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MEASURE AND CATEGORY

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1. Introduction

In [4] Oxtoby remarks that "there is nothing paradoxical in the fact that a set that is small in one sense may be large in some other sense" (Chapter I). The notions of smallness where the author refers to are "measure zero" and "first category". Indeed, there are subsets U of the real line $\mathbf{R}$ which have Lebesgue measure zero (so which are small in one sense) but whose *complement* $\mathbf{R}-U = U'$ is contained in a countable union of nowhere-dense closed sets (so U', the complement, is a set of first category and hence small in another sense). Both these notions, namely measure zero and first category, were intended to be mathematical formalizations of the notion of "negligible, or small, set". So it may not be paradoxical, but it is certainly very unfortunate that there exist such sets, like the set $U \subset \mathbf{R}$ mentioned above, which seem to be small but whose complements also seem to be small (where can one expect a "general point" of $\mathbf{R}$, in U or in it's complement?).

It seems that in applications to physics and engineering, and also in computer simulations, the measure theoretic notions are the relevant ones. On the other hand it seems to be unclear how to generalize the notion of sets of measure zero to Banach-manifolds like $\mathrm{Diff}^r(M)$ (the space of C^r-diffeomorphisms of M to iself) which play an important role in the theory of differentiable dynamical systems, and on which one needs a notion of "small sets".

A concrete example where one sees the paradoxes arrising from these two notions of small sets is the following. It is known that the elements $\phi \in \mathrm{Diff}^r(S^1)$, S^1 is the circle, $r \geq 1$, which have an irrational rotation number form a set of first category ([5]). On the other hand it is known that for any C^1-curve $\gamma\colon [a, b] \to \mathrm{Diff}^r(S^1)$, $r \geq 3$, such that the rotation numbers of $\gamma(a)$ and $\gamma(b)$ are different, the set $M(\gamma) = \{\mu \in [a, b] \mid \gamma(\mu)$ has an irrational rotation number$\}$ has positive Lebesgue measure ([1]). This example is related with questions like: will two (weakly) coupled oscillators "in general" get rationally related frequences due to the interaction?

In this paper we discuss the notation of "null set". In the finite dimensional case a null set is just a set which is both of 1st category and has Lebesgue measure zero. It turns out that this notion of null set can be generalized to the Banach manifolds one considers in dynamical systems. The set of $\phi \in \mathrm{Diff}^r(S^1)$, $r \geqslant 4$, with irrational rotation number, is *not* a null set.

In discussing the various notions of "small sets" we found it usefull, in order to make the discussion more systematic, to introduce the "Fubini property" and the "mapping property". With these we roughly mean the following (the precise definitions must be adapted to the various situations):

A notion of "small sets" has the *Fubini property* if, whenever $N \subset X \times Y$ is a small subset of $X \times Y$, there is a small subset $N_X \subset X$ such that for each $x \in X - N_X$, $N \cap (\{x\} \times Y)$ is a small subset of $\{x\} \times Y$.

A notion of "small sets" has the *mapping property* if, whenever $N \subset Y$ is a small subset, there is a small subset $N \subset C(X, Y)$ ($C(X, Y)$ is a space of mappings from X to Y) such that for $f \in C(X, Y) - N$, $f^{-1}(N)$ is a small subset of X.

For applications in dynamical systems, especially in bifurcation theory, the mapping property is very important. The examples in the next section show that the relation between first category, measure zero and the mapping property is quite unexpected. They also generalize the above mentioned paradox about diffeomorphisms of the circle with irrational rotation number.

This paper is an extension of [9]; part of the examples in the next section was already available in that paper – the present treatment is simpler and the results more general. The discussion of Fubini property, mapping property and null sets is new.

2. Examples

The purpose of this section is to show that for each k, $m \geqslant 1$ and each subset $\boldsymbol{M} \subset \{0, \ldots, m\}$, there is a subset $R_{M,k} \subset \boldsymbol{R}^m$ and a subset $N_n \subset C^k(\boldsymbol{R}^n, \boldsymbol{R}^m)$ for each $n \leqslant m$ ($C^k(\boldsymbol{R}^n, \boldsymbol{R}^m)$ denotes the space of C^k maps from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$) such that:

(i) N_n is a set of first category for all n;

(ii) for $f \in C^k(\boldsymbol{R}^n, \boldsymbol{R}^m) - N_n$, $f^{-1}(R_{M,k})$ has measure zero if $n \notin \boldsymbol{M}$ and has full measure if $n \in \boldsymbol{M}$.

If we denote elements of $C^k(\boldsymbol{R}^n, \boldsymbol{R}^m) - N_n$ by generic maps then the above statement says that the fact that for generic $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^m$, $f^{-1}(A)$ has measure zero (or full measure) does not imply anything for $g^{-1}(A)$, $g\colon \boldsymbol{R}^{n'} \to \boldsymbol{R}^m$ generic, whenever $n \neq n'$.

The above statement follows once we can prove it for the case $\boldsymbol{M}$

$= \{m', m'+1, m'+2, \ldots, m\}$: for the general case one just takes complements unions and intersections of $R_{M,k}$'s with M of the above special form.

It is even enough to prove the above statement only for $M = \{m\}$. To see this, let $R_{M',k} \subset \mathbf{R}^{m'}$ be a subset so that generic C^k-mappings f: $\mathbf{R}^n \to \mathbf{R}^{m'}$ satisfy $f^{-1}(R_{M',k})$ has measure zero if $n < m'$ and full measure if $n = m'$.

Let π: $\mathbf{R}^m \to \mathbf{R}^{m'}$ be a canonical projection. Then $R_{M,k} = \pi^{-1}(R_{M',k})$ corresponds to $M = \{m', m'+1, \ldots, m\}$. In order to see this we observe that for a given projection π: $\mathbf{R}^m \to \mathbf{R}^{m'}$, and considering inverse images of $\pi^{-1}(R_{M',k})$, by maps $f \in C^k(\mathbf{R}^n, \mathbf{R}^m)$, we can just as well consider inverse images of $R_{M',k}$ by maps $\tilde{f} \in C^k(\mathbf{R}^n, \mathbf{R}^{m'})$. This means that for $n \leqslant m'$ we are done. For $n > m'$, or even $n \geqslant m'$, it is well known that for generic maps $\tilde{f} \in C^k(\mathbf{R}^n, \mathbf{R}^{m'})$ (i.e., for maps not belonging to some fixed $N \subset C^k(\mathbf{R}^n, \mathbf{R}^{m'})$ of first category), $K(\tilde{f}) = \{x \in \mathbf{R}^n \mid \operatorname{rank}(d\tilde{f})_x < m'\}$, the critical set of $\tilde{f}$, has Lebesgue measure zero. From this and the rank theorem it follows that for such generic $\tilde{f}$, $\tilde{f}^{-1}(A)$ has measure zero (or full measure) whenever A has measure zero (or full measure). This implies that for generic $f \in C^k(\mathbf{R}^n, \mathbf{R}^m)$, for $n \geqslant m'$, $f^{-1}(\pi^{-1}(R_{M',k}))$ has full measure.

From now on we fix $M = \{m\}$ and k and write R instead of $R_{M,k}$ for the set to be constructed. First we make a set $R_{a,\varepsilon} \subset I^m(a) = \{x_1, \ldots, x_m) \mid |x_i| \leqslant a\}$ such that the Lebesgue measure of $I^m(a) - R_{a,\varepsilon}$ is less than ε, and such that for generic $f \in \mathbf{R}^n \to \mathbf{R}^m$, $n < m$, $f^{-1}(R_{a,\varepsilon}) = \phi$.

Let $\mathbf{B}$ be the Banach space of C^k-maps from $\mathbf{R}^{m-1}$ to $\mathbf{R}^m$, restricted to $I^{m-1}(1) = \{(x_1, \ldots, x_{m-1}) \mid |x_i| \leqslant 1\}$. This Banach space is separable; let $\{f_i\}_{i=1}^{\infty}$ be a dense sequence in $\mathbf{B}$. For each $j \in N$, we take a δ_j-neighbourhood of $f_j(I^{m-1}(1))$ out of $I^m(a)$. We take the δ_j so small that the Lebesgue measure of $I^m(a)$, intersected with this δ_j-neighbourhood, is smaller than $2^{-j} \cdot \varepsilon$. $R_{a,\varepsilon}$ is obtained from $I^m(a)$ by removing all these δ_j-neighbourhoods.

Clearly the Lebesgue measure of $I^m(a) - R_{a,\varepsilon}$ is smaller than ε. $R_{a,\varepsilon}$ is closed by construction so the set of f: $\mathbf{R}^{m-1} \to \mathbf{R}^m$ such that $f(I^{r-1}(1)) \cap R_{a,\varepsilon} = \phi$ is open. By construction, this last set is dense (it contains $\{f_i\}_{i=1}^{\infty}$) Hence there is a subset of first category $N_{m-1,a,\varepsilon,1} \subset C^k(\mathbf{R}^{m-1}, \mathbf{R}^m)$ containing all maps f with $f(I^{m-1}(1)) \cap R_{a,\varepsilon} \neq \phi$. Observe that, for the *same* $R_{a,\varepsilon}$, and any $n < m$, we have that

$$N_{n,a,\varepsilon,1} = \{f \in C^k(\mathbf{R}^n, \mathbf{R}^m) \mid f(I^n(1)) \cap R_{a,\varepsilon} \neq \phi\}$$

is also a set of first category in $C^k(\mathbf{R}^n, \mathbf{R}^m)$. Next we define

$$R = \bigcup_{i=1}^{\infty} R_{i,1/i} \quad \text{and} \quad N_n = \bigcup_{i=1}^{\infty} N_{n,i,1/i,i},$$

where the definition of $N_{n,a,\varepsilon,r}$ is obtained by replacing $I^n(1)$ by $I^n(r)$ in the definition of $N_{n,a,\varepsilon,1}$.

It is now clear that for $n < m$, $f \in C^r(\mathbf{R}^n, \mathbf{R}^m) - N_n$, $f^{-1}(R) = \phi$. It is also clear that N_n is of first category and that R has full measure in $\mathbf{R}^m$. As we saw before, for generic $f \in C^k(\boldsymbol{R}^m, \boldsymbol{R}^m)$, $f^{-1}(R)$ has full measure. This concludes the proof of our claim for $M = \{m\}$ and hence, by the above remarks, of our claim in general.

Note that our proof works both for the strong and for the weak topology on $C^k(\boldsymbol{R}^n, \boldsymbol{R}^m)$.

3. Fubini property

According to the theorem of Fubini, if $N \subset X \times Y$ is a set of measure zero in the product of the measure spaces X and Y, then there is a set $N_X \subset X$ of measure zero such that for every $x \in X - N_X$, $N \cap (\{x\} \times Y)$ is a set of measure zero in $\{x\} \times Y \simeq Y$.

A corresponding property holds for sets of first category (i.e., sets contained in a countable union of nowhere-dense closed sets) instead of set of measure zero, assuming that X and Y are separable topological spaces; this was proved by Kuratowski and Ulam [3].

On $\boldsymbol{R}^n$, or on any finite dimensional differentiable manifold, we have both notions: measure zero (with respect to the Lebesgue measure class) and first category (with respect to the usual topology). In this situation we define a *null set* as a set which is both of first category and has measure zero. We shall now prove that also null-sets have the Fubini property.

PROPOSITION. *Let X and Y be finite dimensional differentiable manifolds and let $N \subset X \times Y$ be a null set in the above sense. Then there is a null set $N_X \subset X$ such that, for any $x \in X - N_X$, $N \cap (\{x\} \times Y)$ is a null set in $\{x\} \times Y \simeq Y$.*

Proof. Since a countable union of null sets is again a null set, we may assume that N is closed (instead of being a countable union of closed sets) and has measure zero (from the fact that N has measure zero it follows that N cannot have interior points; since N is closed this means that N must be nowhere dense). We take a compact set $W \subset Y$ and define

$$X_W = \{x \in X \mid N \cap (\{x\} \times W) \text{ has positive measure}\}.$$

Since Y can be covered by countably many compact subsets, it is enough to show that for any compact W, X_W is a null set (since $N \cap (\{x\} \times W)$ is always closed it is a null set if and only if it has measure zero). From Fubini's theorem we know that X_W has measure zero, so we only need to show that X_W is a set of first category, even only that X_W is a countable union of closed set. We prove that the complement of X_W is a countable intersection of open sets.

We fix a measure m on Y which is of the same measure class as the Lebesgue measure; $U_\varepsilon = \{x \in X \mid m(N \cap (\{x\} \times Y)) < \varepsilon\}$, where we use the "same" measure on $\{x\} \times Y$ as on Y. We claim that $\bigcap_i U = X_W$ and that U_ε contains a neighbourhood of X_W. The first statement follows from the definitions. For $x \in X - X_W$, $N \cap (\{x\} \times W)$ is a compact set of measure zero in $\{x\} \times Y \simeq Y$. Hence, for $\delta > 0$ sufficiently small, the δ-neighbourhood of $N \cap (\{x\} \times W)$ in $(\{x\} \times Y)$ has measure smaller than ε. Since N is closed and W is compact, there is a neighbourhood V of x in X such that, whenever $x' \in V$, $N \cap (\{x'\} \times W)$ is contained in a δ-neighbourhood of $N \cap (\{x\} \times W)$, identifying $\{x\} \times W$ and $\{x'\} \times W$ in the obvious way. This means that $V \subset U_\varepsilon$ and hence that U_ε contains a neighbourhood of $X - X_W$. This means that $X - X_W$ is a countable intersection of open sets and hence that X_W is a null set. As we saw before, this completes the proof of the proposition. □

4. Mapping property

Let X and Y be topological spaces; we assume X to be separable. $C(X, Y)$ denotes a set of continuous (or differentiable) mappings from X to Y; we assume $C(X, Y)$ to have a topology such that for each $x \in X$, the map Ev_x: $C(X, Y) \to Y$, defined by $\mathrm{Ev}_x(f) = f(x)$, is continuous and open.

PROPOSITION. *Let $N \subset Y$ be a set of first category. Then there is a subset $\mathcal{N} \subset C(X, Y)$ of first category such that for any $f \in C(X, Y) - \mathcal{N}$, $f^{-1}(N)$ is a subset of Y first category.*

Proof. Without loss of generality we may assume that N is a closed nowhere-dense set. By assumption there is a countable dense set $\{x_i\}_{i=1}^{\infty}$ in X. Let $\mathcal{N}_i = \{f \in C(X, Y) \mid f(x_i) \in N\}$; from the assumptions on the topology of $C(X, Y)$ it follows that $\mathcal{N}_i$ is a set of first category. Define $\mathcal{N} = \bigcup_i \mathcal{N}_i$. For $f \in C(X, Y) - \mathcal{N}$, $f^{-1}(N)$ is closed (since N is closed) while the complement of $f^{-1}(N)$ contains all points x_i, $i = 1, 2, \ldots$, and hence is dense. This means that $f^{-1}(N)$ is of first category. □

The above proposition means that sets of first category have the mapping property (provided that the above assumptions on the topologies of $C(X, Y)$ and X are satisfied). A corresponding property for sets of measure zero is not known. In fact, even for X, Y finite dimensional vector spaces and $C(X, Y)$ the set of continuous or differentiable mappings from X to Y, one doesn't know how to construct a measure on $C(X, Y)$ for which the mapping property holds.

In order to discuss the mapping property for null sets we first need to extend this notion to spaces like $C(X, Y)$. This will be done in the next section.

Finally we point out that, heuristically, the Fubini property "implies" the mapping property, at least in some cases. For X, Y, $C(X, Y)$ as above and the evaluation map Ev: $X \times C(X, Y) \to Y$, defined by $\mathrm{Ev}(x, f) = f(x)$, is open and continuous, we have: *For each set $N \subset Y$ of first category, $\mathrm{Ev}^{-1}(N)$ is also of first category.* Using the Fubini property we see that there is a subset $N \subset C(X, Y)$ of first category such that for any $f \in C(X, Y) - N$, $(X \times \{f\}) \cap \mathrm{Ev}^{-1}(N)$ is a set of first category in $(X \times \{f\})$ and hence that $f^{-1}(N)$ is a set of first category in X.

The only reason why this last argument does not work for sets of measure zero is that we have no measure on $C(X, Y)$ such that Ev^{-1} transforms sets of measure zero in Y to sets of measure zero in $X \times C(X, Y)$.

5. Generalized null sets

Let X be a space of C^k-vector fields, C^k-diffeomorphisms, or C^k-endomorphisms on a manifold M. X_n denotes the space of n-parameter families of elements in X in the following sense: X_n is the space of C^k-vector fields, C^k-diffeomorphisms, or C^k-endomorphisms on $M \times \mathbf{R}^n$ which are compatible with the projection π: $M \times \mathbf{R}^n \to \mathbf{R}^n$ (in the sense that the composition of the dynamics with π gives the trivial dynamics on $\mathbf{R}^n$). It is clear that the same construction, leading from X to X_n can be carried out when X consists of dynamical systems preserving some extra structure, like Hamiltonian dynamical systems, divergence free dynamical systems etc. However it can also be applied to X_l (l-parameter families of dynamical systems). In that way we obtain $(X_l)_n$. It is clear that $(X_l)_n \simeq X_{l+n}$. We take on all spaces X_n the C^k-Whitney topology for dynamical systems on $M \times \mathbf{R}^n$ (for the following arguments it makes no difference whether we use the weak or the strong topology). It is clear that we can identify the elements of X_n with maps from $\mathbf{R}^n$ into X.

We define a *null set* in X as a subset $N \subset X$ which is of first category and which has the properly that for each $n \geqslant 0$, there is a set of first category $N_n \subset X_n$ such that for $f \in X_n - N_n$, $f^{-1}(N)$ has measure zero.

LEMMA. *For $N \subset X$ and $N_n \subset X_n$ as above, N_n is a null set.*

Proof. Using the identification $(X_n)_{n'} \simeq X_{n+n'}$ we consider $N_{n+n'} \subset (X_n)_{n'}$. By assumption $N_{n+n'}$ is a set of first category. For any $f \in X_{n+n'} - N_{n+n'}$, f: $\mathbf{R}^{n+n'} \to X$ we have that $f^{-1}(N)$ has measure zero. We can also represent f as $\tilde{f}$: $\mathbf{R}^{n'} \to X_n$. We note that $x' \in \mathbf{R}^{n'}$ belongs to $\tilde{f}^{-1}(N_n)$ if and only if $f^{-1}(N) \cap (\mathbf{R}^n \times \{x'\})$ has positive measure in $\mathbf{R}^n \times \{x'\}$. Hence, by Fubini's theorem, for $\tilde{f} \notin (X_n)_{n'} - N_{n+n'}$, $\tilde{f}^{-1}(N_n)$ has measure zero. By assumption N_n is of first category.

In contrast with the examples in Section 2 we have for $N \subset X$ the following.

LEMMA. *Let $N \subset X$ be a set of first category and let $N \subset X_n$ be defined as $N_n = \{f \in X_n \mid f^{-1}(N)$ has positive Lebesgue measure$\}$. If some N_n is of first category and if $0 < n' \leqslant n$, then also $N_{n'}$ is of first category.*

Proof. We first prove that the complement of $N_{n'}$ in $X_{n'}$ is dense. Let $f' \in X_{n'}$. $f \in X_n$ is defined as $f = f' \cdot \pi$ with π: $R^n \to R^{n'}$ a linear projection. Since N_n is a set of first category, there is $\tilde{f}$, arbitrarily near f, with $\tilde{f} \in X_n - N_n$. From this and the Fubini theorem we find an $\tilde{f}'$ near f' with $\tilde{f}'^{-1}(N)$ a set of measure zero. So we only have to show that $U_{n'} = X_{n'} - N_{n'}$ is a countable intersection of closed sets. Without loss of generality we may assume that $N \subset X$ is closed. We obtain neighbourhoods of $U_{n'}$ by taking

$$U_{n',\varepsilon,a} = \{f \in X_{n'} \mid f^{-1}(N) \cap I^{n'}(a) \text{ has measure smaller than } \varepsilon\};$$

$$\bigcap_i U_{n',1/i,i} = U_{n'}$$

(see also the proof of the proposition in Section 3). From the last lemma it follows that, when we take $X = R^n$, the generalized null sets in R^n are just the sets of first category with measure zero. □

It follows from the above results that null sets have the mapping property, at least as far as maps f: $R^n \to X$ in X_n are concerned. The Fubini property does not generalize to the infinite dimensional case for null sets (as far as we know).

6. Generic properties

If X is a topological space, one says that a property P is *generic* for the points (or elements) of X if the set of points, which does *not* have property P, is a set of first category. There are many properties known to be generic for the case that X is a space of mappings or a space of diffeomorphisms (or another space of dynamical systems). The proofs, that these properties are generic, are always based either on transversality or on semi-continuity. For example, the Kupka–Smale theorem ([2], [8]) and its generalizations are based on transversality; on the other hand, the closing lemma ([6], [7]) is based on semi-continuity.

The transversality theorem ([10]) is usually stated as: *Let X be the space of C^1-maps from R^n to R^m, let $S \subset R^m$ be a smooth submanifold, then, for generic $f \in X$, f is transverse with respect to S.* In fact the proof gives a stronger statement: for generic elements $f \in X_{m'}$ (or $\tilde{f}$: $R^{m'} \to X$), $m' \geqslant m$ there is a null set $N_{n'} \subset R^{m'}$ such that for each $q \in R^{m'} - N_{m'}$, $\tilde{f}(q)$ is transverse with respect to S. This implies that the set of elements of $X = C^1(R^n, R^m)$, which are not transverse with respect to S, is a *null set* not only a set of first category. (The fact that we had to restrict to $m' \geqslant m$ is no problem, see the second lemma in Section 5).

The arguments based on semi-continuity on the other hand do not provide null sets as exceptional sets: to prove however that these exceptional sets are no null sets is very hard, it has only been achieved with the C^r-closing lemma on the circle ([5], [1]).

From the above it should be clear that the type of paradoxes, mentioned in the introduction, are only possible with generic properties, for which the exceptional set is not a null set. For properties which hold for all elements not belonging to some null set (like transversality and the conclusions of the Kupka–Smale theorem), I propose to use the term "properties which hold *almost always*". This in analogy with the terminology in probability theory, and because of the fact that this notion is compatible with the measure theoretic notions.

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