated by the relations

$$(i, \alpha) \leqslant (j, \beta) \qquad \text{(written also } (i, \alpha) \to (j, \beta))$$

provided either

(i) $i = j$ and $\alpha \leqslant \beta \leqslant \alpha + 1$, or
(ii) $\beta = \alpha + n_{ij}$.

An infinite partially ordered set P, occurring this way as $P(\Lambda)$ for a Schurian order Λ, will be called a *Schurian partially ordered set*.

For later application we list in Fig. 4 the *critical partially ordered sets* [Kl2].

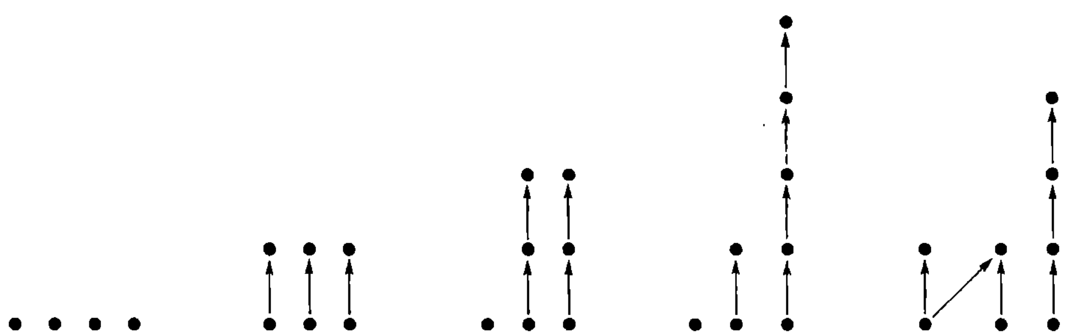

Fig. 4

J) Orders of global dimension 2

One should note that ${}_{\Lambda}\mathfrak{M}^0$ has homological dimension n if and only if Λ has global dimension $n+1$, and hence comparing the representation theory of artinian algebras with the study of lattices over orders, the orders of global dimension 1, i.e. the hereditary orders, play the same role as the semisimple algebras, and the orders of global dimension 2 should play a role similar to that of the hereditary algebras. But contrary to the artinian situation, the orders of global dimension 2 are not so easily described. Moreover, orders of global dimension 2 arise as Auslander orders [ARo].

One can, however, give a criterion for subhereditary orders to be of global dimension 2 [R14]:

(1.21) *Let Λ be an R-order in A such that there exists a maximal order Γ with* $\operatorname{rad}\Gamma \subset \Lambda \subset \Gamma$, *and let $G = \bigoplus_{i=1}^{t} G_i$ be a minimal left Γ-progenerator, decomposed into irreducible Γ-lattices. Then Λ has global dimension at most two if and only if*

(i) $\operatorname{hd}_{\Lambda}(G_i)+\operatorname{hd}_{\Lambda}(G_i^*) \leqslant 1,\ 1 \leqslant i \leqslant t$;

(ii) $\Lambda/\operatorname{rad}\Gamma$ *has global dimension at most two.*

Remark. This result is easily applied to construct explicit examples of orders of global dimension at most two: Let $\mathfrak{A}$ be a $\mathfrak{k}$-algebra of global dimension at most two, and assume that $\mathfrak{A}$ has a faithful representation

$$\tau_0\colon \mathfrak{A} \to \mathfrak{B} = \prod_{i=1}^{t} (\mathfrak{k}_i)_{n_i}$$

such that the simple $\mathfrak{B}$-modules S_i, $1 \leqslant i \leqslant t$, are as $\mathfrak{A}$-modules either projective or injective. Then the R-order Λ constructed in Section C will have global dimension at most two.

So the first step must be to find algebras of global dimension at most two. Natural candidates for $\mathfrak{A}$ will be hereditary $\mathfrak{k}$-algebras. To find other algebras of global dimension two one can make the following construction [R15]:

(1.22) *Let $\mathfrak{A}$ be a $\mathfrak{k}$-algebra with* $\operatorname{gl.dim}\mathfrak{A} \leqslant 2$, *$\mathfrak{B}$ a hereditary $\mathfrak{k}$-algebra and G_0 an $(\mathfrak{A}, \mathfrak{B})$-bimodule with* $\operatorname{hd}_{\mathfrak{A}}(G_0) \leqslant 1$. *If for each indecomposable projective left $\mathfrak{B}$-module P either $G_0 \otimes_{\mathfrak{B}} \operatorname{rad}_{\mathfrak{B}} P = 0$ or $G_0 \otimes_{\mathfrak{B}} \operatorname{rad}_{\mathfrak{B}} P$ is $\mathfrak{A}$-projective, then*

$$\mathfrak{C} = \begin{bmatrix} \mathfrak{A} & G_0 \\ 0 & \mathfrak{B} \end{bmatrix}$$

is a $\mathfrak{k}$-algebra with $\operatorname{gl.dim}\mathfrak{C} \leqslant 2$.

Another type of orders of global dimension 2 can be constructed as *path orders*: Let Λ be a Schurian order.

(1.23) *The single-valued quiver of a Schurian order.* (i) A *single-valued quiver* is a quiver $Q = (\Gamma_0, \Gamma_1)$ together with a map $v\colon \Gamma_1 \to \mathbf{N}_0$, where Γ_1 is the set of arrows in Q, and Γ_0 are the vertices. In addition we assume that each pair of vertices of Q is connected by some oriented path, but there should be no path of length one from $x \in \Gamma_0$ to x, i.e. there should be no loops in Γ.

(ii) The *path order* $\Lambda(Q)$ of Q is defined as follows: Let $\{1, \dots, s\} = \Gamma_0$. Then $\Lambda(Q)$ is an order in $(K)_s$ with diagonal entries R and the (i, j)-entry $\pi^{\alpha(i,j)} \cdot R$, where $\alpha(i, j)$ is the minimal integer m such that there exists an oriented path from i to j of valued path length m. Obviously $\Lambda(Q)$ is a Schurian order, it is two-sidedly indecomposable, but not necessarily basic.

(1.24) EXAMPLE. (ii) The order considered in $\Lambda = \begin{bmatrix} R & R \\ \pi^t & R \end{bmatrix}$ has quiver $1 \overset{t}{\underset{0}{\leftrightarrow}} 2$.

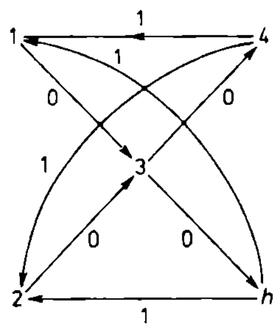

Fig. 5

(ii) Let Q be as in Fig. 5. Then

$$\Lambda(Q) = \begin{bmatrix} R & \pi & R & R & R \\ \pi & R & R & R & R \\ \pi & \pi & R & R & R \\ \pi & \pi & \pi & R & \pi \\ \pi & \pi & \pi & \pi & R \end{bmatrix}.$$

In [RW1–2] a necessary and sufficient condition is given for $\Lambda(Q)$ to have global dimension two, thus characterizing Schurian orders of global dimension two. Since these conditions are rather technical, we only give sufficient conditions on Q to give rise to an order $\Lambda(Q)$ of global dimension two.

Let $Q = (\Gamma_0, \Gamma_1, v)$ be a single-valued quiver, and assume that the subquiver $\bar{Q}$ consisting of the vertices Γ_0 and only those arrows which have valuation zero is a tree. In addition, assume

(a) $v(\Gamma_0) \subset \{0, 1\}$;
(b) Q has no zero cycles, i.e. oriented cycles of total valuation zero;
(c) through each point x there is a 1-cycle (i.e. of valued cycle length one);
(d) each arrow is part of a 1-cycle;
(e) there are no "superfluous" arrows (e.g. in Fig. 6, γ is superfluous);
(f) Q does not contain a full subquiver which can be "shrunk" to (1.24, ii).

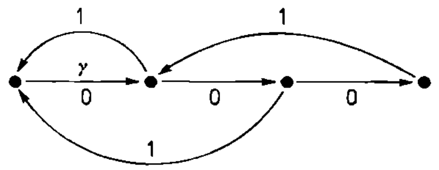

Fig. 6

(1.25) THEOREM [RW1–2]. *Let Q be a single-valued quiver satisfying the above conditions. Then $\Lambda(Q)$ has global dimension two, and is basic Schurian and two-sidedly indecomposable.*

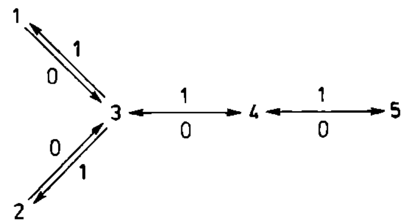

Fig. 7

(1.26) EXAMPLE. If Q is the quiver shown in Fig. 7, then

$$\Lambda(Q) = \begin{bmatrix} R & \pi & R & R & R \\ \pi & R & R & R & R \\ \pi & \pi & R & R & R \\ \pi^2 & \pi^2 & \pi & R & R \\ \pi^3 & \pi^3 & \pi^2 & \pi & R \end{bmatrix} \quad \text{has global dimension two.}$$

§ 2. Auslander–Reiten sequences and Auslander–Reiten quivers

Let Λ be an R-order in A (as in § 1), let $\mathfrak{A}$ be a finite-dimensional $\mathfrak{k}$-algebra, and denote by $\mathfrak{F}(\mathfrak{A})$ the torsion-free $\mathfrak{A}$-modules with respect to a hereditary torsion theory [St].

In the first part, let $\mathfrak{M}^0$ be either ${}_\Lambda\mathfrak{M}^0$ or $\mathfrak{F}(\mathfrak{A})$. By $\operatorname{ind}\mathfrak{M}^0$ we denote the indecomposable objects in $\mathfrak{M}^0$—note that the Krull–Schmidt Theorem holds.

(2.1) DEFINITION. (i) Let $X, Y \in \operatorname{ind}\mathfrak{M}^0$. Then a homomorphism $\psi: X \to Y$ is said to be an *irreducible map* if ψ is not an isomorphism and whenever there is a factorization

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\alpha"'] & & Y \\ & Z \arrow[ur, "\beta"'] & \end{tikzcd}$$

with $Z \in \mathfrak{M}^0$, then either α is a split monomorphism or β is a split epimorphism.

(ii) We say that an exact sequence in $\mathfrak{M}^0$

$$\mathfrak{E}: \quad 0 \to X' \xrightarrow{\psi} X \xrightarrow{\varphi} X'' \to 0$$

is an *Auslander–Reiten* (or *almost split*) *sequence* if

(a) $\mathfrak{E}$ is not split exact;

(b) $X', X'' \in \operatorname{ind} \mathfrak{M}^0$;

(c) whenever there is a homomorphism

$$\beta\colon Z \to X'' \qquad (\alpha\colon X' \to Z) \qquad \text{in } \mathfrak{M}^0$$

which is not a split epimorphism (split monomorphism), then there is a factorization

$$\begin{tikzcd} X \arrow[rr, "\varphi"] & & X'' \\ & Z \arrow[ul] \arrow[ur, "\beta"'] & \end{tikzcd} \qquad \left[\begin{tikzcd} X' \arrow[rr, "\psi"] \arrow[dr, "\alpha"'] & & X \arrow[dl] \\ & Z & \end{tikzcd}\right]$$

We say that $\mathfrak{M}^0$ *has Auslander–Reiten sequences* if, whenever X'' is not projective in $\mathfrak{M}^0$ (X' is not an injective object in $\mathfrak{M}^0$), then there exists an Auslander–Reiten sequence $\mathfrak{E}$. Since then $\mathfrak{E}$ is uniquely determined, we use the following notation (in the Auslander–Reiten sequence $\mathfrak{E}$):

$$X'' = \tau_{\mathfrak{M}^0}(X') = \tau(X'), \qquad X' = \tau_{\mathfrak{M}^0}^{-1}(X'') = \tau^{-1}(X'').$$

(2.2) THEOREM. $\mathfrak{M}^0$ *has Auslander–Reiten sequences.*

(2.3) *Remark.* If $\mathfrak{M}^0 = {}_\Lambda\mathfrak{M}^0$ is the category of Λ-lattices, the existence was established by Auslander–Reiten [Au1, ARe] and in [RS]. For $\mathfrak{M} = \mathfrak{F}(\mathfrak{A})$ in special cases the existence was shown by Bautista–Martínez [BM] and in [R6]. The general case was established by Auslander–Smalø [AS1].

There is a close connection between irreducible maps and Auslander–Reiten sequences: Let

$$0 \to N \xrightarrow{(\varphi_i)} \bigoplus_{i=1}^{t} E_i \xrightarrow{(\psi_i)} M \to 0, \qquad E_i \in \operatorname{ind} \mathfrak{M}^0,$$

be an Auslander–Reiten sequence; then ψ_i and φ_i are irreducible maps. Moreover, for every irreducible map $\sigma\colon X \to M$ ($\tau\colon N \to Y$), $X \simeq E_i$ ($Y \simeq E_j$) for some i (j) and σ (τ) is "essentially" ψ_i (φ_j).

(2.4) *Construction of Auslander–Reiten sequences in* ${}_\Lambda\mathfrak{M}^0$ [R3]. Let M be an indecomposable nonprojective Λ-lattice and let $P \xrightarrow{\psi} M \to 0$ be its projective cover sequence. Then we obtain the exact sequence of right Λ-lattices

$$0 \to \operatorname{Hom}_\Lambda(M, \Lambda) \to \operatorname{Hom}_\Lambda(P, \Lambda) \to \operatorname{tr}(M) \to 0,$$

where $\operatorname{tr}(M) \subset \operatorname{Hom}_\Lambda(\operatorname{Ker}_\psi, \Lambda)$ is the *transpose* of M. We denote the functor $(-)^* = \operatorname{Hom}_R(-, R)$ as *dual*, $D(-)$. Then we obtain the exact sequence

$$0 \to \operatorname{tr}(M)^* \to \operatorname{Hom}_\Lambda(P, \Lambda)^* \to \operatorname{Hom}_\Lambda(M, \Lambda)^* \to 0,$$

and

$$D\operatorname{tr}(M) = \operatorname{tr}(M)^* = \tau^{-1}_{{}_\Lambda\mathfrak{M}^0}(M) =: \tau_\Lambda^{-1}(M)$$

is the *kernel of the Auslander–Reiten sequence.*

In order to compute the Auslander–Reiten sequence itself, we apply $\operatorname{Hom}_\Lambda(M, -)$, and get the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(M, \tau_\Lambda^{-1}(M)) \to \operatorname{Hom}_\Lambda(M, \operatorname{Hom}_\Lambda(P, \Lambda)^*) \xrightarrow{\varrho}$$
$$\operatorname{Hom}_\Lambda(M, \operatorname{Hom}_\Lambda(M, \Lambda)^*) \xrightarrow{\sigma} \operatorname{Ext}^1_\Lambda(M, \tau_\Lambda^{-1}(M)) \to 0.$$

The image of ϱ is—up to natural isomorphism—given by $\operatorname{Hom}_\Lambda(M, M)^*$, which turns out to be an injective lattice over $\operatorname{End}_\Lambda(M)$, and as such has a unique minimal overmodule X. Then $\sigma(X)$ is the simple socle of $\operatorname{Ext}^1_\Lambda(M, \tau_\Lambda^{-1}(M))$ as an $\operatorname{End}_\Lambda(M)$-module. The Auslander–Reiten sequence of M is then represented by an extension coming from this socle.

(2.5) *Remarks.* (i) One should note the difference of construction for lattices over Λ and for modules over an artinian $\mathfrak{k}$-algebra $\mathfrak{A}$: Let X be an indecomposable nonprojective $\mathfrak{A}$-module. Then we take the projective cover sequence of X:

$$P_2 \to P_1 \to X \to 0,$$

and construct the exact sequence

$$0 \to \operatorname{Hom}_\mathfrak{A}(X, \mathfrak{A}) \to \operatorname{Hom}_\mathfrak{A}(P_1, \mathfrak{A}) \to \operatorname{Hom}_\mathfrak{A}(P_2, \mathfrak{A}) \to \operatorname{tr}_\mathfrak{A}(X) \to 0.$$

Then

$$\tau_\mathfrak{A}^{-1}(X) = \operatorname{Hom}_\mathfrak{k}(\operatorname{tr}_\mathfrak{A}(X), \mathfrak{k}).$$

(ii) By studying Auslander–Reiten sequences one obtains not only information on the indecomposable Λ-lattices, but also on the interrelation between them via irreducible maps. (Note that one of the main features of Auslander–Reiten sequences is that if X is not projective (not injective) in $\operatorname{ind}{}_\Lambda\mathfrak{M}^0$, then $\tau_\Lambda(X)$ (resp. $\tau_\Lambda^{-1}(X)$) is indecomposable.)

The study of Auslander–Reiten sequences has led to important results (just to name some):

(2.6) Theorem. (i) [S]. *Λ is of finite lattice type if and only if for any finitely presented functor*

$$F\colon {}_\Lambda\mathfrak{M}^0 \to {}_R\mathfrak{M}^0$$

the Jordan–Zassenhaus theorem holds.

(ii) [Au1], [S]. *Λ is of finite lattice type if and only if for every $M \in {}_{\Lambda}\mathfrak{M}^0$, the functor $\underline{\mathrm{Hom}}_{\Lambda}(-, M)$ has a finite length, where $\underline{\mathrm{Hom}}_{\Lambda}(-, M)$ is the quotient of* $\mathrm{Hom}_{\Lambda}(-, M)$ *modulo homomorphisms which factor via projective Λ-lattices.*

(2.7) THEOREM [BB1]. *Let Λ be an order of finite lattice type and let $M \in \mathrm{ind}\, {}_{\Lambda}\mathfrak{M}^0$ be nonprojective. If*

$$0 \to \tau^{-1}(M) \to \bigoplus_{i=1}^{t} E_i \to M \to 0$$

is its Auslander–Reiten sequence, with $E_i \in \mathrm{ind}\, {}_{\Lambda}\mathfrak{M}^0$, then $t \leqslant 4$, and if $t = 4$, then at least one of the E_i must be a projective and injective Λ-lattice.

Proof. The proof of the above result by Brenner in [BB1] is formulated for artinian algebras only; however, if one replaces the length function in the artinian case by the rank over R in the case of lattices, it carries over to orders.

The Auslander–Reiten quiver of Λ is formed by "glueing together" various Auslander–Reiten sequences. More precisely:

(2.8) DEFINITION. (i) Let $M, N = \mathrm{ind}\, {}_{\Lambda}\mathfrak{M}^0$. Then

$$\mathrm{Irr}(M, N) = \mathrm{rad}\,\mathrm{Hom}_{\Lambda}(M, N)/\mathrm{rad}^2\,\mathrm{Hom}_{\Lambda}(M, N)$$

is called the *bimodule of irreducible maps*; it is an $\mathrm{End}_{\Lambda}(M)$-$\mathrm{End}_{\Lambda}(N)$-bimodule. Since $\mathrm{Irr}(M, N)$ is an artinian module, we have the natural numbers

$$a_{M,N} = \text{length of } \mathrm{Irr}(M, N) \text{ as an } \mathrm{End}_{\Lambda}(M)\text{-module},$$

$$a'_{M,N} = \text{length of } \mathrm{Irr}(M, N) \text{ as an } \mathrm{End}_{\Lambda}(N)\text{-module}.$$

The *Auslander–Reiten quiver* $\mathfrak{A}(\Lambda)$ of Λ has as vertices the isomorphism classes in $\mathrm{ind}\, {}_{\Lambda}\mathfrak{M}^0$, and there exists a bi-valued arrow

$$[M] \xrightarrow{(a_{M,N}, a'_{M,N})} [N]$$

if $\mathrm{Irr}(M, N) \neq 0$ and $a_{M,N}$ and $a'_{M,N}$ are defined as above. (Here $\mathrm{rad}\,\mathrm{Hom}_{\Lambda}(M, N)$ is to be understood as follows: We have the functors $\mathrm{Hom}_{\Lambda}(-, N)$ and $\mathrm{Hom}_{\Lambda}(-, M)$ with radicals (the *radical* of a functor is the intersection of its maximal subfunctors) $\mathrm{rad}\,\mathrm{Hom}_{\Lambda}(-, N)$ and $\mathrm{rad}\,\mathrm{Hom}_{\Lambda}(M, -)$, and the functorial description of Auslander–Reiten sequences shows that

$$[\mathrm{rad}\,\mathrm{Hom}_{\Lambda}(-, N)/\mathrm{rad}^2\,\mathrm{Hom}_{\Lambda}(-, N)](M) \simeq$$

$$[\mathrm{rad}\,\mathrm{Hom}_{\Lambda}(M, -)/\mathrm{rad}^2\,\mathrm{Hom}_{\Lambda}(M, -)](N).)$$

For brevity we write

$$\mathrm{rad}(M, N)/\mathrm{rad}^2_{\bullet}(M, N)$$

for the above isomorphic modules [Au2].

(ii) $M \in \mathrm{ind}\, {}_{\Lambda}\mathfrak{M}^0$. is said to be *stable* if for every $n \in \mathbb{N}$

$$\tau_\Lambda^n(M) = \tau_\Lambda(\tau_\Lambda^{n-1}(M)) \quad \text{and} \quad \tau_\Lambda^{-n}(M) = \tau_\Lambda^{-1}(\tau_\Lambda^{-(n-1)}(M))$$

are defined, i.e. there are no projective or injective objects in the τ-orbit of M. $\mathfrak{A}_s(\Lambda)$ is the full subquiver of $\mathfrak{A}(\Lambda)$ whose vertices are the isomorphism classes of stable modules; $\mathfrak{A}_s(\Lambda)$ is called the *stable Auslander–Reiten quiver.*

(iii) $M \in \operatorname{ind}_\Lambda \mathfrak{M}^0$ is said to be *periodic* if it is not an injective Λ-lattice and $\tau^n(M) \simeq M$ for some $n \in \mathbf{N}$.

It follows directly from the functorial description of Auslander–Reiten sequences that $\operatorname{Irr}(M, N)$ are indeed the irreducible maps; in particular, $a_{M,N}$ is—provided N is not projective—the number of times M occurs in the Auslander–Reiten sequence of N, and $a'_{M,N}$ is—provided M is not an injective Λ-lattice—the number of times N occurs in the Auslander–Reiten sequence of $\tau(M)$. If one has an Auslander–Reiten sequence

$$0 \to N \xrightarrow{\varphi_{ij}} \bigoplus E_i^{(n_i)} \xrightarrow{\psi_{ij}} M \to 0$$

then the $\{\psi_{ij}\}_{1 \leqslant j \leqslant n(i)}$ constitute a basis for $\operatorname{Irr}(N, E_i)$ as an $\operatorname{End}_\Lambda(M)$-module and dually.

We point out one feature which distinguishes the Auslander–Reiten quivers of orders of finite type from those of infinite type: In the finite representation type case, any homomorphism between indecomposable objects is a sum of compositions of irreducible maps.

(2.9) If one applies this to ψ_π: $M \to M$, multiplication by π, then there is a path of irreducible maps from M to M; hence the Auslander–Reiten quiver of an order is never flat. For irreducible maps between lattices we have the following restriction:

(2.10) LEMMA. *Let $M, N \in \operatorname{ind}_\Lambda \mathfrak{M}^0$ and ψ: $M \to N$ an irreducible map. Then either*

(i) *ψ is surjective, or*

(ii) *ψ is injective and* Coker *ψ is torsion-free, i.e. $M \simeq KM\psi \cap N$, or*

(iii) *ψ is a monomorphism onto a maximal submodule.*

This follows immediately from the definition of irreducible maps and holds equally well for $\mathfrak{F}(\mathfrak{A})$, the torsion-free $\mathfrak{A}$-modules.

In the artinian case, since an irreducible map is either a strict monomorphism or a strict epimorphism, there cannot be an irreducible map ψ: $X \to X$. Because of the possibility (iii) this cannot be ruled out in the case of lattices—it occurs for maximal orders. For the Auslander–Reiten quiver this means that there is a *loop* of length one. Since many of the arguments in the artinian case depend strongly on the nonexistence of such loops, it is necessary to find out when orders do have loops in their Auslander–Reiten quiver. This was a question posed to me by M. Auslander around 1976. The answer was given by A. Wiedemann in his 1980 thesis.

(2.11) THEOREM [Wi 4]. *Let Λ be a two-sidedly indecomposable R-order. Then the following statements are equivalent:*

(i) *There is $M \in \operatorname{ind}_{\Lambda}\mathfrak{M}^0$ and an irreducible map $\psi: M \to M$.*

(ii) *The Auslander–Reiten quiver of Λ has the following form:*

$$\circlearrowleft\underset{1}{\bullet} \rightleftarrows \underset{2}{\bullet} \rightleftarrows \underset{3}{\bullet} \cdots \underset{t-1}{\bullet} \rightleftarrows \underset{t}{\bullet}$$

If $R/\pi \cdot R$ is a finite field, then

(iii) ***Λ is Morita equivalent to a Bass order Γ of the following form** [R1, IX, 6.17]: **Γ is a Bass order in the separable skew field D with maximal order Ω and $\Omega/\mathrm{rad}_{\Gamma}\Omega \simeq \Gamma/\mathrm{rad}_{\Gamma}\Gamma[\varepsilon]$, $\varepsilon^2 = 0$, or Γ is maximal.***

The proof is based on the lemma of Harada–Sai and uses

(2.12) THEOREM [Wi4]. *Assume Λ is two-sidedly indecomposable. Let Δ be a connected component of $\mathfrak{A}(\Lambda)$ and assume that the R-ranks of the Λ-lattices in Δ are bounded. Then Λ is of finite lattice type and $\Delta = \mathfrak{A}(\Lambda)$.*

This is known as the *first Brauer–Thrall conjecture*.

(2.13) *Remarks*. (i) An important *application* of (2.12) is the following: Assume one has a two-sidedly indecomposable order Λ and by some technique one has constructed a finite number of indecomposable Λ-lattices which constitute a connected component of the Auslander–Reiten quiver. Then (2.12) implies that one has found all indecomposable Λ-lattices.

(ii) The *second Brauer–Thrall conjecture* asserts that, provided $R/\pi \cdot R$ is infinite and Λ is of infinite type, there are infinitely many integers n_1 such that for each n_1 there are infinitely many indecomposable Λ-lattices of rank n_1. This was verified in [RR1] provided the second Brauer–Thrall conjecture holds in the artinian category $\mathfrak{C}$ (§ 3.1)—which was proved in [NR].

(iii) It is surprising to me that loops in the Auslander–Reiten quiver can only occur in the case of finite lattice type, and moreover, that one can give an explicit description of the orders with loops in their Auslander–Reiten quiver.

(iv) The result of (2.11) allows us to use the results of Riedtmann [Ri1–2] in the version of Happel–Preiser–Ringel to get the structure of connected components of the stable Auslander–Reiten quiver containing a periodic vertex.

In order to do so we have to introduce a considerable amount of notation:

(2.14) DEFINITIONS [HPR]. (i) A *quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ consists of the set of *vertices* Γ_0 and the set of *arrows* Γ_1. We shall always assume that Γ does not have loops or double arrows. If $x \in \Gamma_0$ is a vertex, then we denote by x^+ the set of starting points of arrows with endpoint x, and similarly for x^-. If for all x, the sets x^+ and x^- are finite, the quiver Γ is said to be *locally finite*.

(ii) A *Riedtmann quiver* $\Delta = (\Gamma_0, \Gamma_1, \tau)$ is a quiver (Γ_0, Γ_1) together with an injective function

$$\tau\colon \Gamma_0' \to \Gamma_0,$$

defined on a subset Γ_0' of Γ_0, satisfying $(\tau x)^- = x^+$ [BG].

So given an arrow $\alpha\colon x \to y$ there exists a unique arrow $y \to \tau x$; this arrow will be denoted by $\sigma\alpha$. A Riedtmann quiver Δ is called *stable* provided τ is defined on all of Γ_0 and is also surjective. Any Riedtmann quiver has a *unique maximal stable subquiver*. A vertex $x \in \Gamma_0$ will be called *periodic* if $\tau^n(x) = x$ for some $n \in \mathbf{N}$.

(iii) For a quiver (Γ_0, Γ_1), a function $a\colon \Gamma_1 \to \mathbf{N}\times\mathbf{N}$ will be called a *valuation* and $\Gamma = (\Gamma_0, \Gamma_1, a)$ a *valued quiver*. The image of $\alpha\colon x \to y$ will be denoted by (a_α, a_α') or (a_{xy}, a_{xy}'). A *valued Riedtmann quiver* is a Riedtmann quiver $(\Gamma_0, \Gamma_1, \tau)$ together with a valuation a for (Γ_0, Γ_1) such that $a_{\varrho\alpha} = a_\alpha'$, $a_{\sigma\alpha}' = a_\alpha$ for all $\alpha\colon y \to x$ with $x \in \Gamma_0'$. This valued Riedtmann quiver will be denoted by $\Delta = (\Gamma_0, \Gamma_1, \tau, a)$.

If now $\Gamma = (\Gamma_0, \Gamma_1, a)$ is a valued tree, then on $\mathbf{Z}\Gamma = \mathbf{Z}\times\Gamma$ a valuation is defined by

$$a_{(n,\alpha)} = a_\alpha = a'_{\sigma(n,\alpha)}, \qquad a'_{(n,\alpha)} = a'_\alpha = a_{\sigma(n,\alpha)}.$$

This way $\mathbf{Z}\Gamma$ becomes a valued Riedtmann quiver $\mathbf{Z}\Gamma = \mathbf{Z}(\Gamma_0, \Gamma_1, a)$.

(2.15) THEOREM ([Ri1–2], [HPR]). *Given any stable valued Riedtmann quiver Δ with a periodic vertex, there is a valued oriented tree Γ and a group G of automorphisms of $\mathbf{Z}\Gamma$ such that Δ is isomorphic to $\mathbf{Z}\Gamma/G$. Moreover, Γ is uniquely determined by Δ. It is called the* tree class *of Δ.*

Note that this result depends heavily on the fact that Δ has no loops.

(2.16) DEFINITION. Let Δ be a stable valued Riedtmann quiver, $\Delta = (\Gamma_0, \Gamma_1, a, \tau)$. A *subadditive function* on Δ is a function $f\colon \Gamma_0 \to \mathbf{N}$ satisfying

$$f(x)+f(\tau x) \geqslant \sum_{y\in x^+} f(y)a_{xy}'.$$

f is said to be *additive* provided we always have equality. We say that f is *periodic with respect to* τ if for every x there exists $n \in \mathbf{N}$ such that $f(x) = f(\tau^n x)$.

The following theorem was proved by Happel–Preiser–Ringel for connected stable Riedtmann quivers having a periodic vertex. Webb [We1–2] has noted—for his applications to group rings—that the result also holds for periodic functions.

(2.17) THEOREM ([HPR], [We1–2]). *Let Δ be a stable valued Riedtmann quiver which is locally finite and connected. Assume there is a subadditive function f on Δ which is periodic with respect to τ.*

(i) *The tree class of Δ is either a Dynkin diagram, or a Euclidean diagram, or one of A_∞, A_∞^∞, B_∞, C_∞, D_∞* (cf. the list in Fig. 8).

(ii) *If f is not additive, then the tree class of Δ is a Dynkin diagram or A_∞.*

(iii) *If f is unbounded, then the tree class of Δ is A_∞.*

(2.18) The above-mentioned diagrams are illustrated in Fig. 8.

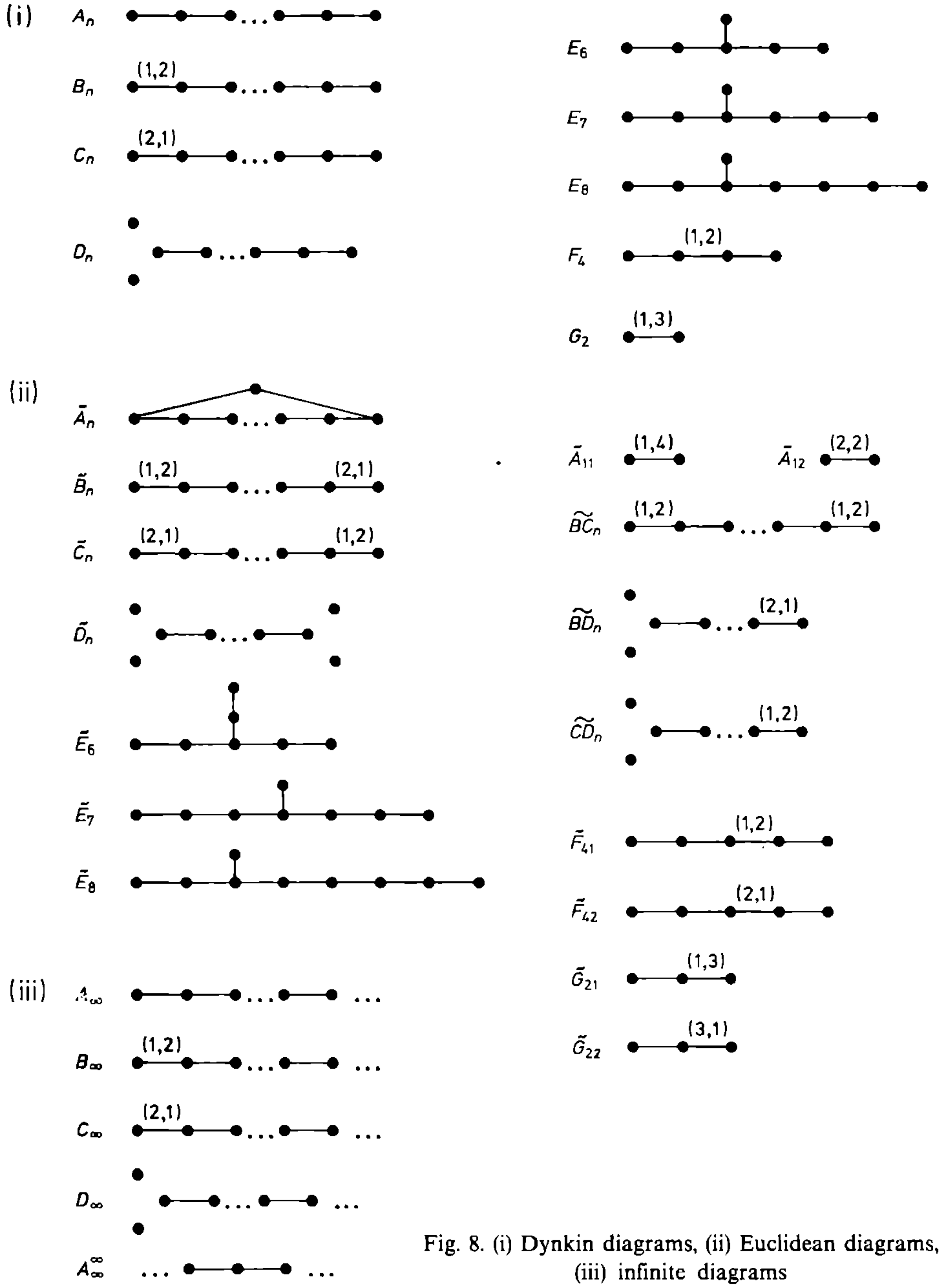

Fig. 8. (i) Dynkin diagrams, (ii) Euclidean diagrams, (iii) infinite diagrams

Using this and (2.11) one obtains

(2.19) THEOREM ([HPR], [Ri1–2], [Wi4]). *Let Λ be an R-order which is two-sidedly indecomposable, and Δ a connected component of the stable Auslander–Reiten quiver $\mathfrak{A}_s(\Lambda)$, containing a periodic vertex.*

(i) *If Δ is finite, then either $\Delta = L_n = \circlearrowleft\bullet \leftrightarrows \bullet \leftrightarrows \bullet \ldots \bullet \leftrightarrows \bullet$ (n vertices), or $\Delta = \mathbf{Z}\Gamma/G$ and Γ – i.e. the tree type of Δ – is a Dynkin diagram.*

(ii) *If Δ is infinite, then $\Gamma = A_\infty$.*

Moreover, all Dynkin diagrams and L_n do occur.

At the end of this section we turn to *preprojective partitions*, a concept which is motivated by the study of artinian algebras. Let $S = \{\Gamma, f_i, {}_iX_j\}$ be a $\mathfrak{k}$-species [DlRi] and let $\mathfrak{A}$ be the tensor algebra of S over $\mathfrak{k}$. Then there is a natural notion of preprojective and preinjective $\mathfrak{A}$-modules: the $\mathfrak{A}$-modules lying in the same connected component – in this case all projectives lie in one connected component – as the projective (injective) $\mathfrak{A}$-modules. The simple-minded generalization to arbitrary $\mathfrak{k}$-algebras does not make sense, since in general projective $\mathfrak{A}$-modules (injective $\mathfrak{A}$-modules) lie in different components. The natural generalization from the tensor algebras of species to arbitrary $\mathfrak{k}$-algebras was given by Auslander–Smalø [AS2] and their results have a natural generalization to orders. (The idea of all this goes back to the proof of the first Brauer–Thrall conjecture for artinian algebras [NR].)

So we start again with definitions:

(2.20) DEFINITIONS. (i) Let $\mathfrak{D}$ be a subcategory of ${}_\Lambda\mathfrak{M}^0$; a *cover* $\mathfrak{C}$ of $\mathfrak{D}$ is a subcategory $\mathfrak{C} \subset \operatorname{ind} {}_\Lambda\mathfrak{M}^0$ such that for every $D \in \mathfrak{D}$ there exists a surjective map

$$\bigoplus_{i=1}^{t} C_i^{(\alpha_i)} \to D, \qquad C_i \in \mathfrak{C}.$$

A *minimal cover* is a cover in which no lattice is superfluous. (If such a minimal cover exists, it is unique.)

(ii) A *preprojective partition* of ${}_\Lambda\mathfrak{M}^0$ is a sequence of subcategories $\mathbf{P}_0$, $\mathbf{P}_1, \ldots, \mathbf{P}_i$, $0 \leqslant i \leqslant \infty$, such that

(α) $\operatorname{ind}_\Lambda\mathfrak{M}^0 = \bigcup_{i \geqslant 0} \mathbf{P}_i$.

(β) $\mathbf{P}_i$ is a *finite* minimal cover for $\bigcup_{j \geqslant i} \mathbf{P}_j$.

(iii) Lattices in $\operatorname{add}(\bigcup_{i<\infty} \mathbf{P}_i)$ are called *preprojective*, where $\operatorname{add}(-)$ denotes the additive category generated by $(-)$ [AS3–4].

The dual notions are those of a *preinjective partition* and of *preinjective lattices.*

The crucial point in this is the word "finite" in (β).

(2.21) THEOREM [AS2]. *${}_\Lambda\mathfrak{M}^0$ has a preprojective partition. And it also has a preinjective partition, which is dual to the preprojective partition.*

The importance of these preprojective (preinjective) partitions is shown by the next result.

(2.22) THEOREM [AS2–3]. (i) $\mathbf{P}_i = \emptyset$ *for some* $i < \infty$ *if and only if* Λ *is of finite lattice type.*

(ii) $\mathbf{P}_\infty = \emptyset$ *if and only if* Λ *is of finite lattice type.*

(iii) *If* $\mathbf{P}_\infty \neq \emptyset$, *then it does not have a finite minimal cover.*

§ 3. Finite type orders and their Auslander–Reiten quivers. Auslander–Reiten quivers of orders as Auslander–Reiten quivers of certain artinian module categories

(The classical situation)

In this section let Λ be an order with associated hereditary order Γ (cf. § 1C). As in (1.10) we put $\mathfrak{D} = \begin{bmatrix} \mathfrak{B} & {}_{\mathfrak{B}}\mathfrak{B}_{\mathfrak{A}} \\ 0 & \mathfrak{A} \end{bmatrix}$, where $\mathfrak{A} = \Lambda/(\operatorname{rad}\Gamma)^n$ and $\mathfrak{B} = \Gamma/(\operatorname{rad}\Gamma)^n$ and $(\operatorname{rad}\Gamma)^n \subset \Lambda$. The $\mathfrak{D}$-modules will be represented as triples $\begin{bmatrix} U \\ V, & \varphi \end{bmatrix}$, where $U \in \operatorname{mod}\mathfrak{A}$, $V \in \operatorname{mod}\mathfrak{B}$ and φ: $\mathfrak{B}\otimes_{\mathfrak{A}} V \to U$ is $\mathfrak{B}$-linear.

(3.1) DEFINITION. The full subcategory $\mathfrak{C}(\mathfrak{D})$ of $\operatorname{mod}\mathfrak{D}$ is defined as follows: The objects are those $\mathfrak{D}$-modules $\begin{bmatrix} U \\ V, & \varphi \end{bmatrix}$ where U is $\mathfrak{B}$-projective, φ is surjective and $\varphi|_{\mathfrak{A}\otimes_{\mathfrak{A}} V}$ is injective. (If it is clear from the context, we shall omit φ.)

We shall denote the *nonisomorphic* Γ*-lattices* by Q_i, $1 \leqslant i \leqslant s$. The *permutation of* Γ is defined as follows: Since Γ is hereditary, Q_i is an injective Γ-lattice and hence has a unique minimal overmodule Q_i^- which is also Γ-projective, and hence $Q_i^- \simeq Q_{\sigma(i)}$. Then σ is a permutation on $\{1, \dots, s\}$.

The next result, which was independently obtained by E. Green and I. Reiner [GR] and [RR1], gives the connection between ${}_\Lambda\mathfrak{M}^0$ and $\operatorname{mod}\mathfrak{D}$.

(3.2) THEOREM. *Let* (Λ, Γ) *and* $\mathfrak{D}$ *be as above. Then the functor* $\mathfrak{F}$: ${}_\Lambda\mathfrak{M}^0 \to \mathfrak{C}(\mathfrak{D})$, *induced by*

$$M \to \begin{bmatrix} \Gamma\cdot M/I\cdot M \\ M/I\cdot M \end{bmatrix}, \quad \textit{where } I = (\operatorname{rad}\Gamma)^n,$$

is a representation equivalence.

(3.3) THEOREM [R19]. $\mathfrak{C}(\mathfrak{D})$ *has enough* $\operatorname{Ext}_{\mathfrak{C}(\mathfrak{D})}$*-projective and* $\operatorname{Ext}_{\mathfrak{C}(\mathfrak{D})}$*-injective objects.*

(i) *The indecomposable* $\operatorname{Ext}_{\mathfrak{C}(\mathfrak{D})}$*-projective objects are:*

(a) $\mathfrak{F}(P)$ *for* P *an indecomposable projective* Λ*-lattice,*

(b) $\mathfrak{F}(\tau(Q_i))$, $1 \leqslant i \leqslant s$, *where*

$$0 \to Q_i \to E \to \tau(Q_i) \to 0$$

is an almost split sequence in ${}_\Lambda\mathfrak{M}^0$, *provided* Q_i *is not an injective* Λ*-lattice.*

(ii) *The indecomposable* $\mathrm{Ext}_{\mathfrak{C}(\mathfrak{D})}$*-injective objects are*:

(a) $\mathfrak{F}(J)$ *for* J *an indecomposable injective* Λ*-lattice,*

(b) $\mathfrak{F}(Q_i)$, $1 \leqslant i \leqslant s$.

(3.4) THEOREM [R19]. $\mathfrak{C}(\mathfrak{D})$ *has almost split sequences, and the Auslander–Reiten quivers of* ${}_\Lambda\mathfrak{M}^0$ *and of* $\mathfrak{C}(\mathfrak{D})$ *are the same up to the Auslander–Reiten translation: The Auslander–Reiten quiver* $\mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$ *is obtained from that of* ${}_\Lambda\mathfrak{M}^0$ *by omitting the translation between* Q_i *and* $\tau(Q_i)$, $1 \leqslant i \leqslant s$.

(3.5) *Remark.* The crucial connection between $\mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$ and $\mathfrak{A}({}_\Lambda\mathfrak{M}^0)$ is given through the following

LEMMATA. 1. *Let* $M, N \in \operatorname{ind}\Lambda$ *and let* $\varphi\colon M \to N$ *be an irreducible map. If* $\mathfrak{F}(\varphi) = 0$, *then* M *is a* Γ*-lattice; moreover,* M *is a direct summand of both* $\operatorname{rad}_\Lambda N$ *and* $I \cdot N$.

2. *Let* $0 \to M' \to M \xrightarrow{\beta} M'' \to 0$ *be an almost split sequence in* ${}_\Lambda\mathfrak{M}^0$. *Then* $\mathfrak{F}(\beta)$ *is right almost split, i.e.* $\mathfrak{F}(\beta)$ *is surjective but not split, and every nonsplit map* $\gamma\colon X \to \mathfrak{F}(M'')$ *factors via* $\mathfrak{F}(\beta)$.

3. *Let* $\mathfrak{E}\colon 0 \to M' \to M \to M'' \to 0$ *be an almost split sequence in* ${}_\Lambda\mathfrak{M}^0$. *If* M' *is not a* Γ*-lattice, then* $\mathfrak{F}(\mathfrak{E})$ *is an almost split sequence in* $\mathfrak{C}(\mathfrak{D})$.

This shows the close relation between $\mathfrak{A}({}_\Lambda\mathfrak{M}^0)$ and $\mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$. I would like to point out that S. Smalø has shown abstractly—without using orders—that $\mathfrak{C}(\mathfrak{D})$ has almost split sequences [Sm].

Auslander–Reiten quivers of subhereditary orders (classical and nonclassical)

We shall state the results in this section not only for classical orders [RR1–2], but in the following more general situation [R2]:

Let R be a commutative noetherian complete local domain with field of quotients K and let A be a finite-dimensional—not necessarily semisimple—K-algebra. An *R-order* Λ *in* A is a subring of A containing the same identity as A such that

(i) Λ is finitely generated as an R-module,

(ii) $K \cdot \Lambda = A$, i.e. Λ contains a K-basis for A.

Given two orders $\Lambda \subset \Gamma$ we shall consider the full subcategory ${}_\Lambda\mathfrak{M}^0(\Gamma)$ of the category of left Λ-modules with

$$\operatorname{ob}({}_\Lambda\mathfrak{M}^0(\Gamma)) = \{X = R\text{-torsion-free finitely generated left } \Lambda\text{-module with } \Gamma \cdot X \text{ projective over } \Gamma\}.$$

Note. We identify $K \otimes_R X = K \cdot X = A \cdot X$, X being R-torsion-free, and hence we can form $\Gamma \cdot X$ inside $A \cdot X$; however, one should observe that $\Gamma \cdot X$ is in general different from $\Gamma \otimes_{\Lambda} X$; as a matter of fact, $\Gamma \cdot X$ is the quotient of $\Gamma \otimes_{\Lambda} X$ modulo its R-torsion submodule; note also that the latter is a Γ-submodule. One sees this by considering the natural map $\Gamma \otimes_{\Lambda} X \to \Gamma \otimes_{\Lambda} K \cdot X \cong K \cdot X$, induced by the inclusion $X \to K \cdot X$. In particular, every Λ-homomorphism in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ between X and Y gives rise to a Γ-homomorphism between $\Gamma \cdot X$ and $\Gamma \cdot Y$. Conceptually one can view X *as a form of the projective* Γ*-module* $\Gamma \cdot X$. It should be noted that this concept can be applied in the more general situation of any ring Γ and a subring Λ if one considers only those left Λ-modules which are Λ-submodules of a free Γ-module.

Without loss of generality we can always assume that Λ is indecomposable as a ring.

(3.6) *Remarks.* 1) In the classical situation, where R is a Dedekind domain and A is separable, Λ is any R-order in A and Γ is a hereditary R-order containing Λ, ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ *is just the category of all* Λ*-lattices.*

2) In the algebraic-geometric situation, where R is a regular and Λ is the local ring of dimension d of an isolated singularity, Γ is the normalization of Λ and ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ contains just the Λ-modules X which become projective when extended to Γ, i.e. $\Gamma \cdot X$ is Γ-projective. (The same applies if Γ is any ring between Λ and its normalization.)

Choose now a two-sided Λ*-ideal* I *such that*

(i) I *is also a two-sided* Γ*-ideal,*

(ii) $I \subset \operatorname{rad} \Lambda$.

We observe that then automatically $I \subset \operatorname{rad} \Gamma$. In fact, $I \subset \operatorname{rad} \Lambda$ and so I is nilpotent modulo $\operatorname{rad} R \cdot \Lambda \subset \operatorname{rad} R \cdot \Gamma \subset \operatorname{rad} \Gamma$, i.e. $I \subset \operatorname{rad} \Gamma$.

With this notation we put

$$\mathfrak{A} = \Lambda/I, \quad \mathfrak{B} = \Gamma/I.$$

Then $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated algebras over the commutative local ring $\bar{R} = R/(R \cap I)$. Moreover, the inclusion $\Lambda \to \Gamma$ induces an $\bar{R}$-algebra injection $\mathfrak{A} \to \mathfrak{B}$, and we identify $\mathfrak{A}$ with a subring of $\mathfrak{B}$. We now construct the *pair category* $\mathfrak{C}^0$ as follows: An object consists of a finitely generated left $\mathfrak{A}$-module U and a finitely generated projective left $\mathfrak{B}$-module V together with an $\mathfrak{A}$-monomorphism σ: $U \to V$ such that

$$\mathfrak{B} \cdot \operatorname{Im} \sigma = V.$$

Morphisms in $\mathfrak{C}^0$ are commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & V \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ U' & \xrightarrow{\sigma'} & V' \end{array}$$

where α is $\mathfrak{A}$-linear and β is $\mathfrak{B}$-linear.

It should be noted that $\mathfrak{C}^0$ can be identified with a certain category of finitely generated modules over the artinian algebra

$$\mathfrak{D} = \begin{bmatrix} \mathfrak{B} & \mathfrak{B} \\ 0 & \mathfrak{A} \end{bmatrix}.$$

We now construct a natural functor

$$\mathfrak{F}\colon {}_{\Lambda}\mathfrak{M}^0(\Gamma) \to \mathfrak{C}^0, \qquad M \to M/I\cdot M \xrightarrow{\sigma} \Gamma\cdot M/I\cdot M,$$

where σ is induced by the inclusion $\iota\colon M \to \Gamma\cdot M$.

Moreover, if $\alpha_1\colon M \to M'$ is a Λ-homomorphism in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$, then it induces a Γ-homomorphism $\beta_1\colon \Gamma\cdot M \to \Gamma\cdot M'$ rendering the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & \Gamma\cdot M \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ M' & \to & \Gamma\cdot M' \end{array}$$

Hence we obtain a morphism in $\mathfrak{C}^0$

$$\begin{array}{ccc} M/I\cdot M & \to & \Gamma\cdot M/I\cdot M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ M'/I\cdot M & \to & \Gamma\cdot M'/I\cdot M' \end{array}$$

It should be noted that $\Gamma\cdot M/I\cdot M$ is $\mathfrak{B}$-projective, $\Gamma\cdot M$ being Γ-projective. Moreover,

$$\mathfrak{B}\cdot(M/I\cdot M) = (\Gamma/I)\cdot(M/I\cdot M) = \Gamma\cdot M/I\cdot M.$$

(3.7) THEOREM [R2]. *The functor $\mathfrak{F}$ induces a representation equivalence between ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ and $\mathfrak{C}^0$; in particular, $\mathfrak{F}$ induces a bijection between the indecomposable objects in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ and in $\mathfrak{C}^0$.*

(3.8) *Remark.* The essential and important point of the theorem is that it allows one to compare the lattices in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ with the finitely generated modules in $\mathfrak{C}^0$, which are modules over the artinian algebra $\mathfrak{D}$.

(3.9) *Remark. The above situation is most transparent if $I = \operatorname{rad}\Gamma$*, i.e. if we have the inclusions

$$\operatorname{rad}\Gamma \subset \operatorname{rad}\Lambda \subset \Lambda \subset \Gamma.$$

We shall assume this from now on. In this case $\mathfrak{B}$ is semisimple over $\mathfrak{k} = R/\operatorname{rad} R$ and $\mathfrak{A}$ is a finite-dimensional $\mathfrak{k}$-algebra. Thus, if we consider the algebra

$$\mathfrak{D} = \begin{bmatrix} \mathfrak{B} & \mathfrak{B} \\ 0 & \mathfrak{A} \end{bmatrix},$$

then $\mathfrak{D}$ has a projective socle. We denote by $\mathfrak{C}(\mathfrak{D})$ the full subcategory of finitely generated left $\mathfrak{D}$-modules which have a projective socle. Then we have:

(3.10) LEMMA. *Let $S_1, \ldots, S_t$ be all the simple nonisomorphic $\mathfrak{B}$-modules. Then*

$$\mathrm{ob}(\mathfrak{C}^0) = \{U \in \mathfrak{C}(\mathfrak{D})\colon\ U \text{ has no simple direct summand}\}.$$

The $\mathfrak{k}$-algebras $\mathfrak{D}$ that arise in the above construction can be easily described as follows:

(3.11) LEMMA. *A $\mathfrak{k}$-algebra $\mathfrak{D}_0$ is Morita equivalent to an algebra*

$$\mathfrak{D} = \begin{bmatrix} \Gamma/\mathrm{rad}\,\Gamma & \Gamma/\mathrm{rad}\,\Gamma \\ 0 & \Lambda/\mathrm{rad}\,\Gamma \end{bmatrix}$$

for R-orders Λ, Γ with

$$\mathrm{rad}\,\Gamma \subset \Lambda \subset \Gamma$$

if and only if $\mathfrak{D}_0$ has a projective left socle, and no simple ring direct factor.

Note. In [RR2], a necessary and sufficient condition is given for the category $\mathfrak{C}(\mathfrak{D})$ to have finitely many indecomposable objects provided $\mathfrak{D}$ is the tensor algebra of a multivalued oriented graph [DlRi], in particular $\mathfrak{D}$ is a hereditary algebra. In this case Λ is called a generalized Bäckström order [R16], the representation theory of which is well understood and will be treated later.

Irreducible maps, almost split sequences and Auslander–Reiten quivers are defined in analogy to the definitions in § 2.

We first turn to $\mathfrak{C}(\mathfrak{D})$, where $\mathfrak{D}$ has a left-projective socle. For a finitely generated left $\mathfrak{D}$-module M we denote by tM the maximal submodule whose socle has no projective submodule, and put $fM = M/tM$. The modules tM are the torsion modules in a hereditary torsion theory [St]. The category $\mathfrak{C}(\mathfrak{D})$ has almost split sequences as was observed in [AS1] and [R6]. In [R16] the following result of C. M. Ringel and the author was proved:

(3.12) THEOREM. *If $\mathfrak{D}$ is left socle projective, then $\mathfrak{C}(\mathfrak{D})$ has almost split sequences, which are constructed as follows: Given X indecomposable in $\mathfrak{C}(\mathfrak{D})$ which is not an injective $\mathfrak{D}$-module.*

(i) *X is* Ext-*injective in $\mathfrak{C}(\mathfrak{D})$ if and only if in the almost split sequence for X in the category of all finitely generated left $\mathfrak{D}$-modules*

$$\tilde{E}\colon\quad 0 \to X \to \tilde{Y} \to \tilde{Z} \to 0$$

we have $f\tilde{Z} = 0$.

(ii) *If X is not* Ext-*injective in $\mathfrak{C}(\mathfrak{D})$, then*

$$0 \to X \to \tilde{Y}/t\tilde{Z} \to f\tilde{Z} \to 0$$

is an almost split sequence in $\mathfrak{C}(\mathfrak{D})$.

We now turn to ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$. For arbitrary R it is not known whether ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ has almost split sequences, except in the following situations:

1. R is Dedekind and Γ hereditary [R6], i.e. the classical situtation.

2. Λ is the coordinate ring of an isolated singularity (M. Auslander [Au3] has shown that in this case the category of Cohen–Macaulay modules has almost split sequences, but our category is different. The existence in the second case was proved in [R2].)

Because of the connection between irreducible maps and almost split sequences we shall have a close look at irreducible maps and our functor $\mathfrak{F}$: ${}_{\Lambda}\mathfrak{M}^0(\Gamma) \to \mathfrak{C}(\mathfrak{D})$ (as introduced above).

(3.13) LEMMA. *Let φ: $M \to N$ be a map between indecomposable modules in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$. Assume $\mathfrak{F}(\varphi) \neq 0$. If $\mathfrak{F}(\varphi)$ is irreducible in $\mathfrak{C}(\mathfrak{D})$, then φ itself is irreducible.*

For $M \in {}_{\Lambda}\mathfrak{M}^0$ we define the *size* of M, $\mathrm{sz}(M)$, to be the number of composition factors of $K \cdot M$ as an A-module.

(3.14) THEOREM (BRAUER–THRALL $1\frac{1}{2}$). *Let Δ be a connected component of the Auslander–Reiten quiver $\mathfrak{A}({}_{\Lambda}\mathfrak{M}^0(\Gamma))$ of ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ such that:*

(i) *The vertices of Δ have bounded size, i.e. the sizes of the modules in Δ have a uniform bound.*

(ii) *$\mathfrak{F}(\Delta)$ contains at least one module for each ring direct factor of $\mathfrak{D}$.*

Then:

(α) $\mathfrak{A}({}_{\Lambda}\mathfrak{M}^0(\Gamma)) = \Delta$.

(β) *$\mathfrak{A}({}_{\Lambda}\mathfrak{M}^0(\Gamma))$ is finite, i.e. ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ has only finitely many indecomposable lattices.*

(3.15) *Remarks.* (1) The hypothesis (ii) is satisfied if, for example, Δ contains all indecomposable projective Λ-lattices or if Δ contains all indecomposable projective Γ-lattices.

(2) The hypothesis (ii) is superfluous if $\dim R = 1$ and Γ is hereditary in a separable algebra A, since in this case ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ is the category of all Λ-lattices, and then one knows the result.

(3) It is likely that the hypothesis (ii) is superfluous in general (cf. [R2]).

In order to discuss the irreducible maps φ in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ we have to restrict the morphisms in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ considerably; since we are mainly interested in indecomposable objects, this is not a severe restriction.

(3.16) DEFINITION. ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$ has the same objects as ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$, but we allow only morphisms φ: $M \to N$, M, $N \in \mathrm{ob}({}_{\Lambda}\mathfrak{M}^0(\Gamma))$, such that $\Gamma \cdot \mathrm{Im}\, \varphi$ is Γ-projective.

(3.17) *Remarks.* (1) We still have a representation equivalence between ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$ and $\mathfrak{C}^0$.

(2) Lemma 3 carries over to ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$, and consequently Theorem (3.14) holds in ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$.

(3) In the classical situation, where $\dim R = 1$ and Γ is hereditary, ${}_{\Lambda}\mathfrak{M}^0(\Gamma) = {}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$.

(3.18) LEMMA. *Let $\varphi\colon M \to N$ be an irreducible morphism between indecomposable objects in ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$. If $\mathfrak{F}(\varphi) = 0$, then $\Gamma \cdot M$ is an indecomposable projective Γ-lattice, and N is Λ-projective.* (The converse is trivially true.)

In this general situation, I cannot say anything in case φ is irreducible in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ (${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$) and $\mathfrak{F}(\varphi) \neq 0$. I would need Lemma 4 for ${}_{\Lambda}\mathfrak{M}^0$ (that the morphism sets in ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$ form abelian groups). However, Lemma 4 does not hold in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ and the morphism sets in ${}_{\Lambda}\mathfrak{M}^0(\Gamma_s)$ do not form abelian groups in general. *The remedy is to turn to the classical situation*:

(3.19) LEMMA. *Let $\dim R = 1$ and assume that Γ is hereditary and A is semisimple. Let $\varphi\colon M \to N$ be an irreducible map between indecomposables in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$. If $\mathfrak{F}(\varphi) \neq 0$, then $\mathfrak{F}(\varphi)$ is irreducible in $\mathfrak{C}(\mathfrak{D})$.*

In the classical situation of generalized Bäckström orders, the predecessors of an indecomposable Γ-lattice Q in the Auslander–Reiten quiver must be injective Λ-lattices, and the successors of Q are projective Λ-lattices [R16].

It is surprising that this result also holds in the very general situation considered here, for the successors.

(3.20) LEMMA. *In ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ let $\varphi\colon Q \to M$ be an irreducible map with Q an indecomposable projective Γ-lattice and M indecomposable in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$. Then M is a projective Λ-lattice.*

I can only prove the corresponding statement for the predecessors under additional assumptions.

(3.21) LEMMA. *Assume that ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ has left almost split sequences.* (This is surely so if $\dim R = 1$ and Γ is hereditary.) *If Q is an indecomposable projective Γ-lattice and $M \in {}_{\Lambda}\mathfrak{M}^0(\Gamma)$ is indecomposable with an irreducible map $\varphi\colon M \to Q$, then M is an* Ext*-injective object in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$.*

Note. The lemma actually only needs $\mathfrak{F}(\varphi)$ to be irreducible.

We assume henceforth that R is one-dimensional, that A is separable and that Λ, Γ are R-orders in A with Γ hereditary, such that

$$\operatorname{rad}\Gamma \subset \Lambda \subset \Gamma,$$

i.e. *Λ is a subhereditary order.*

In that case ${}_{\Lambda}\mathfrak{M}^0(\Gamma) = {}_{\Lambda}\mathfrak{M}^0$ is just the category of all left Λ-lattices. And so the Auslander–Reiten quiver $\mathfrak{A}(\Lambda) = \mathfrak{A}({}_{\Lambda}\mathfrak{M}^0)$ of all Λ-lattices carries an additional structure, namely the partially defined translation coming from almost split sequences, together with a valuation on the arrows. The same holds for $\mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$ (for details we refer to [R16]). The structures of the Auslander–Reiten quivers $\mathfrak{A}(\Lambda)$ and $\mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$ are intimately related.

Let us recall the definition of the permutation associated to a hereditary order Γ (cf. (3.1) above). Let $\{Q\}_{1 \leqslant i \leqslant m}$ be the nonisomorphic indecomposable Γ-lattices. Since Γ is hereditary, they are at the same time injective Γ-lattices; thus Q_i has a unique minimal overlattice $S(Q_i)$, which is again a projective Γ-lattice, and hence

$$S(Q_i) \simeq Q_{\sigma(i)}$$

for some $\sigma(i) \in \{1, \ldots, m\}$. This map σ: $\{1, \ldots, m\} \to \{1, \ldots, m\}$ is a permutation, called the *permutation of* Γ.

(3.22) THEOREM. (i) *If M and N are indecomposable Λ-lattices, and M is not a Γ-lattice, then we have for the spaces of irreducible maps*

$$\mathrm{Irr}_\Lambda(M, N) \simeq \mathrm{Irr}_{\mathfrak{C}(\mathfrak{D})}(\mathfrak{F}(M), \mathfrak{F}(N)).$$

(ii) *If M is a Γ-lattice, say $M = Q_i$, then for an indecomposable projective Λ-lattice P*

$$\mathrm{Irr}_\Lambda(Q_i, P) \simeq \mathrm{Irr}_{\mathfrak{C}(\mathfrak{D})}(Q_{\sigma(i)}/\mathrm{rad}_\Gamma Q_{\sigma(i)}, \mathfrak{F}(P)).$$

(iii) *The Auslander–Reiten quiver of Λ is obtained from that of $\mathfrak{C}(\mathfrak{D})$ by identifying the injective $\mathfrak{D}$-module*

$$E_i = \begin{bmatrix} Q_i/\mathrm{rad}_\Gamma Q_i \\ Q_i/\mathrm{rad}_\Gamma Q_i \end{bmatrix}$$

with the simple projective $\mathfrak{D}$-module

$$S_{\sigma(i)} = \begin{bmatrix} Q_{\sigma(i)}/\mathrm{rad}_\Gamma Q_{\sigma(i)} \\ 0 \end{bmatrix}.$$

(3.23) DEFINITIONS. (i) Let Λ be an R-order in A, and let $Q_1, \ldots, Q_s$ be those indecomposable Λ-lattices which have only projective Λ-lattices as successors (equivalently, have only injective predecessors in $\mathfrak{A}(\Lambda)$). Denote by $\mathfrak{G}$ the full additive subcategory of ${}_\Lambda\mathfrak{M}^0$ generated by $\{Q_i\}_{1 \leqslant i \leqslant s}$.

(ii) Let $\mathfrak{K}$ be the following category: the indecomposable objects are the indecomposable Λ-lattices not in $\mathfrak{G}$, and for each indecomposable $Q \in \mathfrak{G}$, we introduce new objects Q^+ and Q^-.

For X and Y indecomposable Λ-lattices not in $\mathfrak{G}$ we put

$$\mathfrak{K}(X, Y) = \mathrm{Hom}_\Lambda(X, Y)/\mathfrak{G}(X, Y),$$

where $\mathfrak{G}(X, Y)$ is the group of Λ-homomorphisms factoring via an object in $\mathfrak{G}$. Moreover, in addition we put

$$\mathfrak{K}(Q^+, Y) = \mathrm{Hom}_\Lambda(Q, Y)/\mathrm{rad}_{\mathfrak{G}}(Q, Y),$$

$$\mathfrak{R}(X, Q^-) = \mathrm{Hom}_\Lambda(X, Q)/\mathrm{rad}_{\mathfrak{G}}(X, Q),$$
$$\mathfrak{R}(Q^+, Q_0^-) = \mathrm{rad}_{\mathfrak{G}}(Q, Q_0)/\mathrm{rad}^2_{\mathfrak{G}}(Q, Q_0),$$
$$\mathfrak{R}(Q^-, X) = \mathfrak{R}(X, Q^+) = 0,$$

where X, Y are indecomposable objects but $X \neq Q_0^+$ and $Y \neq Q_0^-$ for indecomposable objects $Q, Q_0 \in \mathfrak{G}$. Here $\mathrm{rad}_{\mathfrak{G}}(\ , \ )$ are maps which factor via the radical in $\mathfrak{G}$.

(3.24) *Remark.* It should be noted that in case Λ is a subhereditary order, $\mathfrak{R}$ is just the category $\mathfrak{C}(\mathfrak{D})$, which is by the above definition characterized internally.

We next define the separated Auslander–Reiten quiver of Λ.

(3.25) DEFINITION. The *separated Auslander–Reiten quiver* of Λ, $\mathfrak{A}^s(\Lambda)$, has as vertices:

(i) the indecomposable Λ-lattices which are not in $\mathfrak{G}$,
(ii) for each $Q_i \in \mathfrak{G}$, two new vertices $[Q_i^+]$ and $[Q_i^-]$.

The spaces of irreducible maps between two vertices $[X]$ and $[Y]$ are:

(i) $\mathrm{Irr}_\Lambda(X, Y)$ if X and Y are indecomposable Λ-lattices not in $\mathfrak{G}$,
(ii) $\mathrm{Irr}_{\mathfrak{A}^s(\Lambda)}(X, Q_i^+) = 0$, $\mathrm{Irr}_{\mathfrak{A}^s(\Lambda)}(Q_i^-, Y) = 0$,
(iii) $\mathrm{Irr}_{\mathfrak{A}^s(\Lambda)}(Q_i^+, X) = \mathrm{Irr}_\Lambda(Q_i, X)$, $\mathrm{Irr}_{\mathfrak{A}^s(\Lambda)}(Y, Q_i^-) = \mathrm{Irr}_\Lambda(Y, Q_i)$.

(3.26) THEOREM. *For an R-order Λ of finite lattice type, the following are equivalent*:

(i) *Every oriented cycle in $\mathfrak{A}(\Lambda)$ passes through a lattice in $\mathfrak{G}$.*
(ii) *Λ is subhereditary and $\mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$ has preprojective components. In this case $\mathfrak{R}$ has almost split sequences and $\mathfrak{A}(\mathfrak{R}) = \mathfrak{A}(\mathfrak{C}(\mathfrak{D}))$.*

If Λ satisfies one of these equivalent conditions, we shall call Λ an *almost directed order.*

(3.37) *Remark.* Condition (i) cannot be replaced by the condition that every Auslander–Reiten orbit contains a projective Λ-lattice or an object in $\mathfrak{G}$.

The almost directed orders seem to be the analog for integral representations to the simply connected artinian algebras. The above theorem shows that these give rise to simply connected socle projective categories for $\mathfrak{D}$. However, for $\mathfrak{D}$ the Γ-lattices Q_i^- (in $\mathfrak{R}$) become projective. If one wants to copy some of the results from the artinian situation to subhereditary orders, one has to develop a relative homological algebra for Λ-lattices (Λ is subhereditary for Γ), where also the Γ-lattices are made Λ-projective.

Auslander–Reiten quivers of Bäckström orders

In this section Λ is a generalized Bäckström order with associated hereditary order Γ (cf. § 1D, F). As in (1.10) we put $\mathfrak{D} = \begin{bmatrix} \mathfrak{B} & {}_{\mathfrak{B}}\mathfrak{B}_{\mathfrak{A}} \\ 0 & \mathfrak{A} \end{bmatrix}$, where $\mathfrak{A} = \Lambda/\operatorname{rad}\Gamma$ and $\mathfrak{B} = \Gamma/\operatorname{rad}\Gamma$. Then $\mathfrak{D}$ is a hereditary $\mathfrak{k}$-algebra. As above we put

$$\mathfrak{C}(\mathfrak{D}) = \{X \in \operatorname{mod}\mathfrak{D}\colon \operatorname{soc}_{\mathfrak{D}} X \text{ is projective}\},$$

$$\mathfrak{C}^0(\mathfrak{D}) = \{X \in \mathfrak{C}(\mathfrak{D})\colon X \text{ has no simple direct summand}\}.$$

We also have the functor from (3.2), $\mathfrak{F}\colon {}_{\Lambda}\mathfrak{M}^0 \to \mathfrak{C}^0(\mathfrak{D})$, induced by

$$M \to \begin{bmatrix} \Gamma\cdot M/(\operatorname{rad}\Gamma)\cdot M \\ M/(\operatorname{rad}\Gamma)\cdot M \end{bmatrix}.$$

Before we deal with the generalized Bäckström orders let us look at the category $\mathfrak{C}(\mathfrak{D})$. We assume now that $\mathfrak{D}$ is the $\mathfrak{k}$-tensor algebra of a valued oriented graph T with modulation over $\mathfrak{k}$ [DlRi]. Before we can give a necessary and sufficient condition for $n(\mathfrak{C}(\mathfrak{D}))$—the number of indecomposable objects in $\mathfrak{C}(\mathfrak{D})$—to be finite, we have to introduce some more *notation*:

(3.28) A valued oriented graph T is said to be *contractible* if there exists a pair of vertices a and b linked by an arrow $a \xrightarrow{(1,1)} b$ with valuation (1, 1) in such a way that the graph obtained from T by removing the edge between a and b is the disjoint union of a graph T_1 and a graph T_2, such that a is a source in T_1 and T_2 is a Dynkin diagram of type A_m such that b is a source in T_2. We then say that T is *1-step contractible* to the graph $T_{a=b}$, which is obtained by identifying a with b. We say that T is *contractible* to T' provided T' can be reached from T by a finite number of 1-step contractions.

(3.29) THEOREM [R16–17]. *$n(\mathfrak{C}(\mathfrak{D}))$ is finite if and only if T can be contracted to a finite number of Dynkin diagrams.*

If T is a Dynkin diagram, then the Auslander–Reiten quiver of $\mathfrak{C}(\mathfrak{D})$ is simply connected, and every indecomposable object in $\mathfrak{C}(\mathfrak{D})$ is obtained as an iterated Auslander–Reiten translate—to the right—starting with the indecomposable projectives (these lie in $\mathfrak{C}(\mathfrak{D})$).

Recall from (3.12) how the almost split sequences in $\mathfrak{C}(\mathfrak{D})$ are constructed from those in $\operatorname{mod}\mathfrak{D}$. Since the latter can be constructed algorithmically, the Auslander–Reiten quiver of $\mathfrak{C}(\mathfrak{D})$ is known in the case of finite type. However, one can give a purely combinatorial description of the Auslander–Reiten quiver of $\mathfrak{C}(\mathfrak{D})$ without going first to the Auslander–Reiten quiver of $\operatorname{mod}\mathfrak{D}$, provided there are only finitely many indecomposables in $\mathfrak{C}(\mathfrak{D})$. Let T be a valued oriented tree. Then we define the *disturbed additive function of T*

$$f_T\colon \mathbf{Z}T \to \mathbf{Z}$$

as follows:

(3.30) Let $\mathbf{Z}T$ be the translation quiver of T in the sense of Riedtmann [Ri13]. The vertices of $\mathbf{Z}T$ are labelled (n, i), $n \in \mathbf{Z}$, i a vertex of T. We view T as embedded in $\mathbf{Z}T$ via $(0, i)$. By τ we denote the translation on $\mathbf{Z}T$ defined as τ: $(n, i) \to (n+1, i)$. Let

$$f_T(n, i) = 0 \quad \text{if } n < 0.$$

If $i_1, \ldots, i_s$ are the sources of T, then we put

$$f_T(0, i_j) = 1.$$

For the other points of T we define f_T inductively. So let j be a vertex in T which is not a source, and assume that $f_T(0, i)$ is defined for all i which have an arrow in T to j, $i \to j$. Then we put

$$f_T(0, j) = \sum_{i \to j} f_T(0, i).$$

This defines f_T on T. On the rest of $\mathbf{Z}T$ we define f_T inductively. Let (n, i) be a vertex in $\mathbf{Z}T$ with $n \geqslant 1$ and assume that f_T is defined on all (m, j) such that there is an arrow from (m, j) to (n, i); moreover, we also assume that f_T is defined on $(n-1, i)$. We then put

$$f_T(n, i) = \Big(\sum_{(m,j) \to (n,i)} f_T(m, j) - f_T(n-1, i)\Big) \quad \text{if } f_T(n-1, i) > 0,$$

$$f_T(n, i) = 0 \quad \text{if } f_T(n-1, i) = 0.$$

(3.31) THEOREM [R16]. *Let T be a connected valued oriented tree. Then the following conditions are equivalent:*

(i) *$n(\mathfrak{C}(\mathfrak{D}))$ is finite.*

(ii) *T can be contracted to a Dynkin diagram.*

(iii) *The disturbed additive function f_T has finite support on $\mathbf{Z}T$.*

Moreover, if any of these conditions is satisfied, then $\{(n, i)\colon f_T(n, i) \neq 0\}$ together with the arrows in $\mathbf{Z}T$ is the Auslander–Reiten quiver of $\mathfrak{C}(\mathfrak{D})$. If (n, i) corresponds to $V \in \mathfrak{C}(\mathfrak{D})$, then $f_T(n, i)$ is the number of simple modules in the socle of V.

Auslander–Reiten quivers of Gorenstein orders

In this section let Λ be a Gorenstein order (cf. § 1H) which is indecomposable as a ring. Since projective Λ-lattices are at the same time injective Λ-lattices and conversely, every $M \in \operatorname{ind} \Lambda$ which is not *bijective* (i.e. not injective and projective) is stable in the Auslander–Reiten quiver $\mathfrak{A}(\Lambda)$ (cf. (2.8, ii)). The stable Auslander–Reiten quiver $\mathfrak{A}_s(\Lambda)$ has an isolated point if and only if Λ is a Bäckström order with associated graph a disjoint union of Dynkin diagrams of type A_2, A_3 or B_2. These cases were treated in connection with Bäckström orders. If $\mathfrak{A}_s(\Lambda)$ has no isolated point, then Λ is of finite lattice type if and only if the tree type of $\mathfrak{A}_s(\Lambda)$ is a Dynkin diagram or L_t (cf. (2.29), [R20]). Moreover, all these types do occur.

Assume from now on that Λ *is of finite lattice type*. Then the stable Auslander–Reiten quiver $\mathfrak{A}_s(\Lambda)$ is known by the above remark, and we may assume that $\mathfrak{A}_s(\Lambda)$ is connected. Thus, in order to describe $\mathfrak{A}(\Lambda)$, one has to find the positions in $\mathfrak{A}_s(\Lambda)$ where the indecomposable bijective Λ-lattices B_i, $1 \leqslant i \leqslant s$, have to be added to $\mathfrak{A}_s(\Lambda)$. These positions are called the *configuration points*. We know that in $\mathfrak{A}_s(\Lambda)$ the only irreducible maps to B_i come from the indecomposable direct summands of $\mathrm{rad}_\Lambda B_i$, and the only irreducible maps leaving B_i go to the indecomposable direct summands of B_i^-, the unique minimal overmodule of B_i. In view of (2.11) we may assume that $\mathfrak{A}_s(\Lambda)$ does not have loops. Then the tree type of $\mathfrak{A}_s(\Lambda)$ is a Dynkin diagram.

A. Wiedemann [Wi7] has developed a covering theory for the Auslander–Reiten quiver of an order—this differs considerably from that for the artinian algebras (cf. the remark following (2.8)). Using this he was able to describe all possible configurations in $\mathfrak{A}_s(\Lambda)$, provided $\mathfrak{A}_s(\Lambda)$ has tree class A_n or D_n. Moreover, for each such configuration he has explicitly constructed a Gorenstein order which has this stable Auslander–Reiten quiver and the corresponding configuration points.

In his master thesis Hummel [Hu] has—with the help of a computer—determined all possible configurations in case $\mathfrak{A}_s(\Lambda)$ has tree class E_6, E_7 or E_8. Up to trivial isomorphisms, there are

11 configurations in the case of E_6,
44 configurations in the case of E_7,
138 configurations in the case of E_8.

In each case Hummel has constructed Gorenstein orders with the corresponding Auslander–Reiten quiver.

These results determine completely the Auslander–Reiten quivers of Gorenstein orders of finite lattice type except in case the tree class of $\mathfrak{A}_s(\Lambda)$ is B_n, C_n, F_4 and G_2. Note, however, that these cases cannot occur if for example R is the ring of formal power series over an algebraically closed field.

Simple curve sigularities

The complete local ring of a *plane simple curve singularity* can be viewed as a commutative Gorenstein order over the power series ring in one variable over the complex numbers. More precisely, let $1 \neq G$ be a finite subgroup of $\mathrm{Sl}(2, \mathbf{C})$. Then G acts linearly on the power series ring $\mathbf{C}[\![U, V]\!]$. Let Λ be the subring of fixed points; then Λ has 3 generators X, Y and Z, with the unique relation $f(X, Y)+Z^2$. The equation $f(X, Y)+Z^2=0$ defines in the neighbourhood of the origin a surface with an isolated singularity at the origin. The singularities occurring this way are called *rational double points* or *Kleinian singularities.* The resolution graphs of these singularities are the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8 [Br]. The intersection of such a surface with the plane $Z=0$ is then a *reduced plane curve singularity* in the sense of Arnol'd

[Ar]. Greuel and Knörrer [GK] have characterized the Kleinian singularities among the reduced plane curve singularities as follows:

THEOREM. *The complete local ring Γ of a reduced plane curve singularity has finitely many nonisomorphic torsion-free modules of rank 1 if and only if $\Gamma \simeq \mathbf{C}[\![X, Y]\!]/(f(X, Y))$, where $f(X, Y)+Z^2$ defines a Kleinian singularity.*

It turns out that in the case of a Kleinian singularity the order Γ is of finite lattice type [Au5], and Wiedemann [Wi3] has proved:

(3.32) THEOREM. *Let Λ be a local commutative R-order of finite lattice type such that the stable Auslander–Reiten quiver $\mathfrak{A}_s(\Lambda)$ has tree class A_n, D_n, E_6, E_7 or E_8. Then the Auslander–Reiten quiver $\mathfrak{A}(\Lambda)$ coincides with the Auslander–Reiten quiver of the category of lattices over the complete local ring of a simple curve singularity given by one of the equations $f(X, Y) = 0$ in the table below.*

	Tree class of $\mathfrak{A}_s(\Lambda)$	Type of the corresponding double point with defining polynomial $f(X, Y)$	
1	A_1	A_1	X^2+Y^2
2	A_3	A_3	X^2+Y^4
3	D_m, $m \geqslant 4$	A_{2m-3}	X^2+Y^{2m-2}
4	A_{2m}, $m \geqslant 1$	A_{2m}	X^2+Y^{2m+1}
5	D_n, $n \geqslant 4$, n even	D_n	X^2Y+Y^{n-1}
6	A_{2n-3}, $n \geqslant 5$, n odd	D_n	X^2Y+Y^{n-1}
7	E_6	E_6	X^3+Y^4
8	E_7	E_7	X^3+XY^+
9	E_8	E_8	X^3+Y^5

(3.33) *Remarks.* 1) There seems to be some overlap. This is not so, since $\mathfrak{A}_s(\Lambda)$ with tree class Δ is of the form $\mathbf{Z}\Delta/\alpha$, where α is an admissible automorphism in the sense of Riedtmann [Ri1], and the overlap comes from the fact that these cases differ by α.

2) The discrepancy between the type of the Kleinian singularity and the tree type of the stable Auslander–Reiten quiver in cases 3 and 6 is explained by Dieterich and Wiedemann [DW], where the Auslander–Reiten quivers are described differently, so that they match the type of the singularity. In fact, their description is the one via the McKay graphs.

Group rings

In this section let G be a finite group and let R be an unramified extension of $\hat{\mathbf{Z}}_p$. A *block* of RG is an indecomposable 2-sided direct summand of RG. With each block B one can associate a conjugacy class of p-subgroups of G, called the *defect group* of B. This defect group plays the same role for B as does the Sylow p-subgroup for the whole group ring (for details we refer to [R8]). The general result on the finite lattice type of blocks is the following:

(3.34) THEOREM. *Let B be a block of RG with defect group D. Then B is of finite lattice type if and only if D is cyclic of order p or p^2.*

Historically, E. F. Diederichsen [Dd] showed in 1940 that $\hat{\mathbf{Z}}_p C_p$, C_n the cyclic group of order n, has 3 nonisomorphic indecomposable representations. In 1963 A. Heller and I. Reiner [HR] proved that for a p-group P, $\hat{\mathbf{Z}}_p P$ is of finite lattice type if and only if P is cyclic of order p or p^2. In the latter case there are $4p+1$ indecomposable representations, which they list explicitly. With D. G. Higman's theory of relative projective representations [Hi] it then follows that RG is of finite lattice type if and only if the Sylow p-subgroup of G is cyclic of order p or p^2. It was folklore that the corresponding result holds for blocks. This was proved by Ch. Bessenrodt [Be1–2] and in [R8].

For the description of the lattices in a block of finite lattice type one has:

(3.35) THEOREM. *Let B a block of defect p. If B has e nonisomorphic indecomposable projective lattices, then B has $3e$ nonisomorphic indecomposable representations.*

In case $D = C_G(D)$, the centralizer of the defect group D, the result was proved by J. A. Green [G2] in 1974. The general result—including the structure of B—was obtained independently by H. Jacobinski [Ja4] and in [R8–9].

(3.36) THEOREM. *Let B be a block of RG with defect group cyclic of order p^2, and assume that B has e projective nonisomorphic indecomposable lattices. Then $e|(p-1)$ and B has $(4p-2)e$ indecomposable lattices of vertex p^2 and $2e$ indecomposable lattices of vertex p.*

This result was independently proved by Ch. Bessenrodt [Be 2] in case R/pR splits B/pB and by A. Wiedemann [Wi2] in the general case.

In 1974 M. Butler [Bu1] gave a new description of the indecomposable lattices for $\hat{\mathbf{Z}}_p C_{p^2}$ using diagrammatic methods—namely reducing the problem to the representation theory of the Dynkin diagram D_{2p}. It turns out (cf. below) that the Auslander–Reiten graph is indeed of type D_{2p} [Wi2].

There are several cases where the indecomposable RG-lattices are known though there are infinitely many of them: A. V. Yakovlev [Yak] listed them for $\hat{\mathbf{Z}}_p C_8$ and M. Butler [Bu3] for Klein's 4-group. E Dieterich has completed the classification of p-groups according to finite lattice type, tame lattice type and wild lattice type as follows.

Without going into details, let us recall that an order can be of finite or infinite lattice type, and in the case of infinite lattice type, it can be either of *tame* lattice type – in which case the lattices can be classified – or of *wild* lattice type – in this case there exists a full subcategory $\mathfrak{K}$ of ${}_{\Lambda}\mathfrak{M}^{0}$ together with a representation equivalence between $\mathfrak{K}$ and $\mathfrak{H}(\mathbf{F})$, where $\mathbf{F}$ is a field and $\mathfrak{H}(\mathbf{F})$ is the category of all finitely generated modules over the free associative $\mathbf{F}$-algebra in two – noncommuting – variables. For group rings we have the following classification, where many authors have contributed. For details we refer to [Di1–2].

THEOREM. *Let R be a complete discrete valuation ring and $v(p)$ the exponential valuation with respect to p ($v(p) \in \mathbf{N} \cup \{\infty\}$).*

(i) *The group ring RG is of finite lattice type if and only if one of the following conditions is satisfied*:

1) $v(p) = 0$.
2) $v(p) = 1$ *and* $G = C_{p^e}$, $e \leqslant 2$.
3) $v(p) = 2$ *and* $G = C_p$.
4) $v(p) = 3$ *and* $G = C_3$.
5) $4 \leqslant v(p) \leqslant \infty$ *and* $G = C_2$.

(ii) *The group ring RG is of tame lattice type if and only if one the following conditions is satisfied*:

1) $v(p) = 1$ *and* $G = C_2 \times C_2$ *or* $G = C_8$.
2) $v(p) = 2$ *and* $G = C_4$.
3) $v(p) = 4$ *and* $G = C_3$.
4) $v(p) = \infty$ *and* $G = C_2$.

(iii) *The group ring RG is of wild lattice type in all the remaining cases.*

In the finite and tame cases there is an explicit description of the lattices.

We now come to the Auslander–Reiten quivers of blocks of finite lattice type.

(3.37) THEOREM [Ja3, R8–9]. *A block B of defect one is a Bäckström order. Assume that B has e nonisomorphic projective lattices. Pick an indecomposable lattice M_0 which is not projective, and let*

$$\Pi:\quad \begin{tikzcd} 0 \arrow[r] & M_0 \arrow[r] & P_1 \arrow[dr] & & \cdots \arrow[dr] & & P_1 \arrow[r] & M_0 \arrow[r] & 0 \\ & & & M_{2e} \arrow[ur] & & M_1 \arrow[ur] & & & \end{tikzcd}$$

be a minimal projective resolution of M_0. Then the P_i are indecomposable, and their distribution is obtained by "walking around the Brauer tree" [G1]. *Moreover, Π is the Auslander–Reiten quiver of B after identification of isomorphic modules in Π.*

(3.38) THEOREM [Wi2]. *Let B be a block with cyclic vertex of order p^2. Then the tree class of the stable Auslander–Reiten quiver of B is D_{2p}.*

P. Webb [We2] has given a description of the connected components of the Auslander–Reiten quiver of blocks of infinite lattice type. He has also shown that all lattices in one connected component have the same complexity, and in case the defect group is not cyclic also the same vertex.

Schurian orders of finite type

In this section let Λ be a Schurian R-order with associated partially ordered set $P(\Lambda)$ (cf. § 11). We then have

(3.39) THEOREM [ZK]. *Let Λ be a basic Schurian order. Then Λ is of finite lattice type if and only if $P(\Lambda)$ does not contain a critical partially ordered set* (cf. § 11).

Kleiner [Kl2] has proved that a finite partially ordered set is of finite type if and only if it does not contain a critical subset. The proof of the above theorem is not constructive; in fact, Zavadskij and Kirichenko conjecture: "All indecomposable, admissible lattices over Λ of finite lattice type are in natural one-to-one correspondence with exact indecomposable representations of all exact subsets of $P(\Lambda)$, which are mutually inequivalent."

In order to state the main results of [RW1–2], we have to introduce some *notation.* Let

$$H_m = \begin{bmatrix} R & \pi\cdot R & \pi\cdot R & \dots & \pi\cdot R \\ R & R & \pi\cdot R & \dots & \pi\cdot R \\ \dots & \dots & \dots & \dots & \dots \\ R & R & R & \dots & \pi\cdot R \\ R & R & R & \dots & R \end{bmatrix}_{m\times m}$$

be the "canonical" form of a minimal hereditary order with m nonisomorphic indecomposable lattices, and put

$$J_m = \operatorname{rad} H_m = \begin{bmatrix} \pi\cdot R & \pi\cdot R & \dots & \pi\cdot R & \pi\cdot R \\ R & \pi\cdot R & \dots & \pi\cdot R & \pi\cdot R \\ \dots & \dots & \dots & \dots & \dots \\ R & R & \dots & R & \pi\cdot R \end{bmatrix}_{m\times m}$$

We define $J_m^0 = H_m$ and inductively $J_m^k = J_m^{k-1}\cdot J_m$. If $\Lambda = (\pi^{n_{ij}}\cdot R)_n$, $n_{ij}\in \mathbb{N}_0$, is a Schurian order, then

$$m^*\Lambda =: (J_m^{n_{ij}})_n$$

is a Schurian order in $(K)_{m\cdot n}$. W. Rump has proved the following interesting

(3.40) THEOREM [Ru]. *Let Λ be a Schurian order in $(K)_n$. Then Λ is of* *finite lattice type if and only if this holds for $m^*\Lambda$ for every $m \in \mathbf{N}$. Moreover, each indecomposable Λ-lattice "gives rise" to m nonisomorphic $m^*\Lambda$-lattices.*

Let now $P = (\{1, .., n\} \times \mathbf{Z}, \leqslant)$ be a Schurian partially ordered set (cf. § 11). A *bounded representation*

$$V = \{V(\omega); V(i, \alpha)\} \quad \text{of } P \text{ over } \mathfrak{k}$$

consists of a collection $V(i, \alpha)$, $(i, \alpha) \in P$, of $\mathfrak{k}$-subspaces of the finite-dimensional $\mathfrak{k}$-vector space $V = V(\omega)$, satisfying $V(i, \alpha) = 0$ and $V(i, \beta) = V(\omega)$ for α sufficiently small and for β sufficiently large resp.; moreover, $V(i, \alpha)$ is a subspace of $V(j, \beta)$ provided $(i, \alpha) \leqslant (j, \beta)$. These representations form an additive category $\mathfrak{R}^0(P)$, which has kernels, and in which the Krull–Schmidt theorem is valid. The collection of isomorphism classes of indecomposable objects in $\mathfrak{R}^0(P)$ will be denoted by $\operatorname{ind} P$.

The *dimension vector* of $V = \{V(\omega); V(i, \alpha)\}$ in $\mathfrak{R}^0(P)$ is the infinite-dimensional integral vector

$$\operatorname{Dim} V = \big(\dim_{\mathfrak{k}} V(i, \alpha)\big).$$

The *dimension type* of V is the vector

$$\dim V = \big(\dim_{\mathfrak{k}} V(\omega), x_{i,\alpha}\big), \quad \text{where}$$

$$x_{i,\alpha} = \dim_{\mathfrak{k}} V(i, \alpha) - \dim_{\mathfrak{k}}\Big(\sum_{(j,\beta) < (i,\alpha)} V(j, \beta)\Big).$$

Note that because of the definition of a bounded representation, $\dim V$ has only finitely many entries different from zero.

The automorphism $\hat{\sigma}$: $P \to P$, $(i, \alpha) \to (i, \alpha - 1)$ induces an automorphism σ: $\mathfrak{R}^0(P) \to \mathfrak{R}^0(P)$, $V \to \sigma V$, by putting

$$\sigma V(i, \alpha) = V\big(\hat{\sigma}(i, \alpha)\big) = V(i, \alpha - 1),$$

which restricts to $\operatorname{ind} P$, i.e. respects indecomposability. Inductively one defines $\sigma^{\gamma}(V) = \sigma\big(\sigma^{\gamma-1}(V)\big)$, $\gamma \in \mathbf{N}$. Then one has

(3.41) THEOREM [RW1-2]. (i) *$\mathfrak{R}^0(P)$ has almost split sequences and σ induces an automorphism of the Auslander–Reiten quiver $\mathfrak{A}(\mathfrak{R}^0(P))$ of $\mathfrak{R}^0(P)$.*

(ii) *The following conditions are equivalent:*

(a) *P does not contain a critical subset.*

(b) *Modulo σ, $\operatorname{ind} P$ contains only finitely many nonisomorphic objects.*

(c) *$\mathfrak{A}(\mathfrak{R}^0(P))$ contains a connected component Δ such that for every $V \in \Delta$, $\operatorname{Dim} V$ is uniformly bounded. In this case*

$$\operatorname{Dim} V \leqslant (6; \ldots, 6, 6, 6, \ldots).$$

(iii) *If P does not contain a critical subset, then*

(a) $\operatorname{End}(V) \simeq \mathfrak{k}$ *for each* $V \in \operatorname{ind} P$.

(b) *For each dimension vector (dimension type) there is at most one indecomposable representation in $\mathfrak{R}^0(P)$ having this dimension vector (dimension type).*

(c) *Modulo σ, the number of indecomposable representations in $\mathfrak{R}^0(P)$ does not depend on the field $\mathfrak{k}$.* •

We shall assume for the time being that $R = \mathfrak{k}[\![t]\!]$ and $\pi = t$. Next we define a functor

$$\mathfrak{F}\colon \mathfrak{R}^0(P(\Lambda)) \to {}_\Lambda\mathfrak{M}^0.$$

Before we can do so we have a closer look at a module M in ${}_\Lambda\mathfrak{M}^0$. We note that $M_i = e_i M$, where $e_i = e_{ii}$, $1 \leqslant i \leqslant n$, is a full R-lattice in $e_i KM$, and $M = \bigoplus_{i=1}^n M_i$. Via the multiplication with $e_{ij} \in \Lambda = (K)_n$, all the K-vector spaces $e_i KM$, $1 \leqslant i \leqslant n$, can be identified. In particular, $M_1, \ldots, M_n$ can be viewed as full R-lattices in a common vector space. With this identification, M is characterized by the condition that $t^{ij} \cdot M_j \subset M_i$, $1 \leqslant i, j \leqslant n$. Now we are in a position to define $\mathfrak{F}$: Given $V = \{V(\omega);\ V(i, \alpha)\} \in \mathfrak{R}^0(P(\Lambda))$, we put

$$M_i = \mathfrak{F}(V) = \bigoplus_{\alpha \in \mathbf{Z}} t^\alpha \cdot V(i, \alpha).$$

Then it is easily seen that the family M_i, $1 \leqslant i \leqslant n$, gives rise to a Λ-lattice. Similarly morphisms are defined.

(3.42) THEOREM [RW1–2]. *With the above notation*:

(i) *If* $V \in \operatorname{ind} P(\Lambda)$, *then* $\mathfrak{F}(V) \in \operatorname{ind} \Lambda$.

(ii) $V \in \operatorname{ind} P(\Lambda)$ *is a projective (injective) object if and only if $\mathfrak{F}(V)$ is a projective (injective) Λ-lattice.*

(iii) *If $\varphi\colon V \to V'$ is an irreducible map in $\mathfrak{R}^0(P(\Lambda))$, then $\mathfrak{F}(\varphi)\colon \mathfrak{F}(V) \to \mathfrak{F}(V')$ is an irreducible map in ${}_\Lambda\mathfrak{M}^0$. Moreover, $\mathfrak{F}$ maps almost split sequences in $\mathfrak{R}^0(P(\Lambda))$ to almost split sequences in ${}_\Lambda\mathfrak{M}^0$.*

(iv) *For V, V' in* $\operatorname{ind} P(\Lambda)$, *$\mathfrak{F}(V) \simeq \mathfrak{F}(V')$ if and only if $V \simeq \sigma^\gamma(V')$ for some $\gamma \in \mathbf{Z}$.*

From this one can derive

(3.43) THEOREM [RW1–2]. (i) *$\mathfrak{F}$ induces a map*

$$\mathfrak{F}\colon \mathfrak{A}(\mathfrak{R}^0(P(\Lambda))) \to \mathfrak{A}(\Lambda),$$

and $\operatorname{Im} \mathfrak{F}$ *is a union of connected components of $\mathfrak{A}(\Lambda)$.*

(ii) (a conjecture of Zavadskij and Kirichenko [ZK]) *If Λ is of finite lattice type, then—up to isomorphism—*$\operatorname{ind} \Lambda = \mathfrak{F}(\operatorname{ind} P(\Lambda))$, *and*

$$\mathfrak{A}(\Lambda) \simeq \mathfrak{A}(\mathfrak{R}^0(P(\Lambda)))/\sigma.$$

(3.44) *Remarks.* 1. One can show that the construction of Rump [Ru], passing from Λ to $m^*\Lambda$, just gives an m-fold covering of $\mathfrak{A}(\Lambda)$—replace σ by σ^m.

2. In the case of finite representation type there is an algorithm to construct the indecomposable Λ-lattices explicitly [RW1–2].

3. The passage from rings of power series to arbitrary complete Dedekind domains is done using model-theoretic results of Ax and Kochen [AK] and of Herrmann, Jensen and Lenzing [HJL].

Auslander–Reiten quivers of orders of global dimension 2

For an R-order Λ the condition $\mathrm{gl.dim}\,\Lambda = 2$ is equivalent to the condition that ${}_{\Lambda}\mathfrak{M}^0$ has homological dimension 1, and so in some sense orders of global dimension 2 should behave like hereditary artinian algebras. For hereditary artinian algebras M. Auslander and M. I. Platzeck [AP] have defined a natural "Coxeter transformation", which allows one to construct the Auslander–Reiten quiver and the indecomposable modules. Unfortunately, examples show that such a nice description is not possible for orders of global dimension 2. For a small class of orders we have a positive result though:

(3.45) THEOREM [R7]. *Let Λ be an R-order with the following properties:*

(i) $\mathrm{gl.dim}\,\Lambda \leqslant 2$.

(ii) *There exists a maximal R-order Γ with* $\mathrm{rad}\,\Gamma \subset \Lambda \subset \Gamma$.

(iii) $\Lambda/\mathrm{rad}\,\Gamma$ *is a hereditary $\mathfrak{k}$-algebra.*

Then the following are equivalent:

1. *Λ is of finite lattice type.*

2. *Every $M \in \mathrm{ind}\,{}_{\Lambda}\mathfrak{M}^0$ is of the form $\tau^{-1}(P)$ for some indecomposable projective Λ-lattice P. Here τ^{-1} denotes the Auslander–Reiten translate, i.e. the left-hand side of the Auslander–Reiten sequence.*

Contrary to the artinian case, τ does not act linearly on the Grothendieck group of Λ-lattices.

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