

## LOCAL ANALYTIC INVARIANTS

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Many phenomena in complex analytic geometry have been understood in terms of semicontinuity of local invariants of coherent sheaves. Neither real analytic sets nor the images of (real or complex) analytic mappings are, in general, coherent. The central theme of this article is that, nevertheless, certain natural discrete invariants (like the Hilbert–Samuel function) which are associated to an analytic mapping at each point of its source are upper semicontinuous (in the analytic Zariski topology). Semicontinuity provides both a unified point of view and explicit new techniques for many problems in singularity theory: geometric problems on the images of mappings (semianalytic or subanalytic sets), and analytic problems concerning the singularities of differentiable functions (in particular, the classical division, composition and extension problems of differential analysis).

This article is an exposition of some of our recent results [3], [4], [5] and related questions. We are grateful to the organizers of the Semester on Singularities for the opportunity to present this work, as well as for a very enjoyable meeting.

### 1. Semicontinuity of the Hilbert–Samuel function

Our main problem is introduced here in a general setting; in Section 2 below, it will be reformulated locally in terms of the solution of certain systems of equations.

Let  $K = \mathbf{R}$  or  $\mathbf{C}$ . Let  $X$  and  $Y$  be analytic spaces over  $K$ , and let  $\phi: X \rightarrow Y$  be a morphism. Let  $\phi^*: \mathcal{O}_Y \rightarrow \mathcal{O}_X$  denote the induced homomorphism of the structure sheaves. Suppose that  $\mathcal{F}$  and  $\mathcal{G}$  are coherent  $\mathcal{O}_X$ - and  $\mathcal{O}_Y$ -

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modules, respectively, and that  $\Psi: \mathcal{G} \rightarrow \mathcal{F}$  is a module homomorphism over the ring homomorphism  $\phi^*$ .

Let  $a \in X$ . Then  $\phi^*$  determines a homomorphism of local rings  $\phi_a^*: \mathcal{O}_{Y, \phi(a)} \rightarrow \mathcal{O}_{X, a}$  and  $\Psi$  determines a module homomorphism  $\Psi_a: \mathcal{G}_{\phi(a)} \rightarrow \mathcal{F}_a$  over  $\phi_a^*$ . We write  $\mathcal{O}_a = \mathcal{O}_{X, a}$ , etc., when there is no possibility of confusion. Let  $\hat{\phi}_a^*: \hat{\mathcal{O}}_{\phi(a)} \rightarrow \hat{\mathcal{O}}_a$  and  $\hat{\Psi}_a: \hat{\mathcal{G}}_{\phi(a)} \rightarrow \hat{\mathcal{F}}_a$  denote the induced homomorphisms of the completions.

Our main objective is to understand the way that the *module of formal relations*

$$\mathcal{R}_a = \text{Ker } \hat{\Psi}_a$$

varies with respect to  $a \in X$ . For this purpose, we study associated discrete invariants, such as the *Hilbert–Samuel function*  $H_a$  of  $\hat{\mathcal{G}}_{\phi(a)}/\mathcal{R}_a$ :

$$H_a(k) = \dim_{\mathbf{k}} \frac{\hat{\mathcal{G}}_{\phi(a)}}{\mathcal{R}_a + \mathfrak{m}_{\phi(a)}^{k+1} \cdot \hat{\mathcal{G}}_{\phi(a)}}$$

(where  $\mathfrak{m}_{\phi(a)}$  denotes the maximal ideal of  $\mathcal{O}_{\phi(a)}$ ).

**CONJECTURE.** The Hilbert–Samuel function  $H_a$  is (analytic) Zariski (upper) semicontinuous on  $X$ .

*Zariski semicontinuity* of  $H_a$  means: (1) There are only finitely many distinct  $H_a$  for  $a$  in a compact subset of  $X$ , and (2) given  $a_0 \in X$ , the points  $a$  such that  $H_a(k) \geq H_{a_0}(k)$ , for all  $k$ , form a closed analytic subset of  $X$ .

Zariski semicontinuity of the Hilbert–Samuel function in the *coherent case* (the special case that  $X = Y$  and  $\phi = \text{identity}$ ) is known classically. In general, the module  $\mathcal{R}_a$  does not vary in a coherent way. Nevertheless, we have:

**THEOREM I** ([3, Thm. C]). *The Hilbert–Samuel function  $H_a$  is Zariski semicontinuous in each of the following cases:*

(1) *In the algebraic category. (Here we can use the (algebraic) Zariski topology.)*

(2) *If  $X$  is Cohen–Macaulay (in particular, if  $X$  is smooth) and  $\phi$  is locally finite; i.e., for all  $a \in X$ ,  $\mathcal{O}_a$  is a finite  $\mathcal{O}_{\phi(a)}$ -module via the homomorphism  $\phi_a^*$ .*

(3) *If  $X$  is smooth,  $\Psi = \phi^*: \mathcal{O}_Y \rightarrow \mathcal{O}_X$ , and  $\phi$  is regular in the sense of Gabrielov [9]; i.e., the Krull dimension of  $\mathcal{O}_{\phi(a)}/\text{Ker } \phi_a^*$  is locally constant on  $X$ .*

Case (2) includes the coherent case. The case  $\Psi = \phi^*$  is relevant to the geometry of the image  $\phi(X)$ : In general, if  $b \in Y$ , put

$$\mathcal{R}_{*b} = \bigcap_{a \in \phi^{-1}(b)} \mathcal{R}_a.$$

Suppose  $\Psi = \phi^*$ . Let  $a \in X$  and let  $b = \phi(a)$ . Then  $\mathcal{R}_{*b}$  is the (formal) local

ideal of  $\phi(X)$  at  $b$ . If  $X$  and  $Y$  are smooth, then (with respect to local coordinates  $x$  near  $a$  and  $y = (y_1, \dots, y_n)$  near  $b$ )  $\mathcal{R}_a$  identifies with the ideal of formal power series relations among the Taylor expansions  $\hat{\phi}_{j,a}$  at  $a$  of the components of  $\phi = (\phi_1, \dots, \phi_n): \mathcal{R}_a = \{G(y_1, \dots, y_n) \in \mathbf{K}[[y]]: G(\hat{\phi}_{1,a}(x), \dots, \hat{\phi}_{n,a}(x)) = 0\}$ . Examples (2) and (3) below show why it may be less convenient to study  $\phi(X)$  using its local ideals  $\mathcal{R}_{*b}$  directly, rather than using the ideals of formal relations  $\mathcal{R}_a$ :

EXAMPLES ([3], Rmk. 2.3). (1) If  $\mathbf{K} = \mathbf{C}$  and  $\phi$  is proper, then the Hilbert–Samuel function  $H_b$  of  $\hat{\mathcal{G}}_b/\mathcal{R}_{*b}$  is Zariski semicontinuous on  $Y$ .

(2) On the other hand, if  $\mathbf{K} = \mathbf{R}$  and  $\phi$  is finite (i.e., proper and locally finite), the Hilbert–Samuel function  $H_{\phi(a)}$  need not be Zariski semicontinuous, even as a function of  $a \in X$ . For example, take  $X = \mathbf{R}^2$ ,  $Y = \mathbf{R}^3$  and define  $\phi$  by  $\phi(x_1, x_2) = (x_1, x_2(x_2^3 + x_1 x_2), x_2^3 + x_1 x_2)$ . Let  $\Psi = \phi^*: \mathcal{O}_Y \rightarrow \mathcal{O}_X$ . Then the Hilbert–Samuel function  $H_{\phi(a)}$  is constant on the half-lines  $\{x_2 = 0, x_1 > 0\}$  and  $\{x_2 = 0, x_1 < 0\}$  but has different values on the two half-lines (Figure 1).

(3) If  $\phi$  is not proper, the Hilbert–Samuel function  $H_{\phi(a)}$  need not even be topologically semicontinuous. For example, with  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{C}$ , define  $\phi: \mathbf{K} - \{0\} \rightarrow \mathbf{K}^2$  by  $\phi(t) = (\cos t, \sin t + \sin(1/t))$  and take  $\Psi = \phi^*$ .

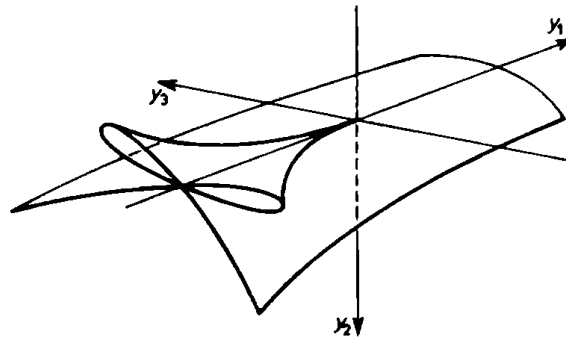


Fig 1.  $y_1 = x_1, y_2 = x_2(x_2^3 + x_1 x_2), y_3 = x_2^3 + x_1 x_2$ .

If  $\phi$  is proper, then, for all  $b \in Y$ , there exists  $a^1, \dots, a^s \in \phi^{-1}(b)$  such that  $\mathcal{R}_{*b} = \bigcap_{i=1}^s \mathcal{R}_{a^i}$  ([3, Prop. 11.1]). Moreover, locally in  $Y$ , there is a bound  $s$  which is uniform with respect to  $b$  ([3, Cor. 11.6]). Because of this, semicontinuity of the Hilbert–Samuel function  $H_a$  applies also to the variation of the  $\mathcal{R}_{*b}$ : Let  $X_\phi^s$  denote the  $s$ -fold fiber product

$$X_\phi^s = \{a = (a^1, \dots, a^s) \in X^s: \phi(a^1) = \dots = \phi(a^s)\}.$$

There is a morphism  $\phi: X_\phi^s \rightarrow Y$  induced by  $\phi$ . If  $a \in X_\phi^s$ ,  $a = (a^1, \dots, a^s)$ , put

$\mathcal{R}_a = \bigcap_{i=1}^s \mathcal{R}_{a_i}$  and let  $H_a$  denote the Hilbert–Samuel function of  $\hat{\mathcal{G}}_{\phi(a)}/\mathcal{R}_a$ . We can prove that  $H_a$  is Zariski semicontinuous on  $X_\phi^s$ , for a given positive integer  $s$ , if and only if  $H_a$  is Zariski semicontinuous on  $X$  ([3, Prop. 9.6]).

Many geometric properties of semialgebraic (respectively, subanalytic) sets  $Z$  are simple consequences of semicontinuity of the Hilbert–Samuel function (or of other local invariants; see Section 3 below) in the case  $\Psi = \phi^*$ , where  $K = \mathbf{R}$  and  $\phi: X \rightarrow Y$  is a proper algebraic (respectively, analytic) mapping such that  $X$  is smooth and  $\phi(X) = Z$ . For example: The set of smooth points of a semialgebraic (respectively, subanalytic) set is semialgebraic (respectively, subanalytic) (Łojasiewicz [18, § 17, Thm. 4], Tamm [30]). The points of a subanalytic set which admit semianalytic neighborhoods form a subanalytic subset (Pawlucki [26]). The points where a real algebraic set is not coherent form a semialgebraic subset (Galbiati [10]).

Assume that  $X$  is smooth. Let  $a \in X$ . Let  $r_1(a)$  denote the generic rank of  $\phi$  near  $a$ . Put  $r_2(a) = \dim \hat{\mathcal{C}}_{\phi(a)}/\text{Ker } \hat{\phi}_a^*$  and  $r_3(a) = \dim \mathcal{C}_{\phi(a)}/\text{Ker } \phi_a^*$  (where  $\dim$  denotes the Krull dimension). Then  $r_1(a) \leq r_2(a) \leq r_3(a)$ . The formal rank  $r_2(a)$  is the degree of the Hilbert–Samuel polynomial of  $\hat{\mathcal{C}}_{\phi(a)}/\text{Ker } \hat{\phi}_a^*$ . In particular, if the Hilbert–Samuel function is Zariski semicontinuous, then so is  $r_2(a)$ .

We say that  $\phi$  is *regular at  $a$*  if  $r_1(a) = r_3(a)$  (as in Theorem I (3)). If  $\phi$  is algebraic, then it is regular at every point  $a$ , by a theorem of Łojasiewicz [18, § 17, Prop. 7]. On the other hand, consider the classical example of Osgood:  $X = \mathbf{K}^2$ ,  $Y = \mathbf{K}^3$  and  $\phi$  is defined by  $\phi(x_1, x_2) = (x_1, x_1 x_2, x_1 x_2 e^{x_2^2})$ . If  $a \in \{x_1 = 0\}$ , then  $r_1(a) = 2$  but  $r_2(a) = r_3(a) = 3$ . Thus  $r_2(a)$  (or  $H_a$ ) detects differences between algebraic and general analytic behavior.

**Description of a subanalytic set.** Although the module of formal relations  $\mathcal{R}_a$  does not, in general, vary in a coherent way, semicontinuity of the Hilbert–Samuel function provides substitute analytical tools:

Recall that a real algebraic set (or a semialgebraic set)  $Z$ , though not necessarily coherent, can be described by Nash functions:  $Z$  admits a semialgebraic stratification with the property that, for every stratum  $\Sigma$ , there are finitely many Nash functions which generate the local ideal of  $Z$  at each point of  $\Sigma$ . For example, the real algebraic set  $z^3 - x^2 y^3 = 0$  is not coherent; the Nash function  $z - x^{2/3} y$  is needed to generate its local ideals at nonzero points  $P$  of the  $x$ -axis (Figure 2).

The images of proper real analytic mappings which are regular at every point form the class of *Nash subanalytic* sets [2], [6]. Every semianalytic set is Nash (by Łojasiewicz [18, § 17, Prop. 7]). Nash subanalytic sets are described by functions which are *Nash analytic* (in the sense of Merrien

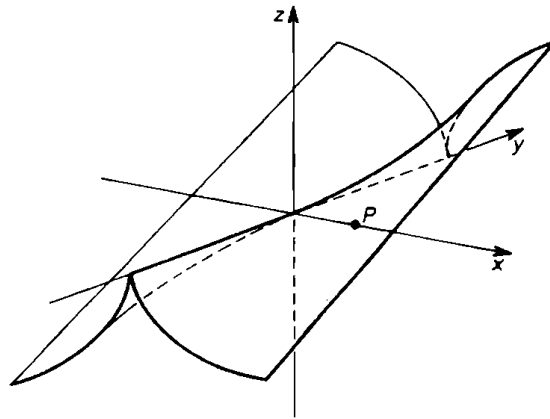


Fig. 2.  $z^3 - x^2 y^3 = 0$ .

[22]), in the same way that semialgebraic sets are described by Nash functions ([6, Thm. 5.1.1]).

Semicontinuity of the Hilbert–Samuel function  $H_a$  provides an analogous description of the formal local ideals of a closed subanalytic set, or of the modules of formal relations  $\mathcal{R}_a$ , in general ([4, Thm. 1.3]): Suppose that  $Y$  is an open subset of  $\mathbf{K}^n$  and that  $\phi$  is proper. If the Hilbert–Samuel function  $H_a$  is Zariski semicontinuous, then there is  $s \in \mathbf{N}$  (at least locally in  $Y$ ) and a locally finite partition  $\{X_\mu\}_{\mu \in \mathbf{N}}$  of  $X_\phi^s$  such that, for each  $\mu$ :

(1)  $X_\mu$  is a relatively compact connected smooth semianalytic subset of  $X_\phi^s$ ;

(2)  $\bar{X}_\mu - X_\mu \subset \bigcup_{\lambda < \mu} X_\lambda$ ;

(3)  $\phi|X_\mu$  has constant rank;

(4) Let  $Y_\mu = \phi(\bigcup_{\lambda \leq \mu} X_\lambda)$ . Then, for all  $a \in X_\mu - \phi^{-1}(Y_{\mu-1})$ ,  $\mathcal{R}_a = \bigcap_{a \in \phi^{-1}(b)} \mathcal{R}_a$ , where  $b = \phi(a)$ ;

(5) For all  $a \in X_\mu$ , there are “special generators”  $G_a^1, \dots, G_a^i$  of  $\mathcal{R}_a \subset \mathbf{K}[[y]]^q$  whose coefficients, as functions of  $a$ , are analytic on  $X_\mu$  and meromorphic through its frontier. If  $a \in X_\mu - \phi^{-1}(Y_{\mu-1})$ , then the  $G_a^i$  depend only on  $b = \phi(a)$ ; say  $G_a^i = G_b^i$ . The  $G_b^i$ , moreover, are induced by functions which are analytic in the variables of  $Y_\mu - Y_{\mu-1}$  and formal with respect to the variables in the normal direction.

## 2. Differentiable functions

The question of whether the Hilbert–Samuel function  $H_a$  is semicontinuous is local in  $X$ . Using a local embedding of  $Y$  in affine space and a local presentation  $\mathcal{O}_Y \rightarrow \mathcal{G} \rightarrow 0$  of  $\mathcal{G}$ , we can replace  $Y$  by an open subspace of  $\mathbf{K}^n$  and  $\mathcal{G}$  by  $\mathcal{O}_Y^q$ . Thus, we can assume that the homomorphism  $\Psi: \mathcal{G} \rightarrow \mathcal{F}$  is given as follows:

Let  $X$  and  $Y$  be smooth analytic spaces (i.e., analytic manifolds). Let  $A$  and  $B$  be  $p \times q$  and  $p \times r$  matrices, respectively, whose entries are analytic functions on  $X$ . Then multiplication by  $A$  or  $B$  induces  $\mathcal{O}_X$ -homomorphisms  $A: \mathcal{O}_X^q \rightarrow \mathcal{O}_X^p$  or  $B: \mathcal{O}_X^r \rightarrow \mathcal{O}_X^p$ . Let  $\Phi: \mathcal{O}_Y^q \rightarrow \mathcal{O}_X^p$  denote the homomorphism over  $\phi^*: \mathcal{O}_Y \rightarrow \mathcal{O}_X$  which is induced by  $A$ . Put  $\mathcal{G} = \mathcal{O}_Y^q$ ,  $\mathcal{F} = \text{Coker } B$ , and let  $\Psi: \mathcal{G} \rightarrow \mathcal{F}$  be the  $\phi^*$ -homomorphism induced by  $\Phi$ . (Locally, any  $\phi^*$ -homomorphism from  $\mathcal{O}_Y^q$  to a coherent  $\mathcal{O}_X$ -module has this form.)

If  $a \in X$ , then  $\mathcal{O}_a$  (respectively,  $\hat{\mathcal{O}}_a$ ) identifies with the ring of convergent (respectively, formal) power series  $K\{x\}$  (respectively,  $K[[x]]$ ), where  $x = (x_1, \dots, x_m)$  are local coordinates near  $a$ . Let  $A_a$  and  $B_a$  denote the matrices of elements of  $\mathcal{O}_a$  induced by  $A$  and  $B$ , respectively. If  $G = (G_1, \dots, G_q) \in \hat{\mathcal{O}}_{\phi(a)}^q$ , we write  $G \circ \hat{\phi}_a$  for  $(\hat{\phi}_a^*(G_1), \dots, \hat{\phi}_a^*(G_q))$ . (In local coordinates,  $G \circ \hat{\phi}_a$  is the formal composition of  $G$  with the Taylor expansion  $\hat{\phi}_a$  of  $\phi$  at  $a$ .) Then

$$\mathcal{R}_a = \text{Ker } \hat{\Psi}_a = \{G \in \hat{\mathcal{O}}_{\phi(a)}^q: A_a \cdot (G \circ \hat{\phi}_a) + B_a \cdot H = 0, \text{ for some } H \in \hat{\mathcal{O}}_a^r\}.$$

Our problems, from this point of view, concern the solution of a system of equations of the form

$$f(x) = A(x) \cdot g(\phi(x)) + B(x) \cdot h(x),$$

where  $f = (f_1, \dots, f_p)$  is given and  $g = (g_1, \dots, g_q)$  and  $h = (h_1, \dots, h_r)$  are the unknown functions.

Semicontinuity of the Hilbert–Samuel function has striking applications to the solution of such systems involving differentiable functions. Suppose that  $X$  and  $Y$  are smooth real analytic spaces. Then  $\phi: X \rightarrow Y$  induces a homomorphism  $\phi^*: \mathcal{C}^\infty(Y) \rightarrow \mathcal{C}^\infty(X)$  between the rings of infinitely differentiable functions. Let  $\Phi: \mathcal{C}^\infty(Y)^q \rightarrow \mathcal{C}^\infty(X)^p$  denote the module homomorphism over  $\phi^*$  defined by  $\Phi(g)(x) = A(x) \cdot g(\phi(x))$ , where  $g = (g_1, \dots, g_q) \in \mathcal{C}^\infty(Y)^q$ , and let  $B \cdot: \mathcal{C}^\infty(X)^r \rightarrow \mathcal{C}^\infty(X)^p$  denote the  $\mathcal{C}^\infty(X)$ -homomorphism induced by multiplication by the matrix  $B$ . Let  $\Theta: \mathcal{C}^\infty(Y)^q \oplus \mathcal{C}^\infty(X)^r \rightarrow \mathcal{C}^\infty(X)^p$  denote the “mixed homomorphism”  $\Theta(g, h) = \Phi(g) + B \cdot h$ , where  $g \in \mathcal{C}^\infty(Y)^q$  and  $h \in \mathcal{C}^\infty(X)^r$ .

For every  $a \in X$ , there is a Taylor series homomorphism  $f \mapsto \hat{f}_a$  from  $\mathcal{C}^\infty(X)^p$  onto  $\hat{\mathcal{C}}_a^p$ . Let  $(\text{Im } \Theta)^\wedge$  denote the elements of  $\mathcal{C}^\infty(X)^p$  which formally belong to  $\text{Im } \Theta$ ; i.e.,  $\{f \in \mathcal{C}^\infty(X)^p: \text{for all } b \in \phi(X), \text{ there exists } G_b \in \hat{\mathcal{O}}_b^q \text{ such that } \hat{f}_a - \hat{\Phi}_a(G_b) \in \text{Im } \hat{B}_a, \text{ for all } a \in \phi^{-1}(b)\}$ . Then  $(\text{Im } \Theta)^\wedge$  is a closed subspace of  $\mathcal{C}^\infty(X)^p$ , in the  $\mathcal{C}^\infty$  topology ([4, Prop. 3.1]), and  $\text{Im } \Theta \subset (\text{Im } \Theta)^\wedge$ .

**THEOREM II** ([3, Thm. D; 4, Thm. 1.1]). *Assume that  $\phi$  is proper. If the Hilbert–Samuel function  $H_a$  is Zariski semicontinuous on  $X$ , then:*

(1) *If  $f \in (\text{Im } \Theta)^\wedge$ , then there exist  $g \in \mathcal{C}^\infty(Y)^q$  and  $h \in \mathcal{C}^\infty(X)^r$  such that*

$$(*) \quad f(x) = A(x) \cdot g(\phi(x)) + B(x) \cdot h(x).$$

*In particular,  $\text{Im } \Theta$  is a closed subspace of  $\mathcal{C}^\infty(X)^p$ .*

(2) *There is a continuous linear mapping  $\text{Im } \Theta \rightarrow \mathcal{C}^\infty(Y)^q \oplus \mathcal{C}^\infty(X)^r$  taking elements  $f \in \text{Im } \Theta$  into solutions  $(g, h)$  of  $(*)$ .*

**THEOREM III** ([3, Rmks. 4.3]). *If the Hilbert–Samuel function  $H_a$  is Zariski semicontinuous on  $X$ , then every formal relation  $G \in \mathcal{R}_a$  is the formal Taylor expansion at  $\phi(a)$  of some “ $\mathcal{C}^\infty$  relation”  $g$ ; i.e., of some  $g \in \mathcal{C}^\infty(Y)^q$  such that  $A \cdot (g \circ \phi) + B \cdot h = 0$ , where  $h \in \mathcal{C}^\infty(X)^r$ .*

*Remark.* Gabrielov [9] proved that if  $r_1(a) = r_2(a)$  (see Section 1 above), then (1)  $r_2(a) = r_3(a)$ , so that  $\phi$  is regular at  $a$ , and (2)  $\mathcal{O}_a \cap \hat{\phi}_a^*(\hat{\mathcal{O}}_{\phi(a)}) = \phi_a^*(\mathcal{O}_{\phi(a)})$ . These assertions are analogues for convergent power series of Theorems III and II (1), respectively (in the case  $A = 1, B = 0$ ). Gabrielov’s assertion (2), however, unlike II (1), actually implies that  $\phi$  is regular at  $a$  ([1], [23]).

Theorem II reduces the main problems of differential analysis, including their functional analytic aspects, to questions of semicontinuity of local invariants in analytic geometry. For example, by Theorem I (1), the conclusions of Theorem II are true if  $\phi: X \rightarrow Y$  is a proper morphism of Nash manifolds and  $A, B$  are matrices of Nash functions on  $X$ . This is the first general results on “modules over a ring of composite differentiable functions”. From Theorems I and II, we also recover all previous results on the classical division, composition and extension problems for  $\mathcal{C}^\infty$  functions:

**Division theorems.** In the coherent case  $X = Y, \phi = \text{identity}, B = 0$ , Theorem II (1) becomes Malgrange’s division theorem:  $A \cdot \mathcal{C}^\infty(X)^q$  is closed in  $\mathcal{C}^\infty(X)^p$  ([19], Ch. VI). The canonical surjection  $\mathcal{C}^\infty(X)^q \rightarrow A \cdot \mathcal{C}^\infty(X)^q$  admits a continuous linear splitting, according to [6, Thm. 0.1.1]; this is the assertion of II (2).

The Malgrange–Mather division theorem [20] is equivalent to the assertions (1) and (2) of Theorem II in the special case that  $X = Y = \mathbb{R}^{n+d}, \phi: X \rightarrow Y$  is the mapping

$$\phi(x, t, \lambda_1, \dots, \lambda_d) = (x, \lambda_1, \dots, \lambda_{d-1}, -t^d - \sum_{j=1}^{d-1} \lambda_j t^{d-j}),$$

where  $x = (x_1, \dots, x_n), B = 0$  and  $A(x, t, \lambda_1, \dots, \lambda_{d-1})$  is the  $1 \times d$  matrix  $(t^{d-1} \ t^{d-2} \ \dots \ 1)$ : Let  $P(t, \lambda)$  denote the polynomial  $t^d + \sum_{j=1}^d \lambda_j t^{d-j}$  with generic coefficients  $\lambda = (\lambda_1, \dots, \lambda_d)$ . Then, given  $f(x, t) \in \mathcal{C}^\infty$ , there exists  $g(x, \lambda) \in \mathcal{C}^\infty(\mathbb{R}^{n+d})^d, g = (g_1, \dots, g_d)$ , such that  $f = A \cdot (g \circ \phi)$  if and only if  $f(x, t) - \sum_{j=1}^d t^{d-j} g_j(x, \lambda)$  divided by  $t^d + \sum_{j=1}^{d-1} \lambda_j t^{d-j} + \lambda_d = P(t, \lambda)$  is a  $\mathcal{C}^\infty$  function; II (1) applies because  $f \in (\mathcal{C}^\infty(\mathbb{R}^{n+d}))^\wedge$  by the formal Weierstrass division theorem. Semicontinuity of the Hilbert–Samuel function in this case is provided by either (1) or (2) of Theorem I.

**Composition theorems.** When  $A = 1$  and  $B = 0$ , II (1) reduces to the composition conjecture  $\phi^* \mathcal{C}^\infty(Y) = (\phi^* \mathcal{C}^\infty(Y))^\wedge$ , established in [2] when  $\phi$  is semiproper and  $\phi(X)$  is Nash subanalytic; this result follows from Theorems I (3) and II. Special cases include Glaeser's classical theorem ([13]), Schwarz's and Luna's theorems on differentiable invariants ([17], [28]), and Tougeron's result ([31]).

**Extension theorems.** Suppose  $A = 1$  and  $B = 0$ . If  $Z = \phi(X)$ , then II (1) asserts that  $\mathcal{C}^\infty(Z)$  (the restrictions to  $Z$  of  $\mathcal{C}^\infty$  functions) embeds as the closed subspace  $\phi^* \mathcal{C}^\infty(Y)$  of  $\mathcal{C}^\infty(X)$ . Thus II (2) provides a continuous linear extension operator  $\mathcal{C}^\infty(Z) \rightarrow \mathcal{C}^\infty(Y)$ . In particular, Theorems I (3) and II prove the existence of extension operators for Nash subanalytic sets, first presented in [6] using [2]. Special cases include the classical extension theorem of Mityagin [24] and Seeley [29] for the half-line, and Mather's splitting theorem for differentiable invariants ([21]).

### 3. Local invariants

A part of our strategy is to exploit the relationships among various conditions on the variation of the module of formal relations  $\mathcal{R}_a$ . Zariski semicontinuity of the Hilbert–Samuel function is equivalent to two other important conditions ([3, Thm. A]):

(1) A uniform version of a lemma of Chevalley [8]. This is interesting even in the coherent case, where it translates into a uniform version of the Artin–Rees theorem in commutative algebra.

(2) Zariski semicontinuity of a “diagram of initial exponents”  $\mathbf{N}(\mathcal{R}_a)$  associated to  $\mathcal{R}_a$ . This diagram gives a combinatorial picture of the module of formal relations, in the spirit of the classical Newton diagram of a formal power series.

Chevalley's lemma estimates the order of vanishing of an element  $G \in \hat{\mathcal{G}}_{\phi(a)}$ , modulo  $\mathcal{R}_a$ , in terms of the order of vanishing of  $\hat{\Psi}_a(G)$ . In precise terms:

**LEMMA.** *Let  $a \in X$ . For each  $k \in \mathbf{N}$ , there exists  $l \in \mathbf{N}$  such that if  $G \in \hat{\mathcal{G}}_{\phi(a)}$  and  $\hat{\Psi}_a(G) \in \mathfrak{m}_a^{l+1} \cdot \hat{\mathcal{F}}_a$ , then  $G \in \mathcal{R}_a + \mathfrak{m}_{\phi(a)}^{k+1} \cdot \hat{\mathcal{G}}_{\phi(a)}$ .*

Let  $a \in X$ . For each  $k \in \mathbf{N}$ , we let  $l(k, a)$  denote the smallest  $l \in \mathbf{N}$  satisfying the conclusion of the lemma.

**The diagram of initial exponents.** Let  $y = (y_1, \dots, y_n)$  and let  $R$  be a submodule of  $K[[y]]^q$ . The diagram of initial exponents  $\mathbf{N}(R)$ , introduced by Hironaka (cf. [7], [11]) is a subset of  $\mathbf{N}^n \times \{1, \dots, q\}$ , defined as follows:

If  $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$ , put  $|\beta| = \beta_1 + \dots + \beta_n$ . The lexicographic ordering on  $(n+2)$ -tuples  $(|\beta|, j, \beta_1, \dots, \beta_n)$ , where  $(\beta, j) \in \mathbf{N}^n \times \{1, \dots, q\}$ ,



induces a total ordering of  $N^n \times \{1, \dots, q\}$ . Let  $G \in K[[y]]^q$ , say  $G = (G_1, \dots, G_q)$ , where each  $G_j = \sum g_{\beta,j} y^\beta$ . Let  $\text{supp } G = \{(\beta, j) : g_{\beta,j} \neq 0\}$  and let  $v(G)$  denote the smallest element of  $\text{supp } G$ . Then  $N(R)$  is defined as  $\{v(G) : G \in R\}$ .

Clearly,  $N(R) + N^n = N(R)$ , where addition is defined by  $(\beta, j) + \gamma = (\beta + \gamma, j)$ ,  $(\beta, j) \in N^n \times \{1, \dots, q\}$ ,  $\gamma \in N^n$ . It follows that there is a smallest finite subset  $V(N(R))$  of  $N(R)$  such that  $N(R) = V(N(R)) + N^n$ .

The "vertices"  $V(N)$  can be used to totally order the set  $\mathcal{D} = \{N \subset N^n \times \{1, \dots, q\} : N + N^n = N\}$  ([3; § 1]): To each  $N \in \mathcal{D}$ , associate the sequence  $v(N)$  obtained by listing the vertices of  $N$  in ascending order and completing this list to an infinite sequence by using  $\infty$  for all the remaining terms. If  $N^1, N^2 \in \mathcal{D}$ , we say that  $N^1 < N^2$  provided that  $v(N^1) < v(N^2)$  with respect to the lexicographic ordering on the set of such sequences.

Clearly, if  $N^1 \supset N^2$ , then  $N^1 \leq N^2$ .

Now, to our module  $\mathcal{R}_a$  we can associate a diagram  $N_a$  which depends on a choice of local coordinates in a neighborhood of  $\phi(a)$ : Assume that  $Y$  is an open subset of  $K^n$  and that  $\mathcal{G} = \mathcal{O}_Y^q$ . If  $b \in Y$ , then  $\hat{C}_b$  identifies with the ring of formal power series  $K[[y]]$  in the coordinates  $y = (y_1, \dots, y_n)$  of  $K^n$ . If  $a \in X$ , we put

$$N_a = N(\mathcal{R}_a).$$

Then it makes sense to talk about Zariski semicontinuity of  $N_a$ , with respect to the total ordering of the set of diagrams  $\mathcal{D}$ .

**THEOREM IV.** *The following conditions are equivalent:*

- (1) *The Hilbert–Samuel function  $H_a$  of  $\hat{\mathcal{G}}_{\phi(a)}/\mathcal{R}_a$  is Zariski semicontinuous on  $X$ .*
- (2) *Uniform Chevalley estimate: Let  $K$  be a compact subset of  $X$ . Then, for every  $k \in \mathbb{N}$ , there exists  $l = l(k, K) \in \mathbb{N}$  such that  $l(k, a) \leq l$  for all  $a \in K$ .*
- (3) *(Assuming that  $Y$  is an open subset of  $K^n$  and  $\mathcal{G} = \mathcal{O}_Y^q$ .) The diagram of initial exponents  $N_a = N(\mathcal{R}_a)$  is Zariski semicontinuous on  $X$ .*

Again consider a submodule  $R$  of  $K[[y]]^q$ , where  $y = (y_1, \dots, y_n)$ . Hironaka's formal division algorithm (cf. [7], [11], [14]) asserts that

$$K[[y]]^q = R \oplus \{G : \text{supp } G \cap N(R) = \emptyset\}.$$

Clearly,  $\{G : \text{supp } G \cap N(R) = \emptyset\}$  is stable with respect to formal differentiation; this plays an important part, for example, in the way we use Theorem IV (3) above to prove Theorem II.

Let  $H$  denote the Hilbert–Samuel function of  $K[[y]]^q/R$ . It follows from the formal division algorithm that  $H(k)$  is the number of pairs  $(\beta, j)$  such that  $(\beta, j) \notin N(R)$  and  $|\beta| \leq k$ . Thus, (3) immediately implies (1) in Theorem IV. The relationship between  $H$  and  $N(R)$  also shows why the Hilbert–Samuel function  $H(k)$  coincides with a polynomial in  $k$ , for  $k$  sufficiently large.

Let  $(\beta_i, j_i)$ ,  $i = 1, \dots, t$ , denote the vertices of  $N(R)$ . It follows from the formal division algorithm that  $R$  has a unique set of generators  $G^1, \dots, G^t$  such that  $\text{supp}(G^i - y^{\beta_i j_i}) \cap N(R) = \emptyset$ , where  $y^{\beta_i j_i}$  denotes the  $q$ -tuple with  $y^{\beta_i}$  in the  $j_i$ -th place and zeros elsewhere. We call  $G^1, \dots, G^t$  the *standard basis* of  $R$ .

Let  $\Sigma$  be an irreducible (germ of  $a$ ) closed analytic subset of  $X$ . If  $N_a$  is Zariski semicontinuous, then there is a proper closed analytic subset  $T$  of  $\Sigma$  such that  $N_a$  is constant on  $\Sigma - T$ . The standard bases of  $\mathcal{R}_a$ ,  $a \in \Sigma - T$ , form the "special generators" we referred to in Section 1 above.

In general, there is a "generic diagram of initial exponents"  $N_\Sigma$  associated to  $\Sigma$ , with the following properties: (1) for all  $a \in \Sigma$ ,  $N_\Sigma \leq N_a$ ; (2)  $N_a = N_\Sigma$  outside a countable union  $U = \bigcup T_k$  of proper closed analytic subsets  $T_k$  of  $\Sigma$  [3, § 8]. The  $T_k$  come from finite order Taylor expansions of our data. To prove our semicontinuity conjecture in general, we need a stabilization result showing that  $U$  is, in fact, analytic. An equivalent approach is to prove that if  $a \in \Sigma - U$ , then any formal relation  $G \in \mathcal{R}_a$  extends to a formal relation outside a proper analytic subset of  $\Sigma$  (in the sense that its coefficients extend analytically).

*Remark.* For certain parametrized families of modules which generalize the coherent case (and include, for example, Tougeron's "familles neothériennes" [32]), it is easy to prove directly that the diagram of initial exponents is Zariski semicontinuous [3, § 7] (cf. [27]). On the other hand, if  $N_a = N(\mathcal{R}_a)$  is Zariski semicontinuous, then the properties of the special generators show that the modules  $\mathcal{R}_a$  form such a parametrized family [3, § 9].

**An invariant diagram.** The diagram of initial exponents  $N_a$  depends on a choice of local coordinates near  $\phi(a)$ . However, there is an invariant diagram  $G_a$  (cf. [14]): Write  $R^\lambda$  to indicate the effect of a coordinate change  $\lambda$  on a submodule  $R$  of  $K[[y]]^q$ ,  $y = (y_1, \dots, y_n)$ . Clearly,  $N(R^\lambda)$  depends only on the linear part of  $\lambda$ , so we can assume that  $\lambda \in GL(n, K)$ . Then  $N(R^\lambda)$  is Zariski semicontinuous on  $GL(n, K)$ , according to the preceding remark. Let

$$G(R) = \min_{\lambda} N(R^\lambda).$$

Then  $G(R) = N(R^\lambda)$  for a generic coordinate system  $\lambda$ .

Put  $G_a = G(\mathcal{R}_a)$ . If  $N_a$  is Zariski semicontinuous, then so is  $G_a$  ([5]).

The special generators arising from the generic diagram  $G_a$  are particularly nice. For example, they are Nash functions in the algebraic case, by a Henselian version of the formal division algorithm ([15]). Thus, for instance, the stratification of semialgebraic sets by Nash functions is a special case of our general description of subanalytic sets in Section 1 above.

**4. An application to ideals of holomorphic functions with  $\mathcal{C}^\infty$  boundary values**

Because our techniques are so explicit, they can be used to give natural sufficient conditions for the conclusions of Theorem II to hold on closed sets which are not necessarily even subanalytic. For example:

Let  $U$  be an open subset of  $K^n$  and let  $X$  be a closed subset of  $U$ . Let  $A$  be a  $p \times q$  matrix with entries in  $\mathcal{O}(U)$  and let  $\mathcal{A} \subset \mathcal{O}_U^p$  denote the sheaf of submodules generated by the columns of  $A$ . The generic diagram  $G_a = G(\mathcal{A}_a)$ ,  $a \in U$ , is Zariski semicontinuous, by the coherent case. Thus there is a unique locally finite filtration of  $U$  by closed analytic subsets,

$$U = \Sigma_0(\mathcal{A}) \supset \Sigma_1(\mathcal{A}) \supset \Sigma_2(\mathcal{A}) \supset \dots,$$

such that (1)  $G_a$  is constant on each  $\Sigma_k - \Sigma_{k+1}$ , where  $\Sigma_k = \Sigma_k(\mathcal{A})$ , and (2)  $G_a < G_b$  when  $a \in \Sigma_k$  and  $b \in \Sigma_{k+1}$ . If each  $\Sigma_k$  is regularly situated with respect to  $X$  (in the sense of Łojasiewicz [18, § 18]), then  $A \cdot \mathcal{E}(X; \cdot)^q$  is a closed submodule of  $\mathcal{E}(X; K)^p$  ([5]). ( $\mathcal{E}(X; K)$  denotes the  $K$ -valued Whitney  $\mathcal{C}^\infty$  functions on  $X$ .)

There are interesting consequences in several complex variables: Suppose  $K = C$ . Let  $\mathcal{A}^\infty(\Omega)$  denote the space of holomorphic functions with  $\mathcal{C}^\infty$  boundary values on a pseudoconvex domain  $\Omega$  with smooth boundary in  $C^n$ . Assume  $\bar{\Omega} \subset U$ . If  $\Omega$  is bounded, then

$$A \cdot \mathcal{A}^\infty(\Omega)^q = \mathcal{A}^\infty(\Omega)^p \cap A \cdot \mathcal{E}(\bar{\Omega}; C)^q,$$

by exactness of the  $\bar{\partial}$ -complex [16] (cf. [25]). It follows that if each  $\Sigma_k$  is regularly situated with respect to  $\bar{\Omega}$ , then  $A \cdot \mathcal{A}^\infty(\Omega)^q$  is a closed submodule of  $\mathcal{A}^\infty(\Omega)^p$ . (For special cases obtained previously, see [12] and its bibliography.) Assume, moreover, that  $\Omega$  is bounded and strictly pseudoconvex. Let  $\mathcal{R} \subset \mathcal{O}_U^q$  denote the sheaf of relations among the columns of  $A$ . If each  $\Sigma_k(\mathcal{A})$  and  $\Sigma_l(\mathcal{R})$  is regularly situated with respect to  $\bar{\Omega}$ , then the canonical surjection  $\mathcal{A}^\infty(\Omega)^q \rightarrow A \cdot \mathcal{A}^\infty(\Omega)^q$  admits a continuous linear splitting ([5]).

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