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DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCXXXVIII

BOLIS BASIT

Some problems concerning different types of vector valued almost periodic functions

WARSZAWA 1995

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Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0012-3862

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1991 *Mathematics Subject Classification*: Primary 43A60; Secondary 28B05, 34K15.

Key words and phrases: almost periodicity, spectrum, indefinite intergral, intergo-differential equations.

Received 8.12.1992; revised version 29.6.1994.

Abstract

We study some problems concerning derivatives and indefinite integrals of elements of wide classes of functions (including almost periodic, recurrent, almost automorphic and Eberlein almost periodic functions). We introduce a notion of a spectrum of a function with respect to a class of functions. This notion enables us to investigate almost periodicity of the bounded solutions of certain functional equations.

Introduction

Let $C_{\mathrm{u}}(\mathbb{J}, X)$ (resp. $C_{\mathrm{ub}}(\mathbb{J}, X)$) be the space of all uniformly continuous (resp. uniformly continuous and bounded) functions defined on $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$ with values in a Banach space X. Let $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ be a class of functions. We address the following problems:

(D) Let $\varphi \in A$ and let its derivative $\varphi' \in C_{\mathrm{u}}(\mathbb{J}, X)$. Characterize A for which $\varphi' \in A$.

(P) Let $\varphi \in A$ and let $P\varphi(t) = \int_0^t \varphi(x)\, dx$. Find the conditions under which $P\varphi \in A$.

(L) Let l be a linear operator defined on a dense submanifold of $C_{\mathrm{ub}}(\mathbb{J}, X)$ and let $\varphi \in A$. Under what conditions is the solution of the operator equation $l\psi = \varphi$ an element of A?

These problems have been addressed for various classes of functions (in particular, almost periodic (a.p.), asymptotically almost periodic (a.a.p.), almost automorphic (a.a.) and Eberlein almost periodic (E.a.p.) functions) [1], [3]–[9], [11]–[16], [18], [21]–[22], [24]–[26], [28]–[30], [32]–[36], [41]–[43], [45], [46], [48].

In this paper we give a unified treatment of each of these problems for all classes which are translation invariant closed subspaces of $C_{\mathrm{ub}}(\mathbb{J}, X)$ containing constants (TICSC). Though our approach covers wide classes of functions (including a.p., a.a.p, a.a. and E.a.p.) our conditions are the best possible for each of the posed problems. The result of Levitan [32] on the integral of a.p. functions, the recent result of Ruess and Summers [42], a private communication of G. Muraz and a partial result of our student Ala'a Hamza led us to extend our paper [4] to larger classes. The conditions of Levitan [32] are weaker than the conditions of Ruess and Summers [42], yet they are sufficient to solve problem (P) for TICSC classes. Previous investigation of problem (L) for many classes from $C_{\mathrm{ub}}(\mathbb{R}, X)$ ([8], [5], [9], [33, Ch. 6]) has revealed the importance of a basic result of Kadets [30]. This result on the integration of almost periodic functions and its generalization to other classes [3] appears to be the key to the development of problem (L). The notion of the spectrum of a function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ with respect to a class $A \subset C_{\mathrm{ub}}(\mathbb{R}, X)$ has been used by Baskakov [9] and the author [5] and was introduced earlier in an abstract setting by Domar [17]. It is decisive in reducing problem (L) to problem (P) (see [33, Ch. 6], [5], [9]). We define what it means for $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ to be a Λ-class or a Π-class. With these classifications we obtain new properties of many classes (including a.p., a.a.p. and E.a.p.) of functions and introduce recurrent (re.), asymptotically recurrent (a.re.), almost automorphic and asymptotically almost automorphic (a.a.a.) functions on $\mathbb{R}^+$. Using these new properties, we extend the functions of many classes defined on $\mathbb{R}^+$ to functions defined on $\mathbb{R}$. The notion of the spectrum of a function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ with respect to a class $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ is given. As in the case $A \subset C_{\mathrm{ub}}(\mathbb{R}, X)$, we can reduce problem (L) to problem (P) for classes $A \subset C_{\mathrm{ub}}(\mathbb{R}^+, X)$.

This paper consists of an introduction and five sections. In Section 1, we give notation, preliminaries, the definitions of a Λ-class and a Π-class and solve problem (D). Section 2 is devoted to the study of the classes of a.p., a.a.p., re., a.re., a.a., a.a.a., E.a.p., ergodic and totally ergodic functions. We give a new property of a.a.p, a.re., a.a.a., E.a.p. and totally ergodic functions defined on $\mathbb{R}^+$, namely, if φ is an element of one of these classes and if $\psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ is such that $\psi|_{\mathbb{R}^+} = \varphi$, then $R_a\psi|_{\mathbb{R}^+}$ is an element of the same class for all $a \in \mathbb{R}$, where $R_a\psi(t) = \psi(t+a)$ for all $a, t \in \mathbb{R}$. We solve not only problem (P) but also the closely related difference problem: If $\psi \in A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ and if $\Delta_s\psi = R_s\psi - \psi \in A$, $s \in \mathbb{J}$, under what conditions does $\psi \in A$? We also discuss problem (P) for the weak classes introduced by Ruess and Summers [42]. In Section 4 we introduce and study the notion of the spectrum of a function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ with respect to a Λ-class $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$. Finally, in Section 5, we solve problem (L) for integro-differential difference equations.

1. Notation, preliminaries, main definitions and solution of problem (D)

In this section we give notation and preliminaries used in the sequel. We also define a Λ-class and a Π-class, study their properties and solve problem (D).

1.1. Notation. Throughout this paper $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$, $\mathbb{N}$ denotes the natural numbers, $\mathbb{C}$ the complex numbers, X any Banach space, X^* its dual Banach space and $L^\infty(\mathbb{J}, X)$ the Banach space of essentially bounded strongly measurable functions defined on $\mathbb{J}$ with values in X, endowed with the norm $\|\varphi\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{J}} \|\varphi(t)\|$, $\varphi \in L^\infty(\mathbb{J}, X)$. By $C_{\mathrm{b}}(\mathbb{J}, X)$, $C_{\mathrm{u}}(\mathbb{J}, X)$, $C_{\mathrm{ub}}(\mathbb{J}, X)$ and $C_0(\mathbb{J}, X)$ we will denote respectively the set of all continuous bounded, uniformly continuous, uniformly continuous bounded and vanishing at infinity continuous functions. If $\{\varphi_n\} \subset C_{\mathrm{ub}}(\mathbb{J}, X)$, then we write $\varphi_n \to \varphi$ as $n \to \infty$ if and only if $\|\varphi_n - \varphi\|_\infty \to 0$ as $n \to \infty$. If $\varphi : \mathbb{J} \to X$, then $\|\varphi\|$ will denote the function having the value $\|\varphi(t)\|$ at the point $t \in \mathbb{J}$ and $R_s\varphi$ the translate of φ defined by $R_s\varphi(t) = \varphi(t+s)$, $t, s \in \mathbb{J}$. If $\varphi \in C_{\mathrm{ub}}(\mathbb{J}, X)$, then $L(\mathbb{J}, \varphi)$ will denote the closed subspace generated by $\{R_s\varphi : s\in\mathbb{J}\}$. We put $L(\mathbb{R},\varphi)=L(\varphi)$, $L(\mathbb{R}^+,\varphi)=L^+(\varphi)$. The dual group of the reals will be denoted by $\widehat{\mathbb{R}}$. If $\varphi \in C_{\mathrm{ub}}(\mathbb{J}, X)$ and $\widehat{\lambda} \in \widehat{\mathbb{R}}$ then $\widehat{\lambda}\varphi$ will denote the function $\widehat{\lambda}|_{\mathbb{J}}\varphi$. The Fourier transform of $f \in L^1(\mathbb{R})$ will be denoted by $\widehat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} f(t)\, dt$. Finally, $S(\mathbb{R})$ will denote rapidly decreasing (at ∞) functions (see [47, p. 146]). If $\varphi \in L^\infty(\mathbb{R}, X)$ or $C_{\mathrm{u}}(\mathbb{R}, X)$ then $T_\varphi(f) = \int_{-\infty}^{\infty} f(t)\varphi(t)\, dt$ for $f \in S(\mathbb{R})$ defines an X-valued distribution T_φ. So $\widehat{T}_\varphi(g) = T_\varphi(\widehat{g})$ for $g \in S(\mathbb{R})$ defines the Fourier transform $\widehat{T}_\varphi$ of T_φ (see [47, p. 146–152]).

1.2. Preliminaries. The following lemma is technical; the first part of it is by no means new and proved only for reference purposes (see [35, p. 1]).

LEMMA 1.2.1. *Let* $\varphi\in L^\infty(\mathbb{R}, X)$, $f\in L^1(\mathbb{R})$. *Then* φf *is a Lebesgue–Bochner integrable function and* $\varphi * f(t)=\int_{-\infty}^{\infty} \varphi(t-s)f(s)\, ds=\int_{-\infty}^{\infty} f(t-s)\varphi(s)\, ds$, $t\in\mathbb{R}$, *defines a function* $\varphi * f \in C_{\mathrm{ub}}(\mathbb{R}, X)$. *If* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$, *then* $\varphi * f \in L(\varphi)$.

Proof. Let $\|\varphi\|_\infty = M$. Then $\|\varphi(t)f(t)\| \le M|f(t)|$ almost everywhere on $\mathbb{R}$. Hence by the Lebesgue dominating theorem, $\|\varphi f\|$ is Lebesgue integrable. By Bochner's theorem [47, p. 133], $f\varphi$ is Lebesgue-Bochner integrable. Hence $R_{-s}\varphi f$ and $\varphi R_{-s} f$ are Lebesgue-Bochner integrable, $s \in \mathbb{R}$. Since $\int_{-\infty}^{\infty} x^*(\varphi(t-s))f(s)\, ds = \int_{-\infty}^{\infty} f(t-s)x^*(\varphi(s))\, ds$, $x^* \in X^*$, by the Hahn–Banach theorem, $\int_{-\infty}^{\infty} \varphi(t-s)f(s)\, ds = \int_{-\infty}^{\infty} f(t-s)\varphi(s)\, ds$. From the inequality $\|\varphi * f(t+h) - \varphi * f(t)\| \le M\int_{-\infty}^{\infty} |f(t+h-s) - f(t-s)|\, ds$ and $\|\varphi * f\| \le M\int_{-\infty}^{\infty} |f(s)|\, ds$ it follows that $\varphi * f \in C_{\mathrm{ub}}(\mathbb{R}, X)$.

Now, let $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. Let $h(t) = 1$, $t \in [a, b]$, and $h(t) = 0$, $t \notin [a, b]$. Then $\varphi * h = \int_a^b R_{-s}\varphi\, ds$. Since this integral is the limit in $C_{\mathrm{ub}}(\mathbb{R}, X)$ of partial sums of the form $\sum_{i=0}^{m-1} R_{-s_i}\varphi\Delta s_i$, where $\Delta s_i = s_{i+1} - s_i$, $a = s_0 < s_1 < \ldots < s_m = b$, we have $\varphi * h \in L(\varphi)$. If $H = \sum_{j=1}^{n} c_j h_j$ is a step function, where h_j is the characteristic function of the interval $[a_j, b_j]$, $j = 1, \ldots, n$, and $\{c_1, \ldots, c_n\} \subset \mathbb{C}$, then $\varphi * H = \sum_{j=1}^{n} \varphi * h_j \in L(\varphi)$. Since $f \in L^1(\mathbb{R})$, there exists a sequence $\{H_n\}$ of step functions such that $\|f - H_n\|_{L^1(\mathbb{R})} \to 0$ as $n \to \infty$. This implies $\varphi * f = \lim_{n\to\infty} \varphi * H_n \in L(\varphi)$ and completes the proof.

In the same way as in the above lemma, one can prove

LEMMA 1.2.2. *Let* $\varphi \in L^\infty(\mathbb{R}^+, X)$ *and* $f \in L^1(\mathbb{R}^+)$. *Let* $\psi(t) = \int_0^\infty \varphi(t+s)f(s)\, ds$. *Then* $\psi \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$. *Moreover, if* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$, *then* $\psi \in L^+(\varphi)$.

Finally, we state the following lemma needed in the sequel:

LEMMA 1.2.3. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *and let* $f_n(t) = (1-\cos nt)/(n\pi t^2)$ *for* $t \neq 0$ *and* $f_n(0) = n/(2\pi)$. *Then* $\varphi * f_n \to \varphi$ *as* $n \to \infty$. *Moreover*, $\operatorname{supp}\widehat{f_n} \subset [-n, n]$ *and* $\varphi * f_n$ *is an infinitely differentiable function for all* $n \in \mathbb{N}$.

For the proof see [33, Proposition 3, p. 87] or [32].

1.3. Main definitions. We introduce $\varLambda$ and $\varPi$-classes and give some of their properties.

Definition 1.3.1. A subset $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ is said to be a *$\varLambda$-class* if and only if it satisfies:

(P.1) A is a closed subspace of $C_{\mathrm{ub}}(\mathbb{J}, X)$.

(P.2) $\widehat{\lambda}\varphi \in A$ for $\varphi \in A$, $\widehat{\lambda} \in \widehat{\mathbb{R}}$.

(P.3) A contains all the constant functions.

(P.$\varLambda$) If $\psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ and $\psi|_{\mathbb{J}} = \varphi \in A$, then $R_a\psi|_{\mathbb{J}} \in A$, $a \in \mathbb{R}$.

Next, we give

Definition 1.3.2. A subset A is said to be a *$\varPi$-class* if and only if it satisfies properties (P.1)–(P.3) and the following:

(P.$\varPi$) For each $\varphi \in A$, there exists $\psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ such that $\psi|_{\mathbb{J}} = \varphi$ and $R_a\psi|_{\mathbb{J}} \in A$, $a \in \mathbb{R}$.

Remark 1.3.3. If $\mathbb{J} = \mathbb{R}$, then property (P.$\varLambda$) is equivalent to property (P.$\varPi$) and says that A is a translation invariant subset of $C_{\mathrm{ub}}(\mathbb{R}, X)$. Hence $A \subset C_{\mathrm{ub}}(\mathbb{R}, X)$ is a $\varLambda$-class if and only if it is a $\varPi$-class.

If $\mathbb{J} = \mathbb{R}^+$, $\varphi \in A$ and if $\psi(t) = \varphi(t)$, $t \ge 0$, and $\psi(t) = \varphi(0)$, $t < 0$, then $\psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$.

Using the above remark we get

Proposition 1.3.4. *Every $\varLambda$-class $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ is a $\varPi$-class.*

Now, we show the following property:

(P.4) If A is a $\varLambda$-class or $\varPi$-class, then $R_s\varphi \in A$ for $\varphi \in A$, $s \in \mathbb{J}$.

Indeed, let $\varphi \in A$. Then by property (P.$\varLambda$) or (P.$\varPi$), there exists $\psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ such that $\psi|_{\mathbb{J}} = \varphi$ and $R_s\psi|_{\mathbb{J}} \in A$, $s \in \mathbb{R}$. This implies $R_s\varphi \in A, s \in \mathbb{J}$.

We end this subsection by the following needed

Remark 1.3.5. A subset $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ is a TICSC-class if and only if it has properties (P.1), (P.3) and (P.4). This means that every $\varLambda$-class or $\varPi$-class is a TICSC-class.

1.4. Problem (D). We solve problem (D) and give two corollaries.

Theorem 1.4.1. *Let $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ satisfy properties* (P.1) *and* (P.4). *Let $\varphi \in A$ and its derivative $\varphi' \in C_{\mathrm{u}}(\mathbb{J}, X)$. Then $\varphi' \in A$.*

Proof. Consider the sequence $n(R_{1/n}\varphi - \varphi) - \varphi' = n\int_0^{1/n}(R_x\varphi' - \varphi')\,dx$. Since $\varphi' \in C_{\mathrm{u}}(\mathbb{J}, X)$, the difference $R_x\varphi' - \varphi' \in C_{\mathrm{ub}}(\mathbb{J}, X)$, $x \in \mathbb{J}$. Moreover, for each $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $\|R_x\varphi' - \varphi'\|_\infty < \varepsilon$, $|x| < 1/n(\varepsilon)$. Hence $\|n\int_0^{1/n}(R_x\varphi' - \varphi')\,dx\|_\infty \le \varepsilon$, $n \ge n(\varepsilon)$. This implies that $n(R_{1/n}\varphi -$

$\varphi) \to \varphi'$ as $n \to \infty$. By (P.4), $R_{1/n}\varphi \in A$ for all $n \in \mathbb{N}$ and by property (P.1), $\{n(R_{1/n}\varphi - \varphi)\} \subset A$ and its limit $\varphi' \in A$.

In the same way, one can prove

COROLLARY 1.4.2. *If* $\varphi, \varphi' \in C_{\mathrm{u}}(\mathbb{J}, X)$, *then* $\varphi' \in C_{\mathrm{ub}}(\mathbb{J}, X)$.

COROLLARY 1.4.3. *Let* $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ *be a* Λ*-class or* Π*-class. If* $\varphi \in A$ *and* $\varphi' \in C_{\mathrm{u}}(\mathbb{J}, X)$, *then* $\varphi' \in A$.

2. Examples of Λ-classes and Π-classes

In this section we study different classes of of almost periodic functions from $C_{\mathrm{ub}}(\mathbb{J}, X)$. We prove new properties of some of these classes, namely, properties (P.Λ) and (P.Π) and classify them into Λ-classes and Π-classes.

2.1. Almost periodic, recurrent and almost automorphic functions. In the following we study the classes of a.p., re. and a.a. functions from $C_{\mathrm{ub}}(\mathbb{J}, X)$. First, we give

DEFINITION 2.1.1. A subset $E \subset \mathbb{R}$ is said to be *relatively dense* if and only if there exists $\{t_1, \dots, t_n\} \subset \mathbb{R}$ such that $\mathbb{R} = \bigcup_{i=1}^{n}(t_i + E)$.

We study the following three types of almost periodicity on $\mathbb{R}$.

DEFINITION 2.1.2. A function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ is said to have

(i) property (BRD) if and only if for each $\varepsilon > 0$, the set $E(\varepsilon, \varphi) = \{\tau : \|\varphi(t+\tau) - \varphi(t)\| < \varepsilon,\ t \in \mathbb{R}\}$ is relatively dense in $\mathbb{R}$;

(ii) property (FRD) if and only if for each $\varepsilon > 0$ and $M > 0$ the set $E(\varepsilon, \varphi, M) = \{\tau : \|\varphi(t+\tau) - \varphi(t)\| < \varepsilon,\ |t| \le M\}$ is relatively dense in $\mathbb{R}$;

(iii) property (LRD) if and only if for each $\varepsilon > 0$ and $M > 0$, there exist $\delta > 0$ and $L > 0$ such that $(E(\delta, \varphi, L) - E(\delta, \varphi, L)) \subset E(\varepsilon, \varphi, M)$.

Now, we give the following

DEFINITION 2.1.3. A function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ is said to be

(i) *almost periodic* if and only if it has property (BRD);

(ii) *recurrent* if and only if its range is relatively compact and it has property (FRD);

(iii) *almost automorphic* if and only if it has properties (FRD) and (LRD).

The sets of a.p., re. and a.a. functions will be denoted respectively by $AP(\mathbb{R}, X)$, $r(\mathbb{R}, X)$ and $AA(\mathbb{R}, X)$.

From Definitions 2.1.2 and 2.1.3 one can obtain:

(2.1.1) The range of $\varphi \in AP(\mathbb{R}, X)$ is relatively compact.

(2.1.2) Property (BRD) implies (FRD) and (LRD).

(2.1.3) If $\varphi \in AP(\mathbb{R},X)$ (resp. $AA(\mathbb{R},X)$), then $\check{\varphi} \in AP(\mathbb{R},X)$ (resp. $AA(\mathbb{R},X)$), where $\check{\varphi}(t) = \varphi(-t),\ t \in \mathbb{R}$.

This implies the following:

(2.1.4) $AP(\mathbb{R},X) \subset r(\mathbb{R},X)$, $AP(\mathbb{R},X) \subset AA(\mathbb{R},X)$.

Definitions 2.1.2 and 2.1.3 imply ([1], [2], [11], [13], [33], [39], [44])

THEOREM 2.1.4. *$AP(\mathbb{R},X)$ and $AA(\mathbb{R},X)$ are Λ-classes.*

The elements of $AP(\mathbb{R},X)$ are called *Bohl–Bohr–Bochner almost periodic functions* (see preface of [33]). We note that if $\varphi : \mathbb{R} \to X$ is continuous and has property (BRD), then $\varphi \in C_{\mathrm{ub}}(\mathbb{R},X)$.

Complex valued almost automorphic functions defined on $\mathbb{R}$ were introduced in 1937 by B. M. Levitan (see references in [33]). They are called *N-almost periodic functions* and were studied on groups by B. J. Levin [31]. In [11] S. Bochner introduced a new approach to almost periodicity which led him to introduce the class of almost automorphic functions. The investigations of W. A. Veech [44], A. Reich [39] and the author [2] showed the relationship between N-almost periodic and almost automorphic functions.

It is known that the class $r(\mathbb{R},X)$ is not linear ([23], [331]). Nevertheless, if $\varphi \in r(\mathbb{R},X)$, then $L(\varphi) \subset r(\mathbb{R},X)$ and the closed linear space $r(\varphi,\mathbb{R},X)$ generated by $L(\varphi) \cup AP(\mathbb{R},X)$ is a subset of $r(\mathbb{R},X)$. Moreover, we have

THEOREM 2.1.5. *$r(\varphi,\mathbb{R},X)$ is a Λ-class for each $\varphi \in r(\mathbb{R},X)$.*

The proof follows from the fact that if $\varphi \in r(\mathbb{R},X)$ and $\psi \in AP(\mathbb{R},X)$, then the pair $(\varphi,\psi) \in r(\mathbb{R},X \times X)$ (see [33, Ch. 7]).

From definitions 2.1.2 and 2.1.3, we deduce the following:

PROPOSITION 2.1.6. *If $\varphi \in C_{\mathrm{ub}}(\mathbb{R},X)$ has property* (FRD) *and $\varphi \neq 0$, then $\varphi|_{\mathbb{R}^+} \notin C_0(\mathbb{R}^+,X)$ and $\varphi|_{\mathbb{R}^-} \notin C_0(\mathbb{R}^-,X)$, where $\mathbb{R}^- = \{t \in \mathbb{R} : t \leq 0\}$.*

Proof. Assuming that $\varphi|_{\mathbb{R}^+} \in C_0(\mathbb{R}^+,X)$, for each $\varepsilon > 0$ there exists $a_\varepsilon > 0$ such that $\|\varphi(t)\| < \varepsilon,\ t \geq a_\varepsilon$. Since φ has property (FRD), for each $t \in \mathbb{R}$ there exists $\tau_t \in E(\varepsilon,\varphi,|t|)$ such that $t + \tau_t \geq a_\varepsilon$. We have $\|\varphi(t)\| \leq \|\varphi(t) - \varphi(t+\tau_t)\| + \|\varphi(t+\tau_t)\| \leq 2\varepsilon$, and hence $\|\varphi(t)\| \leq 2\varepsilon$. Since ε, t are arbitrary, we get $\varphi = 0$, contradicting the hypothesis and proving the first part. The second part can be proved in the same way.

Using the same argument, we get

PROPOSITION 2.1.7. *If $\varphi \in C_{\mathrm{ub}}(\mathbb{R},X)$ has property* (FRD), *then*

(2.1.5) $\|\varphi\|_\infty = \sup_{t\in\mathbb{R}} \|\varphi(t)\| = \sup_{t\in\mathbb{R}^+} \|R_a\varphi(t)\|,\ a \in \mathbb{R}$.

The above two propositions lead to the following

DEFINITION 2.1.8. A function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}^+,X)$ is said to be

(i) *almost periodic* if and only if there exists $\psi \in AP(\mathbb{R},X)$ such that $\psi|_{\mathbb{R}^+} = \varphi$;
(ii) *recurrent* if and only if there exists $\psi \in r(\mathbb{R},X)$ such that $\psi|_{\mathbb{R}^+} = \varphi$;

(iii) *almost automorphic* if and only if there exists $\psi \in AA(\mathbb{R}, X)$ such that $\psi|_{\mathbb{R}^+} = \varphi$.

The sets of all a.p., re. and a.a. functions from $C_{\mathrm{ub}}(\mathbb{R}^+, X)$ will be denoted by $AP(\mathbb{R}^+, X)$, $r(\mathbb{R}^+, X)$ and $AA(\mathbb{R}^+, X)$ respectively.

Using Theorems 2.1.4, 2.1.5, and Propositions 2.1.6, 2.1.7, we obtain

THEOREM 2.1.9. *Each of the following*:

(i) $m_P : AP(\mathbb{R}, X) \to AP(\mathbb{R}^+, X)$, $m_P(\varphi) = \varphi|_{\mathbb{R}^+}$, $\varphi \in AP(\mathbb{R}, X)$,
(ii) $m_r : r(\varphi, \mathbb{R}, X) \to r(\varphi, \mathbb{R}^+, X)$, $m_r(\psi) = \psi|_{\mathbb{R}^+}$, $\psi \in r(\varphi, \mathbb{R}, X)$,
(iii) $m_A : AA(\mathbb{R}, X) \to AA(\mathbb{R}^+, X), m_A(\varphi) = \varphi|_{\mathbb{R}^+}$, $\varphi \in AA(\mathbb{R}, X)$,

is a linear isometric operator.

As a consequence we get

THEOREM 2.1.10. *The spaces* $AP(\mathbb{R}^+, X)$, $r(\varphi, \mathbb{R}^+, X)$ *and* $AA(\mathbb{R}^+, X)$ *are* Π*-classes from* $C_{\mathrm{ub}}(\mathbb{R}^+, X)$.

The concept of almost periodic functions on semigroups was introduced by W. Maak [37] for $X = \mathbb{C}$ and extended by Iseki [27] to the case of vector valued functions (see also Jacobs [28], [29]). Definition 2.1.8(i) is equivalent to Definition 1 from [27, p. 16] in the case $G = \mathbb{R}^+$, $E = X$.

As a consequence of the fact that trigonometric polynomials are dense in $AP(\mathbb{R}, X)$ (cf. [1, Ch. 2] or [33, Ch. 2]) and Theorem 2.1.9(i), we get

THEOREM 2.1.11. $AP(\mathbb{R}, X)$ *is the minimal* Λ*-class from* $C_{\mathrm{ub}}(\mathbb{R}, X)$ *and* $AP(\mathbb{R}^+, X)$ *is the minimal* Π*-class from* $C_{\mathrm{ub}}(\mathbb{R}^+, X)$.

THEOREM 2.1.12. *A function* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *is an element of* $r(\mathbb{R}, X)$ *if and only if there exists* $\psi \in r(\mathbb{R}, X)$ *such that* $R_a\varphi|_{\mathbb{R}^+} \in r(\psi, \mathbb{R}^+, X)$, $a \in \mathbb{R}$. *A function* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *is an element of* $AP(\mathbb{R}, X)$ (*resp.* $AA(\mathbb{R}, X)$) *if and only if* $R_a\varphi|_{\mathbb{R}^+} \in AP(\mathbb{R}^+, X)$ (*resp.* $AA(\mathbb{R}^+, X)$), $a \in \mathbb{R}$.

Proof. If $\varphi \in r(\mathbb{R}, X)$, then $R_a\varphi \in r(\varphi, \mathbb{R}, X)$, $a \in \mathbb{R}$. Hence $R_a\varphi|_{\mathbb{R}^+} \in r(\varphi, \mathbb{R}^+, X)$, $a \in \mathbb{R}$. Conversely, let $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ and $R_a\varphi|_{\mathbb{R}^+} \in r(\psi, \mathbb{R}^+, X)$, $a \in \mathbb{R}$, with $\psi \in r(\mathbb{R}, X)$. Let $m_r^{-1}\varphi|_{\mathbb{R}^+} = \Phi$, where m_r^{-1} is the inverse operator of m_r defined in Theorem 2.1.9. We have

$$R_a\varphi(t) - R_a\Phi(t) = 0,\ t \ge \max\{0, a\},\ a \in \mathbb{R}. \tag{2.1.6}$$

Using the definition of $r(\psi, \mathbb{R}^+, X)$, we conclude that $R_a\Phi|_{\mathbb{R}^+} \in r(\psi, \mathbb{R}^+, X)$, $a \in \mathbb{R}$. This implies that $(R_a\varphi - R_a\Phi)|_{\mathbb{R}^+} \in r(\psi, \mathbb{R}^+, X)$. Using Proposition 2.1.6 and (2.1.6), we conclude that $R_a\varphi = R_a\Phi$, $a \in \mathbb{R}$. This implies that $\varphi = \Phi \in r(\psi, \mathbb{R}, X) \subset r(\mathbb{R}, X)$. In the same way, one can prove the other cases of the theorem.

THEOREM 2.1.13. *Each of the spaces* $AP(\mathbb{R}^+, X)$, $r(\varphi, \mathbb{R}^+, X)$, $AA(\mathbb{R}^+, X)$ *is not a* Λ*-class.*

Proof. Assuming the contrary, let $r(\varphi, \mathbb{R}^+, X)$ be a Λ-class. Further, let $\psi \in r(\varphi, \mathbb{R}^+, X)$ and $\psi \neq \psi(0)$. Let $\Psi(t) = \psi(t)$, $t \geq 0$, and $\Psi(t) = \psi(0)$, $t < 0$. By property (P.Λ), $R_a\Psi|_{\mathbb{R}^+} \in r(\varphi, \mathbb{R}^+, X)$, $a \in \mathbb{R}$. By Theorem 2.1.12, we get $\Psi \in r(\varphi, \mathbb{R}, X)$. This implies that $(\Psi - \psi(0)) \in r(\mathbb{R}, X)$. We have $\Psi(t) - \psi(0) = 0$, $t \leq 0$, and this means that $(\Psi - \psi(0))|_{\mathbb{R}^-} \in C_0(\mathbb{R}^-, X)$, contradicting Proposition 2.1.6 and proving that $r(\varphi, \mathbb{R}^+, X)$ is not a Λ-class. In the same way $AP(\mathbb{R}^+, X)$ and $AA(\mathbb{R}^+, X)$ are not Λ-classes.

Finally, we prove

THEOREM 2.1.14. *Let* $\varphi \in AP(\mathbb{R}, X)$ (*resp.* $r(\varphi, \mathbb{R}, X)$, $AA(\mathbb{R}, X)$) *and* $\varphi'|_{\mathbb{R}^+} \in C_{\mathrm{u}}(\mathbb{R}^+, X)$. *Then* $\varphi' \in AP(\mathbb{R}, X)$ (*resp.* $r(\varphi, \mathbb{R}, X)$, $AA(\mathbb{R}, X)$).

Proof. Consider the sequence $\{n(R_{1/n}\varphi - \varphi)\}$. By Theorem 1.4.1, we have

$$n(R_{1/n}\varphi - \varphi)|_{\mathbb{R}^+} \to \varphi'|_{\mathbb{R}^+} \text{ as } n \to \infty \tag{2.1.7}$$

and $\varphi'|_{\mathbb{R}^+} \in AP(\mathbb{R}^+, X)$ (resp. $r(\varphi, \mathbb{R}^+, X)$, $AA(\mathbb{R}^+, X)$). Using Theorem 2.1.9, we see that $\{n(R_{1/n}\varphi - \varphi)\}$ is a Cauchy sequence from $AP(\mathbb{R}, X)$ (resp. $r(\varphi, \mathbb{R}, X)$ $AA(\mathbb{R}, X)$). Since $AP(\mathbb{R}, X)$ (resp. $r(\varphi, \mathbb{R}, X)$, $AA(\mathbb{R}, X)$) is closed, the sequence $\{n(R_{1/n}\varphi - \varphi)\}$ converges in $C_{\mathrm{ub}}(\mathbb{R}, X)$ and $\lim_{n\to\infty} n(R_{1/n}\varphi - \varphi) = \varphi' \in AP(\mathbb{R}, X)$ (resp. $r(\varphi, \mathbb{R}, X)$, $AA(\mathbb{R}, X)$).

2.2. Asymptotically recurrent and almost automorphic functions. Asymptotically almost periodic functions are introduced by Fréchet [24] and studied by many authors ([14]–[16], [25], [41]–[43], [45], [46]). Here, we introduce also the notions of asymptotically recurrent and asymptotically almost automorphic functions.

DEFINITION 2.2.1. A function $\psi \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$ is said to be

(i) *asymptotically almost periodic* if and only if there exist $\varphi \in AP(\mathbb{R}^+, X)$ and $\xi \in C_0(\mathbb{R}^+, X)$ such that $\psi = \varphi + \xi$.

(ii) *asymptotically recurrent* if and only if there exist $\varphi \in r(\mathbb{R}, X)$ and $\xi \in C_0(\mathbb{R}^+, X)$ such that $\psi = \varphi|_{\mathbb{R}^+} + \xi$.

(iii) *asymptotically almost automorphic* if and only if there exist $\varphi \in AA(\mathbb{R}^+, X)$ and $\xi \in C_0(\mathbb{R}^+, X)$ such that $\psi = \varphi + \xi$.

We adopt the following notation:

(i) $AAP(\mathbb{R}^+, X) = AP(\mathbb{R}^+, X) + C_0(\mathbb{R}^+, X)$.
(ii) $Ar(\varphi, \mathbb{R}^+, X) = r(\varphi, \mathbb{R}^+, X) + C_0(\mathbb{R}^+, X)$, $\varphi \in r(\mathbb{R}, X)$.
(iii) $AAA(\mathbb{R}^+, X) = AA(\mathbb{R}^+, X) + C_0(\mathbb{R}^+, X)$.

We prove

PROPOSITION 2.2.2. *Let* $\psi \in AAP(\mathbb{R}^+, X)$ (*resp.* $Ar(\varphi, \mathbb{R}^+, X)$, $AAA(\mathbb{R}^+, X)$) *and let* $\psi = \varphi + \xi$, *where* $\varphi \in AP(\mathbb{R}^+, X)$ (*resp.* $r(\varphi, \mathbb{R}^+, X)$, $AA(\mathbb{R}^+, X)$) *and* $\xi \in C_0(\mathbb{R}^+, X)$. *Then* $\|\psi\|_\infty \geq \|\varphi\|_\infty$.

Proof. By (2.1.5) we have $\|\psi\|_\infty = \sup_{t\in\mathbb{R}^+}\|\psi(t)\| \geq \sup_{t\in\mathbb{R}^+}\|\psi(t+s)\| = \|R_s\psi\|_\infty$, $s\in\mathbb{R}^+$. Hence $u(s)=\|R_s\psi\|_\infty$ is a decreasing bounded function and its limit $\lim_{s\to\infty}u(s)$ exists. From $\|R_s\psi\|_\infty = \|R_s\varphi + R_s\xi\|_\infty \geq \|R_s\varphi\|_\infty - \|R_s\xi\|_\infty$, $s\in\mathbb{R}^+$, and $\|R_s\xi\|_\infty \to 0$ as $s\to\infty$, we get $\|\psi\|_\infty \geq \lim_{s\to\infty}\|R_s\psi\|_\infty \geq \lim_{s\to\infty}(\|R_s\varphi\|_\infty - \|R_s\xi\|_\infty) = \|\varphi\|_\infty$.

Now, we demonstrate the following

Theorem 2.2.3. *The classes* $AAP(\mathbb{R}^+,X)$, $Ar(\varphi,\mathbb{R}^+,X)$ *and* $AAA(\mathbb{R}^+,X)$ *are closed subspaces of* $C_{\mathrm{ub}}(\mathbb{R}^+,X)$.

Proof. It is sufficient to consider $Ar(\varphi,\mathbb{R}^+,X)$. By Proposition 2.1.6, we have $r(\varphi,\mathbb{R}^+,X)\cap C_0(\mathbb{R}^+,X)=\{0\}$. Hence each element $\Psi\in Ar(\varphi,\mathbb{R}^+,X)$ has a unique decomposition $\Psi=\Phi+\Xi$, where $\Phi\in r(\varphi,\mathbb{R}^+,X)$ and $\Xi\in C_0(\mathbb{R}^+,X)$. Let $\{\Psi_n\}\subset Ar(\varphi,\mathbb{R}^+,X)$ be a convergent sequence in $C_{\mathrm{ub}}(\mathbb{R}^+,X)$. Let $\Psi_n = \Phi_n+\Xi_n$, $n\in\mathbb{N}$, where $\{\Phi_n\}\subset r(\varphi,\mathbb{R}^+,X)$ and $\{\Xi_n\}\subset C_0(\mathbb{R}^+,X)$. Using Proposition 2.2.2, we see that $\{\Phi_n\}$ is a convergent sequence in $C_{\mathrm{ub}}(\mathbb{R}^+,X)$, and hence so is $\{\Xi_n\}$. Since $r(\varphi,\mathbb{R}^+,X)$ and $C_0(\mathbb{R}^+,X)$ are closed subspaces of $C_{\mathrm{ub}}(\mathbb{R}^+,X)$, $\lim_{n\to\infty}\Phi_n=\Phi\in r(\varphi,\mathbb{R}^+,X)$ and $\lim_{n\to\infty}\Xi_n=\Xi\in C_0(\mathbb{R}^+,X)$. Hence $\lim_{n\to\infty}\Psi_n=\Phi+\Xi\in Ar(\varphi,\mathbb{R}^+,X)$ and this proves that $Ar(\varphi,\mathbb{R}^+,X)$ is a closed subspace of $C_{\mathrm{ub}}(\mathbb{R}^+,X)$.

Next, we prove

Theorem 2.2.4. *Each of* $AAP(\mathbb{R}^+,X)$, $Ar(\varphi,\mathbb{R}^+,X)$ *and* $AAA(\mathbb{R}^+,X)$ *is a* Λ*-class. Moreover*, $AAP(\mathbb{R}^+,X)$ *is the minimal* Λ*-class from* $C_{\mathrm{ub}}(\mathbb{R}^+,X)$.

Proof. Since the proof of the first part is the same for each of $AAP(\mathbb{R}^+,X)$, $Ar(\varphi,\mathbb{R}^+,X)$ and $AAA(\mathbb{R}^+,X)$, we restrict ourselves to the case $AAP(\mathbb{R}^+,X)$. First, we show that it has property (P.Λ). Let $\psi\in AAP(\mathbb{R}^+,X)$ and let Ψ be any element of $C_{\mathrm{ub}}(\mathbb{R},X)$ such that $\psi=\Psi|_{\mathbb{R}^+}$. We show that $R_a\Psi|_{\mathbb{R}^+}\in AAP(\mathbb{R}^+,X)$, $a\in\mathbb{R}$. Let $\psi=\varphi+\xi$, where $\varphi\in AP(\mathbb{R}^+,X)$, $\xi\in C_0(\mathbb{R}^+,X)$. By Theorem 2.1.9, $\Phi=m_P^{-1}\varphi\in AP(\mathbb{R},X)$. Put $\Xi=\Psi-\Phi$. We have $(\Psi-\Phi)|_{R^+}=\Xi|_{\mathbb{R}^+}=\xi\in C_0(\mathbb{R}^+,X)$, and hence $R_a\Xi|_{\mathbb{R}^+}\in C_0(\mathbb{R}^+,X)$, $a\in\mathbb{R}$. This proves that $R_a\Psi|_{\mathbb{R}^+} = R_a\Phi|_{\mathbb{R}^+}+R_a\Xi|_{\mathbb{R}^+}\in AAP(\mathbb{R},X)$, $a\in\mathbb{R}$. Hence $AAP(\mathbb{R}^+,X)$ satisfies (P.Λ). The rest of the properties of Λ-classes follow easily from the definition of $AAP(\mathbb{R},X)$. Now, we prove that $AAP(\mathbb{R}^+,X)$ is the minimal Λ-class. Let A be a Λ-class from $C_{\mathrm{ub}}(\mathbb{R}^+,X)$. By Proposition 1.3.4, A is also a Π-class. Hence by Theorem 2.1.11, $AP(\mathbb{R}^+,X)$ is a subset of A. We also prove that $C_0(\mathbb{R}^+,X)\subset A$. First, let $\varphi\in C_0(\mathbb{R}^+,X)$ and $\operatorname{supp}\varphi\subset[0,a]$, for some $a>0$. We show that $\varphi\in A$. Indeed, consider the function $\psi\in C_{\mathrm{ub}}(\mathbb{R},X)$ defined by $\psi(t)=0$, $t\geq 0$, and $\psi(t)=\varphi(t+a)$, $-a\leq t\leq 0$, and $\psi(t)=0, t\leq -a$. Since $\psi|_{\mathbb{R}^+}=0\in C_0(\mathbb{R}^+,X)$, by property (P.$\Lambda$) of A we get $R_a\psi|_{\mathbb{R}^+}=\varphi\in A$. Since the set of all elements of $C_0(\mathbb{R}^+,X)$ with compact support is dense in it and A is closed, $C_0(\mathbb{R}^+,X)\subset A$. Since A is linear, $C_0(\mathbb{R}^+,X)+AP(\mathbb{R}^+,X)=AAP(\mathbb{R}^+,X)\subset A$. This proves that $AAP(\mathbb{R}^+,X)$ is the minimal Λ-class from $C_{\mathrm{ub}}(\mathbb{R}^+,X)$.

2.3. Eberlein almost periodic functions. We prove that the class of all Eberlein almost periodic functions from $C_{\mathrm{ub}}(\mathbb{J},X)$ is a Λ-class. First, we give

Definition 2.3.1. A function $\varphi \in C_{\mathrm{ub}}(\mathbb{J},X)$ is said to be *Eberlein almost periodic* (E.a.p.) if and only if the set $H(\varphi)=\{R_s\varphi : s\in\mathbb{J}\}$ is weakly relatively compact in $C_{\mathrm{ub}}(\mathbb{J},X)$. The set of all E.a.p. functions from $C_{\mathrm{ub}}(\mathbb{J},X)$ will be denoted by $W(\mathbb{J},X)$ (cf. [25], [38], [41]–[43]).

It is well known that a Bohl–Bohr–Bochner function $\varphi\in AP(\mathbb{J},X)$ has the property that $H(\varphi)=\{R_s\varphi : s\in\mathbb{J}\}$ is relatively compact in $C_{\mathrm{ub}}(\mathbb{J},X)$ ([1, Ch. 2], [33, Ch. 2]). Hence we have the following:

$$AP(\mathbb{J},X)\subset W(\mathbb{J},X). \tag{2.3.1}$$

Since the range of each element $\varphi\in C_0(\mathbb{J},X)$ is relatively compact, from [20, Theorem 1.3] and [38] it follows that $C_0(\mathbb{J},X)\subset W(\mathbb{J},X)$. Hence we have

$$AAP(\mathbb{J},X)=C_0(\mathbb{J},X)+AP(\mathbb{J},X)\subset W(\mathbb{J},X). \tag{2.3.2}$$

Now, we prove

Theorem 2.3.2. *$W(\mathbb{J},X)$ is a Λ-class.*

Proof. From [25] one can conclude that $W(\mathbb{J},X)$ is a TICSC-class and has property (P.2). It remains to show that if $\mathbb{J}=\mathbb{R}^+$, then $W(\mathbb{R}^+,X)$ has property (P.Λ). Let $\psi\in W(\mathbb{R}^+,X)$ and let $\Psi\in C_{\mathrm{ub}}(\mathbb{R},X)$ be any function such that $\Psi|_{\mathbb{R}^+}=\psi$. We prove that $R_a\Psi|_{\mathbb{R}^+}\in W(\mathbb{R}^+,X)$, $a\in\mathbb{R}$. Let $\xi=R_a\Psi|_{\mathbb{R}^+}$. We show that the set $H(\xi)=\{R_s\xi : s\in\mathbb{R}^+\}$ is w.r.c. in $C_{\mathrm{ub}}(\mathbb{R}^+,X)$. Let $\{s'_n\}\subset\mathbb{R}^+$. If $\{s'_n\}$ is bounded, then by the Ascoli–Arzelà theorem $\{R_{s'_n}\xi\}$ has a convergent subsequence in $C_{\mathrm{ub}}(\mathbb{R}^+,X)$. If $\{s'_n\}$ is unbounded, then it has a subsequence, say $\{s_n\}$, such that $s_n\to\infty$ as $n\to\infty$. There exists $n_0\in\mathbb{N}$ such that $a+s_{n_0}\ge 0$. This implies $R_{s_n}\xi(t)=\xi(t+s_n)=\Psi(a+t+s_n)|_{\mathbb{R}^+}=\psi(t+a+s_n)$, $n\ge n_0$. Since $\psi\in W(\mathbb{R}^+,X)$, the sequence $\{R_{a+s_n}\psi\}$ has a weakly convergent subsequence in $C_{\mathrm{ub}}(\mathbb{R}^+,X)$, and this implies that $H(R_a\Psi|_{\mathbb{R}^+})$ is w.r.c. in $C_{\mathrm{ub}}(\mathbb{R}^+,X)$ and proves that $W(\mathbb{R}^+,X)$ has property (P.Λ).

Now, we give the following

Example 2.3.3. The function $\varphi(t)=1/t$, $t\ge 1$, $t=1, t<1$ has the following properties:

(a) $\varphi\in C_{\mathrm{ub}}(\mathbb{R},C)$,
(b) $R_a\varphi|_{\mathbb{R}^+}\in C_0(\mathbb{R}^+,C)$, $a\in\mathbb{R}$,
(c) $\varphi\notin W(\mathbb{R},C)$.

This implies that Theorem 2.1.12 is not true for $W(\mathbb{R}^+,X)$.

We conclude this subsection by showing that uniform continuity in Definition 2.3.1 is superfluous (see [20, Theorem 13.1], [25, Theorem 1.16], [43, Proposition 2.1]).

THEOREM 2.3.4. *Let $\varphi \in C_{\mathrm{b}}(\mathbb{J}, X)$ and let $H(\varphi)$ be weakly relatively compact in $C_{\mathrm{b}}(\mathbb{J}, X)$. Then φ is uniformly continuous.*

Proof. It is sufficient to prove the case $\mathbb{J} = \mathbb{R}^+$. Set $S_a\varphi(t) = \int_0^a \varphi(t + x)\,dx$. By Lemma 1.2.2, we have $S_1\varphi \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$. It is easy to verify that the sequence $\{(1/n)\sum_{j=0}^{n-1} R_{j/n}\varphi(t)\}$ converges to $S_1\varphi(t)$ for all $t \in \mathbb{R}^+$. Since $\varphi_n = (1/n)\sum_{j=0}^{n-1} R_{j/n}\varphi \in \operatorname{co} H(\varphi)$ for all $n \in \mathbb{N}$, by the Krein–Šmul'yan theorem, $\{\varphi_n\}$ is weakly relatively compact and hence $\varphi_n \to S_1\varphi$ as $n \to \infty$ weakly and $S_1\varphi \in \overline{\operatorname{co}} H(\varphi)$. In the same way, one can show that $\{nS_{1/n}\varphi\} \subset \overline{\operatorname{co}} H(\varphi)$ and $nS_{1/n}\varphi \to \varphi$ as $n \to \infty$ weakly in $C_{\mathrm{b}}(\mathbb{R}^+, X)$. By Lemma 1.2.2, we have $\psi_n = nS_{1/n}\varphi \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$ for all $n \in \mathbb{N}$. Using Mazur's theorem ([47, p. 120]), there exists a sequence $\{\xi_m\}$ of finite convex combinations from $\{\psi_n\}$ which converges to φ in $C_{\mathrm{b}}(\mathbb{R}^+, X)$. Since ξ_m is a finite sum of elements from $\{\psi_n\}$, we get $\{\xi_m\} \subset C_{\mathrm{ub}}(\mathbb{R}^+, X)$ and hence its limit φ is uniformly continuous.

2.4. Ergodic and totally ergodic functions. In this part we note an important property of the classes of a.p., a.a.p., and E.a.p. functions, namely, the ergodicity property. We give

DEFINITION 2.4.1. A function $\varphi \in C_{\mathrm{ub}}(\mathbb{J}, X)$ is said to be *ergodic* if and only if $\lim_{T\to\infty}(1/T)\int_0^T \varphi(t+s)\,ds$ (for $\mathbb{J} = \mathbb{R}^+$) (resp. $\lim_{T\to\infty}(1/(2T))\int_{-T}^T \varphi(t+s)\,ds$ (for $\mathbb{J} = \mathbb{R}$)) uniformly converges to a constant function $a_\varphi(t) = a_\varphi \in X$. We denote by $E(\mathbb{J}, X)$ the set of all ergodic functions from $C_{\mathrm{ub}}(\mathbb{J}, X)$.

From Definition 2.4.1, one can easily show

THEOREM 2.4.2. *$E(\mathbb{J}, X)$ is a TICSC class of $C_{\mathrm{ub}}(\mathbb{J}, X)$.*

We prove

THEOREM 2.4.3. *$E(\mathbb{J}, X)$ has property* (P.Λ).

Proof. If $\mathbb{J} = \mathbb{R}$, then property (P.Λ) follows directly from Definition 2.4.1. Let $\mathbb{J} = \mathbb{R}^+$, $\psi \in E(\mathbb{R}^+, X)$ and let $\Psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ be such that $\Psi|_{\mathbb{R}^+} = \psi$. We show that $R_a\Psi|_{\mathbb{R}^+} \in E(\mathbb{R}^+, X)$ for all $a \in \mathbb{R}$. Indeed, from the identity

$$\int_0^T \Psi(t+x)\,dt = \int_0^T \Psi(t+x+a)\,dt + \int_0^a \Psi(t+x)\,dt + \int_{a+T}^T \Psi(t+x)\,dt$$

we get $\lim_{T\to\infty}(1/T)\int_0^T R_a\Psi(t+x)\,dt = \lim_{T\to\infty}(1/T)\int_0^T \Psi(t+x)\,dt$ for all $a \in \mathbb{R}$. Hence $R_a\Psi|_{\mathbb{R}^+} \in E(\mathbb{R}^+, X)$ for all $a \in \mathbb{R}$.

We pause to give the following

Remark 2.4.4. If $\varphi \in C_{\mathrm{ub}}(\mathbb{J}, X)$, then $R_s\varphi - \varphi \in E(\mathbb{J}, X)$, $s \in \mathbb{J}$.

Proof. It is sufficient to prove the case $\mathbb{J} = \mathbb{R}^+$. It follows from the identity

$$\int_0^T (\varphi(t+x) - \varphi_s(t+x))\,dt = \int_0^s \varphi(t+x)\,dt + \int_{T+s}^T \varphi(t+x)\,dt$$

that

$$\lim_{T\to\infty}(1/T)\int\limits_0^T(\varphi(t+x)-\varphi_s(t+x))\,dt$$

converges to 0 as $T\to\infty$, uniformly with respect to $x\in\mathbb{R}^+$.

Definition 2.4.5. A function $\varphi\in C_{\mathrm{ub}}(\mathbb{J},X)$ is said to be *totally ergodic* if and only if $\widehat{\lambda}\varphi\in E(\mathbb{J},X)$, $\widehat{\lambda}\in\widehat{\mathbb{R}}$. The set of all totally ergodic functions will be denoted by $TE(\mathbb{J},X)$.

Theorems 2.4.2 and 2.4.3 and Definition 2.4.5 yield

Theorem 2.4.6. *$TE(\mathbb{J},X)$ is a Λ-class from $C_{\mathrm{ub}}(\mathbb{J},X)$.*

We conclude this subsection by

Theorem 2.4.7. *$AAP(\mathbb{J},X)\subset W(\mathbb{J},X)\subset TE(\mathbb{J},X)\subset E(\mathbb{J},X)$.*

Proof. From (2.3.2) we have $AAP(\mathbb{J},X)\subset W(\mathbb{J},X)$, and from Definition 2.4.5 we get $TE(\mathbb{J},X)\subset E(\mathbb{J},X)$. It is sufficient to consider the case $\mathbb{J}=\mathbb{R}^+$. We only need to prove that $W(\mathbb{J},X)\subset E(\mathbb{J},X)$, for $W(\mathbb{J},X)$ is a Λ-class. We follow the method of W. F. Eberlein [20]. The Bohr means

$$\frac{1}{T}\int\limits_0^T\varphi(t+s)\,dt,\qquad \varphi\in C_{\mathrm{ub}}(\mathbb{R}^+,X),\ T>0,$$

is a system of almost invariant integrals (see [20]) for the semigroup $\mathbb{R}^+$. If $\varphi\in W(\mathbb{J},X)$ then by the Krein–Shmul'yan theorem ([19, p. 434]) the closure $\overline{\mathrm{co}}H(\varphi)$ of the convex hull of the set of translates $H(\varphi)$ is weakly compact in $C_{\mathrm{ub}}(\mathbb{J},X)$. Since the Bohr means is a subset of $\overline{\mathrm{co}}H(\varphi)$, by [20, Theorem 3.1], we conclude that $\varphi\in E(\mathbb{J},X)$ (see also [43]).

3. Problems concerning indefinite integral of TICSC-classes

We solve problem (P) for TICSC-classes from $C_{\mathrm{ub}}(\mathbb{J},X)$ and the related difference problem. We discuss sufficient conditions for ergodicity of functions from $C_{\mathrm{ub}}(\mathbb{J},X)$. We apply our results to study the indefinite integral of the weak classes introduced by Ruess and Summers [42].

3.1. Two theorems on indefinite integral and difference problem.

The following theorem extends several results on the indefinite integral of many classes (see [4], [32], [42]).

Theorem 3.1.1. *Let $A\subset C_{\mathrm{ub}}(\mathbb{J},X)$ be a TICSC-class. Let $\varphi\in A$ and let $\psi(t)=\int_0^t\varphi(x)\,dx$. If $\psi\in E(\mathbb{J},X)$, then $\psi\in A$. Moreover, if the range of φ is relatively compact* (*weakly relatively compact*), *then the range of ψ is r.c.* (*w.r.c.*).

Proof. It is sufficient to prove the case $\mathbb{J} = \mathbb{R}^+$. Consider the difference $\Delta_s\psi(t) = R_s\psi(t) - \psi(t) = \int_0^s \varphi(t+x)\,dx$, $s \in \mathbb{J}$. By Lemma 1.2.2, we have $\Delta_s\psi \in L^+(\varphi)$, $s \in \mathbb{R}^+$. Using properties (P.1) and (P.4) of A, $L^+(\varphi) \subset A$ for all $\varphi \in A$. This implies $\Delta_s\psi \in A$ for all $s \in \mathbb{R}^+$. Again, applying Lemma 1.2.2, we get

$$\frac{1}{T}\int_0^T \Delta_s\psi\,ds \in L^+(\varphi) \subset A \text{ for all } T > 0. \tag{3.1.1}$$

Since $\psi \in E(\mathbb{R}^+, X)$, $\lim_{T\to\infty}(1/T)\int_0^T R_s\psi\,ds$ exists and converges to a constant function $a_\psi(t) = a_\psi \in X$. Hence $\lim_{T\to\infty}(1/T)\int_0^T \Delta_s\psi\,ds = a_\psi - \psi$. Using (3.1.1) and property (P.1) of A, we get $a_\psi - \psi \in L^+(\varphi) \subset A$. By property (P.3), A contains all the constant functions, hence $\psi = a_\psi - (a_\psi - \psi) \in A$.

Now, if φ has r.c. (w.r.c.) range, then each element of $L^+(\varphi)$ has r.c. (w.r.c.) range. This implies that $a_\psi - \psi$ has r.c. (w.r.c.) range. Hence ψ has r.c. (w.r.c.) range.

In fact, we established the following (difference problem).

THEOREM 3.1.2. *Let $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ be a TICSC-class. Let $\psi \in E(\mathbb{J}, X)$ and $\Delta_s\psi = R_s\psi - \psi \in A$, $s \in \mathbb{J}$. Then $\psi \in A$. Moreover, if $\Delta_s\psi$ has r.c.* (*w.r.c.*) *range for each $s \in \mathbb{J}$, then ψ has r.c.* (*w.r.c.*) *range.*

Remark 3.1.3. Theorem 3.1.1 was proved by B. M. Levitan for the class $AP(\mathbb{R}, X)$ [32]. We extended Levitan's result to the classes of recurrent and almost automorphic functions [4]. In a private communication G. Muraz obtained this result for the class $W(\mathbb{R}, X)$.

We give the following consequences.

COROLLARY 3.1.4. *Let $\varphi \in AP(\mathbb{R}, X)$* (*resp. $r(\varphi, \mathbb{R}, X)$, $AA(\mathbb{R}, X)$*) *and $\psi(t) = \int_0^t \varphi(x)\,dx$. Let $\psi|_{\mathbb{R}^+} \in E(\mathbb{R}^+, X)$. Then $\psi \in AP(\mathbb{R}, X)$* (*resp. $r(\varphi, R, X)$, $AA(\mathbb{R}, X)$*).

Proof. Since the proof is the same for the three cases, we carry it out for $AP(\mathbb{R}, X)$. Since $AP(\mathbb{R}^+, X)$ is a TICSC-class, by Theorem 3.1.1, $\psi|_{\mathbb{R}^+} \in AP(\mathbb{R}^+, X)$. Let $\Psi = m_P^{-1}\psi|_{R^+}$, where m_P^{-1} is the inverse of the operator m_P defined in Theorem 2.1.9. By Theorem 2.1.14, we have $n(R_{1/n}\psi|_{\mathbb{R}^+} - \psi|_{\mathbb{R}^+}) \to \varphi|_{\mathbb{R}^+}$ as $n \to \infty$. Hence by Theorem 2.1.9, $n(R_{1/n}\Psi - \Psi) \to \varphi$ as $n \to \infty$. This means that $\varphi = \Psi' \in AP(\mathbb{R}, X)$.

COROLLARY 3.1.5. *Let $\varphi \in AAP(\mathbb{R}^+, X)$ and $\psi(t) = \int_0^t \varphi(x)\,dx$. If $\psi \in W(\mathbb{R}^+, X)$, then $\psi \in AAP(\mathbb{R}^+, X)$* (*see* [42]).

Proof. By Theorem 2.4.7, $W(\mathbb{R}^+, X)$ is a subset of $E(\mathbb{R}^+, X)$. Hence from Theorem 3.1.1 it follows that $\psi \in AAP(\mathbb{R}^+, X)$.

Corollary 3.1.6. *Let* $\psi \in AAP(\mathbb{R}^+, X)$ *and let* $\psi = \varphi + \xi$ *where* $\varphi \in AP(\mathbb{R}^+, X)$ *and* $\xi \in C_0(\mathbb{R}^+, X)$. *Let* $\Psi(t) = \int_0^t \psi(x)\,dx$. *Assume that*

(i) Ψ *is bounded and* X *does not contain a subspace isomorphic to the Banach space* c_0 (*of complex sequences converging to zero*) *or the range of* Ψ *is weakly relatively compact.*

(ii) $\int_0^\infty \xi(x)\,dx$ *exists as an improper Riemann integral.*

Then $\Psi \in AAP(\mathbb{R}^+, X)$ (*see* [42]).

Proof. Let $\Xi(t) = \int_0^t \xi(x)\,dx$. Since $\lim_{N\to\infty} \int_N^\infty \xi(x)\,dx = 0$, we get $\Xi \in E(\mathbb{R}^+, X)$. Hence by Theorem 3.1.1, we obtain $\Xi \in AAP(\mathbb{R}^+, X)$ and therefore its range is relatively compact. This implies that $\Phi(t) = \int_0^t \varphi(x)\,dx$ satisfies

(K) Φ is bounded and X does not contain a copy of c_0 or the range of Φ is weakly relatively compact.

Hence by [3] or [30], we get $\Phi \in AP(\mathbb{R}^+, X)$ and therefore $\Psi = \Phi + \Xi \in AAP(\mathbb{R}^+, X)$.

We recall that condition (ii) was used by Fréchet [24] in the case of functions from $C_0(\mathbb{R}^+, C)$ and by Ruess and Summers [42] in the case of functions from $C_0(\mathbb{R}^+, X)$. Condition (K) is due to Kadets [30].

We end this subsection by the following

Corollary 3.1.7. *Let* φ *be an element of one of the classes* $\{AAP(\mathbb{R}^+, X), Ar(\mathbb{R}^+, X), AAA(\mathbb{R}^+, X), W(\mathbb{R}^+, X), TE(\mathbb{R}^+, X)\}$. *Let* $\psi(t) = \int_0^t \varphi(x)\,dx$. *If* $\psi \in E(\mathbb{J}, X)$, *then* ψ *belongs to the same class as* φ.

The proof follows directly from Theorem 3.1.1 and the fact that all these spaces are TICSC-classes.

3.2. Weak classes. In the following we apply our results to the case of weakly asymptotic and weakly Eberlein almost periodic functions introduced by Ruess and Summers [42]. We denote by $C_{\mathrm{wub}}(\mathbb{J}, X)$ the set of all weakly uniformly continuous bounded functions defined on $\mathbb{J}$ with values in X. If $\varphi \in C_{\mathrm{wub}}(\mathbb{J}, X)$ and $x^* \in X^*$, then $x^*(\varphi)$ will denote the function having the value $x^*(\varphi(t))$, $t \in \mathbb{J}$.

Definition 3.2.1. A function $\varphi \in C_{\mathrm{wub}}(\mathbb{J}, X)$ is said to be *weakly asymptotically almost periodic* (w.a.a.p.) if and only if $x^*(\varphi)$ is a.a.p., $x^* \in X^*$. Similarly, one can define weakly Eberlein almost periodic (w.E.a.p.) and weakly ergodic functions.

As a consequence of Theorem 3.1.1, we get

Theorem 3.2.2. *Let* $\varphi \in C_{\mathrm{wub}}(\mathbb{J}, X)$ *be w.a.a.p.* (*w.E.a.p.*). *If* $\psi(t) = \int_0^t \varphi(x)\,dx$ *is weakly ergodic*, *then it is w.a.a.p.* (*w.E.a.p.*).

Since w.E.a.p. functions are weakly ergodic [20, Theorem 5.2], we get

COROLLARY 3.2.3. *Let* $\varphi \in C_{\mathrm{wub}}(\mathbb{J}, X)$ *be a w.a.a.p. function. If* $\psi(t) = \int_0^t \varphi(x)\,dx$ *is w.E.a.p, then it is w.a.a.p* (*see* [42]).

In the following we show that weak ergodicity is not sufficient to solve problem (P) even for almost periodic functions with values in the Banach space c_0. (See also [32].)

EXAMPLE 3.2.4. The function $\varphi : \mathbb{R} \to c_0$ defined by $\varphi(t) = \{(1/n)\cos(t/n)\}$ is almost periodic. Its indefinite integral $\psi(t) = \{\sin(t/n)\}$ is bounded and hence by the Bohl–Bohr theorem ([22, p. 79]), it is weakly almost periodic. This implies that ψ is weakly ergodic. Since the range of ψ is not relatively compact, ψ is not almost periodic. Hence weak ergodicity of the integral is not sufficient to solve problem (P) for almost periodic functions.

3.3. Some conditions for ergodicity. In the following we discuss some conditions sufficient for ergodicity.

PROPOSITION 3.3.1. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{J}, X)$ *and its derivative* $\varphi' \in C_{\mathrm{u}}(\mathbb{J}, X)$. *Then* $\varphi' \in E(\mathbb{J}, X)$.

Proof. It is sufficient to prove the case $\mathbb{J} = \mathbb{R}^+$. By Theorem 1.4.1, we get $\varphi' \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$. From the equality $(1/T)\int_0^T \varphi'(x+t)\,dt = (\varphi(x+T) - \varphi(x))/T$, $T > 0$, we get $\varphi' \in E(\mathbb{R}^+, X)$.

As a consequence, we obtain

PROPOSITION 3.3.2. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{J}, X)$ *and* $\psi(t) = \int_0^t \varphi(x)\,dx$. *If* $\psi \in C_{\mathrm{b}}(\mathbb{J}, X)$, *then* $\varphi \in E(\mathbb{J}, X)$.

Proof. Since φ is bounded, its indefinite integral ψ is uniformly continuous. Hence $\psi \in C_{\mathrm{ub}}(\mathbb{J}, X)$ and by Proposition 3.3.1, $\varphi = \psi' \in E(\mathbb{J}, X)$.

We prove the following criterion:

THEOREM 3.3.3. *Let* $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$ *be a TICSC-class and let* $\varphi \in A$. *Let* $\psi(t) = \int_0^t \varphi(x)\,dx$ *and* $\Phi(t) = \int_0^t \psi(x)\,dx$. *If* $\Phi \in C_{\mathrm{ub}}(\mathbb{J}, X)$, *then* $\psi \in A$.

Proof. Since φ is bounded, ψ is uniformly continuous. By Theorem 1.4.1, we get $\psi = \Phi' \in C_{\mathrm{ub}}(\mathbb{J}, X)$. By Proposition 3.3.2, $\psi \in E(\mathbb{J}, X)$. By Theorem 3.1.1, we get $\psi \in A$.

4. Harmonic analysis for Λ-classes

We introduce the notion of the spectrum of a function $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ with respect to a Λ-class $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$. Throughout this section A will denote a Λ-class from $C_{\mathrm{ub}}(\mathbb{J}, X)$.

4.1. Definition and general properties of the spectrum with respect to a Λ-class $A \subset C_{\mathrm{ub}}(\mathbb{J}, X)$

PROPOSITION 4.1.1. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. *Then the set* $I_A(\varphi) = \{f \in L^1(\mathbb{R}) : \varphi * f|_{\mathbb{J}} \in A\}$ *is a closed ideal of* $L^1(\mathbb{R})$.

Proof. Property (P.1) of A implies that $I_A(\varphi)$ is a closed subspace of $L^1(\mathbb{R})$. Using property (P.Λ), we conclude that $R_a(\varphi * f)|_{\mathbb{J}} = \varphi * R_a f|_{\mathbb{J}} \in A$ for all $a \in \mathbb{R}$ and all $f \in I_A(\varphi)$. This means that $I_A(\varphi)$ is a translation invariant closed subspace of $L^1(\mathbb{R})$. By [40, Theorem 7.1.2], we see that $I_A(\varphi)$ is a closed ideal of $L^1(\mathbb{R})$.

DEFINITION 4.1.2. Let $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. Then the set

$$\sigma_A(\varphi) = \operatorname{hull} I_A(\varphi) = \{\lambda : \widehat{f}(\lambda) = 0 \text{ for all } f \in I_A(\varphi)\}$$

is called the *spectrum of* φ *with respect to* A.

It is easy to prove that $\sigma_A(\varphi)$ is a closed subset of $\mathbb{R}$.

The following notion is introduced by Loomis in the case where A is the complex valued almost periodic functions defined on a locally compact abelian group (see [34]).

DEFINITION 4.1.3. Let $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. A point $\widehat{\lambda} \in \widehat{\mathbb{R}}$ is called *regular for* φ *with respect to a* Λ*-class* A if and only if there exists $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\lambda) \neq 0$ and $\varphi * f|_{\mathbb{J}} \in A$. We denote by $\varrho_A(\varphi)$ the set of all regular points of φ with respect to A.

It is easy to verify that

$$\varrho_A(\varphi) = \mathbb{R} \setminus \sigma_A(\varphi) \text{ for all } \varphi \in C_{\mathrm{ub}}(\mathbb{R}, X). \tag{4.1.1}$$

Before proceeding we notice that if $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$, then the set $I_0(\varphi) = \{f \in L^1(\mathbb{R}) : \varphi * f = 0\}$ is a closed ideal of $L^1(\mathbb{R})$. Moreover, $I_0(\varphi) = \{f \in L^1(\mathbb{R}) : \varphi * R_a f|_{\mathbb{J}} = 0 \text{ for all } a \in \mathbb{R}\}$. Indeed, we have $\varphi * R_a f|_{\mathbb{J}} = 0$ for all $a \in \mathbb{R}$ if and only if $\varphi * f = 0$. This implies $I_0(\varphi) \subset I_A(\varphi)$, $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. Recalling that $\sigma_0(\varphi) = \operatorname{hull} I_0(\varphi)$ is the Beurling spectrum of φ [40, Ch. 7], we get

$$\sigma_A(\varphi) \subset \sigma_0(\varphi) \text{ for all } \varphi \in C_{\mathrm{ub}}(\mathbb{R}, X). \tag{4.1.2}$$

If $\varphi \in C_{\mathrm{u}}(\mathbb{R}, X)$, then T_φ defined in 1.1 is a distribution. We define

$$\sigma_0(\varphi) = \operatorname{supp} \widehat{T}_\varphi. \tag{4.1.3}$$

It is known that if $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ then (4.1.3) holds true. Using the same argument for the corresponding statements on Beurling spectrum ([19, part II, p. 988], [40, Ch. 7]), one can prove

THEOREM 4.1.4. *Let* $\varphi, \psi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *and let* $f \in L^1(\mathbb{R})$. *Then*

$$\sigma_A(\varphi) = \sigma_A(R_a \varphi) \textit{ for all } a \in \mathbb{R}. \tag{4.1.4}$$

$$\sigma_A(\varphi * f) \subset \sigma_A(\varphi) \cap \operatorname{supp} \widehat{f}. \tag{4.1.5}$$

(4.1.6) $\sigma_A(\varphi+\psi) \subset \sigma_A(\varphi) \cup \sigma_A(\psi)$.

(4.1.7) $\sigma_A(\widehat{\lambda}\varphi) = \lambda + \sigma_A(\varphi)$, *where* $\widehat{\lambda}(t) = e^{i\lambda t}$.

4.2. Further properties of $\sigma_A(\varphi)$. In the following, we investigate the case when the set $\sigma_A(\varphi)$ is finite. First, we prove the following

THEOREM 4.2.1. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. *Then* $\sigma_A(\varphi) = \emptyset$ *if and only if* $\varphi|_{\mathbb{J}} \in A$.

Proof. Let $\varphi|_{\mathbb{J}} \in A$. Then by property (P.$\Lambda$), we get $R_a\varphi|_{\mathbb{J}} \in A$ for all $a \in \mathbb{R}$. Since A is a closed linear subspace of $C_{\mathrm{ub}}(\mathbb{J}, X)$, we get $\psi|_{\mathbb{J}} \in A$ for all $\psi \in L(\varphi)$. By Lemma 1.2.1, we obtain $\varphi * f \in L(\varphi)$ and hence $\varphi * f|_{\mathbb{J}} \in A$ for all $f \in L^1(\mathbb{R})$. This implies $I_A(\varphi) = L^1(\mathbb{R})$. By Wiener's tauberian theorem [40, Corollaries 7.2.5], we get $\sigma_A(\varphi) = \emptyset$. Conversely, let $\sigma_A(\varphi) = \emptyset$. Consider the sequence $\{f_n\}$ of Lemma 1.2.3. By (4.1.5) we conclude that $\sigma_A(\varphi * f_n) = \emptyset$ for all $n \in \mathbb{N}$. By the mentioned Wiener's tauberian theorem, we get $I_A(\varphi * f_n) = L^1(\mathbb{R})$. This implies that $\varphi * f_n * f|_{\mathbb{J}} \in A$ for all $f \in L^1(\mathbb{R})$. There exists $h_n \in L^1(\mathbb{R})$ such that $\widehat{h}_n(\lambda) = 1$, $\lambda \in [-n, n]$ and $\operatorname{supp} \widehat{h}_n \subset [-(n+1), n+1]$. Since $\operatorname{supp} \widehat{f}_n \subset [-n, n]$, we get $\varphi * f_n * h_n = \varphi * f_n$ for all $n \in \mathbb{N}$. This implies $(\varphi * f_n)|_{\mathbb{J}} \in A$ for all $n \in \mathbb{N}$. By Lemma 1.2.3, $\varphi * f_n \to \varphi$ as $n \to \infty$. Hence $(\varphi * f_n)|_{\mathbb{J}} \to \varphi|_{\mathbb{J}}$ as $n \to \infty$. By property (P.1) of A, we get $\varphi|_{\mathbb{J}} \in A$.

Next, we give an application of the spectrum to the difference problem.

THEOREM 4.2.2. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. *Then* $\sigma_A(\varphi) \subset \{0\}$ *if and only if the differences* $\Delta_s\varphi|_{\mathbb{J}} \in A$ *for all* $s \in \mathbb{J}$.

Proof. Let $\Delta_s\varphi|_{\mathbb{J}} \in A$ for all $s \in \mathbb{J}$. Let λ be any element of $\mathbb{R}$ such that $\lambda \neq 0$. There exists $g \in L^1(\mathbb{R})$ such that $\widehat{g}(\lambda) \neq 0$. Choose $s_0 \in \mathbb{J}$ such that $e^{i\lambda s_0} \neq 1$. Since $\Delta_{s_0}\varphi|_{\mathbb{J}} \in A$, by Theorem 4.2.1 we get $\Delta_{s_0}(\varphi) * g|_{\mathbb{J}} \in A$. We have $\Delta_{s_0}\varphi * g = \varphi * \Delta_{s_0} g$. Since $\widehat{\Delta_{s_0} g}(\lambda) = (e^{i\lambda s_0} - 1)\widehat{g}(\lambda)$, we get $\widehat{\Delta_{s_0} g}(\lambda) \neq 0$. By Definition 4.1.3, we get $\lambda \in \varrho_A(\varphi)$ and hence by (4.1.1), $\sigma_A(\varphi) \subset \{0\}$. Conversely, let $\sigma_A(\varphi) \subset \{0\}$. By Wiener's tauberian theorem [40, Corollaries 7.2.5], we get $\{f \in L^1(\mathbb{R}) : \widehat{f}(0) = 0\} \subset I_A(\varphi)$. Let $g \in L^1(\mathbb{R})$ be any element and $s \in \mathbb{J}$ be any number. We have $(R_s\varphi - \varphi) * g = \varphi * (R_s g - g)$. Since $\widehat{R_s g}(0) - \widehat{g}(0) = 0$, for all $s \in \mathbb{R}$, we obtain $\varphi * (R_s g - g)|_{\mathbb{J}} \in A$. This implies $g \in I_A(R_s\varphi - \varphi)$ and hence $I_A(R_s\varphi - \varphi) = L^1(\mathbb{R})$. By Theorem 4.2.1, we get $\Delta_s(\varphi)|_{\mathbb{J}} \in A$.

As a consequence, we obtain

COROLLARY 4.2.3. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *and let* $\sigma_A(\varphi) \subset \{0\}$. *If* $\varphi \in E(\mathbb{J}, X)$, *then* $\varphi|_{\mathbb{J}} \in A$.

Proof. By Theorem 4.2.2, we conclude that $\Delta_s\varphi|_{\mathbb{J}} \in A$ for all $s \in \mathbb{J}$. By Theorem 3.1.2, we get $\varphi|_{\mathbb{J}} \in A$.

In the following we extend a result of Favard [21] concerning the indefinite integral of complex valued almost periodic functions. We prove

THEOREM 4.2.4. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *be such that its Beurling spectrum does not contain* 0. *Let* $\varphi|_{\mathbb{J}} \in A$ *and* $\psi(t) = \int_0^t \varphi(x)\,dx$. *Then* $\psi \in A$.

Proof. Since $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$, its indefinite integral $\psi \in C_{\mathrm{u}}(\mathbb{R}, X)$. There exist $\delta > 0$ and $f \in S(\mathbb{R})$ such that $]-\delta, \delta[\, \cap \sigma_0(\varphi) = \emptyset$, $\widehat{f}(0) = 1$ and $\operatorname{supp} \widehat{f} \subset\,]-\delta, \delta[$. This implies that $f * \varphi = 0$. Since $d(\psi * f)/dt = 0$, we conclude that $\psi * f = x$ for some $x \in X$. This implies $f * (\psi - \psi * f) = x - x = 0$ and means that the spectrum $\sigma_0(\psi - x)$ defined by (4.1.3) does not contain 0. Applying [6, Theorem 3.1], we get $\psi - x \in C_{\mathrm{ub}}(\mathbb{R}, X)$. Repeating the same argument for $\varXi(t) = \int_0^t [\psi(x) - x]\,dx$, we conclude that $\varXi \in C_{\mathrm{ub}}(\mathbb{R}, X)$. By Proposition 3.3.1, we obtain $\psi \in E(\mathbb{R}, X)$. Since $\varphi|_{\mathbb{J}} \in A$ and $\psi \in E(\mathbb{R}, X)$, we get $\psi \in A$, by Theorem 3.1.1.

We recall that a subset of $\mathbb{R}$ is called *residual* if and only if it is closed and does not contain nonvoid perfect subsets. A closed subset of $\mathbb{R}$ is residual if and only it is countable. A residual subset of $\mathbb{R}$ without isolated points is void.

The following criterion plays an important role in the study of the behaviour of solutions of many functional equations.

THEOREM 4.2.5. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *and let* $\sigma_A(\varphi)$ *be residual. If* $\varphi|_{\mathbb{J}} \in TE(\mathbb{J}, X)$, *then* $\varphi|_{\mathbb{J}} \in A$.

Proof. We show that $\sigma_A(\varphi)$ does not contain isolated points. Indeed, assuming that λ_0 is an isolated point of $\sigma_A(\varphi)$, there exists $\delta > 0$ such that the open interval $]\lambda_0 - \delta, \lambda_0 + \delta[\, \cap \sigma_A(\varphi) = \{\lambda_0\}$. Choose $f \in L_1(\mathbb{R})$ such that $\widehat{f}(\lambda_0) \neq 0$ and $\operatorname{supp} \widehat{f} \subset\,]\lambda_0 - \delta, \lambda_0 + \delta[$. Since $TE(\mathbb{J}, X)$ is a TICSC-class, $\varphi * f|_J \in TE(\mathbb{J}, X)$. By (4.1.5) and (4.1.7), $\sigma_A(\widehat{\lambda_0}^{-1}\varphi * f) \subset \{0\}$. From the assumptions, we conclude that $\widehat{\lambda_0}^{-1}\varphi * f|_{\mathbb{J}} \in E(\mathbb{J}, X)$. By Corollary 4.2.3, we obtain $\widehat{\lambda_0}^{-1}(\varphi * f)|_{\mathbb{J}} \in A$. By property (P.2) of A, $\varphi * f|_{\mathbb{J}} \in A$. This means that $\lambda_0 \notin \sigma_A(\varphi)$, a contradiction which shows that $\sigma_A(\varphi)$ does not contain isolated points and hence it is void. By Theorem 4.2.1, we conclude that $\varphi|_{\mathbb{J}} \in A$.

We remark that in fact we proved

THEOREM 4.2.6. *Let* $\varphi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ *and let* $\sigma_A(\varphi)$ *be residual. Let* $\widehat{\lambda}^{-1}\varphi|_{\mathbb{J}} \in E(J, X)$ *for all* $\lambda \in \sigma_A(\varphi)$. *Then* $\varphi|_{\mathbb{J}} \in A$.

5. Application to integro-differential difference equations

In this section A will denote a $\varLambda$-class from $C_{\mathrm{ub}}(\mathbb{R}^+, X)$. We solve problem (L) for the following equation:

$$(*) \qquad l(\omega) = \omega^{(n)}(t + t_n) + \sum_{k=0}^{n-1} a_k \omega^{(k)}(t + t_k) + \int\limits_0^\infty g(t + s)\omega(s)\,ds = \varphi(t)$$

where $\{t_0, t_1, \ldots, t_n\} \subset \mathbb{R}^+$, $\{a_0, a_1, \ldots, a_n\} \subset \mathbb{C}$, $g \in L^1(\mathbb{R}^+)$, $\varphi \in A$.

We note that this equation is studied by many authors (e.g. [5], [12], [18]) in the case $A = AP(\mathbb{R}, X)$. To our knowledge, this equation is not studied even for $A = AAP(\mathbb{R}^+, C)$.

5.1. Reduction of equations on $\mathbb{R}^+$ to equations on $\mathbb{R}$. In this subsection we reduce the study of solutions of equation $(*)$ to the study of solutions of an equation defined on $\mathbb{R}$. We denote by $C_{\mathrm{ub}}^{(n)}(-\infty, 0)$ the set of all functions $f :]-\infty, 0] \to \mathbb{C}$ such that $f, f', \ldots, f^{(n)}$ are uniformly continuous bounded functions. We prove

LEMMA 5.1.1. *There exists* $f_0, f_1, \ldots, f_n \in C_{\mathrm{ub}}^{(n)}(-\infty, 0)$ *such that* $f_j^{(k)}(0) = 1$, $k = j$ *and* $f_j^{(k)}(0) = 0, k \neq j$, *where* $k, j = 0, 1, \ldots, n$ *and* $f_j^{(0)} = f_j$, $j = 0, 1, \ldots, n$.

Proof. The functions $f_0(t) = e^{-t^{2n}}$, $f_1(t) = te^{-t^{2n}}, \ldots, f_n(t) = (t^n/n!)e^{-t^{2n}}$ satisfy the needed properties.

LEMMA 5.1.2. *Let* ω *be a bounded solution of the equation* $(*)$ *and let* $f_0, f_1, \ldots, f_n \in C_{\mathrm{ub}}^{(n)}(-\infty, 0)$ *satisfy the conditions of Lemma* 5.1.1. *Put*

(5.1.1) $\Omega(t) = \omega(t)$, $t \geq 0$, *and* $\Omega(t) = \sum_{k=0}^{n} \omega^{(k)}(0) f_k(t)$, $t \leq 0$,

(5.1.2) $G(t) = g(t)$, $t \geq 0$, *and* $G(t) = 0$, $t \leq 0$,

(5.1.3) $\Phi(t) = \varphi(t)$, $t \geq 0$, *and* $\Phi(t) = \Omega^{(n)}(t + t_n) + \sum_{k=0}^{n-1} a_k \Omega^{(k)}(t + t_k) + \int_{-\infty}^{\infty} G(t + s)\Omega(s)\, ds$, $t \leq 0$.

Then Ω *is a bounded solution of the equation*

(5.1.4) $\Omega^{(n)}(t + t_n) + \sum_{k=0}^{n-1} a_k \Omega^{(k)}(t + t_k) + \int_{-\infty}^{\infty} G(t + s)\Omega(s)\, ds = \Phi(t)$.

The proof follows from the definitions of the extended functions Ω, Φ and G.

We end this subsection by the following

LEMMA 5.1.3. *If* ω *is a bounded solution of equation* $(*)$, *then each of the functions* $\omega', \ldots, \omega^{(n)}$ *is bounded and uniformly continuous.*

Proof. By Lemma 5.1.2, Ω is a bounded solution of equation (5.1.4). By [33, Proposition 4, p. 95], we conclude that $\Omega^{(k)} \in C_{\mathrm{ub}}(\mathbb{R}, X)$, $k = 1, 2, \ldots, n$. This implies that $\omega^{(k)} \in C_{\mathrm{ub}}(\mathbb{R}^+, X)$, $k = 1, 2, \ldots, n$.

5.2. Solution of problem (L) for integro-differential difference equations. In the following we prove

THEOREM 5.2.1. *Let the Fourier transform* $\widehat{G}$ *of the function* G *defined by equation* (5.1.2) *be analytic on* $\mathbb{R}$. *Let* ω *be a solution of equation* $(*)$. *If* $\omega \in TE(\mathbb{R}^+, X)$, *then* $\omega \in A$.

Proof. Define

(5.2.1) $z(l) = \{\lambda : (i\lambda)^n e^{i\lambda t_n} + \sum_{k=0}^{n-1} a_k (i\lambda)^k e^{i\lambda t_k} + \widehat{G}(-\lambda) = 0\}$

Since $\widehat{G}$ is analytic, the set $z(l)$ is residual. Let $\lambda_0 \notin z(l)$. There exists $h \in S(\mathbb{R})$ such that $\widehat{h}(\lambda_0) \neq 0$. Since $\omega \in TE(\mathbb{R}^+, X)$, it is a uniformly continuous bounded solution of equation $(*)$. Therefore by Lemma 5.1.2, Ω is a uniformly continuous bounded solution of equation (5.1.4). By [33, Proposition 4, p. 95), $\Omega^{(k)} \in C_{\mathrm{ub}}(\mathbb{R}, X)$, $k = 1, \ldots, n$. Denoting $G(-t)$ by $\check{G}(t)$ and convoluting equation (5.1.4) by h, we get

$$(5.2.2) \qquad R_{t_n}\Omega^{(n)} * h(t) + \sum_{k=0}^{n-1} a_k R_{t_k}\Omega^{(k)} * h(t) - \check{G} * \Omega * h(t) = H * \Omega = \Phi * h$$

where $H(t) = R_{t_n}h^{(n)}(t) + \sum_0^{n-1} a_k R_{t_k} h^{(k)}(t) - \check{G} * h(t)$. The Fourier transform

$$(5.2.3) \qquad \widehat{H}(\lambda) = \widehat{h}(\lambda)[(i\lambda)^n e^{i\lambda t_n} + \sum_{k=0}^{n-1} a_k (i\lambda)^k e^{i\lambda t_k} + \widehat{G}(-\lambda)]$$

shows that $\widehat{H}(\lambda_0) \neq 0$ if and only if $\lambda_0 \notin z(l)$. Since $\Phi \in C_{\mathrm{ub}}(\mathbb{R}, X)$ and $\Phi|_{\mathbb{R}^+} = \varphi \in A$, we get $\psi|_{\mathbb{R}^+} \in A$ for $\psi \in L(\Phi)$. Using Lemma 1.2.1, we obtain $\Phi * h \in L(\Phi)$ and hence $\Phi * h|_{\mathbb{R}^+} \in A$. Since $\widehat{H}(\lambda_0) \neq 0$, by Definition 4.1.3, we get $\lambda_0 \in \varrho_A(\Omega)$. Hence by (4.1.1) we get $\sigma_A(\Omega) \subset z(l)$. Since $z(l)$ is residual, by Theorem 4.2.5 we conclude that $\omega = \Omega|_{\mathbb{R}^+} \in A$.

In fact, the following is true:

THEOREM 5.2.2. *Let the Fourier transform* $\widehat{G}$ *of the function* G *defined by equation* (5.1.2) *be analytic on* $\mathbb{R}$. *Let* ω *be a bounded solution of equation* $(*)$. *If* $\widehat{\lambda^{-1}}\omega \in E(\mathbb{R}^+, X)$ *for all* $\lambda \in z(l)$, *then* $\omega \in A$.

Concluding remark. It is worth noting that the results of Sections 1, 3–5 hold true if we replace the Banach space X by a Fréchet space F. In this case $AAP(\mathbb{J}, F)$, $W(\mathbb{J}, F)$ and $TE(\mathbb{J}, F)$ are Λ-classes.

Acknowledgements. The author would like to thank Alan J. Pryde for useful discussions while this paper was being written.

References

[1] L. Amerio and G. Prouse, *Almost-periodic Functions and Functional Equations*, Van Nostrand, New York, 1971.

[2] B. Basit, *The relationship between almost periodic Levitan functions and almost automorphic functions*, Moscow Univ. Math. Bull. 26 (2) (1971), 74–77.

[3] —, *Generalization of two theorems of M. I. Kadets concerning the indefinite integral of abstract almost periodic functions*, Math. Notes 9 (1971), 181–186.

[4] —, *Note on a theorem of Levitan for the integral of almost periodic functions*, Rend. Istit. Mat. Univ. Trieste 5 (1) (1973), 9–14.

[5] —, *Spectral characterization of abstract functions*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. 28 (1) (1982), 25–34.

[6] —, *On the indefinite integral of abstract functions*, ibid. 29 (3) (1983), 49–54.

[7] B. Basit and M. Emam, *Differences of functions in locally convex spaces and applications to almost periodic and almost automorphic functions*, Ann. Polon. Math. 61 (1983), 193–201.

[8] B. Basit and V. V. Zhikov, *Almost periodic solutions of integro-differential equations in Banach spaces*, Moscow Univ. Math. Bull. 26 (1) (1971), 79–81.

[9] A. G. Baskakov, *Spectral criteria for almost periodicity of solution of functional equations*, Math. Notes 24 (1978), 606–612.

[10] S. Bochner, *Abstrakte fastperiodische Funktionen*, Acta Math. 61 (1933), 149–184.

[11] —, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 1233–1236.

[12] S. Bochner and J. von Neumann, *On compact solutions of operational differential equations*, Ann. of Math. 36 (1935), 255–291.

[13] C. Corduneanu, *Almost Periodic Functions*, Wiley, New York, 1968.

[14] K. De Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. 105 (1961), 61–97.

[15] —, —, *Almost periodic functions on semigroups*, ibid., 99–140.

[16] —, —, *The decomposition of certain group representations*, J. Analyse Math. 15 (1965) 135–192.

[17] Y. Domar, *Harmonic analysis based on certain commutative Banach algebras*, Acta Math. 9 (6) (1956), 1–66.

[18] R. Doss, *On almost periodic solutions of integro-differential difference equations*, Ann. of Math. 81 (1965), 117–123.

[19] N. Dunford and J. T. Schwartz, *Linear Operators*, Parts I and II, Interscience, New York, 1963.

[20] W. F. Eberlein, *Abstract ergodic theorems and weak almost functions*, Trans. Amer. Math. Soc. 67 (1949), 217–240.

[21] J. Favard, *Sur les équations différentielles à coefficients presque-périodiques*, Acta Math. 51 (1927), 31–81.

[22] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Math. 377, Springer, 1974.

[23] P. Flor, *Rhytmische Abbildung abelscher Gruppen II*, Z. Wahrsch. Verw. Gebiete 7 (1967), 17–28.

[24] M. Fréchet, *Les fonctions asymptotiquement presque-périodiques continues*, C. R. Acad. Sci. Paris 213 (1941), 520–522.

[25] S. Goldberg and P. Irwin, *Weakly almost periodic vector-valued functions*, Dissertationes Math. 157 (1979).

[26] H. Günzler, *Integration of almost periodic functions*, Math. Z. 102 (1967), 153–287.

[27] K. Iseki, *Vector valued functions on semigroups I, II, III*, Proc. Japan Acad. (1955), 16–19, 152–155, 699–702.

[28] K. Jacobs, *Ergodentheorie und fastperiodische Funktionen auf Halbgruppen*, Math. Z. 64 (1956), 298–338.

[29] —, *Fastperiodizitätseigenschaften allgemeiner Halbgruppen in Banachraumen*, ibid. 67 (1957), 82–92.

[30] M. I. Kadets, *On the integration of almost periodic functions with values in Banach spaces*, Functional Anal. Appl. 3 (1969), 228–230.

[31] B. J. Levin, *On almost periodic Levitan functions*, Ukrainsk. Mat. Zh. 1 (1949), 49–101 (in Russian).

[32] B. M. Levitan, *Integration of almost periodic functions with values in Banach spaces*, Math. USSR-Izv. 30 (1966), 1101–1110.

[33] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge Univ. Press, Cambridge, 1982.

[34] L. H. Loomis, *Spectral characterization of almost periodic functions*, Ann. of Math. 72 (1960), 362–368.

[35] Yu I. Lyubich, *Introduction to the Theory of Banach Representations of Groups*, Birkhäuser, Basel, 1988.

[36] Yu I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. 88 (1988), 34–37.

[37] W. Maak, *Fastperiodische Funktionen auf Halbgruppen*, Acta Math. 87 (1952), 33–58.

[38] P. Milnes, *On vector valued weakly almost periodic functions*, J. London Math. Soc. (2) 22 (1980), 467–472.

[39] A. Reich, *Präkompakt Gruppen und Fastperiodizität*, Math. Z. 116 (1970), 218–234.

[40] W. Rudin, *Harmonic Analysis on Groups*, Interscience, New York, London, 1962.

[41] W. M. Ruess and W. H. Summers, *Weakly almost periodicity and the strong ergodic limit theorem for contraction semigroups*, Israel J. Math. 64 (2) (1988), 139–157.

[42] —, —, *Integration of asymptotically almost periodic functions and weak asymptotic almost periodicity*, Dissertationes Math. 279 (1989).

[43] —, —, *Ergodic theorems for semigroups of operators*, Proc. Amer. Math. Soc. 114 (2) (1992), 423–432.

[44] W. A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. 87 (1965), 719–751.

[45] Vũ Quôc Phóng and Yu. I. Lyubich, *A spectral criterion of asymptotic almost periodicity for uniformly continuous representations of abelian groups*, J. Soviet Math. 51 (1990), 1263–1266.

[46] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Appl. Math. Sci. 14, Springer, 1975.

[47] K. Yosida, *Functional Analysis*, Springer, 1965.

[48] S. Zaidman, *Almost-Periodic Functions*, Pitman, London, 1985.