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SOME MULTIPLE-PART WIENER–HOPF PROBLEMS IN MATHEMATICAL PHYSICS

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1. Generalized Sommerfeld half-plane problems

1.1. Introduction. In the book by B. Noble (1958), e.g., the famous *Sommerfeld diffraction problem* is studied by means of the so-called *Wiener–Hopf technique*, a function-theoretic method developed for solving certain mixed boundary value problems. This has been generalized to mixed b.v.p.s. for pseudodifferential equations particularly by G. I. Èskin (1973) in his book.

Here we are going to present some generalizations of the classical Sommerfeld problem leading to systems of *two-part Wiener–Hopf functional equations*. Only a few explicit solutions are known which could serve as standard problems for more general mixed boundary value problems with linear boundaries. First we shall describe the Wiener–Hopf method for the Sommerfeld problems with equal boundary conditions on both faces of the half-plane before passing to cases with different conditions and for systems of semi-infinite parallel plates. Lateron, in Chapter 2, we are concernced with multiple-part Wiener–Hopf equations which arise from mixed boundary and boundary-transmission problems in the theory of unsteady plane subsonic flow. Higher dimensional WH-problems exist in the theory of diffraction of electromagnetic or elastodynamic waves. Much less is known in these fields and a general theory of multiple-part WH-equations is still to be developed.

1.2. The classical Sommerfeld problems. Let there be given the screen S as the half-plane $\{(x, y, z) \in \boldsymbol{R}^3\colon x \geqslant 0, y = 0, -\infty < z < \infty\}$ and a plane wave

$$\Phi_{\mathrm{inc}}(x, y) := \exp[ik(x\cos\theta + y\sin\theta)] \tag{1.1}$$

where $k = k_1 + ik_2$; $k_1, k_2 \geqslant 0$ $(k \neq 0)$; $0 < \theta < \pi$, falling upon the screen $\mathfrak{s}$. One is then interested in the reaction of the screen, i.e., one wants to know the reflected, transmitted, and diffracted waves below and above the plane of the screen.

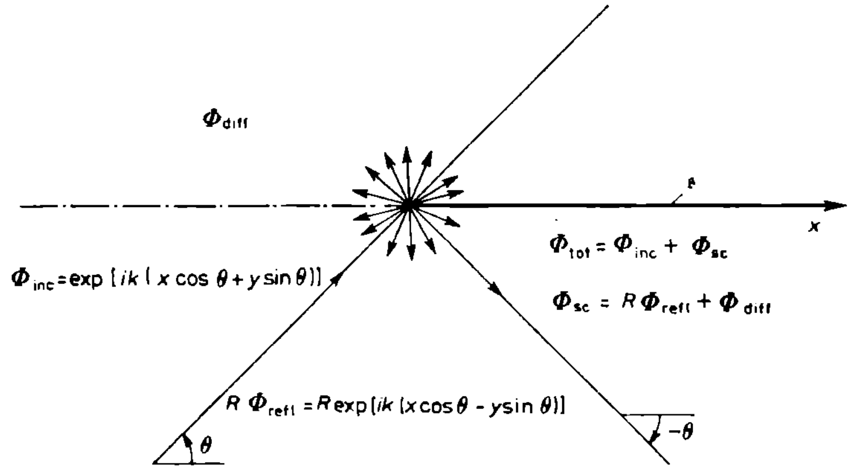

Fig. 1. Plane wave falling upon a half-plane as screen $\mathfrak{s}$.

Depending on the character of the incident wave as an *E-polarized wave* with the vector of electric intensity being parallel to the z-axis, that is parallel to the edge of the screen $\mathfrak{s}$, or a *H-polarized wave* with the magnetic vector $\boldsymbol{H}$ parallel to the edge or – in case of acoustic waves – with a soft or hard boundary we are lead to the following boundary value problems for the total wave potential $\Phi_{\text{tot}}(x, y)$ in $\boldsymbol{R}^2$-space:

(1.2) $$(\Delta + k^2)\Phi_{\text{tot}} = 0 \quad \text{in} \quad \boldsymbol{R}^2$$

with the boundary conditions

(1.3a) $$\lim_{y \to \pm 0} \Phi_{\text{tot}}(x, y) =: \Phi_{\text{tot}}(x, \pm 0) = 0 \qquad \text{for} \quad x > 0$$

or

(1.3b) $$\lim_{y \to \pm 0} \frac{\partial}{\partial y}\Phi_{\text{tot}}(x, y) = \frac{\partial \Phi_{\text{tot}}}{\partial y}(x, \pm 0) = 0 \qquad \text{for} \quad x > 0.$$

Now, if we decompose,

(1.4) $$\Phi_{\text{tot}}(x, y) = \begin{cases} \Phi_{\text{inc}}(x, y) + \Phi_{\text{diff}}(x, y), & y > 0, \\ \Phi_{\text{inc}}(x, y) + R \cdot \Phi_{\text{refl}}(x, y) + \Phi_{\text{diff}}(x, y), & y < 0, \end{cases}$$

we have the additional asymptotic conditions

(1.5) $$\Phi_{\text{diff}}(x, y) = O(1), \qquad \text{grad}\, \Phi_{\text{diff}}(x, y) = O(r^{-\beta})$$

as $r = \sqrt{x^2 + y^2} \to 0$ (*edge condition*) where $0 \leqslant \beta < 1$ and

(1.6a) $$\Phi_{\text{diff}}(x, y), \qquad \text{grad}\, \Phi_{\text{diff}}(x, y) = O(1),$$

(1.6b) $$\frac{\partial \Phi}{\partial r} - ik \cdot \Phi(x, y) = O(r^{-1/2})$$

as $r = \sqrt{x^2 + y^2} \to +\infty$ (*Sommerfeld's radiation condition*).

Representing the scattered field by simple and double layers along the screen as

(1.7) $$\Phi_{\text{diff}}(x, y) = -\frac{1}{2i}\int_0^\infty \left\{H_0^{(1)}(k\sqrt{(x-\xi)^2+y^2})\left[\frac{\partial \Phi_{\text{diff}}}{\partial \eta}(\xi, +0) - \frac{\partial \Phi_{\text{diff}}}{\partial \eta}(\xi, -0)\right] - \right.$$
$$\left. -\frac{\partial}{\partial \eta} H_0^{(1)}(k\sqrt{(x-\xi)^2+(y-\eta)^2})\Big|_{\eta=0}[\Phi_{\text{diff}}(\xi, +0) - \Phi_{\text{diff}}(\xi, -0)]\right\} d\xi$$

shows the following asymptotic behaviour of the total scattered field $\Phi_{\text{sc}}(x, y) := \Phi_{\text{tot}}(x, y) - \Phi_{\text{diff}}(x, y)$:

(1.8) $$\Phi_{\text{sc}}(x, y) = \begin{cases} O(e^{-k_2 x\cos\theta}) & \text{for } x \to +\infty, \\ O(e^{-k_2 r\sin\delta}) & \text{for } r \to +\infty, \end{cases}$$

in $0 < \delta \leqslant \arg(x, y) \leqslant 2\pi - \delta$. We shall apply the Fourier transformation with respect to x

(1.9) $$\hat{\Phi}_{\text{sc}}(\lambda, y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\lambda}\Phi_{\text{sc}}(x, y)\,dx, \qquad \lambda \in C,$$

and see that it exists in the strip $S := \{\lambda \in C\colon -\infty < \operatorname{Re}\lambda < +\infty,\ -k_2\cos\theta < \operatorname{Im}\lambda < k_2\}$. We denote the jump of the normal derivative by $J(x)$ for $x > 0$ in the case of Dirichlet data $-e^{ikx\cos\theta}$ and the jump of the boundary value on the screen by $Q(x)$ in the case of Neumann data $-ik\sin\theta\, e^{ikx\cos\theta}$.

The Helmholtz wave equation (1.2) which holds also for the scattered field is then transformed into

(1.10) $$\left[\frac{d^2}{dy^2} + (k^2 - \lambda^2)\right]\hat{\Phi}_{\text{sc}}(\lambda, y) = 0 \qquad \text{for } y \gtrless 0$$

with the general solution

(1.11) $$\hat{\Phi}_{\text{sc}}(\lambda, y) = \begin{cases} A_1(\lambda)\exp[-y\sqrt{\lambda^2-k^2}] + B_1(\lambda)\exp[y\sqrt{\lambda^2-k^2}] & \text{for } y > 0 \\ A_2(\lambda)\exp[-y\sqrt{\lambda^2-k^2}] + B_2(\lambda)\exp[y\sqrt{\lambda^2-k^2}] & \text{for } y < 0 \end{cases}$$

with the square root $\sqrt{\lambda^2-k^2}$ defined by

(1.12a) $$\sqrt{\lambda^2-k^2} = \sqrt{\lambda-k}\cdot\sqrt{\lambda+k} = |\lambda^2-k^2|^{1/2}\cdot\exp\{\tfrac{1}{2}i[\arg(\lambda-k)+\arg(\lambda+k)]\},$$

(1.12b) $$-3\pi/2 < \arg(\lambda-k) < \pi/2 \qquad \text{and} \qquad -\pi/2 < \arg(\lambda+k) < 3\pi/2,$$

i.e., with branch cuts from k to $+i\infty$ and from $-k$ to $-i\infty$ such that

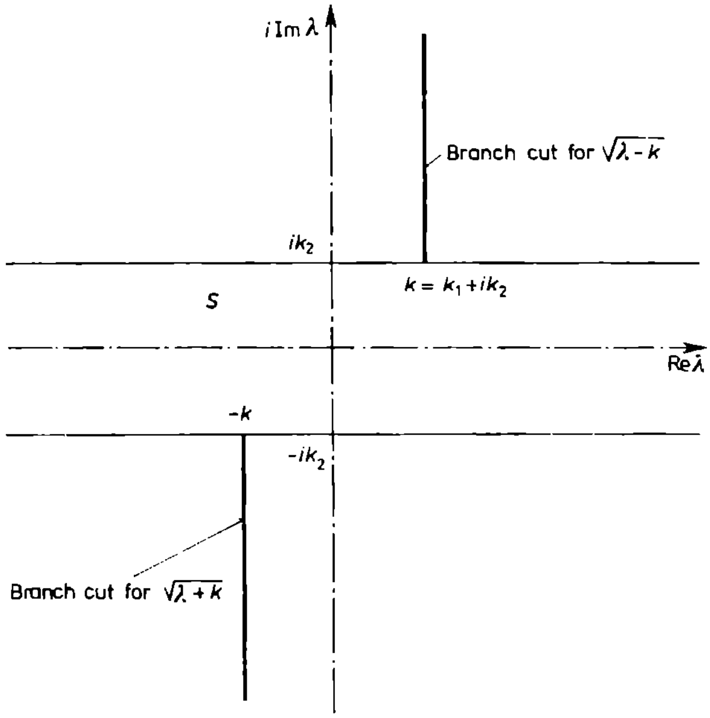

Fig. 2. Branch cuts and strip of holomorphy for $\sqrt{\lambda^2 - k^2} = \gamma$

$\mathrm{Re}\sqrt{\lambda^2-k^2} > 0$ in the strip $-k_2 < \mathrm{Im}\,\lambda < +k_2$. Due to the condition of boundedness we have $B_1(\lambda) \equiv A_2(\lambda) \equiv 0$ and we may put $A_1(\lambda) \equiv B_2(\lambda) \equiv A(\lambda)$ in case of the Dirichlet data when $\Phi_{sc}^{(1)}(x, +0) \equiv \Phi_{sc}^{(1)}(x, -0)$ for all $x \in \boldsymbol{R}$, i.e.,

(1.13a) $$\hat{\Phi}_{sc,D}(\lambda, y) = A_D(\lambda)\exp[-|y|\sqrt{\lambda^2-k^2}]$$

while in the other case we have due to $\dfrac{\partial \Phi_{sc,N}}{\partial y}(x, +0) \equiv \dfrac{\partial \Phi_{sc,N}}{\partial y}(x, -0)$:

(1.13b) $$\hat{\Phi}_{sc,N}(\lambda, y) = \pm A_N(\lambda)\exp[-|y|\sqrt{\lambda^2-k^2}].$$

Applying now the F-transformation to the jumps yields

(1.14a) $$\frac{\partial \hat{\Phi}_{sc,D}}{\partial y}(\lambda, +0) - \frac{\partial \hat{\Phi}_{sc,D}}{\partial y}(\lambda, -0) = \hat{J}_+(\lambda) = -2\sqrt{\lambda^2-k^2}\cdot A_D(\lambda)$$

and

(1.14b) $$\hat{\Phi}_{sc,N}(\lambda, +0) - \hat{\Phi}_{sc,N}(\lambda, -0) = \hat{Q}_+(\lambda) = 2\cdot A_N(\lambda)$$

where the Fourier transforms $\hat{J}_+(\lambda)$ and $\hat{Q}_+(\lambda)$ are holomorphically extendable to the planes $\mathrm{Im}\,\lambda > -k_2\cos\theta$. Denoting the unknown boundary values of $\Phi_{sc,D}(x, 0)$ for $x \leqslant 0$ by $E(x)$ and for $\dfrac{\partial \Phi_{sc,N}}{\partial y}(x, 0)$ for $x < 0$ by $V(x)$ we end up with

(1.15a) $$\hat{\Phi}_{sc,D}(\lambda, 0) = A_D(\lambda) = \hat{E}_-(\lambda) + [i\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1}$$

and

(1.15b) $$\frac{\partial \hat{\Phi}_{sc,N}}{\partial y}(\lambda, 0) = -\sqrt{\lambda^2 - k^2}\cdot A_N(\lambda) = V_-(\lambda) + \frac{k\sin\theta}{\sqrt{2\pi}}(\lambda + k\cos\theta).$$

Substituting these terms into equations (1.14a,b), respectively, we obtain the *Wiener–Hopf functional equations*

(1.16a) $$\hat{E}_-(\lambda) + \frac{1}{2}\frac{\hat{J}_+(\lambda)}{\sqrt{\lambda^2 - k^2}} = -[i\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1},$$

(1.16b) $$\hat{V}_-(\lambda) + \frac{1}{2}\hat{Q}_+(\lambda)\cdot\sqrt{\lambda^2 - k^2} = \frac{k\sin\theta}{\sqrt{2\pi}}(\lambda + k\cos\theta)^{-1}$$

holding for $-k_2\cos\theta < \operatorname{Im}\lambda < k_2$, $-\infty < \operatorname{Re}\lambda < +\infty$.

Multiplying (1.16a) by $\sqrt{\lambda - k}$ and dividing (1.16b) by $\sqrt{\lambda - k}$ and decomposing the resulting functions on the right-hand sides into a plus- and minus-function gives – due to the attenuation of the functions as $|\operatorname{Re}\lambda| \to \infty$

(1.17a) $$\hat{J}_+(\lambda) = 2\sqrt{k/\pi}\cdot\cos\theta/2\cdot\sqrt{\lambda + k}\cdot[\lambda + k\cos\theta]^{-1}$$

and, respectively,

(1.17b) $$\hat{Q}_+(\lambda) = 2i\sqrt{2k/\pi}\cdot\sin\theta/2\cdot[\sqrt{\lambda + k}(\lambda + k\cos\theta)]^{-1} \quad \text{for} \quad \operatorname{Im}\lambda > -k_2\cos\theta.$$

The functions $\hat{E}_-(\lambda)$ and $\hat{V}_-(\lambda)$ are given by

(1.18a) $$\hat{E}_-(\lambda) = -\left[1 + \frac{i\sqrt{2k}}{\sqrt{\lambda - k}}\cos\theta/2\right]\cdot[i\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1}$$

and

(1.18b) $$\hat{V}_-(\lambda) = \frac{k\sin\theta}{\sqrt{2\pi}}[\lambda + k\cos\theta]^{-1}\cdot\left(1 - \frac{i\sqrt{\lambda - k}}{\sqrt{2k}}\,\frac{1}{\cos\theta/2}\right)$$

for $\operatorname{Im}\lambda < k_2$.

After inserting $\hat{J}_+(\lambda)$ and $\hat{Q}_+(\lambda)$ into equations (1.14a,b) and then into formulae (1.13a,b) we arrive at the representations for the Fourier transforms of $\hat{\Phi}_{sc}(\lambda, y)$

(1.19a) $$\hat{\Phi}_{sc,D}(\lambda, y) = -\sqrt{k/\pi}\cdot\cos\theta/2\cdot\frac{\exp[-|y|\sqrt{\lambda^2 - k^2}]}{\sqrt{\lambda - k}(\lambda + k\cos\theta)}$$

and

$$(1.19b) \qquad \hat{\Phi}_{sc,N}(\lambda, y) = \pm i\sqrt{2k/\pi}\cdot \sin\theta/2\cdot \frac{\exp[-|y|\sqrt{\lambda^2-k^2}]}{\sqrt{\lambda+k}(\lambda+k\cos\theta)}.$$

For more details see the book by B. Noble (1958).

1.3. A generalized mixed Sommerfeld half-plane problem. Let now the screen $\mathfrak{s}$ be of such a type that the total field potential Φ_{tot} vanishes on the upper face but the normal derivative $\partial\phi_{tot}/\partial y$ on its lower one. This problem has been studied by A. E. Heins (1980/81), the present author (unpublished), and A. D. Rawlins (1975), treating it by an ad hoc integral representation. Here we shall show the lines of arguments like in Section 1.2.

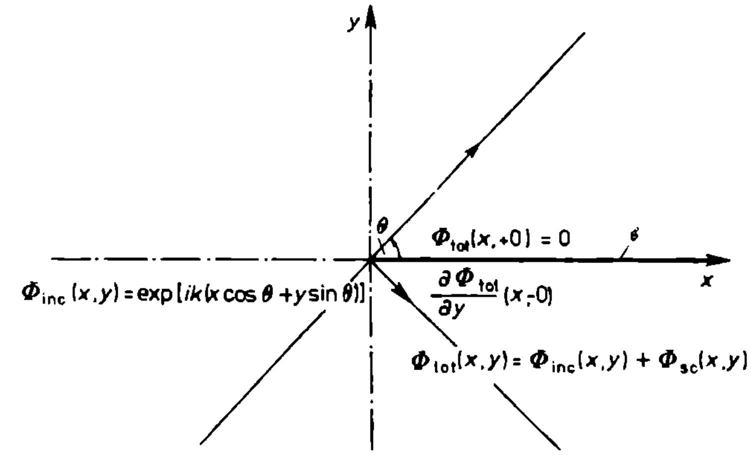

Fig. 3. Sommerfeld half-plane $\mathfrak{s}$ with different boundary conditions

The boundary value problem for the scattered field potential $\Phi_{sc}(x, y)$ then is following:

Find $\Phi_{sc}(x, y) \in C^2(\mathbf{R}^2\setminus\mathfrak{s}) \cap C^1(\mathbf{R}\setminus\{0\})$ *such that*

$$(1.20) \qquad (\Delta + k^2)\Phi_{sc}(x, y) = 0 \quad \text{in} \quad \mathbf{R}^2\setminus\mathfrak{s}$$

where $k = k_1 + ik_2$, k_1, $k_2 \geq 0$, $k \neq 0$, *satisfying the boundary conditions*

$$(1.21a) \qquad \lim_{y\to+0} \Phi_{sc}(x, y) =: \Phi_{sc}(x, +0) = -e^{ikx\cos\theta},$$

for $x > 0$

$$(1.21b) \qquad \lim_{y\to-0} \frac{\partial\Phi_{sc}}{\partial y}(x, y) = \frac{\partial\Phi_{sc}}{\partial y}(x, -0) = -ik\sin\theta\, e^{ikx\cos\theta}$$

with the additional edge and radiation conditions (1.5) *and* (1.6a,b), *respectively.*

Again we apply the Fourier transformation with respect to x and arrive at formula (1.11) for $\hat{\Phi}_{sc}(\lambda, y)$. Due to the condition of boundedness of

$\Phi_{sc}(x, y)$ as $|y| \to \infty$ and consequently of $\hat{\Phi}_{sc}(\lambda, y)$ we are left with formula

$$(1.22) \qquad \hat{\Phi}_{sc}(\lambda, y) = \begin{cases} A(\lambda)\exp[-y\sqrt{\lambda^2-k^2}] & \text{for } y > 0, \\ B(\lambda)\exp[y\sqrt{\lambda^2-k^2}] & \text{for } y < 0 \end{cases}$$

instead of formulae (1.13a,b). Dropping the index "sc" and writing $\hat{\Phi}_{\pm}(\lambda, y)$ for the unilateral Fourier transforms which are holomorphically extendable into the upper and lower half-plane, respectively, we obtain the following relations

$$(1.23a) \qquad \hat{\Phi}_{-}(\lambda, +0) + \hat{\Phi}_{+}(\lambda, +0) = A(\lambda),$$

$$(1.23b) \qquad \hat{\Phi}_{-}(\lambda, -0) + \hat{\Phi}_{+}(\lambda, -0) = B(\lambda)$$

and after differentiating (1.22), $' = \dfrac{\partial}{\partial y}$,

$$(1.23c) \qquad \Phi'_{-}(\lambda, +0) + \Phi'_{+}(\lambda, +0) = -\sqrt{\lambda^2-k^2}\cdot A(\lambda),$$

$$(1.23d) \qquad \Phi'_{-}(\lambda, -0) + \Phi'_{+}(\lambda, -0) = \sqrt{\lambda^2-k^2}\cdot B(\lambda).$$

Due to the continuity of $\Phi(x, y)$ and all its derivatives across $y = 0$, $x < 0$, we need not to distinguish between $+0$ and -0 in $\hat{\Phi}_{-}$ and $\hat{\Phi}'_{-}$. The quantities $\hat{\Phi}_{+}(\lambda, +0)$ and $\hat{\Phi}'_{+}(\lambda, -0)$ are given by the boundary data:

$$(1.24a) \qquad \hat{\Phi}_{+}(\lambda, +0) = [i\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1},$$

$$(1.24b) \qquad \hat{\Phi}'_{+}(\lambda, -0) = k\sin\theta\,[\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1}$$

for $\operatorname{Im}\lambda > -k_2\cos\theta$. Elimination of $A(\lambda)$ and $B(\lambda)$ leads to the following 2×2-*Wiener–Hopf functional system* involving the four unknown functions $\hat{E}(\lambda) := \hat{\Phi}_{-}(\lambda, 0)$, $\hat{V}(\lambda) := \hat{\Phi}'_{-}(\lambda, 0)$, holomorphic for $\operatorname{Im}\lambda < k_2$, and $\Phi'_{+}(\lambda, +0)$ and $\Phi_{+}(\lambda, -0)$, holomorphic for $\operatorname{Im}\lambda > -k_2\cos\theta$:

$$(1.25a) \qquad \sqrt{\lambda^2-k^2}\,\hat{E}(\lambda) + \hat{V}(\lambda) + \hat{\Phi}'_{+}(\lambda, +0) = \frac{\sqrt{\lambda^2-k^2}}{\sqrt{2\pi}(\lambda + k\cos\theta)},$$

$$(1.25b) \qquad \sqrt{\lambda^2-k^2}\,\hat{E}(\lambda) - \hat{V}(\lambda) + \sqrt{\lambda^2-k^2}\cdot\hat{\Phi}_{+}(\lambda, -0) = \frac{k\sin\theta}{\sqrt{2\pi}(\lambda + k\cos\theta)}.$$

We introduce now new vector functions by

$$(1.26a) \qquad \hat{\phi}_{-}(\lambda) := \begin{bmatrix} \hat{\phi}_{1-}(\lambda) \\ \hat{\phi}_{2-}(\lambda) \end{bmatrix} := \begin{bmatrix} \sqrt{\lambda - k}\,\hat{E}(\lambda) \\ \hat{V}(\lambda)/\sqrt{\lambda - k} \end{bmatrix}$$

holomorphic for $\operatorname{Im}\lambda < k_2$

and

$$\hat{\phi}_+(\lambda) := \begin{bmatrix} \hat{\phi}_{1+}(\lambda) \\ \hat{\phi}_{2+}(\lambda) \end{bmatrix} := -\frac{1}{2}\begin{bmatrix} \sqrt{\lambda+k}\cdot\hat{\Phi}_+(\lambda, -0) \\ \hat{\Phi}'_+(\lambda, +0)/\sqrt{\lambda+k} \end{bmatrix} \tag{1.26b}$$

holomorphic for $\operatorname{Im}\lambda > -k_2 \cos\theta$

and may rewrite the system (1.25a,b) in matrix form after diving by $\sqrt{\lambda+k}$ before

$$\hat{\phi}_-(\lambda) = \begin{bmatrix} \sqrt{\dfrac{\lambda-k}{\lambda+k}} & 1 \\ -1 & \sqrt{\dfrac{\lambda+k}{\lambda-k}} \end{bmatrix} \hat{\phi}_+(\lambda) + \hat{r}(\lambda) \tag{1.27}$$

where $\vec{r}(\lambda)$ is the known 2-vector function

$$\hat{r}(\lambda) := \begin{bmatrix} i\sqrt{\lambda-k}+k\sin\theta/\sqrt{\lambda+k} \\ i\sqrt{\lambda+k}-k\sin\theta/\sqrt{\lambda-k} \end{bmatrix}\cdot[2\sqrt{2\pi}(\lambda+k\cos\theta)]^{-1}. \tag{1.28}$$

In order to solve the WH-system (1.27) we have to factorize the 2×2-function matrix $\mathbf{K}(\lambda)$ into $\mathbf{K}^-(\lambda)\cdot[\mathbf{K}^+(\lambda)]^{-1}$ with non-singular holomorphic matrices $\mathbf{K}^\pm(\lambda)$ in the respective half-planes $\operatorname{Im}\lambda \gtrless \mp k_2$. This has been done by Rawlins (1980) and independently by Meister (1980) while Heins (1981) has treated the problem by establishing a system of Wiener–Hopf integral equations of the first kind for the unknown quantities $\dfrac{\partial\Phi_{sc}}{\partial y}(x, +0)$ and $\Phi_{sc}(x, -0)$ on the screen $y=0$, $x>0$. He transforms the equations (1.25a,b) into a system of singular integral equations along the branch cuts of the square root $\sqrt{\lambda^2-k^2}$ which he is able to solve explicitely. This corresponds to a method called the *Wiener–Hopf–Hilbert method* introduced by R. A. Hurd (1976) investigating a generalized Sommerfeld diffraction problem with two different impedance boundary conditions, viz.

$$\frac{\partial\Phi_{tot}}{\partial y}(x, \pm 0) + ik\cdot\alpha_\pm\cdot\Phi_{tot}(x, \pm 0) = 0 \quad \text{on} \quad x>0 \tag{1.29}$$

with real $\alpha_\pm := \sin\eta_\pm$.

Since $\mathbf{K}(\lambda)$ is holomorphic and bounded in the whole complex λ-plane, except the two branch cuts from $\pm k$ to $\pm i\infty$ where the square roots just change only their signs when crossing from one bank to the other we get

$$\mathbf{K}(t\pm 0) = \mathbf{K}_-(t\pm 0)\cdot[\mathbf{K}_+(t)]^{-1} \quad \text{for} \quad t = k+i\varrho,\ \varrho \geqslant 0, \tag{1.30a}$$

and

$$\mathbf{K}(t\pm 0) = \mathbf{K}(t)\cdot[\mathbf{K}_+(t\pm 0)]^{-1} \quad \text{for} \quad t = -k-i\varrho,\ \varrho \geqslant 0. \tag{1.30b}$$

Here we assumed that $\mathbf{K}_{\pm}(\lambda)$ may be continued holomorphically into the cut half-planes $\operatorname{Im}\lambda \leqslant -k_2$ and $\operatorname{Im}\lambda \geqslant +k_2$, respectively. From (1.30a) we may eliminate $\mathbf{K}_+(t)$ and turn up with a vectorial Riemann boundary value problem for $\mathbf{K}_-(\lambda)$ along the upper branch cut C_-:

$$\mathbf{K}_-(t+0) = \mathbf{K}(t+0)\cdot[\mathbf{K}(t-0)]^{-1}\cdot\mathbf{K}_-(t-0) \tag{1.31a}$$

and similarly for $\mathbf{K}_+(\lambda)$ along the lower branch cut C_+:

$$\mathbf{K}_+(t+0) = [\mathbf{K}(t+0)]^{-1}\cdot\mathbf{K}(t-0)\cdot\mathbf{K}_+(t-0). \tag{1.31b}$$

Now we have on C_+:

$$\mathbf{G}(t) := \mathbf{K}(t+0)\cdot[\mathbf{K}(t-0)]^{-1} = \begin{bmatrix} 0 & -\sqrt{\dfrac{t-k}{t+k}} \\ \sqrt{\dfrac{t+k}{t-k}} & 0 \end{bmatrix} \tag{1.32a}$$

and on C_-:

$$\mathbf{H}(t) := [\mathbf{K}(t+0)]^{-1}\cdot\mathbf{K}(t-0) = \begin{bmatrix} 0 & \sqrt{\dfrac{t+k}{t-k}} \\ -\sqrt{\dfrac{t-k}{t+k}} & 0 \end{bmatrix} \tag{1.32b}$$

where the square roots are always taken as boundary values on the right hand banks of the cuts $C_\pm$. Thus it is easy to show that holds $\mathbf{G}(-t) = \mathbf{H}(t)$ for $t \in C_-$. When we extend the branch cuts $C_\pm$ to vertical lines $\operatorname{Re}\lambda = \pm k_1$, $-\infty < \operatorname{Im}\lambda < +\infty$, then we have discontinuous vectorial Riemann boundary value problems with $\mathbf{G}(t)$ and $\mathbf{H}(t)$ replaced by the unit matrices on the complementary parts C'_+ and C'_- of these lines with respect to the branch cuts $C_\pm$.

Writing the matrix relation (1.31a) elementwise we see that

$$K_{11}^-(t+0) = \begin{cases} -\sqrt{\dfrac{t-k}{t+k}}\cdot K_{21}^-(t-0) & \text{for } t\in C_+, \\ K_{11}^-(t-0) & \text{for } t\in C'_+, \end{cases} \tag{1.33a}$$

$$K_{12}^-(t+0) = \begin{cases} -\sqrt{\dfrac{t-k}{t+k}}\cdot K_{22}^-(t-0) & \text{for } t\in C_+, \\ K_{12}^-(t-0) & \text{for } t\in C'_+, \end{cases} \tag{1.33b}$$

$$K_{21}^-(t+0) = \begin{cases} \sqrt{\dfrac{t+k}{t-k}}\cdot K_{11}^-(t-0) & \text{for } t\in C_+, \\ K_{21}^-(t-0) & \text{for } t\in C_-, \end{cases} \tag{1.33c}$$

(1.33d)
$$K_{22}^{-}(t+0) = \begin{cases} \sqrt{\dfrac{t+k}{t-k}} \cdot K_{12}^{-}(t-0) & \text{for } t \in C_+, \\ K_{22}^{-}(t-0) & \text{for } t \in C'_+. \end{cases}$$

The relations (1.33a,b) and (1.33c,d), respectively, are scalar discontinuous Riemann boundary value problems of the same type, viz.

(1.34a)
$$V_1(t+0) = \begin{cases} -\sqrt{\dfrac{t-k}{t+k}}\, V_2(t-0) & \text{for } t \in C_+, \\ V_1(t-0) & \text{for } t \in C'_+ \end{cases}$$

and

(1.34b)
$$V_2(t+0) = \begin{cases} \sqrt{\dfrac{t+k}{t-k}}\, V_1(t-0) & \text{for } t \in C_+, \\ V_2(t-0) & \text{for } t \in C'_+. \end{cases}$$

Thus multiplication and division, respectively, yields

(1.35a)
$$V_1 V_2(t+0) = \begin{cases} -1 & \text{for } t \in C_+, \\ +1 & \text{for } t \in C'_+ \end{cases}$$

and

(1.35b)
$$V_1/V_2(t+0) = \begin{cases} \dfrac{t-k}{t+k} \cdot [V_1/V_2(t-0)]^{-1} & \text{for } t \in C_+, \\ V_1/V_2(t-0) & \text{for } t \in C'_+. \end{cases}$$

Now a special solution to (1.35a) is given by

(1.36a)
$$P(\lambda) = V_1 V_2(\lambda) = 1/\sqrt{\lambda - k}$$

and all other solutions, having polynomial growth at infinity and only an algebraic singularity at $\lambda = k$, are given by linear combinations of

(1.36b)
$$P_m(\lambda) := (\lambda - k)^{m-1/2} \quad \text{with} \quad m \in \mathbf{Z}.$$

In order to solve the *reciprocal Riemann b.v.p.* (1.35b) we take logarithms and get

(1.37)
$$\begin{aligned} S(t+0) &= \log V_1/V_2(t+0) \\ &= \begin{cases} -\log V_1/V_2(t-0) + \log(-(t-k)/(t+k)) & \text{for } t \in C_+, \\ \log V_1/V_2(t-0) & \text{for } t \in C'_+, \end{cases} \end{aligned}$$

with solutions behaving only algebraically as $\lambda \to k$ or to ∞

(1.38)
$$S(\lambda) = -\frac{\sqrt{\lambda - k}}{2\pi i} \int_k^{i\infty} \frac{\log(-(t-k)/(t+k))}{\sqrt{t-k}} \frac{dt}{t-\lambda} + c_n \cdot \frac{(\lambda - k)^n}{\sqrt{\lambda - k}} \quad \text{with} \quad n \in \mathbf{Z}.$$

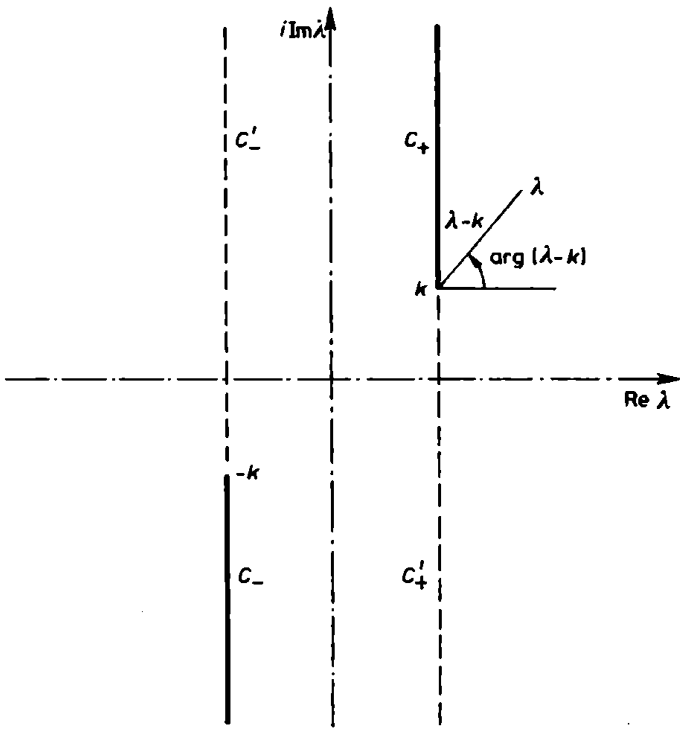

Fig. 4. Lines along which the Riemann b.v.p. is to be solved

The integral $I(\lambda; k)$ may be evaluated explicitly by calculating the *loop integral* along $C_+ \cup C_-$:

$$\frac{1}{2\pi i}\int_{\mathscr{L}} \frac{\log((t-k)/(t+k))}{\sqrt{t-k}}\frac{dt}{t-\lambda} = -\frac{\log(-(\lambda-k)/(\lambda+k))}{\sqrt{\lambda-k}}. \tag{1.39}$$

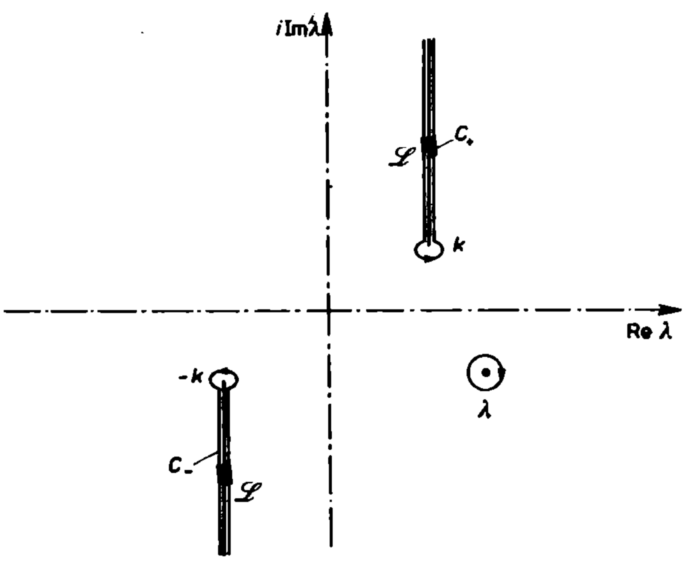

Fig. 5. Calculating the integral $I(\lambda; k)$

We have $\log(-(t-k)/(t+k)) = \log((t-k)/(t+k))+i\pi$ on the +-bank of C_+

$$\text{(1.40a)}\qquad \int_{i\infty}^{k} \frac{\log(-(t-k)/(t+k))}{\sqrt{t-k}}\bigg|_{C_+^-} \frac{dt}{t-\lambda}$$

$$= \int_{k}^{i\infty} \frac{\log(-(t-k)/(t+k))}{\sqrt{t-k}}\bigg|_{C_+^+} \frac{dt}{t-\lambda} - 2\pi i \cdot \int_{k}^{i\infty} \frac{1}{\sqrt{t-k}}\bigg|_{C_+^+} \frac{dt}{t-\lambda},$$

$$\text{(1.40b)}\qquad \int_{-i\infty}^{-k} + \int_{-k}^{-i\infty} \frac{\log(-(t-k)/(t+k))}{\sqrt{t-k}} \frac{dt}{t-\lambda}$$

$$= 2i\pi \int_{-i\infty}^{-k} \frac{1}{\sqrt{t-k}} \cdot \frac{dt}{t-\lambda} = \frac{2\pi i}{\sqrt{\lambda-k}} \cdot \frac{\log\sqrt{t-k}-\sqrt{\lambda-k}}{\sqrt{t-k}+\sqrt{\lambda-k}}\bigg|_{t=-i\infty}^{-k}.$$

Thus get

$$\text{(1.40c)}\qquad -\frac{\log((k-\lambda)/(k+\lambda))}{\sqrt{\lambda-k}}$$

$$= \frac{\pi i}{\sqrt{\lambda-k}} + \frac{1}{\pi i}\int_{k}^{i\infty} \frac{\log((t-k)/(t+k))}{\sqrt{t-k}}\bigg|_{C_+^+} \cdot \frac{dt}{t-\lambda} + \frac{1}{\sqrt{\lambda-k}} \log \frac{\sqrt{2k}-i\sqrt{\lambda-k}}{\sqrt{2k}+i\sqrt{\lambda-k}}$$

from which relation we obtain

$$\text{(1.41)}\qquad I(\lambda;k) = +\frac{1}{2}\log\frac{k-\lambda}{k+\lambda} + \frac{\pi i}{2} - \frac{1}{2}\log\frac{\sqrt{2k}-i\sqrt{\lambda-k}}{\sqrt{2k}+i\sqrt{\lambda-k}}$$

$$= \log\sqrt{\frac{\lambda-k}{\lambda+k}} + \log\sqrt{\frac{\sqrt{2k}-i\sqrt{\lambda-k}}{\sqrt{2k}+i\sqrt{\lambda-k}}} + \pi i.$$

Thus we get from formula (1.38)

$$\text{(1.42)}\qquad V_1/V_2(\lambda) = -\sqrt{\lambda-k}\cdot[\sqrt{2k}+i\sqrt{\lambda-k}]^{-1}\cdot \exp\{c_n(\lambda-k)^{n-1/2}\}$$

where only in the case of $n=0$ there exists a solution with algebraic behavior near $\lambda = k$ and $\lambda = \infty$. After multiplying then equations (1.36b) and (1.42) we obtain

$$\text{(1.43a)}\qquad [V_1(\lambda)]^2 = -a_m(\lambda-k)^m[\sqrt{2k}+i\sqrt{\lambda-k}]^{-1}, \qquad m\in \mathbf{Z},$$

and after dividing

$$[V_2(\lambda)]^2 = -a_m(\lambda-k)^{m-1}[\sqrt{2k}+i\sqrt{\lambda-k}], \qquad m\in \mathbf{Z}. \tag{1.43b}$$

After taking square roots with both possible signs and substituting into equations (1.34a,b) one sees that in

$$V_1(\lambda) = i\sqrt{a_m}(\lambda-k)^{m/2}\cdot[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}, \tag{1.44a}$$

$$V_2(\lambda) = \mp i\sqrt{a_m}(\lambda-k)^{(m-1)/2}[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2} \tag{1.44b}$$

we have to take m odd for the upper and m even for the lower sign. We thus obtain

$$K_{11}^-(\lambda) = i\sqrt{a_1}\cdot(\lambda-k)^{1/2}\cdot[\sqrt{2k}+i\sqrt{\lambda-k}]^{-1/2}, \tag{1.45a}$$

$$K_{12}^-(\lambda) = i\sqrt{a_0}\cdot[\sqrt{2k}+i\sqrt{\lambda-k}]^{-1/2}, \tag{1.45b}$$

$$K_{21}^-(\lambda) = -i\sqrt{a_1}\cdot[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}, \tag{1.45c}$$

$$K_{22}^-(\lambda) = i\sqrt{a_0}\cdot(\lambda-k)^{-1/2}[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}. \tag{1.45d}$$

The determinant is given by

$$\det \mathbf{K}^-(\lambda) = +\sqrt{a_1 a_0}\cdot(-1-1) = -2\sqrt{a_1 a_0} \tag{1.46}$$

so we could choose, due to $\det \mathbf{K}(\lambda) = 2$ and

$$\mathbf{K}(\lambda) = \mathbf{K}^-(\lambda)\cdot[\mathbf{K}^+(\lambda)]^{-1}, \qquad i\sqrt{a_1}\cdot i\sqrt{a_0} = -1,$$

e.g., $a_1 = a_0 = 1$, and can calculate the matrix $\mathbf{K}^+(\lambda)$ as

$$K_{11}^+(\lambda) = \frac{i}{2}\{[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}+[\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2}\}, \tag{1.47a}$$

$$K_{12}^+(\lambda) = \frac{i}{2\sqrt{\lambda-k}}\{[\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2}-[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}\}, \tag{1.47b}$$

$$K_{21}^+(\lambda) = \frac{i}{2}\sqrt{\frac{\lambda-k}{\lambda+k}}\{[\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2}-[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}\}, \tag{1.47c}$$

$$K_{22}^+(\lambda) = \frac{i}{2\sqrt{\lambda+k}}\{[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}+[\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2}\}. \tag{1.47d}$$

These functions are holomorphic for $\operatorname{Im}\lambda > -k_2$ since their branch cuts are only from $-k$ to $-i\infty$. When λ passes from one bank to the other of the upper branch cut C_+ $\sqrt{\lambda-k}$ goes over into $-\sqrt{\lambda-k}$ but the change of signs cancels.

With the explicit knowledge of $\mathbf{K}^\pm(\lambda)$ and their inverses, respectively, we may solve the two-part WH-system (1.27) by multiplying from the left with

$[\mathbf{K}^-(\lambda)]^{-1}$ and then splitting $[\mathbf{K}^-(\lambda)]^{-1}\cdot\hat{\bar{r}}(\lambda)=:\hat{\bar{s}}(\lambda)$ additively into $\hat{\bar{s}}_+(\lambda)+\hat{\bar{s}}_-(\lambda)$ according

$$\hat{\bar{s}}_\pm(\lambda):=\pm\frac{1}{2\pi i}\int_{i\delta_\pm-\infty}^{i\delta_\pm+\infty}\frac{\hat{\bar{s}}(\tau)\,d\tau}{\tau-\lambda} \tag{1.48}$$

for $\operatorname{Im}\lambda>\delta_+>-k_2\cos\theta$ and $\operatorname{Im}\lambda<\delta_-<k_2$

such that

$$\hat{\bar{\phi}}_-(\lambda)=\mathbf{K}^-(\lambda)\hat{\bar{s}}_-(\lambda)\qquad\text{for}\quad\operatorname{Im}\lambda<\delta_-<k_2 \tag{1.49a}$$

and

$$\hat{\bar{\phi}}(\lambda)=\mathbf{K}^+(\lambda)\hat{\bar{s}}_+(\lambda)\qquad\text{for}\quad\operatorname{Im}\lambda>\delta_+>-k_2\cos\theta \tag{1.49b}$$

for which formula we may calculate $\hat{E}(\lambda)$, $\hat{V}(\lambda)$ by (1.26a) and $\hat{\Phi}_+(\lambda,-0)$, $\hat{\Phi}'_+(\lambda,+0)$ by (1.26b).

Since $\hat{\bar{s}}(\tau)$ behaves like $O(|\tau|^{-1/2})$ due to (1.28) as $|\operatorname{Re}\tau|\to\infty$ and the first row of $[\mathbf{K}^-(\lambda)]^{-1}$ like $O(|\tau|^{-1/4})$ and the second like $O(|\tau|^{1/4})$ as $|\operatorname{Re}\tau|\to\infty$ we get $\hat{s}_{-1}(\lambda)=O(|\lambda|^{-3/4})$ and $\hat{s}_{-2}(\lambda)=O(|\lambda|^{-1/4})$ as $|\operatorname{Re}\lambda|\to\infty$, respectively, which yields according to formulae (1.45)

$$\hat{\bar{\phi}}_-(\lambda)=O(|\lambda|^{-1/2}) \tag{1.50}$$

which gives

$$\hat{E}(\lambda)=O(|\lambda|^{-1}),\qquad\hat{V}(\lambda)=O(1)\qquad\text{as}\quad\lambda\to\infty\qquad\text{in}\quad\operatorname{Im}\lambda<\delta_+ \tag{1.51}$$

the last formula actually should be $o(1)$ in order to be a Fourier transform. Similarly we obtain $\hat{\bar{\phi}}_+(\lambda)=O(|\lambda|^{-1/2})$ as $\lambda\to\infty$ in $\operatorname{Im}\lambda>\delta_+$. A careful investigation, like Rawlins (1975) and Heins (1980) did, gives

$$\Phi^+_{\text{tot}}(x,-0)=O(x^{1/4})\qquad\text{as}\quad x\to+0 \tag{1.52a}$$

and

$$\frac{\partial\Phi^+_{\text{tot}}}{\partial y}(x,+0)=O(x^{-3/4})\qquad\text{as}\quad x\to+0. \tag{1.52b}$$

Remarks. Several authors studied generalizations to the above mixed half-plane problem in connexion with the effect of an acoustic wave in a subsonic stream upon a semi-infinite vortex layer extending downstream of the half-plane. See, e.g., papers by Jones and Morgan (1972, 1973, 1974) or of electro-magnetic waves acting in anisotropic, dielectric materials, like in Hurd and Przezdziecki (1967, 1976, 1977) and the literature quoted there. Now we may also assume two different media in the half-spaces $y>0$ and $y<0$, with wave-numbers k_u and k_l, respectively, such that on $y=0$, $x<0$, transmissions conditions have to hold while on the screen $\mathfrak{s}$, $y = 0$, $x > 0$, we have boundary conditions as before. See, e.g., van Splunter and van den Berg (1979) for the case of a strip $\mathfrak{s}$ instead of a half-plane and equal Dirichlet boundary conditions on both banks of $\mathfrak{s}$. The case of mixed boundary conditions will be published by the present author elsewhere.

1.4. Some generalized mixed boundary value problems for systems of semi-infinite plates. In the book by B. Noble quoted at the beginning one finds also the discussion of the diffraction of plane waves by two parallel plates

(1.53) $$\mathfrak{s}_1 \cup \mathfrak{s}_2 := \{(x, y) \in \boldsymbol{R}^2:\ x \geqslant 0,\ y = \pm a\}$$

and of a circular semi-infinite pipe $\mathscr{R}$ of radius a in $\boldsymbol{R}^3$-space:

(1.54) $$\boldsymbol{R} := \{(x, y, z) \in \boldsymbol{R}^3:\ z \leqslant 0,\ x^2 + y^2 = \varrho^2 = a^2\}$$

or of a periodic infinite system considered by Heins (1947) or by Meister (1970) with finite blade length and skew incident electro-magnetic fields.

Here we want to formulate the Wiener–Hopf functional systems in the Fourier transform λ-plane in the case of mixed (Dirichlet–Neumann) conditions on a pair and an infinite periodic system of semi-infinite thin plates without stagger in the geometry.

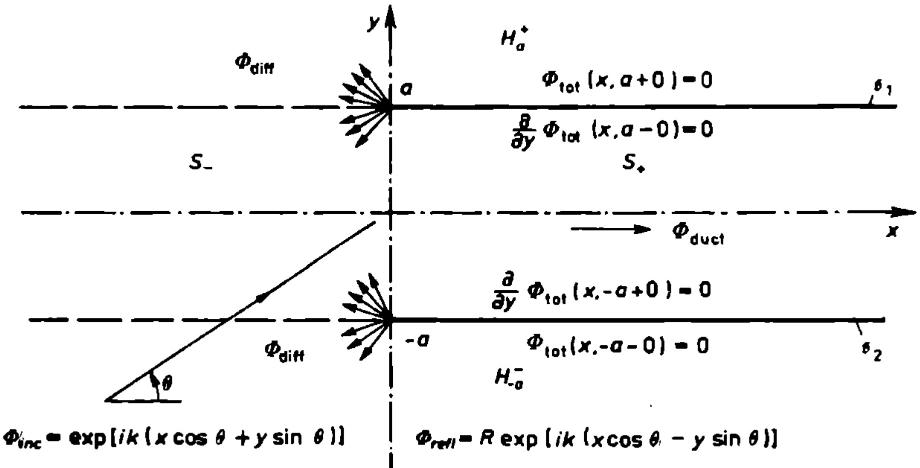

Fig. 6. Plane wave falling upon a pair of plates with different boundary behavior

There are of course several possible cases, viz. one with equal behavior of both plates and the other one with the boundary conditions interchanged such that the semi-infinite strip $S_+ := \{(x, y) \in \boldsymbol{R}^2:\ x \geqslant 0,\ |y| < a\}$ acts like a duct with Neumann (or Dirichlet) inner boundary data.

Applying the Fourier transforms with respect to x in the three y-regions corresponding to the half-spaces H_a^+, H_{-a}^- and the strip $S = S_- \cup S_+$ we obtain for bounded solutions as $|y| \to \infty$:

(1.55) $$\hat{\Phi}(\lambda, y) = \begin{cases} A^{(+)}(\lambda)\cdot\exp[-y\sqrt{\lambda^2-k^2}] & \text{for } y > a, \\ A_1(\lambda)\cdot\exp[-y\sqrt{\lambda^2-k^2}] + B_1(\lambda)\cdot\exp[y\sqrt{\lambda^2-k^2}] & \text{for } |y| < a, \\ B^{(-)}(\lambda)\cdot\exp[y\sqrt{\lambda^2-k^2}] & \text{for } y < -a. \end{cases}$$

Now splitting the boundary values of $\Phi(x, y)$ on $y = \pm a$ like in section 1.3 and denoting by $\hat{\Phi}_\pm(\lambda, \pm a)$, $\hat{\Phi}'_\pm(\lambda, \pm a)$ the corresponding unilateral Fourier transforms we arrive at the following relations; where $\gamma(\lambda) := \sqrt{\lambda^2 - k^2}$:

(1.56a) $$\hat{\Phi}_-(\lambda, a+0) + \hat{\Phi}_+(\lambda, a+0) = A^{(+)}(\lambda)\cdot e^{-a\gamma},$$

(1.56b) $$\hat{\Phi}_-(\lambda, a-0) + \hat{\Phi}_+(\lambda, a-0) = A_1(\lambda)e^{-a\gamma} + B_1(\lambda)e^{a\gamma},$$

(1.56c) $$\hat{\Phi}_-(\lambda, -a+0) + \hat{\Phi}_+(\lambda, -a+0) = A_1(\lambda)e^{a\gamma} + B_1(\lambda)e^{-a\gamma},$$

(1.56d) $$\hat{\Phi}_-(\lambda, -a-0) + \hat{\Phi}_+(\lambda, -a-0) = B^{(-)}(\lambda)e^{-a\gamma}$$

and similar relations for $\hat{\Phi}'(\lambda, \pm a \pm 0)$:

(1.57a) $$\hat{\Phi}'_-(\lambda, a+0) + \hat{\Phi}'_+(\lambda, a+0) = -\gamma\cdot A^{(+)}(\lambda)e^{-a\gamma},$$

(1.57b) $$\hat{\Phi}'_-(\lambda, a-0) + \hat{\Phi}'_+(\lambda, a-0) = -\gamma\{A_1(\lambda)e^{-a\gamma} - B_1(\lambda)e^{a\gamma}\},$$

(1.57c) $$\hat{\Phi}'_-(\lambda, -a+0) + \hat{\Phi}'_+(\lambda, -a+0) = -\gamma\{A_1(\lambda)e^{a\gamma} - B_1(\lambda)e^{-a\gamma}\},$$

(1.57d) $$\hat{\Phi}'_-(\lambda, -a-0) + \hat{\Phi}'_+(\lambda, -a-0) = \gamma\cdot B^{(-)}(\lambda)e^{-a\gamma}.$$

Now we make use of the continuity of all derivatives of $\Phi(x, y)$ for $x < 0$, so we need not to distinguish the boundary values of $\hat{\Phi}_-(\lambda, \pm a)$ and $\hat{\Phi}'_-(\lambda, \pm a)$ from above and below, respectively. Due to the incident field we have

(1.58a) $$\hat{\Phi}_+(\lambda, \pm(a+0)) = \frac{-1}{\sqrt{2\pi}}\int_0^\infty e^{i\lambda x}\cdot\exp[ik(x\cos\theta \pm a\sin\theta)]\,dx$$
$$= \exp(\pm ika\sin\theta)\cdot[i\sqrt{2\pi}\cdot(\lambda + k\cos\theta)]^{-1}$$

and

(1.58b)
$$\hat{\Phi}'_+(\lambda, \pm(a-0)) = \frac{-1}{\sqrt{2\pi}}\int_0^\infty e^{i\lambda x}\cdot ik\sin\theta\cdot\exp[ik(x\cos\theta \pm a\sin\theta)]\,dx$$
$$= k\sin\theta\cdot\exp(\pm ika\sin\theta)\cdot[\sqrt{2\pi}\cdot(\lambda + k\cos\theta)]^{-1}$$

holomorphic for $\operatorname{Im}\lambda > -k_2\cos\theta$.

The elimination of the four factors $A^{(+)}$, A_1, B_1, $B^{(-)}$ in equations (1.56) and (1.57) leads to a system of four Wiener–Hopf functional equations for the remaining four unknown plus-functions $\hat{\Phi}_+(\lambda, a-0)$, $\hat{\Phi}_+(\lambda, -a+0)$, $\hat{\Phi}'_+(\lambda, a+0)$, and $\hat{\Phi}'_+(\lambda, -a-0)$ as well as for the four unknown minus-functions

$$\hat{E}_{\pm a}(\lambda) := \hat{\Phi}_-(\lambda, \pm a) \qquad \text{and} \qquad \hat{V}_{\pm a}(\lambda) := \hat{\Phi}'(\lambda, \pm a),$$

(1.59a)
$$\hat{E}_a(\lambda) + \frac{1}{\gamma}\hat{V}_a(\lambda) + \frac{1}{\gamma}\hat{\Phi}'_+(\lambda, a+0) = -\hat{\Phi}_+(\lambda, a+0),$$

(1.59b)
$$\hat{E}_{-a}(\lambda) - \frac{1}{\gamma}\hat{V}_{-a}(\lambda) - \frac{1}{\gamma}\hat{\Phi}_+(\lambda, -a-0) = -\hat{\Phi}_+(\lambda, -a-0),$$

(1.59c)
$$\hat{E}_a(\lambda) - \frac{1}{\gamma}\hat{V}_a(\lambda) + \hat{\Phi}_+(\lambda, a-0) - $$
$$- e^{-2a\gamma}\hat{E}_{-a}(\lambda) + \frac{1}{\gamma}e^{-2a\gamma}\hat{V}_{-a}(\lambda) - e^{-2a\gamma}\hat{\Phi}_+(\lambda, -a+0)$$
$$= \frac{1}{\gamma}\hat{\Phi}'_+(\lambda, a-0) - \frac{e^{-2a\gamma}}{\gamma}\hat{\Phi}'_+(\lambda, -a+0),$$

(1.59d)
$$\hat{E}_{-a}(\lambda) + \frac{1}{\gamma}\hat{V}_{-a}(\lambda) + \hat{\Phi}_+(\lambda, -a+0) - $$
$$- e^{-2a\gamma}\hat{E}_a(\lambda) - \frac{e^{-2a\gamma}}{\gamma}\hat{V}_a(\lambda) - e^{-2a\gamma}\hat{\Phi}_+(\lambda, a-0)$$
$$= -\frac{1}{\gamma}\hat{\Phi}'_+(\lambda, -a+0) + \frac{e^{-2a\gamma}}{\gamma}\hat{\Phi}'_+(\lambda, a-0).$$

After adding the equations (1.59a) and (1.59b) and (1.59c) and (1.59d) and afterwards subtracting the corresponding we obtain

(1.60a)
$$[\hat{E}_a(\lambda) \pm \hat{E}_{-a}(\lambda)] + \frac{1}{\gamma}\hat{V}_a(\lambda) \mp \hat{V}_{-a}(\lambda)] +$$
$$+ \frac{1}{\gamma}[\hat{\Phi}'_+(\lambda, a+0) \mp \hat{\Phi}'_+(\lambda, -a-0)] = -\hat{\Phi}_+(\lambda, a+0) \mp \hat{\Phi}_+(\lambda, -a-0),$$

(1.60b)
$$(1 \mp e^{-2a\gamma})[\hat{E}_a(\lambda) \pm \hat{E}_{-a}(\lambda)] - \frac{1}{\gamma}(1 \pm e^{-2a\gamma})[\hat{E}_a(\lambda) \mp \hat{V}_{-a}(\lambda)] +$$
$$+ [\hat{\Phi}_+(\lambda, a-0) \pm \hat{\Phi}_+(\lambda, -a+0)](1 \mp e^{-2a\gamma})$$
$$= \frac{1 \pm e^{-2a\gamma}}{\gamma}[\hat{\Phi}'_+(\lambda, a-0) \mp \hat{\Phi}'_+(\lambda, -a+0)].$$

We multiply these equations $\sqrt{\lambda-k}$ and introduce

$$\hat{\hat{\phi}}_-(\lambda) := \begin{bmatrix} \sqrt{\lambda-k}\cdot[\hat{E}_a(\lambda)+\hat{E}_{-a}(\lambda)] \\ \dfrac{1}{\sqrt{\lambda-k}}[\hat{V}_a(\lambda)-\hat{V}_{-a}(\lambda)] \end{bmatrix} \tag{1.61a}$$

holomorphic for $\operatorname{Im}\lambda < +k_2$,

$$\hat{\hat{\phi}}_+(\lambda) = \begin{bmatrix} \sqrt{\lambda+k}[\hat{\Phi}_+(\lambda, a-0)+\hat{\Phi}_+(\lambda, -a+0)] \\ \dfrac{1}{\sqrt{\lambda+k}}[\hat{\Phi}'_+(\lambda, a+0)-\hat{\Phi}'_+(\lambda, -a-0)] \end{bmatrix} \tag{1.61b}$$

holomorphic for $\operatorname{Im}\lambda > -k_2\cos\theta$, and

$$\hat{\hat{\psi}}_-(\lambda) := \begin{bmatrix} \sqrt{\lambda-k}\cdot[\hat{E}_a(\lambda)-\hat{E}_{-a}(\lambda)] \\ \dfrac{1}{\sqrt{\lambda-k}}[\hat{V}_a(\lambda)+\hat{V}_{-a}(\lambda)] \end{bmatrix} \tag{1.61c}$$

holomorphic for $\operatorname{Im}\lambda < +k_2$,

$$\hat{\hat{\psi}}_+(\lambda) = \begin{bmatrix} \sqrt{\lambda+k}[\hat{\Phi}(\lambda, a-0)-\hat{\Phi}_+(\lambda, -a+0)] \\ \dfrac{1}{\sqrt{\lambda+k}}[\hat{\Phi}'_+(\lambda, a+0)+\hat{\Phi}'_+(\lambda, -a-0)] \end{bmatrix} \tag{1.61d}$$

holomorphic for $\operatorname{Im}\lambda > -k_2\cos\theta$.

This leads to the following two 2×2-WH-systems:

$$\begin{bmatrix} 1 & \sqrt{\dfrac{\lambda-k}{\lambda+k}} \\ 1-e^{-2a\gamma} & -(1+e^{-2a\gamma})\sqrt{\dfrac{\lambda-k}{\lambda+k}} \end{bmatrix}\cdot\hat{\hat{\phi}}_-(\lambda)+\begin{bmatrix} 0 & 1 \\ (1-e^{-2a\gamma})\sqrt{\dfrac{\lambda-k}{\lambda+k}} & 0 \end{bmatrix}\times \tag{1.62a}$$

$$\times\hat{\hat{\phi}}_+(\lambda) = \hat{r}^{(+)}(\lambda) \quad \text{(known)}$$

and

$$\begin{bmatrix} 1 & \sqrt{\dfrac{\lambda-k}{\lambda+k}} \\ 1+e^{-2a\gamma} & -(1-e^{-2a\gamma})\sqrt{\dfrac{\lambda-k}{\lambda+k}} \end{bmatrix}\cdot\hat{\hat{\psi}}_-(\lambda)+\begin{bmatrix} 0 & 1 \\ (1+e^{-2a\gamma})\sqrt{\dfrac{\lambda-k}{\lambda+k}} & 0 \end{bmatrix}\times \tag{1.62b}$$

$$\times\hat{\hat{\psi}}_+(\lambda) = \hat{r}^{(-)}(\lambda) \quad \text{(known)}$$

or

$$(1.63)\quad \begin{bmatrix}\hat{\Phi}_-(\lambda)\\ \hat{\psi}_-(\lambda)\end{bmatrix} e^{-a\gamma} \begin{bmatrix}\sinh(a\gamma)\sqrt{\dfrac{\lambda-k}{\lambda+k}} & \cosh(a\gamma)\\ -\sinh(a\gamma) & \sinh(a\gamma)\sqrt{\dfrac{\lambda+k}{\lambda-k}}\end{bmatrix} \cdot \begin{bmatrix}\hat{\Phi}_+(\lambda)\\ \hat{\psi}_+(\lambda)\end{bmatrix}$$

$$= \begin{bmatrix}\vec{s}^{(+)}(\lambda)\\ \vec{s}^{(-)}(\lambda)\end{bmatrix} := \begin{bmatrix}\dfrac{1}{2}[(1\pm e^{-2a\gamma})r_1^{(\pm)}(\lambda)+r_2^{(\pm)}(\lambda)]\\ \dfrac{1}{2}\sqrt{\dfrac{\lambda+k}{\lambda-k}}[(1\mp e^{-2a\gamma})r_1^{(\pm)}(\lambda)-r_2^{(\pm)}(\lambda)]\end{bmatrix}$$

with the right-hand sides known. These two systems reduce to the system (1.27) as $a \to +\infty$. Here the two 2×2-function matrices have to be factorized in front of $\hat{\Phi}_+(\lambda)$ and $\hat{\psi}_+(\lambda)$. This has not yet been effectively done.

Finally we want to discuss the effect of scattering of a plane wave incident upon a periodic system of semi-infinite plates with Neumann and Dirichlet boundary conditions on the same sides of the screens s_n, $n\in \boldsymbol{Z}$.

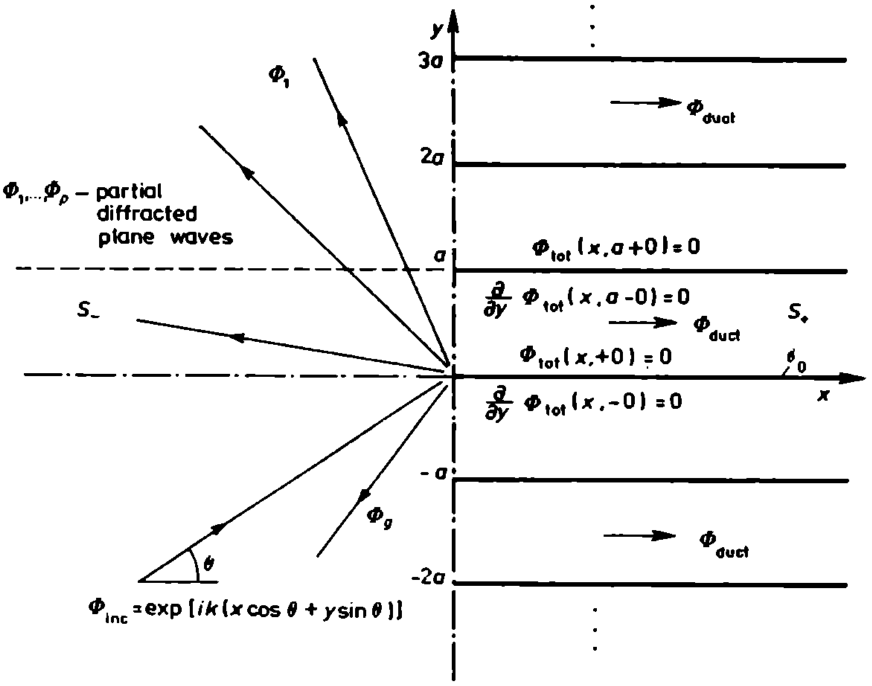

Fig. 7. Plane wave falling upon a vertical cascade of plates

Due to the quasi-periodic behavior of the incident wave, i.e.,

$$(1.64)\qquad \Phi_{\text{inc}}(x, y+na) = e^{ikna\sin\theta}\,\Phi_{\text{inc}}(x, y)$$

we are looking only for such scattered waves $\Phi_{\text{sc}}(x, y)$ which exhibit the same behavior. Then we may confine ourselves to the standard strip

$S := \{(x, y) \in \boldsymbol{R}^2:\ -\infty < x < +\infty, 0 < y < a\}$. Due to this periodic structure we have to expect plane reflected waves in front of $(x < 0)$ the plate system and ducted waves between two adjacent plates as it was shown in Heins (1947) and Meister (1970) for pure Dirichlet or Neumann data on the plates $\mathfrak{s}_n$. Here again we shall assume a positive imaginary part k_2 of the wave-number in order to avoid separation of the plane waves incorporated in Φ_{sc} in order to apply the classical Fourier transformation. Here we assume only the asymptotic behavior $\Phi_{sc}(x, y)$,

(1.65) $$\operatorname{grad} \Phi_{sc}(x, y) = O(\exp(-k_2 \cos\theta \cdot |x|)) \quad \text{as} \quad |x| \to \infty.$$

Denoting the values of $\Phi_{sc}(x, y)$ on $x < 0$, $y = 0$ and $= a$ by $E(x)$ and $e^{ika\sin\theta} E(x)$ and $V(x)$ and $e^{ika\sin\theta} V(x)$, respectively, the representation of the Fourier-transform $\hat{\Phi}(\lambda, y)$ of the scattered field in the standard strip S leads to

(1.66) $$\hat{\Phi}(\lambda, y) = A(\lambda) e^{-y\gamma} + B(\lambda) e^{y\gamma}$$

and the boundary values on $y = 0$ and $y = a$, respectively,

(1.67a) $$\hat{E}(\lambda) + \hat{\Phi}_+(\lambda, +0) = A(\lambda) + B(\lambda),$$

(1.67b) $$e^{ika\sin\theta} \hat{E}(\lambda) + \hat{\Phi}_+(\lambda, a-0) = A(\lambda) e^{-a\gamma} + B(\lambda) e^{a\gamma},$$

(1.68a) $$\hat{V}(\lambda) + \hat{\Phi}'_+(\lambda, 0) = -\gamma[A(\lambda) - B(\lambda)],$$

(1.68b) $$e^{ika\sin\theta} \hat{V}(\lambda) + \hat{\Phi}'_+(\lambda, a-0) = -\gamma[A(\lambda) e^{-a\gamma} - B(\lambda) e^{a\gamma}].$$

After eliminating $A(\lambda)$, $B(\lambda)$ and inserting the known transforms of the boundary data according to

(1.69a) $$\hat{\Phi}_+(\lambda, +0) = [i\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1},$$

(1.69b) $$\hat{\Phi}'_+(\lambda, a-0) = k\sin\theta \cdot e^{iak\sin\theta} [\sqrt{2\pi}(\lambda + k\cos\theta)]^{-1}$$

holomorphic for $\operatorname{Im}\lambda > -k_2\cos\theta$, we obtain

(1.70a) $$(1 - \exp(a\gamma + ika\sin\theta)) \hat{E}(\lambda) - \frac{1}{\gamma}(1 - \exp(a\gamma + ika\sin\theta)) \hat{V}(\lambda) -$$
$$- e^{a\gamma} \cdot \hat{\Phi}_+(\lambda, a-0) - \frac{1}{\gamma} \hat{\Phi}'_+(\lambda, +0) = -\hat{\Phi}_+(\lambda, +0) - \frac{e^{a\gamma}}{\gamma} \hat{\Phi}'_+(\lambda, a-0),$$

(1.70b) $$(1 - \exp(-a\gamma + ika\sin\theta)) \hat{E}(\lambda) + \frac{1}{\gamma}(1 - \exp(-a\gamma + ika\sin\theta)) \hat{V}(\lambda) -$$
$$- e^{-a\gamma} \cdot \hat{\Phi}_+(\lambda, a-0) + \frac{1}{\gamma} \hat{\Phi}'_+(\lambda, +0)$$
$$= -\hat{\Phi}_+(\lambda, +0) + \frac{e^{-a\gamma}}{\gamma} \hat{\Phi}'_+(\lambda, a-0)$$

where the right-hand sides are known functions. After adding and subtracting the last equations we get

(1.71a) $$[1-e^{ika\sin\theta}\cosh(a\gamma)]\hat{E}(\lambda)+e^{ika\sin\theta}\gamma^{-1}\sinh(a\gamma)\hat{V}(\lambda)-$$
$$-\cosh(a\gamma)\cdot\hat{\Phi}_{+}(\lambda,a-0)=-\hat{\Phi}_{+}(\lambda,+0)-\gamma^{-1}\cdot\sinh(a\gamma)\cdot\hat{\Phi}'_{+}(\lambda,a-0)$$

and

(1.71b) $$e^{iak\sin\theta}\cdot\gamma\sinh(a\gamma)\hat{E}(\lambda)+[1-e^{ika\sin\theta}\cosh(a\gamma)]\hat{V}(\lambda)+$$
$$+\hat{\Phi}'_{+}(\lambda,+0)+\gamma\cdot\sinh(a\gamma)\cdot\hat{\Phi}_{+}(\lambda,a-0)=\cosh(a\gamma)\cdot\hat{\Phi}'_{+}(\lambda,a-0).$$

When we introduce the unknown function vectors

(1.72a) $$\hat{\vec{\phi}}_{-}(\lambda):=\begin{bmatrix}\sqrt{\lambda-k}\cdot\hat{E}(\lambda)\\ \hat{V}(\lambda)/\sqrt{\lambda-k}\end{bmatrix}$$

holomorphic for $\operatorname{Im}\lambda<-k_2$ and

(1.72b) $$\hat{\vec{\phi}}_{+}(\lambda):=\begin{bmatrix}\sqrt{\lambda+k}\,e^{-iak\sin\theta}\cdot\hat{\Phi}_{+}(\lambda,a-0)\\ \hat{\Phi}'_{+}(\lambda,+0)/\sqrt{\lambda+k}\end{bmatrix}$$

holomorphic for $\operatorname{Im}\lambda>-k_2$. We may rewrite the system (1.71a, b) as the following 2×2-WH functional system

(1.73) $$\hat{\vec{\phi}}_{-}(\lambda)+\mathbf{K}(\lambda;a,\theta)\cdot\hat{\vec{\phi}}_{+}(\lambda)=[\mathbf{M}(\lambda;a,\theta)]^{-1}\vec{r}(\lambda)=\vec{s}(\lambda)$$

where we have

(1.74a) $$\mathbf{K}(\lambda;a,\theta):=\frac{1}{2}\begin{bmatrix}\sqrt{\dfrac{\lambda-k}{\lambda+k}}\left[1-\dfrac{i\cdot\sin(ak\sin\theta)}{\cosh(a\gamma)-\cos(ak\sin\theta)}\right] & \dfrac{1}{2}\left[\coth\dfrac{a}{2}(\gamma-ik\sin\theta)+\coth\dfrac{a}{2}(\gamma+ik\sin\theta)\right]\\ -\dfrac{1}{2}\left[\coth\dfrac{a}{2}(\gamma-ik\sin\theta)+\coth\dfrac{a}{2}(\gamma+ik\sin\theta)\right] & \sqrt{\dfrac{\lambda+k}{\lambda-k}}\left[1+\dfrac{i\cdot\sin(ak\sin\theta)}{\cosh(a\gamma)-\cos(ak\sin\theta)}\right]\end{bmatrix}$$

and

(1.74b) $$\mathbf{M}(\lambda):=\begin{bmatrix}1-e^{iak\sin\theta}\cosh(a\gamma) & \sqrt{\dfrac{\lambda-k}{\lambda+k}}\,e^{iak\sin\theta}\sinh(a\gamma)\\ e^{iak\sin\theta}\cdot\sinh(a\gamma) & \sqrt{\dfrac{\lambda-k}{\lambda+k}}\,[1-e^{iak\sin\theta}\cosh(a\gamma)]\end{bmatrix}.$$

The matrix $\mathbf{K}(\lambda;a,\theta)$ has not yet been explicitly factorized. Here

(1.74c) $$\vec{s}(\lambda):=\begin{bmatrix}\dfrac{i\gamma[\cosh(a\gamma)-e^{-iak\sin\theta}]+k\sin\theta\cdot\sinh(a\gamma)}{\sqrt{2\pi}\,[\cosh(a\gamma)-\cos(ak\sin\theta)](\lambda+k\cos\theta)\sqrt{\lambda+k}}\\ \dfrac{i\gamma\sinh(a\gamma)-k\sin\theta\,[\cosh(a\gamma)-e^{ika\sin\theta}]}{\sqrt{2\pi}\,[\cosh(a\gamma)-\cos(ak\sin\theta)](\lambda+k\cos\theta)\sqrt{\lambda-k}}\end{bmatrix}$$

is known.

1.5. Diffraction by plane screens in R^3-space. Generalizing the classical Sommerfeld half-plane problem described in Chapter 1 many people have studied the problem of diffraction of acoustic and electromagnetic waves by plane thin screens Σ in R^3-space. Here we are interested only in the diffraction by a quarter-plane $\Sigma := \{(x, y, z) \in R^3 : x, y \geqslant 0, -\infty < z < \infty\}$ by an incident wave $\Phi_{inc}(x, y, z)$ particularly being a plane acoustic wave

$$\Phi_{inc}(x, y, z) = \exp[ik(x \cos \Phi_1 + y \cos \Phi_2 + z \cos \Phi_3)]. \tag{1.75}$$

Again we assume that the total field or its normal derivative vanish on both sides of the screen thus leading to the following boundary value problem:

Find $\Phi = \Phi_{sc}(x, y, z) \in C^2(R^3 \setminus \Sigma)$ being a solution to Helmholtz's equation

$$(\Delta_3 + k^2)\Phi(x, y, z) = 0 \quad \text{in} \quad R^3 \setminus \Sigma \tag{1.76}$$

satisfying, for $x, y \in \mathring{\Sigma}$, one of the following boundary conditions

$$\Phi(x, y, \pm 0) = -\Phi_{inc}(x, y, 0) \tag{1.77a}$$

or

$$\frac{\partial}{\partial z}\Phi(x, y, \pm 0) = -\frac{\partial}{\partial z}\Phi_{inc}(x, y, 0) \tag{1.77b}$$

and additionally the asymptotic conditions

$$\Phi(x, y, z) = O(1), \qquad \text{grad}\, \Phi = O(\varrho_j^{-\beta}) \tag{1.78}$$

as $\varrho_1 := \sqrt{x_2^2 + x_3^2}$ or $\varrho_2 := \sqrt{x_1^2 + x_3^2} \to 0$ with $0 \leqslant \beta < 1$ ("edge conditions") and

$$\Phi(x, y, z) = O(1), \qquad \left(\frac{\partial}{\partial r} - ik\right)\Phi(x, y, z) = O\left(\frac{e^{-k_2 r}}{r}\right) \tag{1.79}$$

as $r := \sqrt{x^2 + y^2 + z^2} \to \infty$, $k = k_1 + ik_2 \in C^{++}$.

If we try to solve these problems similarly like in Chapter 1 we shall now apply a two-dimensional Fourier transformation F_2 with respect to the variables x and y

$$\hat{\Phi}(\lambda_1, \lambda_2, z) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\lambda_1 x + \lambda_2 y)]\, \Phi(x, y, z)\, dx dy \tag{1.80}$$

and arrive, similarly to (1.13a,b) in case of $k_2 > 0$ at

$$\Phi_D(\lambda_1, \lambda_2, z) = A_D(\lambda_1, \lambda_2) \exp[-|z|\, \gamma(\lambda_1, \lambda_2)] \tag{1.81a}$$

for Dirichlet data on Σ and

(1.81b) $$\hat{\Phi}_N(\lambda_1, \lambda_2, z) = \pm A_N(\lambda_1, \lambda_2)\exp[-|z|\,\gamma(\lambda_1, \lambda_2)]$$

for Neumann data on Σ where

(1.82) $$\gamma(\lambda_1, \lambda_2) := \sqrt{\lambda_1^2+\lambda_2^2-k^2} \qquad \text{with} \quad \operatorname{Re}\gamma > 0$$

and the branch cuts taken in C^2-space along $\sqrt{\lambda_1^2+\lambda_2^2} = \pm(k+i\varrho)$; $\varrho \geqslant 0$; such that the "poly-strip" $\mathfrak{s} := \{(\lambda_1, \lambda_2) \in C^2\colon (\operatorname{Im}\lambda_1)^2+(\operatorname{Im}\lambda_2)^2 < k_2^2\}$ is a domain of holomorphy. Denoting the four unilateral Fourier transforms by $++$, $-+$, $--$ and $+-$, respectively, and keeping in mind the continuity of all derivatives of $\Phi(x, y, z)$ off the screen Σ we get

(1.83a) $$A_D(\lambda_1, \lambda_2) = -\hat{\Phi}_{\text{inc}}(\lambda_1, \lambda_2, 0)+\hat{\Phi}_{-+}(\lambda_1, \lambda_2, 0)+\hat{\Phi}_{--}(\lambda_1, \lambda_2, 0)+\hat{\Phi}_{+-}(\lambda_1, \lambda_2, 0)$$

and

(1.83b) $$\gamma\cdot A_N(\lambda_1, \lambda_2) = -\frac{\partial}{\partial z}\hat{\Phi}_{\text{inc}}(\lambda_1, \lambda_2, 0)+\frac{\partial}{\partial z}\hat{\Phi}_{-+}(\lambda_1, \lambda_2, 0)+\frac{\partial}{\partial z}\hat{\Phi}_{--}(\lambda_1, \lambda_2, 0)+ +\frac{\partial}{\partial z}\hat{\Phi}_{+-}(\lambda_1, \lambda_2, 0),$$

respectively. After introducing the unknown jumps of the potential

$$2\cdot\hat{J}_{++}(\lambda_1, \lambda_2) := \left[\frac{\partial}{\partial z}\hat{\Phi}_{++}(\lambda_1, \lambda_2, +0)-\frac{\partial}{\partial z}\hat{\Phi}_{++}(\lambda_1, \lambda_2, 0)\right]$$

and

$$2\cdot\hat{Q}_{++}(\lambda_1, \lambda_2) := [\hat{\Phi}_{++}(\lambda_1, \lambda_2, +0)-\hat{\Phi}_{++}(\lambda_1, \lambda_2, -0)]$$

across the screen Σ we arrive at the two special *four part Wiener–Hopf functional* relations

(1.84a) $$\gamma^{-1}(\lambda_1, \lambda_2)\cdot\hat{J}_{++}(\lambda_1, \lambda_2)-\hat{\Phi}_{-+}(\lambda_1, \lambda_2, 0)- -\hat{\Phi}_{--}(\lambda_1, \lambda_2, 0)-\hat{\Phi}_{+-}(\lambda_1, \lambda_2, 0) = -\hat{\Phi}_{\text{inc},++}(\lambda_1, \lambda_2, 0)$$

and

(1.84b) $$\gamma(\lambda_1, \lambda_2)\cdot\hat{Q}_{++}(\lambda_1, \lambda_2)-\frac{\partial}{\partial z}\hat{\Phi}_{-+}(\lambda_1, \lambda_2, 0)-\frac{\partial}{\partial z}\hat{\Phi}_{--}(\lambda_1, \lambda_2, 0) -\frac{\partial}{\partial z}\hat{\Phi}_{+-}(\lambda_1, \lambda_2, 0) = -\frac{\partial\hat{\Phi}_{\text{inc},++}}{\partial z}(\lambda_1, \lambda_2, 0)$$

involving four unknown transform functions each being holomorphic in pairs of complex half-planes like $\operatorname{Im}\lambda_1 > -q_1$, $\operatorname{Im}\lambda_2 > -q_2$ such that $q_1^2+q_2^2 < k_2^2$ and so forth.

The above formulated problems have been treated, e.g., in Radlow (1961, 1965). They may easily be generalized to almost arbitrary plane domains Σ representing the screens – or several disjoint of them. The functional relations (3.10a,b) are actually two-part problems with respect to the projectors $\hat{P}_\Sigma$ and $\hat{P}_{\mathbf{R}^2\setminus\Sigma}$ corresponding to the Fourier transforms of the *space projector* $P_\Sigma = \chi_\Sigma$ and $Q_\Sigma := P_{\mathbf{R}^2\setminus\Sigma} = \chi_{\mathbf{R}^2\setminus\Sigma}$.

In order to solve equations (1.84a,b) we have to factorize $\gamma(\lambda_1, \lambda_2)$ with respect to these two projectors into

$$\gamma(\lambda_1, \lambda_2) = \gamma_\Sigma(\lambda_1, \lambda_2)\cdot\gamma_{\mathbf{R}^2\setminus\Sigma}(\lambda_1, \lambda_2) \tag{1.85}$$

where in the case of Σ being the lst quarter-plane in $\mathbf{R}^2$ the factor γ_Σ is holomorphic and different from zero in the cartesian product of the half-planes $\operatorname{Im}\lambda_1 > -q_1$, $\operatorname{Im}\lambda_2 > -q_2$. Thus multiplying or dividing equations (1.84a) or (1.84b) by $\gamma_{\mathbf{R}^2\setminus\Sigma}(\lambda_1, \lambda_2)$ we have to decompose additively

$$\begin{aligned}\gamma_\Sigma^{-1}(\lambda_1, \lambda_2)\cdot\hat{J}_\Sigma(\lambda_1, \lambda_2)-\gamma_{\mathbf{R}^2\setminus\Sigma}(\lambda_1, \lambda_2)\cdot\hat{P}_{\mathbf{R}^2\setminus\Sigma}\Phi(\lambda_1, \lambda_2, 0)\\ = -\hat{\Phi}_{\mathrm{inc}}(\lambda_1, \lambda_2, 0)\cdot\gamma_{\mathbf{R}^2\setminus\Sigma}(\lambda_1, \lambda_2)\end{aligned} \tag{1.86a}$$

or

$$\begin{aligned}\gamma_\Sigma(\lambda_1, \lambda_2)\cdot\hat{Q}_\Sigma(\lambda_1, \lambda_2)-\gamma_{\mathbf{R}^2\setminus\Sigma}^{-1}(\lambda_1, \lambda_2)\cdot\hat{P}_{\mathbf{R}^2\setminus\Sigma}\frac{\partial}{\partial z}\hat{\Phi}(\lambda_1, \lambda_2, 0)\\ = -\frac{\partial}{\partial z}\hat{\Phi}_{\mathrm{inc}}(\lambda_1, \lambda_2, 0)\cdot\gamma_{\mathbf{R}^2\setminus\Sigma}^{-1}(\lambda_1, \lambda_2)\end{aligned} \tag{1.86b}$$

from which formulae we get

$$\hat{J}_\Sigma(\lambda_1, \lambda_2) = -\gamma_\Sigma(\lambda_1, \lambda_2)\cdot\hat{P}_\Sigma[\gamma_{\mathbf{R}^2\setminus\Sigma}(\tau_1, \tau_2)\hat{\Phi}_{\mathrm{inc}}(\tau_1, \tau_2, 0)](\lambda_1, \lambda_2) \quad \text{for } \operatorname{Im}\lambda_1 > -q_1,\ \operatorname{Im}\lambda > -q_2 \tag{1.87a}$$

and

$$\hat{Q}_\Sigma(\lambda_1, \lambda_2) = -\gamma_\Sigma^{-1}(\lambda_1, \lambda_2)\cdot\hat{P}_\Sigma\left[\gamma_{\mathbf{R}\setminus\Sigma}^{-1}(\tau_1, \tau_2)\frac{\partial}{\partial z}\hat{\Phi}_{\mathrm{inc}}(\tau_1, \tau_2, 0)\right](\lambda_1, \lambda_2) \quad \text{for } \operatorname{Im}\lambda_1 > -q_1,\ \operatorname{Im}\lambda > -q_2. \tag{1.87b}$$

These transform functions then may be inserted into

$$-2\gamma(\lambda_1, \lambda_2)A_D(\gamma_1, \gamma_2) = 2\hat{J}_\Sigma(\lambda_1, \lambda_2) \qquad \text{and} \qquad 2A_N(\lambda_1, \lambda_2) = 2\hat{Q}_\Sigma(\lambda_1, \lambda_2)$$

in order to obtain representations of $\hat{\Phi}_D(\lambda_1, \lambda_2, z)$ and $\hat{\Phi}_N(\lambda_1, \lambda_2, z)$, respectively. The main problem which remains is the effective factorization of $\gamma(\lambda_1, \lambda_2)$ with respect to the class of two-dimensional Fourier transforms whose original functions have supports in $\boldsymbol{R}^2$-space on Σ and $\boldsymbol{R}^2 \setminus \Sigma$, respectively. The problem of diffraction of electromagnetic waves by a right-angled dielectric wedge or more general semiinfinite obstacles all of them filled with slightly absorbent materials has been dealt with by several others. Meister and Speck (1979) gave already a detailed account on the problems. So details are deleted here.

2. Plane subsonic flow past oscillating profiles

2.1. Introduction. Flutter phenomena for single airfoils in free space or in wind-tunnels, for elastic blades in a turbo-machine, or for hydrofoils have been observed since the thirties. Many authors have tried to develop models of increasing complexity to describe theoretically the aerodynamic and aeroelastic effects on the vibrating foils and in the surrounding flowing gas or fluid. A general theory predecting the forces acting on an arbitrarily moving profile in two-dimensional subsonic flow of an inviscid or even viscous gas is still missing, since the corresponding mathematical problem will be a non-linear mixed initial-boundary value problem for a system of flow quantities.

So it is quite clear that most of the authors in the past made strong assumptions which resulted in linearizations concerning the governing differential equations and the geometry of the profiles. Here we want to mention just a few authors who developed the linear theory of unsteady profile flow: Küssner (1940), Küssner and Schwarz (1940), and Söhngen (1940) (among others) studied harmonically oscillating and arbitrarily deflected single profiles which were assumed to be thin, slightly cambered and performing movements of small amplitudes – compared to their chord-length l_0. They assumed the fluid to be inviscid and incompressible, so they arrived at a mixed boundary value problem in two-dimensional potential theory for the composition of the interval $[-l_0, l_0]$, representing the chord of the profile, and its wake (l_0, L) with $l_0 < L < \infty$. The distribution of bound and free vortices on the chord and wake, respectively, lead them to a system of singular and Abel-type integral equations for the unknown vortex densities.

These methods have been generalized to the case of infinite systems of such kinds of profiles in a cascade, e.g., Billington (1949), Söhngen (1953, 1955), Chaskind (1958), and Meister (1958, 1960). The latter used conformal mapping techniques and the methods for solving Hilbert's functiontheoretic boundary value problems for periodic functions in the complex plane.

Allowing for flow separation at fixed points on the profile several authors studied the two-dimensional flow of inviscid fluids with function-theoretic methods. like. e.g.. Sisto (1955). Wu (1957). Parkin (1957). Tulin

(1963), Woods (1963) and Davies (1970) for single profiles and Sisto (1967), Meister (1970) for cascades.

As early as in (1938) Possio started to investigate the two-dimensional subsonic flow past an oscillating thin profile by establishing a singular integral equation resulting from a double-layer representation of the acceleration potential solving the time-independent wave equation. A different approach has been worked out by Timman in his thesis (Timman (1946)) and in Timman and van de Vooren (1949) using Mathieu functions' expansions of the solution. In 1970 a pupil of mine (Speck (1970)) derived a system of integral equations after applying the Wiener-Hopf technique. This will be displayed here lateron.

Concerning the subsonic flow past profiles in a wind-tunnel Runyan and Watkins (1953), Jones (1953) and Miles (1956) set up integral equations for the vortex of pessure densities on the chord of the profile. The present author started in 1962 his investigations of subsonic plane flows past harmonically oscillating thin profiles in cascades applying a generalization of the Wiener–Hopf technique. For details concerning unsteady cascade flows see the survey paper by Meister (1967).

2.2. Formulation of the mixed boundary value problems. Denoting the velocity of the basic flow far in front of the profiles by U, the speed of sound in the moving gas by a_0, the Mach number by $M = U/a_0$. The velocity potential Φ_0 for the small perturbations satisfies the wave equation

$$\Delta_0 \Phi_0 - \frac{1}{a_0^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_0}\right)^2 \Phi_0(x_0, y_0, t) = 0 \tag{2.1}$$

in the domain $D_0 \subset \mathbf{R}_0^2$ exterior to the profiles and their wakes and, possibly, between the wind-tunnel walls or the free surface and the bottom of a stream. In the case of harmonic time-dependence a time-factor $e^{-i\omega t}$ is assumed. The perturbation pressure field may then be calculated according

$$p(x_0, y_0, t) = -\varrho_0 U\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_0}\right)\Phi_0(x_0, y_0, t) \tag{2.2}$$

with the density ϱ_0 of the gas in the basic steady flow.

Before introducing a reduced potential $\Phi(x, y)$ with respect to dimensionless coordinates x, y we shall describe roughly the changing boundary conditions on the projections of the thin profile and its wake in case of a single one. On $[-l_0, \theta l_0]$ with a fixed $-1 \leqslant \theta \leqslant +1$ the flow is attached to the oscillating profile which means the normal derivatives of Φ_0 are known. θl_0 shall denote the point of flow separation on the upper side of the profile such that a thin cavity bubble is assumed to extend downstream from θl_0 to c_0 on the upper and from l_0 to c_0 on the lower side. The bubble terminates at $c_0 \geqslant l_0$ and is followed by a thin vortex sheet extending from c_0 to $+\infty$

This model corresponds to that one introduced in Woods (1963) in the case of an incompressible gas.

It is assumed that the pressure be prescribed along the bounding lines of the bubble and that there are no discontinuities in the pressure and the normal velocities across the free vortex wake behind the bubble.

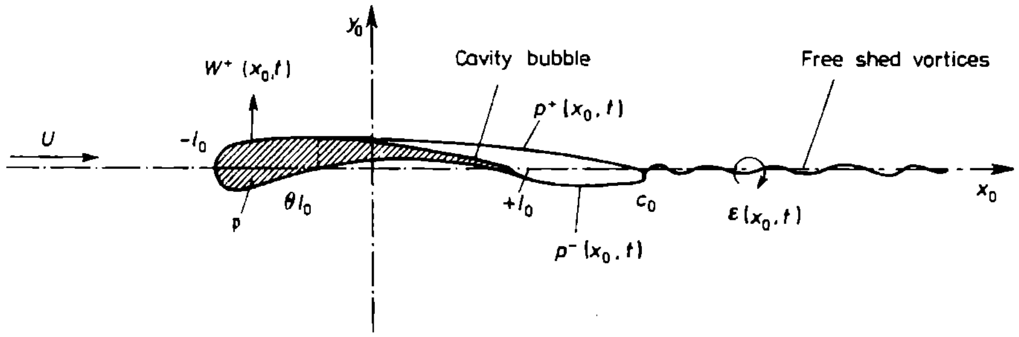

Fig. 8. Geometry of the flow past the profile p

By means of the following substitutions we arrive at the mixed boundary value problem for the reduced velocity potential in case for a cascade of profiles ($n\in \boldsymbol{Z}$) or a single one ($n = 0$) in free space or in a wind-tunnel ($n = 0$, but $\partial\Phi/\partial y(x, \pm 1/2) = 0$ for $x\in \boldsymbol{R}$ additionally):

(2.3) $$x_0 = T_0 x\cdot\sqrt{1-M^2},\quad y_0 = T_0 y,$$

(2.4) $$\omega^* := \omega T_0/U \quad \text{and} \quad k := \omega^* M/\sqrt{1-M^2},$$

(2.5) $$\Phi_0(x_0, y_0, t) = T_0\cdot \mathrm{Re}\left[\Phi(x, y)\cdot \exp(-i(kMx+\omega t))\right].$$

Determine $\Phi\in C^2(D)\cap C^1(D')$ *where* D *is the region exterior to the cascade and its wakes* $C := \{(x, y)\in \boldsymbol{R}^2:\ -l\leqslant x\leqslant +\infty,\ y = n,\ n\in \boldsymbol{Z}\}$ *and* $\bar{D}'$ *its closure with exception of the "corner points"* $x = -l,\ =\theta l,\ c,\ y = n\}$, $l := l_0/T_0\sqrt{1-M^2}$, $c := c_0/T_0\sqrt{1-M^2}$, *such that*

(2.6) $$(\Delta+k^2)\Phi(x, y) = 0 \qquad \text{in } D,$$

(2.7a) $$\frac{\partial\Phi}{\partial y}(x, n\pm 0)$$
$$= g_n^{\pm}(x) := w_n^{\pm}(T_0\sqrt{1-M^2}\cdot x)\, e^{+ikMx} \qquad \text{for} \quad -l < x < \theta l,$$

(2.7b) $$\frac{\partial\Phi}{\partial y}(x, n-0) = g_n^-(x) \qquad \text{for} \quad \theta l\leqslant x < l,$$

(2.8a)
$$\left(-\frac{ik}{M}+\frac{\partial}{\partial x}\right)\Phi(x, n+0) = f_n^+(x) := -\frac{\sqrt{1-M^2}}{\varrho_0 U^2}\, e^{+ikMx}\cdot p_n^+(T_0\sqrt{1-M^2}\cdot x)$$
$$\text{for} \quad \theta l < x < c,$$

(2.8b) $$\left(-\frac{ik}{M}+\frac{\partial}{\partial x}\right)\Phi(x, n-0) = f_n^-(x) \qquad for \quad l < x < c,$$

(2.9a)
$$\left(-\frac{ik}{M}+\frac{\partial}{\partial x}\right)\Phi(x, n+0) = \left(-\frac{ik}{M}+\frac{\partial}{\partial x}\right)\Phi(x, n-0) \qquad for \quad c < x < \infty,$$

and

(2.9b) $$\frac{\partial}{\partial y}\Phi(x, n+0) = \frac{\partial}{\partial y}\Phi(x, n-0)$$

and additionally the following asymptotic conditions shall hold:

(2.10a) $$\Phi(x, y) = O(1),$$

(2.10b) $$\text{grad } \Phi(x, y) = O(r_n^{-\beta})$$

for $r_n := \sqrt{(x+l)^2+n^2} \to 0$ (*"leading edge condition"*),

(2.11) $$\Phi(x, y) \quad and \quad \text{grad } \Phi(x, y) = O(1)$$

$(x, y) \to (\theta l, n+0)$ *and* $(x, y) \to (l, n-0)$ (*"condition of smooth flow separation"*),

(2.12a) $$\Phi(x, y) = O(1)$$

(1.12b) $$\text{grad } \Phi(x, y) = O(|\log r_n'|)$$

as $r_n' := \sqrt{(x-c)^2+n^2} \to 0$ (*"closing condition at the end of cavity bubble"*),

(2.13) $$\Phi(x, y) \quad and \quad \text{grad } \Phi(x, y) = O(1)$$

as $|x| \to \infty$ *or* $r = \sqrt{x^2+y^2} \to \infty$ *and* Φ *shall not contain terms corresponding to incoming waves* (*"radiation condition"*).

After integrating the boundary and transmission conditions (2.8a,b) and (2.9a), respectively, we end up with the Dirichlet conditions on parts of the linear boundary:

(2.14a) $$\Phi(x, n+0) = F_n^+(x) := C_n^+ e^{+ikx/M} + \int_c^x f_n^+(\xi) e^{-ik(\xi - x)/M} d\xi$$
$$\text{for} \quad \theta l \leqslant x \leqslant c,$$

(2.14b) $$\Phi(x, n-0) = F_n^-(x) := C_n^- e^{+ikx/M} + \int_c^x f_n^-(\xi) e^{-ik(\xi - x)/M} d\xi$$
$$\text{for} \quad l \leqslant x \leqslant c$$

where the constants $C_n^\pm$ must be fixed by the conditions (2.11) while (2.9a) gives rise to

(2.15) $$\Phi(x, n+0) - \Phi(x, n-0) = De^{+ikx/M} \qquad \text{for} \quad c < x < \infty$$

with the D to be fixed by condition (2.12b).

Since partially the boundary values of Φ and those of $\partial\Phi/\partial y$, respectively, are known we shall introduce their restrictions to the five x-intervals and try to solve the corresponding first and second boundary value problems with prescribed data on $y=n\pm 0$, $x\in \boldsymbol{R}$, and then equate both solutions thus obtaining a system of relations among the known and unknown boundary values on the both sides of the lines $y=n$, $n\in \boldsymbol{Z}$ (or $=0$). For this purpose we introduce

(2.16) $$E_n(x):=\Phi(x,n) \quad \text{for} \quad x<-l,$$

(2.17a) $$F_{1n}(x):=\Phi(x,n-0)$$
(2.17b) $$F_{1n}(x)+\gamma_{1n}(x):=\Phi(x,n+0) \quad \text{for} \quad -l\leqslant x<\theta l,$$

(2.18a) $$F_{2n}(x):=\Phi(x,n-0),$$
(2.18b) $$F_{2n}(x)+\gamma_{2n}(x):=\Phi(x,n+0) \quad \text{for} \quad \theta l\leqslant x<l,$$

(2.19a) $$F_{3n}(x):=\Phi(x,n-0),$$
(2.19b) $$F_{3n}(x)+\gamma_{3n}(x):=\Phi(x,n+0) \quad \text{for} \quad l\leqslant x<c,$$

(2.20a) $$F_{4n}(x):=\Phi(x,n-0),$$
(2.20b) $$F_{4n}(x)+\gamma_{4n}(x):=\Phi(x,n+0) \quad \text{for} \quad c\leqslant x<\infty$$

thus the $\gamma_{jn}(x)$ describing the jumps of the potential across the corresponding jth intervall. Similarly we set

(2.21) $$V_n(x):=\frac{\partial}{\partial y}\Phi(x,n) \quad \text{for} \quad x<-l,$$

(2.22a) $$H_{1n}(x):=\frac{\partial}{\partial y}\Phi(x,n-0),$$
(2.22b) $$H_{1n}(x)+\mu_{1n}(x):=\frac{\partial}{\partial y}\Phi(x,n+0) \quad \text{for} \quad -l<x<\theta l,$$

(2.23a) $$H_{2n}(x):=\frac{\partial}{\partial y}\Phi(x,n-0),$$
(2.23b) $$H_{2n}(x)+\mu_{2n}(x):=\frac{\partial}{\partial y}\Phi(x,n+0) \quad \text{for} \quad \theta l<x<l,$$

(2.24a) $$H_{3n}(x):=\frac{\partial}{\partial y}\Phi(x,n-0),$$
(2.24b) $$H_{3n}(x)+\mu_{3n}(x):=\frac{\partial}{\partial y}\Phi(x,n+0) \quad \text{for} \quad l<x<c,$$

(2.25) $$H_{4n}(x):=\frac{\partial}{\partial y}\Phi(x,n\pm 0) \quad \text{for} \quad c<x<\infty.$$

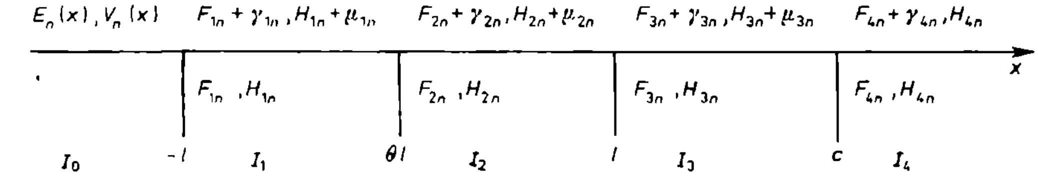

Fig. 9. Intervals with differently prescribed data

2.3. Application of the Fourier transformation. Let

$$\hat{f}(\lambda) := (\mathcal{F}f)(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x)\,dx \tag{2.26}$$

denote the classical Fourier-transform which may be extended to the Schwartz space $\mathcal{S}'$ of tempered distributions such that the well-known rule holds for any $\phi \in \mathcal{S}'$

$$\mathcal{F}\frac{\partial\phi}{\partial x}(\lambda) = (-i\lambda)(\mathcal{F}\Phi)(\lambda) = (-i\lambda)\cdot\Phi(\lambda). \tag{2.27}$$

Taking now regular distributions, that are functions not growing faster than polynomials, with respect to x and depending at least twice continuously differentiable on y we may transform the Helmholtz equation (2.6) into the following one

$$\left[(-i\lambda)^2 + \frac{\partial^2}{\partial y^2} + k^2\right]\hat{\Phi}(\lambda, y) = 0 \tag{2.28}$$

for $\lambda \in \boldsymbol{R}$ and $n < y < n+1$, $n \in \boldsymbol{Z}$, or $y \gtrless 0$.

To simplify the calculations let us take here the case of a single profile. The condition (2.13) then leads to the following two representations for the image function $\hat{\Phi}(\lambda, y)$ in $y > 0$ and < 0, respectively, according to the Dirichlet or Neumann data given on $y = 0$:

(2.29a)

$$\hat{\Phi}_+(\lambda, y) = \{\hat{E}(\lambda) + \sum_{j=1}^{4} [\hat{F}_j(\lambda) + \hat{\gamma}_j(\lambda)]\}\cdot\exp(-y\sqrt{\lambda^2-k^2}), \qquad y > 0,$$

and

$$\hat{\Phi}_-(\lambda, y) = \{\hat{E}(\lambda) + \sum_{j=1}^{4} \hat{F}_j(\lambda)\}\cdot\exp(y\sqrt{\lambda^2-k^2}), \qquad y < 0 \tag{2.29b}$$

or

(2.30a)

$$\hat{\Phi}_+(\lambda, y) = -\{\hat{V}(\lambda) + \sum_{j=1}^{4} [\hat{H}_j(\lambda) + \hat{\mu}_j(\lambda)]\}\frac{\exp(-y\sqrt{\lambda^2-k^2})}{\sqrt{\lambda^2-k^2}}, \qquad y > 0,$$

and

(2.30b) $$\hat{\Phi}_-(\lambda, y) = \{\hat{V}(\lambda)+\sum_{j=1}^{4}\hat{H}_j(\lambda)\}\frac{\exp(y\sqrt{\lambda^2-k^2})}{\sqrt{\lambda^2-k^2}}, \qquad y<0,$$

where $\hat{\mu}_4(\lambda) \equiv 0$ and the square root is defined according

(2.31)
$$\sqrt{\lambda^2-k^2} = \sqrt{\lambda-k}\cdot\sqrt{\lambda+k} = |\lambda^2-k^2|^{1/2}\cdot\exp\{\tfrac{i}{2}[\arg(\lambda-k)+\arg(\lambda+k)]\}$$

with

(2.32) $$-3\pi/2 < \arg(\lambda-k) \leqslant \pi/2 \qquad \text{and} \qquad -\pi/2 < \arg(\lambda+k) \leqslant 3\pi/2,$$

respectively, such that $\sqrt{\lambda^2-k^2} \sim |\lambda|$ as $|\lambda| \to \infty$ as a real variable.

Equating now the right-hand sides of equations (2.29a) and (2.30a) as well as (2.29b) and (2.30b) yields the following *generalized Wiener–Hopf system of functional equations* by additional summing and subtracting, respectively:

(2.33a) $$2\hat{E}(\lambda)+2\hat{F}_1(\lambda)+\hat{\gamma}_1(\lambda)+\hat{F}_2(\lambda)+2\hat{F}_4(\lambda)+$$
$$+[\hat{\mu}_2(\lambda)+\hat{\mu}_3(\lambda)]/\sqrt{\lambda^2-k^2} = \hat{r}_1(\lambda) := -[\hat{F}_2(\lambda)+\hat{\gamma}_2(\lambda)]-$$
$$-[\hat{F}_3(\lambda)+\hat{\gamma}_3(\lambda)]-\hat{F}_3(\lambda)-\hat{\gamma}_4(\lambda)-\hat{\mu}_1(\lambda)/\sqrt{\lambda^2-k^2}$$

and

(2.33b) $$2\hat{V}(\lambda)/\sqrt{\lambda^2-k^2}+\hat{\gamma}_1(\lambda)-\hat{F}_2(\lambda)+$$
$$+[\hat{\mu}_2(\lambda)+2\hat{H}_3(\lambda)+\hat{\mu}_3(\lambda)+2\hat{H}_4(\lambda)]/\sqrt{\lambda^2-k^2} = \hat{r}_2(\lambda) := -[\hat{F}_2(\lambda)+\hat{\gamma}_2(\lambda)]-$$
$$-[\hat{F}_3(\lambda)+\hat{\gamma}_3(\lambda)]+\hat{F}_3(\lambda)-\hat{\gamma}_4(\lambda)-[2\hat{H}_1(\lambda)+\hat{\mu}_1(\lambda)+2\hat{H}_2(\lambda)]/\sqrt{\lambda^2-k^2}$$

both on $\boldsymbol{R}_\lambda$ with known right-hand sides $\hat{r}_1, \hat{r}_2$. On the left-hand sides there are collected all unknown transforms of functions with supports in the intervalls $I_0, I_j; j = 1, \ldots, 4; c\boldsymbol{R}_x$. In case of finite l and c all the transforms labelled by 1, 2 or 3 are entire functions of exponential type in the complex λ-plane while the functions $\hat{E}(\lambda)$, $\hat{V}(\lambda)$ are holomorphic in the lower half-plane $\operatorname{Im}\lambda < 0$ and $\hat{H}_4(\lambda)$, $\hat{\gamma}_4(\lambda)$ holomorphic in the upper half-plane $\operatorname{Im}\lambda > 0$, respectively.

Since the boundary values of Φ and those of their normal derivatives are not elements of the same space – particularly due to the different asymptotic behavior near the "switch-points" $-l$, $0l$, $+l$, c, new unknown transform functions have to be introduced first to study the system (2.33a,b) as a vectorial five-part Wiener–Hopf system on a Banach space $\mathscr{X}$ of functions on $\boldsymbol{R}_\lambda$. If the wave-number k in (2.4) is assumed to have a positive imaginary part $k_2 > 0$ then it is reasonable to sharpen the asymptotic condition (2.13) to $O(e^{-k_2 r})$ for $r \to \infty$ in $0 < \arg(x+iy) < 2\pi$ and $O(1)$ for

$x \to +\infty$ which gives rise to $\hat{E}(\lambda)$, $\hat{V}(\lambda) = o(1)$ for $\lambda \to \infty$ in $\operatorname{Im} \lambda < k_2$ and $\hat{H}_4(\lambda)$, $\hat{\gamma}_4(\lambda) = o(1)$ in $\operatorname{Im} \lambda > 0$ due to the Riemann–Lebesgue Lemma but a more careful investigation using Abel-type theorems of the Fourier transformation says

$$\hat{E}(\lambda), \qquad \hat{\gamma}_4(\lambda) = O(|\lambda|^{-1}) \tag{2.34a}$$

and

$$\hat{V}(\lambda) = O(|\lambda|^{\beta-1}), \qquad \hat{H}_4(\lambda) = O(|\lambda|^{-1} \log|\lambda|) \tag{2.34b}$$

as $|\lambda| \to \infty$ in the appropriate half-planes. Since $\sqrt{\lambda - k}$ is holomorphic in $\operatorname{Im} \lambda < k_2$ and $\sqrt{\lambda + k}$ in $\operatorname{Im} \lambda > -k_2$, respectively, we have

$$\psi_-^{(1)}(\lambda) := e^{i\lambda l} \cdot \hat{E}(\lambda) \cdot \sqrt{\lambda - k} = O(|\lambda|^{-1/2}), \tag{2.35a}$$

$$\psi_-^{(2)}(\lambda) := e^{i\lambda l} \cdot \hat{V}(\lambda) / \sqrt{\lambda - k} = O(|\lambda|^{\beta - 3/2}) \tag{2.35b}$$

in $\operatorname{Im} \lambda < k_2$ while

$$\psi_+^{(1)}(\lambda) := e^{-i\lambda c} \cdot \hat{F}_4(\lambda) \cdot \sqrt{\lambda + k} = O(|\lambda|^{-1/2}) \tag{2.36a}$$

$$\psi_+^{(2)}(\lambda) := e^{-i\lambda c} \cdot \hat{H}_4(\lambda) / \sqrt{\lambda + k} = O(|\lambda|^{-3/2} \cdot \log|\lambda|). \tag{2.36b}$$

So we see that these four functions belongs to $L^q(\boldsymbol{R}_\lambda)$ where $q > 2$. This leads us to study the Wiener–Hopf system of the modified boundary values, e.g., in spaces $L^p(\boldsymbol{R}_x)$ for $1 \leqslant p < 2$ $(1/p + 1/q = 1)$. The functions bearing an index 1, 2 or 3 in equations (2.33a,b) have to be modified in a similar way but there are always two possibilities existing depending on the decision whether the entire functions of exponential type should become plus or minus functions that they have an algebraic behavior as $\lambda \to \infty$ in the upper or lower half-plane, respectively, and exhibit an exponential growth in the complementary half-plane, e.g.,

$$e^{-i\lambda l} \cdot \hat{\gamma}_1(\lambda) \cdot \sqrt{\lambda + k} = O(|\lambda|^{-1/2}) \text{ as } \lambda \to \infty \text{ in } \operatorname{Im} \lambda \geqslant 0 \tag{2.37a}$$

and

$$e^{-i\lambda \theta l} \cdot \hat{\gamma}_1(\lambda) \cdot \sqrt{\lambda - k} = O(|\lambda|^{-1/2}) \text{ as } \lambda \to \infty \text{ in } \operatorname{Im} \lambda \leqslant 0. \tag{2.37b}$$

After all we end up with a five-part-system

$$\vec{\Psi}_-(\lambda) + \sum_{j=1}^{3} \mathbf{A}_j(\lambda) \cdot \vec{\Psi}_j(\lambda) + \mathbf{A}_4(\lambda) \cdot \vec{\Psi}_+(\lambda) = S(\lambda) \tag{2.38}$$

on $\boldsymbol{R}_\lambda$ where $\vec{\Psi}_\pm(\lambda) := (\hat{P}_\pm \vec{\Psi})(\lambda)$; $\vec{\Psi}_j(\lambda) := (\hat{P}_j \vec{\Psi})(\lambda)$ with continuous mutually disjoint projectors $\hat{P}$ on $\mathscr{F}L^p(\boldsymbol{R}_x)$ being the transforms of the "space-projectors" $P_\pm \stackrel{\wedge}{=} \chi'_\pm$, $\hat{P}_j \stackrel{\wedge}{=} \chi'_j$ with the characteristic functions for $\boldsymbol{R}_\pm$, $J_1 := [0, (1+\theta)l]$, $J_2 := [(1+\theta)l, 2l]$, $J_3 := [2l, l+c]$, respectively.

We shall not list the details of the general case here, since one may find them in Meister (1978). So we shall confine ourselves to special three-part problems.

2.4. Two special cases of three-part mixed boundary value problems

(i) First let us choose $\theta = 1$ and $c = l$ then we arrive at the three-part Wiener–Hopf problem corresponding to a thin oscillating profile of finite length $2l_0$. This has been treated, e.g., by F.-O. Speck in his master's thesis (Speck 1970). All quantities with indices 2 or 3 drop out in equations (2.33a,b) and the last one reads

$$(2.39)\quad 2\hat{V}(\lambda)/\sqrt{\lambda^2-k^2}+\hat{\gamma}_1(\lambda)+2\hat{H}_4(\lambda)/\sqrt{\lambda^2-k^2} = \hat{\gamma}(\lambda)$$
$$:= \frac{iD}{\sqrt{2\pi}}\frac{\exp[i(\lambda-k/M)]}{\lambda+k/M}-\frac{2\hat{H}_1(\lambda)+\hat{\mu}_1(\lambda)}{\sqrt{\lambda^2-k^2}}$$

where the functions on the left-hand side are unknown. Putting, in a first step,

$$(2.40a)\quad \hat{\phi}^-(\lambda) := e^{i\lambda l}\,\hat{V}(\lambda)/\sqrt{\lambda-k},$$

$$(2.40b)\quad \hat{\phi}^+(\lambda) := \tfrac{1}{2}e^{i\lambda l}\,\hat{\gamma}_1(\lambda)\cdot\sqrt{\lambda+k}$$

we arrive at a classical two-part WH-functional relation after multiplying (2.39) by $\frac{1}{2}e^{i\lambda l}\sqrt{\lambda+k}$ and pushing the $\hat{H}_4$-term to the right-hand side:

$$(2.41)\quad \hat{\phi}^-(\lambda)+\hat{\phi}^+(\lambda) = \hat{s}_1(\lambda)-e^{i\lambda l}\hat{H}_4(\lambda)/\sqrt{\lambda-k}.$$

This may be solved quite simply just by an additive decomposition of the function on the right-hand side. When $s(\lambda)$ denotes a function holomorphic in the strip $\mathfrak{s}_\alpha := \{\lambda\in C:\ -\infty<\operatorname{Re}\lambda<+\infty,\ -\alpha<\operatorname{Im}\lambda<+\alpha\}$ then we denote by

$$(2.42)\quad \hat{s}^\pm(\lambda) = (\hat{P}_\pm\hat{s})(\lambda) := \pm\frac{1}{2\pi i}\int_{\pm\delta i-\infty}^{\pm\delta i+\infty}\frac{\hat{s}(\tau)\,d\tau}{\tau-\lambda}$$
$$\text{for}\quad \operatorname{Im}\lambda \gtrless \mp\delta,\ 0<\delta<\alpha,$$

its additive components which are uniquely defined for $\hat{s}(\lambda) = o(1)$ as $|\operatorname{Re}\lambda| \to \infty$. Applying the projection operator $\hat{P}_\pm$ onto (2.41) we end up with the two equations

$$(2.42a)\quad \hat{\phi}^+(\lambda) = \hat{s}_1^+(\lambda)-\big(\hat{P}_+\big(e^{il\tau}\hat{H}_4(\tau)/\sqrt{\tau-k}\big)\big)(\lambda)$$

holomorphic for $\operatorname{Im}\lambda > -\delta > -k_2$, and

$$(2.42b)\quad \hat{\phi}^-(\lambda) = \hat{s}_1^-(\lambda)-\big(\hat{P}_-\big(e^{il\tau}\hat{H}_4(\tau)/\sqrt{\tau-k}\big)\big)(\lambda)$$

holomorphic for $\operatorname{Im}\lambda < \delta < k_2$.

In a second step we solve again a two-part problem by introducing

(2.43a) $$\hat{\psi}^-(\lambda) := \tfrac{1}{2}e^{-il\lambda}\hat{\gamma}_1(\lambda)\cdot\sqrt{\lambda-k},$$

(2.43b) $$\hat{\psi}^+(\lambda) := e^{-i\lambda l}\hat{H}_4(\lambda)/\sqrt{\lambda+k}$$

multiplying (2.39) now by $\frac{1}{2}e^{-i\lambda l}\sqrt{\lambda-k}$, and pushing the term with $\hat{V}(\lambda)$ to the right-hand side. Additive decomposition then yields the two relations

(2.44a) $$\hat{\psi}^+(\lambda) = \hat{s}_2^+(\lambda) - \big(\hat{P}_+\big(e^{-il\tau}\hat{V}(\tau)/\sqrt{\tau+k}\big)\big)(\lambda)$$

holomorphic for $\operatorname{Im}\lambda > -\delta > -k_2$, and

(2.44b) $$\hat{\psi}^-(\lambda) = \hat{s}_2^-(\lambda) - \big(\hat{P}_-\big(e^{-il\tau}\hat{V}(\tau)/\sqrt{\tau+k}\big)\big)(\lambda).$$

Now the equations (2.42b) and (2.44a) constitute a coupled system of integral equations for $\hat{\phi}^-(\lambda)$ and $\hat{\psi}^+(\lambda)$ due to equations (2.40a) and (2.43b).

This *alternating system* may be written in the following form

(2.45a) $$\hat{\phi}^-(\lambda) + \hat{P}_-\left(e^{2il\tau}\sqrt{\frac{\tau+k}{\tau-k}}\hat{\psi}^+(\tau)\right)(\lambda) = \hat{s}_1^-(\lambda),$$

(2.45b) $$\hat{\psi}^+(\lambda) + \hat{P}_+\left(e^{-2il\tau}\sqrt{\frac{\tau-k}{\tau+k}}\hat{\phi}^-(\tau)\right)(\lambda) = \hat{s}_2^+(\lambda)$$

where the $\hat{s}_1^-$ and $\hat{s}_2^+$ are known up to the constant D connected with the amplitude of shed vortices in the wake $x > l$. After inserting the first equation into the second one, we arrive at

(2.46) $$\hat{\psi}^+(\lambda) - \hat{P}_+\left(e^{-2il\tau}\sqrt{\frac{\tau-k}{\tau+k}}\hat{P}_- e^{2il\sigma}\sqrt{\frac{\sigma+k}{\sigma-k}}\hat{\psi}^+(\sigma)\right)(\lambda) = \hat{h}^+(\lambda)$$

$$:= \hat{s}^+(\lambda) - \hat{P}_+\left(e^{-2il\tau}\sqrt{\frac{\tau-k}{\tau+k}}\hat{s}_1^-(\tau)\right)(\lambda)$$

or – due to $\hat{P}_+ + \hat{P}_- = \hat{I}$ the identity operator –

(2.47) $$\left(\hat{P}_+ e^{-2il\tau}\sqrt{\frac{\tau-k}{\tau+k}}\hat{P}_+\right)\left(\hat{P}_+ e^{2il\sigma}\sqrt{\frac{\sigma+k}{\sigma-k}}\hat{P}_+\right)\hat{\psi}^+ = \hat{h}^+(\lambda)$$

holding for $\operatorname{Im}\lambda > -\delta > -k_2$ and constituting a product of two Wiener–Hopf operators (WHOs) in the Fourier transform space, with reciprocal "Symbols" $e^{-2il\tau}\sqrt{\dfrac{\tau-k}{\tau+k}}$. Investigations concerning general WHOs, or Toeplitz operators as they are also called, with such kind and even more general piecewise continuous and almost periodic symbols on the real line were performed, e.g., Sarason (1973, 1977) and Dudučava (1979). An explicit inversion of equation (2.47) is not known to the present author.

The system (2.45a,b) may be solved by on iteration technique, corresponding to the Schwarzschild procedure for the diffraction by a wide slit, when l is not too small. For this purpose the line integrals defining $\hat{P}_{\pm}$ in formula (2.42) are deformed into loop integrals along the branch cuts from k to $i\infty$ and $-k$ to $-i\infty$, respectively, since only, the square roots entering

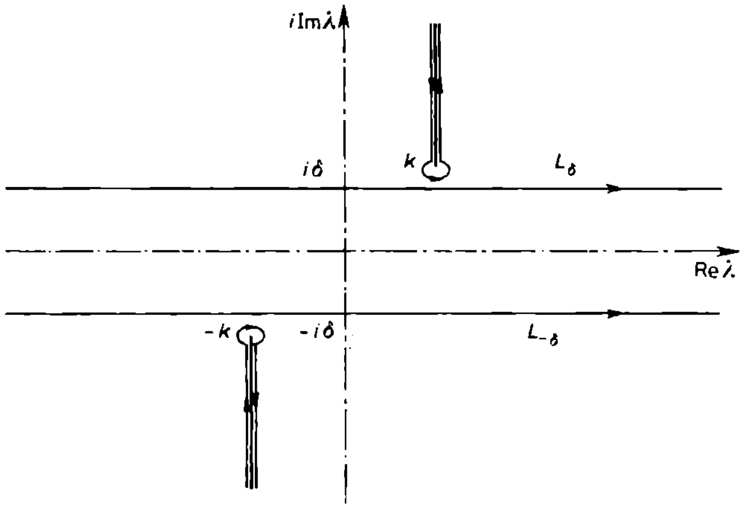

Fig. 10. Deformation of integration lines $L_{\pm\delta}$

into the integrals are not holomorphic in the half-planes $\mathrm{Im}\,\lambda \geqslant k_2$ and $\mathrm{Im}\,\lambda \leqslant -k_2$, respectively. We finally end up with

(2.48a) $$\hat{\phi}^-(\lambda) - \frac{e^{2ilk}}{\pi i} \int_{\varrho=0}^{\infty} e^{-2l\varrho} \sqrt{\frac{\varrho - 2ik}{\varrho}} \cdot \frac{\hat{\psi}^+(k+i\varrho)\,d\varrho}{\varrho + i(\lambda - k)} = \hat{s}_1^-(\lambda)$$

holding for $\mathrm{Im}\,\lambda < \delta < k_2$,

(2.48b) $$\hat{\psi}^+(\lambda) + \frac{e^{2ilk}}{\pi i} \int_{\varrho=0}^{\infty} e^{-2l\varrho} \cdot \sqrt{\frac{\varrho - 2ik}{\varrho}} \cdot \frac{\hat{\phi}^-(-k-i\varrho)\,d\varrho}{\varrho - i(\lambda + k)} = \hat{s}_2^+(\lambda)$$

holding for $\mathrm{Im}\,\lambda > \delta > -k_2$.

The solution to this system is known when the values of $\hat{\phi}^-(-k-i\varrho)$ and $\hat{\psi}^+(k+i\varrho)$, $\varrho \geqslant 0$, are known, respectively, which are to be calculated from the *alternating system of integral equations with Hille–Tamarkin-kernels*:

(2.49a) $$\hat{\phi}^-(-k-i\sigma) - \int_0^{\infty} K(\sigma, \varrho)\hat{\psi}^+(k+i\varrho)\,d\varrho = \hat{s}_1^-(-k-i\sigma),$$

(2.49b) $$\hat{\psi}^+(k+i\sigma) + \int_0^{\infty} K(\sigma, \varrho)\hat{\phi}^-(-k-i\varrho)\,d\varrho = \hat{s}_2^+(k+i\sigma)$$

where $K(\sigma, \varrho)$, $\varrho, \sigma > 0$ is given by

$$K(\sigma, \varrho) := \frac{e^{2ilk}}{\pi i} e^{-2l\varrho} \cdot \sqrt{\frac{\varrho - 2ik}{\varrho}} (\varrho + \sigma - 2ik)^{-1} \tag{2.50}$$

which makes sense also for real $k > 0$. Since it is not square integrable the kernel defines a linear and compact operator only from $L^q(\mathbf{R}_+)$ to $L^p(\mathbf{R}_+)$ for $p > 1$ and $q > 2$. For $l \to +\infty$ it becomes zero so that the operator norm will be < 1 for $l > l(k)$ yielding then a contraction mapping. For these l (2.49a) may be solved by the iteration technique.

(ii) Let us now consider the special case of $\theta = -1$ and $c = +\infty$ corresponding to a flow which separates at the leading edge with a narrow cavity extending to infinity. Now all quantities which are labelled 1 or 4 have to be dropped in eqs. (2.33a,b). Again we are lead to a three-part WH functional system. Introducing, in the first step, the functions

$$\hat{\phi}_1^-(\lambda) := 2e^{i\lambda l} \hat{E}(\lambda) \cdot \sqrt{\lambda - k}, \tag{2.51a}$$

$$\hat{\phi}_2^-(\lambda) := 2e^{i\lambda l} \hat{V}(\lambda) / \sqrt{\lambda - k} \tag{2.51b}$$

being holomorph for $\operatorname{Im} \lambda < k_2$ and atenuating as $\lambda \to \infty$ there, while

$$\hat{\phi}_1^+(\lambda) = e^{i\lambda l} \cdot \hat{F}_2(\lambda) \sqrt{\lambda + k}, \tag{2.52a}$$

$$\hat{\phi}_2^+(\lambda) := e^{i\lambda l} \cdot \hat{\mu}_2(\lambda) / \sqrt{\lambda + k} \tag{2.52b}$$

being holomorphic for $\operatorname{Im} \lambda > -k_2$ and tending to zero as $\lambda \to \infty$ there, we may rewrite the equations (2.33a,b) as

$$\begin{bmatrix} \hat{\phi}_1^-(\lambda) \\ \hat{\phi}_2^-(\lambda) \end{bmatrix} + \begin{bmatrix} \sqrt{\frac{\lambda - k}{\lambda + k}} & 1 \\ -1 & \sqrt{\frac{\lambda + k}{\lambda - k}} \end{bmatrix} \begin{bmatrix} \hat{\phi}_1^+(\lambda) \\ \hat{\phi}_2^+(\lambda) \end{bmatrix} + \tag{2.53}$$

$$+ \begin{bmatrix} e^{i\lambda l} \hat{\mu}_3(\lambda) / \sqrt{\lambda + k} \\ e^{i\lambda l} [2\hat{H}_3(\lambda) + \hat{\mu}_3(\lambda)] / \sqrt{\lambda - k} \end{bmatrix} = \begin{bmatrix} e^{i\lambda l} \sqrt{\lambda - k} \cdot \hat{r}_1(\lambda) \\ e^{i\lambda l} \sqrt{\lambda + k} \cdot \hat{r}_2(\lambda) \end{bmatrix} =: \vec{s}(\lambda).$$

Now the 2×2-function matrix $\mathbf{L}(\lambda)$ has to be factored according $\mathbf{L}^-(\lambda) \cdot [\mathbf{L}^+(\lambda)]^{-1}$ where $\mathbf{L}^{\pm}(\lambda)$ are holomorphic and non-singular in the respective half-planes $\operatorname{Im} \lambda \gtrless \mp k_2$. But this is the matrix which appeared already in the generalized Sommerfeld half-plane problem in Chapter 1. So, after multiplying (2.53) from the left by $\mathbf{L}^-(\lambda)^{-1}$ and then decomposing additively we arrive at the system corresponding to the equations (2.42a,b):

(2.52a)
$$\begin{bmatrix}\phi_1^+(\lambda)\\ \phi_2^+(\lambda)\end{bmatrix} = \mathbf{L}^+(\lambda)\cdot(\hat{P}_+[\mathbf{L}^-(\tau)]^{-1}\hat{\hat{s}}(\tau))(\lambda) - $$
$$-\mathbf{L}^+(\lambda)\cdot\left(\hat{P}_+[\mathbf{L}^-(\tau)]^{-1}\begin{bmatrix} e^{i\tau l}\cdot\hat{\mu}_3(\tau)/\sqrt{\tau+k}\\ e^{i\tau l}\cdot[2\hat{H}_3(\tau)+\hat{\mu}_3(\tau)]/\sqrt{\tau-k}\end{bmatrix}\right)(\lambda)$$
for $\operatorname{Im}\lambda > -\delta > -k_2$

and

(2.52b)
$$\begin{bmatrix}\hat{\phi}_1^-(\lambda)\\ \hat{\phi}_2^+(\lambda)\end{bmatrix} = \mathbf{L}^-(\lambda)\cdot(\hat{P}_-[\mathbf{L}^-(\tau)]^{-1}\hat{\hat{s}}(\tau))(\lambda) - $$
$$-\mathbf{L}^-(\lambda)\left(\hat{P}[\mathbf{L}^-(\tau)]^{-1}\begin{bmatrix} e^{i\tau l}\hat{\mu}_3(\tau)/\sqrt{\tau+k}\\ e^{i\tau l}[2\hat{H}_3(\tau)+\hat{\mu}_3(\tau)]/\sqrt{\tau-k}\end{bmatrix}\right)(\lambda)$$
for $\operatorname{Im}\lambda < \delta < k_2$.

Since the quantities $\hat{H}_3$ and $\hat{\mu}_3$ are still unknown we need a second system containing these and the $\hat{\phi}_1^-$ and $\hat{\phi}_2^-$. To this end we introduce the functions

(2.53a)
$$\hat{\psi}_1^-(\lambda) := e^{-i\lambda l}\hat{F}_2(\lambda)\cdot\sqrt{\lambda-k},$$

(2.53b)
$$\hat{\psi}_2^-(\lambda) := e^{-i\lambda l}\cdot\hat{\mu}_2(\lambda)/\sqrt{\lambda-k}$$

being holomorphic for $\operatorname{Im}\lambda < k_2$, while

(2.54a)
$$\hat{\psi}_1^+(\lambda) := e^{-i\lambda l}\hat{\mu}_3(\lambda)/\sqrt{\lambda+k},$$

(2.54b)
$$\hat{\psi}_2^+(\lambda) := 2e^{-i\lambda l}\cdot\hat{H}_3(\lambda)/\sqrt{\lambda+k}$$

being holomorphic for $\operatorname{Im}\lambda > -k_2$. Then we get the following WH-system

(2.55)
$$\begin{bmatrix} 1 & \sqrt{\dfrac{\lambda-k}{\lambda+k}}\\ -\sqrt{\dfrac{\lambda+k}{\lambda-k}} & 1\end{bmatrix}\begin{bmatrix}\hat{\psi}_1^-(\lambda)\\ \hat{\psi}_2^-(\lambda)\end{bmatrix} + \begin{bmatrix} 1 & 0\\ \sqrt{\dfrac{\lambda+k}{\lambda-k}} & \sqrt{\dfrac{\lambda+k}{\lambda-k}}\end{bmatrix}\begin{bmatrix}\hat{\psi}_1^+(\lambda)\\ \hat{\psi}_2^+(\lambda)\end{bmatrix} +$$
$$+\begin{bmatrix} 2e^{-i\lambda l}\hat{E}(\lambda)\cdot\sqrt{\lambda-k}\\ 2e^{-i\lambda l}\hat{V}(\lambda)/\sqrt{\lambda-k}\end{bmatrix} = \begin{bmatrix} e^{-i\lambda l}\sqrt{\lambda-k}\,\hat{\gamma}_1(\lambda)\\ e^{-i\lambda l}\sqrt{\lambda+k}\,\hat{\gamma}_2(\lambda)\end{bmatrix}$$

or after multiplication by the inverse of the first matrix

(2.56)
$$\begin{bmatrix}\hat{\psi}_1^-(\lambda)\\ \hat{\psi}_2^-(\lambda)\end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1\\ 2\sqrt{\dfrac{\lambda+k}{\lambda-k}} & \sqrt{\dfrac{\lambda+k}{\lambda-k}}\end{bmatrix}\begin{bmatrix}\hat{\psi}_1^+(\lambda)\\ \hat{\psi}_2^+(\lambda)\end{bmatrix} +$$
$$+e^{-i\lambda l}\begin{bmatrix}\hat{E}(\lambda)\sqrt{\lambda-k} - \dfrac{\hat{V}(\lambda)}{\sqrt{\lambda+k}}\\ \hat{E}(\lambda)\sqrt{\lambda+k} + \dfrac{\hat{V}(\lambda)}{\sqrt{\lambda-k}}\end{bmatrix} = e^{-i\lambda l}\begin{bmatrix}[\hat{\gamma}_1(\lambda)-\hat{\gamma}_2(\lambda)]\sqrt{\lambda-k}\\ [\hat{\gamma}_1(\lambda)+\hat{\gamma}_2(\lambda)]\sqrt{\lambda+k}\end{bmatrix} =: \hat{\hat{s}}^*(\lambda).$$

Now by factoring the matrix $\mathbf{M}(\lambda)$ in the first line into $\mathbf{M}^-(\lambda)[\mathbf{M}^+(\lambda)]^{-1}$ then multiplying the last equation by $[\mathbf{M}^-(\lambda)]^{-1}$ and finally splitting additively, we arrive at the second system

(2.57a) $$\begin{bmatrix} \hat{\psi}_1^-(\lambda) \\ \hat{\psi}_2^-(\lambda) \end{bmatrix} = \mathbf{M}^-(\lambda)(\hat{P}_-[\mathbf{M}^-(\tau)]^{-1}\hat{\hat{s}}^*(\tau))(\lambda) -$$
$$-\mathbf{M}^-(\lambda)\left(\hat{P}_-[\mathbf{M}^-(\tau)]^{-1}e^{-i\tau l}\begin{bmatrix} \hat{E}(\tau)\sqrt{\tau-k} - \hat{V}(\tau)/\sqrt{\tau+k} \\ \hat{E}(\tau)\sqrt{\tau+k} + \hat{V}(\tau)/\sqrt{\tau-k} \end{bmatrix}\right)(\lambda)$$
$$\text{for} \quad \operatorname{Im}\lambda < \delta < k_2,$$

and

(2.57b) $$\begin{bmatrix} \hat{\psi}_1^+(\lambda) \\ \hat{\psi}_2^+(\lambda) \end{bmatrix} = \mathbf{M}^+(\lambda)(\hat{P}_+[\mathbf{M}^-(\tau)]^{-1}\hat{\hat{s}}^*(\tau))(\lambda) -$$
$$-\mathbf{M}^+(\lambda)\left(\hat{P}_+[\mathbf{M}^-(\tau)]^{-1}e^{-i\tau l}\begin{bmatrix} \hat{E}(\tau)\sqrt{\tau-k} - \hat{V}(\tau)/\sqrt{\tau+k} \\ \hat{E}(\tau)\sqrt{\tau+k} + \hat{V}(\tau)/\sqrt{\tau-k} \end{bmatrix}\right)(\lambda)$$
$$\text{for} \quad \operatorname{Im}\lambda > -\delta > -k_2.$$

The two systems (2.52b) and (2.57b) contain the four unknown functions $\hat{\phi}_{1,2}^-(\lambda)$ and $\hat{\psi}_{1,2}^+(\lambda)$ using the vector notations

(2.58a) $$\hat{\hat{X}}_-(\lambda) := [\mathbf{L}^-(\lambda)]^{-1}\begin{bmatrix} \hat{\phi}_1^-(\lambda) \\ \hat{\phi}_2^-(\lambda) \end{bmatrix},$$

(2.58b) $$\hat{\hat{Y}}_+(\lambda) := [\mathbf{M}^+(\lambda)]^{-1}\begin{bmatrix} \hat{\psi}_1^+(\lambda) \\ \hat{\psi}_2^+(\lambda) \end{bmatrix},$$

(2.59a) $$\hat{\hat{X}}_-(\lambda) + \hat{P}_-\{e^{2i\tau l}[\mathbf{L}^-(\tau)]^{-1}\mathbf{C}(\tau)\mathbf{M}^+(\tau)\hat{\hat{Y}}_+(\tau)\}(\lambda)$$
$$= (\hat{P}_-[\mathbf{L}^-(\tau)]^{-1}\hat{\hat{s}}(\tau))(\lambda) \quad \text{for} \quad \operatorname{Im}\lambda < \delta < k_2,$$

and

(2.59b) $$\hat{\hat{Y}}_+(\lambda) + \hat{P}_+\{e^{-2i\tau l}[\mathbf{M}^+(\tau)]^{-1}\cdot[\mathbf{C}(\tau)]^{-1}\cdot\mathbf{L}^-(\tau)\hat{\hat{X}}_-(\tau)\}(\lambda)$$
$$= (\hat{P}_+[\mathbf{M}^-(\tau)]^{-1}\hat{\hat{s}}^*(\tau))(\lambda) \quad \text{for} \quad \operatorname{Im}\lambda > -\delta > -k_2,$$

where

$$\mathbf{C}(\tau) := \begin{bmatrix} 1 & 0 \\ \sqrt{\dfrac{\tau+k}{\tau-k}} & \sqrt{\dfrac{\tau+k}{\tau-k}} \end{bmatrix}$$

such that with

(2.60) $$\mathbf{K}(\tau; k, l) := e^{2i\tau l}\cdot[\mathbf{L}^-(\tau)]^{-1}\cdot\mathbf{C}(\tau)\mathbf{M}^+(\tau)$$

the system (2.59a,b) is an *alternating reciprocal one*. By substituting (2.59b) into (2.59a) we arrive at

$$
\begin{aligned}
&\hat{\hat{X}}_-(\lambda)-(\hat{P}_-\mathbf{K}(\tau)\hat{P}_+[\mathbf{K}(\sigma)]^{-1}\hat{P}_-\hat{\hat{X}}_-(\sigma))(\lambda)\\
&\quad=(\hat{P}_-\mathbf{K}(\tau)\hat{P}_-)(\hat{P}_-[\mathbf{K}(\sigma)]^{-1}\hat{P}_-)\hat{\hat{X}}_- = \hat{\hat{h}}_-(\lambda)\\
&\quad:=(\hat{P}_-[\mathbf{L}^-(\tau)]^{-1}\hat{\hat{s}}(\tau))(\lambda)-(\hat{P}_-\mathbf{K}(\tau)\hat{P}_+[\mathbf{M}^-(\sigma)]^{-1}\hat{\hat{s}}(\sigma))(\lambda)\\
&\qquad\qquad\text{for}\quad \operatorname{Im}\lambda<\delta<k_2.
\end{aligned}
\tag{2.61}
$$

This 2×2-system contains the product of two WHOs with inverse symbols like in the case (i) which was a scalar one. It can be shown that the operator in the first line of (2.61) is a Fredholm–Riesz one on certain spaces $L^p(\boldsymbol{R}_\lambda)$. An explicit inverse is unknown up to now. Using deformations of the integration paths in the projection operators $\hat{P}_\pm$ one may transform this system into one with Hille–Tamarkin-kernels. We omit details here.

Remark. 1. The present author discussed also the cases of oscillating profiles in a wind-tunnel, in a cascade or hydrofoils in a similar way. In these cases the simple square root $\sqrt{\lambda^2-k^2}$ has to be replaced by more complicated functions which are meromorphic in the whole λ-plane. For the details cf. Meister (1978).

2. Meister and Sommer (1979) discussed also the case of arbitrary time-dependence applying the Mikusiński calculus with respect to the time t.

3. In the case of a rotating blade-row with oscillating blades in an annular channel the corresponding boundary value problems have been derived by the author and his co-workers (1982). The mixed boundary value problems may be transformed into multiple-part WH-systems for an infinite number of unknown transform functions after applying an eigenfunction expansion with respect to vanishing normal derivatives on the inner and outer cylinder of the annular channel. The details of these investigations are published elsewhere.

3. Diffraction of elastic waves by three-dimensional semi-infinite bodies

Now we are going to study the behavior of elastic waves propagating in $\boldsymbol{R}^3$-space which is divided into N semi-infinite regions $\bar{G}_j$ filled by different homogeneous isotropic absorbing materials. We assume that Gauss' theorem may be applied to every bounded subdomain $\bar{G}_j(R):=\bar{G}_j\cap K_R(0)$. Let a primary field characterized by the displacement vector field $\vec{U}_{\text{pr}}(x)$, the stress tensor field $\mathbf{S}_{\text{pr}}(x)$, and the body force field $\vec{K}_{\text{pr}}(x)$ be given in $\bar{G}_1$ with the density ϱ_1, the Lamé coefficients λ_1, μ_1, and the damping coefficients η_1. Our problem then is the following one:

Find

$\vec{U}_j(x) \in C^2(\mathring{G}_j) \cap C^1(\bar{G}_j') \cap C(\bar{G}_j)$,

$$j = 1, \ldots, N, \ \mathbf{S}(x) = (\sigma_{kl}(x))_{k,l=1,2,3} \in C^1(\bar{G}_j') \cap C^1(\bar{G}_j)$$

where $\bar{G}_j'$ denotes the closed regions with the exception of edges and vertices of their boundaries. These fields shall satisfy the equations of a harmonically time dependent motion

$$\operatorname{div} \mathbf{S}_j + (\omega^2 \varrho_j + i\omega\eta_j) \vec{U}_j + \vec{K}_j = \vec{O} \tag{3.1}$$

where $\vec{K}_j = \vec{O}$ for $j \neq 1$. Hooke's law is assumed to hold, i.e.,

$$\mathbf{S}_j = \lambda_j \operatorname{div} \vec{U}_j \cdot (\delta_{kl})_{k,l=1,2,3} + \mu_j \cdot \left(\frac{\partial U_k^{(j)}}{\partial x_l} + \frac{\partial U_l^{(j)}}{\partial x_k} \right)_{k,l=1,2,3}. \tag{3.2}$$

Across the common non-void boundaries $\partial G_j \cap \partial G_{j-1}$; $j = 1, \ldots, N$; the total displacement vectors and surface tensions shall pass continuously, i.e.,

$$\vec{U}_{j,\text{tot}}|_{\partial G_{j-1}} = \vec{U}_{j-1,\text{tot}}|_{\partial G_{j-1}} \tag{3.3a}$$

and

$$\mathbf{S}_{j,\text{tot}}\, \boldsymbol{n}_j|_{\partial G_j} + \mathbf{S}_{j-1,\text{tot}}\, \boldsymbol{n}_{j-1}|_{\partial G_{j-1}} = \vec{O} \tag{3.3b}$$

where $\boldsymbol{n}_j$ denotes the outward directed unit normal vector to ∂G_j such that $\boldsymbol{n}_{j-1} = -\boldsymbol{n}_j$ at common boundary points. (These are the simplest transmission conditions.)

In the neighborhood of edges and vertices we claim that $\mathbf{S}_j$, $\nabla \mathbf{S}_j$, $\nabla \otimes \vec{U}_j$ behave in such a way that the integrals occurring in the following context converge absolutely. Additionally as $r \to \infty$ we claim

$$\vec{U}_j, \quad \nabla \otimes \vec{U}_j, \quad \mathbf{S}_j, \quad \operatorname{div} \mathbf{S}_j = O(e^{-qr}) \tag{3.4}$$

with some $q > 0$.

We are going to derive an N-part composite Wiener–Hopf system for 10-vectors having

$$U_h^{(j)}, \quad h = 1, 2, 3, \qquad \sigma_{kl}^{(j)}, \quad 1 \leqslant k, l \leqslant 3, \qquad \theta_j := \operatorname{div} \vec{U}_j$$

as components.

Let $\mathbf{T}$ denote any symmetric continuous bounded tensor field with $\operatorname{div} \mathbf{T} \in L^1(\bar{G})$, assume the existence of a continuously differentiable bounded vector field with $\nabla \otimes \vec{V} := \left(\frac{\partial V_l}{\partial x_k} \right)_{k,l=1,2,3} \in L^1(\bar{G})$, $\bar{G}$ a bounded regular domain, then the following integral theorems hold:

$$\int_B \operatorname{div} \mathbf{T}\, dx = \oint_{\partial B} \mathbf{T} \boldsymbol{n}\, do, \tag{3.5}$$

$$\int_B \nabla \otimes \vec{V}\, dx = \oint_{\partial B} \boldsymbol{n} \otimes \vec{V}\, do \tag{3.6}$$

where $\boldsymbol{n}\otimes\vec{V} := (n_k V_l)_{k,l=1,2,3}$ denotes the tensor product of $\vec{V}$ and the surface normal vector $\boldsymbol{n}$. Now we choose $\mathbf{T} := \mathbf{S}e^{ix\xi}$ and obtain

$$\int_B e^{ix\xi}[\operatorname{div}\mathbf{S}+i\mathbf{S}\xi]\,dx = \oint_{\partial B} e^{ix\xi}\mathbf{S}\boldsymbol{n}\,do. \tag{3.7}$$

Taking $\mathbf{S} = \mathbf{S}_{\text{tot}}$ and $B = \bar{G}_j \cap K_R(o)$ and letting $R \to \infty$ we obtain the result due to the relations (3.3b) and (3.4):

$$\sum_{j=1}^{N} \int_{\bar{G}_j} e^{ix\xi}[\operatorname{div}\mathbf{S}_j + i\mathbf{S}_j\xi]\,dx + \int_{\bar{G}_1} e^{ix\xi}[\operatorname{div}\mathbf{S}_{\text{pr}} + i\mathbf{S}_{\text{pr}}\xi]\,dx = \vec{O}. \tag{3.8}$$

Denoting the three-dimensional Fourier transform by F and $P_j\mathbf{S}_j = \chi_{\bar{G}_j'}\mathbf{S}$ we get

$$\sum_{j=1}^{N} (FP_j \operatorname{div}\mathbf{S} + i(FP_j\mathbf{S})\boldsymbol{\xi}) = -FP_1 \operatorname{div}\mathbf{S}_{\text{pr}} - i(FP_1\mathbf{S}_{\text{pr}})\boldsymbol{\xi}. \tag{3.9}$$

Inserting the left-hand side of (3.1) we obtain

$$\begin{aligned} -\sum_{j=1}^{N} (\omega^2\varrho_j + i\omega\eta_j)FP_j U + i\sum_{j=1}^{N}(FP_j\mathbf{S})\boldsymbol{\xi} \\ = -FP_1\operatorname{div}\mathbf{S}_{\text{pr}} - i(FP_1\mathbf{S}_{\text{pr}})\boldsymbol{\xi} + FP_1\vec{K}_1 =: FP_1\vec{R}_{\text{pr}} \end{aligned} \tag{3.10}$$

where the vector field $\vec{R}_{\text{pr}}$ is known in $\bar{G}_1$.

Now we insert $\vec{U}_{j,\text{tot}}e^{ix\xi}$ for $\vec{V}$ in (3.6) and obtain

$$\int_{\bar{G}_j} e^{ix\xi}[\nabla\otimes\vec{U}_{j,\text{tot}} + i\boldsymbol{\xi}\otimes\vec{U}_{j,\text{tot}}]\,dx = \oint_{\partial G_j} e^{ix\xi}(\boldsymbol{n}_j\otimes\vec{U}_{j,\text{tot}})\,do. \tag{3.11}$$

Summing from 1 to N and splitting $\vec{U}_{1,\text{tot}}$ into $\vec{U}_1 + \vec{U}_{\text{pr}}$ and taking into account the relations (3.3a) we arrive at

$$\sum_{j=1}^{N} \int_{\bar{G}_j} e^{ix\xi}[\nabla\otimes\vec{U}_j + i\boldsymbol{\xi}\otimes\vec{U}_j]\,dx = -\int_{\bar{G}_1} e^{ix\xi}[\nabla\otimes\vec{U}_{\text{pr}} + i\boldsymbol{\xi}\otimes\vec{U}_{\text{pr}}]\,dx. \tag{3.12}$$

Now after adding the equations of the transposed tensor fields $(\nabla\otimes\vec{U}_j)^T$ and $i(\boldsymbol{\xi}\otimes\vec{U}_j)^T$, respectively, we make use of Hooke's law and arrive at

$$\begin{aligned} \sum_{j=1}^{N}\left(\frac{1}{\mu_j}FP_j\mathbf{S} - \frac{\lambda_j}{\mu_j}FP_j\theta\mathbf{I} + iFP_j\vec{U}\otimes\boldsymbol{\xi} + i\boldsymbol{\xi}\otimes FP_j\vec{U}\right) \\ = -FP_1\big(\nabla\otimes\vec{U}_{\text{pr}} + (\nabla\otimes\vec{U}_{\text{pr}})^T\big) - iF(P_1\vec{U}_{\text{pr}}\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes P_1\vec{U}_{\text{pr}}) =: FP_1\vec{W}_{\text{pr}} \end{aligned} \tag{3.13}$$

which is a known tensor field. Here we have introduced $\theta := \operatorname{div}\vec{U}$.

The final equation we are looking for is derived from the most common form of Gauss' theorem, viz.

$$\begin{aligned} \int_{\bar{G}_j} \operatorname{div}(\vec{U}_{j,\text{tot}}e^{ix\xi})\,dx &= \int_{\bar{G}_j} e^{ix\xi}[\operatorname{div}\vec{U}_{j,\text{tot}} + i\langle\vec{U}_{j,\text{tot}},\boldsymbol{\xi}\rangle]\,dx \\ &= \int_{\partial G_j} e^{ix\xi}\langle\vec{U}_{j,\text{tot}},\boldsymbol{n}_j\rangle\,do. \end{aligned} \tag{3.14}$$

Splitting $\vec{U}_{1,\text{tot}}$ into $\vec{U}_1$ and $\vec{U}_{\text{pr}}$ and summing we obtain

$$(3.15)\quad \sum_{j=1}^{N} (FP_j\theta + i\langle FP_j\vec{U},\xi\rangle) = -FP_1\theta_{\text{pr}} - i\langle FP_1\vec{U}_{\text{pr}},\xi\rangle =: FP_1\vec{r}$$

being a known function.

The equations (3.9), (3.13) and (3.15) constitute the announced system of Wiener–Hopf equations for 10-vectors $(FP_j\theta, FP_j\vec{U}, FP_j\mathbf{S})$. The corresponding N 10×10-matrices of functions on $\boldsymbol{R}^3_\xi$ are only partially occupied by elements different from zero. A careful investigation gives the following N matrices $\mathbf{M}^{(j)}(\xi;\lambda_j,\mu_j)$ from which we write down one representative

$$(3.16)\quad \mathbf{M}(\xi;\lambda,\mu) = \begin{bmatrix} 1 & i\xi_1 & i\xi_2 & i\xi_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 & i\xi_1 & i\xi_2 & i\xi_3 & 0 & 0 & 0 \\ 0 & 0 & M_{33} & 0 & 0 & i\xi_1 & 0 & i\xi_2 & i\xi_3 & 0 \\ 0 & 0 & 0 & M_{44} & 0 & 0 & i\xi_1 & 0 & i\xi_2 & i\xi_3 \\ -\lambda/\mu & 2i\xi_1 & 0 & 0 & \mu^{-1} & & & & & \\ 0 & i\xi_2 & i\xi_1 & 0 & & \mu^{-1} & & & 0 & \\ 0 & i\xi_3 & 0 & i\xi_1 & & & \mu^{-1} & & & \\ -\lambda/\mu & 0 & 2i\xi_2 & 0 & & & & \mu^{-1} & & \\ 0 & 0 & i\xi_3 & i\xi_2 & & & 0 & & \mu^{-1} & \\ -\lambda/\mu & 0 & 0 & 2i\xi_3 & & & & & & \mu^{-1} \end{bmatrix}$$

where $M_{ll} = -(\varrho\omega^2 + i\omega\eta)$ for $l = 2, 3, 4$.

After taking suitable linear combinations of the equations of this Wiener–Hopf system it is possible to decouple the system in such a way that there results a 3×3-system containing only the unknown three components of the displacement vectors $\vec{U}^{(j)}$ while the six components from the stress tensors may be calculated from the remaining six equations. The matrices corresponding to the equations of motion

$$\mu_j\Delta\vec{U}^{(j)} + (\lambda_j+\mu_j)\,\text{grad div}\,\vec{U}^{(j)} + (\varrho_j\omega^2 + i\eta_j\omega)\vec{U}^{(j)} + \vec{K} = \vec{O}$$

are given by

$$(3.17)\quad \mathbf{A}_j := \begin{bmatrix} \mu_j|\xi|^2 + (\lambda_j+\mu_j)\xi_1^2 - (\omega^2\varrho_j + i\omega\eta_j) & (\lambda_j+\mu_j)\xi_1\xi_2 & (\lambda_j+\mu_j)\xi_1\xi_3 \\ (\lambda_j+\mu_j)\xi_2\xi_1 & \mu_j|\xi|^2 + (\lambda_j+\mu_j)\xi_2^2 - (\omega^2\varrho_j + i\omega\eta_j) & (\lambda_j+\mu_j)\xi_2\xi_3 \\ (\lambda_j+\mu_j)\xi_3\xi_1 & (\lambda_j+\mu_j)\xi_3\xi_2 & \mu_j|\xi|^2 + (\lambda_j+\mu_j)\xi_3^2 - (\omega^2\varrho_j + i\omega\eta_j) \end{bmatrix}$$

the determinant of which is

$$\det \mathbf{A}_j = \big(\mu_j|\xi|^2-(\omega^2\varrho_j+i\eta_j\omega)\big)^2\big((\lambda_j+2\mu_j)|\xi|^2-(\omega^2\varrho_j+i\omega\eta_j)\big). \tag{3.18}$$

Let

$$\sum_{j=1}^{N}\mathbf{A}_j(\xi)\hat{P}_j\hat{\vec{U}} = \hat{P}_1\vec{Q} \qquad \text{(be known)} \tag{3.19}$$

where the 3×3-function matrices $\mathbf{A}_j(\xi)$ are simultaneously unitarily equivalent to

$$\mathbf{D}_j(\xi) =: \mathbf{T}^*(\xi)\mathbf{A}_j(\xi)\mathbf{T}(\xi) \tag{3.20}$$

where we have the N diagonal matrices

$$\mathbf{D}_j(\xi) := \begin{bmatrix} \mu_j|\xi|^2-(\omega^2\varrho_j+i\omega\eta_j) & 0 & 0 \\ 0 & \mu_j|\xi|^2-(\omega^2\varrho_j+i\omega\eta_j) & 0 \\ 0 & 0 & (\lambda_j+2\mu_j)|\xi|^2-(\omega^2\varrho_j+i\omega\eta_j) \end{bmatrix}, \qquad j=1,\dots,N \tag{3.21}$$

and the orthogonal matrix

$$\mathbf{T}(\xi) := \begin{bmatrix} \dfrac{-\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \dfrac{-\xi_1\xi_3}{|\xi|\sqrt{\xi_1^2+\xi_2^2}} & \dfrac{\xi_1}{|\xi|} \\ \dfrac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & \dfrac{-\xi_2\xi_3}{|\xi|\sqrt{\xi_1^2+\xi_2^2}} & \dfrac{\xi_2}{|\xi|} \\ 0 & \dfrac{\sqrt{\xi_1^2+\xi_2^2}}{|\xi|} & \dfrac{\xi_2}{|\xi|} \end{bmatrix}. \tag{3.22}$$

With the representation of the $\mathbf{A}_j(\xi)$ by (3.17) and multiplication then by $\mathbf{T}^{-1}(\xi) = \mathbf{T}^*(\xi)$ gives with the transformed components $\mathbf{T}^*(\xi)\hat{P}_j\cdot\hat{\vec{U}} =: \hat{P}'_j\cdot\hat{\vec{U}}$ which are also mutually orthogonal to each other due to

$$\begin{aligned} \langle \hat{P}'_j\hat{\vec{U}}, \hat{P}'_k\hat{\vec{U}}\rangle &= \langle \mathbf{T}^*\hat{P}_j\hat{\vec{U}}, \mathbf{T}^*\hat{P}_k\hat{\vec{U}}\rangle \\ &= \int_{\mathbb{R}^3_\xi} (\mathbf{T}^*\hat{P}_j\hat{\vec{U}}(\xi), \mathbf{T}^*\hat{P}_k\hat{\vec{U}}(\xi))_{\mathbb{R}^3}\,d\xi \\ &= \int_{\mathbb{R}^3_\xi} (\mathbf{T}(\xi)\mathbf{T}^*(\xi)\hat{P}_j\hat{\vec{U}}(\xi), \hat{P}_k\hat{\vec{U}}(\xi))_{\mathbb{R}^3}\,d\xi \\ &= \int_{\mathbb{R}^3_\xi} (\hat{P}_j\hat{\vec{U}}(\xi), \hat{P}_k\hat{\vec{U}}(\xi)) = \delta_{jk}\cdot\|\hat{\vec{U}}_j\|^2_{L^2}, \end{aligned} \tag{3.23}$$

$$\sum_{j=1}^{N}\mathbf{D}_j(\xi)\hat{P}'_j\hat{\vec{U}} = \mathbf{T}^*\hat{P}_1\vec{Q} \qquad \text{(known).} \tag{3.24}$$

This 3-system of N-part Wiener–Hopf equations decomposes into three scalar ones due to the diagonal form of the matrices $\mathbf{D}_j(\xi)$.

Now we shall invoke a general theorem from Meister and Speck (1977, 1979) which constitutes a *sufficient* condition for continuously inverting a N-part Wiener–Hopf operator (WHO):

THEOREM. *Let* $\mathbf{B}_j := F^{-1}\mathbf{\Phi}_j(\xi)\cdot F \in \mathscr{L}(\mathscr{H})$ *be a translation invariant operator acting on the Hilbert space* $\mathscr{H} := (L^2(\mathbf{R}^n))^m$ *for* $n, m \in N$ *with symbol matrices* $\mathbf{\Phi}_j(\xi) \in \mathrm{GL}(m, C)$ *and* $\mathbf{\Phi}_j \in (L^\infty(\mathbf{R}^n_\xi))^{m\times m}$. *If* $|\lambda_{j,\max}(\xi)|$ *denotes the* (ξ- *and j-dependent) maximal absolut eigenvalues of the Hermitean matrices*

$$(3.25)\qquad (\mathbf{I}-\mathbf{\Phi}_j(\xi))^*(\mathbf{I}-\mathbf{\Phi}_j(\xi)) \in (L^\infty(\mathbf{R}^n_\xi))^{m\times m}$$

then

$$(3.26)\qquad W_N u := \sum_{j=1}^{N} \mathbf{B}_j P_j \vec{u} = \vec{v} \in L^2(\mathbf{R}^n_x)^m$$

is continuously invertible, i.e., $W_N^{-1} \in \mathscr{L}(\mathscr{H})$ *if*

$$(3.27)\qquad \sum_{j=1}^{N} \|\lambda_{j,\max}(\xi)\|_{L^\infty(\mathbf{R}^n_\xi)} < 1$$

(e.g., *which is the case for*

$$(3.28)\qquad \Lambda_j := \|\lambda_{j,\max}(\xi)\|_{L^\infty(\mathbf{R}^n_\xi)} < 1/N \qquad \forall j = 1, \ldots, N.$$

Proof. $W_N \vec{u} := \sum_{j=1}^{N} F^{-1}\mathbf{\Phi}_j(\xi)\cdot F\cdot P_j \vec{u} = \vec{v}$ will be multiplied by the unitary operator F: $L^2(\mathbf{R}^n_x)^m \to L^2(\mathbf{R}^n_\xi)^m$. Denoting $\hat{P}_j := FP_jF^{-1}$ and $\hat{\vec{u}} := F\vec{u}$, $\hat{\vec{v}} := F\vec{v}$ we get

$$(3.29)\qquad FW_N \vec{u} := \hat{W}_N \hat{\vec{u}} := \sum_{j=1}^{N} \mathbf{\Phi}_j(\xi)\hat{P}_j \hat{\vec{u}} = F\vec{v} = \hat{\vec{v}}.$$

The N orthogonal projectors P_j adding up to the identity keep this property as transformed ones $\hat{P}_j$. Now we write the last equation as

$$(3.30)\qquad \hat{W}_N \hat{\vec{u}} = \hat{\vec{u}} - \sum_{j=1}^{N} (\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{P}_j \hat{\vec{u}} = \hat{\vec{v}}.$$

A sufficient condition for the continuous invertibility of $\hat{W}_N$ is simply the following one

$$(3.31)\qquad \Big\|\sum_{j=1}^{N} (\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{P}_j\Big\|_{\mathscr{L}(\hat{\mathscr{H}})} < 1$$

in order to apply Banach's fixed point principle.

In order to estimate the operator norm in the last inequality one has –

keeping in mind that the operators on $\hat{\mathscr{H}} := F\mathscr{H}$ are just multiplication matrices:

$$\|\sum_{j=1}^{N} (\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{P}_j\|_{\mathscr{L}(\hat{\mathscr{H}})} := \sup_{\|\hat{u}\|_{\hat{\mathscr{H}}}=1} \|\sum_{j=1}^{N} (\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{P}_j\hat{u}\|_{\hat{\mathscr{H}}}. \tag{3.32}$$

So we estimate the term below the sup-sign

$$\|\sum_{j=1}^{N} (\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{P}_j\hat{u}\|_{\hat{\mathscr{H}}}^2 \leqslant \big(\sum_{j=1}^{N} \|(\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{P}_j\hat{u}\|\big)^2 \tag{3.33}$$

$$\leqslant \big(\sum_{j=1}^{N} \|(\mathbf{I}-\mathbf{\Phi}_j(\xi))\|_{\mathscr{L}(\hat{\mathscr{H}})}^2 \cdot \sum_{j=1}^{N} \|\hat{P}_j\hat{u}\|_{\hat{\mathscr{H}}}^2\big)$$

$$\Big(\text{put } \|\hat{u}\|_{\hat{\mathscr{H}}}^2 = \sum_{j=1}^{N} \|\hat{P}_j\hat{u}\|^2\Big)$$

and

$$\|(\mathbf{I}-\mathbf{\Phi}_j(\xi))\|_{\mathscr{L}(\hat{\mathscr{H}})}^2 = \sup_{\|\hat{v}\|_{\hat{\mathscr{H}}}=1} \|(\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{v}\|_{\hat{\mathscr{H}}}^2 \tag{3.34}$$

$$= \sup_{\|\hat{v}\|_{\hat{\mathscr{H}}}=1} \langle(\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{v}, (\mathbf{I}-\mathbf{\Phi}_j(\xi)\hat{v})\rangle_{\hat{\mathscr{H}}}$$

$$= \sup_{\|\hat{v}\|_{\hat{\mathscr{H}}}=1} \langle(\mathbf{I}-\mathbf{\Phi}_j(\xi))^*(\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{v}, \hat{v}\rangle_{\hat{\mathscr{H}}}$$

$$\leqslant \sup_{\|\hat{v}\|_{\hat{\mathscr{H}}}=1} \int_{\mathbf{R}_\xi^n} \big((\mathbf{I}-\mathbf{\Phi}_j(\xi))^*(\mathbf{I}-\mathbf{\Phi}_j(\xi))\hat{v}(\xi)\cdot\hat{\bar{v}}(\xi)\big)_{\mathbf{R}^3}\,d\xi$$

$$\leqslant \sup_{\|\hat{v}\|_{\hat{\mathscr{H}}}=1} \int_{\mathbf{R}_\xi^n} |\lambda_{j,\max}(\xi)|\cdot\hat{v}(\xi)\cdot\hat{\bar{v}}(\xi)\,d\xi \leqslant \Lambda_{j,\max}\cdot\|\hat{v}\|_{\hat{\mathscr{H}}}^2 = \Lambda_{j,\max}$$

where

$$\Lambda_{j,\max} := \sup_{\xi\in\mathbf{R}^n} |\lambda_{j,\max}(\xi)| = \|\lambda_{j,\max}(\xi)\|_{L^\infty(\mathbf{R}_\xi^n)} \qquad \text{for } j=1,\ldots,N.$$

Now we see immediately that

$$\sum_{j=1}^{N} \|(\mathbf{I}-\mathbf{\Phi}_j(\xi))\|_{\mathscr{L}(\hat{\mathscr{H}})}^2 \leqslant \sum_{j=1}^{N} \Lambda_{j,\max} < 1 \tag{3.35}$$

is a sufficient condition here.

Remark. In the case of the $\mathbf{\Phi}_j(\xi)$ being diagonal function matrices we may weaken the condition to the corresponding modulus $|\lambda_j^{(l)}(\xi)|$ and $\Lambda_j^{(l)} := \sup_{\xi\in\mathbf{R}^n} |\lambda_j^{(l)}(\xi)|$ and then

$$\sum_{j=1}^{N} \Lambda_j^{(l)} < 1 \qquad \text{for } l=1,\ldots,m. \tag{3.36}$$

In order to apply this result let us multiply (3.24) by the diagonal matrix

$$\psi(\xi):=\begin{bmatrix}(\mu|\xi|^2-\varkappa)^{-1} & 0 & 0\\ 0 & (\mu|\xi|^2-\varkappa)^{-1} & 0\\ 0 & 0 & ((\lambda+2\mu)|\xi|^2-\varkappa)^{-1}\end{bmatrix} \tag{3.37}$$

with auxiliary parameters $\varkappa$, λ, μ. We then put

$$\mathbf{I}-\mathbf{\Phi}_j(\xi):=\mathbf{I}-\psi(\xi)\cdot\mathbf{D}_j(\xi) \tag{3.38}$$

$$=\begin{bmatrix}\dfrac{(\mu-\mu_j)|\xi|^2-(\varkappa-\varkappa_j)}{\mu|\xi|^2-\varkappa} & 0 & 0\\ 0 & \dfrac{(\mu-\mu_j)|\xi|^2-(\varkappa-\varkappa_j)}{\mu|\xi|^2-\varkappa} & 0\\ 0 & 0 & \dfrac{(\lambda-\lambda_j+2\mu-2\mu_j)|\xi|^2-(\varkappa-\varkappa_j)}{(\lambda+2\mu)|\xi|^2-\varkappa}\end{bmatrix}$$

and then will have the eigenvalues of the Hermitean matrices $(\mathbf{I}-\mathbf{\Phi}_j(\xi))^*\times\times(\mathbf{I}-\mathbf{\Phi}_j(\xi))$ given as the diagonal elements of

$$\mathbf{L}_j(\xi):=\bar{\psi}(\xi)\cdot\psi(\xi)\bar{\mathbf{D}}_j(\xi)\mathbf{D}_j(\xi) \tag{3.39}$$

$$=\begin{bmatrix}\dfrac{|(\mu-\mu_j)|\xi|^2-(\varkappa-\varkappa_j)|^2}{|\mu|\xi|^2-\varkappa|^2} & 0 & 0\\ 0 & \dfrac{|(\mu-\mu_j)|\xi|^2-(\varkappa-\varkappa_j)|^2}{|\mu|\xi|^2-\varkappa|^2} & 0\\ 0 & 0 & \dfrac{|(\lambda-\lambda_j+2\mu-2\mu_j)|\xi|^2-(\varkappa-\varkappa_j)|^2}{|(\lambda+2\mu)|\xi|^2-\varkappa|^2}\end{bmatrix}.$$

In order to fix $\varkappa$, λ, μ to meet the condition, we have to calculate the maximum values of each element with respect to $\xi\in\mathbf{R}^3$, i.e., $t:=|\xi|^2\geqslant 0$. The asymptotic values as $t\to+0$ and $t\to+\infty$, respectively, for this elements $f_j^{(l)}(t)$ are given by

$$\lim_{t\to+0} f_j^{(l)}(t)=|1-\varkappa_j/\varkappa|^2,\qquad l=1,2,3, \tag{3.40a}$$

$$\lim_{t\to+\infty} f_j^{(1,2)}(t)=|1-\mu_j/\mu|^2, \tag{3.40b}$$

$$\lim_{t\to+\infty} f_j^{(3)}(t)=\left|1-\frac{\lambda_j+2\mu_j}{\lambda+2\mu}\right|^2. \tag{3.40c}$$

These are finite and non-negative so the maxima do exist and are positive. They may be found by differentiating with respect to $t \geqslant 0$, e.g. The exact sufficient conditions on the parameters $\mu_j, \lambda_j, \varkappa_j, j = 1, \ldots, N$; and λ, μ, $\varkappa$ have to be worked out in order to ensure the condition for the applicability of Banach's fixed point theorem to be valid.

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