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Method of construction of the evasion strategy for differential games with many pursuers

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Introduction

Since 1960 it has been possible to observe great progress in the theory of differential games. There are two basic trends in this field. The aim of the first is the construction of general models of such games and investigations of sufficient conditions of the existence of the value of the game, optimal strategies etc. Typical results of such studies are contained in [10], [16], [17], [27], [32], [33], [38].

The second line of investigations aims at finding some constructive methods, at least for some classes of such games. The question how to determine the value of the game, an optimal or satisfactory strategy etc., is typical for this trend. A number of papers have been devoted to the study of some very special examples of differential games, see e.g. [8], [26], [29], [35], [36], [39]. Especially a great many beautiful examples can be found in Isaacs [14].

Quite general but in some sense constructive methods have been established for the class of so-called pursuit or evasion games. Methods of efficient pursuit (see e.g. [7], [9], [21], [26], [31]) or efficient evasion (see e.g. [6], [15], [22], [23]) have been found.

The subject of the above-mentioned papers is a game governed by a differential equation $z' = f(z, u, v)$, where $f\colon \boldsymbol{R}^n \times U \times V \to \boldsymbol{R}^n$ and U, V are non-empty, compact subsets of $\boldsymbol{R}^k$ and $\boldsymbol{R}^l$ respectively. The aim of player E (evader), who controls parameter u, is to avoid the coincidence of the solution z with a fixed linear subspace M of $\boldsymbol{R}^n$. The aim of the second player P (pursuer) which controls parameter v is opposite.

An effective method of evasion for linear differential games has been given by L. S. Pontrjagin under the assumption $\dim(M) \leqslant n-2$. The essence of this method lies, roughly speaking, in reducing the problem of avoiding M to the problem of avoiding one point contained in the orthogonal complement W of M. The method has also been extended to the non-linear case (see e.g. [6], [15]).

In games with more than one pursuer the set M is the union of an adequate number of linear subspaces. In this situation Pontrjagin's method fails to work directly. For this reason sufficient conditions for the existence of an evasion strategy with many pursuers need some extra assumptions (see e.g. [5], [6], [11], [12]). Many examples of such games have been presented in [4], [28], [29], [35], [36], together with some direct arguments leading straight to the solutions of the above-mentioned examples.

In this paper we propose a constructive method of evasion for differential games with many pursuers. The method can be applied to the above-mentioned examples as well as to many others for which the known theory fails to work. It is worth mentioning that we do not assume that the evader knows at any moment t the decision of the pursuer $v(t)$, which assumption was crucial for the results presented in [11], [12], [15], [20], [22], [23]. Some of our ideas have already been used in [28], [29], [30].

In the first two sections we follow our paper [30] introducing the basic definitions and proving the main lemma. In Section 3 a certain sufficient condition of the existence of the evasion strategy for a differential game with many pursuers is obtained. In Section 4 we deal with the problem of evasion along each trajectory. It is interesting to observe that, very often, if the evader E wins along each trajectory in a separate game against one pursuer P_i, $i = 1, 2, \ldots, n$, then he also wins in the game against all pursuers P_1, $P_2, \ldots, P_n$. Section 5 is devoted to the study of games with several evaders and pursuers. It is assumed that all evaders have to use the same control during the whole game. Other possibilities of using of our construction are briefly mentioned in Section 6, where an evasion game with incomplete information is considered. Incomplete information means that the evader is not able to evaluate exactly the position of the pursuer.

1. Preliminaries

1.1. Notation

For $n \in \boldsymbol{N}$ and $a, b \in \boldsymbol{R}^n$ denote by $\langle a, b\rangle$ the Euclidean scalar product of the vectors a, b and put $\|a\| = \sqrt{\langle a, a\rangle}$. Define $\operatorname{comp}(\boldsymbol{R}^n)$ to be the family of all non-empty and compact subsets of $\boldsymbol{R}^n$.

For $A, B \subset \boldsymbol{R}^n$, $\hat{a} \in \boldsymbol{R}^n$, $r \geqslant 0$, introduce the following notation:

2^A denotes the family of all subsets of A,

$\operatorname{conv} A$ denotes the closed, convex hull of the set A,

$\operatorname{Int} A$ denotes the interior of A,

$B^n(\hat{a}, r) = \{a \in \boldsymbol{R}^n \colon \|a - \hat{a}\| \leqslant r\}$,

$A + B = \{a + b \colon a \in A,\ b \in B\}$.

If z is an arbitrary function, then Dz will denote the domain of that function. For $t \in [0, \infty)$, $t^* \in (t, \infty)$ and for two arbitrary functions x, y defined at least on $[t, t^*)$ and $[t^*, \infty)$, respectively, denote by $x\langle t^*\rangle y$ a function z, with $Dz = [t, \infty)$, such that $z(s) = x(s)$, $s \in [t, t^*)$ and $z(s) = y(s)$, $s \in [t^*, \infty)$.

We will consider measurable functions defined only on the intervals $[t, \infty)$, where $t \geqslant 0$.

Finally, denote by μ the Lebesgue measure on $\boldsymbol{R}$.

1.2. Control systems. Strategies

Let us fix two natural numbers m, $\hat{m}$ and a set $W \in \operatorname{comp}(\boldsymbol{R}^{\hat{m}})$. Let W_t denote the set of all measurable functions $w\colon [t, \infty) \to W$ for $t \geqslant 0$.

Suppose that for any $t \geqslant 0$, $a \in \boldsymbol{R}^m$ and $w \in W_t$ the symbol $Z(t, a, w)$ denotes a continuous function from $[t, \infty)$ into $\boldsymbol{R}^m$ satisfying the conditions:

$$Z(t, a, w)(t) = a, \qquad Z(t, a, w)(t^*) = Z(0, a, \tilde{w})(t^* - t),$$
$$\text{where } \tilde{w}(s) = w(t+s),\ s \geqslant 0,$$
$$Z(t, a, w\langle t^*\rangle w^*) = Z(t, a, w)\langle t^*\rangle Z\big(t^*, Z(t, a, w)(t^*), w^*\big)$$
$$\text{for any } t^* \in (t, \infty) \text{ and } w^* \in W_{t^*}.$$

DEFINITION 1.1. The function Z is a *control system over* $(\boldsymbol{R}^m, W)$ if for any $(t, a) \in [0, \infty) \times \boldsymbol{R}^m$ and any $T > 0$ functions from the set $Z(t, a, W_t)$ are equicontinuous on the interval $[t, t+T]$.

For each $t \geqslant 0$ define Ξ_t to be the set of all closed and well-ordered sets $A \subset [t, \infty)$ such that $\min A = t$ and $\sup A = \infty$. For $A \in \Xi_t$ and $a \in A$ define $a_A = \min\{s \in A\colon a < s\}$.

Throughout this paper the symbol X (sometimes with indices) will stand for a control system of the player E and the symbol Y (also sometimes with indices) for a control system of the player P.

Now we are going to define the evader's and the pursuer's strategies (see also Definitions 1.1 and 1.2 of [29]).

Suppose X is a control system over $(\boldsymbol{R}^k, U)$, $U \in \operatorname{comp}(\boldsymbol{R}^{\hat{k}})$, Y is a control system over $(\boldsymbol{R}^l, V)$, $V \in \operatorname{comp}(\boldsymbol{R}^{\hat{l}})$ and $(t_0, a_0, b_0) \in [0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$ is a fixed situation.

DEFINITION 1.2. A pair $(e, \#e)$ is a *strategy of the player* E if $e\colon Y(t_0, b_0, V_{t_0}) \to U_{t_0}$, $\#e\colon Y(t_0, b_0, V_{t_0}) \to \Xi_{t_0}$ and the following condition is satisfied:

(ES) if $y, \tilde{y} \in Y(t_0, b_0, V_{t_0})$, $y|_{[t_0, \xi]} = \tilde{y}|_{[t_0, \xi]}$ and $\xi \in \#e(y)$, then also $\xi \in \#e(\tilde{y})$ and $e(y)|_{[t_0, \xi_{\#e(y)})} = e(\tilde{y})|_{[t_0, \xi_{\#e(\tilde{y})})}$.

The set of all such strategies will be denoted by $E(X, Y, a_0, b_0, t_0)$.

Remark 1.1. In many papers there are a large number of different definitions of the strategy for the evader. The above one has some common ideas with that from [24]. I had tried to use some other, more "constructive" strategies before I decided to apply the above one. However, my attempts led to various technical difficulties. Moreover, the strategies used in this paper can also be applied by the pursuer in some special games, in which he has to avoid capture by some additional players $Q_1, Q_2, \ldots, Q_n$, see Example 2 of [28].

DEFINITION 1.3. A function $p: U_{t_0} \to V_{t_0}$ is a *strategy of player P* when the following statement holds:

(SP) for any $t \geq t_0$ and u, $\tilde{u} \in U_{t_0}$, if $u(s) = \tilde{u}(s)$ almost everywhere on $[t_0, t]$, then also $p(u)(s) = p(\tilde{u})(s)$ almost everywhere on $[t_0, t]$.

The set of all such strategies will be denoted by $P(U, V, t_0)$.

Note that functions satisfying condition (SP) are so-called non-anticipating functions, see [38].

Remark 1.2. Employing analogous arguments to those used in the proof of Theorem 1 of [24], one can state that the set of all outcomes resulting from arbitrary two strategies $(e, \#e) \in E(X, Y, a_0, b_0, t_0)$ and $p \in P(U, V, t_0)$ is non-empty and consists only of a single pair of trajectories.

Now we turn to the definition of a particular case of the so-called ε-strategy, see [24].

Suppose that functions q and δ are defined on $[t_0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$ and satisfy the conditions

$$q(t, a, b) \in U_t \quad \text{and} \quad \delta(t, a, b) \in (0, \infty),$$

for any $(t, a, b) \in [t_0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$.

PROPOSITION 1.1. *There exists a unique strategy* $(e, \#e) \in E(X, Y, a_0, b_0, t_0)$ *such that for any* $y \in Y(t_0, b_0, V_{t_0})$ *and* $\xi \in \#e(y)$ *we have*

$$\xi_{\#e(y)} = \xi + \delta(\xi, x(\xi), y(\xi)),$$

$$x(t) = X\big(\xi, x(\xi), q(\xi, x(\xi), y(\xi))\big)(t), \qquad t \in [\xi, \xi_{\#e(y)}],$$

where $x = X(t_0, a_0, e(y))$.

The proof of this theorem is analogous to the proof of Theorem 1 of [24].

DEFINITION 1.4. The function e from Proposition 1.1 will be denoted by $[q, \delta, a_0, b_0, t_0]$ and we will say that the set $\#e(y)$ is *determined* by $[q, \delta, a_0, b_0, t_0]$ and y for $y \in Y(t_0, b_0, V_{t_0})$.

Further, to make the notation simpler, we will write $[q, \delta, a_0, b_0, t_0] \in E(X, Y, a_0, b_0, t_0)$.

Finally, let us introduce final definitions, see also [11]. Call $\varphi: \boldsymbol{R}^k \times \boldsymbol{R}^l \to [0, \infty)$ a payoff functional if it is uniformly continuous. Next, denote by w_φ a non-decreasing modulus of continuity of this functional, that is,

$$|\varphi(a, b) - \varphi(\tilde{a}, \tilde{b})| \leq w_\varphi(\|a - \tilde{a}\| + \|b - \tilde{b}\|), \qquad a, \tilde{a} \in \boldsymbol{R}^k,\ b, \tilde{b} \in \boldsymbol{R}^l.$$

DEFINITION 1.5. A strategy $(e, \#e) \in E(X, Y, a_0, b_0, t_0)$ *guarantees the result* d in the game $(X, Y, a_0, b_0, t_0; \varphi)$ on the interval $[t_0, t_0+T]$ if

$$\varphi\big(X(t_0, a_0, e(y))(t), y(t)\big) \geqslant d$$

for $y \in Y(t_0, b_0, V_{t_0})$ and $t \in [t_0, t_0+T]$.

DEFINITION 1.6. The player E *wins in the game* $(X, Y, a_0, b_0, t_0; \varphi)$ on the interval $[t_0, t_0+T]$ if there exists a strategy $(e, \#e) \in E(X, Y, a_0, b_0, t_0)$ which guarantees a result $d > 0$ in this game on the interval $[t_0, t_0+T]$.

DEFINITION 1.7. The player E *wins locally in the game* (X, Y, φ) if for any $(t, a, b) \in [0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$ such that $\varphi(a, b) > 0$ and any $T > 0$ he wins in the game $(X, Y, a, b, t; \varphi)$ on the interval $[t, t+T]$.

Remark 1.3. It is easy to see that, if E wins locally in the game (X, Y, φ), then for any $(t, a, b) \in [0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$ such that $\varphi(a, b) > 0$ there exists a strategy $(e, \#e) \in E(X, Y, a, b, t)$ satisfying the condition

$$\varphi\big(X(t, a, e(y))(s), y(s)\big) > 0,$$

for all $y \in Y(t, b, V_t)$ and $s \in [t, \infty)$.

Thus, in this case, one can say that E wins in the game (X, Y, φ).

Remark 1.4. One can say that the above strategy is a winning strategy of the evader. Conversely, one can also say that $p \in P(U, V, t)$ is a winning strategy of the pursuer in the situation (t, a, b) if for any $u \in U_t$ there is an $s \in [t, \infty)$ such that $\varphi\big(X(t, a, u)(s), Y(t, b, p(u))(s)\big) = 0$.

From Remark 1.1 we conclude that there is no situation (t, a, b) in which both the evader and the pursuer would simultaneously have winning strategies.

2. Main lemma

In this section we will prove an auxilliary lemma. Using this lemma in the next sections, we will be able to reduce the investigations of very difficult evasion games with many pursuers to the study of a separate game against a single pursuer.

DEFINITION 2.1. Suppose $(e, \#e) \in E(X, Y, a_0, b_0, t_0)$ and $T > 0$. A function $M\colon [t_0, t_0+T) \times Y(t_0, b_0, V_{t_0}) \to 2^{\boldsymbol{R}^k \times \boldsymbol{R}^l}$ is *connected with* $(e, \#e)$ on the interval $[t_0, t_0+T]$ if

$$M(t, y) = M(t, \tilde{y}) = M(\xi, y)$$

for all $y, \tilde{y} \in Y(t_0, b_0, V_{t_0})$, $\xi \in \#e(y) \cap [t_0, t_0+T)$ and $t \in [\xi, \xi_{\#e(y)})$, whenever $y|[t_0, \xi] = \tilde{y}|_{[t_0, \xi]}$.

For each $t \geqslant t_0$ define $X[t_0, a_0; t]$ to be the set $\{a \in \boldsymbol{R}^k$: there is an

$x \in X(t_0, a_0, U_{t_0})$ such that $x(t) = a\}$ and define $Y[t_0, b_0; t]$ in a similar way.

Now suppose $(e_0, \# e_0) \in E(X, Y, a_0, b_0, t_0)$, $t_0 + T \in \# e_0(y)$ and

$$\xi_{\# e_0(y)} - \xi \geqslant c > 0, \qquad \xi \in \# e_0(y) \cap [t_0, t_0 + T], \qquad y \in Y(t_0, b_0, V_{t_0}),$$

φ: $\boldsymbol{R}^k \times \boldsymbol{R}^l \to [0, \infty)$ is a payoff functional, M is connected with $(e_0, \# e_0)$ on the interval $[t_0, t_0 + T]$ and η, H are positive constants.

Let us introduce the following hypotheses:

(H1) There is a $\zeta > 0$ satisfying the implication: if $u, \tilde{u} \in U_{t_0}$ and $\mu(\{t \in [t_0, t_0 + T]: u(t) \neq \tilde{u}(t)\}) \leqslant \zeta$, then $\|X(t_0, a_0, u)(t) - X(t_0, a_0, u)(t)\| \leqslant \eta$ for $t \in [t, t_0 + T]$.

(H2) There are a $\hat{\varphi} > 0$ and a non-decreasing function $\hat{\sigma}$: $[0, \infty) \to [0, \infty)$ with the following property:

if $y \in Y(t_0, b_0, V_{t_0})$, $\xi \in \# e_0(y) \cap [t_0, t_0 + T)$, $t \in [\xi, \xi_{\# e_0(y)})$, $a \in X[t_0, a_0; t]$, $b \in Y[t_0, b_0; t]$, $\|a - X(t_0, a_0, e_0(y))(t)\| \leqslant \eta$, $(a, b) \in M(t, y)$ and $\varphi(a, b) < \hat{\varphi}$, then

$$\varphi(X(t, a, e_0(y))(s), \tilde{y}(s)) \geqslant \hat{\sigma}(s - t),$$

for all $\tilde{y} \in Y(t, b, V_t)$ and $s \in [t, t + H] \cap [\xi, \xi_{\# e_0(y)}]$.

(H3) There exists a non-decreasing function σ^*: $[0, \infty) \to [0, \infty)$ and for any $h \in (0, H]$ one can find $\varphi^* > 0$ satisfying the following condition: if $y \in Y(t_0, b_0, V_{t_0})$, $\xi \in \# e_0(y) \cap [t_0, t_0 + T)$, $t \in [\xi, \xi_{\# e_0(y)})$, $a \in X[t_0, a_0; t]$, $b \in Y[t_0, b_0; t]$, $\|a - X(t_0, a_0, e_0(y))(t)\| \leqslant \eta$, $(a, b) \notin M(t, y)$ and $\varphi(a, b) < \varphi^*$, then we have $h^* \in (0, h]$ and $u^* \in U_t$ such that

$$\varphi(X(t, a, u^*)(s), \tilde{y}(s)) \geqslant \sigma^*(s - t), \qquad s \in [t, t + h^*] \cap [\xi, \xi_{\# e_0(y)}],$$

$(X(t, a, u^*)(t + h^*), \tilde{y}(t + h^*)) \in M(t + h^*, y)$, provided $t < \xi_{\# e_0(y)} - h^*$ for all $\tilde{y} \in Y(t, b, V_t)$.

Denote by w a non-decreasing modulus of continuity such that

$$\|x(s) - x(t)\| + \|y(s) - y(t)\| \leqslant w(|s - t|)$$

for $x \in X(t_0, a_0, U_{t_0})$, $y \in Y(t_0, b_0, V_{t_0})$ and $s, t \in [t_0, t_0 + T]$.

Define

$$n^* = \min\{n \in N: \ T \leqslant nc\} \quad \text{and} \quad m^* = \min\{m \in N: \ T \leqslant mH\}.$$

For

$$h = \min\{c/(2m^*), \zeta/(n^* m^*)\}$$

(where ζ is taken from (H1)) take φ^* in accordance with (H3),

$$\bar{\varphi} = \min\{\hat{\varphi}, \varphi^*\} \text{ and } j^* \in \mathbf{N} \text{ such that } w_\varphi\big(w(T/(j^* m^*))\big) \leqslant \tfrac{1}{2}\bar{\varphi},$$

where w_φ denotes a non-decreasing modulus of continuity of φ.

For $r \geqslant 0$, let

$$\tilde{\sigma}(r) = \min_{s \in [0,H]} \max\{\sigma^*(s), r - w_\varphi(w(s))\}$$

and

$$\sigma(r) = \min_{s \in [0,H]} \max\{\hat{\sigma}(s), \tilde{\sigma}(r) - w_\varphi(w(s))\}.$$

LEMMA 2.1. *Under the above assumptions, there exists a strategy* $(e, \# e) \in E(X, Y, a_0, b_0, t_0)$ *such that, if* $y \in Y(t_0, b_0, V_{t_0})$ *and* $x = X(t_0, a_0, e(y))$, *then*

$$t_0 + T \in \# e(y), \quad \zeta_{\# e(y)} - \xi \geqslant c/(j^* m^*), \quad \xi \in \# e(y) \cap [t_0, t_0 + T),$$

$$\|x(s) - X(t_0, a_0, e_0(y))(s)\| \leqslant \eta, \quad s \in [t_0, t_0 + T],$$

$$\varphi(x(s), y(s)) \geqslant \min\{\sigma(\varphi(a_0, b_0)), \tfrac{1}{2}\bar{\varphi}\}, \quad s \in [t_0, (t_0)_{\# e(y)}],$$

$$\varphi(x(\xi), y(\xi)) \geqslant \min\{\hat{\sigma}(c/(2m^*)), \tfrac{1}{2}\bar{\varphi}\}, \quad \xi \in \# e_0(y) \cap (t_0, t_0 + T],$$

$$\varphi(x(s), y(s)) \geqslant \sigma\big(\min\{\hat{\sigma}(c/(2m^*)), \tfrac{1}{2}\bar{\varphi}\}\big), \quad s \in [(t_0)_{\# e(y)}, t_0 + T].$$

Proof. To prove this lemma we first find, for arbitrary $y \in Y(t_0, b_0, V_{t_0})$, the appropriate strategy $[q(y), \delta(y), a_0, b_0, t_0] \in E(X, Y, a_0, b_0, t_0)$.

For $t \in [t_0, t_0 + T]$, $y \in Y(t_0, b_0, V_{t_0})$, $a \in X[t_0, a_0; t]$, $b \in Y[t_0, b_0; t]$ satisfying the conditions:

$$\|a - X(t_0, a_0, e_0(y))(t)\| \leqslant \eta, \quad (a, b) \notin M(t, y) \quad \text{and} \quad \varphi(a, b) < \bar{\varphi}$$

take $\xi \in \# e_0(y)$ such that $\xi \leqslant t < \xi_{\# e_0(y)}$ and denote by $Q(t, a, b, y)$ the set of all pairs $(u^*, h^*) \in U_t \times (0, h]$ such that

$$\varphi(X(t, a, u^*)(s), \tilde{y}(s)) \geqslant \sigma^*(s - t), \quad s \in [t, t + h^*] \cap [\xi, \xi_{\# e_0(y)}],$$

$(X(t, a, u^*)(t + h^*), \tilde{y}(t + h^*)) \in M(t + h^*, y)$, provided $t < \xi_{\# e_0(y)} - h^*$ for all $\tilde{y} \in Y(t, b, V_t)$. It is clear from (H3) that the set $Q(t, a, b, y)$ is non-empty. Since M is connected with $(e_0, \# e_0)$ on the interval $[t_0, t_0 + T]$, we have $(t, a, b, y) \in DQ$ iff $(t, a, b, \tilde{y}) \in DQ$ for all $y, \tilde{y} \in Y(t_0, b_0, V_{t_0})$ such that $y|_{[t_0, t]} = \tilde{y}|_{[t_0, t]}$. Furthermore, in that case, $Q(t, a, b, y) = Q(t, a, b, \tilde{y})$.

It follows from the above that there exists a selector $Q^* = (U^*, H^*)$ of Q with the properties:

$$Q^*(t, a, b, y) \in Q(t, a, b, y) \quad \text{and} \quad Q^*(t, a, b, y) = Q^*(t, a, b, \tilde{y}),$$

for $y, \tilde{y} \in Y(t_0, b_0, V_{t_0})$, $t \in [t_0, t_0 + T)$, $a \in \mathbf{R}^k$, $b \in \mathbf{R}^l$, whenever $(t, a, b, y) \in DQ$ and $y|_{[t_0, t]} = \tilde{y}|_{[t_0, t]}$.

Now, fix arbitrary $y \in Y(t_0, b_0, V_{t_0})$, $(t, a, b) \in [t_0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$ and take $\xi \in \# e_0(y)$ such that $\xi \leqslant t < \xi_{\# e_0(y)}$.

If $t < t_0 + T$, $a \in X[t_0, a_0; t]$, $b \in Y[t_0, b_0; t]$, $\|a - X(t_0, a_0, e_0(y))(t)\| \leqslant \eta$, $(a, b) \notin M(t, y)$ and $\varphi(a, b) < \bar{\varphi}$, then we put

$$q(y)(t, a, b) = U^*(t, a, b, y)\langle s^* \rangle e_0(y),$$

where $s^* = \min\{t + H^*(t, a, b, y), \xi_{\# e_0(y)}\}$.

In all the other cases

$$q(y)(t, a, b) = e_0(y)|_{[t, \infty)}.$$

The function $\delta(y)$ we define in the following way:

If $\varphi(a, b) \geqslant \bar{\varphi}$, then we put

$$\delta(y)(t, a, b) = \min\{\xi_{\# e_0(y)} - t, (\xi_{\# e_0(y)} - \xi)/(j^* m^*)\};$$

if, conversely, $\varphi(a, b) < \bar{\varphi}$, then take

$$\delta(y)(t, a, b) = \min\{\xi_{\# e_0(y)} - t, (\xi_{\# e_0(y)} - \xi)/m^*\}.$$

Let $e(y) = [q(y), \delta(y), a_0, b_0, t_0](y)$ and let $\# e(y)$ be the set determined by $[q(y), \delta(y), a_0, b_0, t_0]$ and y for $y \in Y(t_0, b_0, V_{t_0})$.

It is easy to see that

$$q(y)(s, a, b) = q(\tilde{y})(s, a, b) \quad \text{and} \quad \delta(y)(s, a, b) = \delta(\tilde{y})(s, a, b)$$

for all $y, \tilde{y} \in Y(t_0, b_0, V_{t_0})$, with $y|_{[t_0, t]} = \tilde{y}|_{[t_0, t]}$, $t \geqslant t_0$ and $(s, a, b) \in [t_0, t] \times \boldsymbol{R}^k \times \boldsymbol{R}^l$.

It follows from the definition of $\delta(y)$ that $\# e_0(y) \subset \# e(y)$, for $y \in Y(t_0, b_0, V_{t_0})$, so that $(e, \# e)$ satisfies condition (SE) of Definition 1.2 and, moreover, $t_0 + T \in \# e(y)$ for $y \in Y(t_0, b_0, V_{t_0})$.

Next, we will prove the remaining properties of $(e, \# e)$. Let $y \in Y(t_0, b_0, V_{t_0})$ be fixed. From the definition of $\delta(y)$ we conclude that

$$\xi_{\# e(y)} - \xi \geqslant c/(j^* m^*) \quad \text{for } \xi \in \# e(y) \cap [t_0, t_0 + T).$$

Note that for $\xi \in \# e_0(y) \cap [t_0, t_0 + T)$ and $\alpha \in \# e(y) \cap [\xi, \xi_{\# e_0(y)})$ if

$$e(y)|_{[\alpha, \alpha_{\# e(y)})} \neq e_0(y)|_{[\alpha, \alpha_{\# e(y)})},$$

then, by the definitions of q and δ,

$$\varphi(X(t_0, a_0, e(y))(\alpha), y(\alpha)) < \bar{\varphi} \quad \text{and} \quad \alpha_{\# e(y)} - \alpha = (\xi_{\# e_0(y)} - \xi)/m^*.$$

Inasmuch as

$$\{s \in [\alpha, \alpha_{\# e(y)}]: e(y)(s) \neq e_0(y)(s)\} \subset [\alpha, \alpha + H^*(\alpha, X(t_0, a_0, e(y))(\alpha), y(\alpha), y)],$$

for the α considered above, we have

$$\mu(\{s\in[\xi, \xi_{\#e_0(y)}]: e(y)(s)\neq e_0(y)(s)\})\leqslant m^* h,$$

which implies, by the inequality $T/(\xi_{\#e_0(y)}-\xi)\leqslant T/c\leqslant n^*$, that

$$\mu(\{s\in[t_0, t_0+T]: e(y)(s)\neq e_0(y)(s)\})\leqslant n^* m^* h\leqslant \zeta.$$

Now it is clear that the third condition of Lemma 2.1 is satisfied.

We are going to estimate $\varphi(X(t_0, a_0, e(y))(s), y(s))$ for $s\in[t_0, t_0+T]$.

Suppose $\varphi(a_0, b_0)<\bar{\varphi}$. Let $\#e(y)\cap[t_0, t_0+T]=\{t_0, t_1, \dots, t_j\}$, where $t_i<t_{i+1}$, $i=0, 1, \dots, j-1$, and $x=X(t_0, a_0, e(y))$.

If $(a_0, b_0)\notin M(t_0, y)$, then $e(y)(s)=(u^*\langle t_0+h^*\rangle e_0(y))(s)$, for $s\in[t_0, t_1)$, where $u^*=U^*(t_0, a_0, b_0, y)$ and $h^*=H^*(t_0, a_0, b_0, y)$.

By (H3),

$$\varphi(x(s), y(s))\geqslant \sigma^*(s-t_0), \qquad s\in[t_0, t_0+h^*]$$

and by (H2),

$$\varphi(x(s), y(s))\geqslant \hat{\sigma}(s-(t_0+h^*)), \qquad s\in[t_0+h^*, t_1],$$

because of the inequalities

$$h^*\leqslant h\leqslant c/(2m^*)<c/m^*\leqslant t_1-t_0.$$

On the other hand,

$$\begin{aligned}\varphi(x(s), y(s)) &\geqslant \varphi(a_0, b_0)-|\varphi(x(s), y(s))-\varphi(a_0, b_0)|\\ &\geqslant \varphi(a_0, b_0)-w_\varphi(\|x(s)-a_0\|+\|y(s)-b_0\|)\\ &\geqslant \varphi(a_0, b_0)-w_\varphi(w(s-t_0)),\end{aligned}$$

for $s\in[t_0, t_0+T]$. Therefore,

$$\begin{aligned}\varphi(x(s), y(s)) &\geqslant \min_{\tau\in[t_0, t_0+H]} \max\{\sigma^*(\tau-t_0), \varphi(a_0, b_0)-w_\varphi(w(\tau-t_0))\}\\ &=\tilde{\sigma}(\varphi(a_0, b_0)), \qquad s\in[t_0, t_0+h^*],\end{aligned}$$

and, by the inequality $t_1-t_0\leqslant T/m^*\leqslant H$,

$$\begin{aligned}\varphi(x(s), y(s)) &\geqslant \min_{\tau\in[t_0+h^*, t_1]} \max\{\hat{\sigma}(\tau-(t_0+h^*)), \varphi(x(t_0+h^*), y(t_0+h^*))-\\ &\qquad -w_\varphi(w(\tau-(t_0+h^*)))\}\\ &\geqslant \min_{\tau\in[0, H]} \max\{\hat{\sigma}(\tau), \tilde{\sigma}(\varphi(a_0, b_0))-w_\varphi(w(\tau))\}=\sigma(\varphi(a_0, b_0)),\end{aligned}$$

for $s\in[t_0+h^*, t_1]$.

From the above considerations: $\varphi(x(s), y(s))\geqslant \sigma(\varphi(a_0, b_0))$ on the entire interval $[t_0, t_1]$.

If $\varphi(a_0, b_0) < \bar{\varphi}$ and $(a_0, b_0) \in M(t_0, y)$ then, taking account of the inequality $\tilde{\sigma}(r) \leqslant r$, $r \geqslant 0$, we can establish the same estimation.

Now, suppose $\varphi(a_0, b_0) \geqslant \bar{\varphi}$. In view of the formula

$$\delta(y)(t_0, a_0, b_0) = ((t_0)_{\# e_0(y)} - t_0)/(j^* m^*)$$

we have

$$\varphi(x(s), y(s)) \geqslant \varphi(a_0, b_0) - w_\varphi(w(s-t_0)) \geqslant \bar{\varphi} - w_\varphi(w(T/(j^* m^*))) \geqslant \tfrac{1}{2}\bar{\varphi},$$

for $s \in [t_0, t_1]$. Finally, in all the cases:

$$\varphi(x(s), y(s)) \geqslant \min\{\sigma(\varphi(a_0, b_0)), \tfrac{1}{2}\bar{\varphi}\}, \qquad s \in [t_0, t_1],$$

and, moreover,

$$\varphi(x(t_1), y(t_1)) \geqslant \min\{\hat{\sigma}(t_1 - (t_0 + h^*)), \tfrac{1}{2}\bar{\varphi}\} \geqslant \min\{\hat{\sigma}(c/(2m^*)), \tfrac{1}{2}\bar{\varphi}\}.$$

Applying the above procedure to the next intervals $[t_1, t_2]$, ..., we complete the proof.

3. Avoidance of many pursuers

In this section we give a sufficient condition for the existence of the evasion strategy in a differential game with several pursuers.

DEFINITION 3.1. For arbitrary payoff functionals φ_i: $\boldsymbol{R}^m \times \boldsymbol{R}^{m_i} \to [0, \infty)$ and for $(a, b_i) \in \boldsymbol{R}^m \times \boldsymbol{R}^{m_i}$, $i = 1, 2, \ldots, j$, define

$$(\varphi_1 \wedge \ldots \wedge \varphi_j)(a, b_1, \ldots, b_j) = \min\{\varphi_i(a, b_i): i = 1, 2, \ldots, j\}.$$

DEFINITION 3.2. Suppose X is a control system over $(\boldsymbol{R}^k, U)$, Y is a control system over $(\boldsymbol{R}^l, V)$ and $(t, a, b) \in [0, \infty) \times \boldsymbol{R}^k \times \boldsymbol{R}^l$ is a fixed situation.

A strategy $(e, \# e) \in E(X, Y, a, b, t)$ is *piecewise constant* on the interval $[t, t+T]$ if $e(y)(s) = e(y)(\xi)$ for all $\xi \in \# e(y) \cap [t, t+T]$, $s \in [\xi, \xi_{\# e(y)})$ and $y \in Y(t, b, V_t)$.

Suppose $n \in \boldsymbol{N}$, $n \geqslant 1$, $U \in \operatorname{comp}(\boldsymbol{R}^{\hat{k}})$, $V^j \in \operatorname{comp}(\boldsymbol{R}^{\hat{l}_j})$, f: $\boldsymbol{R}^k \times U \to \boldsymbol{R}^k$, g_j: $\boldsymbol{R}^k \times V^j \to \boldsymbol{R}^k$ are continuous and satisfy the Lipschitz condition

$$\|f(a, u) - f(b, u)\| \leqslant L^* \|a-b\|, \qquad \|g_j(a, v) - g_j(b, v)\| \leqslant L^* \|a-b\|,$$

for all $a, b \in \boldsymbol{R}^k$, $u \in U$, $v \in V^j$ and $j = 1, 2, \ldots, n$.

Denote by M^* a linear subspace of $\boldsymbol{R}^k$ with codim $M^* \geqslant 2$ and by W a two-dimensional subspace of $(M^*)^\perp$, which means that $\dim W = 2$ and $\langle a, b \rangle = 0$ whenever $a \in W$ and $b \in M^*$.

If h, h^*: $\boldsymbol{R}^k \times U \to \boldsymbol{R}^k$ and h^* has a part derivative with respect to the first variable, then we define (see also [25])

$$(\Delta_h h^*)(a, u) = \frac{\partial h^*}{\partial a}(a, u) h(a, u) \quad \text{for } (a, u) \in \boldsymbol{R}^k \times U,$$

and we will use similar notation for functions defined on $\boldsymbol{R}^k \times V^j$, $j = 1, 2, \ldots, n$.

Assume that $\varkappa \in N$ and that the functions f, g_j have continuous derivatives with respect to the first variables up to the order $\varkappa$ for $j = 1, 2, \ldots, n$, and define the sequences

$$(\Delta^{-1} f)(a, u) = a, \quad (\Delta^{i+1} f)(a, u) = \big(\Delta_f(\Delta^i f)\big)(a, u),$$
$$(\Delta^{-1} g_j)(b, v) = b, \quad (\Delta^{i+1} g_j)(b, v) = \big(\Delta_{g_j}(\Delta^i g_j)\big)(b, v)$$

for $a, b \in \boldsymbol{R}^k$, $u \in U$, $v \in V^j$, $i = -1, 0, \ldots, \varkappa - 1$ and $j = 1, \ldots, n$.

Let $\hat{\Pi}$ denote the orthogonal projection from $\boldsymbol{R}^k$ onto W and let $\hat{\Pi}(\Delta^i f)(a, U)$, $\hat{\Pi}(\Delta^i g_j)(a, V^j)$ consist only of single points for $i = -1, 0, \ldots, \varkappa - 1$ and $j = 1, 2, \ldots, n$, whereas

$$(3.1) \qquad \hat{\Pi}(\Delta^\varkappa g_j)(b, V^j) \subset \operatorname{Int} \operatorname{conv} \hat{\Pi}(\Delta^\varkappa f)(a, U)$$

for all $a, b \in \boldsymbol{R}^k$ such that $a - b \in M^*$ and $j = 1, 2, \ldots, n$.

For $j \in \{1, 2, \ldots, n\}$, $t \in [0, \infty)$, a, $b \in \boldsymbol{R}^k$, $u \in U_t$ and $v \in V_t^j$ denote by $\mathfrak{X}(t, a, u)$ and $\mathfrak{Y}_j(t, b, v)$ the solutions of the relevant differential equations:

$$\mathfrak{x}' = f(\mathfrak{x}, u), \quad \mathfrak{x}(t) = a \quad \text{and} \quad \mathfrak{y}' = g_j(\mathfrak{y}, v), \quad \mathfrak{y}(t) = b.$$

For $t \in [0, \infty)$, $a \in \boldsymbol{R}^k$, $(a^0, a^1, \ldots, a^\varkappa) \in W^{\varkappa+1}$, where $a^i \in W$, $i = 0, 1, \ldots, \varkappa$, and for $u \in U_t$ denote by $X(t, a, a^0, \ldots, a^\varkappa, u)$ the trajectory $(\mathfrak{x}, x^0, \ldots, x^\varkappa)$, where $\mathfrak{x} = \mathfrak{X}(t, a, u)$, $x^i(t) = a^i$, $(x^i)' = x^{i+1}$, $i = 0, 1, \ldots, \varkappa - 1$, $x^\varkappa(t) = a^\varkappa$ and $(x^\varkappa)' = \hat{\Pi}(\Delta^\varkappa f)(\mathfrak{x}, u)$.

The symbols $Y_j(t, b, b^0, \ldots, b^\varkappa, v)$, $v \in V^j$, $j = 1, 2, \ldots, n$, we define analogously.

For $i = 0, 1, \ldots, \varkappa$, $a, b \in \boldsymbol{R}^k$, $(a^0, \ldots, a^\varkappa) \in W^{\varkappa+1}$, $(b^0, \ldots, b^\varkappa) \in W^{\varkappa+1}$ define

$$\varphi^i(a, a^0, \ldots, a^\varkappa, b, b^0, \ldots, b^\varkappa) = \max\{\operatorname{dist}(a-b, M^*), \|a^i - b^i\|\},$$

where $\operatorname{dist}(a-b, M^*) = \min\{\|a-b-z\|: z \in M^*\}$.

Definition 3.3. For each $a \in \boldsymbol{R}^k$ and $\hat{a} \in W$, $\|\hat{a}\| = 1$, define $U^e(a, \hat{a})$ to be the set of all $\hat{u} \in U$ satisfying the condition

$$\langle \hat{\Pi}(\Delta^\varkappa f)(a, \hat{u}), \hat{a} \rangle = \max_{u \in U} \langle \hat{\Pi}(\Delta^\varkappa f)(a, u), \hat{a} \rangle.$$

Now, let us fix arbitrary $t_0 \in [0, \infty)$, $T \in (0, \infty)$, $\bar{a}_0 = (a_0, a_0^0, \ldots, a_0^{\varkappa}) \in \boldsymbol{R}^k \times W^{\varkappa+1}$, $\bar{b}_{j0} = (b_{j0}, b_{j0}^0, \ldots, b_{j0}^{\varkappa}) \in \boldsymbol{R}^k \times W^{\varkappa+1}$ such that $\{a_0^i\} = \hat{\Pi}(\Delta^{i-1} f)(a_0, U)$, $\{b_{j0}^i\} = \hat{\Pi}(\Delta^{i-1} g_j)(b_{j0}, V^j)$, $i = 0, 1, \ldots, \varkappa$, and

$$a_0 - b_{j0} \notin M^*, \quad j = 1, 2, \ldots, n.$$

Note that

$$\hat{\Pi}\mathfrak{x}(s) = a^0 + (s-t)a^1 + \ldots + \frac{(s-t)^{\varkappa}}{\varkappa!} a^{\varkappa} + \int_t^s \int_t^{s_\varkappa} \ldots \int_t^{s_1} \hat{\Pi}(\Delta^{\varkappa} f)(\mathfrak{x}(\tau), u(\tau))\, d\tau\, ds_1 \ldots ds_{\varkappa}$$

and

$$\hat{\Pi}\mathfrak{y}(s) = b_j^0 + (s-t)b_j^1 + \ldots + \frac{(s-t)^{\varkappa}}{\varkappa!} b_j^{\varkappa} + \int_t^s \int_t^{s_\varkappa} \ldots \int_t^{s_1} \hat{\Pi}(\Delta^{\varkappa} g_j)(\mathfrak{y}(\tau), v(\tau))\, d\tau\, ds_1 \ldots ds_{\varkappa}$$

for all $\bar{a} = (a, a^0, \ldots, a^{\varkappa}) \in X[t_0, \bar{a}_0; t]$, $\bar{b}_j = (b_j, b_j^0, \ldots, b_j^{\varkappa}) \in Y_j[t_0, \bar{b}_{j0}; t]$, $\mathfrak{x} = \mathfrak{X}(t, a, u)$, $\mathfrak{y} = \mathfrak{Y}_j(t, b_j, v)$, $u \in U_t$, $v \in V_t^j$, $s \in [t, \infty)$ and $j = 1, 2, \ldots, n$.

Moreover, there are constants L, $R > 0$ such that

$$\|x(s) - \bar{a}_0\| \leqslant R, \quad \|y(s) - \bar{b}_0\| \leqslant R,$$

$$\|x(s) - x(t)\| \leqslant L|s-t|, \quad \|y(s) - y(t)\| \leqslant L|s-t|, \tag{3.2}$$

for $x \in X(t_0, \bar{a}_0, U_{t_0})$, $y \in Y_j(t_0, \bar{b}_{j0}, V_{t_0}^j)$, $s, t \in [t_0, t_0 + T + 1]$ and $j = 1, 2, \ldots, n$.

Taking account of formula (3.1), one can find constants $\varrho \in (0, \infty)$ and $\gamma \in (0, L]$ such that, if $a \in \boldsymbol{R}^k$, $\|a - a_0\| \leqslant R$, $\hat{a} \in W$, $\|\hat{a}\| = 1$ and $\hat{u} \in U^e(a, \hat{a})$, then

$$\langle \hat{\Pi}(\Delta^{\varkappa} f)(\tilde{a}, \hat{u}) - \hat{\Pi}(\Delta^{\varkappa} g_j)(\tilde{b}, v), \hat{a} \rangle \geqslant 2\gamma \tag{3.3}$$

for all $\tilde{a} \in B^k(a, 2\varrho)$, $\tilde{b} \in B^k(b_0, R)$, $\operatorname{dist}(\tilde{a} - \tilde{b}, M^*) \leqslant 2\varrho$ and $j = 1, 2, \ldots, n$.

Finally, for $j \in \{1, 2, \ldots, n\}$, $t \in [0, \infty)$, $\bar{b}_i \in \boldsymbol{R}^k \times W^{\varkappa+1}$, $v_i \in V_t^i$, $i = 1, 2, \ldots, j$, define

$$(Y_1 \times \ldots \times Y_j)(t, \bar{b}_1, \ldots, \bar{b}_j, v_1, \ldots, v_j) = Y_1(t, \bar{b}_1, v_1) \times \ldots \times Y_j(t, \bar{b}_j, v_j).$$

3.1. First step

Let $\delta_0 \in (0, \min\{1, \varrho/(2L+\gamma)\}]$ be such that $T/\delta_0 \in \boldsymbol{N}$. Suppose q: $(\boldsymbol{R}^k \times W^{\varkappa+1}) \times (\boldsymbol{R}^k \times W^{\varkappa+1}) \to U$ and δ: $(\boldsymbol{R}^k \times W^{\varkappa+1}) \times (\boldsymbol{R}^k \times W^{\varkappa+1}) \to (0, \infty)$ are such that for any $\bar{a} = (a, a^0, \ldots, a^\varkappa) \in \boldsymbol{R}^k \times W^{\varkappa+1}$ and $\bar{b} = (b, b^0, \ldots, b^\varkappa) \in \boldsymbol{R}^k \times W^{\varkappa+1}$, $\delta(\bar{a}, \bar{b}) = \delta_0$, there exists an $\hat{a} \in W$, $\|\hat{a}\| = 1$ for which $q(\bar{a}, \bar{b}) \in U^e(a, \hat{a})$ and $\hat{a} = (a^\varkappa - b^\varkappa)/\|a^\varkappa - b^\varkappa\|$ provided $a^\varkappa \neq b^\varkappa$.

Let $e_1(y_1) = [q, \delta, \bar{a}_0, \bar{b}_{10}, t_0](y_1)$ and let $\# e_1(y_1)$ denote the set determined by $[q, \delta, \bar{a}_0, \bar{b}_{10}, t_0]$ and y_1 (see Definition 1.4) for $y_1 \in Y_1(t_0, \bar{b}_{10}, V^1_{t_0})$.

It is not very difficult to show that there is a $d_1 > 0$ such that the strategy $(e_1, \# e_1)$ guarantees the result d_1 in the game $(X, Y_1, \bar{a}_0, \bar{b}_{10}, t_0; \varphi^\varkappa)$ on the interval $[t_0, t_0 + T]$.

Let $j \in \{1, 2, \ldots, n-1\}$. Assume that we have found positive constants c_i, d_i, η_i and, piecewise on the interval $[t_0, t_0 + T]$, strategies $(e_i, \# e_i) \in E(X, Y_1 \times \ldots \times Y_i, \bar{a}_0, \bar{b}_{10}, \ldots, \bar{b}_{i0}, t_0)$ which guarantee the results d_i in the games $(X, Y_1 \times \ldots \times Y_i, \bar{a}_0, \bar{b}_{10}, \ldots, \bar{b}_{i0}, t_0; \varphi^\varkappa \wedge \ldots \wedge \varphi^\varkappa)$ (i-times), $i = 1, 2, \ldots, j$, and assume that the following conditions are satisfied for all $y_i \in Y_i(t_0, \bar{b}_{i0}, V^i_{t_0})$, $i = 1, 2, \ldots, j$:

(a) $\eta_1 = \min\{\frac{1}{2} d_1, \frac{1}{4}\varrho\}$, $\eta_{i+1} = \frac{1}{2}\min\{d_{i+1}, \eta_i\}$, $i = 1, 2, \ldots, j-1$,

(b) $\alpha_{\# e_i(y_1, \ldots, y_i)} - \alpha \geqslant c_i$, $\alpha \in \# e_i(y_1, \ldots, y_i) \cap [t_0, t_0 + T)$, $i = 1, 2, \ldots, j$.

(c) $\# e_i(y_1, \ldots, y_i) \subset \# e_{i+1}(y_1, \ldots, y_{i+1})$, $i = 1, 2, \ldots, j-1$,

(d) $\|X(t_0, \bar{a}_0, e_{i+1}(y_1, \ldots, y_{i+1}))(t) - X(t_0, \bar{a}_0, e_i(y_1, \ldots, y_i))(t)\| \leqslant \eta_i$, $t \in [t_0, t_0 + T]$, $i = 1, 2, \ldots, j-1$,

(e) if $i \in \{1, 2, \ldots, j\}$, $\alpha \in \# e_i(y_1, \ldots, y_i) \cap [t_0, t_0 + T)$ and the statement: "there are $v \in \boldsymbol{N}$, $1 \leqslant v < i$ and $\beta \in \# e_v(y_1, \ldots, y_v)$ such that $\alpha \in [\beta, \beta_{\# e_v(y_1, \ldots, y_v)})$ and $e_i(y_1, \ldots, y_i)(\alpha) = e_v(y_1, \ldots, y_v)(\beta)$" is false, then $e_i(y_1, \ldots, y_i)(\alpha) \in U^e(\mathfrak{x}(\alpha), \hat{a})$ for some $\hat{a} \in W$, $\|\hat{a}\| = 1$, where $\mathfrak{x}$ denotes the first component of the trajectory $X(t_0, \bar{a}_0, e_i(y_1, \ldots, y_i))$.

It is easy to see that

(E) for any $(y_1, \ldots, y_j) \in (Y_1 \times \ldots \times Y_j)(t_0, \bar{b}_{10}, \ldots, \bar{b}_{j0}, V_{t_0} \times \ldots \times V^j_{t_0})$ and any $\xi \in \# e_j(y_1, \ldots, y_j) \cap [t_0, t_0 + T)$ there are $v \leqslant j$, $\beta \in \# e_v(y_1, \ldots, y_v)$ and $\hat{a} \in W$, $\|\hat{a}\| = 1$, such that

$$(3.4) \quad \xi \in [\beta, \beta_{\# e_v(y_1, \ldots, y_v)}), \quad e_j(y_1, \ldots, y_j)(\xi) = e_v(y_1, \ldots, y_v)(\beta),$$
$$e_v(y_1, \ldots, y_v)(\beta) \in U^e(\mathfrak{x}(\beta), \hat{a}),$$

where $\mathfrak{x}$ denotes the first component of $X(t_0, \bar{a}_0, e_v(y_1, \ldots, y_v))$.

Take $H = \min\{1, \varrho/(2L+\gamma)\}$, $c = c_j$, $\eta = \eta_j$ and

$$\varphi(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) = \varphi^\varkappa(\bar{a}, \bar{b}_{j+1})$$

for $\bar{a}, \bar{b}_i \in \boldsymbol{R}^k \times W^{\varkappa+1}$, $i = 1, 2, \ldots, j+1$.

Let us fix, for a moment,

$$(y_1, \ldots, y_{j+1}) \in (Y_1 \times \ldots \times Y_{j+1})(t_0, \bar{b}_{10}, \ldots, \bar{b}_{j+1,0}, V^1_{t_0} \times \ldots \times V^{j+1}_{t_0})$$

and define

$$e_0(y_1, \ldots, y_{j+1}) = e_j(y_1, \ldots, y_j), \qquad \# e_0(y_1, \ldots, y_{j+1}) = \# e_j(y_1, \ldots, y_j).$$

For $t \in [t_0, t_0+T)$ find $\xi \in \# e_j(y_1, \ldots, y_j)$ such that $\xi \leqslant t < \xi_{\# e_j(y_1,\ldots,y_j)}$ and let $M(t, y_1, \ldots, y_{j+1})$ be the set of all points $(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) \in (\boldsymbol{R}^k \times \times W^{\varkappa+1})^{j+2}$, $\bar{a} = (a, a^0, \ldots, a^\varkappa)$, $\bar{b}_i = (b_i, b_i^0, \ldots, b_i^\varkappa)$, $i = 1, 2, \ldots, j+1$, for which

$$\operatorname{dist}(a - b_{j+1}, M^*) \geqslant \eta_j \quad \text{or} \quad \langle a^\varkappa - b_{j+1}^\varkappa, \hat{a} \rangle \geqslant 0,$$

where $\hat{a} \in W$ is taken for $y_1, \ldots, y_j$ and ξ in accordance with (E).

LEMMA 3.1. *Under the above assumptions, Hypotheses* (H1), (H2) *and* (H3) *of Lemma* 2.1 *are satisfied.*

Proof. Routine calculations involving the well-known Gronwall inequality (see e.g. the proof of Theorem 9 of [19], p. 46) show that (H1) holds true.

Verification of Hypothesis (H2). Take $\hat{\varphi} = \eta_j$ and $\hat{\sigma}(t) = 2\gamma t$, for $t \geqslant 0$. Let

$$(y_1, \ldots, y_{j+1}) \in (Y_1 \times \ldots \times Y_{j+1})(t_0, \bar{b}_{10}, \ldots, \bar{b}_{j+1,0}, V_{t_0}^1 \times \ldots \times V_{t_0}^{j+1}),$$

$$\xi \in \# e_0(y_1, \ldots, y_{j+1}) \cap [t_0, t_0+T), \qquad t \in [\xi, \xi_{\# e_0(y_1,\ldots,y_{j+1})}),$$

$$\bar{a} = (a, a^0, \ldots, a^\varkappa) \in X[t_0, \bar{a}_0; t] \quad \text{and} \quad \bar{b}_i = (b_i, b_i^0, \ldots, b_i^\varkappa) \in Y_i[t_0, \bar{b}_{i0}; t], \quad i = 1, 2, \ldots, j+1,$$

be such that

$$\|\bar{a} - X(t_0, \bar{a}_0, e_0(y_1, \ldots, y_{j+1}))(t)\| \leqslant \eta,$$

$$(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) \in M(t, y_1, \ldots, y_{j+1}) \quad \text{and} \quad \varphi(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) < \hat{\varphi}.$$

Denote $x = (\mathfrak{x}, x^0, \ldots, x^\varkappa) = X(t, \bar{a}, e_0(y_1, \ldots, y_{j+1}))$ and

$$x_i = (\mathfrak{x}_i, x_i^0, \ldots, x_i^\varkappa) = X(t_0, \bar{a}_0, e_i(y_1, \ldots, y_i)), \qquad i = 1, 2, \ldots, j.$$

Since $\operatorname{dist}(a - b_{j+1}, M^*) \leqslant \varphi(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) < \hat{\varphi} = \eta_j$, so $\langle a^\varkappa - b_{j+1}^\varkappa, \hat{a} \rangle \geqslant 0$, where $\hat{a}$, ν and β are chosen for $(y_1, \ldots, y_j)$ and ξ in accordance with (E).

Note that

$$\begin{aligned}
\|\mathfrak{x}(s) - \mathfrak{x}_\nu(s)\| &\leqslant \|\mathfrak{x}(s) - \mathfrak{x}(t)\| - \|\mathfrak{x}(t) - \mathfrak{x}_\nu(t)\| - \|\mathfrak{x}_\nu(t) - \mathfrak{x}_\nu(s)\| \\
&\leqslant L(s-t) + \|a - \mathfrak{x}_\nu(t)\| + L(s-t) \\
&\leqslant 2LH + \|a - \mathfrak{x}_j(t)\| + \|\mathfrak{x}_j(t) - \mathfrak{x}_{j-1}(t)\| + \ldots \\
&\qquad \ldots + \|\mathfrak{x}_{\nu+1}(t) - \mathfrak{x}_\nu(t)\| \\
&\leqslant 2LH + \eta_j + \eta_{j-1} + \ldots + \eta_\nu < 2LH + 2\eta_1 < \tfrac{3}{2}\varrho
\end{aligned}$$

and

$$\|x_v(s)-x_v(\beta)\| \leqslant L(s-\beta) \leqslant LH < \tfrac{1}{2}\varrho \quad \text{for } s\in[t, t+H].$$

Let us fix $(v_1, \ldots, v_{j+1})\in V_t^1 \times \ldots \times V_t^{j+1}$, $(\tilde{y}_1, \ldots, \tilde{y}_{j+1}) = (Y_1 \times \ldots \times Y_{j+1})(t, \bar{b}_1, \ldots, \bar{b}_{j+1}, v_1, \ldots, v_{j+1})$, $\tilde{y}_i = (\tilde{\mathfrak{y}}_i, \tilde{y}_i^0, \ldots, \tilde{y}_i^\varkappa)$, $i = 1, 2, \ldots, j+1$.

Clearly, $\operatorname{dist}(\mathfrak{x}(s)-\tilde{\mathfrak{y}}_{j+1}, M^*) \leqslant \operatorname{dist}(a-b_{j+1}, M^*)+2L(s-t) \leqslant \eta_j+2LH < 2\varrho$, $s\in[t, t+H]$, which implies, by (3.4) and (3.3), that

$$\langle \hat{\Pi}(\Delta^\varkappa f)(\mathfrak{x}(s), e_0(y_1, \ldots, y_{j+1})(s))-\hat{\Pi}(\Delta^\varkappa g_{j+1})(\tilde{\mathfrak{y}}_{j+1}(s), v_{j+1}(s)), \hat{a}\rangle \geqslant 2\gamma$$

for $s\in[t, t+H]$. Thus,

$$\begin{aligned}\varphi(x(s), \tilde{y}_1(s), \ldots, \tilde{y}_{j+1}(s)) &\geqslant \|x^\varkappa(s)-\tilde{y}^\varkappa_{j+1}(s)\| \geqslant \langle x^\varkappa(s)-\tilde{y}^\varkappa_{j+1}(s), \hat{a}\rangle \\ &= \langle a^\varkappa - b^\varkappa_{j+1}, \hat{a}\rangle + \\ &\quad + \Big\langle\int_t^s \big(\hat{\Pi}(\Delta^\varkappa f)(\mathfrak{x}(\tau), e_0(y_1, \ldots, y_{j+1})(\tau))- \\ &\quad - \hat{\Pi}(\Delta^\varkappa g_{j+1})(\tilde{\mathfrak{y}}_{j+1}(\tau), v_{j+1}(\tau))\big)d\tau, \hat{a}\Big\rangle \\ &\geqslant 0+2\gamma(s-t) = \hat{\sigma}(s-t), \qquad s\in[t, t+H].\end{aligned}$$

Verification of Hypothesis (H3). Define, for each $t \geqslant 0$, $\sigma^*(t) = \gamma t$ and fix an arbitrary $h\in(0, H]$. Let

$$\varphi^* = \min\{\eta_j, \tfrac{1}{2}h(2L+\gamma)\exp(-2\pi(2L+\gamma)/\gamma)\}, \qquad \bar{\varphi} = \min\{\hat{\varphi}, \varphi^*\},$$
$$n^* = \min\{n\in N\colon\ T \leqslant nc_j\}, \qquad m^* = \min\{m\in N\colon\ T \leqslant mH\}$$

(see the notation before Lemma 2.1) and let $i^*, j^*\in N$ be such that

$$2LT \leqslant \tfrac{1}{2}\bar{\varphi}j^* m^* \quad \text{and} \quad (2L+\gamma)T \leqslant i^* j^* m^* \varphi^*.$$

Suppose that $y_i = (\mathfrak{y}_i, y_i^0, \ldots, y_i^\varkappa)\in Y_i(t_0, \bar{b}_{i0}, V^i_{t_0})$, $i = 1, \ldots, j+1$, $\xi\in \#e_0(y_1, \ldots, y_{j+1})\cap[t_0, t_0+T)$, $t\in[\xi, \xi_{\#e_0(y_1,\ldots,y_{j+1})})$, $\bar{a} = (a, a^0, \ldots, a^\varkappa) \in X[t_0, \bar{a}_0; t]$, $\bar{b}_i = (b_i, b_i^0, \ldots, b_i^\varkappa)\in Y_i[t_0, \bar{b}_{i0}; t]$ satisfy the conditions

$$\|\bar{a}-X(t_0, \bar{a}_0, e_0(y_1, \ldots, y_{j+1}))(t)\| \leqslant \eta,$$
$$(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1})\notin M(t, y_1, \ldots, y_{j+1}) \quad \text{and} \quad \varphi(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) < \varphi^*.$$

Denote

$$h_* = (\xi_{\#e_0(y_1,\ldots,y_{j+1})}-\xi)/(i^* j^* m^*)$$

and

$$x_0 = (\mathfrak{x}_0, x_0^0, \ldots, x_0^\varkappa) = X(t_0, \bar{a}_0, e_0(y_1, \ldots, y_{j+1})).$$

For $(y_1, \ldots, y_{j+1})$ and ξ choose $\nu \leqslant j$, β and $\hat{a}$ in accordance with (E). Of course, $\langle a^\varkappa - b_{j+1}^\varkappa, \hat{a}\rangle < 0$ since $\operatorname{dist}(a-b_{j+1}, M^*) \leqslant \varphi(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{j+1}) < \varphi^* \leqslant \eta_j$.

For $s \in [t, \infty)$ denote by $B(s)$ the closed, convex hull of the set of all point $\tilde{b} \in W$ for which there exists a trajectory $\tilde{y} = (\tilde{\mathfrak{y}}, \tilde{y}^0, \ldots, \tilde{y}^\varkappa)$ with the properties

$$\tilde{\mathfrak{y}} \in \mathfrak{Y}(t, b_{j+1}, V_t^{j+1}), \quad (\tilde{y}^i)' = \tilde{y}^{i+1}, \quad \tilde{y}^i(t) = b_{j+1}^i, \quad i = 0, \ldots, \varkappa - 1,$$
$$\tilde{y}^\varkappa(t) = b_{j+1}^\varkappa, \quad (\tilde{y}^\varkappa)' \in \hat{\Pi}(\Delta^\varkappa g_{j+1})(\tilde{\mathfrak{y}}, V^{j+1}) + W \cap B^k(0, \gamma), \quad \tilde{b} = \tilde{y}^\varkappa(s).$$

Denote by Γ the set of all points $\hat{b} \in W$ lying on the shorter arc of the unit circle $\{w \in W\colon \|w\| = 1\}$, between $(a^\varkappa - b_{j+1}^\varkappa)/\|a^\varkappa - b_{j+1}^\varkappa\|$ and $\hat{a}$, and satisfying the inequality $\langle a^\varkappa - b_{j+1}^\varkappa, \hat{b}\rangle \leqslant 0$.

Let A stand for the set of all pairs $(s, a^*) \in [t, \infty) \times (W - B(s))$ for which there is an $\alpha = \alpha(s, a^*) \in \Gamma$ satisfying the inequality

$$\max_{\tilde{b} \in B(s)} \langle \tilde{b} - a^*, \alpha\rangle = 0.$$

Define $B^*\colon A \to 2^W$ and $U^*\colon A \times \boldsymbol{R}^k \to U$ by the formulas

$$\text{(3.5)} \quad \begin{aligned} & B^*(s, a^*) \subset B(s), \\ & \langle b^* - a^*, \alpha(s, a^*)\rangle = \max_{\tilde{b} \in B(s)} \langle \tilde{b} - a^*, \alpha(s, a^*)\rangle, \quad b^* \in B^*(s, a^*), \\ & U^*(s, a^*, a) \in U^e(a, \alpha(s, a^*)), \quad (s, a^*, a) \in A \times \boldsymbol{R}^k, \end{aligned}$$

for all $(s, a^*) \in A$.

We are going to define an auxiliary trajectory $x_* = (\mathfrak{x}_*, x_*^0, \ldots$ $\ldots, x_*^\varkappa) \in X(t, \bar{a}, U_t)$.

Of course, it should be $x_*(t) = \bar{a}$. Assume that x_* has been defined on $[t, t_l] = [t, t + lh_*]$ for some $l = 0, 1, \ldots$

If $(t_l, x_*^\varkappa(t_l)) \notin A$, then we extend x_* onto the entire interval $[t, \infty)$ in any admissible way.

If, conversely, $(t_l, x_*^\varkappa(t_l)) \in A$, then we define x_* on the interval $[t_l, t_{l+1}]$ by the formulas

$$\begin{aligned} \mathfrak{x}_*'(s) &= f(\mathfrak{x}_*(s), U^*(t_l, x_*^\varkappa(t_l), \mathfrak{x}_*(t_l))), \\ (x_*^i)' &= x_*^{i+1}, \quad i = 0, 1, \ldots, \varkappa - 1, \\ (x_*^\varkappa)'(s) &= \hat{\Pi}(\Delta^\varkappa f)(\mathfrak{x}_*(s), U^*(t_l, x_*^\varkappa(t_l), \mathfrak{x}_*(t_l))). \end{aligned}$$

For $s \in [t, \infty)$ such that $(s, x_*^\varkappa(s)) \in A$ denote by $y_*(s)$ the element of $B^*(s, x_*^\varkappa(s))$ satisfying the condition

$$\|y_*(s) - x_*^\varkappa(s)\| = \min_{b^* \in B^*(s, x_*^\varkappa(s))} \|b^* - x_*^\varkappa(s)\|.$$

Finally, for $s \in Dy_*$ define $\theta(s)$ to be the angle (on W) between $a^\varkappa - b^\varkappa_{j+1}$ and $x^\varkappa_*(s) - y_*(s)$.

Now, let us fix an $l^* \in N$ for which

$$t_l \leqslant t+h \quad \text{and} \quad (t_l, x^\varkappa_*(t_l)) \in A, \quad l = 0, 1, \ldots, l^*-1,$$

and fix an arbitrary $\tilde{y} = (\tilde{\mathfrak{y}}, \tilde{y}^0, \ldots, \tilde{y}^\varkappa)$ from the definition of $B(\cdot)$.

Inasmuch as

$$\|x_*(s) - x_*(t_l)\| \leqslant L(s-t_l) < \varrho, \quad s \in [t_l, t_{l+1}], \quad l = 0, 1, \ldots, l^*-1,$$

and

$$\operatorname{dist}\big(\mathfrak{x}_*(s) - \tilde{\mathfrak{y}}(s), M^*\big) \leqslant \operatorname{dist}(a - b_{j+1}, M^*) + 2L(s-t) \leqslant 2\varrho, \quad s \in [t, t+H],$$

we have, by (3.5) and (3.3),

$$\big\langle (x^\varkappa_*)'(s) - (\tilde{y}^\varkappa)'(s), \alpha\big(t_l, x^\varkappa_*(t_l), \mathfrak{x}_*(t_l)\big)\big\rangle \geqslant \gamma$$

for almost all $s \in [t_l, t_{l+1}]$ and $l = 0, 1, \ldots, l^*-1$. Therefore,

$$\begin{aligned}\theta(t_{l+1}) - \theta(t_l) &\geqslant \sin\big(\theta(t_{l+1}) - \theta(t_l)\big)\\ &\geqslant \gamma h_* / \big(\|a^\varkappa - b^\varkappa_{j+1}\| + (2L+\gamma)(t_{l+1} - t_l)\big)\\ &\geqslant \gamma h_* / \big(\varphi^* + (2L+\gamma)(l+1) h_*\big)\end{aligned}$$

and

$$x^\varkappa_*(s) \notin B(s)$$

for $s \in [t_l, t_{l+1}]$, $l = 0, 1, \ldots, l^*-1$. Using the above inequality, we can show by induction that

$$\theta(t_l) \geqslant \frac{\gamma}{4L+2\gamma} \ln \frac{\varphi^* + l(2L+\gamma) h_*}{\varphi^* + (2L+\gamma) h_*}, \quad l = 0, 1, \ldots, l^*.$$

In addition, by the definition of A, we must have $\theta(t_{l^*-1}) \leqslant \pi$, which implies that $l^* h_* \leqslant h$. Hence, there is a smallest $l^* \in N$ such that $l^* h_* \leqslant h$ and $(t_l, x^\varkappa_*(t_l)) \in A$, $l = 0, \ldots, l^*-1$, whereas $(t_{l^*}, x^\varkappa_*(t_{l^*})) \notin A$. Define

$$h^* = l^* h_* \quad \text{and} \quad u^* \in U_t$$

by the formula

$$u^*(s) = U^*\big(t_l, x^\varkappa_*(t_l), \mathfrak{x}_*(t_l)\big), \quad s \in [t_l, t_{l+1}), \quad l = 0, \ldots, l^*-1.$$

Then

$$\langle x^\varkappa_*(t+h^*) - y_*(t+h^*), \hat{a}\rangle \geqslant 0 \quad \text{and} \quad x^\varkappa_*(s) \notin B(s), \quad s \in [t, t+h^*]. \tag{3.6}$$

The above inequality implies that

$$\langle x^\varkappa_*(t+h^*) - \tilde{y}^\varkappa(t+h^*), \hat{a}\rangle > 0$$

for $\tilde{y} = (\tilde{\eta}, \tilde{y}^0, \ldots, \tilde{y}^{\varkappa}) \in Y_{j+1}(t, \bar{b}_{j+1}, V_t^{j+1})$, but this means that

$$(x_*(t+h^*), \tilde{y}_1(t+h^*), \ldots, \tilde{y}_{j+1}(t+h^*)) \in M(t+h^*, y_1, \ldots, y_{j+1})$$

for all $\tilde{y}_i \in Y_i(t, \bar{b}_i, V_t^i)$, $i = 1, 2, \ldots, j+1$.

Finally, we conclude from (3.6) that

$$\varphi(x_*(s), \tilde{y}_1(s), \ldots, \tilde{y}_{j+1}(s)) \geqslant \|x_*^{\varkappa}(s) - \tilde{y}_{j+1}(s)\| \geqslant \gamma(s-t) = \sigma^*(s-t)$$

for the above $\tilde{y}_1, \ldots, \tilde{y}_{j+1}$ and $s \in [t, t+h^*]$. The proof of Lemma 3.1 is finished.

In view of Lemma 2.1, there are d_{j+1}, $c_{j+1}^* > 0$ and

$$(e_{j+1}, \# e_{j+1}) \in E(X, Y_1 \times \ldots \times Y_{j+1}, \bar{a}_0, \bar{b}_{10}, \ldots, \bar{b}_{j+1,0}, t_0),$$

which guarantees the result d_{j+1} in the game

$$(X, Y_1 \times \ldots \times Y_{j+1}, \bar{a}_0, \bar{b}_{10}, \ldots, \bar{b}_{j+1,0}, t_0; \varphi^{\varkappa} \wedge \ldots \wedge \varphi^{\varkappa}) \quad (j+1\text{-times})$$

on the interval $[t_0, t_0+T]$ and satisfies the conditions:

$$\xi_{\# e_{j+1}(y_1, \ldots, y_{j+1})} - \xi \geqslant c_{j+1}^*, \quad \xi \in \# e_{j+1}(y_1, \ldots, y_{j+1}) \cap [t_0, t_0+T),$$

$$\# e_j(y_1, \ldots, y_j) \subset \# e_{j+1}(y_1, \ldots, y_{j+1}),$$

$$\|X(t_0, \bar{a}_0, e_{j+1}(y_1, \ldots, y_{j+1}))(t) - X(t_0, \bar{a}_0, e_j(y_1, \ldots, y_j))(t)\| \leqslant \eta_j, \quad t \in [t_0, t_0, +T]$$

for $y_i \in Y_i(t_0, \bar{b}_{i0}, V_{t_0}^i)$, $i = 1, 2, \ldots, j+1$.

For $(y_1, \ldots, y_{j+1}) \in (Y_1 \times \ldots \times Y_{j+1})(t_0, \bar{b}_{10}, \ldots, \bar{b}_{j+1,0}, V_{t_0}^1 \times \ldots \times V_{t_0}^{j+1})$ and $\xi \in \# e_{j+1}(y_1, \ldots, y_{j+1})$, if $\xi \in [t_0, t_0+T)$, $(x(\xi), y_1(\xi), \ldots, y_{j+1}(\xi)) \notin M(\xi, y_1, \ldots, y_{j+1})$ and $\varphi(x(\xi), y_1(\xi), \ldots, y_{j+1}(\xi)) < \varphi^*$, where $x = X(t_0, \bar{a}_0, e_{j+1}(y_1, \ldots, y_{j+1}))$, take h_* and l^* in the same way as in the verification of (H3) and define

$$S(\xi, y_1, \ldots, y_{j+1}) = \{\xi, \xi+h_*, \ldots, \xi+l^* h_*\}.$$

In all the other cases take $S(\xi, y_1, \ldots, y_{j+1}) = \{\xi\}$. Let $\# S(y_1, \ldots, y_{j+1})$ be the union of all sets $S(\xi, y_1, \ldots, y_{j+1})$ for $\xi \in \# e_{j+1}(y_1, \ldots, y_{j+1})$.

It is not difficult to check that $(e_{j+1}, \# S)$ belongs to $E(X, Y_1 \times \ldots \times Y_{j+1}, \bar{a}_0, \bar{b}_{10}, \ldots, \bar{b}_{j+1,0}, t_0)$ and satisfies conditions (a), …, (e).

Thus, we have proved by induction the following result:

PROPOSITION 3.1. *The player E wins locally in the game* $(X, Y_1 \times \ldots \times Y_n, \varphi^{\varkappa} \wedge \ldots \wedge \varphi^{\varkappa})$ (*n-times*).

3.2. Continuation

Let $\varkappa \geqslant 1$. To make the notation simpler we assume that $n = 1$. The procedure in the case of $n > 1$ is similar.

For $i \in \{0, 1, \ldots, \varkappa\}$, $t \in [t_0, t_0+T]$, $\bar{a} = (a, a^0, \ldots, a^\varkappa) \in X[t_0, \bar{a}_0; t]$, $\bar{b} = (b, b^0, \ldots, b^\varkappa) \in Y[t_0, \bar{b}_0; t]$, $u \in U_t$, $v \in V_t$, define

$$z_i(t, \bar{a}, \bar{b}, u, v) = x^i - y^i,$$

where $x = (\mathfrak{x}, x^0, \ldots, x^\varkappa) = X(t, \bar{a}, u)$ and $y = (\mathfrak{y}, y^0, \ldots, y^\varkappa) = Y(t, \bar{b}, v)$, and

$$\tilde{z}_i(t, \bar{a}, \bar{b})(s) = a^i - b^i + (s-t)(a^{i+1} - b^{i+1}) + \ldots + \frac{(s-t)^{\varkappa - i}}{(\varkappa - i)!}(a^\varkappa - b^\varkappa),$$

$$s \in [t, t_0 + T + 1].$$

Denote by Z_i the set of all functions $z_i(t, \bar{a}, \bar{b}, u, v)$ and $\tilde{z}_i(t, \bar{a}, \bar{b})$ for $t \in [t_0, t_0+T]$, $\bar{a} \in X[t_0, \bar{a}_0; t]$, $\bar{b} \in Y[t_0, \bar{b}_0; t]$, $u \in U_t$, $v \in V_t$ and $i = 0, 1, \ldots, \varkappa$.

There is a $K \geqslant L$ such that $\|z(s)\|$, $\|z'(s)\| \leqslant 2K$ for all $z \in Z_i$, $i = 0, 1, \ldots, \varkappa$, and $s \in Dz \cap [t_0, t_0 + T + 1]$.

Recall that $\varphi^i(\bar{a}, \bar{b}) = \max\{\text{dist}(a-b, M^*), \|a^i - b^i\|\}$ for $\bar{a} = (a, a^0, \ldots, a^\varkappa)$, $\bar{b} = (b, b^0, \ldots, b^\varkappa) \in \boldsymbol{R}^k \times W^{\varkappa+1}$ and $i = 0, 1, \ldots, \varkappa$.

Now, suppose that $i \in \{1, 2, \ldots, \varkappa\}$, $(e_0, \#e_0) \in E(X, Y, \bar{a}_0, \bar{b}_0, t_0)$ guarantees the result $d > 0$ in the game $(X, Y, \bar{a}_0, \bar{b}_0, t_0; \varphi^i)$ on the interval $[t_0, t_0+T]$ and satisfies the conditions

$$t_0 + T \in \#e_0(y) \quad \text{and} \quad \xi_{\#e_0(y)} - \xi \geqslant c > 0$$
$$\text{for } \xi \in \#e_0(y) \cap [t_0, t_0+T),\ y \in Y(t_0, \bar{b}_0, V_{t_0}).$$

Take

$$\varphi = \varphi^{i-1}, \qquad \eta = \min\{\varrho, \tfrac{1}{2}d\}, \qquad H = \min\{1, \eta/(4K)\}$$

and

$$M(t, y) = \{(\bar{a}, \bar{b}) \in (\boldsymbol{R}^k \times W^{\varkappa+1}) \times (\boldsymbol{R}^k \times W^{\varkappa+1}) \colon \text{dist}(a-b, M^*) \geqslant \eta$$
$$\text{or } \langle a^{i-1} - b^{i-1}, a^i - b^i \rangle \geqslant 0\}$$

for $t \in [t_0, t_0+T)$ and $y \in Y(t_0, \bar{b}_0, V_{t_0})$.

We are going to check Hypotheses (H1), (H2) and (H3). Hypothesis (H1) has been verified in Section 3.1.

Verification of Hypothesis (H2). Let $\hat{\varphi} = \eta$ and $\hat{\sigma}(t) = \frac{1}{2}\eta t$ for $t \geqslant 0$. Suppose $y \in Y(t_0, \bar{b}_0, V_{t_0})$, $\xi \in \#e_0(y) \cap [t_0, t_0+T)$, $t \in [\xi, \xi_{\#e_0(y)})$, $\bar{a} = (a, a^0, \ldots, a^\varkappa) \in X[t_0, \bar{a}_0; t]$, $\bar{b} \in (b, b^0, \ldots, b^\varkappa) \in Y[t_0, \bar{b}_0; t]$ satisfy the conditions

$$\|\bar{a} - X(t_0, \bar{a}_0, e_0(y))(t)\| \leqslant \eta, \quad (\bar{a}, \bar{b}) \in M(t, y) \quad \text{and} \quad \varphi(\bar{a}, \bar{b}) < \hat{\varphi}.$$

Let $x_0 = (\mathfrak{x}_0, x_0^0, \ldots, x_0^\varkappa) = X(t_0, \bar{a}_0, e_0(y))$, $x = (\mathfrak{x}, x^0, \ldots, x^\varkappa) = X(t, \bar{a}, e_0(y))$ and $\tilde{y} = (\tilde{\mathfrak{y}}, \tilde{y}^0, \ldots, \tilde{y}^\varkappa) \in Y(t, \bar{b}, V_t)$.

Clearly, $\langle a^{i-1} - b^{i-1}, a^i - b^i \rangle \geqslant 0$ because $(\bar{a}, \bar{b}) \in M(t, y)$ and $\operatorname{dist}(a-b, M^*) \leqslant \varphi(\bar{a}, \bar{b}) < \eta$. Moreover, $\|a^i - b^i\| \geqslant \|b^i - x_0^i(t)\| - \|x_0^i(t) - a^i\| \geqslant d - \eta \geqslant \eta$. Thus,

$$\begin{aligned}
\varphi(x(s), \tilde{y}(s)) &\geqslant \|x^{i-1}(s) - \tilde{y}^{i-1}(s)\| \\
&\geqslant \langle x^{i-1}(s) - \tilde{y}^{i-1}(s), a^i - b^i \rangle (\|a^i - b^i\|)^{-1} \\
&= \Big[\langle a^{i-1} - b^{i-1}, a^i - b^i \rangle + \int_t^s \langle x^i(\tau) - \tilde{y}^i(\tau), a^i - b^i \rangle \, d\tau\Big] (\|a^i - b^i\|)^{-1} \\
&\geqslant 0 + \int_t^s \langle a^i - b^i + x^i(\tau) - \tilde{y}^i(\tau) - (a^i - b^i), a^i - b^i \rangle (\|a^i - b^i\|)^{-1} \, d\tau \\
&\geqslant (\|a^i - b^i\| - L(s-t))(s-t) \geqslant (\eta - LH)(s-t) \geqslant \tfrac{1}{2}\eta(s-t) = \hat{\sigma}(s-t)
\end{aligned}$$

for $s \in [t, t+H]$.

Verification of Hypothesis (H3). For each $t \geqslant 0$ define

$$\sigma^*(t) = 3\gamma^2 \left(\sqrt{4K^2 - \gamma^2} + \sqrt{K^2 - \gamma^2}\right)^{-1} \frac{t^{2+\varkappa-i}}{(2+\varkappa-i)!}$$

and choose an arbitrary $h \in (0, H]$. Let $\hat{h} \in (0, \infty)$ be such that

$$\eta - 2K\hat{h} = (\eta + 2K\hat{h})\left(1 - (\gamma/(2K))^2\right)^{1/2}.$$

Now, take

$$h^* = \min\{h, \hat{h}, \eta^2/(24K^2)\} \quad \text{and} \quad \varphi^* = \min\{\eta, \eta^2 h^*/(6K)\}.$$

We need a few lemmas.

LEMMA 3.2. *Given* $t \in [t_0, t_0 + T]$, $\bar{a} = (a, a^0, \ldots, a^\varkappa) \in X[t_0, \bar{a}_0; t]$, $\bar{b} = (b, b^0, \ldots, b^\varkappa) \in Y[t_0, \bar{b}_0; t]$, $u \in U_t$, $v \in V_t$ *such that* $\|a^i - b^i\| \geqslant \eta$ *and* $\varphi(\bar{a}, \bar{b}) < \varphi^*$. *Then,*

$$\langle z(t+h^*), z'(t+h^*) \rangle \geqslant 0,$$

where z *denotes* $z_{i-1}(t, \bar{a}, \bar{b}, u, v)$ *or* $\tilde{z}_{i-1}(t, \bar{a}, \bar{b})$.

Proof. Observing that $z(t) = a^{i-1} - b^{i-1}$ and $z'(t) = a^i - b^i$, we get

$$\begin{aligned}
\langle z(s), z'(s) \rangle &= \Big\langle z(t) + (s-t)z'(t) + \int_t^s \int_t^{\bar{s}} z''(\tau)\, d\tau\, d\bar{s},\ z'(t) + \int_t^s z''(\tau)\, d\tau \Big\rangle \\
&= \langle z(t), z'(t) \rangle + \Big\langle z(t), \int_t^s z''(\tau)\, d\tau \Big\rangle + (s-t)\langle z'(t), z'(t) \rangle + \\
&\quad + (s-t)\Big\langle z'(t), \int_t^s z''(\tau)\, d\tau \Big\rangle + \Big\langle \int_t^s \int_t^{\bar{s}} z''(\tau)\, d\tau\, d\bar{s},\ z'(t) \Big\rangle +
\end{aligned}$$

$$+\left\langle \int_t^s \int_t^{\tilde{s}} z''(\tau)\,d\tau\,d\tilde{s},\ \int_t^s z''(\tau)\,d\tau \right\rangle$$

$$\geqslant -2K\varphi^* - 2K\varphi^*(s-t) + \eta^2(s-t) - 4K^2(s-t)^2 -$$

$$-2K^2(s-t)^2 - 2K^2(s-t)^3$$

$$\geqslant -2K\varphi^* + (\eta^2 - 2K\varphi^*)(s-t) - 8K^2(s-t)^2$$

$$\geqslant -2K\varphi^* + \tfrac{2}{3}\eta^2(s-t) - 8K^2(s-t)^2$$

$$\geqslant -2K\varphi^* + \left(\tfrac{2}{3}\eta^2 - 8K^2(s-t)\right)(s-t)$$

$$\geqslant -2K\varphi^* + \tfrac{1}{3}\eta^2(s-t), \qquad s \in [t, t+h^*].$$

Hence, $\langle z(t+h^*), z'(t+h^*)\rangle \geqslant -2K\varphi^* + \frac{1}{3}\eta^2 h^* \geqslant 0$.

LEMMA 3.3. *Suppose* $z \in Z_{i-1}$, $t \in [t_0, t_0+T+1]$, $[t, t_0+T+1] \subset Dz$ *and* $\|z(t)\| \geqslant \frac{1}{2}\eta$. *Then,*

$$|\langle z(s)-z(t), z'(t)\rangle| \geqslant \frac{\eta - 2K|s-t|}{\eta + 2K|s-t|}\|z(s)-z(t)\|\,\|z'(t)\|, \qquad s \in Dz.$$

Proof. Because of the formula

$$z(s)-z(t) = (s-t)z'(t) + \int_t^s \int_t^{\tilde{s}} z''(\tau)\,d\tau\,d\tilde{s}, \qquad s \in Dz,$$

we have

$$|\langle z(s)-z(t), z'(t)\rangle| \geqslant \|z'(t)\|^2|s-t| - K(s-t)^2\|z'(t)\|, \qquad s \in Dz,$$

and

$$\|z(s)-z(t)\|\,\|z'(t)\| \leqslant \|z'(t)\|^2|s-t| + K(s-t)^2\|z'(t)\|, \qquad s \in Dz.$$

Therefore,

$$|\langle z(s)-z(t), z'(t)\rangle| \geqslant \frac{\|z'(t)\| - K|s-t|}{\|z'(t)\| + K|s-t|}\left(\|z'(t)\|^2|s-t| + K(s-t)^2\|z'(t)\|\right)$$

$$\geqslant \frac{\frac{1}{2}\eta - K|s-t|}{\frac{1}{2}\eta + K|s-t|}\|z(s)-z(t)\|\,\|z'(t)\|, \qquad s \in Dz.$$

LEMMA 3.4. *Given* $t \in [t_0, t_0+T]$, $\bar{a} = (a, a^0, \ldots, a^\varkappa) \in X[t_0, \bar{a}_0; t]$, $\bar{b} = (b, b^0, \ldots, b^\varkappa) \in Y[t_0, \bar{b}_0; t]$, $\hat{u} \in U$, $v \in V_t$ *and* $\hat{a} \in W$, $\|\hat{a}\| = 1$, *such that* $\operatorname{dist}(a-b, M^*) \leqslant \varrho$ *and* $\hat{u} \in U^e(a, \hat{a})$ (see Definition 3.3). *Then*

$$\langle z(s)-\tilde{z}(s), \hat{a}\rangle \geqslant \frac{\gamma}{K}\|z(s)-\tilde{z}(s)\|, \qquad s \in [t, t+H],$$

where $z = z_{i-1}(t, \bar{a}, \bar{b}, \hat{u}, v)$ *and* $\tilde{z} = \tilde{z}_{i-1}(t, \bar{a}, \bar{b})$.

Proof. It is easy to see that

$$z(s)-\tilde{z}(s)$$
$$=\int_{t^*}^{s}\int_{t}^{s_{1+\varkappa-i}}\dots\int_{t}^{s}\left[\hat{\Pi}(\Delta^{\varkappa}f)(\mathfrak{x}(\tau),\hat{u})-\hat{\Pi}(\Delta^{\varkappa}g)(\mathfrak{y}(\tau),v(\tau))\right]d\tau\,ds_1\dots ds_{1+\varkappa-i}$$

for $s\in[t, t_0+T+1]$. Using (3.3) and observing that

$$\left\|\hat{\Pi}(\Delta^{\varkappa}f)(\mathfrak{x}(s),\hat{u})-\hat{\Pi}(\Delta^{\varkappa}g)(\mathfrak{y}(s),v(s))\right\| = \|z_{\varkappa}'(t,\bar{a},\bar{b},u,v)(s)\| \leqslant 2K$$

for almost all $s\in[t, t_0+T+1]$, we obtain

$$\text{(3.7)}\qquad \langle z(s)-\tilde{z}(s),\hat{a}\rangle \geqslant 2\gamma\frac{(s-t)^{2+\varkappa-i}}{(2+\varkappa-i)!}\quad\text{and}\quad \|z(s)-\tilde{z}(s)\| \leqslant 2K\frac{(s-t)^{2+\varkappa-i}}{(2+\varkappa-i)!}$$

for $s\in[t, t+H]$, which implies the required inequality.

LEMMA 3.5. *Given $\hat{a}$, $a^{\perp}$, $a_*\in W$ for which we have $\|\hat{a}\|=1$, $a^{\perp}\neq 0$, $\langle a^{\perp},\hat{a}\rangle=0$ and $\langle\hat{a},a_*\rangle=\|a_*\|$.*

Under these assumptions, if $\tilde{a}$, $a^*\in W$ are such that

$$\langle a^*,\hat{a}\rangle \geqslant \frac{\gamma}{K}\|a^*\|,\qquad |\langle\tilde{a}-a_*,a^{\perp}\rangle| \geqslant \left(1-(\gamma/(2K))^2\right)^{1/2}\|\tilde{a}-a_*\|\,\|a^{\perp}\|,$$

then

$$\|\tilde{a}+a^*\| \geqslant \tfrac{3}{2}\gamma\left(\sqrt{4K^2-\gamma^2}+\sqrt{K^2-\gamma^2}\right)^{-1}\|a^*\|.$$

Proof. We can prove this lemma by using simple trigonometric calculations.

Now we turn to the main task. Let $y\in Y(t_0,\bar{b}_0,V_{t_0})$, $\xi\in\#e_0(y)\cap\cap[t_0,t_0+T)$, $t\in[\xi,\xi_{\#e_0(y)})$, $\bar{a}=(a,a^0,\dots,a^{\varkappa})\in X[t_0,\bar{a}_0;t]$, $\bar{b}=(b,b^0,\dots,b^{\varkappa})\in Y[t_0,\bar{b}_0;t]$ satisfy the conditions

$$\left\|\bar{a}-X(t_0,\bar{a}_0,e_0(y))(t)\right\| \leqslant \eta,\qquad (\bar{a},\bar{b})\notin M(t,y)\qquad\text{and}\qquad \varphi(\bar{a},\bar{b})<\varphi^*.$$

Of course, $\|a^i-b^i\|\geqslant d-\eta\geqslant\eta$, $\operatorname{dist}(a-b,M^*)<\varphi^*\leqslant\eta$ and $\langle a^{i-1}-b^{i-1},a^i-b^i\rangle<0$.

It follows from Lemma 3.2 that there exists a $t^*\in[t,t+h^*]$ for which $\langle\tilde{z}(t^*),\tilde{z}'(t^*)\rangle=0$, where $\tilde{z}=\tilde{z}_{i-1}(t,\bar{a},\bar{b})$.

Take $\hat{a}\in W$, $\|\hat{a}\|=1$, such that $\langle\hat{a},\tilde{z}'(t^*)\rangle=0$ and $\langle\hat{a},\tilde{z}(t^*)\rangle=\|z(t^*)\|$. It is easy to see that $\|\tilde{z}'(t^*)\|\geqslant\frac{1}{2}\eta$.

Thus, by Lemma 3.3, the definition of $\hat{h}$ and the inequality $h^*\leqslant\hat{h}$, we have

$$\text{(3.8)}\qquad |\langle\tilde{z}(s)-\tilde{z}(t^*),\tilde{z}'(t^*)\rangle| \geqslant \sqrt{1-(\gamma/2K)^2}\,\|\tilde{z}(s)-\tilde{z}(t^*)\|\,\|\tilde{z}'(t^*)\|$$
$$\text{for } s\in[t,t^*+h^*]\cap[t,t_0+T+1].$$

Let $u^* \in U^e(a, \hat{a})$, $v \in V_t$ and $s \in [t, t+h^*]$ be fixed. According to Lemma 3.4 and formula (3.8), for $a^\perp = \tilde{z}'(t^*)$, $a_* = \tilde{z}(t^*)$, $a^* = z(s) - \tilde{z}(s)$, $\tilde{a} = \tilde{z}(s)$, where $z(s) = z_{i-1}(t, \bar{a}, \bar{b}, u^*, v)(s)$, the assumptions of Lemma 3.5 are satisfied. Hence, by (3.7),

$$\|z(s)\| = \|\tilde{z}(s) + (z(s) - \tilde{z}(s))\|$$
$$\geqslant 3\gamma^2 (\sqrt{4K^2 - \gamma^2} + \sqrt{K^2 - \gamma^2})^{-1} \frac{(s-t)^{2+\varkappa-i}}{(2+\varkappa-i)!} = \sigma^*(s-t).$$

It is clear from the above that

$$\varphi(X(t, \bar{a}, u^*)(s), Y(t, \bar{b}, v)(s)) \geqslant \sigma^*(s-t), \qquad s \in [t, t+h^*].$$

Furthermore, it follows from Lemma 3.2 that

$$(X(t, \bar{a}, u^*)(t+h^*), Y(t, \bar{b}, v)(t+h^*)) \in M(t+h^*, y).$$

The verification of Hypothesis (H3) is finished.

In the light of Proposition 3.1 and Lemma 2.1 we have just proved by induction the following theorem:

THEOREM 3.1. *The player E wins locally in the game* $(X, Y_1 \times \ldots \times Y_n, \varphi^0 \wedge \ldots \wedge \varphi^0)$ *(n-times).*

Remark 3.1. Note that $\varphi^0(\bar{a}, \bar{b}) = \operatorname{dist}(a-b, M^*)$ for all $\bar{a} = (a, a^0, \ldots, a^\varkappa)$, $\bar{b} = (b, b^0, \ldots, b^\varkappa) \in \mathbf{R}^k \times W^{\varkappa+1}$.

Remark 3.2. Theorem 3 of [6] does not follow from our Theorem 3.1. However, we have assumed $\dim W = 2$ instead of $\dim W_j \geqslant 3$, $j = 1, \ldots, v$, assumed there. Moreover, let us consider the following example:

EXAMPLE 3.1. Suppose $n = 2$, $k = 4$, $U = B^2(0, R_E)$, $V^j = B^2(0, R_P)$, $R_P < R_E$, $M^* = \{(a_1, a_2) \in \mathbf{R}^2 \times \mathbf{R}^2 \colon a_1 = 0\}$, $f(a_1, a_2, u) = (a_2, u)$, $g_j(a_1, a_2, v) = (a_2, v)$ for $a_1, a_2 \in \mathbf{R}^2$, $u \in U$, $v \in V^j$, $j = 1, 2, \ldots, n$.

The assumptions of Theorem 3.1 are satisfied with $\varkappa = 1$ whereas Condition 5 of [6] requires the inequality $R_P < R_E/\sqrt{2}$.

4. Evasion along each trajectory

Let X denote a control system over $(\mathbf{R}^k, U)$ and Y denote a control system over $(\mathbf{R}^l, V)$. The following definitions will be useful in this section:

DEFINITION 4.1. A strategy $(e, \#e) \in E(X, Y, a, b, t)$ satisfies the condition $(u, \varepsilon, c, d, \varphi)$ on the interval $[t, t+T]$ if $u \in U_t$, $\varepsilon, c, d \in (0, \infty)$, $\varphi \colon \mathbf{R}^k \times \mathbf{R}^l \to [0, \infty)$ is a payoff functional and if, for any $y \in Y(t, b, V_t)$,

$$t+T \in \#e(y), \qquad \xi_{\#e(y)} - \xi \geqslant c, \qquad \xi \in \#e(y) \cap [t, t+T),$$

$$\|X(t, a, e(y))(s) - X(t, a, u)(s)\| \leq \varepsilon, \quad s \in [t, t+T],$$
$$\varphi(X(t, a, e(y))(s), y(s)) \geq d, \quad s \in [t, t+T].$$

DEFINITION 4.2. The player E *wins locally along each trajectory* in the game (X, Y, φ) if for any $\varepsilon, r, R, T_1 \in (0, \infty)$ and any $T_2 \in [T_1, \infty)$ there exist $c, d > 0$ such that for arbitrary $t \in [0, \infty)$, $a \in B^k(0, R)$, $b \in B^l(0, R)$, with $\varphi(a, b) \geq r$, $T \in [T_1, T_2]$ and $u \in U_t$ one can find a strategy $(e, \#e) \in E(X, Y, a, b, t)$ satisfying the condition $(u, \varepsilon, c, d, \varphi)$ on the interval $[t, t+T]$.

Remark 4.1. If the player E wins locally along each trajectory in the game (X, Y, φ), then he wins locally (see Definition 1.7) in this game.

DEFINITION 4.3. Let Z denote a control system over $(\mathbf{R}^m, W)$. We will say that Z is *equicontinuous on bounded sets* if for any $R, T > 0$ the trajectories from $Z(0, a, W_0)$, $a \in B^m(0, R)$, are equicontinuous in the interval $[0, T]$.

PROPOSITION 4.1. *Suppose that the control systems X over $(\mathbf{R}^k, U)$ and Y_i over $(\mathbf{R}^{l_i}, V^i)$ are equicontinuous on bounded sets for $i = 1, 2, \ldots, n$. If the player E wins locally along each trajectory in the games (X, Y_i, φ_i), $i = 1, 2, \ldots, n$, then he also wins locally along each trajectory in the game $(X, Y_1 \times \ldots \times Y_n, \varphi_1 \wedge \ldots \wedge \varphi_n)$.*

Proof. Suppose $n = 2$. This assumptions involves no loss of generality. Choose arbitrary $\varepsilon, r, R, T_1 \in (0, \infty)$ and $T_2 \in [T_1, \infty)$.

For X, Y_1, φ_1 and for these constants take c, d in accordance with Definition 4.2. Fix $T \in [T_1, T_2]$, $t_0 \in [0, \infty)$, $a_0 \in B^k(0, R)$, $b_{i0} \in B^{l_i}(0, R)$ such that $\varphi_i(a_0, b_{i0}) \geq r$, $i = 1, 2$, and $u \in U_{t_0}$. Let $(e_1, \#e_1) \in E(X, Y_1, a_0, b_{10}, t_0)$ satisfy the condition $(u, \varepsilon/2, c, d, \varphi_1)$ on the interval $[t_0, t_0+T]$.

There exists an $R^* \geq R$ such that

$$X[0, a; t] \subset B^k(0, R^*) \quad \text{and} \quad Y_2[0, b_2; t] \subset B^{l_2}(0, R^*)$$

for any $t \in [0, T_2]$, $a \in B^k(0, R)$ and $b_2 \in B^{l_2}(0, R)$.

For $n = \min\{m \in \mathbf{N}: T_2 \leq mc\}$ define the sequence $\varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_n$ with the following properties:

$$\varepsilon_n \leq \varepsilon/4, \quad w_{\varphi_1}(2\varepsilon_n) \leq \tfrac{1}{2} d,$$

if $i \in \{1, 2, \ldots, n-1\}$, $t \in [t_0, t_0+T_2]$, $a, \bar{a} \in B^k(0, R^*)$ and $\|a - \bar{a}\| \leq 2\varepsilon_i$, then $\|X(t, a, \tilde{u})(s) - X(t, \bar{a}, \tilde{u})(s)\| \leq \varepsilon_{i+1}$ for $\tilde{u} \in U_t$ and $s \in [t, t_0+T_2]$.

Now, let us fix arbitrary $(y_1, y_2) \in (Y_1 \times Y_2)(t_0, b_{10}, b_{20}, V^1_{t_0} \times V^2_{t_0})$. Let $\{t_0, t_1, \ldots, t_m\} = \#e_1(y_1) \cap [t_0, t_0+T]$. Obviously, $m \leq n$ and $t_m = t_0 + T$.

We are going to define $e(y_1, y_2)$ and $\#e(y_1, y_2)$.

For $X, Y_2, \varphi_2, \varepsilon_1, r, R^*, c$ and T_2 choose c_1, d_1 in accordance with Definition 4.2, $(e^2_1, \#e^2_1) \in E(X, Y_2, a_0, b_{20}, t_0)$ satisfying the condition

$(e_1(y_1), \varepsilon_1, c_1, d_1, \varphi_2)$ on the interval $[t_0, t_1]$ and put

$$e(y_1, y_2)(s) = e_1^2(y_2)(s), \quad s \in [t_0, t_1),$$
$$\# e(y_1, y_2) \cap [t_0, t_1] = \# e_1^2(y_2) \cap [t_0, t_1].$$

Assume that c_j, d_j and $e(y_1, y_2)|[t_{j-1}, t_j)$, $\# e(y_1, y_2) \cap [t_{j-1}, t_j]$ have been defined for $j = 1, 2, \ldots, i$, where $i < m$, and take

$$e(y_1, y_2)(s) = e_{i+1}^2(y_2)(s), \quad s \in [t_i, t_{i+1}),$$
$$\# e(y_1, y_2) \cap [t_i, t_{i+1}] = \# e_{i+1}^2(y_2) \cap [t_i, t_{i+1}],$$

where $(e_{i+1}^2, \# e_{i+1}^2) \in E\big(X, Y_2, X(t_0, a_0, e(y_1, y_2))(t_i), y_2(t_i), t_i\big)$ satisfies the condition $\big(e_1(y_1)|_{[t_i, \infty)}, \varepsilon_{i+1}, c_{i+1}, d_{i+1}, \varphi_2\big)$ on the interval $[t_i, t_{i+1}]$, and c_{i+1}, d_{i+1} are chosen for X, Y_2, φ_2, ε_{i+1}, d_i, R^*, c and T_2 in accordance with Definition 4.2. We have defined $e(y_1, y_2)$ on $[t_0, t_0 + T)$ and $\# e(y_1, y_2) \cap [t_0, t_0 + T]$.

Now, for $s \geqslant t_0 + T$ define $e(y_1, y_2)(s) = u(s)$ and take $\# e(y_1, y_2) \cap \cap [t_0 + T, \infty) = \{t_0 + T, t_0 + 2T, \ldots\}$.

It is not difficult to verify that $(e, \# e) \in E(X, Y_1 \times Y_2, a_0, b_{10}, b_{20}, t_0)$.

Fix arbitrary $(y_1, y_2) \in (Y_1 \times Y_2)(t_0, b_{10}, b_{20}, V_{t_0}^1 \times V_{t_0}^2)$.

Clearly, $t_0 + T \in \# e(y_1, y_2)$ and $\xi_{\# e(y_1, y_2)} - \xi \geqslant \min\{c_1, \ldots, c_m\}$ for $\xi \in \# e(y_1, y_2) \cap [t_0, t_0 + T)$.

By the properties of the sequence $\varepsilon_1, \ldots, \varepsilon_n$, it can easily be shown by induction that

$$\big\|X(t_0, a_0, e(y_1, y_2))(t) - X(t_0, a_0, e_1(y_1))(t)\big\| \leqslant 2\varepsilon_i$$

for $t \in [t_0, t_i]$ and $i = 1, 2, \ldots, m$. This implies, by the inequalities $2\varepsilon_i \leqslant 2\varepsilon_m \leqslant 2\varepsilon_n \leqslant \varepsilon/2$, $i = 1, 2, \ldots, m$, that

$$\begin{aligned}
&\big\|X(t_0, a_0, e(y_1, y_2))(t) - X(t_0, a_0, u)(t)\big\| \\
&\qquad \leqslant \big\|X(t_0, a_0, e(y_1, y_2))(t) - X(t_0, a_0, e_1(y_1))(t)\big\| + \\
&\qquad\quad + \big\|X(t_0, a_0, e_1(y_1))(t) - X(t_0, a_0, u)(t)\big\| \\
&\qquad \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad t \in [t_0, t_0 + T].
\end{aligned}$$

Furthermore,

$$\varphi_1\big(X(t_0, a_0, e(y_1, y_2))(t), y_1(t)\big) \geqslant \tfrac{1}{2} d$$

and

$$\varphi_2\big(X(t_0, a_0, e(y_1, y_2))(t), y_2(t)\big) \geqslant \min\{d_1, \ldots, d_m\}$$

for $t \in [t_0, t_0 + T]$. The constants c_i and d_i, $i = 1, 2, \ldots, m$, could be chosen independently of t_0, a_0, b_{10}, b_{20} and T. Therefore the proof is finished.

4.1. Some general criteria

Let us fix $U \in \mathrm{comp}(R^{\hat{k}})$, $V \in \mathrm{comp}(R^{\hat{l}})$, continuous functions $f\colon R^2 \times U \to R^2$, $g\colon R^2 \times V \to R^2$ and a natural number $k \geqslant 1$. Suppose that the functions f and g satisfy the conditions

$$\|f(a, u) - f(b, u)\| \leqslant L^* \|a-b\|, \qquad \|g(a, v) - g(b, v)\| \leqslant L^* \|a-b\|,$$

$$(4.1) \qquad g(a, V) \subset \operatorname{Int} \operatorname{conv} f(a, U)$$

for $a, b \in R^2$, $u \in U$ and $v \in V$, and define functions F and G by the formulas

$$F(x, u) = \big(x^2, x^3, \ldots, x^k, f(x^1, u)\big), \qquad G(x, v) = \big(x^2, x^3, \ldots, x^k, g(x^1, v)\big),$$

where $x = (x^1, \ldots, x^k) \in R^{2k}$, $x^i \in R^2$, $i = 1, 2, \ldots, k$, $u \in U$ and $v \in V$.

For $t \in [0, \infty)$, $a = (a^1, \ldots, a^k) \in R^{2k}$, $b = (b^1, \ldots, b^k) \in R^{2k}$, where a^i, $b^i \in R^2$, $i = 1, 2, \ldots, k$, for $u \in U_t$ and $v \in V_t$ denote by $X(t, a, u)$ the solution of the differential equation

$$x' = F(x, u), \qquad x(t) = a,$$

defined on $[t, \infty)$, and by $Y(t, b, v)$ the solution of the equation

$$y' = G(y, v), \qquad y(t) = b,$$

defined also on $[t, \infty)$.

Definition 4.4. A strategy $(e, \#e) \in E(X, Y, a, b, t)$ is *c-extremal* on the interval $[t, t+T]$ if for any $y \in Y(t, b, V_t)$ the following conditions are fulfilled:

$$t+T \in \#e(y), \qquad c = \xi_{\#e(y)} - \xi \qquad \text{and} \qquad e(y)(s) = e(y)(\xi)$$
$$\text{for all } \xi \in \#e(y) \cap [t, t+T) \text{ and } s \in [\xi, \xi_{\#e(y)}),$$

for any $\xi \in \#e(y) \cap (t, t+T)$ there is an $\hat{a} \in R^2$, $\|\hat{a}\| = 1$, such that

$$\langle f(x^1(\xi), e(y)(\xi)), \hat{a}\rangle = \max_{u \in U} \langle f(x^1(\xi), u), \hat{a}\rangle,$$

where $x = (x^1, \ldots, x^k) = X\big(t, a, e(y)\big)$.

Now, let us fix arbitrary R, $T_2 \in (0, \infty)$. There are positive constants L, R^* such that

$$\|x(s) - a_0\| \leqslant R^*, \qquad \|y(s) - b_0\| \leqslant R^*$$

and

$$\|x(s) - x(t)\| \leqslant L|s-t|, \qquad \|y(s) - y(t)\| \leqslant L|s-t|$$

for all $x \in X(t_0, a_0, U_{t_0})$, $y \in Y(t_0, b_0, V_{t_0})$, a_0, $b_0 \in B^{2k}(0, R)$, $t_0 \in [0, \infty)$ and s, $t \in [t_0, t_0+T_2]$. Hence, according to (4.1), one can find $\varrho \in (0, \infty)$ and $\gamma \in (0, L]$ such that, if a, $\hat{a} \in \mathbf{R}^2$, $\|\hat{a}\| = 1$, $\|a\| \leq \mathbf{R}^*$, $\hat{u} \in U$ and $\langle f(a, \hat{u}), \hat{a} \rangle = \max_{u \in U} \langle f(a, u), \hat{a} \rangle$, then

$$\langle f(\tilde{a}, \hat{u}) - g(\tilde{b}, v), \hat{a} \rangle \geq 2\gamma,$$

for all $\tilde{a} \in B^2(a, 2\varrho)$ and $\tilde{b} \in B^2(\tilde{a}, 2\varrho)$.

Finally, take any $\eta \in (0, \varrho]$, $H = \min\{1, \varrho/(2L+\gamma)\}$ and

$$\varphi(a, b) = \max\{\|a^1 - b^1\|, \|a^k - b^k\|\}$$

for $a = (a^1, \ldots, a^k)$, $b = (b^1, \ldots, b^k) \in \mathbf{R}^{2k}$, a^i, $b^i \in \mathbf{R}^2$, $i = 1, \ldots, k$.

LEMMA 4.1. *If $t_0 \geq 0$, a_0, $b_0 \in B^{2k}(0, R)$, $T \in (0, T_2]$ and $(e_0, \# e_0) \in E(X, Y, a_0, b_0, t_0)$ is c-extremal on the interval $[t_0, t_0 + T]$ with $c \leq H$, then there exists a function M such that Hypotheses* (H1), (H2) *and* (H3) *are satisfied.*

Proof. The proof is very similar to (and even easier than) the proof of Lemma 3.1. For details see [30].

Let q: $\mathbf{R}^{2k} \times \mathbf{R}^{2k} \to U$ satisfy the conditions

$$\langle F(a, q(a, b)), b - a \rangle = \max_{u \in U} \langle F(a, u), b - a \rangle$$

for $a = (a^1, \ldots, a^k)$, $b = (b^1, \ldots, b^k) \in \mathbf{R}^{2k}$, and there is an $\hat{a} \in \mathbf{R}^2$, $\|\hat{a}\| = 1$, such that

$$\langle f(a^1, q(a, b)), \hat{a} \rangle = \max_{u \in U} \langle f(a^1, u), \hat{a} \rangle,$$

whenever $a^k = b^k$.

For $T \in (0, T_2]$ and $i \in N$, $i \geq 1$, define the constant function δ_i^T: $\mathbf{R}^{2k} \times \mathbf{R}^{2k} \to (0, \infty)$ by the formula

$$\delta_i^T(a, b) = T/i, \quad (a, b) \in \mathbf{R}^{2k} \times \mathbf{R}^{2k}.$$

Suppose $t_0 \geq 0$, a_0, $b_0 \in B^{2k}(0, R)$ and $(e, \# e) \in E(X, Y, a_0, b_0, t_0)$. For each $y \in Y(t_0, b_0, V_{t_0})$ define

$$e_i^{q,T}(y) = [q, \delta_i^T, a_0, a_0, t_0]\big(X(t_0, a_0, e(y))\big)$$

and $\# e_i^{q,T}(y)$ to be the set determined by the strategy $[q, \delta_i^T, a_0, a_0, t_0] \in E(X, X, a_0, a_0, t_0)$ and the trajectory $X(t_0, a_0, e(y))$.

LEMMA 4.2. *For any $\varepsilon > 0$ there is an $i_0 \in N$ such that, if $T \in (0, T_2]$, then $(e_i^{q,T}, \# e_i^{q,T}) \in E(X, Y, a_0, b_0, t_0)$ is T/i-extremal on the interval $[t_0, t_0 + T]$ and*

$$\|X(t_0, a_0, e_i^{q,T}(y))(t) - X(t_0, a_0, e(y))(t)\| \leq \varepsilon$$

for all $(e, \# e) \in E(X, Y, a_0, b_0, t_0)$, $i \geq i_0$ and $t \in [t_0, t_0 + T]$.

Proof. The first part of this statement is obvious. The second one follows from Lemmas 1 and 2 of [1].

LEMMA 4.3. *For any* ε, $r>0$ *and* $T_1\in(0, T_2]$ *there are* c, $d>0$ *such that for arbitrary* $t_0\geqslant 0$, $a_0, b_0\in B^{2k}(0, R)$, *with* $\varphi(a_0, b_0)\geqslant r$, $T\in[T_1, T_2]$ *and* $u\in U_t$ *one can find a strategy* $(e, \# e)\in E(X, Y, a_0, b_0, t_0)$ *satisfying the condition* $(u, \varepsilon, c, d, \varphi)$ *on the interval* $[t_0, t_0+T]$.

Proof. Choose arbitrary ε, $r>0$ and $T_1\in(0, T_2]$. Let $t_0\geqslant 0$, a_0, $b_0\in B^{2k}(0, R)$ with $\varphi(a_0, b_0)\geqslant r$, $T\in[T_1, T_2]$ and $u\in U_t$ be fixed.

Denote by $(\hat{e}, \#\hat{e})$ the strategy from $E(X, Y, a_0, b_0, t_0)$ defined in the following way:

$$\hat{e}(y)=u, \qquad \#\hat{e}(y)=\{t_0, t_0+T, t_0+2T, \ldots\} \qquad \text{for } y\in Y(t_0, b_0, V_{t_0}).$$

Let $i_0\in \boldsymbol{N}$ be taken for R, T_2 and $\varepsilon/2$ in accordance with Lemma 4.2. Without loss of generality we may assume that $T/i_0\leqslant H$.

Now, take $(e_0, \# e_0)=(\hat{e}^{q,T}_{i_0}, \#\hat{e}^{q,T}_{i_0})$. The strategy $(e_0, \# e_0)$ satisfies the assumptions of Lemma 4.1, and so by the condition $\varphi(a_0, b_0)\geqslant r$, we have a $d>0$ and $(e, \# e)\in E(X, Y, a_0, b_0, t_0)$ for which

$$\left\|X\left(t_0, a_0, e(y)\right)(t)-X\left(t_0, a_0, e_0(y)\right)(t)\right\|\leqslant \varepsilon/2$$

and

$$\varphi\left(X\left(t_0, a_0, e(y)\right)(t), y(t)\right)\geqslant d$$

for $y\in Y(t_0, b_0, V_{t_0})$ and $t\in[t_0, t_0+T]$. Moreover, it is not difficult to check that the constant d may be found independently of t_0, a_0, b_0, T and u. Since T/i_0 is also independent of t_0, a_0, b_0 and u, the proof is complete.

Remark 4.2. If $k=1$, then the player E wins locally along each trajectory in the game (X, Y, φ) and, by Proposition 4.1, he also wins in the game $(X, Y\times\ldots\times Y, \varphi\wedge\ldots\wedge\varphi)$ (n-times). This case has been partially investigated in [37].

Further, using analogous methods to those used in Section 3.2, one can get the following result:

THEOREM 4.1. *The player* E *wins locally along each trajectory in the game* (X, Y, ψ), *where*

$$\psi(a, b)=\|a^1-b^1\| \quad \textit{for } a=(a^1, \ldots, a^k),\ b=(b^1, \ldots, b^k)\in \boldsymbol{R}^{2k}.$$

Remark 4.3. In view of Proposition 4.1, using the above theorem, we can obtain the same result for an evasion game with many pursuers.

Remark 4.4. The case of $k=2$, U, $V^i\in\operatorname{comp}(\boldsymbol{R}^2)$, $V^i\subset\operatorname{Int}\operatorname{conv} U$, $f(x, u)=u$, $g_i(x, v)=v$, $x\in\boldsymbol{R}^2$, $u\in U$, $v\in V^i$, $i=1, 2, \ldots, n$, has been investigated in [36].

To make sure that assumption (4.1) (just as (3.1)) is essential, we will present the following example:

EXAMPLE 4.1. Suppose $k = 1$, $U = B^2(0, R_E)$, $V = B^2(0, R_P)$, $f(a, u) = u$, $g(a, v) = v$, $\varphi(a, b) = \|a-b\|$ for $a, b \in \mathbf{R}^2$, $u \in U$ and $v \in V$.

If $R_P < R_E$, then the player E wins locally along each trajectory in the game (X, Y, φ), thus he also wins in a game with many pursuers $(X, Y \times \ldots \times Y, \varphi \wedge \ldots \wedge \varphi)$ (n-times).

If $R_P = R_E$ and $n \geqslant 3$, then there are situations $(0, a, b_1, \ldots, b_n)$ in which the pursuer P wins, see [26].

I was told by prof. A. Bielecki that this pursuit problem was solved by J. Perkal in 1950's but unfortunately it was not published.

EXAMPLE 4.2 (Lions versus Man in the circle). Let X, Y be defined as in the previous example and let $R > 0$ be fixed. A little refinement results of this section and the method of Example 1 of [28] yield the following:

For any $n \in N$ there exist d, $T > 0$ such that for arbitrary $a \in B^2(0, R)$ and $b_i \in \mathbf{R}^2$, with $a \neq b_i$, $i = 1, 2, \ldots, n$, there exists a strategy $(e, \#e) \in E(X, Y \times \ldots \times Y, a, b_1, \ldots, b_n, 0)$ such that, if $y_i \in Y(0, b_i, V_0)$, $i = 1, 2, \ldots, n$ and $x = X(0, a, e(y_1, \ldots, y_n))$, then

$$\begin{aligned} x(t) &\neq y_i(t), && t \in [0, T], \\ \|x(t) - y_i(t)\| &\geqslant d, && t \in [T, \infty), \\ \|x(t)\| &\leqslant R, && t \in [0, \infty) \end{aligned}$$

for $i = 1, 2, \ldots, n$.

The case of $n = 1$ has been investigated in detail in [8] and [18].

4.2. Some special examples

Suppose that X is a control system over $(\mathbf{R}^k, U)$, Y is a control system over $(\mathbf{R}^l, V)$, $\varphi\colon \mathbf{R}^k \times \mathbf{R}^l \to [0, \infty)$ is a payoff functional and $\hat{M} \subset \mathbf{R}^k \times \mathbf{R}^l$.

Let X and Y be equicontinuous on bounded sets and, for any $T, R > 0$, let there exist H, $\hat{\varphi} > 0$ and strictly increasing functions $\hat{\sigma}$, σ^*: $[0, H] \to [0, \infty)$ such that the following hypotheses are satisfied:

($\hat{\text{H}}$1) For any $\eta > 0$ there is a $\zeta > 0$ such that, if $a \in B^k(0, R)$, $u, \tilde{u} \in U_0$ and $\mu(\{t \in [0, T]\colon u(t) \neq \tilde{u}(t)\}) \leqslant \zeta$, then $\|X(0, a, u)(t) - X(0, a, \tilde{u})(t)\| \leqslant \eta$, $t \in [0, T]$.

($\hat{\text{H}}$2) If $a \in B^k(0, R)$, $b \in B^l(0, R)$, $(a, b) \in \hat{M}$ and $\varphi(a, b) < \hat{\varphi}$, then

$$\varphi(x(t), y(t)) \geqslant \hat{\sigma}(t), \quad x \in X(0, a, U_0), \quad y \in Y(0, b, V_0), \quad t \in [0, H].$$

($\hat{\text{H}}$3) For any $h \in (0, H]$ there exist $h^* \in (0, h]$ and $\varphi^* > 0$ such that, if $a \in B^k(0, R)$, $b \in B^l(0, R)$, $(a, b) \notin \hat{M}$ and $\varphi(a, b) < \varphi^*$, then one can find $u^* \in U_0$ such that

$\varphi(X(0, a, u^*)(t), y(t)) \geqslant \sigma^*(t)$, $t \in [0, h^*]$, $(X(0, a, u^*)(h^*), y(h^*)) \in \hat{M}$ for each $y \in Y(0, b, V_0)$.

PROPOSITION 4.2. *If Hypotheses* ($\hat{\text{H}}1$), ($\hat{\text{H}}2$) *and* ($\hat{\text{H}}3$) *hold true, then the player E wins locally along each trajectory in the game* (X, Y, φ).

Proof. Let us fix arbitrary ε, r, R, $T_1 > 0$ and $T_2 \geqslant T_1$. There is an $R^* \geqslant R$ such that

$$X[0, a; t] \subset B^k(0, R^*) \quad \text{and} \quad Y[0, b; t] \subset B^l(0, R^*)$$

for $a \in B^k(0, R)$, $b \in B^l(0, R)$ and $t \in [0, T_2]$.

Choose $t_0 \geqslant 0$, $a_0 \in B^k(0, R)$, $b_0 \in B^l(0, R)$, with $\varphi(a_0, b_0) \geqslant r$, $T \in [T_1, T_2]$ and $u \in U_t$. For R^*, T_2 take H, $\hat{\varphi} > 0$, functions $\hat{\sigma}$, σ^*: $[0, H] \to [0, \infty)$ and, additionally, for $\eta = \varepsilon$ take $\zeta > 0$ in accordance with ($\dot{\text{H}}1$) and define

$$e_0(y) = u, \qquad \# e_0(y) = \{t_0, t_0 + T, t_0 + 2T, \ldots\},$$
$$M(t, y) = \hat{M} \qquad \text{for } y \in Y(t_0, b_0, V_{t_0}),\ t \in [t_0, t_0 + T).$$

To end the proof it is enaugh to observe that the assumptions of Lemma 2.1 are satisfied.

4.2.1. The homicidal Chauffeur game. Let $U = \{u \in \boldsymbol{R}^2\colon \|u\| \leqslant K\}$ and $V \in \text{comp}(\boldsymbol{R}^{\hat{l}})$. Suppose that g_1: $\boldsymbol{R}^2 \times \boldsymbol{R}^l \to \boldsymbol{R}^2$, g_2: $\boldsymbol{R}^2 \times \boldsymbol{R}^l \times V \to \boldsymbol{R}^l$ are continuous and Lipschitzian with respect to the first two variables, g_1 has a continuous derivative and

(f) for each $R > 0$ there exists a $\gamma \in (0, \infty)$ such that $\gamma \leqslant \|g_1(b_1, b_2)\|$ when $b_1 \in B^2(0, R)$ and $b_2 \in B^l(0, R)$.

For $t \geqslant 0$, a, $b_1 \in \boldsymbol{R}^2$, $b_2 \in \boldsymbol{R}^l$, $u \in U_t$ and $v \in V_t$ denote by $X(t, a, u)$ the solution of the differential equation

$$x' = u, \qquad x(t) = a,$$

defined on $[t, \infty)$, and by $Y(t, b_1, b_2)$ the solution (y_1, y_2) of the equation

$$y_1' = g_1(y_1, y_2), \qquad y_1(t) = b_1, \qquad y_2' = g_2(y_1, y_2, v), \qquad y_2(t) = b_2,$$

defined also on $[t, \infty)$. Finally, let

$$\varphi(a, b_1, b_2) = \|a - b_1\|$$

for a, $b_1 \in \boldsymbol{R}^2$, $b_2 \in \boldsymbol{R}^l$.

We are going to prove that E wins locally along each trajectory in the game (X, Y, φ).

Without loss of generality we may assume that $K < \gamma$.

Define $\hat{M} = \{(a, b_1, b_2) \in \boldsymbol{R}^2 \times \boldsymbol{R}^2 \times \boldsymbol{R}^l\colon \langle a - b_1, g_1(b_1, b_2)\rangle \leqslant 0\}$.

Hypothesis ($\hat{\mathrm{H}}$1) is easy to verify.

Verification of Hypothesis ($\hat{\mathrm{H}}$2). There exists an $R^* \geq R$ such that $Y[0, b_1, b_2; t] \subset B^{2+l}(0, R^*)$ for all $b_1 \in B^2(0, R)$, $b_2 \in B^l(0, R)$ and $t \in [0, T]$. Let $\hat{\alpha}$, $\hat{R} > 0$ be such that

$$\|g_1(b_1, b_2)\| \leq \hat{R}, \quad \|g_2(b_1, b_2, v)\| \leq \hat{R},$$

$$\left\|\frac{\partial g_1}{\partial b_1}(b_1, b_2)\right\| \leq \hat{\alpha}, \quad \left\|\frac{\partial g_1}{\partial b_2}(b_1, b_2)\right\| \leq \hat{\alpha}$$

for $b_1 \in B^2(0, R^*)$, $b_2 \in B^l(0, R^*)$ and $v \in V$. Let

$$\alpha \in \hat{\alpha}\hat{R}, \quad H = (\gamma - K)/(2\alpha), \quad \hat{\sigma}(t) = (\gamma - K)t - \alpha t^2, \quad t \in [0, H].$$

Note that for $b_1 \in B^2(0, R)$, $b_2 \in B^l(0, R)$ and $v \in V$, if $(y_1, y_2) = Y(0, b_1, b_2, v)$, then

$$y_1''(s) = \frac{\partial g_1}{\partial b_1}(y_1(s), y_2(s))g_1(y_1(s), y_2(s)) + \frac{\partial g_1}{\partial b_2}(y_1(s), y_2(s))g_2(y_1(s), y_2(s), v(s))$$

for almost all $s \geq 0$, which implies that $\|y''(s)\| \leq 2\alpha$ for almost all $s \in [0, T]$. From the above and from the formula

$$y_1(s) = y_1(0) + sy_1'(0) + \int_0^s \int_0^t y_1''(\tau)\, d\tau\, dt$$

for

$$y_0(s) = \int_0^s \int_0^t y_1''(\tau)\, d\tau\, dt,$$

we have

$$\|y_0(s)\| \leq \alpha s^2, \quad s \in [0, T],$$

and

$$y_1(s) = y_1(0) + sy_1'(0) + y_0(s), \quad s \in [0, \infty).$$

Now, assume that a, $b_1 \in B^2(0, R)$, $b_2 \in B^l(0, R)$, $u \in U_0$, $v \in V_0$, $(a, b_1, b_2) \in \hat{M}$ and $x = X(0, a, u)$, $(y_1, y_2) = Y(0, b_1, b_2, v)$.

Taking into account the inequality $\langle b_1 - a, g_1(b_1, b_2)\rangle \geq 0$, we obtain

$$\begin{aligned}
\|y_1(s) - x(s)\| &\geq \langle y_1(s) - x(s), g_1(b_1, b_2)\rangle/(\|g_1(b_1, b_2)\|)^{-1} \\
&= \langle y_1(0) + sy_1'(0) + y_0(s) - a - \int_0^s u(\tau)\, d\tau, g_1(b_1, b_2)\rangle/(\|g_1(b_1, b_2)\|)^{-1} \\
&\geq \langle b_1 - a, g_1(b_1, b_2)\rangle/(\|g_1(b_1, b_2)\|)^{-1} + \\
&\quad + s\langle y_1'(0), g_1(b_1, b_2)\rangle/(\|g_1(b_1, b_2)\|)^{-1} - \alpha s^2 - Ks \\
&\geq \gamma s - Ks - \alpha s^2 = \hat{\sigma}(s), \quad s \in [0, H].
\end{aligned}$$

Verification of Hypothesis ($\hat{\text{H}}$3). For an arbitrary $h \in (0, H]$ take

$$h^* = \min\{h, K/(2\alpha), \gamma^2/(10\alpha\hat{R}), \qquad \varphi^* = \gamma^2 h^*/\hat{R} - 5\alpha(h^*)^2,$$
$$\sigma^*(t) = Kt - \alpha t^2, \qquad t \in [0, H].$$

Assume that a, $b_1 \in B^2(0, R)$, $b_2 \in B^l(0, R)$, $(a, b_1, b_2) \notin \hat{M}$, $\|a - b_1\| < \varphi^*$, $u \in U_0$, $v \in V_0$, $x^* = X(0, a, u^*)$, where $u^* \in U$, $\|u^*\| = K$, $\langle u^*, g_1(b_1, b_2)\rangle = 0$, $\langle a - b_1, u^*\rangle \geqslant 0$, and $(y_1, y_2) = Y(0, b_1, b_2, v)$. Let y_0 be taken for y_1, as in the verification of ($\hat{\text{H}}$2),

$$\begin{aligned}\|y_1(s) - x^*(s)\| &\geqslant \langle x^*(s) - y_1(s), u^*/K\rangle \\ &= \langle a - b_1, u^*/K\rangle + s\langle u^* - y_1'(0), u^*/K\rangle - \langle y_0(s), u^*/K\rangle \\ &\geqslant sK - \alpha s^2 = \sigma^*(s), \qquad s \in [0, H].\end{aligned}$$

Moreover, in view of the formula

$$\|y_1'(s) - y_1'(0)\| \leqslant 2\alpha s$$

for almost all $s \in [0, T]$, we have

$$\begin{aligned}&\langle x^*(s) - y_1(s), g_1(y_1(s), y_2(s))\rangle = \langle x^*(s) - y_1(s), y_1'(s)\rangle \\ &= \langle a + su^* - b_1 - sy_1'(0) - y_0(s), y_1'(s)\rangle \\ &= \langle a - b_1, y_1'(s)\rangle + s\langle u^* - y_1'(0), y_1'(s)\rangle - \langle y_0(s), y_1'(s)\rangle \\ &\leqslant \|a - b_1\|\hat{R} + s\langle u^* - y_1'(0), y_1'(0) + y_1'(s) - y_1'(0)\rangle + \alpha\hat{R}s^2 \\ &\leqslant \|a - b_1\|\hat{R} + s[\langle u^*, y_1'(s) - y_1'(0)\rangle - \gamma^2 - \langle y_1'(0), y_1'(s) - y_1'(0)\rangle] + \alpha\hat{R}s^2 \\ &\leqslant \|a - b_1\|\hat{R} + s(2K\alpha s - \gamma^2 + 2\hat{R}\alpha s) + \alpha\hat{R}s^2 \leqslant \|a - b_1\|\hat{R} + s(4\hat{R}\alpha s - \gamma^2) + \hat{R}\alpha s^2 \\ &= \|a - b_1\|\hat{R} + 5\hat{R}\alpha s^2 - \gamma^2 s, \qquad s \in [0, H].\end{aligned}$$

Therefore $(x^*(h^*), y_1(h^*), y_2(h^*)) \in \hat{M}$, because of the inequality

$$\|a - b_1\|\hat{R} + 5\alpha\hat{R}(h^*)^2 - \gamma^2 h^* \leqslant \varphi^*\hat{R} + 5\alpha\hat{R}(h^*)^2 - \gamma^2 h^* \leqslant 0.$$

EXAMPLE 4.3. The homicidal Chauffeur game, see [14]. Let $\hat{l} = l = 1$, $V = [-\hat{R}, \hat{R}]$, where $\hat{R} > 0$, and

$$g_1(b_1, b_2) = \gamma(\cos b_2, \sin b_2), \qquad g_2(b_1, b_2, v) = v$$

for $b_1 \in \boldsymbol{R}^2$, $b_2 \in \boldsymbol{R}$, $v \in V$.

Obviously, our initial assumptions are satisfied. Thus, the player E wins locally along each trajectory in the homicidal Chauffeur game. By Proposition 4.1, he also wins in a game with several pursuers.

4.2.2. The game of $1 + n$ cars. Suppose $U = [-R_E, R_E]$, $V = [-R_P, R_P]$ and v_E, $v_P > 0$. For $t \geqslant 0$, (a_1, a_2), $(b_1, b_2) \in \boldsymbol{R}^2$, a_3, $b_3 \in \boldsymbol{R}$, $u \in U_t$ and $v \in V_t$

denote by $(x_1, x_2, x_3) = X(t, a_1, a_2, a_3, u)$ and by $(y_1, y_2, y_3) = Y(t, b_1, b_2, b_3, v)$ the solutions of the relevant differential equations

$$\begin{aligned} x_1' &= v_E \cos x_3, & x_1(t) &= a_1, & y_1' &= v_P \cos y_3, & y_1(t) &= b_1, \\ x_2' &= v_E \sin x_3, & x_2(t) &= a_2, & y_2' &= v_P \sin y_3, & y_2(t) &= b_2, \\ x_3' &= u, & x_3(t) &= a_3, & y_3' &= v, & y_3(t) &= b_3, \end{aligned}$$

and take

$$\varphi(a_1, a_2, a_3, b_1, b_2, b_3) = \left((a_1-b_1)^2+(a_2-b_2)^2\right)^{1/2}.$$

These formulas describe the so-called *game of two cars*, see [14].

Suppose $v_E > v_P$, $v_E R_E \geqslant v_P R_P$ and define

$$\hat{M} = \{(a, b) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3:\ \langle (a_1, a_2)-(b_1, b_2), (\cos a_3, \sin a_3)\rangle \geqslant 0\},$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

Using suitable results of [29], one can easily show that Hypotheses ($\hat{\text{H}}1$), ($\hat{\text{H}}2$) and ($\hat{\text{H}}3$) are satisfied. Thus, the player E wins locally along each trajectory in the above game. By Proposition 4.1, he also wins in a game with many pursuers.

There are numerous papers concerning the game of two cars. For instance, see [2], [3], [14], [34].

5. Differential games of many evaders and pursuers

5.1. General example

Let us fix $n \in N$, $n \geqslant 1$, $u \in \text{comp}(\boldsymbol{R}^2)$, $V^i \in \text{comp}(\boldsymbol{R}^2)$, functions f_i: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$, $i = 1, 2, \ldots, n$, and natural numbers $k_1, k_2, \ldots, k_n \geqslant 1$. Suppose that $\|f_i(a)-f_i(b)\| \leqslant L^* \|a-b\|$ and

$$V^i \subset \text{Int conv } U \tag{5.1}$$

for all a, $b \in \boldsymbol{R}^2$ and $i = 1, 2, \ldots, n$.

For $i = 1, 2, \ldots, n$, $x_i = (x_i^1, \ldots, x_i^{k_i}) \in \boldsymbol{R}^{2k_i}$, where $x^j \in \boldsymbol{R}^2$, $j = 1, 2, \ldots, k_i$, $u \in U$ and $v \in V^i$ define

$$F_i(x_i, u) = \left(x_i^2, x_i^3, \ldots, x_i^{k_i}, f_i(x_i^1)+u\right),$$

$$G_i(x_i, v) = \left(x_i^2, x_i^3, \ldots, x_i^{k_i}, f_i(x_i^1)+v\right),$$

and define the control systems X_i, Y_i as in Section 4.1.

DEFINITION 5.1. Suppose that X is a control system over $(\boldsymbol{R}^k, U)$, $(e, \# e) \in E(X, Y, a, b, t)$ and $T > 0$. The strategy $(e, \# e)$ is *extremal on the interval* $[t, t+T]$ if $e(y)(t) = e(y)(\xi) \in U^e$ for $y \in Y(t, b, V_t)$ and

$\xi \in \#e(y) \cap [t, t+T)$, where U^e denotes the set of all extremal points of conv U.

From this point on we follow Section 3. Let us fix arbitrary $t_0 \geqslant 0$, $T > 0$, $a_{i0} = (a_{i0}^1, \ldots, a_{i0}^{k_i}) \in \boldsymbol{R}^{2k_i}$, $b_{i0} = (b_{i0}^1, \ldots, b_{i0}^{k_i}) \in \boldsymbol{R}^{2k_i}$ such that $a_{i0}^1 \neq b_{i0}^1$, $i = 1, 2, \ldots, n$.

There are constants L, $R > 0$ such that

$$\|x(s) - a_{i0}\| \leqslant R, \qquad \|y(s) - b_{i0}\| \leqslant R,$$
$$\|x(s) - x(t)\| \leqslant L|s-t|, \qquad \|y(s) - y(t)\| \leqslant L|s-t|$$

for $x \in X_i(t_0, a_{i0}, U_{t_0})$, $y \in Y_i(t_0, b_{i0}, V_{t_0}^i)$, s, $t \in [t_0, t_0 + T]$, $i = 1, 2, \ldots, n$.

It is easy to see that there exist constants $\varrho \in (0, \infty)$ and $\gamma \in (0, L]$ such that, if $\hat{a} \in \boldsymbol{R}^2$, $\|\hat{a}\| = 1$, $\hat{u} \in U$ and $\langle \hat{u}, \hat{a} \rangle = \max\{\langle u, \hat{a} \rangle : u \in U\}$, then

$$\langle f_i(a) + \hat{u} - f_i(b) - v, \hat{a} \rangle \geqslant 2\gamma$$

for $a \in B^2(0, R)$, $b \in B^2(a, 2\varrho)$, $v \in V^i$, $i = 1, 2, \ldots, n$.

Take arbitrary $\eta \in (0, \varrho]$, $H = \min\{1, \varrho/(2L+\gamma)\}$, and define

$$\varphi_i((a_i^1, \ldots, a_i^{k_i}), (b_i^1, \ldots, b_i^{k_i})) = \max\{\|a_i^1 - b_i^1\|, \|a_i^{k_i} - b_i^{k_i}\|\}$$

for $(a_i^1, \ldots, a_i^{k_i})$, $(b_i^1, \ldots, b_i^{k_i}) \in \boldsymbol{R}^{2k_i}$.

Now, we can prove by induction, as Lemma 3.1, that E wins in the game $(X_1 \times \ldots \times X_n, Y_1 \times \ldots, Y_n, a_{10}, \ldots, a_{n0}, b_{10}, \ldots, b_{n0}, t_0; \varphi)$ on the interval $[t_0, t_0 + T]$, where

$$\varphi(a_1^1, \ldots, a_1^{k_1}, \ldots, a_n^1, \ldots, a_n^{k_n}, b_1^1, \ldots, b_1^{k_1}, \ldots, b_n^1, \ldots, b_n^{k_n})$$
$$= \min\{\varphi_i(a_i^1, \ldots, a_i^{k_i}, b_i^1, \ldots, b_i^{k_i}) : i = 1, 2, \ldots, n\}.$$

(At every step we will obtain extremal strategies on the interval $[t_0, t_0 + T]$.)

Continuing the procedure from Section 3.2, we can show that E wins in the game $(X_1 \times \ldots \times X_n, Y_1 \times \ldots \times Y_n, a_{10}, \ldots, a_{n0}, b_{10}, \ldots, b_{n0}; \psi)$ on the interval $[t_0, t_0 + T]$, where

$$\psi(a_1^1, \ldots, b_n^{k_n}) = \min\{\|a_i^1 - b_i^1\| : i = 1, 2, \ldots, n\}.$$

In the next section we will use a slightly different approach to a certain special game.

5.2. A squadron of ships amidst iceberges

First, in accordance with a commonly accepted practice in game theory, we will present a certain story:

"A squadron of ships, moving in a given formation, meets a group of iceberges on its way. The task of the squadron is to pass through the dangerous area without changing the formation".

We will treat an iceberg as a pursuer since its behaviour is dificult to foresee, see also [34]. Thus, the above task can be put in the following way:

The evaders $E_1, \ldots, E_n$ are moving on the plane. The task of each evader E_i is to avoid meeting the pursuers P_j, $j = 1, 2, \ldots, m$. However, we also require the trajectories of all the evaders to be in line exact to a parallel shift.

Now, we are going to give a formal description of this problem. Let U, V and X, Y be the same as in Section 4.2.2 and let $X_i = X$, $Y_j^i = Y$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Fix $a^i = (a_1^i, a_2^i)$, $b^j = (b_1^j, b_2^j) \in \boldsymbol{R}^2$, $a_3^i, b_3^j \in \boldsymbol{R}$ such that $a^i \neq b^j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $a_3^1 = a_3^2 = \ldots = a_3^n$.

As we know, see Section 4.2.2, Remarks 4.1 and 1.3, there exists a strategy $(e, \# e) \in E(X_1, Y_1^1 \times \ldots \times Y_1^n \times \ldots \times Y_m^1 \times \ldots \times Y_m^n, (a^1, a_3^1), (b^1 + a^1 - a^1, b_3^1), \ldots, (b^1 + a^1 - a^n, b_3^1), \ldots, (b^m + a^1 - a^1, b_3^m), \ldots, (b^m + a^1 - a^n, b_3^m), 0)$ such that for any $(y_1^1, \ldots, y_m^n) \in (Y_1^1 \times \ldots \times Y_m^n)(0, (b^1 + a^1 - a^1, b_3^1), \ldots, (b^m + a^1 - a^n, b_3^m), V_0 \times \ldots \times V_0)$, if

$$x = (x_1, x_2, x_3) = X_1\big(0, (a^1, a_3^1), e(y_1^1, \ldots, y_m^n)\big),$$

then

$$\big(x_1(t), x_2(t)\big) \neq \big((y_j^i)_1(t), (y_j^i)_2(t)\big)$$

for $t \geqslant 0$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. It follows from the above that, if

$$x^i(t) = x(t) + (a_1^i - a_1^1, a_2^i - a_2^1, 0) \quad \text{and} \quad \tilde{y}_j^i(t) = y_j^i(t) + (a_1^i - a_1^1, a_2^i - a_2^1, 0),$$

then

$$\big(x_1^i(t), x_2^i(t)\big) \neq \big((\tilde{y}_j^i)_1(t), (\tilde{y}_j^i)_2(t)\big), \qquad t \geqslant 0,$$

and

$$x^i \in X_i\big(0, (a^i, a_3^i), U_0\big), \qquad \tilde{y}_j^i \in Y_j\big(0, (b^j, b_3^j), V_0\big)$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

It is now easy to see that the task considered above can be realized.

6. On evasion games with incomplete information

We will consider here a game in which the evader can only approximately determine the position of the pursuer. It may be due to inaccuracy of measurement or to delay of information. Suppose that the evader has a strategy allowing him to avoid meeting the pursuer in a game without any perturbations. It seems natural to ask when such a strategy would be efficient in the case where some perturbations appear. Similar problems were also considered in [4], [13].

Since the accuracy of estimation of the pursuer's position may depend on the relative positions of the two players, the following description of this situation seems natural.

Let X be a control system over $(\boldsymbol{R}^k, U)$ and Y be a control system over $(\boldsymbol{R}^l, V)$. Assume that $a \in \boldsymbol{R}^k$, $b, \tilde{b} \in \boldsymbol{R}^l$, $t \geqslant 0$ and that $\tilde{p}$: $X(t, a, U_t) \times \times Y(t, b, V_t) \to Y(t, \tilde{b}, V_t)$ is a non-anticipating function. The meaning of $\tilde{p}$ is as follows. If the evader moves along a trajectory $x \in X(t, a, U_t)$ and the pursuer moves along a trajectory $y \in Y(t, b, V_t)$, then the evader "gets the impression" that the pursuer moves along the trajectory $\tilde{y} = \tilde{p}(x, y) \in Y(t, \tilde{b}, V_t)$. (One can additionally assume that the values of the function $\tilde{p}$ depend on a certain random parameter. However, when the assumption as in (6.1) is maintained, this seems to be unnecessary.)

Let φ: $\boldsymbol{R}^k \times \boldsymbol{R}^l \to [0, \infty)$ be a payoff functional, $r \in (0, \infty)$ and

$$(6.1) \qquad \|\tilde{p}(x, y)(s) - y(s)\| \leqslant r$$

for $x \in X(t, a, U_t)$, $y \in Y(t, b, V_t)$ and $s \in [t, \infty)$. The constant r estimates the distance between the pursuer's trajectory $\tilde{y}$ observed by the evader and the pursuer's real trajectory y. For example, if the evader gets information on the position of the pursuer delayed by $h > 0$ and the trajectories from $Y(t, b, V_t)$ are equicontinuous with a modulus of the continuity w, then one can assume that

$$\tilde{p}(x, y)(s) = b \quad \text{for } s \in [t, t+h],$$
$$\tilde{p}(x, y)(s) = y(s-h) \quad \text{for } s \in (t+h, \infty),$$
$$r = w(h).$$

Let us assume that there is a strategy $(e, \#e) \in E(X, Y, a, b, t)$ which satisfies the condition

$$\varphi\big(X(t, a, e(\tilde{y})), (s), \tilde{y}(s)\big) \geqslant d > 0 \quad \text{for } \tilde{y} \in Y(t, \tilde{b}, V_t),\ s \in [t, t+T].$$

If the pursuer moves along the trajectories from $Y(t, \tilde{b}, V_t)$, then this strategy guarantees the evader the result d on the interval $[t, t+T]$. For small perturbations the strategy $(e, \#e)$ may also be a "good" one for the trajectories from $Y(t, b, V_t)$. It should be explained, however, how the evader can use this strategy when the pursuer moves in reality along the trajectories from the set $Y(t, b, V_t)$. The way the strategy $(e, \#e)$ "works" in the situation (t, a, b) may be formally described as follows.

First, note that for a given $y \in Y(t, b, V_t)$ the function $\tilde{p}(\cdot, y)$: $X(t, a, U_t) \to Y(t, \tilde{b}, V_t)$ is non-anticipating; thus it may be shown that there exists exactly one $\tilde{y} \in Y(t, \tilde{b}, V_t)$ which is a solution of the equation

$$(6.2) \qquad \tilde{y} = \tilde{p}\big(X(t, a, e(\tilde{y})), y\big).$$

Therefore we may define the functions $\tilde{e}$: $Y(t, b, V_t) \to U_t$ and $\#\tilde{e}$: $Y(t, b, V_t) \to \Xi_t$ by the formulas $\tilde{e}(y) = e(\tilde{y})$ and $\#\tilde{e}(y) = \#e(\tilde{y})$, where $\tilde{y}$ is the solution of (6.2).

It may be shown that $(\tilde{e}, \#\tilde{e}) \in E(X, Y, a, b, t)$, and so the control $\tilde{e}(y)$ from the above formulas can be accepted as the evader's response to the pursuer's trajectory y.

Now we are going to check what the strategy $(\tilde{e}, \#\tilde{e})$ guarantees the evader E:

$$\varphi\big(X\big(t, a, \tilde{e}(y)\big)(s), y(s)\big) = \varphi\big(X\big(t, a, e(\tilde{y})\big)(s), y(s)\big)$$
$$\geqslant \varphi\big(X\big(t, a, e(\tilde{y})\big)(s), \tilde{y}(s)\big) - w_\varphi\big(\|\tilde{y}(s) - y(s)\|\big) \geqslant d - w_\varphi(r)$$

for $y \in Y(t, b, V_t)$ and $s \in [t, t+T]$. Thus, if $w_\varphi(r) < d$, then the strategy $(\tilde{e}, \#\tilde{e})$ guarantees also a positive result in the game $(X, Y, a, b, t; \varphi)$ on the interval $[t, t+T]$.

7. Final remarks

7.1. We have only considered evasion games in which the players can move over their "own" sets of trajectories. We have not considered games described by differential equations of the form $z' = f(z, u, v_1, \dots, v_n)$. It seems that, after suitable modifications, the application of the method here presented to such games is possible.

7.2. The evasion game with many pursuers is a special case of the so-called *N-person game*. In these games the Nash equilibrium problem, Pareto equilibrium problem, etc. are those most frequently considered. It seems that the following problem is also worthy of notice.

Let us consider a certain N-person game (not necessarily an evasion game) which the player E can play with each of the players $P_1, \dots, P_n$ separately and gets the results d_i, $i = 1, 2, \dots, n$, respectively. What result can E obtain playing against all the players simultaneously?

Obviously, in such a general situation one can only say that the result cannot be greater than $\min\{d_1, d_2, \dots, d_n\}$. Thus, this problem should be posed only with reference to certain particular classes of N-person games.

The above formulation of the game problem for the evasion games with many pursuers has proved useful. Example 2 of [28] allows us to pressume that it is not the only such possibility.

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