

MODULI OF PLANE CURVE SINGULARITIES WITH C^* -ACTION

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1. Introduction

The aim of this paper is to describe the moduli space of plane curve singularities with a fixed topological type, i.e. of the topological type of $(X_0, 0)$ defined by the zero set of a weighted homogeneous polynomial f with weights w_1 and w_2 and degree d in C^2 , C the field of complex numbers. If $w_1 = b$ and $w_2 = a$ and a and b are relatively prime, this is the moduli space of irreducible plane curve singularities with the semi-group $\langle a, b \rangle$. There was already an approach by Washburn (cf. [9]) but it turned out to be wrong in general.

We use the following approach: Let $\mathfrak{X} \rightarrow \mathfrak{S}$ be a good representative of the versal μ -constant deformation, μ the Milnor number of $(X_0, 0)$. Because of the C^* -action \mathfrak{S} can be chosen as a Zariski open subset of C^n and \mathfrak{X} a hypersurface in $\mathfrak{S} \times C^2$. \mathfrak{S} already "contains" all singularities we are hunting for. Along the integral manifolds of the kernel V of the Kodaira–Spencer map the family $\mathfrak{X} \rightarrow \mathfrak{S}$ is analytically trivial. To obtain a moduli space we have to look for the quotient of \mathfrak{S} by the group $G = \exp V$. Obviously, a good quotient can only exist on the strata \mathfrak{S}_τ of fixed orbit dimension. $\{\mathfrak{S}_\tau\}$ turns out to be the stratification defined by fixing the Tjurina number τ , i.e. the dimension of the base of the versal deformation of the singularity in the corresponding fibre of the family.

Notice that the τ -constant stratum in the whole versal deformation of $x^a + y^b$ is in general not contained in the μ -constant stratum \mathfrak{S} .

The quotient \mathfrak{S}_τ/G always exists in the analytic category (cf. [4]). In the

case of $w_1 = b$ and $w_2 = a$, a and b being relatively prime, it is the moduli space of all plane curve singularities with semi-group $\langle a, b \rangle$ and Tjurina number τ . If a and b are not prime, the corresponding moduli space is a quotient of \mathfrak{S}/G by a finite group.

We prove that on the open stratum $\mathfrak{S}_{\tau_{\min}}$ the quotient $\mathfrak{S}_{\tau_{\min}}/G$ in the sense of Mumford (cf. [5]) does exist and is a quasi-smooth algebraic variety, i.e. locally an open subset of a weighted projective space. For a and b relatively prime the dimension of this variety has already been computed by Delorme [2]. We give an algorithm to compute the dimension in general and some explicit formulas. Except the case $a = b$ and even we prove that the quotient of $\mathfrak{S}_{\tau_{\min}}/G$ by the action of the finite group is an algebraic variety. In the exceptional case we do not know whether this finite group acts algebraically.

The other cases of possible weights are treated similarly. It turns out that $\tau \geq \tau'$ implies $\dim \mathfrak{S}_{\tau}/G \leq \dim \mathfrak{S}_{\tau'}/G$, i.e. especially $\dim \mathfrak{S}_{\tau_{\min}}/G = m + \tau_{\min} - \mu$ is the modality of $x^a + y^b$ with respect to the μ -constant stratum, under the action of the contact group (m the modality with respect to right equivalence, cf. [1]). We give an example showing that this is not true for surface singularities in C^3 . In this example \mathfrak{S}_{τ} may be empty for some τ , $\tau_{\min} \leq \tau \leq \mu$. This is however not possible for curves.

2. The Kodaira–Spencer map of the versal μ -constant family

In this chapter we will study the Kodaira–Spencer map of the versal μ -constant family $\mathfrak{X} \rightarrow \mathfrak{S}$ of the germ of the singularity $(X_0, 0)$ defined by the polynomial $f = x^a + y^b$, $a \leq b$, in $(C^2, 0)$, f is quasi-homogeneous with weights b, a and degree $d = ab$. Let $B = \{m_1, \dots, m_{\mu}\}$ be a monomial base of $R_0 = C[x, y]/\Delta f$, $\Delta f = (\partial f/\partial x, \partial f/\partial y) = (x^{a-1}, y^{b-1})$, ordered by degree and the lexicographic order. This order can be realized by changing the degree-function a little bit:

$$\deg(x^e y^{e'}) = eb + e'a, \quad dg(x^e y^{e'}) = e(b - 1/z) + e'a$$

with a suitable $z \geq b$ such that $(zb - 1, za) = 1$. If $(a, b) = 1$, we could use $dg = \deg$. We may choose $m_1 = 1$ and the Hessian $m_{\mu} = x^{a-2} y^{b-2}$.

If it does not lead to any confusion we will not distinguish between the monomials and their exponents.

Denote by (\cdot, \cdot) the bilinear form defined by the coefficient of the Hessian m_{μ} of the product in R_0 : Let $f, g \in R_0$ and $fg = a_1 m_1 + \dots + a_{\mu} m_{\mu}$, then $(f, g) := a_{\mu}$. Denote by $B^0 = \{n_1, \dots, n_{\mu}\}$ the dual basis with respect to this bilinear form, i.e. $n_i = m_{\mu}/m_i$. Denote

$$B_u = \{n_1, \dots, n_r, \deg(n_i) > d\}$$

(notice that for $q = \gcd(a, b)$, $r = (a-3)(b-3)/2 + [b/a] - (q+1)/2$ plus 1 if $a = b$ and plus 0 otherwise).

$$\begin{aligned}
 B_l &= \{m_1, \dots, m_r\} = B_u^0, \\
 B_0 &= \{n_{r+1}, \dots, n_{r+s}, \deg(n_i) = d\}, \\
 F_0 &= f + \sum_{i=r+1}^s t_i n_i \quad \text{and } D \text{ its discriminant,} \\
 F &= f + \sum_{i=1}^s t_i n_i
 \end{aligned}$$

(notice that $r = s$ and $D = 1$ in case $(a, b) = 1$)

$$\begin{aligned}
 H &= C[t_1, \dots, t_s]_D \quad \text{and} \quad H_0 = C[t_{r+1}, \dots, t_s]_D, \\
 \mathfrak{H} &= \text{Spec } H \quad \text{and} \quad \mathfrak{X} = \text{Spec } H[x, y]/F.
 \end{aligned}$$

Because of the C^* -action $\mathfrak{X} \rightarrow \mathfrak{H}$ is a good representative of the versal μ -constant deformation of $(X_0, 0)$. Obviously, there is a C^* -action on \mathfrak{H} and the Lie algebra of derivations $\text{Der } H$ defined by

$$\deg(t_i) = d - \deg(n_i) \quad \text{and} \quad \deg(\partial/\partial t_1) = -\deg(t_1).$$

The Kodaira–Spencer map is given by the map

$$\begin{aligned}
 g: \text{Der}(H) &\rightarrow H[x, y]/(F, \Delta F), \\
 \Delta F &= (\partial F/\partial x, \partial F/\partial y), \quad g(d) = \text{cls}(dF) \quad (\text{cf. [4]}).
 \end{aligned}$$

We will study the kernel $V = \text{Ker}(g)$ of the Kodaira–Spencer map.

It is not difficult to see that V is a graded Lie algebra. Denote by V_+ the Lie algebra of all vector fields of V of degree > 0 , by V_0 the Lie algebra generated (as H_0 -module) by V_+ and the Euler vector field of $\text{Der } H$.

PROPOSITION 2.1. *V_0 is an H_0 -module of finite rank and V_0 generates V as an H -module. V_+ is nilpotent and V_0 is solvable.*

Proof. $H[x, y]/\Delta F =: R$ is a free H -module generated by the elements of B . The multiplication by F is an H -linear map. Denote by $K = (k_{ij})$ the matrix of this map with respect to the bases B and B^0 . The image of F is contained in the submodule generated by the elements of B_u . Denote by d_i the vector field $\sum_{j=1}^r k_{ji} \partial/\partial t_j$. By the definition of V the vector fields d_i , $i = 1, \dots, r$, generate V ; notice that $d_i = 0$ if $i > r$. By definition, d_i is homogeneous of degree $\deg(m_i)$, i.e. an element of V_0 . □

For homogeneous vector fields d and d' we have

$$\deg([d, d']) \geq \deg(d) + \deg(d'),$$

but the degree of a vector field of V_0 is bounded by $\deg(m_r)$; notice that the elements of H have negative degree, i.e. V_+ is nilpotent. Finally, V_+

$= [V_0, d_1]$, $d \cdot d_1$ is the Euler vector field on H and $[d_1, d] = \text{deg}(d/d) d$.

Remark 2.2. It is not difficult to prove that

$$[d_i, d_j] = (\text{deg}(d_j) - \text{deg}(d_i))/d \cdot d_k + d$$

if $m_i m_j = m_k$, d a homogeneous vector field of the submodule generated by d_{k+1}, \dots, d_r .

The way to study the kernel of the Kodaira–Spencer map and the action on H is to study the Lie algebra V_0 and the matrix corresponding to its generators d_i . Because of $m_i F = 0$ in R if $i > r$ and the image of the multiplication by F being contained in the submodule generated by n_1, \dots, n_r we may consider F to be a map on these submodules.

Denote by M_l the submodule of R generated by B_l and by M_u the submodule of R generated by B_u . For technical reasons we will consider the map $E: M_l \rightarrow M_u$ corresponding to the multiplication by $-dF$.

Let us denote by $C(t)$ the matrix corresponding to E with respect to the bases B_l and B_u and by $CL(t)$ the matrix of the linear terms with respect to t of $C(t)$.

The following lemma will give some simple properties of these matrices.

LEMMA 2.3. 1) $CL(t)$ is symmetric;

2) $C_{ij}(t) = 0$ iff $CL_{ij}(t) = 0$;

3) let $j_i = \max \{j, C_{ji} \neq 0\}$, then $j_i \geq j_{i+1}$;

4) for $j \leq j_i$ there are integers $k(i, j)$ with the following properties: $k(i, j) < k(i, j+1)$, $k(i, j) < k(i+1, j)$;

5) if $j = j_i$, then $CL_{ij} = (\text{deg}(n_{k(i,j)}) - d)t_{k(i,j)}$.

Proof. 1) is clear because of the choice of the bases. Now, $x^e y^{e'} \in B_l$ iff $eb + e'a < ab - 2b - 2a$, i.e. $m_i m_j \in B_l$ implies $m_i m_k \in B_l$ for $k \leq j$. Let j_i be maximal such that $m_i m_{j_i} \in B_l$. For $j \leq j_i$ define $k(i, j)$ by $m_i m_j = m_{k(i,j)}$. Then we have also $m_i n_{k(i,j)} = n_j$. By the definition of E ,

$$Em_i = \sum_{j=1}^r (\text{deg}(n_j) - d) d_j n_j,$$

i.e.

$$Em_i = \sum_{j=1}^r (\text{deg}(n_j) - d) t_j n_j m_i = \sum_{j=1}^r C_{ji} n_j.$$

If $j \leq j_i$ then $n_j = m_i n_{k(i,j)}$, i.e.

$$CL_{ji} = (\text{deg}(n_{k(i,j)}) - d)t_{k(i,j)}.$$

Suppose now $C_{ji} \neq 0$ for $j > j_i$ and choose j maximal with this property; then $0 = Em_i m_j = C_{ji} n_1$ which is a contradiction.

Remark. In the higher dimensional case, B_l cannot be characterized simply by degree. This is the reason why 2) of the lemma fails.

PROPOSITION 2.4 (T. Yano). *There is a basis B'_i of M_i with the following properties:*

- 1) $m'_i = m_i + h_i$, h_i homogeneous of degree m_i in the submodule generated by h_{i+1}, \dots, h_r ;
- 2) the matrix of E with respect to B'_i is symmetric with linear part $CL(t)$.

Proof: Consider, on $M_i \times M_u$, the pairing (\cdot, \cdot) defined by the coefficient of the Hessian of the product of two elements of M_i , resp. M_u . Let $h, g \in R$ and $hg = c_1 n_1 + \dots + c_r n_r$; then $(h, g) = c_1 \cdot (\cdot, \cdot)$ has the following properties:

- (i) $(m_i, n_i) = 1$;
- (ii) $(m_i, m_j) = 0$ if $i + j < r$;
- (iii) if $(m_i, n_j) \neq 0$, then it is homogeneous of degree $\deg(m_i) + \deg(n_j) - \deg(n_1)$.

Denote by K the matrix of this pairing with respect to the bases B_i and B_u . Obviously, the map E induces a symmetric bilinear form on M_i defined by (\cdot, E) with a symmetric matrix G with respect to the basis B_i .

Notice that $(m_i, Em_j) = 0$ if $m_i \cdot m_j \notin B_i$! Now
 (iv) $G = KC(t)$.

This implies that $C(t)K^{-1}$ is symmetric. The base change induced by K^{-1} on M_i has the required properties (because of (i) to (iv)). □

Remark. A base change in M_i corresponds to a choice of other generators of the kernel of the Kodaira–Spencer map.

COROLLARY 2.5. *There is a basis B'_u of M_u and an automorphism w of H with the following properties:*

- 1) w is homogeneous;
- 2) $n'_i = n_i + h_i$, h_i homogeneous of degree n_i in the submodule generated by n_1, \dots, n_{i-1} ;
- 3) the matrix of E with respect to B_i and B'_u is $CL(w(t))$.

Proof. With the notations of the proposition the base change on M_u induced by K has the required property 2) and the matrix of E with respect to B_i and B'_u is G . Now $(m_i, Em_j) = 0$ if $m_i m_j$ is not in B_i . On the other hand, if $Em_i = \sum_{j=1}^r C_{ji} n_j$ and $m_i m_j = m_k$, $k = k(i, j)$, then $(m_i, Em_j) = C_{k_i}$. But $C_{k_i} = (\deg(n_k) - d)t_k + \text{higher order terms}$ and $(\deg(n_k) - d)t_k$ is the ij -th element in $CL(t)$.

Let us consider the example $f = x^5 + y^{11}$:

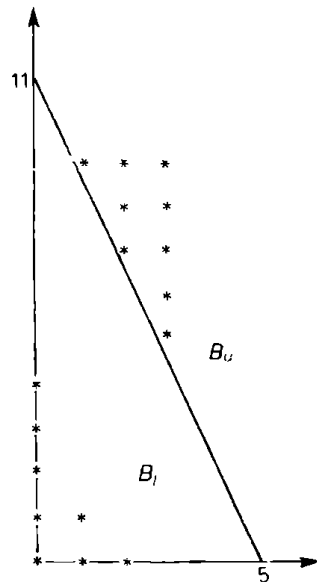
$$B_i = \{1, y, y^2, x, y^3, xy, y^4, xy^2, x^2\}.$$

$$B_u = \{x^3 y^9, x^3 y^8, x^3 y^7, x^2 y^9, x^3 y^6, x^2 y^8, x^3 y^5, x^2 y^7, xy^9\},$$

cf. the corresponding Newton diagram of f :

$$r = 9, \quad \mu = 40,$$

$$H = C[t_1, \dots, t_9].$$



If we choose a base B'_i given by the proposition we get the following symmetric matrix $C' = (C'_{ji})$ and generators of the kernel of the Kodaira-Spencer map $\sum C'_{ji} \frac{\partial}{\partial t_j}$.

$$\begin{bmatrix} 23t_1 & 18t_2 & 13t_3 & 12t_4 & 8t_5 & 7t_6 & 3t_7 & 2t_8 & t_9 \\ 18t_2 & 13t_3 + E & 8t_5 + D & 7t_6 + C & 3t_7 + B & 2t_8 + A & 0 & 0 & 0 \\ 13t_3 & 8t_5 + D & 3t_7 + F & 2t_8 & 0 & 0 & 0 & 0 & 0 \\ 12t_4 & 7t_6 + C & 2t_8 & t_9 & 0 & 0 & 0 & 0 & 0 \\ 8t_5 & 3t_7 + B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7t_6 & 2t_8 + A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3t_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2t_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = -9t_9^2/11, \quad B = -7t_9t_8/11, \quad C = 3t_9t_7^2/11,$$

$$H = -8t_9t_6/11 + 2t_9^2t_7^2/11,$$

$$E = -117t_9t_4/11 + 16t_8t_7t_5/11 - 7t_9t_8t_7t_6/11 + t_9^2t_8t_7^2/121,$$

$$F = t_9t_8.$$

We are now interested in the flattening stratification of $H[x, y]/(F, \Delta F)$ as an H -module. This stratification coincides obviously with the stratification given by the rank of $C(t)$: For $t \in \mathfrak{H}$ denote by $\mu(F(t))$, resp. $\tau(F(t))$, the Milnor number, resp. the Tjurina number, of the singularity in the fibre \mathfrak{X}_t of the family.

Remark 2.6. $\mu = \mu(F(t)) = \tau(F(t)) + rk(C(t))$, $t \in \mathfrak{H}$.

Let us denote by \mathfrak{S}_τ the reduced flattening stratification of \mathfrak{H} , i.e. the underlying set is the set of all $t \in \mathfrak{H}$ with fixed Tjurina number τ .

Remark. In general \mathfrak{S}_τ is not the τ -constant stratum of the whole versal deformation of f : The family $x^5 + y^{11} + tx^2y^7 + 2t'x^4y^2 + t'^2x^3y^4$ is for small $t \neq 0$ and arbitrary t' τ -constant ($\tau = 34$) but not μ -constant ($\mu(t' = 0) = 40$, $\mu(t' \neq 0) = 39$).

In our example we get the following stratification:

$$\mathfrak{S}_{34}: 4t_8^2 - 3t_9t_7 - t_9^2t_8 =: g \neq 0;$$

$$\mathfrak{S}_{35}: g = 0 \text{ and } 2t_8 + A \neq 0 \text{ or } 3t_7 + B \neq 0 \text{ or } 4t_8(7t_6 + C)(8t_5 + D) - t_9(8t_5 + D)^2 - (3t_7 - t_9t_2)(7t_6 + C)^2 =: h \neq 0;$$

$$\mathfrak{S}_{36}: 2t_8 + A = 3t_7 + B = h = 0 \text{ and } (7t_6 + C)^2 - t_9(13t_3 + E) \neq 0 \text{ or } 11(7t_6 + C)(8t_5 + D) - 9t_9^2(13t_3 + E) \neq 0 \text{ or } (8t_5 + D)(9t_9(7t_6 + C) - 11(8t_5 + D)) \neq 0 \text{ or } (7t_6 + C)(9t_9(7t_6 + C) - 11(8t_5 + D)) \neq 0;$$

$$\mathfrak{S}_{37}: \text{the polynomials defining } \mathfrak{S}_{36} \text{ equal to zero and } t_9 \neq 0 \text{ or } 7t_6 + C \neq 0 \text{ or } 8t_5 + D \neq 0 \text{ or } 13t_3 + E \neq 0;$$

$$\mathfrak{S}_{38}: t_9 = \dots = t_5 = t_3 = 0 \text{ and } t_4 \neq 0 \text{ or } t_2 \neq 0;$$

$$\mathfrak{S}_{39}: t_9 = \dots = t_2 = 0 \text{ and } t_1 \neq 0;$$

$$\mathfrak{S}_{40}: t_9 = \dots = t_1 = 0.$$

Notice that \mathfrak{S}_{34} and \mathfrak{S}_{35} are smooth.

Especially, \mathfrak{S}_τ is not empty for $\tau_{\min} \leq \tau \leq \mu$. We will prove that this is always true in our situation.

Remark. \mathfrak{S}_τ may be empty in the surface case:

$$f = x^3 + y^{10} + z^{19}, \quad \mu = 324, \quad \tau_{\min} = 246, \quad \mathfrak{S}_{247} = \emptyset,$$

\mathfrak{S}_{248} is an open set in a hypersurface in \mathfrak{H} .

THEOREM 2.7. $\tau(F(t))$, $t \in \mathfrak{H}$ takes every possible value between τ_{\min} and $\mu = \mu(f)$.

Proof. Because of Corollary 2.5 it is enough to show that for the linear matrix every rank between 0 and the maximal rank is possible. Any minor of the linear matrix $CL(t)$ has the form $M(t) = (d_{a(i,j)}t_{a(i,j)})$ with $a(i, j) \in \{0, \dots, r\}$, $i, j = 1, \dots, m$, $d_{a(i,j)} = \deg(n_{a(i,j)}) - d$ if $a(i, j) \neq 0$ and $d_0 = 0$. The $a(i, j)$ satisfy the following properties (cf. 2.3):

$$a(i, j) \neq 0 \text{ implies } 0 \neq a(i, j-1) < a(i, j) \text{ and } 0 \neq a(i-1, j) < a(i, j).$$

Then

$$m(f, \tau) = \dim \mathfrak{S}_\tau - rk(C(t) | \mathfrak{S}_\tau).$$

Denote by $m(f) = \max \{m(f, \tau)\}$ the modality of f with respect to the μ -constant stratum, under the action of the contact group.

- THEOREM 2.10. 1) $m(f) = \dim \mathfrak{H} + \tau_{\min} - \mu$;
 2) $\tau \leq \tau'$ implies $m(f, \tau) \geq m(f, \tau')$.

Proof. We have to prove 2). Analysing the proof of Theorem 2.7, it is enough to prove the following lemma for any linear matrix similar to $CL(t)$:

LEMMA 2.11. For $i, j = 1, \dots, m$ let $a(i, j) \in \{0, \dots, r\}$ satisfying the property: $a(m-i+1, i) \neq 0$ for $i = 1, \dots, m$ and $a(i, j) \neq 0$ implies $0 \neq a(i-1, j) < a(i, j)$ and $0 \neq a(i, j-1) < a(i, j)$.

Let $d_{a(i,j)} \in \mathbb{C}$ such that $d_0 = 0$ and $d_{a(i,j)} \neq 0$ if $a(i, j) \neq 0$. Let I_p be the ideal generated by the $(p+1)$ -minors of the matrix $M(t) := (d_{a(i,j)} t_{a(i,j)})$. Then

$$\dim C[t_1, \dots, t_r] / I_p \leq r - m + p.$$

Proof. Use induction on m and on the number of different $t_{a(i,j)}$'s contained in the matrix. If $p+1 = m$, the assumption is true, because $\det(M(t)) \neq 0$ (Lemma 2.8).

Let $s = \max \{a(i, j), a(i, j) \neq 0\}$ and let U be a component of the zero set of I_p .

Case 1. U is contained in the hypersurface defined by $t_s = 0$. Consider the matrix $M'(t)$ obtained from $M(t)$ by deleting the last row and the last column. By the definition of s and the properties of the $a(i, j)$, the matrix $M'(t, t_s = 0)$ satisfies the properties required in the lemma and we may apply the induction hypothesis.

Case 2. U is not contained in the hypersurface defined by t_s . Suppose t_s occurs l times in the matrix $M(t)$. Then, on the open set defined by $t_s \neq 0$ the rank of $M(t)$ is at least l . In particular, this implies $l \leq p$. Consider the matrix $M'(t)$ obtained by deleting the rows and columns in which t_s occurs. For $t_s \neq 0$ it is easy to see that $rk(M(t)) = rk(M'(t)) + l$, therefore

$$\{t, rk(M(t)) \leq p\} = \{t, rk(M'(t)) \leq p - l\}.$$

Now $M'(t)$ satisfies the properties required in the lemma and does not contain t_s . We can apply the induction hypotheses, and the lemma is proved.

Remark. Theorem 2.10 fails also in the case of surface singularities. Consider the same example as before: $f = x^3 + y^{10} + z^{19}$, $m(f, 246) = 88$ and $m(f, 248) = 89$.

We will describe the open set $S_{\tau_{\min}}$ more precisely in terms of the matrix C . A decreasing filtration $F^* = F^p M_u$ on M_u is introduced, induced by a

filtration of the monomial base B compatible with the dual filtration on M_l . (The dual filtration F^* on M_l is defined by:

$$m_i \in F^p M_l \quad \text{iff} \quad n_i \in F^{l-p} M_u).$$

Let us denote the matrix corresponding to

$$\text{gr}_p E: \text{gr}_p M_l \rightarrow \text{gr}_p M_u$$

by C^p and the radical of the ideal generated by the maximal minors of C^p by I_p .

LEMMA 2.12. *There is a filtration F^* of M_u satisfying the following properties:*

- 1) $F^0 M_u = M_u \supseteq F^1 M_u \supseteq F^{i+1} M_u \supseteq F^l M_u = 0$;
- 2) $n_i \in F^p M_u$ implies $n_j \in F^p M_u$ if $j < i$;
- 3) $\text{rk}(\text{gr}_p M_l) \leq \text{rk}(\text{gr}_p M_u)$ if $2p \leq l+1$ (rk as H -module);
- 4) the elements of C^p are invariant with respect to the action of V_+ ;
- 5) $I^p = I^{l-p}$ for all p ;
- 6) $I^p \subseteq I^{p-1}$ or $I^p \subseteq I^{p-2}$ if $2p \leq l+1$;
- 7) $m_i \in F^p$ implies $m_i m_j \in F^{p+1}$ if $m_i m_j \in B_l$;
- 8) $m_i \in F^p$ implies $m_i \cdot n_j \in F^p$;
- 9) $S_{\tau_{\min}}$ is the open set defined by $I_{l'}$ or $I_{l'} \cap I_{l'-1}$, $l' := [(l+1)/2]$.

Consider the example $x^5 + y^{11}$. In this case, F^* is the (x, y) -adic filtration:

$\text{gr}_0 M_u$ is generated by $xy^9, x^2 y^7, x^3 y^5$;

$\text{gr}_1 M_u$ is generated by $x^2 y^8, x^3 y^6$;

$\text{gr}_2 M_u$ is generated by $x^2 y^9, x^3 y^7$;

$\text{gr}_3 M_u$ is generated by $x^3 y^8$;

$\text{gr}_4 M_u$ is generated by $x^3 y^9$.

For the graded pieces C^p of the matrix we get:

$$C^0 = (t_9, 2t_8, 3t_7),$$

$$C^1 = (A', B'), \quad A' = 2t_8 + A, \quad B' := 3t_7 + B,$$

$$C^2 = \begin{bmatrix} A' & B' \\ t_9 & 2t_8 \end{bmatrix}, \quad C^3 = \begin{bmatrix} B' - \frac{9}{11} t_8 A' \\ A' \end{bmatrix}, \quad \begin{bmatrix} B' - \frac{9}{11} t_8 A' \\ A' \\ t_9 \end{bmatrix},$$

$$I_0 = I_4 = (t_7, t_8, t_9),$$

$$I_1 = I_3 = (A', B'),$$

$$I_2 = (t_8 A' - t_9 B'),$$

$\tau_{\min} = 34$ and \mathfrak{S}_{34} defined by $t_8 A' - t_9 B' \neq 0$.

Construction of a suitable filtration of the monomial base:

LEMMA 2.13. *There is a map $dh: B \rightarrow N$ having the following properties:*

- 1) dh is injective;
- 2) if $e_1, e_2 \in M_l$ (resp. M_u) and $\deg(e_1) < \deg(e_2)$, then $dh(e_1) < dh(e_2)$;
- 3) for $e_1 \in B_l$ and $e_2 \in B_u$ we have: if for a suitable $e_3 \in B_u$ with $dh(e_3) = dh(e_1) + dh(e_2)$, then we have $e_3 = e_1 + e_2$.
- 4) $dh(e) + dh(e^0) = dh(n_1) =: d_*$,
- 5) Let $k_* := dh((0, 1))$, then

$$0 < \# B_l \cap dh^{-1}(\{k+1, \dots, k+k_*\}) < a-1$$

if $k+k_* \leq d_*$.

- 6) 5) also holds for B_u .

Proof. Define

$$dh(e) := \begin{cases} z \cdot dg(e) & \text{if } \deg(e) < ab, \\ z \cdot dg(e) - a(zb-1) & \text{if } \deg(e) \geq ab. \end{cases}$$

1) It is not difficult to see that dh has an obvious extension to a bijection

$$dh': \{1, \dots, za-1\} \times \{1, \dots, zb-2\} \rightarrow \{1, \dots, za(zb-1)\}$$

defined by

$$dh'(e, e') := (zb-1)e + zae' \text{ mod } za(zb-1).$$

Hence

$$dh'(ze, ze') = z \cdot dh(e, e').$$

2) is clear by definition.

3) Suppose $e_1 \in B_l$; $e_2, e_3 \in B_u$ and $dh(e_3) = dh(e_1) + dh(e_2)$, then we have $dh(e_3) = dh(e_1 + e_2)$ and $\deg e_3 = \deg(e_1 + e_2)$ and this implies $e_3 = e_1 + e_2$.

4) is a consequence of 3).

5) For a fixed number l we regard the following sequence of monomials $(l, 0), (l, 1), \dots, (l, t)$ from B , then the values of dh on the sequence increase by $k_* = dh(0, 1) = za$ as long as $\deg(l, t) < ab$. Suppose $\deg((l, t+1)) > ab$; then $dh(l, t+1)$ is smaller than k_* . Therefore exactly one monomial $(0, t_0) \in B_l$, exactly one monomial $(a-2, t_{a-2}) \in B_u$ and at most one monomial $(l, t_1), 0 < l < a-2$, belong to I_k ,

$$I_k := dh^{-1}(\{k+1, \dots, k+k_*\}). \quad \square$$

LEMMA 2.14. *If $k < (d_* - k_*)/2$, then*

$$\# B_l \cap I_k \leq \# B_u \cap I_k.$$

Proof. Suppose $B_l \cap I_k$ contains l monomials, which necessarily are of the form $(0, t_0), (1, t_1), \dots, (l-1, t_{l-1})$. Then $B_u \cap I_k$ contains either $a-l-1$ monomials $(l, t_l), \dots, (a-2, t_{a-2})$ or $a-l-2$ monomials $(l+1, t_{l+1}), \dots, (a-2, t_{a-2})$.

The second case occurs iff $k < dh(l, b-1) \leq k + k_*$. Hence the statement of the lemma is equivalent to the inequality:

$$dh([a/2], b-1) > (d_* + k_*)/2.$$

A direct computation shows that

$$2 \cdot dh([a/2], b-1) - d_* - k_* = z(b-a) + a - 1 > 0,$$

if a is odd (resp. $z(2b-a) + a - 2 > 0$ if a is even). □

Let

$$c(e) := \# \{h \in B_u, dh(h) \leq dh(e)\} - \# \{h \in B_l, dh(h) \leq dh(e)\},$$

then $c(e) = c(e^0) + 1$ if $e \in B_u$. Let $c = \max c(e)$: then a maximal anti-diagonal of $C(t)$ containing no zeros has length $r - c$; hence $\text{rank } C(t) = r - c$.

Choose $e_* \in B_u$ with $dh(e_*) < d_*/2$, such that $c = c(e_*)$ and $dh(e_*)$ maximal. Let $e_0 \in B_l$ such that $dh(e_0) > dh(e_*)$ and $dh(e_0)$ minimal.

LEMMA 2.15. $dh(e_0^0) - dh(e_0) < k_*$.

Proof. Let $I := dh^{-1}(\{dh(e_0), \dots, dh(e_0) + r_* - 1\})$ and suppose $dh(e_0) \leq (d_* - k_*)/2$; then, by Lemma 2.14, we have

$$\# B_l \cap I \leq \# B_u \cap I.$$

Let $h \in B_u \cap I$ maximal; then $c(h) = c$, and hence $dh(h) > d_*/2$, i.e. $dh(h^0) < d_*/2$ and $c(h^0) = c - 1$ for the dual h^0 .

Let $h_0 \in B_l$ be maximal with $dh(h_0) < dh(h)$; then $c(h_0^0) = c$, hence $dh(h_0^0) \leq dh(e_*)$ and $dh(h^0) \leq dh(e_0)$. We get

$$dh(e_0) + k_* > dh(h) \geq dh(e_0^0)$$

and

$$dh(e_0^0) - dh(e_0) < k_*. \quad \square$$

Let us illustrate this construction by another example, which does not lead to the (x, y) -adic filtration:

$f = x^5 + y^{12}$; take $z = 10$. There is a unique exponent (q, q') , such that $q(zb-1) \equiv 1 \pmod{za}$ and $q'(za) \equiv 1 \pmod{zb-1}$; $q = 29, q' = 50$. Then, for a monomial (e, e') we have:

$$dh(e, e') = n/10 \quad \text{iff} \quad (ze, ze') = (nq \pmod{za}, nq' \pmod{zb-1})$$

i	n_i	j	m_i	$c(e, e')$	$dh(e, e')$
		1	(0, 0)	-1	0
10	(3, 5)			0	12
9	(1, 10)			1	24
8	(2, 8)			2	43
		2	(0, 1)	1	50
7	(3, 6)			2	62
6	(2, 9)			3	93
		3	(0, 2)	2	100
5	(3, 7)			3	112
		4	(1, 0)	2	119
4	(2, 10)			3	143
		5	(0, 3)	2	150
3	(3, 8)			3	162
		6	(1, 1)	2	169
		7	(0, 4)	1	200
2	(3, 9)			2	212
		8	(1, 2)	1	219
		9	(2, 0)	0	238
		10	(0, 5)	-1	250
1	(3, 10)			0	262

$$c = 3; \quad k_* = 50; \quad c_0 = (3, 7); \quad c_* = (1, 0); \quad c_0^0 = (0, 3); \quad c_*^0 = (2, 10).$$

We are now ready to construct the filtration. We start with a filtration on B . Let

$$S^i := dh^{-1}(\{dh(c_0) - ik_*, \dots, dh(c_0^0) - ik_*\})$$

and

$$R^i = dh^{-1}(\{dh(c_0^0) - (i+1)k_* - 1, \dots, dh(c_0) - ik_* - 1\}).$$

Because of $(S^0)^0 = S^0$ and $(R^0)^0 = R^1$ we get

$$(S^i)^0 = S^{-i} \quad \text{and} \quad (R^i)^0 = R^{1-i}.$$

Notice that it is possible that

$$c_*^0 = c_0, \quad \text{i.e. } S^i = \emptyset \text{ for all } i,$$

or

$$c_*^0 = c_0 + (0, 1), \quad \text{i.e. } R^i = \emptyset \text{ for all } i.$$

In these cases we will get the (x, y) -adic filtration on M_u .

Denote by $F^* B$ the filtration defined by S^i and R^i as follows: $F^0 B := B$, suppose $F^i B$ is defined and the minimal element $h \in F^i B$ belongs to S^j (resp. to R^j), $j < 0$, then

$$F^{i+1} := \begin{cases} F^i - S^j, & \text{if } R^{j+1} \cap B_u \neq \emptyset \text{ and } R^{j+1} \cap B_l \neq \emptyset, \\ F^i - S^j - R^{j+1}, & \text{otherwise;} \end{cases}$$

(resp.

$$F^{i+1} := \begin{cases} F^i - R^j, & \text{if } S^j \cap B_u \neq \emptyset \text{ and } S^j \cap B_l \neq \emptyset, \\ F^i - R^j - S^j, & \text{otherwise).} \end{cases}$$

To obtain symmetry we define for $j = 0$

$$F^{i+1} := F^i - S^0 \quad (\text{resp. } F^i - R^0)$$

and if $j > 0$

$$F^{i+1} := \begin{cases} F^i - S^j, & \text{if } S^j \cap B_u \neq \emptyset \text{ and } S^j \cap B_l = \emptyset, \\ F^i - S^j - R^{j+1}, & \text{otherwise} \end{cases}$$

for $h \in R^j$ it is defined in a similar way. Let l be minimal such that $F^l = 0$, then, because of the duality $B_u^0 = B_l$, we have

$$(F^i B)^0 = B - F^{l-i}.$$

F^* induces filtrations on B_u and B_l and by duality we have $(F^i B_u)^0 = B_l - F^{l-i} B_l$.

To illustrate the filtration F^* let us consider the following example:

$$\begin{aligned} S^0 &= \{m_4, n_4\}, & R^0 &= \{m_3, n_5\}, \\ S^{-1} &= \{n_6\}, & R^{-1} &= \{m_2, n_7\}, \\ S^{-2} &= \{n_8, n_9\}, & R^{-2} &= \{m_1, n_{10}\}, \\ \text{gr}_0 F &= \{m_1, n_{10}, n_9, n_8\}, \\ \text{gr}_1 F &= \{m_2, n_7, n_6\}, \\ \text{gr}_2 F &= \{m_3, n_5\}, \\ \text{gr}_3 F &= \{m_4, n_4\}, \\ \text{gr}_4 F &= \{m_5, n_3\}, \\ \text{gr}_5 F &= \{m_6, m_7, n_2\}, \\ \text{gr}_6 F &= \{m_8, m_9, m_{10}, n_1\}. \end{aligned}$$

Let us denote the induced filtration on M_l and M_u also by F^* . This filtration has the properties required in Lemma 2.12.

Proof of Lemma 2.12. 1) and 2) are obvious.

3) $\text{gr}_p M_l$ is generated by $\{x^e y^{e'}, dh(e, e') \in \text{gr}_p B_l\}$ (similarly for $\text{gr}_p M_u$). But $\text{gr}_p B_u$ is $S^j \cap B_u$ or $R^j \cap B_u$ or $(S^j \cup R^j) \cap B_u$ or $(S^j \cup R^{j+1}) \cap B_u$ for some j , similarly for $\text{gr}_p B_l$. Now, by definition

$$\# S^0 \cap B_u = \# S^0 \cap B_l \quad \text{and} \quad \# R^0 \cap B_u \geq \# R^0 \cap B_l.$$

Using Lemma 2.13 we have

$$\# S^j \cap B_u \geq \# S^j \cap B_l, \quad \# R^j \cap B_u \geq \# R^j \cap B_l \quad \text{if } j < 0.$$

4) holds because of the fact that

$$dh(c_1) - dh(c_2) < k_*$$

for any two elements c_1 and c_2 of $\text{gr}_p B$ and that t_q is invariant under the action of V_+ if $q > j_2$ (Lemma 2.4): if an element of C^p depends on t_q with $dh(n_q) > k_*$ (we may assume the linear part to depend on t_q) and this element is in the i th column and the j th row of $C(t)$, then

$$m_j \cdot n_q = n_i \quad \text{and } m_j, n_i \in \text{gr}_p B$$

but then

$$dh(n_q) = dh(n_i) - dh(m_j) < k_*$$

5) is a consequence of the duality and the fact that the change of the matrix $C(t)$ to the symmetric matrix does not change I_{l-p} .

6) is a consequence of Lemma 2.13: Suppose $\text{gr}_p B = S^j$, $j \leq 0$, and $\text{gr}_{p-2} B = S^{j-1}$; then

(i) $\text{gr}_{p-2} B_u = \{e - (0, 1), e \in \text{gr}_p B_u\} \cup L$, L is empty or contains just one element;

(ii) $\text{gr}_{p-2} B_l = \{e - (0, 1), e \in \text{gr}_p B_l\} \setminus T$, T is empty or contains just one element.

Furthermore

$$\# \text{gr}_p B_u \geq \# \text{gr}_p B_l = : d_p.$$

I_{p-2} is the radical of the ideal generated by the d_{p-2} -minors of the matrix C^{p-2} . Let $m_{i(1)}, \dots, m_{i(d-2)}$ generate $\text{gr}_{p-2} M_l$. Suppose I_{p-2} vanishes at a point t ; then the leading forms of $m_{i(1)}E, \dots, m_{i(d-2)}E$ with respect to the graduation, i.e. in $\text{gr}_{p-2} M_u$, are dependent.

Now, because of (ii) the leading forms of $ym_{i(1)}E, \dots, ym_{i(d-2)}E$ define rows of the matrix C^p . By (i) they depend on t , too. This implies that the corresponding d_{p-2} -minors of C^p vanish at t . But $d_p \geq d_{p-2}$ implies that I_p vanishes at t . All the other cases are similar.

7) and 8) are obvious by the definition of F^* .

9) By the choice of e_* , $c(e_*) = c$, the matrix $C(t)$ has a maximal rank $r - c$ at a general point. The rank decreases if the rank of a graded piece decreases. Because of 5) and 6), $S_{\tau_{\min}}$ is defined by $I_{l'} \cap I_{l'-1}$ if $S^0 \neq \emptyset$ and $R^0 \cap B_l \neq \emptyset$ or by $I_{l'}$ if $S^0 \neq \emptyset$ or $R^0 \cap B_l = \emptyset$ ($l' := [(l+1)/2]$).

In the example $x^5 + y^{12}$ we get the following linear matrix:

$$\left[\begin{array}{l} 26t_1 \ 21t_2 \ 16t_3 \ 14t_4 \ 11t_5 \ 9t_6 \ 6t_7 \ 4t_8 \ 2t_9 \ t_{10} \} C^0 \\ 21t_2 \ 16t_3 \ 11t_5 \ 9t_6 \ 6t_7 \ 4t_8 \ t_{10} \} C^1 \\ 16t_3 \ 11t_5 \ 6t_7 \ 4t_8 \ t_{10} \} C^2 \\ 14t_4 \ 9t_6 \ 4t_8 \ 2t_9 \} C^3 \\ 11t_5 \ 6t_7 \ t_{10} \} C^4 \\ \left. \begin{array}{l} 9t_6 \ 4t_8 \\ 6t_7 \ t_{10} \end{array} \right\} C^5 \\ \left. \begin{array}{l} 4t_8 \\ 2t_9 \\ t_{10} \end{array} \right\} C^6 \end{array} \right]$$

$$I_2 = (t_{10}), \quad I_3 = (2t_9 - 3t_{10}^2/5);$$

$S_{\tau_{\min}}$ is the open set defined by $t_{10}(2t_9 - 3t_{10}^2/5)$.

From the last results one can obtain formulas for the maximal rank of the matrix $C(t)$. If $(a, b) = 1$, we represent

$$\frac{b}{a} = [r_1, \dots, r_k] = r_1 + \frac{1}{r_2 + \frac{1}{r_3 + \dots}}$$

as a continued fraction, then define l_i and t_i inductively:

$$l_k = 0, \quad t_k = 1;$$

$$l_{i-1} = l_i + t_i r_i \text{ and } t_{i-1} = 0 \text{ if } t_i = 1 \text{ and } l_i \text{ even;}$$

$$t_i = 1 \text{ otherwise.}$$

COROLLARY (cf. [2]).

$$\text{rank } C(t) = (a-2)(b-2)/4 - (l_0-2)/4 + t_1(r_1 + t_2 - 2)/2.$$

If, for instance, $b = r \cdot a \pm 1$, we get

$$\text{rank } C(t) = \begin{cases} (a-2)(b-3)/4, & a \text{ even,} \\ (a-1)(b-r-3)/4, & a \text{ odd.} \end{cases}$$

COROLLARY (Herφy). If $(a, b) = q$ and q even

$$rk - \max C(t) = (a-2)(b-2)/4 - q/2 + \begin{cases} 1 & \text{if } q = a, \\ 0 & \text{otherwise.} \end{cases}$$

In the cases not covered by these formulas the definition of the function $c(e)$ and the construction of e_* in Lemma 2.15 gives an algorithm to compute the maximal rank.

3. Moduli of irreducible plane curve singularities with semi-group $\langle a, b \rangle$

We shall now construct the moduli space of irreducible plane curve singularities with the semi-group $\Gamma = \langle a, b \rangle$ and minimal Tjurina number τ .

There was already an approach by Washburn (cf. [9]) but his construction is wrong in general, also his dimension formula. He uses as filtration F^* the (x, y) -adic filtration on M_u and M_l . But the corresponding graduation is not compatible with the multiplication $E: M_l \rightarrow M_u$. This is true only in very special cases (cf. Lemma 2.13), not true for $a = 5, b = 12$.

THEOREM 3.1. *Let $\Gamma = \langle a, b \rangle, (a, b) = 1$, be a semi-group. There exists a fine moduli space with universal family $\pi: X_{\tau_{\min}}^{\Gamma} \rightarrow T_{\tau_{\min}}^{\Gamma}$ parametrizing all plane curve singularities with the semi-group $\bar{\Gamma}$ and minimal Tjurina number τ_{\min} .*

1) $T_{\tau_{\min}}^{\Gamma}$ is a quasi-smooth scheme, i.e. locally an open subset in a weighted projective space of dimension $(a-4)(b-4)/4 + l_0/4 + (2-t_1)(r_1-2)/2 - t_1 t_2/2$.

2) $X_{\tau_{\min}}^{\Gamma}$ is an algebraic space and there is an affine covering $\{U_i\}$ of $T_{\tau_{\min}}^{\Gamma}$ such that $\pi^{-1}(U_i)$ are affine schemes.

Proof. Suppose $a < b$. For short, let $\tau := \tau_{\min}$. Let $(X_0, 0)$ be a germ of a plane curve with singularity at 0, having Γ as semi-group. Consider $\mathfrak{X} \rightarrow \mathfrak{H}$ to be the versal μ -constant deformation of the singularity defined by $x^a + y^b$.

There is a $t \in \mathfrak{H}$ such that $(X_0, 0) \simeq (\mathfrak{X}_t, 0)$ (cf. [1]).

LEMMA 3.2. *If for $t^1, t^2 \in \mathfrak{H} (\mathfrak{X}_{t^1}, 0) \simeq (\mathfrak{X}_{t^2}, 0)$, then t^1 and t^2 are in an analytically trivial subfamily of $\mathfrak{X} \rightarrow \mathfrak{H}$.*

Proof. The C^* -action induces a canonical filtration on the automorphism group E of $C[[x, y]]$:

$$E_l = \{\varphi \in E, \deg(\varphi(x) - x) \geq l + a, \deg(\varphi(y) - y) \geq l + b\},$$

$$\deg(\varphi) := l \quad \text{iff} \quad \varphi \in E_l - E_{l+1}.$$

Suppose $(X_{t^1}, 0) \simeq (X_{t^2}, 0)$, i.e. there is an $\varphi \in E$ and a unit $u \in C[[x, y]]$ such that

$$F(x, y, t^1) = u(x, y) F(\varphi(x), \varphi(y), t^2).$$

We will prove that $\deg \varphi \geq 0$.

If this is true t^1, t^2 are in an analytically trivial subfamily of $\mathfrak{X} \rightarrow \mathfrak{H}$ induced by the C^* -action. Consider the map induced by φ and the corresponding map $\bar{\varphi}$ of the normalizations.

$$\begin{array}{ccc} \varphi: C[[x, y]]/F(x, y, t^1) & \longrightarrow & C[[x, y]]/F(x, y, t^2) \\ \parallel & & \parallel \\ C[[t^a + \text{higher order}, t^b + \dots]] & \longrightarrow & C[[t^a + \dots, t^b + \dots]] \\ \text{ni} & & \text{ni} \\ \bar{\varphi}: C[[t]] & \longrightarrow & C[[t]] \\ \varphi(t) = t \cdot h(t), & & h(t) \text{ a unit in } C[[t]]. \end{array}$$

Then it is clear that $\text{deg } \varphi \geq 0$.

Using the lemma, we get $T_\tau^f = \mathfrak{S}_\tau/V$, \mathfrak{S}_τ the open stratum in the flattening stratification of \mathfrak{H} with respect to the kernel of the Kodaira–Spencer map V . By general results,

$$X_\tau^f \rightarrow T_\tau^f = \mathfrak{S}_\tau/V$$

exists in the category of algebraic spaces (cf. [4], [6]).

We will prove that \mathfrak{S}_τ/V is locally an open subset in a weighted projective space. V is a graded Lie algebra generated as an H -module by the elements d_i , $\text{deg } d_i = \text{deg } m_i$ (cf. Proposition 2.1). It is enough to study $\mathfrak{S}_\tau/V_0 = (\mathfrak{S}_\tau/V_+)/C^*$ ($\exp V_+$ is a normal subgroup in $\exp V_0$ and $\exp V_0/\exp V_+ = C^*$).

LEMMA 3.3. *Let R be a commutative algebra over a field k . $d_1, \dots, d_q \in \text{Der}_k A$ with the following properties:*

- (i) $[d_i, d_j] = 0$ for all i, j ;
- (ii) d_i nilpotent, i.e. for all $a \in R$ there is an $n(a)$ such that $d_i^{n(a)}(a) = 0$;
- (iii) there are $z_1, \dots, z_q \in R$ such that $d_i z_j$ is invariant with respect to the action of the Lie algebra $\sum d_i k =: L$ and $\det(d_i z_j)$ is invertible in R .

Then $R^L[z_1, \dots, z_q] = R$ and $\text{Spec } R \rightarrow \text{Spec } R^L$ is a geometric quotient.

We will prove Lemma 3.3 at the end of this section.

Now we study the action of V_+ on S_τ . Using 4), 6) and 9) of Lemma 2.12, we can cover S_τ by invariant affine open sets defined by the product of suitable minors of C^p , $p \leq l = [(l+1)/2]$. Let $U = \text{Spec } C[t]_h$, $h = h_1 \dots h_l$, be one of these open sets, h_i minors of C^i . Let $1 = i_1, \dots, i_{t(1)}, \dots, i_{t(1)+1}, \dots, i_{t(2)}, \dots, i_{t(l)}$ define the columns and $j_1, \dots, j_{s(1)}, \dots, j_{s(l)}$ the rows of $C(t)$ corresponding to these minors, $t(l) = s(l) = \mu - \tau - 1 = rkC(t) - 1$. Because of Lemma 2.12 7) and 3) and Remark 2.2, $[d_{j_{s(i)}}, d_k]$ is in the $C[t]_h$ -module generated by $d_{j_{s(i)+1}}, \dots, d_{j_{s(l)}}$. Starting with $d_{j_{s(l-1)+1}}, \dots, d_{j_{s(l)}}$, we apply Lemma 3.3 l times and get

$$C[t]_h^{V_+}[t_{i_1}, \dots, t_{i_{t(l)}}] = C[t]_h.$$

We may choose homogeneous invariant functions $g_{i_{t(l)+1}}, \dots, g_{i_q} \in C[t]$ generating $C[t]_h^{V_+}$ determined by $g_{i_k}/h = t_{i_k} \text{ mod } (t_{i_1}, \dots, t_{i_{t(l)}})$:

$$U/V_+ = \text{Spec } C[g_{i_{t(l)+1}}, \dots, g_{i_q}]_h.$$

$x^a + y^b + \sum_{j>t(l)} g_{i_j}/h \cdot n_{i_j}$ is the corresponding family.

U/V_0 is the open set defined by h in the corresponding weighted projective space.

Now it is clear that the quotients of the invariant affine open sets covering S_τ by V_0 glue to a quasi-smooth scheme T_τ^f . The corresponding families glue in the étal topology.

Let us consider our example $x^5 + y^{11}$: For shortness let $A' = 2t_8 + A$ and $B' = 3t_7 + B$. $S_\tau = \text{Spec } C[t]_{t_8 A' - t_9 B'}$. Let us consider U defined by $h := t_9 A' - (t_8 A' - t_9 B')$. Then $i_v = v$, $v = 1, \dots, 4$, $i_5 = 6$.

$$C[t]_h^{V+} = C[t_9, t_8, t_7, A't_5 - B't_6]_h$$

with the corresponding family $x^5 + y^{11} + t_9 xy^9 + t_8 x^2 y^7 + t_7 x^3 y^5 + (t_5 - t_6 B'/A') x^3 y^6$.

Similarly we get the invariants on the other open sets. S_τ/V is the open set $D_+(t_8 A' - t_9 B')$ in $\mathbf{P}_{(1:2:3:10)}^3 =: \text{Proj } C[t_9, t_8, t_7, w]$. $T_\tau^f = S/V_0$ is covered by the open sets $U_1 = D_+(A') \cap T_\tau^f$ and $U_2 = D_+(B') \cap T_\tau^f$. On U_1 , resp. U_2 , we have the universal families $x^5 + y^{11} + t_9 xy^9 + t_8 x^2 y^7 + t_7 x^3 y^5 + w/A' x^3 y^6$, resp. $x^5 + y^{11} + t_9 xy^9 + t_8 x^2 y^7 + t_7 x^3 y^5 + w/B' x^2 y^8$.

Proof of Lemma 3.3. We assume that $d_i z_j = \delta_{ij}$,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Otherwise, let $Z = (z_{ij})$ be the inverse of the matrix $(d_i z_j)$. The z_{ij} are invariant under the action of L . Let $z_i = \sum_j z_{ji} z_j$; then

$$d_i(z_j) = \delta_{ij}.$$

Denote by

$$R_n = \{y, d_1^{n(1)} \dots d_q^{n(q)} y = 0, \text{ if } n(1) + \dots + n(q) \geq n\};$$

$R_1 = R^L$ and $y \in R_n$ imply $d_i y \in R_{n-1}$.

Assume now $R_{n-1} \subseteq R^L[z_1, \dots, z_q]$ and let $y \in R_n$. Then

$$d_i y = \sum_{n(1) + \dots + n(q) \leq n-1} h_n^i z_1^{n(1)} \dots z_q^{n(q)}$$

with $h_n^i \in R^L$. $d_k d_i y = d_i d_k y$ implies

$$n(k) h_{n(1), \dots, n(i)-1, \dots, n(q)}^i = n(i) h_{n(1), \dots, n(k)-1, \dots, n(q)}^k$$

for all n , $n(i) \leq n$. If n is given, $n(i) \leq n$ and $n(k) > 0$, then let

$$h_n := h_{n(1), \dots, n(k)-1, \dots, n(q)}^k / n(k)$$

and

$$y' = \sum_n h_n z_1^{n(1)} \dots z_q^{n(q)}.$$

Then $y - y' \in R^L$.

4. Moduli of reducible plane curve singularities of quasi-homogeneous type

We will give an idea of construction of the moduli space of all plane curve singularities having the topological type of a quasi-homogeneous plane curve singularity, more precisely connected by a topologically trivial family to a singularity defined by a non-degenerated quasi-homogeneous polynomial of degree d with respect to the weights w_1, w_2 . The following three cases will occur:

(i) k branches with semigroup $\Gamma = \langle a_0, b_0 \rangle$; $(a_0, b_0) = 1$; $(a, b) = k(a_0, b_0)$

$$w_1 = d/ka_0, \quad w_2 = d/kb_0; \quad f = x^a + y^b;$$

(ii) k branches with semigroup Γ and one smooth branch:

$$w_1 = d/ka_0, \quad w_2 = (d - w_1)/kb_0; \quad f = x^a + xy^b;$$

(iii) k branches with semigroup Γ and two smooth branches:

$$w_1 = (d - w_2)/ka_0, \quad w_2 = (d - w_1)/kb_0; \quad f = x^a y + xy^b.$$

The set of non-degenerated quasi-homogeneous (qh) polynomials of type $(w_1, w_2; d)$ is represented by the points of an affine open set $\mathfrak{H}_0(w_1, w_2; d)$ in an affine $(k+1)$ -space (the open set in the space of coefficients defined by $D \neq 0$, D the discriminant of the generic qh-polynomial).

Two qh-singularities of the same type are isomorphic iff there is a qh-coordinate change $x \rightarrow p(x, y)$, $y \rightarrow p'(x, y)$, p and p' qh-polynomials of degree w_1 , resp. w_2 , transforming the associated equations. Denote by \mathfrak{G}_0 the group of these coordinate changes. The qh-coordinate changes form the following algebraic groups:

$$\begin{aligned} & \text{Gl}_2 \text{ if } a_0 = b_0 = 1, \\ & \mathbf{C}^* \times \mathbf{C}^* \text{ if } a_0, b_0 > 1, \\ & \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C} \text{ if } a_0 \text{ or } b_0 = 1. \end{aligned}$$

The first two groups are reductive and the geometric quotient $H_0 = \mathfrak{H}_0/\mathfrak{G}_0$ therefore exists as an affine scheme, and is a coarse moduli space. In the third case this is also true although the group is not reductive. In general (for more than two variables) a geometric quotient \mathfrak{H}_0 probably exists only in the category of algebraic spaces.

Choose qh-polynomials $B_u = \{n_1, \dots, n_r\}$ of degree $> d$ and linearly independent in $\mathbf{C}[x, y]/\Delta f$ for all $f \in \mathfrak{H}_0(w_1, w_2; d)$.

We will restrict ourselves to the case (i). The other cases are treated in a similar way. Denote by

$$F_0 = \sum_{i=0}^k u_i x^{a-i a_0} y^{i b_0}$$

a generic non-degenerated qh-form whose coefficients belong to

$$\mathfrak{H}_0 = \text{Spec } C[u_0, \dots, u_k]_D, \quad D = \text{discr}(F_0).$$

(In case $1 = a_0 < b_0$ any non-degenerated qh-form is represented in the family

$$F_0 = x^a + xy^b + \sum_{i=1}^{k-2} u_i x^{a-i} y^{ib_0}.$$

Two functions f, g in the family F_0 define isomorphic singularities iff there is a coordinate change $\varphi: x \rightarrow cx, y \rightarrow c'y$ such that $f(\varphi) = g, c^a = 1, c^b \cdot c^{b_0} = 1$. These transformations form a finite group denoted by \mathfrak{G}_0 too, hence the geometric quotient $\mathfrak{H}_0/\mathfrak{G}_0$ exists as an affine scheme).

Now the family $F = F_0 + \sum_{B_\mu} t_i n_i$ is a versal μ -constant family at every point $\alpha \in \mathfrak{H}_0$ and any singularity being topological of this qh-type is represented in that family.

LEMMA 4.1. *The action of the group \mathfrak{G}_0 on \mathfrak{H}_0 lifts to a rational action on the parameter space $\mathfrak{H} = \mathfrak{H}_0 \times A^r$ of the family F .*

Proof. In case \mathfrak{G}_0 is finite or $C^* \times C^*$, this is clear because the element of \mathfrak{G}_0 are represented by diagonal matrices. We have to consider the homogeneous case more carefully: Given a homogeneous coordinate transformation $x \rightarrow h_1(x, y), y \rightarrow h_2(x, y)$, then, by induction on the degree, one gets a unique representation

$$F(h_1, h_2) - F_0(h_1, h_2) = \sum q_i n_i \text{ mod } (x, y)^2 \Delta F, \quad q_i \in \mathfrak{H}_0[t].$$

On the other hand, there is an action of the algebraic group $\exp V_0$ on \mathfrak{H} and the orbits are the maximal integral submanifolds of V , the kernel of the Kodaira–Spencer map.

Similar to the irreducible case we have:

LEMMA 4.2. *The geometric quotient $\mathfrak{G}_r/\exp V_0$ exists as an algebraic space. For $\tau = \tau_{\min}$ this is locally an open set in a weighted projective space (a quasi-smooth scheme).*

To get a coarse moduli space for all singularities of a fixed qh-topological type we have to factor \mathfrak{H} by the equivalence relation induced by the contact equivalence of functions:

$$(u, t) \sim (u', t') \quad \text{iff} \quad F(x, y, u', t') = c(x, y) \varphi(F(x, y, u, t)),$$

$c(x, y)$ a unit and φ an automorphism of $C[[x, y]]$.

We want to show that the group \mathfrak{G} , the subgroup of $\text{Aut}(\mathfrak{H})$ generated by $\exp V_0$, and the automorphisms of the lifted action of the qh-coordinate transformation \mathfrak{G}_0 , induces the equivalence relation on \mathfrak{H} .

LEMMA 4.3. *Two functions in the family F are contact equivalent iff the associated points in \mathfrak{H} belong to the same \mathfrak{G} -orbit.*

Proof. Let $F(x, y, u', t) = cF(\varphi, u, t)$, $\varphi \in \text{Aut}(C[[x, y]])$ and suppose that $c(0, 0) = 1$; then by Lemma 3.2 the automorphism φ has positive degree and the qh-leading form φ_0 of φ defines an element $g \in \mathfrak{G}_0$, because $F(x, y, u', 0) = F(\varphi_0, u, 0)$. Let $t_1 = gt$ be the image of t by the lifted action of g . Then $F(x, y, u', t_1)$ and $F(x, y, u', t)$ belong to an analytical trivial family defined by $c(\lambda^{w_1} x, \lambda^{w_2} y) F(\varphi_\lambda, u, t)$,

$$\begin{aligned} \varphi_\lambda(x) &:= \varphi_{\lambda,1}, & \varphi_\lambda(y) &:= \varphi_{\lambda,2}, \\ \varphi_{\lambda,i}(x, y) &:= \varphi_i(\lambda^{w_1} x, \lambda^{w_2} y)/\lambda^{w_i}, & i &= 1, 2, \\ \varphi_1 &:= \varphi(x), & \varphi_2 &:= \varphi(y). \end{aligned}$$

COROLLARY 4.4. *Let $w_1 = w_2 = 1$ (homogeneous case) and suppose that d is odd, then $\mathfrak{S}_{\tau_{\min}}/\mathfrak{G}$ exists as an affine scheme.*

Proof. $\exp V_+$ is a normal subgroup of \mathfrak{G} and $\mathfrak{G}/\exp V_+ = \mathfrak{G}_0 = \text{Gl}_2$. Furthermore, $\mathfrak{S}_{\tau_{\min}}$ is affine, because it is defined by the non-vanishing of exactly one minor given by $C^{(d-3)/2}$. The geometric quotient of the affine variety $\mathfrak{S}_{\tau_{\min}}/\exp V_+$ by Gl_2 exists because Gl_2 is reductive. \square

Now let us state the general result:

THEOREM 4.5. *Fix the quasi-homogeneous type $(w; d)$ and the Tjurina number τ ; then a coarse moduli space $T_{(w;d),\tau}$ of all plane curve singularities with that qh-topological type and Tjurina number τ exists in the category of algebraic spaces. For $\tau = \tau_{\min}$ the moduli space T is a scheme (except may be in the homogeneous case, i.e. $w_1 = w_2 = d$, if d is even).*

Idea of proof. We have to look for a geometric quotient of $(\mathfrak{S}_i/\exp V_0)$ by $\mathfrak{G}/\exp V_+$. If that group is finite the statement is obvious by Lemma 4.2. This is fulfilled in the case $1 = a_0 < b_0$, where $\mathfrak{G}/\exp V_+ = \mathfrak{G}_0$ is finite. It works similarly in the case $a_0, b_0 > 1$, if we take

$$F_0 := x^a + y^b + \sum_{i=1}^{k-1} u_i x^{k-ia_0} y^{ib_0},$$

which also represents all classes of qh-functions. On the parameter space $\mathfrak{H}_0 = \text{Spec } C[u_1, \dots, u_{k-1}]_D$ the corresponding group \mathfrak{G}_0 is the finite group of ab -th roots of unity.

In the homogeneous case $\mathfrak{G}/\exp V_0 = \text{Sl}_2$ and the quotient exists in the case d is odd (Lemma 4.4). If d is even, it is not clear if there exists an affine invariant covering in $\mathfrak{S}/\exp V_0$ with respect to the action of Sl_2 . There is a more general construction which also in the homogeneous case reduces the problem to an action of a finite group \mathfrak{G}_0 (this works in higher dimensions

too), but the action of that finite group is only defined in the category of algebraic spaces.

Some remarks on the homogeneous case: A homogeneous form of degree d can be transformed into

$$\begin{aligned}
 F_0 &= xy(x^{d-2} + u_1 x^{d-3} y + \dots + u_{d-3} xy^{d-3} + y^{d-2}) \\
 &= xy(x - v_1 y) \dots (x - v_{d-2} y), \quad v_1 \dots v_{d-2} = 1.
 \end{aligned}$$

$\mathfrak{H}'_0 := \text{Spec } \mathbb{C}[v_1, \dots, v_{d-3}]_D$ is an étal covering of $\mathfrak{H}_0 := \text{Spec } \mathbb{C}[u_1, \dots, u_{d-3}]_D$ induced by the elementary symmetric functions in v .

On $\pi: \mathfrak{H}'_0 \rightarrow \mathfrak{H}_0$ acts the group of $d(d-2)$ th-roots of unity by

$$x \rightarrow [n/(d(d-2))]x, \quad y \rightarrow [-(d-1)n/(d(d-2))]y,$$

here $[n/m] := \exp(2\pi i(n/m))$.

We take the quotient according to this finite group action and denote it by the same letters for simplicity.

Now homogeneous coordinate changes from Gl_2 applied to elements of the family F_0 induce equivalence relations on \mathfrak{H}'_0 and on \mathfrak{H}_0 , denoted by R' and R .

We want to realize these equivalence relations by the action of a finite group \mathfrak{G}_0 .

LEMMA 4.6. *There is a rational action of a finite group \mathfrak{G}_0 on \mathfrak{H}'_0 inducing the equivalence relation R' .*

Idea of proof. Let $\text{Gl}_2 := \text{Spec } \mathbb{C}[U, V, S, T]_{(UT-VS)}$. Let Z be the zero set of the equations

$$\begin{aligned}
 F_0(U, V) &= 0, \\
 F_0(S, T) &= 0, \\
 U \partial F_0 / \partial x(S, T) + V \partial F_0 / \partial y(S, T) &= 1, \\
 S \partial F_0 / \partial x(U, V) + T \partial F_0 / \partial y(U, V) &= 1
 \end{aligned}$$

in $\text{Gl}_2 \times \mathfrak{H}'_0$. We have two maps pr_2 and o from $\text{Gl}_2 \times \mathfrak{H}'_0$ to the affine space of all homogeneous d -forms in x, y ; pr_2 the projection onto the second factor and o the orbit map. The image of Z by $\text{pr}_2 \times o$ is in $\mathfrak{H}'_0 \times \mathfrak{H}'_0$ (resp. in $\mathfrak{H}_0 \times \mathfrak{H}_0$) and coincides by construction with the geometric realizations \mathfrak{R}' (resp. \mathfrak{R}) of the equivalence relations. Now the main step is to prove that \mathfrak{R}' splits into irreducible components Z_i isomorphic to \mathfrak{H}'_0 via pr_2 . On the set of the sections q_i of pr_2 restricted to \mathfrak{R}' we define a group structure by

$$q_i * q_j := q_j \text{pr}_1 q_i,$$

which is again a section.

This group, denoted by \mathfrak{G}_0 , acts on \mathfrak{H}'_0 by

$$q_i: \mathfrak{v} \rightarrow \text{pr}_1 q_i(\mathfrak{v})$$

and this action induces R' .

EXAMPLE. For $d = 4$ we have $\mathfrak{G}_0 = \mathbb{Z}/3$.

But already in this simple case the action of \mathfrak{G}_0 is not compatible with π , i.e. gives no rational action on \mathfrak{H}_0 (only an action in the category of algebraic spaces), i.e. \mathfrak{R} does not split completely into irreducible components isomorphic to \mathfrak{H}_0 .

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