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APPROXIMATION BY SOLUTIONS OF GENERAL BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS OF ARBITRARY ORDER

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1. Introduction

The following report is a historical survey, including the newest results, about approximation by solutions of general boundary value problems for elliptic equations.

In 1960 H. Beckert [2] proved the following result. Let L be an elliptic differential operator of the second order with sufficiently smooth coefficients, $\Omega \subset \mathbf{R}^n$ a bounded domain with a smooth boundary $\partial\Omega$, $\Gamma \subset \Omega$ an $(n-1)$-dimensional smooth surface such that $\Omega \setminus \Gamma$ is connected, V an open subset of the boundary $\partial\Omega$ and

$$L_V(\Omega) = \{u \in C^2(\Omega) \cap C(\bar{\Omega}):\ Lu = 0 \text{ in } \Omega,\ u|_{\partial\Omega \setminus V} = 0\}.$$

Further it is supposed that the homogeneous Dirichlet problem for the equation $Lu = 0$ has only the trivial solution (condition (U)). Then any $g_0 \in W_2^1(\Gamma)$ and $g_1 \in L_2(\Gamma)$ can be simultaneously approximated by the restriction to Γ of a function from $L_V(\Omega)$ resp. of its normal derivative; i.e. for a given $\varepsilon > 0$ we can find $u \in L_V(\Omega)$ such that

$$\|g_0 - u|_\Gamma\|_{W_2^1(\Gamma)} + \|g_1 - D_n u|_\Gamma\|_{L_2(\Gamma)} < \varepsilon.$$

Moreover, it is proved that for the case of the biharmonic operator $L = \Delta\Delta$, any $g_0 \in W_2^3(\Gamma)$ can be approximated in an analogous manner. In other words, by variation of the boundary values on a fixed (possibly small) part V of the boundary one can generate a dense set of solutions in the corresponding spaces.

A. Göpfert [7], [8] generalized this result in 1966 resp. 1971 for the case where Γ is a closed $(n-1)$-dimensional surface (i.e. $\Omega \setminus \Gamma$ is not connected), for the case where the condition (U) does not hold, for the second and third boundary value problems, for special elliptic systems and for parabolic equations of the second order.

If the boundary $\partial\Omega$ is regular in the sense of N. Wiener and $\Omega\backslash\Gamma$ is connected, for the case $L = \Delta$ G. Anger [1] proved in 1971 the density of $L_V(\Gamma) := L_V(\Omega)|_\Gamma$ in $C(\Gamma)$. Finally, in 1981 G. Wanka [13] proved the density in $C(\Gamma)$ for a general elliptic operator of the second order with sufficiently smooth coefficients – also for the case of a closed surface Γ and without condition (U).

In the following we shall give very general and almost final results of the described kind established since 1983 for the most general situation of elliptic boundary value problems of higher order equations.

Theorems of another type for approximation of solutions of elliptic equations of arbitrary order are given by F. E. Browder [5], [6].

2. Approximation in Sobolev spaces

First we formulate our assumptions.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.

(1) Let

$$L = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

($m > 0$ an integer, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $D_i = \partial/\partial x_i$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$) be a properly elliptic differential operator with coefficients in $C^\infty(\mathbf{R}^n)$, i.e. the polynomial

$$L^0(x, \xi + \tau\eta) = \sum_{|\alpha| = 2m} a_\alpha(x)(\xi + \tau\eta)^\alpha$$

with respect to τ, which corresponds to the principal part

$$L^0 = \sum_{|\alpha| = 2m} a_\alpha(x) D^\alpha$$

of the differential operator, for any pair $\xi, \eta \in \mathbf{R}^n$, $\xi \neq 0$, $\eta \neq 0$ of linearly independent vectors and $x \in \mathbf{R}^n$ has exactly m roots with positive imaginary part.

(2) For the adjoint operator

$$L^* u = \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha \left(\overline{a_\alpha(x)}\, u\right)$$

we assume "uniqueness in the small". This means that if u is a solution of $L^* u = 0$ in a connected open set Ω vanishing on a nonempty open subset $\Omega' \subset \Omega$, then u must be identically zero in Ω. This condition is fulfilled for instance if the coefficients of L are analytic.

(3) On the boundary $\partial\Omega$ we consider a normal system of boundary

operators $B = (B_1, \ldots, B_m)$ with infinitely differentiable coefficients and $m_j = \operatorname{ord} B_j \leqslant 2m-1$ ($j = 1, \ldots, m$; $m_i \neq m_j$ for $i \neq j$). Further we suppose that L is covered on $\partial\Omega$ by the B_j. For the definition of these notions see [12].

(4) Let $\Gamma \subset \Omega$ be a C^∞-manifold of dimension $\leqslant n-1$ in the interior of Ω such that $\Omega \backslash \Gamma$ is connected.

If $V \subset \partial\Omega$ is a fixed open subset of the boundary and $G \subset \Omega \backslash \Gamma$ an open set, we write

$$L_V(\Omega) := \{u \in C^\infty(\bar{\Omega}): \; Lu = 0 \text{ in } \Omega, \; Bu|_{\partial\Omega \backslash V} = 0\},$$

$$L_G(\Omega) := \{u \in C^\infty(\bar{\Omega}): \; \operatorname{supp} Lu \subset G, \; Bu|_{\partial\Omega} = 0\}.$$

Let $L_V(\Gamma)$ resp. $L_G(\Gamma)$ be the space of restrictions of functions from $L_V(\Omega)$ resp. $L_G(\Omega)$ to Γ. The aim of the paper is to give results about the density of $L_V(\Gamma)$ resp. $L_G(\Gamma)$ in suitable function spaces $H(\Gamma)$. It is sufficient to show $\overline{L_G(\Gamma)} = H(\Gamma)$. Namely, if we suppose that this is already proved, we consider a smooth domain $\Omega_1 \supset \Omega$ such that:

(i) $\partial\Omega \backslash V \subset \partial\Omega_1$.

(ii) The coefficients of B can be extended to $\partial\Omega_1 \backslash \partial\Omega$ in such a way that for the new system of boundary operators on $\partial\Omega_1$ condition (3) is satisfied.

We choose an open set $G \subset \Omega_1 \backslash \Omega$ and consider the space $L_G(\Omega_1)$. By our assumption we have $\overline{L_G(\Omega_1)|_\Gamma} = H(\Gamma)$. Since $L_G(\Omega_1)|_\Omega \subset L_V(\Omega)$ we obtain $\overline{L_V(\Gamma)} = H(\Gamma)$.

In [14] the following result is proved. $W_2^{2m-1}(\Gamma)$ is the classical Sobolev space over Γ.

THEOREM 1. *Under conditions* (1)–(4) *the density* $\overline{L_V(\Gamma)} = \overline{L_G(\Gamma)} = W_2^{2m-1}(\Gamma)$ *holds.*

One must show that any continuous linear functional $l \in (W_2^{2m-1}(\Gamma))'$ with $l(u) = 0$ for every $u \in L_G(\Gamma)$ vanishes identically. To show this, in [14] the functions $u \in L_G(\Omega)$ are represented by means of the Green function.

A refined technique founded upon the same idea leads to the following result (see [11] or for a little more general result [10]), where we suppose $\dim \Gamma = n-1$ and $W_p^{2m-1/p}(\Gamma)$ denotes the usual trace space of the Sobolev space $W_p^{2m}(\Omega)$.

THEOREM 2. *Under conditions* (1)–(4), $\overline{L_V(\Gamma)} = \overline{L_G(\Gamma)} = W_p^{2m-1/p}(\Gamma)$ ($1 < p < \infty$).

K. Beyer [4] first showed for the special case of the Dirichlet problem for the Laplace operator that the restriction of the differentiation index s in $W_p^s(\Gamma)$ only depends on the method of proof. Using the abstract properties of the general Green operator of the problem K. Beyer obtained the following result (see [4]).

THEOREM 3. *Let* $L = \Delta$, *let* B *be the Dirichlet boundary condition and*

$\Gamma \subset \Omega$ a closed surface of dimension $n-1$. Then for any integer $s \geqslant 1$ the density $\overline{L_V(\Gamma)} = \overline{L_G(\Gamma)} = W_2^{s-1/2}(\Gamma)$ holds.

This result can be proved in a much more general form (see [9]). $W_p^s(\Gamma)$ ($0 \leqslant s < \infty$, $1 < p < \infty$ real numbers) are the Sobolev–Slobodetskii spaces.

THEOREM 4. *Under conditions (1)–(4), $\overline{L_V(\Gamma)} = \overline{L_G(\Gamma)} = W_p^s(\Gamma)$ for all real numbers $0 \leqslant s < \infty$ and $1 < p < \infty$.*

Now we want to approximate also by normal derivatives. For the sake of brevity we suppose $\dim \Gamma = n-1$. Let $D^k u|_\Gamma$ be the normal derivative of order k on Γ,

$$R_l u = \{u|_\Gamma, Du|_\Gamma, \ldots, D^l u|_\Gamma\},$$

$$R_l L_G(\Omega) = \{R_l u\colon\ u \in L_G(\Omega)\},$$

$$R_l L_V(\Omega) = \{R_l u\colon\ u \in L_V(\Omega)\}.$$

The problem is then the following. Given a vector function

$$f = (f_k)_{k=0}^l \in \prod_{k=0}^{l} W_p^{s_k}(\Gamma)$$

($0 \leqslant s_k < \infty$, s_k real numbers, $1 < p < \infty$). Does there exist a sequence (u_i), $u_i \in L_G(\Omega)$ resp. $\in L_V(\Omega)$, such that

$$\sum_{k=0}^{l} \|D^k u_i|_\Gamma - f_k\|_{W_p^{s_k}(\Gamma)} \to 0$$

as $i \to \infty$? The following theorem (see [9]) shows that the answer is affirmative if $l = 2m-1$.

THEOREM 5. *Assuming (1)–(4) we have*

$$\overline{R_{2m-1} L_G(\Omega)} = \overline{R_{2m-1} L_V(\Omega)} = \prod_{k=0}^{2m-1} W_p^{s_k}(\Gamma)$$

for all real numbers $0 \leqslant s_k < \infty$ and $1 < p < \infty$.

The proof uses continuity properties of the Green operator, imbedding theorems and so on.

The following theorems of the first author from 1984 are not yet published.

THEOREM 6. *Assuming (1)–(4) we have*

$$\overline{R_l L_G(\Omega)} \neq \prod_{k=0}^{l} W_p^{s_k}(\Gamma)$$

for $l \geqslant 2m$ and for all real numbers $0 \leqslant s_k < \infty$ and $1 < p < \infty$.

Now we suppose that the C^∞-manifold Γ of dimension $n-1$ splits up the domain into two parts Ω_i and Ω_a with $\Omega_i \cap \Omega_a = \emptyset$, $\Omega = \Omega_i \cup \Omega_a \cup \Gamma$, $\partial\Omega_i = \Gamma$, $\partial\Omega_a = \partial\Omega \cup \Gamma$, $G \subset \Omega_a$. We put the additional condition:

(5) It is supposed that the Dirichlet problem

$$Lu = 0 \quad \text{in } \Omega_i,$$
$$D^{j-1} u|_\Gamma = 0 \quad (j = 1, \ldots, m)$$

has only the trivial solution.

THEOREM 7. *Under conditions* (1)–(3) *and* (5) *we have*

$$\overline{R_{m-1} L_G(\Omega)} = \overline{R_{m-1} L_V(\Omega)} = \prod_{k=0}^{m-1} W_p^{s_k}(\Gamma)$$

for all real numbers $0 \leqslant s_k < \infty$ *and* $1 < p < \infty$. *The density does not hold for* R_l *with* $l \geqslant m$.

3. Uniform approximation

In order to use methods of potential theory to prove results about uniform approximation we need the following condition:

(6) We suppose that there exists a global locally integrable fundamental solution $\Phi(x, y)$ for the operator L, i.e.

$$\varphi(z) = \int \Phi(x, z) L^* \varphi(x) dx, \qquad \varphi(x) = \int \Phi(x, z) L \varphi(z) dz$$

for every $\varphi \in C_0^\infty(\boldsymbol{R}^n)$.

$W^{m-1}(\Gamma)$ denotes the vector space of Whitney–Taylor fields of order $m-1$ on Γ, i.e. the space of vectors $g = (g_\alpha)_{|\alpha| \leqslant m-1}$ of functions $g_\alpha \in C(\Gamma)$ such that there exists $\varphi \in C^{m-1}(\boldsymbol{R}^n)$ with $D^\alpha \varphi|_\Gamma = g_\alpha$ ($|\alpha| \leqslant m-1$). On account of our smoothness conditions for Γ the space $W^{m-1}(\Gamma)$ is a Banach space with the norm

$$\|g\|_{W^{m-1}(\Gamma)} = \sum_{|\alpha| \leqslant m-1} \sup_{x \in \Gamma} |g_\alpha(x)|.$$

THEOREM 8. *Assuming* (1)–(3), (5), (6) *we have*

$$\overline{L_G(\Gamma)} = \overline{L_V(\Gamma)} = W^{m-1}(\Gamma).$$

For the proof we refer to [15], [16]. In order to show that every continuous linear functional l on $W^{m-1}(\Gamma)$ with $l(u) = 0$ for $u \in L_G(\Gamma)$ vanishes identically, the representation of $u \in L_G(\Omega)$ by means of the Green function and methods of potential theory for elliptic equations of higher order (see [12]) are used. If the boundary value problem

$$Lu = 0 \quad \text{in } \Omega, \qquad Bu|_{\partial\Omega} = 0$$

has a nontrivial solution, then the Green function exists in a generalized sense (see Yu. M. Berezanskii and Ya. A. Roitberg [3], or [17]). From the technical point of view the proof is then much more complicated.

By using the classical results of potential theory for the Laplace operator, the assertion of Theorem 8 can be proved under very weak conditions on Γ (see [18]). We suppose:

(7) The closed surface Γ satisfies the following cone condition. For every $x \in \Gamma$ there is a cone $C_x \subset \Omega_a$ with vertex at x.

THEOREM 9. *Let $L = \Delta$, let B be the Dirichlet boundary condition, and let condition* (7) *be satisfied. Then we have*

$$\overline{L_V(\Gamma)} = \overline{L_G(\Gamma)} = C(\Gamma).$$

By explicitly using linear combinations of the fundamental solution with a special choice of poles, related results about uniform approximation are proved in [12].

We conclude the report with some unpublished results of the first author from 1984 about uniform approximation, with proofs very much related to those of Theorems 5 and 7.

THEOREM 10. *Let conditions* (1)–(3) *and* (5) *be satisfied and let s_k be integers with $0 \leqslant s_k < \infty$. Then*

$$\overline{R_{m-1}\, L_V(\Omega)} = \overline{R_{m-1}\, L_G(\Omega)} = \prod_{k=0}^{m-1} C^{s_k}(\Gamma).$$

THEOREM 11. *Let conditions* (1)–(4) *be satisfied and let s_k be integers with $0 \leqslant s_k < \infty$. Then*

$$\overline{R_{2m-1}\, L_V(\Omega)} = \overline{R_{2m-1}\, L_G(\Omega)} = \prod_{k=0}^{2m-1} C^{s_k}(\Gamma).$$

Theorems 5 and 7 resp. 10 and 11 are also of interest because modified results (with restrictions on s_k) clearly also hold if the smoothness of Γ is limited in a suitable way.

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