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CONTINUOUS LINEAR RIGHT INVERSES FOR DIFFERENTIAL OPERATORS

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The study of continuous linear right inverses for partial differential operators was initiated by a question of L. Schwartz in the early fifties. Nevertheless, satisfactory solutions for this group of problems were obtained in many cases only rather recently.

The general situation is quite simple: We are given a partial differential operator $P(D)$ with constant coefficients (or more generally, a convolution operator) on a locally convex space E of (generalized) functions, and we want to solve the inhomogeneous equation

$$(*)\qquad P(D)f = g, \qquad g \in E,$$

in such a way that the operator

$$(**)\qquad R : E \to E, \qquad g \to f =: R(g),$$

is linear and continuous. That f should be a solution of $(*)$ just means that

$$(P(D) \circ R)(g) = g \qquad \text{for any } g \in E,$$

that is, R is a continuous linear right inverse for $P(D)$ in E.

The existence of a continuous linear right inverse for $P(D)$ is interesting for several reasons: One aim is to obtain simple and explicit solution formulas for the inhomogeneous equation $(*)$, generalizing e.g. the convolution with an elementary solution of $P(D)$. Also, one often obtains additional information on the structure of the kernel of $P(D)$ such as expansions with respect to a basis or the existence of a topological isomorphism to a well known sequence space (e.g. a power series space). Thirdly, we can solve the vector valued version of the equation $(*)$, i.e. if $g = g_\lambda$ depends in a certain way (e.g. holomorphically) on a parameter λ, we can

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solve $(*)$ by $f_\lambda := R(g_\lambda)$ with the same dependence on λ. Finally, continuous linear right inverses are important for nonlinear problems connected with the implicit function theorem of Nash and Moser, since the application of this theorem relies on the existence of right inverses for the linearized equation which satisfy rather specific continuity estimates (so called tame estimates).

It should be remarked that R in $(**)$ can always be chosen linear (by means of a Hamel basis in E). It is also well known that R can be chosen continuous (but nonlinear) by a result of E. Michael [Mi].

Generally speaking, there are four groups of problems which have been studied so far:

1. E is the spaces of C^∞-functions or distributions (respectively, of ultradifferentiable functions and ultradistributions) on an open set $\Omega \subset \mathbb{R}^N$ (Trèves [Tr2], Cohoon [C1, C2], Vogt [V1, V2], Meise, Vogt [MV], Meise, Taylor, Vogt [MTV1–MTV8], Braun, Meise, Vogt [BMV], Palamodov [Pa2]).
2. E is a weighted space of (generalized) functions (Langenbruch [L1–L5]).
3. E is a space of holomorphic or real analytic functions (Trèves [Tr1], Schwerdtfeger [S], Taylor [T], Meise [M1, M2], Meise, Taylor [MT1–MT3], Meise, Momm, Taylor [MMT], Momm [Mo1–Mo6], Langenbruch, Momm [LM], Korobeĭnik, Melikhov [KM], Langenbruch [L6]).
4. $P(D)$ is the Cauchy–Riemann system (Palamodov [Pa1], Taylor [T], Meise, Taylor [MT2], Momm [Mo1, Mo2, Mo7], Langenbruch [L5, L7–L9]).

Of course, these four parts of the right inverse problem are not separated strictly, and the quoted papers are not restricted to the group of problems where they appear. In particular, many of the papers in group 3 treat problems in the first two groups as well.

The results as well as the methods needed in the proofs of the right inverse problem are rather manifold and the needed tools contain a considerable part of analysis: One needs tools from the theory of partial differential equations (e.g. the Ehrenpreis principle (Ehrenpreis [E]), construction of fundamental solutions), from functional analysis (locally convex spaces and duality theory, theory of power series spaces and splitting theory (see e.g. Meise, Vogt [MV2]), the Ext functor), from algebraic geometry and algebra (the division theorem of Bierstone and Schwartz [BS], the Malgrange preparation theorem (see e.g. Hörmander [Hö2], Hilbert's Nullstellensatz) and from complex function theory (the solution of the weighted $\overline{\partial}$-problem of Hörmander [Hö1], function theory on algebraic varieties).

In this survey, we will concentrate on parts of the function theoretic problem in group 4 and on its consequences, as studied in Langenbruch [L7–L9]. In doing so we will in general not aim at the best possible answers (for which we refer to the literature), but rather present only the main ideas of the proofs.

The paper is divided into three sections: In the first section, the splitting of the $\overline{\partial}$-complex on weighted spaces of locally L_2-functions is characterized and the consequences for the right inverses of partial differential operators are studied. The splitting condition is evaluated in section 2, and some applications of the splitting of the $\overline{\partial}$-complex are discussed in section 3.

1. Splitting of the $\overline{\partial}$-complex. In this section we will study the existence of continuous linear right inverses for the Cauchy–Riemann operator (or in several variables, the splitting of the $\overline{\partial}$-complex). To motivate this study and the general assumptions which are needed, we first consider the consequences of this question for partial differential operators. Other applications (in function theory and in the theory of ultradifferentiable functions) are given in the third section. When function theory is applied to solve partial differential equations (or convolution equations), the common *ansatz* is the following:

We are given a space E of (generalized) functions, which is the dual space of some (Fréchet or dual of a Fréchet) space F. On F we have defined the Fourier transformation

$$\mathfrak{F}(f)(z) := \widehat{f}(z) = \langle_x f, e^{-i\langle x,z\rangle}\rangle \qquad \text{for } f \in F \text{ and } z \in \mathbb{C}^N,$$

where $\langle x, z\rangle = \sum x_k z_k$. We usually have the following fact:

Paley–Wiener type theorem. *$\mathfrak{F} : F \to H_F$ is a topological isomorphism.*

Here H_F is a weighted space of entire functions defined by an increasing system

$$\mathfrak{W} := \{W_n \mid n \geq 1\}$$

of weight functions in the following way:

$$H_F = H(\mathfrak{W}) := \Big\{ f \in H(\mathbb{C}^N) \mid \exists n \geq 1 : \\ \|f\|_n := \Big(\int |f(z)|^2 \exp(-2W_n(z))\, dz \Big)^{1/2} < \infty \Big\}.$$

Here and in the following we concentrate on the case where F is the dual of a Fréchet Schwartz space. The case of a Fréchet Schwartz space needs only minor modifications (e.g. *decreasing* weight systems). Let the corresponding space of locally square integrable functions be defined by

$$L(\mathfrak{W}) := \{ f \in L^2_{\mathrm{loc}}(\mathfrak{W}) \mid \exists n \geq 1 : \|f\|_n < \infty\}$$

and let the q-forms in $d\overline{z}$ with coefficients in $L(\mathfrak{W})$ be denoted by $L_q(\mathfrak{W})$. Then the natural domain of definition for the $\overline{\partial}$-operator acting on the forms is

$$\widetilde{L}_q(\mathfrak{W}) = \{ f \in L_q(\mathfrak{W}) \mid \overline{\partial}(f) \in L_{q+1}(\mathfrak{W})\}.$$

We now need the following technical assumptions:

$\exists A > 0\ \forall n \geq 1\ \exists k \geq 1,\ B :$

$$\text{(1.1)} \qquad \sup\{W_n(z+\xi) \mid \|\xi\| \leq (1+\|z\|)^{-A}\} \\ \leq \inf\{W_k(z+\xi) \mid \|\xi\| \leq (1+\|z\|)^{-A}\} + B,$$

$$\text{(1.2)} \qquad W_n(z) + \ln(1+|z|) \leq W_k(z) + B.$$

Then we have:

1.1. Proposition. *Let $\mathfrak{W}$ satisfy* (1.1) *and* (1.2) *and let $\mathfrak{W}$ correspond to F* ($= E'_b$) *as above. If*

$$\overline{\partial} : \widetilde{L}_0(\mathfrak{W}) \to \widetilde{L}_1(\mathfrak{W})$$

has a continuous linear right inverse R (*on its range space*), *then any partial differential operator $P(D)$ with constant coefficients has a continuous linear right inverse in E.*

Sketch of proof (For details in a special case see Langenbruch [L5, Proposition 3.1]). Since $\mathfrak{F}$ maps $P(D)$ into the multiplication operator with $P(z)$ on $H_F = H(\mathfrak{W})$, we have by dualization and Fourier transformation: $P(D)$ has a continuous linear right inverse in E if and only if the multiplication operator with $P(-z)$ has a continuous linear *left* inverse in $H(\mathfrak{W})$.

For simplicity, we assume that x_1 is noncharacteristic. Then $P(-z)$ locally has a suitable estimate from below (see Hörmander [Hö2, pp. 293–294]) such that a local linear continuous left inverse L_a for $P(-z)$ can easily be defined by the integral formula

$$L_a(f)(z) := \frac{1}{2\pi i} \int\limits_{|a_1 - s| = \gamma} \frac{f(s, z_2, \ldots, z_N)}{P(-s, -z_2, \ldots, -z_N)(s - z_1)}\, ds.$$

Let $B_\delta(\xi) := \{z \in \mathbb{C}^N \mid |z - \xi| < \delta\}$. Then it can be proved that

$$L_a : H(B_\varepsilon(a)) \to H(B_\delta(a))$$

is linear and continuous, where

$$\varepsilon = \varepsilon(a) \le (1 + \|a\|)^{-A} \quad \text{for large } |a| \tag{1.3}$$

and

$$\delta = \delta(a) \ge (1 + |a|)^{-B} \quad \text{for some } B \text{ and large } |a|. \tag{1.4}$$

We can now choose a sequence a_n and a C^∞-resolution φ_n subordinate to $\{B_\delta(a_n) \mid n \in \mathbb{N}\}$ such that

$$\|\nabla \varphi_n\| = O(1/\delta(a_n)) = O(\|a_n\|^B) \tag{1.5}$$

by (1.4). Then

$$\widetilde{L}(f) := \sum \varphi_n(L_{a_n}(f))$$

is continuous and linear from $H(\mathfrak{W})$ into $\widetilde{L}(\mathfrak{W})$ by (1.1)–(1.3) and (1.5). Also,

$$\widetilde{L}(P(-z)f) \quad \text{for any } f \in H(\mathfrak{W}),$$

since L_a is a local left inverse for $P(-z)$. Therefore, a continuous linear left inverse for $P(-z)$ in $H(\mathfrak{W})$ is defined by

$$L = (\mathrm{Id} - R \circ \overline{\partial}) \circ \widetilde{L},$$

where R is the right inverse of $\overline{\partial}$ from the assumption.

Proposition 1.1 shows that *weighted* spaces of functions are most interesting as far as the splitting of the $\overline{\partial}$-complex is concerned. In fact, there is no hope for a splitting of this complex without growth conditions for the coefficients, since the Cauchy–Riemann system is elliptic, and hence has no continuous linear right inverse in C^∞ $(\mathbb{C}^N)$ by a classical result of Grothendieck (Trèves [Tr2]). Notice, however, that any other $\overline{\partial}$-operator in this complex (with C^∞-coefficients) has a right inverse by Palamodov [Pa1].

The study of the splitting of the $\overline{\partial}$-complex for weighted spaces of (generalized) functions was initiated by Taylor [T] in his research on continuous linear extension operators for holomorphic functions on analytic subvarieties. He used the theory of analytically uniform spaces and the Ehrenpreis principle [E] to solve the problem for $\mathfrak{W} = \{n|z|^a \mid n \geq 1\}$, $a \geq 1$. Then Meise, Taylor [MT2] applied the splitting theory of Vogt [MV2] to solve the case of radial Hörmander algebras, i.e. $\mathfrak{W} = \{nW(|z|) \mid n \geq 1\}$. Radial weight systems (also on Reinhardt domains) were studied by Momm [Mo1, Mo2, Mo7] using a generalized Taylor series expansion. The results of Momm are in this special case close to the solution of the splitting problem for the $\overline{\partial}$-complex, which I will discuss now. This solution is based on Hörmander's $\overline{\partial}$-techniques [Hö1].

Generalizing (1.1) and (1.2) we use the following notation: Let $\Omega \subset \mathbb{C}^N$ be pseudoconvex. An increasing system $\mathfrak{W} = \{W_n \mid n \geq 1\}$ of measurable functions on Ω is called an *increasing weight system* (IWS) if there is a function $r : \Omega \to \mathbb{R}_+$ such that $r(z) < \operatorname{dist}(z, \partial\Omega)$ and such that for any $n \geq 1$ there are $I(n)$ and $A(n)$ such that for any n:

$$\sup\{W_n(z+\xi) \mid \|\xi\| \leq r(z)\} \leq \inf\{W_{I(n)}(z+\xi) \mid \|\xi\| \leq r(z)\} + A(n), \tag{1.6}$$

$$W_n(z) + \ln(1/r(z)) + \ln(1+|z|) \leq W_{I(n)}(z) + A(n). \tag{1.7}$$

We now say that the $\overline{\partial}$-complex for an IWS $\mathfrak{W}$

$$0 \to H(\mathfrak{W}) \to \widetilde{L}_0(\mathfrak{W}) \xrightarrow{\overline{\partial}} \widetilde{L}_1(\mathfrak{W}) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \widetilde{L}_N(\mathfrak{W}) \to 0 \tag{$*$}$$

splits if *any* of the $\overline{\partial}$-operators has a continuous linear right inverse on its range.

The following theorem now relates the splitting of $(*)$ to the existence of certain plurisubharmonic (psh.) functions satisfying specific estimates:

1.2. Theorem (Langenbruch [L7]). *Let $\mathfrak{W}$ be an IWS. Then the following conditions are equivalent*:

(i) *The $\overline{\partial}$-complex for $\mathfrak{W}$ splits.*

(ii) *For any $t \in \Omega$ there are $f_t \in H(\Omega)$ and for any $n \geq 1$ there are $J(n) \geq n$ and $B(n)$ such that*

$$\ln|f_t(z)| - \ln|f_t(t)| \leq W_{J(n)}(z) - W_n(t) + B(n). \tag{1.8}$$

(iii) (1.8) *holds for some psh. functions g_t instead of* $\ln|f_t|$.

Sketch of proof. (a) The necessity of (1.8) is easily seen for $N = 1$ (in the case of several variables we have to use double complex methods, i.e. the Koszul complex): With $\varphi_t \in D(\mathbb{C})$ such that

$$\varphi_t = 1 \quad \text{near } t, \quad \operatorname{supp} \varphi_t \subset \{z \mid |z-t| < r(t)\}, \quad \|\nabla \varphi_t\|_\infty = O(1/r(t)) \tag{1.9}$$

we set

$$f_t(z) = \varphi_t(z) - (z-t) R \circ \overline{\partial}(\varphi_t/(\cdot - t))(z),$$

where R is the right inverse for $\overline{\partial}$ existing by (i). Obviously,

$$f_t(t) = 1 \quad \text{and} \quad \overline{\partial} f_t = \overline{\partial}\varphi_t - (\cdot - t)(\overline{\partial} \circ R)(\overline{\partial}\varphi_t/(\cdot - t)) = 0$$

and (1.8) directly follows from the continuity of R, (1.9), (1.6) and (1.7).

(b) The sufficiency of (iii) is again easier for $N = 1$: On Ω we choose a C^∞-resolution of the identity $\varphi_n = \varphi_{t_n}$ with (1.9) and set

$$R(f) = \sum R_k(\varphi_k f), \tag{1.10}$$

where R_k are "local" right inverses for $\overline{\partial}$ on certain Hilbert spaces, which are defined by means of the psh. functions g_t from (iii) as follows: Let

$$F_k := \Big\{ f \in L^2_{\mathrm{loc}}(\Omega) \,\Big|\, |f|_k := \Big(\int |f(z)|^2 \exp(-2\widetilde{g}_k(z))\, dz \Big)^{1/2} < \infty \Big\}$$

and

$$E_k := \{ f \in L^2_{\mathrm{loc}}(\Omega) \mid \|f\|_k^2 := |f(1 + |\cdot|)^{-1}|_k^2 + |\overline{\partial} f|_k^2 < \infty \},$$

where $\widetilde{g}_k(z) := \sup\{ g_t(z) \mid |t - t_k| \le r(t_k)\}$. Then the solution of the weighted $\overline{\partial}$-problem of Hörmander [Hö1] shows that for $f \in F_k$ the equation $\overline{\partial} g = f$ has a solution $g \in E_k$ such that $\|g\|_k \le 2|f|_k$. Since $\overline{\partial} : E_k \to F_k$ is continuous and E_k is a Hilbert space, there is a linear right inverse

$$R_k : F_k \to E_k$$

for $\overline{\partial}$ with norm at most 2.

From this, the continuity of R in $\widetilde{L}(\mathfrak{W})$ now immediately follows (use (1.8), (1.6) and (1.7)). R is a right inverse for $\overline{\partial}$, since

$$\overline{\partial} R(f) = \sum (\overline{\partial} \circ R_k)(\varphi_k f) = \sum \varphi_k f = f.$$

For several variables, the $\overline{\partial}$ operators are not surjective and the cutting off procedure leads outside the range of $\overline{\partial}$. So formula (1.10) is too simple then. In fact, we have to reason by induction on the decreasing order of the forms using in (1.10) also a projection onto the range of $\overline{\partial}$, which then exists by the induction hypothesis. This also affects the definition of the spaces E_k and F_k (see Langenbruch [L7] for the details).

From the proof of Theorem 1.2 we see that the module of continuity for the right inverse of $\overline{\partial}$ (i.e. the choice of indices in the continuity estimates for $\overline{\partial}$ with respect to the norms in $\widetilde{L}_q(\mathfrak{W})$) is essentially $J(n)$ from (1.8) (modulo finite compositions with $I(n)$ from (1.6) and (1.7); $I(n)$ is usually small compared with $J(n)$). So we can determine how well $H(\mathfrak{W})$ is complemented in $\widetilde{L}(\mathfrak{W})$ and we also obtain precise continuity estimates for the right inverses of the partial differential operators in Proposition 1.1. This has some importance for the Nash–Moser technique, since rather strong continuity estimates (so-called tame estimates) are needed there. It can also be used to obtain right inverses for partial differential operators with variable coefficients by means of a perturbation argument (Langenbruch [L5]).

An appropriate version of Theorem 1.2 also holds for a decreasing weight system, i.e. in the case where the coefficient spaces in the $\overline{\partial}$-complex are Fréchet spaces (Langenbruch [L7]).

2. Examples and evaluation. We first consider radial weight systems, i.e. the case where

$$\Omega = \mathbb{C}^N \quad \text{and} \quad \mathfrak{W} = \{W_n(|z|) \mid n \geq 1\}.$$

We also assume that W_n is logarithmically convex, i.e.

$$Y_n(x) := W_n(\exp(x)) \text{ is convex and increasing} \tag{2.1}$$

and that

$$\lim_{x\to\infty} (Y_n(x)/x) = \infty \text{ for any } n \geq 1. \tag{2.1$'$}$$

In fact, (2.1$'$) is a consequence of (1.8) (Langenbruch [L8]). Without loss of generality, we can also assume (2.1) since (1.8) already implies that we can choose any W_n radial and psh. (Langenbruch [L7, Remark 1.2]), hence logarithmically convex.

As an extension of this observation, it is noticed in Langenbruch [L9] that a slight variant of (1.8) (and also the existence of an equivalent weight system consisting of psh. functions) is equivalent to the *exactness* of the $\overline{\partial}$-complex for $\mathfrak{W}$.

For radial weight systems the evaluation of (1.8) is, in fact, a problem on the convex functions Y_n and their Young conjugates

$$Y_n^*(x) = \sup\{xt - Y(t) \mid t \geq 0\}.$$

Moreover, the plurisubharmonic functions g_t in (1.8) have the simple form $g_t = a_t(\ln|z| - \ln|t|)$ then. We have

2.1. Theorem (Langenbruch [L8]). *Let* $W_n(z) = W_n(|z|)$ *satisfy* (2.1) *and* (2.1$'$). *Then* $\mathfrak{W}$ *satisfies* (1.8) *if and only if there are* $b_t > 0$ *such that for any* $n \geq 1$ *and* $t \geq 0$,

$$b_t t \geq Y^*_{J(n)}(b_t) + Y_n(t) - A(n). \tag{2.2}$$

(2.2) is intensively studied in Langenbruch [L8]. Instead of giving a survey of these results, I will only discuss the special case of entire functions of exponential type here, which was studied already by Taylor [T], Meise, Taylor [MT2] and Momm [Mo2], to show how $J(n)$ can be determined, i.e. how the module of continuity for the right inverse of the corresponding $\overline{\partial}$-operator can be estimated. This was not possible with the methods from loc. cit.

2.2. EXAMPLE. Let $\mathfrak{W} := \{n|z|^a \mid n \geq 1\}$, $a > 0$. Then the $\overline{\partial}$-complex for $\mathfrak{W}$ splits. For any $\varepsilon > 0$ the right inverses for the $\overline{\partial}$-operators can be chosen such that

$$\|R(f)\|_{K(n)} \leq C_n \|g\|_n \quad \text{with} \quad K(n) = e^{\varepsilon n} \tag{2.3}$$

and this choice is best possible.

Proof. We have $Y_n(x) = ne^{ax}$ and $Y_n^*(x) = (x/a)(\ln x - \ln(ean))$ for large x. We set $b_t = c_t a e^{at}$ and get the following estimate, which is equivalent to (2.2):

$$1 + \ln J(n) \geq \ln c_t + (n - A(n)e^{-at})/c_t. \tag{2.4}$$

For $c_t \equiv C$ this is satisfied with $J(n) = (C/e)\exp(n/C)$. On the other hand, from (2.4) with $n = 1$ we see that $c_t \in [1/C, C]$ for some $C > 0$ and large t. Hence, $J(n) \geq \exp(\varepsilon n)$ for some ε, again by (2.4). (2.3) now follows from the remarks after Theorem 1.2, since (1.6) and (1.7) are satisfied with $r \equiv 1$ and $I(n) = n + \delta$, $\delta > 0$. So finite compositions with $I(n)$ do not matter.

When Theorem 2.1 is applied to radial Hörmander algebras (i.e. $\mathfrak{W} = \{nW(|z|) \mid n \geq 1\}$ with $W(z)$ satisfying (2.1) and (2.1′)) the result of Meise, Taylor [MT2] is obtained with control of the continuity estimates.

That the choice of $J(n)$ can change very much by a (seemingly) small change of the weight system, is seen from the following

2.3. EXAMPLE. Let $\mathfrak{W} := \{(1 - 1/n)|z|^a \mid n \geq 1\}$, $a > 0$. Then the statement of Example 2.2 holds for $K(n) = n + k$, $k \in \mathbb{N}$.

Proof. Instead of (2.4) we obtain

$$1 + \ln(1 - 1/J(n)) \geq \ln c_t + (1 - 1/n - A(n)e^{-at})/c_t. \tag{2.5}$$

(2.5) is obviously satisfied for $c_t = 1$ and $J(n) = n + 1$.

The splitting of the $\overline{\partial}$-complex for weight systems like in Example 2.2 and 2.3 has been characterized in Momm [Mo2, Mo7] by means of linear topological invariants of the (DN)-type (see section 3 for the importance of these invariants for the splitting of exact sequences).

Radial weight systems on the unit (poly)disc can also be treated as above (Langenbruch [L8], see Momm [Mo1, Mo2, Mo7, Mo8] for earlier results in this direction).

Many weight systems which are needed in analysis are sums of two-weight systems. One advantage of the splitting condition (1.8) is that it is additive. This means the following: If (1.8) is satisfied with psh. g_t (respectively, G_t) for the IWS $\mathfrak{W}_1$ (and $\mathfrak{W}_2$, respectively), then (1.8) is satisfied for $\mathfrak{W}_1 + \mathfrak{W}_2$ by $g_t + G_t$.

In fact, for $\mathfrak{W}_2$ we can allow in (1.8) errors which are small compared with $\mathfrak{W}_1$. So sums of weight systems can be handled rather easily.

Instead of treating the general situation, we again prefer to discuss a typical example, to show the main arguments more clearly:

Let $\mathfrak{V} = \{V_n(|x|) \mid n \geq 1\}$ be a decreasing system of convex functions on $\mathbb{R}^N$ satisfying

$$V_{I(n)}(|x|) + |x| \leq V_n(|x|) + A(n) \tag{2.6}$$

and

$$V_{I(n)}(|x| + 1) \leq V_n(|x|) + A(n). \tag{2.7}$$

Let $C^\infty(\mathfrak{V})$ be the weighted space of C^∞-functions defined by

$$C^\infty(\mathfrak{V}) := \{f \in C^\infty(\mathbb{R}^N) \mid f^{(a)}(x) = O(\exp(V_n(|x|)) \text{ for any } n \text{ and any } a\}.$$

We then have the following Paley–Wiener theorem:

$$C^\infty(\mathfrak{V})' \simeq H(\mathfrak{W}) = H(\mathfrak{L} + \mathfrak{V}^*),$$

where $\mathfrak{L} = \{n \ln(1 + |z|) \mid n \geq 1\}$ and where $\mathfrak{V}^* := \{V_n^*(|\mathrm{Im}\, z|) \mid n \geq 1\}$ is the system of Young conjugates (see e.g. Langenbruch [L5] for related results). $\mathfrak{W}$ is an IWS, since $\mathfrak{V}^*$ satisfies (1.6) by (2.6) for $r(|z|) = 1$ ((1.7) is trivial). We then have the following

2.4. Example. Let $\mathfrak{L}$ and $\mathfrak{V}^*$ be as above. Then the following conditions are equivalent:

(i) The $\overline{\partial}$-complex splits for $\mathfrak{W} = \mathfrak{L} + \mathfrak{V}^*$.
(ii) The $\overline{\partial}$-complex splits for $\mathfrak{R} := \{V_n^*(\ln_+ |z|) \mid n \geq 1\}$.

Proof. (ii)$\Rightarrow$(i). Since the $\overline{\partial}$-complex splits for $\mathfrak{R}$, we know, by Theorem 2.1, that (2.2) is satisfied for $Y_n := V_n^*$. By the remark before that theorem, (1.8) is satisfied for $\mathfrak{R}$ with $g_t(z) := a_t(\ln(|z|) - \ln(|t|))$. Therefore, the function $G_t(z) := g_{\exp(|\mathrm{Im}\, t|)}(\exp(|\mathrm{Im}\, z|))$ is psh. and satisfies (1.8) for $\mathfrak{V}^*$, i.e.

$$G_t(z) - G_t(t) \leq V_{J(n)}^*(|\mathrm{Im}\, z|) - V_n^*(|\mathrm{Im}\, t|) + A(n). \tag{2.8}$$

Choose an entire function φ such that

$$\varphi(0) = 1 \quad \text{and} \quad |\varphi(z)| \leq \exp(|\mathrm{Im}\, z| - n \ln(1 + |z|) + C_n) \quad \text{for any } n.$$

Then $\psi_t(z) := \ln |\varphi(z - t)|$ satisfies

$$\psi_t(z) = \psi_t(z) - \psi_t(t) \leq |\mathrm{Im}\, z| + |\mathrm{Im}\, t| - n \ln(1 + |t|) + \ln(1 + |z|) + C_n. \tag{2.9}$$

By (2.7), this can be interpreted as (1.8) being valid for $\mathfrak{L}$ with small errors compared with $\mathfrak{V}^*$, i.e. one can take k and j such that

$$V_n^*(|\mathrm{Im}\, t|) + |\mathrm{Im}\, t| \leq V_k^*(|\mathrm{Im}\, t|) + C_n' \quad \text{for any } t \tag{2.10}$$

and

$$v_{J(k)}^*(|\mathrm{Im}\, z|) + |\mathrm{Im}\, z| \leq V_j^*(|\mathrm{Im}\, z|) + C_n' \quad \text{for any } z. \tag{2.11}$$

So by (2.8)–(2.11) we get

$$
\begin{aligned}
G_t(z) &+ \psi_t(z) - G_t(t) - \psi_t(t) \\
&\le V^*_{J(k)}(|\operatorname{Im} z|) + |\operatorname{Im} z| + n\ln(1+|z|) \\
&\quad - V^*_k(|\operatorname{Im} t|) + |\operatorname{Im} t| - n\ln(1+|t|) + C''_n \\
&\le V^*_j(|\operatorname{Im} z|) + j\ln(1+|z|) - V^*_n(|\operatorname{Im} t|) - n\ln(1+|t|) + B_n.
\end{aligned}
$$

This is (1.8) for $\mathfrak{W} = \mathfrak{L} + \mathfrak{V}^*$.

(i)⇒(ii). Let g_t satisfy (1.8) for $\mathfrak{W} = \mathfrak{L} + \mathfrak{V}^*$. To get rid of the logarithmic terms in (1.8), i.e. to suppress the influence of $\mathfrak{L}$, we again use ψ_t as above for $t \in i\mathbb{R}$, interchange z and t in (2.9) and use (2.10) with $2|\operatorname{Im} t|$ instead of $|\operatorname{Im} t|$ to get

$$
\begin{aligned}
g_t(z) &+ \psi_t(z) - g_t(t) - \psi_t(t) \\
&\le V^*_{J(k)}(|\operatorname{Im} z|) + |\operatorname{Im} z| - V^*_k(|\operatorname{Im} t|) + |\operatorname{Im} t| + J(k)\ln(1+|t|) + C''_n \\
&\le V^*_{J(k)}(|\operatorname{Im} z|) - V^*_k(|\operatorname{Im} t|) + 2|\operatorname{Im} t| + B_n \\
&\le V^*_j(|\operatorname{Im} z|) - V^*_n(|\operatorname{Im} t|) + B'_n.
\end{aligned}
$$

Since the right hand side only depends on $|\operatorname{Im} z|$, we can take on the left hand side the supremum over all z with $|\operatorname{Im} z|$ fixed. This gives psh. functions depending only on $|\operatorname{Im} z|$. Hence these functions are convex and we can substitute the variable $|\operatorname{Im} z|$ by $\ln|z|$, obtaining again psh. functions which satisfy (1.8) for $\{V^*_n(\ln_+|z|) \mid n \ge 1\}$.

So the splitting problem for the nonradial weight system $\mathfrak{W} = \mathfrak{L} + \mathfrak{V}^*$ is reduced to the splitting problem for a radial weight system, which can be treated as before.

A systematic study of the argument in the above proof is contained in Langenbruch [L8]. In this way, the results obtained in Langenbruch [L2, L3, L5] for various types of nonradial weight systems can be improved. Nonradial Hörmander algebras were first studied in Meise, Taylor [MT2] by using the linear topological invariant (DN) introduced by Vogt and splitting theory for power series spaces of infinite type (see Meise, Vogt [MV2] and section 3).

3. Consequences. In section 1 we already noticed the connection of the splitting of the $\overline{\partial}$-complex and the general existence of right inverses for partial differential operators with constant coefficients. In this section we will sketch some further applications of the splitting of the $\overline{\partial}$-complex in complex function theory, structure theory of Fréchet spaces and the Whitney problem for (ultra)differentiable functions.

We start with function theory: B. A. Taylor [T] was the first who studied the splitting of the weighted $\overline{\partial}$-complex, and his motivation came from complex interpolation. The problem in its simplest form is the following: Let $\mathfrak{W}$ be an IWS such that $H(\mathfrak{W})$ is an algebra. Let (z_k) be a sequence of distinct complex numbers which are the (simple) zeroes of an entire function $F \in H(\mathfrak{W})$. Let

$$\Lambda(\mathfrak{W}) := \Big\{(c_k) \in \mathbb{C}^{\mathbb{N}} \;\Big|\; \sum_k |c_k| \exp(-W_n(z_k)) < \infty \text{ for some } n\Big\}.$$

Then the restriction mapping

$$\varrho : H(\mathfrak{W}) \to \Lambda(\mathfrak{W})$$

is linear and continuous and the question is to find a continuous linear right inverse

$$E : \Lambda(\mathfrak{W}) \to H(\mathfrak{W}).$$

If we interpret $(c_k) \in \Lambda(\mathfrak{W})$ as a holomorphic function on the zero-dimensional complex variety $V := \{(z_k) \mid k \in \mathbb{N}\}$, this means: E is a continuous linear extension operator for holomorphic functions preserving bounds.

To prove the surjectivity of ϱ, one usually takes the *ansatz*

$$\widetilde{f} = \sum \varphi_k c_k$$

for suitable C^∞-functions φ_k, such that on $\operatorname{supp} \overline{\partial}\varphi_k$ we have some estimate for F from below (which often follows from the fact that $FH(\mathfrak{W})$ is closed in $H(\mathfrak{W})$ (see e.g. Momm [Mo9]) or holds for general $F \in H(\mathfrak{W})$ by a minimum modulus theorem and the special form of $\mathfrak{W}$). To obtain a holomorphic extension f, one has to solve the equation

$$\overline{\partial} v = \sum \overline{\partial}\varphi_k / F \quad \text{ in } L(\mathfrak{W}) \tag{3.1}$$

and set

$$f := \widetilde{f} - Fv.$$

The solution of (3.1) is thus the only (in general) nonlinear part of the construction. So if the $\overline{\partial}$-equation can be solved in a linear and continuous way, an extension operator E as above exists. Moreover, precise continuity estimates can be given for E if they are known for the right inverse of the $\overline{\partial}$-operator.

The extension problem is closely connected to the question of which ideals in $H(\mathfrak{W})$ are complemented. Indeed, the existence of E is equivalent to the fact that the kernel of ϱ (i.e. the ideal $H(\mathfrak{W}, V) = \{f \in H(\mathfrak{W}) \mid f|_V = 0\}$) is complemented in $H(\mathfrak{W})$. Similarly, one sees that any "reasonable" ideal I (e.g. any localized ideal generated by slowly decreasing functions) is complemented if the $\overline{\partial}$-complex splits. A central tool in this topic is the sequence space representation for the quotient $H(\mathfrak{W})/I$ (see Meise [M2], Meise, Taylor [MT2], Meise, Momm, Taylor [MMT]).

The connection of the complementation problem for ideals I in $H(\mathfrak{W})$ to the right inverse problem for partial differential operators is obvious: If I is the principal ideal generated by a polynomial $P(-z)$ (or the Fourier transform of a convolution operator), then the complementation of I just means that $P(D)$ has a continuous linear right inverse in the corresponding space of generalized functions (in the setting of Proposition 1.1).

If the $\overline{\partial}$-complex does not split, then the above problem turns into the question of whether an extension operator exists for an individual variety V, respectively, whether an individual ideal I is complemented. Of course, also the right inverse problems 1)–3) for partial differential operators from the introduction can be formulated in this way.

In Langenbruch, Momm [LM] it is shown that in certain cases the complementability of an individual ideal I is equivalent to the fact that (1.8) holds for any t in the set of common zeroes of I.

An adequate discussion of the manifold and interesting results pertaining to the above problems is beyond the scope of this paper and we refer the interested reader to the literature (Berenstein, Taylor [BT], Taylor [T], Schwerdtfeger [S], Meise [M1, M2], Meise, Taylor [MT1–MT3], Meise, Momm, Taylor [MMT], Momm [Mo1–Mo8], Langenbruch [L7], Langenbruch, Momm [LM]).

A further application in function theory deals with algebras $H(\mathfrak{W})$ and their generation: A result of Hörmander [Hö3] states that under suitable technical assumptions, a set $G_1, \ldots, G_m$ of functions in $H(\mathfrak{W})$ generates $H(\mathfrak{W})$ as an algebra if and only if there is j such that

$$\max\{|G_k(z)| \mid k \le m\} \ge C \exp(-W_j(z)) \qquad \text{for any } z \in \Omega.$$

Now we can show the following theorem (see Langenbruch [L7]): If the $\overline{\partial}$-complex splits for $\mathfrak{W}$, then this generation can be achieved by means of linear continuous operators, i.e. there are continuous linear operators T_j in $H(\mathfrak{W})$ such that

$$f = \sum T_j(f) G_j \qquad \text{for any } f \in H(\mathfrak{W}).$$

We now consider applications in the structure theory of nuclear Fréchet spaces. The importance of structural considerations for problems of right inverses comes from the general splitting theorem of Vogt below. This central result is connected with power series spaces which are defined as follows: For an increasing and unbounded sequence α_n and $r = 0$ (*finite type*) or $r = \infty$ (*infinite type*) let

$$\Lambda_r(\alpha_n) = \Big\{(x_n) \in \mathbb{C}^{\mathbb{N}} \Bigm| \sum_n |x_n| \exp(\varrho\alpha_n) < \infty \text{ for any } \varrho < r\Big\}.$$

Splitting Theorem (Vogt [V3]). *Let E, F, G be (nuclear) Fréchet spaces and let E (respectively, G) be isomorphic to a quotient (respectively, to a subspace) of a power series space of infinite type. Then any exact sequence*

$$0 \to E \to F \xrightarrow{q} G \to 0$$

is split, i.e. q has a continuous linear right inverse.

This splitting theorem and the structure theory of power series spaces is now available in the book of Meise, Vogt [MV2]. By now, there are also variants of this theorem which use power series spaces of finite type and which allow a precise calculation of continuity estimates for the right inverses of q (Poppenberg, Vogt [PV]).

To apply such splitting theorems in concrete analytical situations, it is important to known which spaces of analysis are isomorphic to (subspaces or quotients of) power series spaces. Such sequence space representations have been proved for many spaces used in analysis (Vogt [V5] and the cited literature connected with ultradifferentiable functions). By the results of Vogt [V3] and Vogt, Wagner [VW] this means that one has to prove the linear topological invariants (Ω) for F and (DN) for G. These are defined as follows:

Let $\| \|_n$ be an increasing system of seminorms defining the topology of a (nuclear) Fréchet space E and let

$$\|y\|_n^* := \sup\{|y(x)| \mid \|x\|_n \leq 1\}$$

be the corresponding dual norms in E'.

E has *property* (Ω) if and only if (Meise, Vogt [MV2])

$$\forall p \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall k \in \mathbb{N}\ \exists 0 < \vartheta < 1,\ C > 0:$$
$$\|y\|_q^* \leq C \|y\|_p^{*1-\vartheta} \|y\|_k^{*\vartheta} \quad \text{for any } y \in E'.$$

E has *property* (DN) if and only if (Meise, Vogt [MV2])

$$\exists p \in \mathbb{N}\ \forall k \in \mathbb{N}\ \exists n \in \mathbb{N},\ C > 0: \quad \|x\|_k^2 \leq C \|x\|_p \|x\|_n \quad \text{for any } x \in E.$$

Since we did not use the splitting theorem to prove the splitting of the $\overline{\partial}$-complex (as e.g. in Meise, Taylor [MV2]), we can prove (DN) and (Ω) for $H(\mathfrak{W})$ in the following way:

If (1.8) holds (i.e. the $\overline{\partial}$-complex is split for $\mathfrak{W}$) and if $L(\mathfrak{W})'$ has (DN) or (Ω) then the complemented subspace $H(\mathfrak{W})'$ of $L(\mathfrak{W})'$ also has (DN) or (Ω). Since we have cut-off functions in $L(\mathfrak{W})$, (DN) and (Ω) are easily proved for $L(\mathfrak{W})'$ for weight systems like $\{W(z) + nV(z) \mid n \geq 1\}$. So the nontrivial property (DN) for $H(\mathfrak{W})'$ is reduced to the splitting problem. Again, the most interesting cases are the nonradial weight systems, since for radial weight systems one can often use power series expansion to prove (DN). We only mention two typical examples. The properties $(\underline{\mathrm{DN}})$ and $(\overline{\Omega})$ mentioned below substitute the invariants (DN) and (Ω) in the classification of subspaces and quotients of power series spaces of *finite type* (Vogt [V4]).

3.1. Examples. (a) Let $\mathfrak{W}_1 = \{(1 - 1/n)|\mathrm{Im}\, z|^\alpha + (1 - 1/n)|z|^\beta \mid n \geq 1\}$, $\alpha > 1$, $\beta > 0$. Then $H(\mathfrak{W}_1)'$ has $(\underline{\mathrm{DN}})$ and $(\overline{\Omega})$.

(b) Let $\mathfrak{W}_2 := \{|\mathrm{Im}\, z|^\alpha + n|z|^\beta \mid n \geq 1\}$, $\alpha \geq 1$, $\beta > 0$. Then $H(\mathfrak{W}_1)'$ has (DN) and (Ω).

(c) Let $\mathfrak{W}_3 := \{|\mathrm{Im}\, z| - n|z|^\beta \mid n \geq 1\}$, $1 > \beta > 0$. Then $H(\mathfrak{W}_1)$ has (DN) and (Ω).

Proof. (i) The $\overline{\partial}$-complex for the systems $\{n|z|^\beta \mid n \geq 1\}$ and $\{(1-1/n)|z|^\beta \mid n \geq 1\}$ splits by Examples 2.1 and 2.3. Since $\alpha > 1$ in (a), this also shows that (1.8) is satisfied for $\{(1 - 1/n)|\mathrm{Im}\, z|^\alpha \mid n \geq 1\}$. (1.8) is trivial for $\{|\mathrm{Im}\, z|^\alpha\}$. Since (1.8) is additive, (1.8) holds in cases (a) and (b), and the corresponding $\overline{\partial}$-complexes are split. Since the properties (DN) and (Ω) (respectively, $(\underline{\mathrm{DN}})$ and

$(\overline{\Omega})$) are proved for $L(\mathfrak{W})' = L(-\mathfrak{W})$ as for power series spaces (see e.g. Meise, Vogt [MV2]), the conclusion follows.

(ii) It is shown in Langenbruch [L9] that the $\overline{\partial}$-complex for the *decreasing* system $\mathfrak{W}_3$ splits. So properties (DN) and (Ω) follow for $H(\mathfrak{W})$, since they are satisfied by $L(\mathfrak{W})$.

From Example 3.1(a) and the characterization of power series spaces of finite type (Vogt [V2]) we conclude that $H(\mathfrak{W}_1)$ is isomorphic to the dual of a power series spaces of finite type, in particular, $H(\mathfrak{W}_1)$ has a basis (see Haslinger [Ha] and Langenbruch [L10] for related results).

Examples 3.1(b) and 3.1(c) are related to ultradifferentiable functions:

For $\alpha = 1$ and $\beta < 1$, $H(\mathfrak{W}_2)$ is isomorphic to the dual of the Gevrey class

$$\gamma^{1/\beta}(I) = \{f \in C^\infty(I) \mid \sup\{|f^{(a)}(x)|k^a a!^{-1/\beta} \mid x \in I,\ a \in \mathbb{N}_0\} < \infty \text{ for any } k \geq 1\},$$

where $I = [-1, 1]$. So we conclude that $\gamma^{1/\beta}(I)$ has (DN) and (Ω). Also, $H(\mathfrak{W}_3)$ is isomorphic to $\gamma_0^{1/\beta}(I) := \{f \in \gamma^{1/\beta}(\mathbb{R}) \mid \operatorname{supp}(f) \subset I\}$. So we conclude that $\gamma_0^{1/\beta}(I)$ has (DN) and (Ω).

Notice that the decreasing weight system $\mathfrak{W}_3$ consists of functions which are not psh. So already the exactness of the corresponding $\overline{\partial}$-complex is nontrivial. The exactness of this complex was characterized in Langenbruch [L9] by the existence of so-called optimal cut-off functions which are important for the Whitney problem for arbitrary closed sets (Bruna [B], Bonet, Braun, Meise, Taylor [BBMT], Petzsche [P], Franken [F]).

The method of Examples 3.1(b) and 3.1(c) can be used for the common classes of ultradifferentiable functions (see Braun, Meise, Taylor [BMT], Komatsu [K]) to obtain linear topological invariants of the above type (Langenbruch [L9]). Using different arguments these invariants were obtained before by Meise, Taylor [MT4, MT5, MT7], Petzsche [Pe], Langenbruch [L11, L12].

These results are interesting in view of the Whitney problem, especially for the existence of a linear and continuous extension operator in classes of (ultra)differentiable functions. The Whitney problem has been solved by various methods. A by no means complete list of corresponding papers is contained in the references (Bonet, Meise, Taylor [BoMT], Bonet, Braun, Meise, Taylor [BBMT], Bruna [B], Ehrenpreis [E], Franken [F], Langenbruch [L13], Meise, Taylor [MT5–MT7], Petzsche [Pe]).

In Langenbruch [L9] we observed that the splitting condition (1.8) is connected to the Whitney problem. We finish this paper by showing this relation for the case of the Gevrey classes $\gamma^{1/\beta}$.

In this case, the Whitney problem in its simplest form is the following:

Is the restriction mapping $\varrho : \gamma^{1/\beta}(2I) \to \gamma^{1/\beta}(I)$ surjective for $I = [-1, 1]$?

Meise, Taylor [MT6] used Fourier transformation and Baernstein's lemma to prove that we have to show: $\forall n \geq 1\ \exists k \geq 1,\ C \geq 1\ \forall f \in H(\mathfrak{W}_2)$:

$$\ln|f(z)| \leq 2|\mathrm{Im}\, z| + n|z|^\beta \text{ on } \mathbb{C} \Rightarrow \ln|f(z)| \leq C + |\mathrm{Im}\, z| + k|z|^\beta \text{ on } \mathbb{C}. \tag{3.2}$$

(3.2) is now easily seen: We already noticed that the $\overline{\partial}$-complex for $\mathfrak{W}_3$ splits (see Example 3.1(c)). The splitting condition (1.8) (for the case of decreasing weight systems) shows the existence of $f_t \in H(\mathfrak{W}_3)$, $J(n)$ and $B(n)$ such that $f_t(t) = 1$ and

$$\ln|f_t(z)| \leq |\mathrm{Im}\, z| - |\mathrm{Im}\, t| - n|z|^\beta + J(n)|t|^\beta + B(n). \tag{3.3}$$

We set

$$B := \{f f_t \exp(|\mathrm{Im}\, t| - J(n+1)|t|^\beta) \mid t \in \mathbb{C},\ f \text{ as in the assumption of (3.2)}\}.$$

From (3.2) and (3.3) we get, for any $g \in B$,

$$|g(z)| \leq \exp(3|\mathrm{Im}\, z| - |z|^\beta + B(n)) \qquad \text{on } \mathbb{C}.$$

So the inverse Fourier transform $\mathfrak{F}^{-1}(B)$ is a bounded set of continuous functions with compact support in $3I$. On the other hand, any $f \in H(\mathfrak{W}_2)$ is the Fourier transform of an ultradistribution with support in I, and by (3.3) we conclude that $\mathfrak{F}^{-1}(B)$ consists of ultradistributions with support on $2I$. Since the supports in the sense of ultradistributions and of continuous functions coincide, we conclude that $\mathfrak{F}^{-1}(B)$ is a bounded set of continuous functions with support in $2I$. We therefore have with a suitable constant C, for any function in B,

$$|f(z)|\,|f_t(z)| \exp(|\mathrm{Im}\, t| - J(n+1)|t|^\beta) \leq \exp(2|\mathrm{Im}\, z| + C) \qquad \text{for any } z \text{ and } t.$$

We now set $z = t$ in this inequality and obtain the conclusion of (3.2) from the fact that $f_t(t) = 1$.

So the surjectivity of ϱ immediately follows from the splitting condition for $\mathfrak{W}_3$. The sequence

$$0 \to \gamma_0^{1/\beta}([-2,-1]) \times \gamma_0^{1/\beta}([1,2]) \to \gamma_0^{1/\beta}([-2,2]) \xrightarrow{\varrho} \gamma^{1/\beta}([-1,1]) \to 0$$

is therefore exact. We conclude from Examples 3.1(b) and 3.1(c) (i.e. from the splitting of the corresponding $\overline{\partial}$-complexes) that the last space has (DN) and that the first space has (Ω). So the sequence is split by the above splitting theorem of Vogt and we have proved the existence of a continuous linear extension operator $E : \gamma^{1/\beta}([-1,1]) \to \gamma^{1/\beta}([-2,2])$.

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