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SOME THEOREMS ON COMPACT CONFORMALLY SYMMETRIC RIEMANNIAN SPACES

BY

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1. Introduction. An n-dimensional ($n > 3$) Riemannian manifold is said to be *conformally symmetric* [2] if its Weyl's conformal tensor

$$(1)\qquad C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik})$$

satisfies

$$(2)\qquad C^h{}_{ijk,l} = 0,$$

where the comma indicates covariant differentiation with respect to the metric.

It follows easily from (1) and (2) that every conformally flat ($n > 3$) as well as every symmetric (in the sense of E. Cartan) Riemannian manifold ($n > 3$) is necessarily conformally symmetric. The converse of this is, in general, not true [5].

The purpose of the present paper is to obtain necessary and sufficient conditions for compact Riemannian manifolds to be conformally symmetric, as well as necessary and sufficient conditions for a compact conformally symmetric manifold to be (locally) Cartan-symmetric.

Throughout the paper we assume that the considered manifolds are connected and have positive-definite metrics.

2. Conditions for a compact Riemannian manifold to be conformally symmetric. In the sequel we need the following result ([1], equation (3.7), and [3], p. 91):

LEMMA 1. *The Weyl conformal tensor satisfies the condition*

$$(3)\qquad C_{hijk,l} + C_{hikl,j} + C_{hilj,k} = \frac{1}{n-3}(g_{ik}C^r{}_{hjl,r} + g_{hj}C^r{}_{ikl,r} + g_{il}C^r{}_{hkj,r} + g_{hk}C^r{}_{ilj,r} + g_{ij}C^r{}_{hlk,r} + g_{hl}C^r{}_{ijk,r}).$$

With help of Lemma 1 and the equations

$$(4)\qquad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \qquad C^r{}_{ijr} = C^r{}_{irk} = C^r{}_{rjk} = 0,$$

which are immediate consequences of (1), we prove the following lemma:

LEMMA 2. *The Weyl conformal tensor satisfies the condition*

$$(5)\qquad g^{lm} C^{hijk} C_{hijk,lm} = \frac{2(n-2)}{n-3} C^{hijk} C^r{}_{ijk,rh} - 2C^{hijk} S_{hij}{}^r{}_{kr},$$

where

$$S_{hijlkm} = C_{rijl} R^r{}_{hkm} + C_{hrjl} R^r{}_{ikm} + C_{hirl} R^r{}_{jkm} + C_{hijr} R^r{}_{lkm}.$$

Proof. Writing

$$P_{hijkl} = C_{hijk,l} + C_{hikl,j} + C_{hilj,k},$$

we obtain

$$(6)\qquad C^{hijk} C_{hijk,lm} + C^{hijk} C_{hikl,jm} + C^{hijk} C_{hilj,km} = C^{hijk} P_{hijkl,m}.$$

Since

$$C_{hikl,jm} = C_{hikl,mj} - S_{hikljm} = C_{lkih,mj} - S_{hikljm},$$
$$C_{hilj,km} = C_{hilj,mk} - S_{hiljkm} = C_{ljhi,mk} - S_{hiljkm},$$

which follows easily from (4), equation (6) gives

$$(7)\qquad C^{hijk} C_{hijk,lm} + C^{hijk} C_{lkih,mj} + C^{hijk} C_{ljhi,mk} = C^{hijk} P_{hijkl,m} + C^{hijk} S_{hikljm} + C^{hijk} S_{hiljkm}.$$

But, in view of (4), we find

$$C^{hijk} C_{lkih,mj} = C^{kjih} C_{lkih,mj} = C^{jkhi} C_{ljhi,mk}, \qquad C^{hijk} C_{ljhi,mk} = C^{jkhi} C_{ljhi,mk},$$

which reduces (7) to the form

$$C^{hijk} C_{hijk,lm} + 2C^{jkhi} C_{ljhi,mk} = C^{hijk} P_{hijkl,m} + C^{hijk} S_{hikljm} + C^{hijk} S_{hiljkm}.$$

The last relation, in view of

$$C^{hijk} S_{hikljm} = -C^{hikj} S_{hikljm} = -C^{hijk} S_{hijlkm}, \qquad C^{hijk} S_{hiljkm} = -C^{hijk} S_{hijlkm},$$

yields now

$$C^{hijk} C_{hijk,lm} = C^{hijk} P_{hijkl,m} - 2C^{jkhi} C_{ljhi,mk} - 2C^{hijk} S_{hijlkm},$$

whence, by contraction with g^{lm}, we get

$$(8)\qquad g^{lm} C^{hijk} C_{hijk,lm} = g^{lm} C^{hijk} P_{hijkl,m} - 2C^{jkhi} C^r{}_{jhi,rk} - 2C^{hijk} S_{hij}{}^r{}_{kr}.$$

But it follows from (4) and Lemma 1 that

$$g^{lm} C^{hijk} P_{hijkl,m} = \frac{2}{n-3} C^{hijk} C^r{}_{ijk,rh}.$$

Substituting the last result into (8) we obtain

$$g^{lm}C^{hijk}C_{hijk,lm} = \frac{2}{n-3}C^{hijk}C^{r}{}_{ijk,rh} - 2C^{jkhi}C^{r}{}_{jhi,rk} - 2C^{hijk}S_{hij}{}^{r}{}_{kr},$$

which leads immediately to (5), since

$$C^{jkhi}C^{r}{}_{jhi,rk} = -C^{hijk}C^{r}{}_{ijk,rh}.$$

THEOREM 1. *A compact Riemannian manifold of dimension $n > 3$ is conformally symmetric if and only if the conditions*

(9) $$C^{r}{}_{ijk,r} = 0 \quad \text{and} \quad S_{hijklm} = 0$$

hold.

Proof. The necessity of the conditions follows immediately from (2) and the Ricci identity.

Applying to $Q = C^{hijk}C_{hijk}$ the Laplace operator Δ, we obtain

$$\tfrac{1}{2}\Delta Q = g^{lm}C^{hijk}{}_{,m}C_{hijk,l} + g^{lm}C^{hijk}C_{hijk,lm},$$

which, in virtue of Lemma 2, yields

(10) $$\tfrac{1}{2}\Delta Q = g^{lm}C^{hijk}{}_{,l}C_{hijk,m} + \frac{2(n-2)}{n-3}C^{hijk}C^{r}{}_{ijk,rh} - 2C^{hijk}S_{hij}{}^{r}{}_{kr}.$$

Suppose now that conditions (9) are satisfied. Then, as an immediate consequence of (10), we find

$$\tfrac{1}{2}\Delta Q = g^{lm}C^{hijk}{}_{,l}C_{hijk,m}.$$

Since the metric is positive-definite by the assumption, the last equation gives $\Delta Q \geqslant 0$ everywhere on the manifold. Hence, using Hopf's theorem ([6], p. 30), we obtain $\Delta Q = 0$ and, therefore, $C_{hijk,l} = 0$. Thus the proof of the theorem is complete.

THEOREM 2. *Let M be a compact Riemannian manifold of dimension $n > 3$. If, for M, the conditions*

(11) $$R_{ij,k} = R_{ik,j} \quad \text{and} \quad S_{hijklm} = 0$$

hold, then M is conformally symmetric and has a constant scalar curvature.

Proof. Differentiating (1) covariantly, summing for h, l and taking into account the well-known formulas

(12) $$R^{r}{}_{j,r} = \tfrac{1}{2}R_{,j} \quad \text{and} \quad R^{r}{}_{ijk,r} = R_{ij,k} - R_{ik,j},$$

we obtain

(13) $$C^{r}{}_{ijk,r} = \frac{n-3}{n-2}\left[(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik})\right].$$

Suppose now that the condition $R_{ij,k} = R_{ik,j}$ is satisfied. Then, in view of the first equation of (12), we get $R_{,j} = 0$ and, therefore, $R = \text{const}$. The last result, together with (13), gives now $C^r{}_{ijk,r} = 0$, which shows that conditions (9) hold. The manifold is thus conformally symmetric by means of Theorem 1.

Remark. Theorems 1 and 2 remain true if we replace $S_{hijklm} = 0$ by the condition $C^{hijk} S_{hij}{}^r{}_{kr} \leqslant 0$.

3. Conditions for a compact conformally symmetric manifold to be Cartan-symmetric. First, we prove the following lemma:

LEMMA 3. *The Ricci tensor of a Riemannian manifold satisfies the condition*

$$\tfrac{1}{2}\Delta Q^* = g^{lm} R^{ij}{}_{,l} R_{ij,m} + \tfrac{1}{2} R^{ij} R_{,ij} + R^{ij} T^r{}_{ijr} + g^{lm} R^{ij} Z_{ijlm}, \tag{14}$$

where

$$Q^* = R^{ij} R_{ij}, \qquad Z_{ijlm} = R_{ij,lm} - R_{il,jm},$$
$$T_{lijm} = R_{rl} R^r{}_{imj} + R_{ri} R^r{}_{lmj} = R_{li,jm} - R_{li,mj}. \tag{15}$$

Proof. Applying to Q^* the Laplace operator, we obtain

$$\tfrac{1}{2}\Delta Q^* = g^{lm} R^{ij}{}_{,l} R_{ij,m} + g^{lm} R^{ij} R_{ij,lm}. \tag{16}$$

On the other hand, it follows easily from (15) that

$$R_{ij,lm} = R_{il,jm} + Z_{ijlm},$$

whence

$$R_{ij,lm} = R_{li,mj} + T_{lijm} + Z_{ijlm}. \tag{17}$$

Transvecting now (17) with $g^{lm} R^{ij}$, using (16) and the first formula of (12), we easily obtain (14). Thus the proof of the lemma is complete.

LEMMA 4. *The Ricci tensor of a compact Riemannian manifold is covariantly constant if and only if the conditions*

$$R_{ij,l} = R_{il,j} \quad \textit{and} \quad T_{ijlm} = 0 \tag{18}$$

hold.

Proof. Suppose that the Ricci tensor is covariantly constant. Then we have $R_{ij,lm} = 0$, whence

$$R_{ij,lm} - R_{ij,ml} = 0, \tag{19}$$

and, in consequence, $T_{ijlm} = 0$. This, together with $R_{ij,l} = 0$, leads immediately to (18). The necessity of conditions (18) is thus proved.

If now conditions (18) are satisfied, then, in view of (14), we find

$$\tfrac{1}{2}\Delta Q^* = g^{lm}R^{ij}{}_{,l}R_{ij,m}+\tfrac{1}{2}R^{ij}R_{,ij}. \tag{20}$$

But the first equation of (18), in view of $R^r{}_{j,r} = \frac{1}{2}R_{,j}$, implies $R_{,j} = 0$ which, together with (20), yields

$$\tfrac{1}{2}\Delta Q^* = g^{lm}R^{ij}{}_{,l}R_{ij,m}.$$

Since the metric is positive-definite and the manifold is compact, Hopf's theorem gives $\Delta Q^* = 0$ and, therefore, $R_{ij,l} = 0$. This completes the proof.

Since every conformally symmetric manifold with covariantly constant Ricci tensor is Cartan-symmetric, in view of Lemma 4, we have

COROLLARY 1. *A compact conformally symmetric manifold is Cartan--symmetric if and only if conditions* (18) *hold.*

It follows easily from (13) that the assumptions $C^r{}_{ijk,r} = 0$ and $R = \text{const}$ imply $R_{ij,k} = R_{ik,j}$. Hence, in view of Corollary 1, we have

COROLLARY 2. *Let M be a compact conformally symmetric manifold with a constant scalar curvature. If, for M, the condition $T_{ijlm} = 0$ holds, then M is Cartan-symmetric.*

THEOREM 3. *A compact conformally symmetric manifold, which is not conformally flat, is Cartan-symmetric if and only if the equation $T_{ijlm} = 0$ holds.*

Proof. Głodek proved [4] that, for a conformally symmetric manifold,

$$g^{lm}R_{,l}R_{,m}C_{hijk} = 0,$$

whence

$$g^{lm}R_{,l}R_{,m}C^{hijk}C_{hijk} = 0. \tag{21}$$

But it follows easily from (2) that $C^{hijk}C_{hijk} = \text{const}$. Therefore, as an immediate consequence of (21), we have $R = \text{const}$.

Our theorem follows now from Corollaries 1 and 2.

4. Conditions for a Riemannian manifold to be Cartan-symmetric. We can now prove the following characteristics of a compact Cartan-symmetric manifold:

THEOREM 4. *A compact Riemannian manifold ($n > 3$) is Cartan-symmetric if and only if the conditions*

$$S_{hijklm} = 0, \quad R_{ij,k} = R_{ik,j}, \quad T_{ijkl} = 0 \tag{22}$$

hold.

Proof. If conditions (22) are satisfied, the manifold is Cartan-symmetric by means of (13), $R_{,j} = 0$, Theorem 1 and Corollary 2.

If now $R_{hijk,l} = 0$, then the manifold is Cartan-symmetric and we have

$$C_{hijk,lm} - C_{hijk,ml} = 0,$$

whence $S_{hijklm} = 0$.

The last result, together with Corollary 1, leads to (22). Thus the proof of the theorem is complete.

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