POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCLIV

BOGDAN ZIEMIAN

Generalized analytic functions with applications to singular ordinary and partial differential equations

WARSZAWA 1996

Bogdan Ziemian
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: ziemian@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

P R I N T E D   I N   P O L A N D

ISSN 0012-3862

CONTENTS

1991 *Mathematics Subject Classification*: 34A20, 35A20, 35C20, 46F12, 30D15, 46F15.
Received 19.12.1994.

Introduction

The starting point for the paper is the classical one-dimensional theory of functions which are representable in the form of right-sided Laplace integrals: $\int_0^\infty e^{-y\alpha}\, d\mu(\alpha)$. In logarithmic coordinates ($s = -\ln x$) these functions are of the form $(*)$ $\int_0^\infty x^\alpha\, d\mu(\alpha)$ for $x > 0$, where $d\mu$ is a certain measure on $(0, \infty)$, and can be regarded as a "continuous" analogue of the "discrete" power series $\sum_{j=0}^\infty a_j x^j$ representing functions analytic in a neighbourhood of zero. Such functions appear naturally as solutions to singular equations (with irregular singular points) and as shown in Section 15 practically all special functions are of that form (in suitable variables). The analysis of special functions shows, however, that it is desirable to extend the class of functions $(*)$ by replacing the measure $d\mu$ by more general objects which may be "evaluated" at the functions $\alpha \mapsto x^\alpha$ for fixed x. However, one should be careful to avoid objects supported by the point ∞ ([Ka], [M]) since they would represent infinitely flat functions and one would loose the "analytic" character of functions of the form $(*)$ (see Property A in Section 9). The right (and practically unique) choice is the class of Laplace distributions introduced and studied in Section 4. A Laplace distribution T evaluated at test functions $\alpha \mapsto x^\alpha$ (written $(**)$ $f(x) = T[x^\alpha]$) defines a function f which generalizes $(*)$ and is called here a generalized analytic function (GAF for short) (Definition 9.1). For a given GAF f the distribution T can be found as the difference of boundary values (i.e. the jump) of the Mellin transformation (i.e. the right-sided Laplace transformation in logarithmic coordinates) and can be regarded as a generalization of the classical Borel transformation (see Section 12). This motivates the study of the Mellin transformation in the setting of generalized functions (Section 3) and the study of hyperfunctions (Section 2.2), which provides general methods for dealing with the boundary values of holomorphic functions. On the other hand, the study of Borel summability of solutions to singular differential equations started by Jean Ecalle in the 70-ties (now being continued i.a. by B. Malgrange, M. Sato, T. Kawai, F. Pham) led him to introduce the class of so-called resurgent functions (Section 14). These form a subclass of GAFs consisting (roughly speaking) of those functions whose Borel transforms continue analytically to multivalued functions with a discrete set of branching points. The operations of taking jumps at the branching points lead to the definition of the so-called alien derivatives of resurgent functions (Sections 5.1 and 14.1).

Alien derivatives turned out to be a powerful tool in the problems of analytic classification of nonlinear ordinary differential equations (Appendix II.1) and in proving the Poincaré–Dulac conjecture on the discreteness of limit cycles for an analytic vector field on the plane ([E3]) (to mention only two of a wide range of applications). The effectiveness of alien derivatives is due to the fact that the resurgent functions which arise as solutions to problems formulated in terms of analytic functions have the property that from the jumps of their Borel transforms one can recover the Borel transforms themselves, up to a simple relation (called by Ecalle the bridge equation). This phenomenon (illustrated in Section 15 by the example of the confluent hypergeometric equation) justifies the name "resurgent" given to those functions by J. Ecalle.

Generalized analytic functions behave in a simple way under fundamental algebraic and differential operations and analytic changes of variable as shown in Section 13. From the geometric point of view (of microlocal analysis) they form a natural subclass of distributions conormal with respect to zero whose symbols (in the sense of Hörmander and Weinstein) have a very explicit form (see Appendix I).

The main application of GAFs given in the paper is to so-called elliptic Fuchsian type partial differential equations. Examples of such equations are supplied by elliptic equations at infinity (in logarithmic coordinates). The fundamental example here is the Laplace operator (at infinity) (see Example 17.1) and one expects that Fuchsian type PDEs with parameters may serve as a local model of Laplace–Beltrami operators on symmetric spaces, near the boundary points. Systems of such equations were studied (by means of algebraic microlocal analysis) by Kashiwara [Schl] under the name of holonomic systems with regular singularities.

The main result of the paper (Theorem 17.2) deals with the "constant" coefficient case. It states that under natural geometric conditions on the principal Mellin symbol all solutions to a Fuchsian type equation in 2 variables are GAFs as functions of one variable, the other being regarded as parameter. The result, which is based on earlier work of the author (collected in the book [SZ]), rests on Phragmén–Lindelöf type theorems (Section 6.1) which are used to eliminate infinitely flat functions. The proofs of the latter theorems rely on Paley–Wiener type theorems for Mellin analytic functionals (Section 6) which allow one to represent certain entire functions as the Mellin transforms of analytic functionals (with bounded carriers). The main tool used to prove Theorem 17.2 is the study of the modified Cauchy transformation (Sections 8 and 17) which arises as the Mellin transform of the cut-off function localizing the solution near the origin. This, in turn, leads to the study of Nilsson type integrals $\widetilde{\mathcal{C}}_{\zeta_1}(\zeta_2)$ with holomorphic parameters ([N], [L], [An]). The methods used work in any dimension n; however, in the case $n = 2$ explicit formulas for the Borel transforms (in one variable, with the other regarded as parameter) can be given (Theorem 17.3) for solutions to Fuchsian type partial differential equations.

Inspection of the form of solutions thus obtained shows that they are in fact GAFs in 2 variables; hence the need for the theory of GAFs in several variables, presented in Section 18 (we have deliberately postponed until that section the presentation of the multidimensional theory in order to avoid technical complications of hyperfunction theory in several variables). Further, it is seen that the solutions are resurgent functions, which opens new horizons in their study by means of alien analysis, as well as new perspectives in the investigation of the corresponding nonlinear equations (see Appendix II.2).

The reader interested in systematic presentation of the methods of (multidimensional) Mellin transformation and the "smooth" version of the results presented in the paper is referred to [SZ]. One will also find there an extensive bibliography of the subject.

During the preparation of the paper the author benefited partially from the grant 210459101 of the Committee for Scientific Research.

I. Preliminaries

1. A review of classical results in the theory of Laplace integrals. Practically all the results stated in this section can be found (together with proofs) in the monograph [Wi] by Widder. To underline the analogy with power series and having in mind further developments we choose to present them in the logarithmic coordinates $s = -\ln x$.

Let $a \in \mathbb{C}$ and denote by $(\widetilde{\mathbb{C}\setminus\{0\}}, a)$ the universal covering space of $\mathbb{C}\setminus\{0\}$ with a base point a, i.e. the set of homotopy classes of continuous curves in $\mathbb{C}\setminus\{0\}$ with one end point at a. Actually $(\widetilde{\mathbb{C}\setminus\{0\}}, a)$ is nothing but the Riemann surface for the complex logarithmic function $\ln w$, $w = x + iy$. In the sequel we shall assume that $a \in \mathbb{R}_+$ is sufficiently small and we often neglect it in the notation. In particular, it follows that for every fixed $w \in (\widetilde{\mathbb{C}\setminus\{0\}}, a)$ the general exponential function $\mathbb{C} \ni z \mapsto w^z \in \mathbb{C}$ is well defined and holomorphic on $\mathbb{C}$.

Consider the function

$$f(w) = \int_0^\infty w^\alpha T(\alpha)\, d\alpha, \tag{1.1}$$

where $T(\alpha)$ is integrable on $(0, r)$ for any $r > 0$.

The simplest natural example are the functions

$$\frac{1}{(-\ln x)^{j+1}} = \int_0^\infty x^\alpha \frac{\alpha^j}{j!}\, d\alpha, \qquad j = 0, 1, \ldots \tag{1.2}$$

(see §15 for other examples).

Lemma 1.1. *If* (1.1) *converges for some* $\gamma \in \mathbb{C}$ *then it converges for all* $w \in (\widetilde{\mathbb{C}\setminus\{0\}}, a)$ *with* $0 < |w| < |\gamma|$.

Proof. Suppose that (1.1) converges for some $\gamma \in \mathbb{C}$. Take

$$B(\alpha) = \int_0^\alpha \gamma^\varrho T(\varrho)\, d\varrho.$$

We have

$$\gamma^\alpha T(\alpha) = \frac{d}{d\alpha}\int_0^\alpha \gamma^\varrho T(\varrho)\, d\varrho \qquad \text{(at Lebesgue points).}$$

Consequently,

$$\begin{aligned}\int_0^R w^\alpha T(\alpha)\, d\alpha &= \int_0^R (w/\gamma)^\alpha \gamma^\alpha T(\alpha)\, d\alpha = \int_0^R (w/\gamma)^\alpha\, dB(\alpha)\\ &= (w/\gamma)^R B(R) + (\ln w - \ln\gamma)\int_0^R (w/\gamma)^\alpha B(\alpha)\, d\alpha\end{aligned}$$

(since $(d/d\alpha)(w/\gamma)^\alpha = (\ln w - \ln\gamma)(w/\gamma)^\alpha)$. If $|w| < |\gamma|$ the first term goes to zero because $B(R)$ has a finite limit as $R \to \infty$. The other term is absolutely convergent. In fact,

$$\Big|\int_0^\infty (w/\gamma)^\alpha B(\alpha)\,d\alpha\Big| \le M\int_0^\infty |w/\gamma|^\alpha\,d\alpha < \infty,$$

where $M = \sup_{0\le\alpha<\infty}|B(\alpha)|$.

Lemma 1.1 allows us to define the *convergence radius* $\delta_{\rm c}$ of (1.1) as the supremum of $|\gamma|$ for γ such that (1.1) converges.

In analogy with the classical formula $1/\varrho = \overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}$ for the convergence radius of the power series $\sum_{n=0}^\infty a_n x^n$ we have the following lower bound for $\delta_{\rm c}$.

LEMMA 1.2. *Let $\varrho \ge 0$ be defined by the upper limit*

$$\overline{\lim_{\alpha\to\infty}}\frac{\ln|T(\alpha)|}{\alpha} = -\ln\varrho. \tag{1.3}$$

Then $\delta_{\rm c} \ge \varrho$.

Proof. Let $0 < \delta < 1$ be arbitrary. Then from (1.3) we have $|T(\alpha)| \le C(\varrho\delta)^{-\alpha}$ for large $\alpha > N$.

Hence

$$\int_N^\infty |w|^\alpha|T(\alpha)|\,d\alpha \le C\int_N^\infty \left|\frac{w}{\varrho\delta}\right|^\alpha d\alpha,$$

which is finite for $|w| < \varrho\delta$, i.e. for $|w| < \varrho$.

However, contrary to the case of power series we cannot claim in general that $\delta_{\rm c} = \varrho$. Actually, in this respect, it is more convenient to consider an expression slightly more general than (1.1):

$$f(w) = \int_0^\infty w^\alpha\,d\mu(\alpha), \tag{1.4}$$

where the function $\mu(\alpha)$ is of bounded variation on every interval $(0, r)$, $r > 0$. It will also be convenient to assume that $\mu(\alpha)$ is normalized, i.e.

$$\mu(0) = 0, \quad \mu(\alpha) = \frac{\mu(\alpha_+) + \mu(\alpha_-)}{2} \quad \text{for } \alpha > 0,$$

where $\mu(\alpha_\pm)$ devotes the right (left) limit at α. The integral (1.4) can be understood as a Stieltjes integral or an integral with respect to the measure $d\mu(\alpha)$ whose distribution function is $\mu(\alpha)$. The expression (1.4) has the advantage over (1.1) that it incorporates the "discrete" case of power series $\sum_{n=0}^\infty a_n w^n$. Namely, in that case we have $d\mu(\alpha) = \sum_{n=0}^\infty a_n\delta_{(n)}$, where $\delta_{(n)}$ is the Dirac delta at n.

Returning to the problem of the convergence radius we have in the present setting:

THEOREM 1.1 ([Wi], Th. 2.4 a). *If*

$$\lim_{\alpha\to\infty}\frac{\ln|\mu(\alpha)|}{\alpha} = -\ln\varrho \quad \textit{with } \varrho \ne 1$$

then $\delta_{\rm c} = \varrho$ for the integral (1.4).

This leaves open the case of $\varrho = 1$, which is not surprising since the function w^α in the integral (1.4) changes drastically its properties as $|w|$ crosses 1.

Analogously to δ_{c} let us introduce the *radius of absolute convergence* of (1.1) as the maximum of those $\gamma > 0$ such that $\int_0^\infty |\gamma|^{-\alpha}\,|d\mu(\alpha)|$ converges. For the radius of absolute convergence (denoted by δ_{a}) we clearly have

$$\delta_{\mathrm{a}} \le \delta_{\mathrm{c}}.$$

However, contrary to the situation for power series, in general one does not have $\delta_{\mathrm{a}} = \delta_{\mathrm{c}}$ as the following example shows.

Example 1.1. For the function

$$\int_0^\infty x^\alpha e^\alpha \sin e^\alpha \, d\alpha = \int_1^\infty s^{\ln x} \sin s \, ds$$

we have $\delta_{\mathrm{c}} = 1$ and $\delta_{\mathrm{a}} = 1/e$.

Indeed, the integral converges absolutely for $x < 1/e$ since $|x^\alpha e^\alpha \sin e^\alpha| \le (xe)^\alpha$ and does not converge absolutely for $x = 1/e$ since

$$\int_1^\infty \frac{|\sin s|}{s}\, ds = +\infty.$$

On the other hand, the equality $\delta_{\mathrm{c}} = 1$ is clear from the properties of the integral $\int_1^\infty \frac{\sin s}{s^a}\, ds$.

Analogously to Theorem 1.1 we have

Theorem 1.2 ([Wi]). *If*

$$\varlimsup_{\alpha\to\infty} \frac{\ln v(\alpha)}{\alpha} = -\ln \varrho \qquad \textit{with } \varrho \ne 1$$

then $\delta_{\mathrm{a}} = \varrho$, *where* $v(\alpha)$ *is the variation of* μ *in the interval* $(0,\alpha)$ (*if* $\mu(\alpha) = \int_0^\alpha \phi(s)\, ds$ *then* $v(\alpha) = \int_0^\alpha |\phi(s)|\, ds$).

There are numerous theorems relating the behaviour of the function f at zero and 1 (if $\delta_{\mathrm{c}} = 1$) to the behaviour of $T(\alpha)$ at 0_+ and ∞. They are known as *Abel theorems* generalizing the classical result of Abel for the radial limit of a convergent power series (Abel summability). As an example we quote the following.

Theorem 1.3 ([Wi]). *If*

$$f(x) = \int_0^\infty x^\alpha \, d\mu(\alpha)$$

and the integral is convergent for $0 < x < 1$ *and if for some* A *and* $\gamma \ge 0$,

$$\mu(t) \to At^\gamma/\Gamma(\gamma+1) \quad \textit{as } t \to 0_+ \ (t\to\infty, \ \textit{resp.})$$

then

$$f(x) \to A/(-\ln x)^\gamma \qquad \textit{as } x \to 0_+ \ (x \to 1, \ \textit{resp.}).$$

Various versions of the "converse" result are known as *Tauberian theorems*. An up to date account of (multidimensional) Tauberian theorems can be found in the monograph [VDZ].

2. Boundary values of holomorphic functions in one variable. Let Ω be an open set in $\mathbb{R}$. An open set V in $\mathbb{C}$ is called a *complex neighbourhood* of Ω if $V \cap \mathbb{R} = \Omega$. Let $F \in \mathcal{O}(V \setminus \Omega)$. We are interested in conditions that the function F should satisfy, when approaching the set Ω, to extend holomorphically to V. The simplest condition is given by the classical Painlevé theorem which states that a function F holomorphic on $V \setminus \Omega$ and continuous on V extends holomorphically to V. The condition of continuity of F on Ω can be formulated as the existence and equality of the limits

$$\lim_{z\to x,\, \operatorname{Im} z>0} F(z) = \lim_{z\to x,\, \operatorname{Im} z<0} F(z) \quad \text{for } x \in \Omega, \tag{2.1}$$

i.e. the equality of the boundary values of F from above and below.

We now proceed to a generalization of the notion of boundary values.

2.1. *Distributions as boundary values of holomorphic functions.* We consider functions $F \in \mathcal{O}(V \setminus \Omega)$ satisfying the so-called condition of polynomial growth (near $\mathbb{R}$):

DEFINITION 2.1. We say that a function $F \in \mathcal{O}(V \setminus \Omega)$ is of *polynomial growth* near $\mathbb{R}$ if for every compact set $K \subset \Omega$ there exist constants $N = N(K)$ and $C = C(K)$ such that

$$|F(\alpha + i\beta)| \le C|\beta|^{-N} \quad \text{for } \alpha \in K \text{ and } 0 < |\beta| \le \eta \tag{2.2}$$

for some small η.

We intend to give sense to the limits (2.1) for functions of polynomial growth.

Let $F \in \mathcal{O}(V \setminus \Omega)$ be of polynomial growth. Since the limits (2.1) need not exist in the pointwise sense, consider the limits of the integral means

$$\lim_{\beta\to 0_+} \int_\Omega F(\alpha + i\beta)\phi(\alpha)\, d\alpha, \qquad \lim_{\beta\to 0_+} \int_\Omega F(\alpha - i\beta)\phi(\alpha)\, d\alpha, \tag{2.3}$$

where ϕ belongs to the space $C_0^\infty(\Omega)$ of compactly supported smooth functions on Ω.

The theorem below can be regarded as a generalization of the classical Painlevé theorem.

THEOREM 2.1. *If $F \in \mathcal{O}(V \setminus \Omega)$ is of polynomial growth then for every $\phi \in C_0^\infty(\Omega)$ the limits* (2.3) *exist. Moreover*, *if the limits* (2.3) *coincide for every $\phi \in C_0^\infty(\Omega)$ then F extends to a holomorphic function on V.*

We shall denote those limits by

$$\begin{aligned} F(\cdot + i0)[\phi] &= \lim_{\beta\to 0_+} \int_\Omega F(\alpha + i\beta)\phi(\alpha)\, d\alpha, \\ F(\cdot - i0)[\phi] &= \lim_{\beta\to 0_+} \int_\Omega F(\alpha - i\beta)\phi(\alpha)\, d\alpha, \end{aligned} \tag{2.4}$$

and call them the *boundary values* of F from above and below, respectively. The difference of the boundary values (2.4) is denoted by $b(F)[\phi]$ or $[F][\phi]$ and is called the *jump* of F across $\mathbb{R}$. Thus

$$b(F)[\phi] := [F][\phi] := F(\cdot + i0)[\phi] - F(\cdot - i0)[\phi] \quad \text{for } \phi \in C_0^\infty(\Omega).$$

P r o o f o f T h e o r e m 2.1. We may assume that Ω is an open interval. Let K be a fixed closed subinterval of Ω. Choose $0 < \eta < 1$ so that

$$A = \{z = \alpha + i\beta \in \mathbb{C} : \alpha \in K,\ 0 < \beta \le \eta\} \subset V_+ = V \cap \{\operatorname{Im} z > 0\}$$

and (2.2) holds for $\alpha + i\beta \in A$. Let $\mathring{z}$ be a fixed point in A with $\operatorname{Im} \mathring{z} = \eta$. Define

$$J_+F(z) = \int_{\gamma_z} F(\theta)\, d\theta, \qquad z \in V_+,$$

where γ_z is the curve in V_+ joining $\mathring{z}$ to $z = \alpha + i\beta$ and consisting of two segments $[\mathring{z}, \alpha + i\eta]$ and $[\alpha + i\eta, \alpha + i\beta]$. Clearly $\frac{d}{dz} J_+F(z) = F(z)$ for $z \in V_+$ and from (2.2) and the fact that F is bounded on $K + i\eta$ we get for $z \in A$,

$$|J_+F(z)| \le \Big| \int_{\alpha+i\beta}^{\alpha+i\eta} F(\theta)\, d\theta + \int_{\alpha+i\eta}^{\mathring{z}} F(\theta)\, d\theta \Big| \le \widetilde{C} + C \int_{\beta}^{\eta} t^{-N}\, dt \qquad \text{for } z \in A.$$

Hence there exists a constant $C_1(K)$ such that for $\alpha + i\beta \in A$,

$$|J_+F(\alpha + i\beta)| \le \begin{cases} C_1 \beta^{-N+1} & \text{if } N > 1, \\ C_1 |\ln \beta| & \text{if } N = 1. \end{cases}$$

Iterating the above operation $N + 1$ times we find $J_+^{N+1}F \in \mathcal{O}(V_+)$ such that $(d/dz)^{N+1} J_+^{N+1} F = F$ on V_+ and $J_+^{N+1}F(\alpha + i\varepsilon)$ converges uniformly, as $\varepsilon \to 0_+$, to a continuous function $F_+^{N+1}(\alpha)$ on K. Thus $J_+^{N+1}F$ extends to a continuous function on $\overline{A} = \{z \in \mathbb{C} : \alpha \in K,\ 0 \le \beta \le \eta\}$.

Let $\phi \in C_0^\infty(\Omega)$, $\operatorname{supp} \phi \subset K$. Integrating by parts we get from the Lebesgue theorem

$$\begin{aligned} \lim_{\varepsilon \to 0_+} \int_{\Omega} F(\alpha + i\varepsilon)\phi(\alpha)\, d\alpha &= \lim_{\varepsilon \to 0_+} (-1)^{N+1} \int_{\Omega} J_+^{N+1} F(\alpha + i\varepsilon)(d/d\alpha)^{N+1}\phi(\alpha)\, d\alpha \\ &= (-1)^{N+1} \int_{\Omega} F_+^{N+1}(\alpha)(d/d\alpha)^{N+1}\phi(\alpha)\, d\alpha. \end{aligned}$$

Thus the limit $F(\alpha + i0)$ exists. By symmetry one defines J_-F and $J_-^{N+1}F$ on $V_- = V \cap \{\operatorname{Im} z < 0\}$ and one proves the existence of the limit $F(\alpha - i0)$. Now if $F(\alpha - i0) = F(\alpha + i0)$ it follows (see Exercise 10.1 in [Sz]) that

$$F_+^{N+1}(\alpha) - F_-^{N+1}(\alpha) = w(\alpha),$$

where w is a polynomial of degree at most N. Hence by the classical Painlevé theorem we infer that the function

$$F^{N+1}(z) = \begin{cases} J_+^{N+1}F(z) & \text{for } z \in V_+, \\ F_+^{N+1}(\alpha) & \text{for } z = \alpha, \\ J_-^{N+1}F(z) + w(z) & \text{for } z \in V_-, \end{cases}$$

is holomorphic on V. Hence $(d/dz)^{N+1} F^{N+1}$ is the desired extension of $F \in \mathcal{O}(V \setminus \Omega)$ to V.

Let us have a closer look at how the boundary values $F(\cdot \pm i0)[\phi]$ depend on the test function ϕ. To fix ideas let us consider the case of $F(\cdot + i0)[\phi]$. First, we observe the linearity:

$$F(\cdot + i0)[\phi_1 + \phi_2] = F(\cdot + i0)[\phi_1] + F(\cdot + i0)[\phi_2] \qquad \text{for } \phi_1, \phi_2 \in C_0^\infty(\Omega).$$

Thus we are dealing with a linear functional (which we denote by $F(\cdot + i0)$). Second, it follows from the proof of Theorem 2.1 that for every compact set $K \subset \Omega$ there exist constants $C(K)$ and $N(K)$ such that

$$(2.5) \qquad |F(\cdot + i0)[\phi]| \le C \sum_{j=0}^{N} \sup_{\alpha \in K} \left| \frac{d^j}{d\alpha^j} \phi(\alpha) \right| \qquad \text{for } \phi \in C_0^\infty(\Omega),\ \operatorname{supp} \phi \subset K.$$

The estimate (2.5) means that the linear functional $F(\cdot + i0)$ is continuous in the sense of the norm on the right hand side of (2.5). Recall that such a functional is called a *distribution on* Ω (see e.g. [Sz]). The space of distributions on Ω is denoted by $D'(\Omega)$ (= the dual space of $C_0^\infty(\Omega)$ with the topology in $C_0^\infty(\Omega)$ given by the norms on the right hand side of (2.5)). Thus in the terminology introduced above, Theorem 2.1 can be verbalized as follows:

For every function $F \in \mathcal{O}(V \setminus \Omega)$ *of polynomial growth* (*near* $\mathbb{R}$) *the jump* $b(F)$ *of* F *is a distribution on* Ω. *If in addition* $b(F) = 0$ *then* $F \in \mathcal{O}(V)$.

The converse result is also true (see [Br]): every distribution u on Ω can be represented as the jump of a holomorphic function F on $V \setminus \Omega$ (where V is a complex neighbourhood of Ω). Such a function F is called a *defining function* for u. We then use the notation $u = [F]$.

2.2. *Hyperfunctions in one variable.* In Theorem 2.1 we dealt with the boundary values $F(\cdot \pm i0) = \lim_{\beta \to 0_+} F(\cdot \pm i\beta)$ of a holomorphic function $F \in \mathcal{O}(V \setminus \Omega)$, where Ω is an open set in $\mathbb{R}$ and V its open complex neighbourhood (i.e. Ω is relatively closed in V). We assumed that the function F was of polynomial growth near Ω. We also defined the jump of F across $\mathbb{R}$ as the difference of the boundary values from above and from below and we denoted it by

$$(2.6) \qquad [F] = F(\cdot + i0) - F(\cdot - i0).$$

We now give a meaning to the boundary values of F and to the jump of F without imposing any growth restriction on F near Ω. The intuition is provided again by the Painlevé theorem: in order that F extend holomorphically from $V \setminus \Omega$ to V it is sufficient that the limits $F(\cdot \pm i0)$ exist (e.g. in the sense of distributions) and were equal. The idea consists in considering the assertion of the Painlevé theorem as the definition of the coincidence of the boundary values in the case where F has no growth restrictions:

Let $F \in \mathcal{O}(V \setminus \Omega)$. We denote by $[F]$ the equivalence class of F modulo $\mathcal{O}(V)$ (i.e. the set $F + \mathcal{O}(V)$) and call it the *hyperfunction* on Ω with *defining function* F. Clearly a defining function is not unique and is determined modulo holomorphic functions on V. The space of hyperfunctions on Ω is denoted by $B(\Omega)$, i.e. we have

$$(2.7) \qquad B(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V).$$

Now, we show that the use of the symbol $[F]$ to denote the hyperfunction corresponding to a defining function $F \in \mathcal{O}(V \setminus \Omega)$ is compatible with the interpretation (2.6). To this end we define the (hyperfunction) boundary values of F. Under the notation $V_\pm$ for the sets $V \cap \{\pm \operatorname{Im} z > 0\}$ we put

$$F^+(z) = \begin{cases} F(z), & z \in V_+, \\ 0, & z \in V_-, \end{cases} \qquad F^-(z) = \begin{cases} 0, & z \in V_+, \\ -F(z), & z \in V_-, \end{cases}$$

and define

$$F(\cdot + i0) = [F^+] \in B(\Omega), \qquad F(\cdot - i0) = [F^-] \in B(\Omega).$$

The hyperfunctions $F(\cdot \pm i0)$ are called the *boundary values* of F. Clearly we have

$$[F] = F(\cdot + i0) - F(\cdot - i0).$$

It is proved by applying the Mittag-Leffler type theorem (see e.g. [Ka], [Ko]) that the space (2.7) is independent of the choice of the complex neighbourhood V of Ω.

Hyperfunctions can be localized to open sets and it is possible to define the support of a hyperfunction as a maximal closed set outside which the hyperfunction is zero. Hyperfunctions behave in many respects as "usual" functions: one may add hyperfunctions, multiply them by analytic functions, differentiate and integrate, essentially by performing those operations on the level of defining functions. Hyperfunctions with compact support K (denoted by B_K) can be treated, analogously to the case of distributions, as continuous linear functionals on a suitable space of test functions. In our case it is the space of functions analytic on K:

$$\mathcal{A}(K) = \varinjlim_{K \subset V} \mathcal{O}(V)$$

(recall that the inductive limit $\varinjlim_{K \subset V} \mathcal{O}(V)$ is the space $\bigcup_{V \supset K} \mathcal{O}(V)$ with the topology of almost uniform convergence on each fixed V). We define $\mathcal{A}'(K)$, the dual of $\mathcal{A}(K)$, as the space of linear continuous functionals on $\mathcal{A}(K)$ (continuity of a linear functional u means here that if $\phi_n \to 0$ in $\mathcal{A}(K)$ then $u[\phi_n] \to 0$). The space $\mathcal{A}'(K)$ is called the *space of analytic functionals*.

THEOREM 2.2 (Köthe [Ka]). *For every compact set $K \subset \mathbb{R}$ there exists a natural isomorphism*

$$B_K \simeq \mathcal{A}'(K)$$

given by the assignment $B_K \ni f \mapsto If \in \mathcal{A}'(K)$, where for $f = [F]$,

$$If[\phi] = -\int_\Gamma F(z)\phi(z)\,dz \qquad \text{for } \phi \in \mathcal{A}(K),$$

where Γ is a smooth curve encircling the set K once in the anticlockwise direction and contained in the set of holomorphy of F and of (*the holomorphic extension of*) *ϕ. The inverse mapping is given by the Cauchy transformation*: *if $u \in \mathcal{A}'(K)$ then $I^{-1}u = [Cu]$, where*

$$Cu(z) = \frac{-1}{2\pi i} u\left[\frac{1}{z - \alpha}\right].$$

Proof. Most of the assertions are obvious and we restrict ourselves to checking that $I \cdot I^{-1} = \mathrm{id}$. This amounts to proving that

$$-\int_\Gamma Cu(z)\phi(z)\,dz = u[\phi] \qquad \text{for } \phi \in \mathcal{A}(K).$$

Subject to a suitable choice of the curve Γ we have by the Cauchy integral formula

$$-\int_\Gamma Cu(z)\phi(z)\,dz = \frac{1}{2\pi i}\int_\Gamma u\left[\frac{1}{z-\alpha}\right]\phi(z)\,dz = u\left[\frac{1}{2\pi i}\int_\Gamma \frac{\phi(z)}{z-\alpha}\,dz\right] = u[\phi].$$

In the computation we have commuted the integration over a parameter with the action of an analytic functional. A rigorous justification of that fact requires consideration of the Riemann sums for the integral and uses the linearity and continuity of the analytic functional u.

3. Mellin analytic functionals, Mellin hyperfunctions and Mellin distributions. Set $I = (0,t]$ for some $t > 0$, and

$$I^\theta = \{z \in \mathbb{C} : 0 < |z| \le t,\ |\arg z| \le \theta\}.$$

In other words, I^θ is a sector of radius t and opening angle 2θ. If $|\theta| > \pi$ then I^θ is understood to be a sector in the universal covering space $(\widetilde{\mathbb{C}\setminus\{0\}}, a)$ of $\mathbb{C}\setminus\{0\}$ with base point $a \in \mathbb{R}_+$. We also consider the directional compactification $\overline{I}^\theta$ of I^θ defined by adjoining to I^θ the arc $S_\theta = \{z \in \mathbb{C} : |z| = 1,\ |\arg z| \le \theta\}$ interpreted as the set of all directions at zero that can be reached by the points in I^θ.

Any subset (of $\widetilde{\mathbb{C}\setminus\{0\}}$) of the form $\{w \in \widetilde{\mathbb{C}\setminus\{0\}} : M_1 < \arg z < M_2\}$ for some $0 < M_1 < M_2$ is called an *open* (infinite) *sector*. Its intersection with an open neighbourhood of zero is called an *open local sector* (at zero). An open set $V \subset \widetilde{\mathbb{C}\setminus\{0\}}$ is called a *local sectorial set* if there exists $r > 0$ such that $V \cap \{|w| \le r\}$ coincides with the open cone (with vertex at zero) over the set $V \cap \{|w| = r\}$. A subset $K \subset V$ is called a *proper sectorial subset* if K is a local sectorial set and the closure of K in $\widetilde{\mathbb{C}\setminus\{0\}}$ is contained in V.

Definition 3.1. Let $b \in \mathbb{R}$. The space of *Mellin analytic test functions* on $\overline{I}^\theta$ is the inductive limit

$$\underset{\sim}{\mathcal{M}}_{(b)}(\overline{I}^\theta) = \varinjlim_{V \supset I^\theta} \{\phi \in \mathcal{O}(V) : \text{there exists } \delta > 0 \text{ with } \sup_{z\in K} |z^{b-\delta+1}\phi(z)| < \infty$$
$$\text{on any bounded proper sectorial set } K \subset V\},$$

where V is a local sectorial neighbourhood of I^θ in $\widetilde{\mathbb{C}\setminus\{0\}}$.

The dual space $\underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I}^\theta)$ of $\underset{\sim}{\mathcal{M}}_{(b)}(\overline{I}^\theta)$ is called the space of *Mellin analytic functionals* carried by $\overline{I}^\theta$.

Another equivalent way of defining Mellin analytic functionals is as follows: Let V be an open local sectorial neighbourhood of I^θ. For $b \in \mathbb{R}$ set

$$\widetilde{\mathcal{M}}_{(b)}(V) = \{F \in \mathcal{O}(V) : \text{for every } \varepsilon > 0,\ \sup_{z\in K} |z^{-b+\varepsilon}F(z)| < \infty$$
$$\text{on every bounded proper sectorial set } K \subset V\}.$$

Theorem 3.1. *There exists a canonical isomorphism*

$$\underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I}^\theta) \simeq \widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)/\widetilde{\mathcal{M}}_{(b)}(V),$$

where V is any local sectorial neighbourhood of I^θ.

P r o o f. The isomorphism is given as follows (cf. Köthe Theorem 2.2): Take $F \in \widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)$ and define

$$f[\phi] = -\int_\Gamma F(z)\phi(z)\,dz \quad \text{for } \phi \in \underset{\sim}{\mathcal{M}}_{(b)}(\bar{I}^\theta), \tag{3.1}$$

where Γ is a curve with end points at zero, encircling I^θ once in the anticlockwise direction, contained in V and in the set of holomorphy of ϕ. The integral is well defined since $F(z)\phi(z)$ is integrable on the support of Γ. Moreover, it follows from (the integrable variant of) the Cauchy formula that if F is changed by a function in $\widetilde{\mathcal{M}}_{(b)}(V)$ the resulting f will be the same. This shows that we found a canonical assignment

$$\widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)/\widetilde{\mathcal{M}}_{(b)}(V) \to \underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta),$$

which, as can be seen, is independent of the sectorial neighbourhood V. To prove that the assignment is 1-1 and onto we indicate the inverse: Let $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$. Then for every $\varepsilon > 0$, $f \in \underset{\sim}{\mathcal{M}}'_{b-\varepsilon}(\bar{I}^\theta)$, where $\underset{\sim}{\mathcal{M}}'_c(\bar{I}^\theta)$ is the dual of the space

$$\underset{\sim}{\mathcal{M}}_c(\bar{I}^\theta) = \varinjlim_{V \supset I^\theta} \{\phi \in \mathcal{O}(V) : \sup_{z\in K} |z^{c+1}\phi(z)| < \infty \text{ on any proper bounded sectorial subset } K \subset V\}.$$

Consider the logarithmic defining function

$$F(z) = \frac{-1}{2\pi i} f[G(z,\zeta)], \tag{3.2}$$

here for fixed $z \notin I^\theta$, $G(z,\zeta)$ denotes the function

$$G(z,\zeta) = \frac{1}{\zeta}\frac{e^{-(\ln z - \ln\zeta)^2}}{\ln z - \ln \zeta}.$$

To see that (3.2) makes sense we show that the function $\zeta \mapsto G(z,\zeta)$ belongs to $\underset{\sim}{\mathcal{M}}_{(\infty)}(\bar{I}^\theta)$. To this end fix a $z \notin I^\theta$ and let V be a local sector such that $z \notin V \supset I^\theta$. Then on any bounded proper sectorial set $K \subset V$ we have

$$\sup_{\zeta\in K} |\zeta^{c+1} G(z,\zeta)| \le B|z|^c \tag{3.3}$$

for any fixed $c \in \mathbb{R}$. Now it follows easily that $F \in \mathcal{O}(\widetilde{\mathbb{C}\setminus\{0\}} \setminus I^\theta)$. Further, since f maps bounded families of functions onto bounded functions it follows that F is bounded on every bounded subsector in $\widetilde{\mathbb{C}\setminus\{0\}} \setminus I^\theta$. Thus if $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$ we see that the function F defined by (3.2) belongs to $\widetilde{\mathcal{M}}_{(b)}(\widetilde{\mathbb{C}\setminus\{0\}} \setminus I^\theta)$. Thus we assign to f the class of F modulo $\widetilde{\mathcal{M}}_{(b)}(V)$, where V is any local sectorial neighbourhood of I^θ. To prove that this mapping is inverse to the one defined at the beginning of the proof it is enough to show that for $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$,

$$f[\phi] = \frac{1}{2\pi i}\int_\Gamma f[G(z,\zeta)]\phi(z)\,dz \quad \text{for } \phi \in \underset{\sim}{\mathcal{M}}_{(b)}(\bar{I}^\theta),$$

where Γ is a suitable curve. Interchanging f with the integral over Γ we see that it suffices to show that

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\phi(z)G(z,\zeta)\,dz=\phi(\zeta)$$

for ζ in a local sectorial neighbourhood of I^θ. But this is clear since the function $z\mapsto G(z,\zeta)$ has a unique simple pole at $z=\zeta$ with residue 1 and by (3.3) the integral is bounded on Γ.

DEFINITION 3.2. For $\theta=0$ (then $I^\theta=I$) the spaces

$$\underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I})\simeq\widetilde{\mathcal{M}}_{(b)}(V\setminus I)/\widetilde{\mathcal{M}}_{(b)}(V),$$

where V is any sectorial neighbourhood of I, are called the spaces of *Mellin hyperfunctions.*

It can be proved that for any fixed $b\in\mathbb{R}$ any hyperfunction on $\mathbb{R}_+$ with support in I can be imbedded in the space of Mellin hyperfunctions $\underset{\sim}{\mathcal{M}}'_{(b)}$. That imbedding, however, is non-unique due to the occurrence of Mellin hyperfunctions supported by the point zero. Therefore the situation is unsatisfactory from the point of view of local Mellin analysis of singularities (presented in [SZ]).

A special case of Mellin analytic functionals is the analytic functionals carried by sets of the form

$$I^\theta_r=\{z\in\mathbb{C}:r\le|z|\le t,\ |\arg z|\le\theta\}$$

for some $r>0$. They are defined as members of the dual $\mathcal{A}'(I^\theta_r)$ of the space

$$\mathcal{A}(I^\theta_r)=\varinjlim_{U\supset I^\theta_r}\mathcal{O}(U),$$

where U ranges over open complex neighbourhoods of I^θ_r.

A proof similar to that of Theorem 2.2 shows that

$$\mathcal{A}'(I^\theta_r)\simeq\mathcal{O}(U\setminus I^\theta_r)/\mathcal{O}(U),$$

where U is a complex neighbourhood of I^θ_r.

$\mathcal{A}'(I^\theta_r)$ can be regarded in an obvious way as a subspace of $\underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$ for any $b\in\mathbb{R}$. Conversely, given a $u\in\underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$ for a fixed $b\in\mathbb{R}$ we shall prove that $u\in\mathcal{A}'(I^\theta_r)$ if there exists a defining function G for u such that

(i) $G\in\mathcal{O}(V\setminus I^\theta_r)$, where V is a sectorial neighbourhood of I^θ,
(ii) for any $\varepsilon>0$ and any proper sectorial subset $K\subset V$,

$$|G(z)|\le C_{\varepsilon,K}|z|^{b-\varepsilon}\quad\text{for } z\in K.$$

Indeed, we have the following

LEMMA 3.1. *Let G satisfy* (i) *and* (ii) *and consider $u=[G]$ ($[G]$ is the equivalence class of G modulo $\widetilde{\mathcal{M}}_{(b)}(V)$) regarded as a linear functional on $\underset{\sim}{\mathcal{M}}_{(b)}(\bar{I}^\theta)$. Then u extends to a* (*unique*) *analytic functional in $\mathcal{A}'(I^\theta_r)$.*

Proof. According to Theorem 3.1,

$$u[\phi]=-\int\limits_{\Gamma}G(z)\phi(z)\,dz\quad\text{for }\phi\in\underset{\sim}{\mathcal{M}}_{(b)}(I^\theta),$$

where Γ is a pertinent curve as in the proof of Theorem 3.1. In particular, for Γ we can take the curve $\Gamma_{\tilde\theta}$ consisting of the two radii $\{se^{i\tilde\theta} : 0 < s \le t\varrho\}$, $\{se^{-i\tilde\theta} : 0 < s \le t\varrho\}$ and of the arc $\{t\varrho e^{i\kappa} : |\kappa| \le \widetilde\theta\}$, where $\widetilde\theta > \theta$ is close enough to θ and $\varrho > 1$ is close to 1. Choosing as K the sector $\{z \in \mathbb{C} : 0 < |z| \le \widetilde r,\ |\arg z| \le \widetilde\theta\}$, where $\widetilde r < r$, we have the estimate

$$|G(z)\phi(z)| \le C_{\varepsilon,K}|z|^{-1+\delta} \quad \text{for } z \in K \text{ and some } \delta > 0.$$

Hence the integral over the arc $\Gamma(s) = \{se^{i\kappa} : |\kappa| \le \widetilde\theta\}$ gives

$$\Big|\int_{\Gamma(s)} G(z)\phi(z)\,dz\Big| \le C \int_{\Gamma(s)} s^{-1+\delta}\,dz = \widetilde C s^{\delta}\int_{-\tilde\theta}^{\tilde\theta} d\kappa = C' s^{\delta},$$

which goes to zero as $s \to 0_+$. In a similar way the integral over the radii $\{\tau e^{\pm i\widetilde\theta} : 0 < \tau < s\}$ tends to zero as $s \to 0_+$. This shows that the curve Γ_θ can be modified to any curve encircling I_r^θ and contained in $V \setminus I_r^\theta$. Thus u extends to a functional in $\mathcal{A}'(I_r^\theta)$.

Remark 3.1. Note that the assumption (ii) is essential. If we had $u = [G] \in \widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)/\widetilde{\mathcal{M}}_{(b)}(V)$ satisfying (i) then $\widetilde u = [G] \bmod \mathcal{O}(V \setminus I^\theta)$ is clearly in $\mathcal{A}'(I_r^\theta)$ but $\widetilde u$ may differ from u by a Mellin analytic functional carried by the arc S_θ.

We shall now consider the space of Mellin distributions. To this end define (cf. [SZ]) for $b \in \mathbb{R}$ the space $\mathcal{M}'_{(b)}(I)$ of *Mellin distributions* as the dual of the space

$$\mathcal{M}_{(b)}(I) = \varinjlim_{a<b} \mathcal{M}_a(I),$$

where

$$\mathcal{M}_a(I) = \{\phi \in C^\infty(I) : \sup_{x\in I}|x^{a+1}(xd/dx)^\alpha\phi(x)| < \infty \text{ for all } \alpha \in \mathbb{N}_0\}.$$

We also set

$$\mathcal{M}'_{(-\infty)}(I) = \bigcup_{b\in\mathbb{R}} \mathcal{M}'_{(b)}(I), \qquad \mathcal{M}'_{(\infty)}(I) = \bigcap_{b\in\mathbb{R}} \mathcal{M}'_{(b)}(I).$$

Mellin distributions are characterized by the following structure theorem (see [SZ], Th. 8.2, or [Ły1] for the proof):

THEOREM 3.2. *Let $b \in \mathbb{R} \cup \{\infty\}$. In order that u belong to $\mathcal{M}'_{(b)}(I)$ it is necessary and sufficient that for every $a < b$ there exist $k_a \in \mathbb{N}_0$ and bounded functions $h_{\lambda,a}$ on I for $0 < \lambda < k_a$ such that*

$$u = \sum_{\lambda=0}^{k_a} (xd/dx)^\lambda (x^a h_{\lambda,a}) \quad \textit{in } \mathcal{M}'_a(I).$$

The principal feature of Mellin distributions is that they are concentrated on I (i.e. there are no Mellin distributions supported by $\{0\}$), which is due to the fact that the space $C^\infty_{(0)}(I) := C_0^\infty(\mathbb{R}_+) \cap C^\infty(I)$ is dense in $\mathcal{M}_{(b)}(I)$ for any $b \in \mathbb{R}$ (cf. [SZ]).

Below we consider Mellin distributions as Mellin hyperfunctions and describe their defining functions.

THEOREM 3.3. *A Mellin hyperfunction $u \in \widetilde{\mathcal{M}}_{(b)}(I)$ is a Mellin distribution in $\mathcal{M}'_{(b)}(I)$ if and only if $u = [F]$, where F satisfies the following conditions*:

(i) *$F\in\mathcal{O}(V\setminus I)$, where V is a local sectorial neighbourhood of I,*

(ii) *for any $\varepsilon>0$ there exists $k\in\mathbb{N}_0$ such that for any bounded proper sectorial subset K of V,*

$$|F(z)|\le C_{\varepsilon,K}|z|^{b-\varepsilon}/|\arg z|^k \quad \textit{for } z\in K\setminus I.$$

P r o o f. The proof essentially resembles that of Theorem 3.1 and therefore we outline only the new points. Let $u\in\mathcal{M}'_{(b)}(I)$ and consider the defining function

$$F(z)=\frac{-1}{2\pi i}u[G(z,x)],$$

where as before

$$G(z,x)=\frac{1}{x}\frac{e^{-(\ln z-\ln x)^2}}{\ln z-\ln x}.$$

To get the estimate of $F(z)$ we proceed as in the proof of Theorem 3.1: $u\in\mathcal{M}'_{(b)}(I)$, hence $u\in\mathcal{M}'_{b-\varepsilon}(I)$ for every $\varepsilon>0$. Since multiplication by $x^{b-\varepsilon}$ is an isomorphism of $\mathcal{M}'_0(I)$ onto $\mathcal{M}'_{b-\varepsilon}(I)$ and on the level of defining functions it corresponds to multiplication by $z^{b-\varepsilon}$ it is clearly enough to consider $u\in\mathcal{M}'_0(I)$. By the Structure Theorem 3.2 for $\mathcal{M}'_0(I)$ there exist $k\in\mathbb{N}_0$ and $C>0$ such that

$$|u[\phi]|\le C\sum_{j=0}^{k}\sup_{0<x\le t}|x(xd/dx)^j\phi(x)| \quad \text{for } \phi\in\mathcal{M}_0(I). \tag{3.4}$$

We clearly have $F\in\mathcal{O}((\widetilde{\mathbb{C}\setminus\{0\}})\setminus I)$ and from (3.4),

$$|F(z)|\le C\sup_{0<x\le t}\frac{1}{|\ln z-\ln x|^{k+1}}\le\widetilde{C}\sup_{0<x\le t}\frac{1}{(\ln|z|/|x|+|\arg z|)^{k+1}}\le\frac{C}{|\arg z|^{k+1}}.$$

Conversely, let F satisfy (i) and (ii). Fix $\varepsilon>0$ and consider the function $F_\varepsilon(z)=z^{-\mathring{b}+\varepsilon}F(z)$. Take $\mathring{z}\in V\setminus I$ and consider the operation

$$JF_\varepsilon(z)=\int_{\Gamma_z}\frac{F_\varepsilon(\zeta)}{\zeta}\,d\zeta,$$

where Γ_z is a curve joining z to $\mathring{z}$ and contained in $V\setminus I$. It is clear that $JF_\varepsilon\in\mathcal{O}(V\setminus I)$ and one can prove (e.g. by the change of variables $\zeta=e^{-w}$; see [SZ], Th. 13.4) that if F satisfies the estimate

$$|F_\varepsilon(z)|\le C_k|z|^{b-\mathring{b}}/|\arg z|^k$$

on proper bounded sectorial subsets K of V then $|J^{k+1}F_\varepsilon(z)|<\widetilde{C}|z|^{b-\mathring{b}}$. Thus $[J^{k+1}F_\varepsilon]$ is a function in $\mathcal{M}'_{b-\mathring{b}}(I)$. Consequently,

$$[F_\varepsilon]=(xd/dx)^{k+1}[J^{k+1}F_\varepsilon]\in\mathcal{M}'_0(I).$$

Since $[F]=x^{\mathring{b}-\varepsilon}[F_\varepsilon]$ for $\varepsilon>0$ it follows that $[F]\in\mathcal{M}'_{(b)}(I)$.

4. Laplace distributions. The space of *Laplace distributions* (of type $\omega\in\mathbb{R}$) supported by $\overline{\mathbb{R}}_+$, denoted by $L'_{(\omega)}(\overline{\mathbb{R}}_+)$, is defined as the image under the mapping

$$\overline{\mathbb{R}}_+\ni s\mapsto\mu(s)=e^{-s}\in(0,1]$$

of the space $\mathcal{M}'_{(\omega)}((0,1])$ of Mellin distributions:

$$L'_{(\omega)}(\overline{\mathbb{R}}_+) = \{u \circ \mu : u \in \mathcal{M}'_{(\omega)}((0,1])\}.$$

We shall describe this space explicitly. To this end let $a \in \mathbb{R}$ and denote by $L_a(\overline{\mathbb{R}}_+)$ the space of those functions $\phi \in C^\infty(\overline{\mathbb{R}}_+)$ such that for every $\alpha \in \mathbb{N}_0$ the seminorm

$$\gamma_{a,\alpha}(\phi) = \sup_{s \in \overline{\mathbb{R}}_+} \left| e^{-as} \frac{d^\alpha}{ds^\alpha} \phi(s) \right|$$

is finite. We endow $L_a(\overline{\mathbb{R}}_+)$ with the convergence topology given by the sequence of the seminorms $\{\gamma_{a,\alpha}\}_{\alpha \in \mathbb{N}_0}$. Next, for $\omega \in \mathbb{R} \cup \{\infty\}$ we define

$$L_{(\omega)}(\overline{\mathbb{R}}_+) = \varinjlim_{a<\omega} L_a(\overline{\mathbb{R}}_+)$$

equipped with the inductive limit topology. It follows from the corresponding fact for the Mellin distributions that the set $C^\infty_{(0)}(\overline{\mathbb{R}}_+) = C^\infty_0(\mathbb{R}) \cap C^\infty(\overline{\mathbb{R}}_+)$ is dense in $L_{(\omega)}(\overline{\mathbb{R}}_+)$. Hence the dual space $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is a subspace of $D'(\overline{\mathbb{R}}_+)$ (= the dual of $C^\infty_{(0)}(\overline{\mathbb{R}}_+)$). We also set

$$L'_{(-\infty)}(\overline{\mathbb{R}}_+) = \bigcup_{\omega \in \mathbb{R}} L'_{(\omega)}(\overline{\mathbb{R}}_+), \qquad L'_{(\infty)}(\overline{\mathbb{R}}_+) = \bigcap_{\omega \in \mathbb{R}} L'_{(\omega)}(\overline{\mathbb{R}}_+).$$

The following characterization of Laplace distributions follows immediately from the Structure Theorem 3.2 for Mellin distributions.

THEOREM 4.1. *Let $\omega \in \mathbb{R}$. A distribution $T \in D'(\overline{\mathbb{R}}_+)$ is in $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ if and only if for every $\kappa > 0$ there exist $m_\kappa \in \mathbb{N}_0$ and measurable functions $T_{\lambda,\kappa}$ on $\mathbb{R}$ with support in $\overline{\mathbb{R}}_+$ for $0 \le \lambda \le m_\kappa$ such that*

$$T = \sum_{\lambda=0}^{m_\kappa} \frac{d^\lambda}{d\alpha^\lambda} T_{\lambda,\kappa} \quad \text{in } L'_{(\omega-\kappa)}(\overline{\mathbb{R}}_+), \tag{4.1}$$

where

$$|T_{\lambda,\kappa}(\alpha)| \le C_\kappa e^{(-\omega+\kappa)\alpha} \qquad \text{for } 0 \le \alpha < \infty \tag{4.2}$$

with some constant $C_\kappa > 0$.

Remark 4.1. Observe that if $\omega > 0$ the sum in (4.1) cannot in general be reduced to a single summand (as can be done e.g. for tempered distributions in $S'(\mathbb{R})$). The example is provided by $T = \delta_{(0)} \in L'_{(\infty)}(\overline{\mathbb{R}}_+)$. Indeed, if $\delta_{(0)} = \frac{d^m}{d\alpha^m} T_m$ then we must have $T_m(\alpha) = \alpha^m/m!$ for $\alpha > 0$ and T_m does not satisfy (4.2) for $\omega > 0$. On the other hand, for every $\omega \in \mathbb{R}$ we have

$$\delta_{(0)} = -\omega e^{-\omega\alpha} Y + \frac{d}{d\alpha} e^{-\omega\alpha} Y,$$

where Y is the Heaviside function.

DEFINITION 4.1. We say that a Laplace distribution $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is of *order* $m \in \mathbb{N}_0$ if in the representation (4.1) we have $m_\kappa \le m$ for $\kappa > 0$.

It follows from Theorem 4.1 that for every $\kappa > 0$ each Laplace distribution $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is of a finite order as a Laplace distribution in the space $L'_{(\omega-\kappa)}(\overline{\mathbb{R}}_+)$.

Laplace distributions can be regarded as a special case of Fourier hyperfunctions (cf. [Ka]) which we recall briefly below:

Let W be a tubular neighbourhood of $\overline{\mathbb{R}}_+$ in $\mathbb{C}$ (i.e. W is an open set in $\mathbb{C}$ containing $(\overline{\mathbb{R}}_+)_\varepsilon = \{z \in \mathbb{C} : \mathrm{dist}(z, \mathbb{R}_+) < \varepsilon\}$ for some $\varepsilon > 0$). Let $\omega \in \mathbb{R}$ and define

$$\widetilde{\mathcal{O}}_{(\omega)}(W) = \{H \in \mathcal{O}(W) : \sup_{\zeta \in K} |e^{(\omega-\delta)\zeta} H(\zeta)| < \infty \text{ for every } \delta > 0 \text{ and}$$
$$\text{every closed (in } \mathbb{C}) \text{ tubular subset } K \text{ of } W\}.$$

With an analogously defined space $\widetilde{\mathcal{O}}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ one sets

$$\mathcal{Q}_{(\omega)}(\overline{\mathbb{R}}_+) = \widetilde{\mathcal{O}}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\widetilde{\mathcal{O}}_{(\omega)}(W).$$

The quotient space $\mathcal{Q}_{(\omega)}(\overline{\mathbb{R}}_+)$ is called the space of *Fourier hyperfunctions* on $\overline{\mathbb{R}}_+$ of type ω.

We now proceed to embed Laplace distributions in the space of Fourier hyperfunctions. Let $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ and define

$$\Psi(\zeta) = \frac{-1}{2\pi i} T\left[\frac{e^{-(\zeta-w)^2}}{\zeta - w}\right] \quad \text{for } \zeta \notin \overline{\mathbb{R}}_+ \tag{4.3}$$

(this is understood as the value of T on the test function $\overline{\mathbb{R}}_+ \ni w \mapsto e^{-(\zeta-w)^2}/(\zeta - w)$ for a fixed $\zeta \notin \overline{\mathbb{R}}_+$).

LEMMA 4.1. *The function Ψ is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{R}}_+$ and for every $\delta > 0$ there exists $k \in \mathbb{N}_0$ such that for any $\varepsilon, \eta > 0$,*

(i) $|\Psi(\alpha + i\beta)| \le C_{\varepsilon,\delta} e^{\alpha(-\omega+\delta)}/|\beta|^k$ *for* $\alpha + i\beta \in (\overline{\mathbb{R}}_+)_\varepsilon$,

(ii) $|\Psi(\alpha + i\beta)| \le C_{\varepsilon,\eta,\delta} e^{\alpha(-\omega+\delta)}/|\beta|$ *for* $\alpha + i\beta \notin (\overline{\mathbb{R}}_+)_\varepsilon,\ |\beta| < \eta$.

Proof. Analogous to that of Theorem 3.3.

Let W be an open tubular neighbourhood of $\overline{\mathbb{R}}_+$. Denote by $\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)$ the space of all $\Psi \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+)$ satisfying (i) in the lemma for $\alpha + i\beta \in W$ and $\omega = a$. We equip $\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)$ with the inductive limit topology given by the family of norms

$$\Theta^k_{\kappa,K}(\Psi) = \sup_{\alpha+i\beta \in K} |e^{\alpha\kappa} \Psi(\alpha + i\beta) \beta^k|$$

as K runs through closed subtubular neighbourhoods of W and $\kappa < a$. Note that by the 3-line theorem [C] the space $\widetilde{\mathcal{O}}^k_{(a)}(W)$ is closed in $\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)$. Then the quotient space

$$\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)/\widetilde{\mathcal{O}}_{(a)}(W)$$

is a Hausdorff topological space and so is the inductive limit as $k \to \infty$. We have the following result which can be regarded as a variant of Theorem 3.1 (cf. also the Painlevé Theorem 2.1):

THEOREM 4.2. *There exists a natural topological isomorphism*

$$\varprojlim_{a<\omega} \varinjlim_{k \in \mathbb{R}} \widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)/\widetilde{\mathcal{O}}_{(a)}(W) \simeq L'_{(\omega)}(\overline{\mathbb{R}}_+).$$

The isomorphism is given as follows: If $\Psi \in \widetilde{\mathcal{O}}^k_\kappa(W \setminus \overline{\mathbb{R}}_+)$ for some $k \in \mathbb{N}_0$ and $\kappa < \omega$ we define

$$T[\phi] = \lim_{\beta\to 0_+} \int_{\mathbb{R}_+} \Psi(\alpha + i\beta)\phi(\alpha)\,d\alpha - \lim_{\beta\to 0_+} \int_{\mathbb{R}_+} \Psi(\alpha - i\beta)\phi(\alpha)\,d\alpha,$$

where $\phi \in L_\theta(\overline{\mathbb{R}}_+)$, $\kappa < \theta < \omega$. The inverse mapping is $T \mapsto [\Psi]$, where Ψ is the function (4.3).

4.1. *Convolution of Laplace distributions.* The convolution of Laplace distributions is defined in a completely analogous way to the case of "usual" distributions. Namely, if $T_1 \in L'_{(\omega_1)}(\overline{\mathbb{R}}_+)$ and $T_2 \in L'_{(\omega_2)}(\overline{\mathbb{R}}_+)$ we define $T_1 * T_2 \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$, where $\omega = \min(\omega_1, \omega_2)$, by the formula

$$T_1 * T_2[\phi] = T_1[T_2[\phi(\alpha + \alpha_1)]] \quad \text{for } \phi \in L_{(\omega)}(\overline{\mathbb{R}}_+). \tag{4.4}$$

The correctness of the above definition follows from

LEMMA 4.2. *If $\phi \in L_{(\omega)}(\overline{\mathbb{R}}_+)$ and $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ then the formula*

$$\psi(\alpha) = T[\phi(\alpha + \alpha_1)], \quad \alpha \in \overline{\mathbb{R}}_+$$

(*here T is applied to the test function $\overline{\mathbb{R}}_+ \ni \alpha_1 \overset{\phi_\alpha}{\mapsto} \phi(\alpha + \alpha_1)$, where $\alpha \in \overline{\mathbb{R}}_+$ is a parameter*) *defines a function $\psi \in L_{(\omega)}(\overline{\mathbb{R}}_+)$.*

Proof. First observe that for every fixed $\alpha \in \overline{\mathbb{R}}_+$, $\phi_\alpha \in L_{(\omega)}(\overline{\mathbb{R}}_+)$. Fix $\kappa > 0$. Then $\phi_\alpha \in L_{(\omega-\kappa)}(\overline{\mathbb{R}}_+)$ and by Theorem 4.1 we have for $\omega - \kappa < a < \omega$ and any $\gamma \in \mathbb{N}_0$,

$$\left| e^{-a\alpha} \frac{d^\gamma}{d\alpha^\gamma} T[\phi_\alpha] \right| \le C_\kappa \sum_{\lambda=0}^{m_\kappa} \int_0^\infty \left| \frac{d^{\lambda+\gamma}}{d\alpha_1^{\lambda+\gamma}} \phi(\alpha + \alpha_1) \right| e^{(-\omega+\kappa)\alpha_1 - a\alpha}\,d\alpha_1,$$

which is finite by the definition of the seminorms in $L_{(\omega-\kappa/2)}(\overline{\mathbb{R}}_+)$. Since $\kappa > 0$ can be arbitrarily small, this proves the assertion.

Note that in the case where T_1, T_2 are locally integrable functions supported by $\overline{\mathbb{R}}_+$ (of exponential growth at ∞), $T_1 * T_2$ is again a locally integrable function supported by $\overline{\mathbb{R}}_+$ (of exponential growth) and is given by

$$T_1 * T_2(\alpha) = \int_0^\alpha T_1(\alpha - \alpha_1) T_2(\alpha_1)\,d\alpha_1 \quad \text{for } \alpha \ge 0.$$

Next observe that the operation of convolution commutes with translations. This means that if $T_1 \in L'_{(\omega_1)}(L_1)$ and $T_2 \in L'_{(\omega_2)}(L_2)$, where $L_1 = \zeta_1 + \overline{\mathbb{R}}_+$ and $L_2 = \zeta_2 + \overline{\mathbb{R}}_+$ for some fixed $\zeta_1, \zeta_2 \in \mathbb{C}$ and we define $T_1 * T_2$ as an element of $L'_{(\omega)}(L_1 + L_2)$ again by formula (4.4) with $\phi \in L_{(\omega)}(L_1 + L_2)$, then

$$T_1 * T_2 = (T_1(\cdot + \zeta_1) * T_2(\cdot + \zeta_2))(\cdot - \zeta_1 - \zeta_2) \tag{4.5}$$

(here $T(\cdot - \zeta)$ denotes the translate of the distribution T by the vector ζ and is defined by the duality $T(\cdot - \zeta)[\phi] = T[\phi(\alpha + \zeta)]$). The formula (4.5) allows us to carry over to the general case of $L = \zeta + \overline{\mathbb{R}}_+$ the properties of convolution derived for the standard case of $L = \overline{\mathbb{R}}_+$.

Sometimes it is convenient to represent convolution of Laplace distributions in terms of their defining functions obtained via Theorem 4.2. To this end take $T_1 \in L'_{(\omega_1)}(L_1)$ and $T_2 \in L'_{(\omega_2)}(L_2)$. Fix $\kappa > 0$ and represent T_1 and T_2 as

$$T_1 = [\Psi_1], \quad \Psi_1 \in \widetilde{\mathcal{O}}^{k_1}_{(\omega_1-\kappa)}(V_1 \setminus L_1), \quad T_2 = [\Psi_2], \quad \Psi_2 \in \widetilde{\mathcal{O}}^{k_2}_{(\omega_2-\kappa)}(V_2 \setminus L_2),$$

where k_1, k_2 are some nonnegative integers and V_1 (V_2, resp.) is a tubular neighbourhood of $\zeta_1 + \mathbb{R}$ (of $\zeta_2 + \mathbb{R}$, resp.). For fixed z sufficiently close to $L_1 + L_2$ with $z \notin L_1 + L_2$ define

$$\Psi_1 * \Psi(z) = \int_{\Gamma_\varepsilon} \Psi_1(z-\gamma)\Psi_2(\gamma)\,d\gamma, \tag{4.6}$$

where Γ_ε is the curve encircling L_2 defined by

$$\Gamma_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, L_2) = \varepsilon\}$$

and oriented anticlockwise, with $\varepsilon > 0$ so small that $\Gamma_\varepsilon \subset V_2 \cap (-V_1 + z)$. We shall establish the following

LEMMA 4.3. *If $\Psi_1 \in \widetilde{\mathcal{O}}^{k_1}_{(a_1)}(V_1 \setminus L_1)$ and $\Psi_2 \in \widetilde{\mathcal{O}}^{k_2}_{(a_2)}(V_2 \setminus L_2)$ then $\Psi_1 * \Psi_2$ given by* (4.6) *belongs to $\widetilde{\mathcal{O}}^{k_1+k_2}_{(\min(a_1,a_2))}(V_3 \setminus (L_1 + L_2))$, where V_3 is a tubular neighbourhood of $L_1 + L_2$.*

Proof. We begin with the standard case where $L_1 = L_2 = \overline{\mathbb{R}}_+$ and $V_1 = V_2 = (\mathbb{R})_\delta$ for some $\delta > 0$. Then it follows from the definition of $\widetilde{\mathcal{O}}^k_{(a)}$ that for every $\kappa_1, \kappa_2 > 0$,

$$\begin{aligned} &|\Psi_1(\alpha + i\beta)| \le C_1 e^{(-a_1+\kappa_1)\alpha}/|\beta|^{k_1} \quad \text{for } \alpha + i\beta \in V_1, \\ &|\Psi_2(\alpha + i\beta)| \le C_2 e^{(-a_2+\kappa_2)\alpha}/|\beta|^{k_2} \quad \text{for } \alpha + i\beta \in V_2, \end{aligned} \tag{4.7}$$

where C_1, C_2 are certain constants depending on κ_1 and κ_2 respectively. Fix $z \in \mathbb{C} \setminus \mathbb{R}$ so small that the function $\gamma \mapsto \Psi_1(z-\gamma)$ is holomorphic on the set $(\overline{\mathbb{R}}_+)_\varepsilon$, where $\varepsilon = |\operatorname{Im} z|/2$.

Then it follows from (4.7) that for the points $\gamma = a + ib$ with $|b| = \varepsilon$ we have the estimates

$$\begin{aligned} |\Psi_1(z-a-ib)\Psi_2(a+ib)| &\le C_1C_2 \frac{e^{(-a_2+\kappa_2)a} e^{(-a_1+\kappa_1)(\operatorname{Re} z - a)}}{|\operatorname{Im} z - b|^{k_1} \varepsilon^{k_2}} \\ &\le C \frac{e^{(-a_1+\kappa_1)\operatorname{Re} z}}{|\operatorname{Im} z|^{k_1+k_2}} e^{(a_1-a_2+\kappa_2-\kappa_1)a} \end{aligned} \tag{4.8}$$

for a suitable constant C. Suppose $a_1 \le a_2$. If we choose $\kappa_2 < \kappa_1$ then the integral in (4.6) will be absolutely convergent and will define a function $\Psi_1 * \Psi_2(z)$ holomorphic in $\{0 < |\operatorname{Im} z| < \widetilde{\varepsilon}\}$ for some small $\widetilde{\varepsilon} > 0$. Moreover, it follows from (4.8) that on that set,

$$|\Psi_1 * \Psi_2(z)| \le C_3 e^{(-a_1+\kappa)\operatorname{Re} z}/|\operatorname{Im} z|^{k_1+k_2}.$$

Now, if $\operatorname{Re} z < 0$ we take $\varepsilon = |\operatorname{Re} z|/2$ in (4.6) and observe that the function $\Psi_1 * \Psi_2$ so defined is holomorphic for $\operatorname{Re} z < 0$ and $|\operatorname{Im} z| < \widetilde{\varepsilon}$. This proves the lemma in the case where $a_1 \le a_2$. To settle the case of $a_2 \le a_1$ observe that for fixed $\alpha \in \mathbb{R}_-$ and $\varepsilon = -\alpha/2$, if $a_1 = a_2$ we obtain, by the change of variable $\theta = \alpha - \gamma$,

$$\int_{\Gamma_\varepsilon} \Psi_1(\alpha-\gamma)\Psi_2(\gamma)\,d\gamma = \int_{\Gamma_\varepsilon} \Psi_2(\alpha-\theta)\Psi_1(\theta)\,d\theta$$

modulo a function holomorphic in the strip $|\mathrm{Im}\, z| < \eta$ for some $\eta > 0$. Thus the argument given above applies with Ψ_1 and Ψ_2 interchanged, proving the lemma in the standard case $L_1 = L_2 = \overline{\mathbb{R}}_+$. The general case of $L_1 = \zeta_1 + \overline{\mathbb{R}}_+$ and $L_2 = \zeta_2 + \overline{\mathbb{R}}_+$ reduces to the standard case by the translation invariance of the convolution:

$$\Psi_1 * \Psi_2(z) = (\Psi_1(\cdot + \zeta_1) * (\Psi_2(\cdot + \zeta_2))(z - \zeta_1 - \zeta_2).$$

PROPOSITION 4.1. *Under the notation preceding Lemma* 4.3 *we have*

$$T_1 * T_2 = [\Psi_1 * \Psi_2],$$

where the right-hand side is the inductive limit as $\kappa \to 0_+$ *of the equivalence class of* $\Psi_1 * \Psi_2$ *given by* (4.6) *modulo* $\widetilde{\mathcal{O}}_{(\omega-\kappa)}(V_3)$, *and the identity is understood in the sense of Theorem* 4.2.

Proof. By Lemma 4.3 we see that for every $\kappa > 0$, $\Psi_1 * \Psi_2 \in \widetilde{\mathcal{O}}^{k_1+k_2}_{(\widetilde{\omega})}(V_3 \setminus (L_1 + L_2))$, where $\widetilde{\omega} = \omega - \kappa$. Take $\phi \in L_{(\widetilde{\omega})}(L_1 + L_2)$ holomorphic in a tubular neighbourhood of $L_1 + L_2$ and satisfying there the estimate $|\phi(z)| \le C_a e^{a\,\mathrm{Re}\, z}$ for some $a < \widetilde{\omega}$. Let $\varepsilon > 0$ be so small that the curve Γ_ε is contained in the set where both ϕ and $\Psi_1 * \Psi_2$ are holomorphic. Observe that for $z \in \Gamma_\varepsilon$ we may take the curve $\Gamma_{\varepsilon/2}$ in (4.6) and by the change of variables $\theta_1 = z - \gamma$, $\theta_2 = \gamma$ we get

$$\begin{aligned}\int_{\Gamma_\varepsilon} \Psi_1 * \Psi_2(z)\phi(z)\, dz &= \int_{\Gamma_\varepsilon} \phi(z) \int_{\Gamma_{\varepsilon/2}} \Psi_1(z-\gamma)\Psi_2(\gamma)\, d\gamma\, dz \\ &= \int_{\Gamma_\varepsilon} \Big(\int_{\Gamma_{\varepsilon/2}} \phi(\theta_1 + \theta_2)\Psi_2(\theta_2)\, d\theta_2 \Big) \Psi_1(\theta_1) d\theta_1 \\ &= T_1[T_2[\phi(\theta_1 + \theta_2)]],\end{aligned}$$

in view of Theorem 4.2. As the test functions ϕ considered here are dense in $L_{(\widetilde{\omega})}(L_1+L_2)$ the proof is complete.

It follows from the preceding considerations that the space $L'_{(-\infty)} = \varinjlim_{\omega\in\mathbb{R}} L'_{(\omega)}$ of Laplace distributions forms a convolution algebra. Moreover, this algebra does not have nonzero zero divisors as the following variant of the Titchmarsh theorem ([T]) asserts.

THEOREM 4.3. *Let* $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$, $T \ne 0$. *Then there exists at most one Laplace distribution* $T^{-1} \in L'_{(-\omega)}(\overline{\mathbb{R}}_+)$ *such that* $T * T^{-1} = \delta_{(0)}$.

5. Ecalle distributions. Let Σ be a discrete set in $\mathbb{C}$. Let $\widetilde{\mathbb{C}\setminus\Sigma} = (\widetilde{\mathbb{C}\setminus\Sigma}, a)$ denote the universal covering space of $\mathbb{C}\setminus\Sigma$ with base point $a \notin \Sigma$ and let $\Pi : \widetilde{\mathbb{C}\setminus\Sigma} \to \mathbb{C}\setminus\Sigma$ be the natural projection which to a curve assigns the (nonfixed) end point. For any $\varepsilon > 0$ denote by $\widetilde{\Sigma}_\varepsilon$ the set

$$\widetilde{\Sigma}_\varepsilon = \{z \in \widetilde{\mathbb{C}\setminus\Sigma} : |\Pi z - \sigma| < \varepsilon \text{ for } \sigma \in \Sigma\}.$$

DEFINITION 5.1. Let $k \in \mathbb{N}_0$ and $\omega \in \mathbb{R}$. We denote by $\widetilde{\mathcal{O}}^k_{(\omega)}(\widetilde{\mathbb{C}\setminus\Sigma})$ the space of $\Phi \in \mathcal{O}(\widetilde{\mathbb{C}\setminus\Sigma})$ such that

(i) for every $\varepsilon, \kappa > 0$ and $\theta_1 < \theta_2 \in \mathbb{R}$,

$$|\Phi(z)| \le C_{\varepsilon,\kappa,\theta_1,\theta_2} e^{(-\omega+\kappa)|z|} \quad \text{for } z \in \widetilde{\mathbb{C}\setminus\Sigma} \setminus \widetilde{\Sigma}_\varepsilon,\ \theta_1 < \arg z < \theta_2,$$

(ii) for some fixed $\varepsilon_0 > 0$ and any $\theta_1 < \theta_2 \in \mathbb{R}$,

$$|\Phi(z)| \le C^\sigma_{\kappa,\theta_1,\theta_2}/|\Pi z - \sigma|^k \quad \text{for } |\Pi z - \sigma| < \varepsilon_0,\ \theta_1 < \arg(z-\sigma) < \theta_2,$$

and for any $\kappa > 0$,

$$\sum_{\sigma\in\Sigma} C^\sigma_{k,\theta_1,\theta_2} e^{(-\omega+\kappa)|\sigma|} < \infty.$$

DEFINITION 5.2. A Laplace distribution $T \in L'_{(\omega)}(L)$ of order k, with $\omega \in \mathbb{R}$ and $L = \zeta_0 + \overline{\mathbb{R}}_+$ is said to be an *Ecalle distribution* of type Σ if there exists a function $\Phi \in \widetilde{\mathcal{O}}^k_{(\omega)}(\widetilde{\mathbb{C}\setminus\Sigma})$ such that T restricted to $\operatorname{Int} L = \zeta_0 + \mathbb{R}_+$ coincides with the distributional boundary value $\lim_{\substack{\beta\to\operatorname{Im}\zeta_0\\ \beta>\operatorname{Im}\zeta_0}} \Phi(\cdot + i\beta)$ of Φ from above (as in Theorem 4.2). We call Φ the *variation* of T and denote by $\Phi = \operatorname{var} T$.

The space of Ecalle distributions is denoted by $\mathcal{R}^k_{(\omega)}(L,\Sigma)$. We also set

$$\mathcal{R}^{-\infty}_{(-\infty)}(L,\Sigma) = \bigcup_{k,\omega} \mathcal{R}^k_{(\omega)}(L,\Sigma) \quad \text{and} \quad \mathcal{R} = \bigcup_{L,\Sigma} \mathcal{R}^{-\infty}_{(-\infty)}(L,\Sigma).$$

Remark 5.1. The condition of being an Ecalle distribution is of very restrictive nature. Namely, outside the end point ζ_0, the Ecalle distribution is completely determined by its restriction to the neighbourhood of any point in $(\mathbb{C}\setminus\Sigma)\cap(\zeta_0 + \overline{\mathbb{R}}_+)$.

The space of Ecalle distributions is closed under convolution in the following sense:

PROPOSITION 5.1. *Let $T_j \in \mathcal{R}^{k_j}_{(\omega_j)}(L_j,\Sigma_j)$, $j = 1,2$. Then $T_1 * T_2 \in \mathcal{R}^{k_1+k_2}_{(\min(\omega_1,\omega_2))}(L_1 + L_2, \Sigma_1 + \Sigma_2)$.*

Proof. First we observe that there exists a defining function Ψ_j of T_j modulo $\mathcal{O}_{(\omega_j)}(\mathbb{C})$ such that $\Psi_j \in \widetilde{\mathcal{O}}^{k_j}_{(\omega_j)}(\widetilde{\mathbb{C}\setminus\Sigma_j})$ for $j = 1,2$. The proof then proceeds along the lines of that of Lemma 4.3 with the curve Γ_ε suitably modified (see also [E1].

5.1. *Alien derivatives of Ecalle distributions.* Following Ecalle [E1] we introduce below a family of operations Δ_σ, $\sigma \in \Sigma$, which measure how far the variation of an Ecalle distribution $T \in \mathcal{R}(L,\Sigma)$ is from being a singlevalued function. Since Δ_σ are derivations of the convolution algebra $\mathcal{R}(\overline{\mathbb{R}}_+,\mathbb{N}_0)$:

$$\Delta_\sigma(T_1 * T_2) = (\Delta_\sigma T_1) * T_2 + T_1 * (\Delta_\sigma T_2), \tag{5.1}$$

and are different from the standard derivation (= multiplication by z), Ecalle calls them *alien derivations*.

The presentation below essentially follows that of [M] with slight modifications to match it to our class of distributions.

We begin with the standard case of the space $\mathcal{R}^k_{(\omega)}(L,\Sigma)$, where $L = \mathbb{R}_+$ and $\Sigma \cap L \subset \mathbb{N}_0$. Choose a base point $a \in (0,1/2)$ and let $n > 0$. Let W be a small tubular neighbourhood of $\mathbb{R}_+$ such that $W \cap \Sigma \subset \mathbb{N}_0$. Denote by γ_n a curve in $(\widetilde{W\setminus\mathbb{N}_0}, a)$ which joins a to the point $b = n - a$ and encircles the points $1,\ldots,n-1$ either from above or from below. Let $T \in \mathcal{R}^k_{(\omega)}(\overline{\mathbb{R}}_+,\mathbb{N}_0)$. We define an Ecalle distribution $\Delta_{\gamma_n} T \in \mathcal{R}^k_{(\omega)}(\overline{\mathbb{R}}_+,\mathbb{N}_0)$ as follows. Take $\Phi = \operatorname{var} T$. Let Ψ be the function in a neighbourhood of $n-1$ obtained by analytic continuation of Φ restricted to a neighbourhood of a, along γ_n. Translate Ψ by $-n$ and extend to a function on $W \setminus \mathbb{R}_+$. Denote by $\widetilde{T}$ the difference of the boundary values of Ψ across $\mathbb{R}$. Then $\Delta_{\gamma_n} T$ coincides with $\widetilde{T}$ on $(-\infty, 1)$ and is extended to $\mathbb{R}$ by the boundary value from above of the analytic continuation of the distribution $\widetilde{T}$ restricted to $(0,1)$.

Next we define

$$\Delta_n = \sum_{[\gamma_n]} \mathcal{E}(\gamma_n)\Delta_{\gamma_n},$$

where the sum is taken over homotopy classes $[\gamma_n]$ of curves as above, and $\mathcal{E}(\gamma_n) = p!(n-1-p)!/n!$, where $p = 0, 1, \ldots, n-1$ is the number of points encircled by γ_n from above.

It is proved in [M] that Δ_n, $n \in \mathbb{N}$, are derivations, i.e. satisfy (5.1) with $\sigma = n$. Sometimes it is convenient to complete Δ_n, $n \in \mathbb{N}$, with the operation $\Delta_0 T$ defined as follows. Take $\Phi = \operatorname{var} T$ restricted to $(0,1)$ and extend it to a function on $W \setminus \mathbb{R}_+$. Then $\Delta_0 T$ coincides with the difference $\widetilde{T}$ of the boundary values of Φ on $(-\infty, 1)$ and is extended to $\mathbb{R}$ by the boundary value from above of the analytic continuation of $\widetilde{T}$ restricted to $(0,1)$. Note, however, that $\Delta_0 T$ so defined is not a derivation but an algebra homomorphism:

$$\Delta_0(T_1 * T_2) = \Delta_0 T_1 * \Delta_0 T_2.$$

Next observe that the situation will remain practically the same if instead of being equal to $\mathbb{N}_0$, the intersection $\Sigma \cap \overline{\mathbb{R}}_+$ is an arbitrary discrete set. Thus for any $\sigma \in \Sigma \cap \overline{\mathbb{R}}_+$ we have the operations

$$\Delta_\sigma : \mathcal{R}^k_{(\omega)}(\overline{\mathbb{R}}_+, \Sigma) \to \mathcal{R}^k_{(\omega)}(\overline{\mathbb{R}}_+, \Sigma - \sigma).$$

It is clear from the definition of an alien derivative that it commutes with translations. Thus we define the dotted alien derivations $\dot{\Delta}_\sigma$ by the formula $\dot{\Delta}_\sigma = \delta_{(\sigma)} * \Delta_\sigma$. Then

$$\dot{\Delta}_\sigma : \mathcal{R}^k_{(\omega)}(\overline{\mathbb{R}}_+, \Sigma) \to \mathcal{R}^k_{(\omega)}(\sigma + \overline{\mathbb{R}}_+, \Sigma).$$

Similarly if instead of $\overline{\mathbb{R}}_+$ we have a half-line $L = \zeta + \overline{\mathbb{R}}_+$ we define, for $\sigma \in L \cap \Sigma$,

$$\Delta^L_\sigma := \delta_{(\zeta)} * (\Delta_{\sigma-\zeta}(\delta_{(-\zeta)} * \cdot)), \qquad \dot{\Delta}^L_\sigma := \delta_{(\sigma)} * \Delta^L_\sigma.$$

Then

$$\Delta^L_\sigma : \mathcal{R}^k_{(\omega)}(L, \Sigma) \to \mathcal{R}^k_{(\omega)}(L, \Sigma - \sigma), \qquad \dot{\Delta}^L_\sigma : \mathcal{R}^k_{(\omega)}(L, \Sigma) \to \mathcal{R}^k_{(\omega)}(\sigma + L, \Sigma), \tag{5.2}$$

and for $T_j \in \mathcal{R}^k_{(\omega_j)}(L_j, \Sigma_j)$, $L_j = \zeta_j + \overline{\mathbb{R}}_+$, $j = 1, 2$, if $\sigma \in (L_1 + L_2) \cap (\Sigma_1 + \Sigma_2)$ and $\sigma \neq \zeta_1 + \zeta_2$, we have

$$\Delta^{L_1+L_2}_\sigma (T_1 * T_2) = (\Delta^{L_1}_{\sigma-\zeta_2} T_1) * T_2 + T_1 * (\Delta^{L_2}_{\sigma-\zeta_1} T_2). \tag{5.3}$$

6. Paley–Wiener type theorems for Mellin analytic functionals. The content of this section is basically a reformulation of the results of [Ka], [Mo]; see also [Pl].

We begin with the definition of the Mellin transform of a Mellin analytic functional $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(I^\theta)$. To this end note that for any fixed $\theta \geq 0$ the function $I^\theta \ni w \mapsto w^{-z-1}$ belongs to $\underset{\sim}{\mathcal{M}}_{(b)}(\bar{I}^\theta)$ if and only if $\operatorname{Re} z < b$. Thus we define

$$\mathcal{M}f(z) = f[w^{-z-1}] \quad \text{for } \operatorname{Re} z < b$$

and call it the *Mellin transform* of f. It is a standard fact that $\mathcal{M}f$ is holomorphic for $\operatorname{Re} z < b$ and we also denote by $\mathcal{M}f$ any holomorphic extension of that function.

We have the following Paley–Wiener type results for the Mellin transforms of Mellin analytic functionals.

Proposition 6.1. *Let* $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$. *Then for every* $\varepsilon > 0$ *there exists a constant* C_ε *such that*

$$|\mathcal{M}f(\alpha + i\beta)| \le C_\varepsilon e^{(\theta+\varepsilon)|\beta|}(te^\varepsilon)^{-\alpha} \quad \text{for } \alpha \le b - \varepsilon.$$

Proof. Choose $F \in \widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)$ such that f is represented as

$$f[\phi] = -\int_\gamma F(w)\phi(w)\,dw, \qquad \phi \in \underset{\sim}{\mathcal{M}}_{(b)}(\bar{I}^\theta),$$

for a suitable curve γ. Then it follows from the definition of the Mellin transformation and of the space $\widetilde{\mathcal{M}}_{(b)}$ that for every $\varepsilon > 0$,

$$|\mathcal{M}f(\alpha + i\beta)| \le \Big|\int_\gamma F(w)w^{-\alpha-i\beta-1}\,dw\Big| \le \widetilde{C}_{2\varepsilon}\int_\gamma |w^{b-2\varepsilon-\alpha-1}|\,dw.$$

To estimate the last integral we take

$$\gamma = \{w = se^{\pm i(\theta+\varepsilon)} : 0 < s \le te^\varepsilon\} \cup \{w = (te^\varepsilon)e^{i\varrho} : -\varepsilon - \theta \le \varrho \le \theta + \varepsilon\}.$$

The result follows easily after simple computations.

A similar reasoning leads to the following:

Proposition 6.2. *Let* $f \in \mathcal{A}'(I_r^\theta)$. *Then* $\mathcal{M}f \in \mathcal{O}(\mathbb{C})$ *and for every* $\varepsilon > 0$ *there exists a constant* C_ε *such that*

$$|\mathcal{M}f(\alpha + i\beta)| \le \begin{cases} C_\varepsilon e^{(\theta+\varepsilon)|\beta|}(te^\varepsilon)^{-\alpha} & \text{for } \alpha \le 0, \\ C_\varepsilon e^{(\theta+\varepsilon)|\beta|}(re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0 \end{cases}$$

(the 0 here can be replaced by any fixed real number; e.g. by -1 as in [SZ]).

Our next goal is to present a result converse to Propositions 6.1 and 6.2. First, however, we must familiarize ourselves with the so called exponential partition of unity:

Let

$$\xi_-(z) = \frac{e^{(2\theta+1)iz}}{1 + e^{(2\theta+1)iz}}, \qquad \xi_+(z) = \frac{1}{1 + e^{(2\theta+1)iz}}.$$

Then $\xi_+ + \xi_- \equiv 1$ and $\xi_\pm$ are meromorphic functions on $\mathbb{C}$ with simple poles at the points of $S = \big\{\frac{2k+1}{2\theta+1}\pi : k \in \mathbb{Z}\big\}$. Further, outside any sectorial neighbourhood of $\mathbb{R}_- \cup \mathbb{R}_+$ we have the estimate

$$|\xi_\mp(\alpha + i\beta)| \le Ce^{-(2\theta+1)|\beta|} \quad \text{for } \beta \in \overline{\mathbb{R}}_\pm, \tag{6.1}$$

and for every $\varepsilon > 0$,

$$|\xi_\mp(\alpha + i\beta)| \le C_\varepsilon e^{\varepsilon|\beta|} \quad \text{for } \beta \in \overline{\mathbb{R}}_\mp. \tag{6.2}$$

Theorem 6.1. *Let* $F \in \mathcal{O}(\{\operatorname{Re} z < b\})$ *and suppose there exist* $t > 0$ *and* $\theta > 0$ *such that for any* $\varepsilon > 0$,

$$|F(\alpha + i\beta)| \le C_\varepsilon e^{(\theta+\varepsilon)|\beta|}(te^\varepsilon)^{-\alpha} \quad \text{for } \alpha \le b - \varepsilon. \tag{6.3}$$

Choose any $\mathring{a}$ *such that* $[\mathring{a}, b) \cap S = \emptyset$. *Then the pair of functions*

$$G_\pm(w) = \pm\int_{\operatorname{Re} z = \mathring{a}} F(z)w^z\xi_\pm(z)\,dz \tag{6.4}$$

defined for $w \in \widetilde{\mathbb{C}\setminus\{0\}}$ *such that* $\theta < \pm\arg w < \theta+1$ *extends holomorphically to a function* $G \in \widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)$, *where*

$$V = \{w \in \widetilde{\mathbb{C}\setminus\{0\}} : -\theta-1 < \arg w < \theta+1\}.$$

The class $g = [G]$ *of* G *modulo* $\widetilde{\mathcal{M}}_{(b)}(V)$ *is independent of the exponential partition of unity and defines a Mellin analytic functional* $g \in \underset{\sim}{\mathcal{M}}'_{(b)}(\bar{I}^\theta)$ *such that* $\mathcal{M}g = F$.

Proof. The complete proof of this result (in several variables) based on [Ka] can be found in [Pl]. Below we sketch some essential points. One first proves that

$$G_\pm \in \widetilde{\mathcal{M}}_{(b)}(\{\theta < \pm\arg w < \theta+1\}).$$

This is done by estimating the expression (6.4) by using (6.1)–(6.3) and noting that the $G_\pm$ are independent of the choice of $\mathring{a}$, which allows us to take $\mathring{a} < b$ arbitrarily close to b.

To extend $G_\pm$ to a function in $\widetilde{\mathcal{M}}_{(b)}(V \setminus I^\theta)$ one deforms the path of integration in (6.4) as follows: Let $\phi \in [\pi/2, \pi)$ and define

$$\mathbb{R}_+^\phi = a_0 + \mathbb{R}_+ e^{i\phi}, \quad \mathbb{R}_-^\phi = a_0 + \mathbb{R}_- e^{-i\phi}.$$

We estimate the integral

$$G_-^\phi(w) = -\int\limits_{\mathbb{R}_+^\phi \cup \mathbb{R}_-^\phi} F(z)\xi_-(z) w^z\,dz$$

as before and find that $G_-^\phi \in \mathcal{O}(V_-^\phi)$, where

$$V_-^\phi = \{w : -\theta-1 < \arg w + \ln(t/|w|)\cot\phi,\ \arg w - \ln(t/|w|)\cot\phi < -\theta\}.$$

Note that this coincides with the previous result for $\phi = \pi/2$, since $V_-^0 = \{w : -\theta-1 < \arg w < -\theta\}$.

For G_+^ϕ analogously defined one proves that $G_+^\phi \in \mathcal{O}(V_+^\phi)$, where

$$V_+^\phi = \{w : \theta < \arg w - \ln(t/|w|)\cot\phi,\ \arg w + \ln(t/|w|)\cot\phi < \theta+1\}.$$

The next step consists in proving that $G_\pm^\phi = G_\pm$ on $V_\pm^\phi \cap V_\pm$. This is done by comparing the integrals

$$\Big|\int\limits_{\mathring{a}+i[0,r]} F(z)\xi_-(z)w^z\,dz - \int\limits_{\mathring{a}+i[0,re^{i\phi}]} F(z)\xi_-(z)w^z\,dz\Big|$$
$$= \Big| r\int_0^\phi F(\mathring{a}+re^{i\gamma})\xi_-(\mathring{a}+re^{i\gamma}) w^{\mathring{a}+re^{i\gamma}}\,d\gamma\Big|$$
$$\le C_\varepsilon r\int_0^\phi \exp r\Big(\Big(\ln\frac{t}{|w|} - \varepsilon\Big)\cos\gamma - (\arg w - \theta - 1 - \varepsilon)\sin\gamma\Big)\,d\gamma.$$

If $w \in V_-^\phi \cap V_-^0$ and ε is small enough, we see that the exponent is negative for $0 \le \gamma \le \phi$. Thus the integral goes to zero as $r \to \infty$. The case of G_+^ϕ is completely analogous. Taking the union of the sets $V_\pm^\phi$ over $\phi \in [\pi/2, \pi)$ we see from the above that the $G_\pm$ extend holomorphically to the set $\{w \in \widetilde{\mathbb{C}\setminus\{0\}} : |w| > t\}$. It remains only to show that the extensions of G_+ and G_- coincide. To this end observe that

$$G_+^\phi(w) - G_-^\phi(w) = \int\limits_{\mathbb{R}_+^\phi \cup \mathbb{R}_-^\phi} F(z) w^z \, dz. \tag{6.5}$$

Clearly the integral is defined on the set $\{|\arg w| \le \ln(t/|w|)\tan\phi\}$ and is independent of ϕ. By means of suitable estimates we prove that

$$\int\limits_{\mathbb{R}_\pm^\phi} F(z) w^z \, dz = \pm i \int\limits_{-\infty}^{\mathring{a}} F(\alpha) w^\alpha \, d\alpha,$$

which shows that the difference in (6.5) is zero. This proves the theorem.

Now we proceed to the proof of a result converse to Proposition 6.2.

Theorem 6.2. *Let $F \in \mathcal{O}(\mathbb{C})$ and suppose there exist $0 < r < t$ and $\theta_1, \theta_2 > 0$ such that for every $\varepsilon > 0$,*

$$|F(\alpha + i\beta)| \le \begin{cases} C_\varepsilon e^{(\theta_1+\varepsilon)|\beta|} (te^\varepsilon)^{-\alpha} & \text{for } \alpha \le 0, \\ C_\varepsilon e^{(\theta_2+\varepsilon)|\beta|} (re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0. \end{cases}$$

Then there exists a unique analytic functional $g \in \mathcal{A}'(I_r^\theta)$, where $\theta = \min(\theta_1, \theta_2)$, such that $\mathcal{M}g = F$.

Proof. By Theorem 6.1 we clearly have $F = \mathcal{M}g$, where $g \in \mathcal{M}'_{(0)}(\bar{I}^{\theta_1})$. Next, we proceed as in the proof of Theorem 6.1, replacing the imaginary line $i\mathbb{R}$ by $\mathbb{R}_\phi^+ \cup \mathbb{R}_\phi^-$ with $0 < \phi \le \pi/2$. We observe that as before the functions $G_\pm^\phi$ extend the functions $G_\pm$ to the set $\{w \in \widetilde{\mathbb{C} \setminus \{0\}} : |w| < r\}$. Next by taking the union of the sets of holomorphy of the functions $G_\pm^\phi$ for $\phi \in (0, \pi)$ we observe that the $G_\pm$ extend to a function G holomorphic on $V = (\widetilde{\mathbb{C} \setminus 0}) \setminus I_r^\theta$, where $\theta = \min(\theta_1, \theta_2)$. To prove that $g \in \mathcal{A}'(I_r^\theta)$ it remains now, by Theorem 3.3, to check that the extension G satisfies the condition

$$|G(z)| \le C_{\varepsilon,K} |z|^{-k} \quad \text{for } z \in K$$

for any $\varepsilon > 0$ and any proper sectorial subset $K \subset V$. This is done by estimating the function $G_-^\phi(w)$ for $0 < \phi \le \pi/2$ similarly to the proof of Theorem 6.1.

Next we recall analogous results in the case of Mellin distributions. The proofs can be found e.g. in [SZ]. The theorems below are only a slight modification of the classical Paley–Wiener theorem:

Theorem 6.3. *In order that a function $F(z)$ be the Mellin transform of a distribution $u \in \mathcal{M}'_{(\omega)}(I)$ it is necessary and sufficient that F be holomorphic in the set $\{z \in \mathbb{C}^n : \operatorname{Re} z < \omega\}$ and that for every $b < \omega$ there exist a constant $s = s(b)$ such that for every $\varepsilon > 0$,*

$$|F(\alpha + i\beta)| \le C_\varepsilon \langle\beta\rangle^s (te^\varepsilon)^{-\alpha} \quad \text{for } \alpha \le b,$$

where $C_\varepsilon = C_\varepsilon(b)$ is a constant (here $\langle\beta\rangle := (1 + |\beta|)$).

Theorem 6.4. *In order that a function $F(z)$ be the Mellin transform of a compactly supported distribution $u \in D'([r, t])$ with $0 < r < t < \infty$ it is necessary and sufficient* *that F be an entire function on $\mathbb{C}$ and that there exist $s \in \mathbb{R}$ such that for every $\varepsilon > 0$,*

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon \langle\beta\rangle^s (te^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0,\\ C_\varepsilon \langle\beta\rangle^s (re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0. \end{cases} \tag{6.6}$$

Moreover, u is a smooth function supported by $[r,t]$ if and only if for each $j \in \mathbb{N}$,

$$|F(\alpha+i\beta)| \le \begin{cases} C_j \dfrac{1}{\langle\beta\rangle^j} t^{-\alpha} & \text{for } \alpha \le 0,\\ C_j \dfrac{1}{\langle\beta\rangle^j} r^{-\alpha} & \text{for } \alpha \ge 0. \end{cases} \tag{6.7}$$

for some constants C_j.

Remark 6.1. Observe that if $F \in \mathcal{O}(\mathbb{C})$ satisfies (6.6) then $\frac{d^p}{dz^p}F(z)$ satisfies (6.6) for every $p \in \mathbb{N}_0$.

Proof. This follows from (6.6) by applying the Cauchy formula

$$\frac{d^p}{dz^p}F(z) = \frac{1}{2\pi i}\int\limits_{|\zeta-z|=\kappa} \frac{F(\zeta)}{(\zeta-z)^{p+1}}\,d\zeta, \qquad p \in \mathbb{N}_0,$$

with some fixed $\kappa > 0$.

6.1. *Phragmén–Lindelöf type theorems.* Below we present results on the automatic improvement of estimates for certain entire functions, which can be regarded as variants of the classical Phragmén–Lindelöf theorem ([C]). The proofs are based on the Paley–Wiener type theorems presented in Section 6 which translate the growth properties of an entire function to the information on the carriers (supports) of the corresponding Mellin analytic functionals.

THEOREM 6.5. *If $F \in \mathcal{O}(\mathbb{C})$ and there exist $0 < r < t$ and $\theta_1, \theta_2 \ge 0$ such that for every $\varepsilon > 0$,*

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon e^{(\theta_1+\varepsilon)|\beta|} (te^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0,\\ C_\varepsilon e^{(\theta_2+\varepsilon)|\beta|} (re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0, \end{cases}$$

then for every $\varepsilon > 0$,

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon e^{(\theta+\varepsilon)|\beta|} (te^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0,\\ C_\varepsilon e^{(\theta+\varepsilon)|\beta|} (re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0, \end{cases}$$

where $\theta = \min(\theta_1, \theta_2)$.

Proof. This follows by combining Theorem 6.2 and Proposition 6.2.

THEOREM 6.6. *If $F \in \mathcal{O}(\mathbb{C})$ and there exist $0 < r < t$, $\theta > 0$ and $s \in \mathbb{R}$ such that for any $\varepsilon > 0$,*

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon \langle\beta\rangle^s (te^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0,\\ C_\varepsilon e^{(\theta+\varepsilon)|\beta|} (re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0, \end{cases}$$

then there exists $\widetilde{s} \in \mathbb{R}$ such that for every $\varepsilon > 0$,

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon \langle\beta\rangle^{\tilde{s}} (te^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0,\\ C_\varepsilon \langle\beta\rangle^{\tilde{s}} (re^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0. \end{cases}$$

P r o o f. By Theorem 6.2 (with $\theta_1 = 0$) there exists a unique hyperfunction u supported by $[r,t]$ such that $\mathcal{M}u = F$. Regarding u as a Mellin hyperfunction on $[0,t]$ we observe that its standard logarithmic defining function Ψ has the following properties:

(i) $\Psi \in \mathcal{O}(V \setminus [r,t])$,
(ii) for any $\varepsilon > 0$,

$$|\Psi(w)| \le C_{\varepsilon,K}|w|^{-\varepsilon} \quad \text{for } |w| < r,\ w \in K \subset V \setminus [r,t],$$

where V is a sectorial neighbourhood of $\mathbb{R}_+$. On the other hand, by the Paley–Wiener Theorem 6.3 for Mellin distributions we get $u \in \mathcal{M}'_{(0)}((0,t])$. Hence by Theorem 3.3 we derive that the function Ψ satisfies the condition

(ii′) For any $\varepsilon > 0$,

$$|\Psi(w)| \le C_{\varepsilon,K}|w|^{-\varepsilon}/|\operatorname{Arg} w|^k \quad \text{for } w \in K \setminus (0,t],$$

where K is any proper sectorial subset of V and k depends on s (e.g. $k = s+2$). Thus it follows from the above by combining (ii) and (ii′), and noting that

$$|\operatorname{Arg} w| = \left|\arcsin \frac{\operatorname{Im} w}{|w|}\right| \ge \frac{\operatorname{Im} w}{|w|},$$

that

(i) $\Psi \in \mathcal{O}(V \setminus [r,t])$,
(ii) $|\Psi(w)| \le C/|\operatorname{Im} w|^k$ for $\operatorname{Im} w$ close to zero

(since $|w|$ is bounded away from zero on $[r,t]$). By the Painlevé Theorem 2.1 this means that $u \in D'([r,t])$. The desired estimate now follows from Theorem 6.4.

7. The cut-off functions and their Mellin transforms. The simplest cut-off function is the characteristic function of an interval $(0,r]$, where $r > 0$. We denote it by χ_r. The Mellin transform of χ_r is readily computed:

$$\mathcal{M}(\chi_r)(z) = \int_0^r x^{-z-1}\,dx = \frac{r^{-z}}{-z} \quad \text{for } z \ne 0.$$

Sometimes it is convenient to use smooth cut-off functions. To define them, fix $r > 0$ and let $\widetilde{r} = r\delta$ for some $0 < \delta < 1$. Take a function $\chi \in C^\infty(\mathbb{R}_+)$ such that $\chi(x) \equiv 1$ for $0 < x < \delta$, $\chi(x) = 0$ for $x \ge 1$ and $0 \le \chi(x) \le 1$ for all x. Define

$$\chi_{\tilde{r},r}(x) = \chi(x/r) \quad \text{for } x > 0.$$

Clearly $\chi_{\tilde{r},r} \in C^\infty(\mathbb{R}_+)$, $0 \le \chi_{\tilde{r},r} \le 1$, $\chi_{\tilde{r},r}(x) \equiv 1$ for $0 < x \le \widetilde{r}$ and $\chi_{\tilde{r},r}(x) \equiv 0$ for $x \ge r$.

In a sense χ_r can be regarded as a limit case of $\chi_{\tilde{r},r}$ as $\widetilde{r} \to r$. To underline this we set $\chi_{r,r} = \chi_r$.

The Mellin transform of $\chi_{\tilde{r},r}$ resembles, in many respects, that of χ_r. Namely we have

PROPOSITION 7.1. *The Mellin transform of the cut-off function* $\chi_{\tilde{r},r} = \chi(x/r)$ *has the following properties*:

(i) $\mathcal{M}\chi_{\tilde{r},r}(z) = r^{-z}\mathcal{M}\chi(z)$,

(ii) $\mathcal{M}\chi_{\tilde r,r}(z) = r^{-z}/(-z) + \widetilde G_{\tilde r,r}(z)$, *where* $\widetilde G_{\tilde r,r} \in \mathcal{O}(\mathbb{C})$ *and for any* $p \in \mathbb{N}$,

$$\left|\frac{d^p}{dz^p}\widetilde G_{\tilde r,r}(z)\right| \le \begin{cases} C_p\langle \ln r\rangle^p r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle & \text{for } \operatorname{Re} z \le 0,\\ C_p\langle \ln r\rangle^p \widetilde r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle & \text{for } \operatorname{Re} z > 0,\end{cases}$$

for some constant C_p *independent of* r,

(iii) *for every* $\varepsilon > 0$ *and* $p, j \in \mathbb{N}$ *there exists a constant* $C_{\varepsilon,j,p}$ (*depending on* χ *but independent of* r) *such that for* $|z| \ge \varepsilon$,

$$\left|\frac{d^p}{dz^p}\mathcal{M}\chi_{\tilde r,r}(z)\right| \le \begin{cases} C_{\varepsilon,j,p}\langle \ln r\rangle^p r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \text{for } \operatorname{Re} z \le 0,\\ C_{\varepsilon,j,p}\langle \ln r\rangle^p \widetilde r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \text{for } \operatorname{Re} z > 0.\end{cases}$$

P r o o f. (i) is verified for $\operatorname{Re} z < 0$ by the change of variable $\widetilde x = x/r$ in the integral

$$\mathcal{M}\chi_{\tilde r,r}(z) = \int_0^r \chi(x/r)x^{-z-1}\,dx,$$

and extended by analytic continuation. To prove (iii) define $\chi'(x) = x\frac{d}{dx}\chi(x)$. Then χ' is smooth on $\mathbb{R}_+$ and supported by the bounded interval $[\delta, 1]$. Hence it follows from the second part of Theorem 6.4 that for $j \in \mathbb{N}$ the function $F = \mathcal{M}\chi'$ satisfies the estimate

$$|F(z)| \le \begin{cases} C_j\dfrac{1}{\langle \operatorname{Im} z\rangle^j} & \text{for } \operatorname{Re} z \le 0,\\ C_j\dfrac{1}{\langle \operatorname{Im} z\rangle^j}\delta^{-\operatorname{Re} z} & \text{for } \operatorname{Re} z > 0.\end{cases}$$

Combining this estimate with the Cauchy formula

$$\frac{d^p}{dz^p}F(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = \kappa}\frac{F(\zeta)}{(\zeta - z)^{p+1}}\,d\zeta$$

for any $\kappa > 0$, we get

$$\left|\frac{d^p}{dz^p}F(z)\right| \le \begin{cases} C_{j,p}\dfrac{1}{\langle \operatorname{Im} z\rangle^j} & \text{for } \operatorname{Re} z \le 0,\\ C_{j,p}\dfrac{1}{\langle \operatorname{Im} z\rangle^j}\delta^{-\operatorname{Re} z} & \text{for } \operatorname{Re} z > 0.\end{cases} \tag{7.1}$$

Next, it follows from the operational properties of the Mellin transformation that

$$\mathcal{M}\Big(x\frac{d}{dx}\chi\Big)(z) = z\mathcal{M}\chi(z),$$

which combined with (i) gives

$$\mathcal{M}\chi_{\tilde r,r}(z) = \frac{r^{-z}}{z}F(z) \quad \text{for } z \in \mathbb{C}\setminus\{0\}. \tag{7.2}$$

The proof of (iii) now follows by the Leibniz rule from (7.2) and the estimate (7.1). Finally, (ii) is easily derived from (7.2) and (iii) in view of the identity

$$F(0) = \mathcal{M}\chi'(0) = \int_\delta^1 \frac{d}{dx}\chi(x)\,dx = -\chi(\delta) = -1.$$

8. Modified Cauchy transformation in dimension 1. We start with certain well-known facts about the classical Cauchy and Hilbert transformations in $L^2(\mathbb{R})$ in a slightly changed setting; namely the Cauchy transformation is not considered relative to the real axis $\mathbb{R}$ but relative to a fixed pure imaginary line $\overset{*}{\alpha} + i\mathbb{R}$ for some $\overset{*}{\alpha} \in \mathbb{R}$.

Let $T \in L^2(\mathbb{R})$ and fix $\overset{*}{\alpha} \in \mathbb{R}$. The *right* and *left Cauchy transforms* of T (relative to the line $\overset{*}{\alpha} + i\mathbb{R}$) are defined as

$$C^+T(z) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{T(\gamma)}{z - \overset{*}{\alpha} - i\gamma}\,d\gamma \quad \text{for } \operatorname{Re} z > \overset{*}{\alpha},$$

$$C^-T(z) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{T(\gamma)}{z - \overset{*}{\alpha} - i\gamma}\,d\gamma \quad \text{for } \operatorname{Re} z < \overset{*}{\alpha}.$$

The following is a classical fact in the theory of the Hilbert transformation:

THEOREM 8.1. *If $T \in L^2(\mathbb{R})$ then $C^\pm T(\alpha + i\,\cdot) \in L^2(\mathbb{R})$ and the following limits exist in $L^2(\mathbb{R})$:*

$$H^\pm T(\beta) = \lim_{\alpha \to \overset{*}{\alpha}_\pm} C^\pm T(\alpha + i\beta).$$

Moreover,

$$H^-T - H^+T = T.$$

The functions H^+T, $H^-T \in L^2(\mathbb{R})$ are called the *right* and *left Hilbert transforms* of T.

We want to establish analogues of these facts in S' instead of L^2. To this end we introduce modified Cauchy kernels which replace the *classical Cauchy kernel* $-1/z$. To justify the definition below recall that by the example in Section 7,

$$-1/z = \big(\mathcal{M}\chi_{(0,1]}\big)(z) \quad \text{for } z \neq 0,$$

where $\chi_{(0,1]}$ is the characteristic function of the interval $(0, 1]$.

Further, recall that in Proposition 7.1 we considered the Mellin transforms of functions χ which are restrictions to $\mathbb{R}_+$ of functions $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\widetilde{\chi} \equiv 1$ in a neighbourhood of zero. We established there that $\mathcal{M}\chi$ differs from $-1/z$ by an entire function and $z\mathcal{M}\chi(z)$ rapidly decreases along the imaginary axis. Therefore we call $G = \mathcal{M}\chi$ a *modified Cauchy kernel* (determined by χ).

DEFINITION 8.1. Fix $\overset{*}{\alpha} \in \mathbb{R}$ and let $T \in S'(\mathbb{R})$. The (holomorphic) functions

$$\mathcal{C}^\pm T(z) = \frac{1}{2\pi} T[G(z - \overset{*}{\alpha} - i\gamma)] \quad \text{for } \pm\operatorname{Re} z > \pm\overset{*}{\alpha} \tag{8.1}$$

are called the *right* and *left modified Cauchy transforms* of T. In the sequel we often omit the word *modified* since the function G and the point $\overset{*}{\alpha}$ will remain fixed.

THEOREM 8.2. *Let $T \in S'(\mathbb{R})$ and fix $\overset{*}{\alpha} \in \mathbb{R}$. Then in the sense of the convergence in $S'(\mathbb{R})$ the following limits exist*:

$$\mathcal{H}^\pm T = \lim_{\alpha \to \overset{*}{\alpha}_\pm} \mathcal{C}^\pm T(\alpha + i\,\cdot).$$

Moreover,

$$\mathcal{H}^-T - \mathcal{H}^+T = T. \tag{8.2}$$

The theorem will be proved in a more general setting in Section 17.

Theorem 8.2 can be applied to the study of holomorphic extensions of left Cauchy transforms as follows:

Let F be holomorphic in an open set $U \subset \mathbb{C}$. Let $\mathring{\alpha} \in \mathbb{R}$ and suppose that the function $\beta \mapsto F(\mathring{\alpha} + i\beta)$, defined for $\beta \in \mathbb{R}$ such that $\mathring{\alpha} + i\beta \in U$, extends to a distribution in $S'(\mathbb{R})$ which we denote by $F_{\mathring{\alpha}}$.

COROLLARY 8.1. *Under the assumptions on F given above the function*

$$\psi(z) = \begin{cases} \mathcal{C}^- F_{\mathring{\alpha}}(z) & \text{for } \operatorname{Re} z < \mathring{\alpha}, \\ \mathcal{C}^+ F_{\mathring{\alpha}}(z) + F(z) & \text{for } z \in \{\operatorname{Re} z > \mathring{\alpha}\} \cap U, \end{cases}$$

extends to a holomorphic function on $\{\operatorname{Re} z < \mathring{\alpha}\} \cup U$.

Proof. It is easy to see that $\mathcal{C}^{\pm} F_{\mathring{\alpha}}(\alpha + i\beta)$ satisfies (2.2) in α. The rest follows immediately from (8.2) in view of the distributional Painlevé Theorem 2.1.

We end this section with a property of the classical Cauchy transformation which will find application in Section 17.

PROPOSITION 8.1. *Let T be a distribution on $\mathbb{R}$ of order $\widetilde{p}$ with bounded support. Suppose T restricted to an interval $(0, \mathring{b})$ with $\mathring{b} > 0$ is a differentiable function such that for $j = 0, 1$,*

$$|(\partial/\partial\gamma)^j T(\gamma)| \le C/\gamma^p \quad \text{for } \gamma \in (0, \mathring{b}), \tag{8.3}$$

where $C > 0$ and $p \ge 0$ are some constants. Then the (*classical*) *left Cauchy transform*

$$C^- T(z) = -\frac{1}{2\pi} T\left[\frac{1}{z - i\gamma}\right]$$

defined and holomorphic for $\operatorname{Re} z < 0$ extends to a continuous function on the set $\{z \in \mathbb{C} : \operatorname{Re} z \le 0,\ 0 < \operatorname{Im} z < \mathring{b}\}$ and for every $\mathring{\alpha} < 0$ there exists $\widetilde{C} > 0$ such that

$$|C^- T(\alpha + i\beta)| \le \widetilde{C}/\beta^{\hat{p}} \quad \text{for } \mathring{\alpha} \le \alpha < 0 \text{ and small } \beta > 0,$$

where $\widehat{p} = \max(p, \widetilde{p} + 1)$.

The proof can be found in [SZ] and [Zie2].

II. The theory of generalized analytic functions

Now we proceed to the main theme of the paper. Namely, we introduce the concept of a generalized analytic function and investigate the properties of such functions. The concept of a generalized analytic function arises on the one hand as a natural generalization of the class of functions (1.1) and (1.4) considered within the framework of the classical theory of Laplace integrals (Section 1); on the other hand, it is closely related to the recent developments in the theory of Borel summability and in particular to the theory of resurgent functions by Jean Ecalle (see Appendix II.1).

9. Definition of a generalized analytic function. Let Z be a closed subset of $\mathbb{C}$ such that there exist (different) points $\zeta_k \in \mathbb{C}$, $k = 0, \ldots, b$, so that

$$Z \subset \mathbf{L} := \bigcup_{k=0}^{b} \zeta_k + \overline{\mathbb{R}}_+.$$

For $k = 0, \ldots, b$ define

$$L_k = \zeta_k + \overline{\mathbb{R}}_+, \quad Z_k = Z \cap L_k.$$

Definition 9.1. A function $f : (0, \varrho) \to \mathbb{C}$ is called a *generalized analytic function* (GAF, for short) of type (Z, m), $m \in \mathbb{N}_0$, of convergence radius not less than ϱ if there exist Laplace distributions $T_k \in L'_{(\ln \varrho)}(L_k)$ of order at most m with $\operatorname{supp} T_k \subset Z_k$ for $k = 0, \ldots, b$ such that f extends to a function on $\widetilde{\mathbb{C} \setminus \{0\}}$ of the form

$$f(w) = \sum_{k=0}^{b} T_k[w^z] \quad \text{for } w \in (\widetilde{\mathbb{C} \setminus \{0\}}, a), \quad |w| < \varrho \tag{9.1}$$

(here, for $k = 0, \ldots, b$, and a fixed w with $|w| < \varrho$, w^z denotes the Laplace test function $L_k \ni z \mapsto w^z \in \mathbb{C}$).

It is easy to see that if $|w| < \varrho$ then for every $k = 0, \ldots, b$ the function $L_k \ni z \mapsto w^z$ belongs to $L_{(\ln \varrho)}(L_k)$, and hence the function f is well defined for such w. Moreover, it follows in a standard way from the continuity of the Laplace distributions (see [SZ], Theorem 7.1, for a similar proof) that f is holomorphic on $\widetilde{D}(\varrho) = \{w \in (\widetilde{\mathbb{C} \setminus \{0\}}, a) : |w| < \varrho\}$.

The decomposition (9.1) is unique. The proof is based on the study of the Mellin transform of f and will follow from Theorem 10.1.

Under the notation of Definition 9.1, for $k = 0, \ldots, b$ define

$$f_k(w) = T_k[w^z] \quad \text{for } |w| < \varrho.$$

After multiplication of f_k by $w^{-\zeta_k}$ it can be reduced to the standard GAF

$$u(w) = T[w^z] \quad \text{for } |w| < \varrho, \tag{9.2}$$

where $T \in L'_{(\ln \varrho)}(\overline{\mathbb{R}}_+)$. Often it is enough to investigate only the standard case and the results obtained will easily extend to the general case. For instance, it follows from Theorem 4.1 that for any fixed $\kappa > 0$ we may represent u as

$$u(w) = \sum_{\lambda=0}^{m_\kappa} (\ln w)^\lambda \int_0^\infty w^\alpha T_{\lambda,\kappa}(\alpha)\, d\alpha \qquad \text{for } w \in \widetilde{D}(e^{-\kappa}\varrho) := \{w \in \mathbb{C} \setminus \{0\} : |w| < e^{-\kappa}\varrho\}, \tag{9.3}$$

where $T_{\lambda,\kappa}$ are measurable functions on $\mathbb{R}_+$ such that

$$|T_{\lambda,\kappa}(\alpha)| \le C_\kappa (e^\kappa/\varrho)^\alpha \quad \text{for } 0 \le \alpha < \infty. \tag{9.4}$$

In particular, (9.3) allows us to prove again in an easy way that u is well defined and holomorphic on $\widetilde{D}(\varrho)$.

Owing to the fact that $C^\infty_{(0)}(\overline{\mathbb{R}}_+)$ is dense in $L_{(\omega)}(\overline{\mathbb{R}}_+)$, generalized analytic functions possess the following important "analyticity" property:

PROPERTY A. If $T_k \equiv 0$ on L_k in the sense of $D'(L_k)$, $k = 0, \ldots, b$, then the generalized analytic function f given by (9.1) is zero on $(0, \varrho)$. In other words, if f given by (9.1) is flat of order r for all r then $f \equiv 0$ on $(0, \varrho)$.

Remark. One can replace the Laplace distributions T_k, in the definition of GAF, by more general objects, e.g. by the Fourier hyperfunctions considered by Morimoto [Mo] (see [Ły2] for details). However, the resulting function f will in general not satisfy Property A. This is due to the existence of Fourier hyperfunctions supported by the point $\{\infty\}$ which correspond to nonzero functions flat of infinite order.

Further analysis of GAF requires the application of the Mellin transformation.

10. The Mellin transform of a generalized analytic function. Since the Mellin transformation can only be applied to functions with support bounded on the right, in order to compute the Mellin transform of a GAF u, it is necessary to multiply it by a suitable cut-off function $\chi_{\tilde{r},r}$ or χ_r. We have the following result, which extends that of Proposition 7.1.

THEOREM 10.1. *Let u be a generalized analytic function of type (Z, m) of convergence radius not less than $\varrho > 0$. Define*

$$\mathbf{L}_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbf{L}) < \varepsilon\} \quad \textit{for } \varepsilon > 0,$$

where $\mathbf{L}$ is as in Section 9. Then for any cut-off function $\chi_{\tilde{r},r}$ with $r < \varrho$,

(i) *$\mathcal{M}(\chi_{\tilde{r},r}u)$ extends to a holomorphic function on $\mathbb{C} \setminus Z$,*

(ii) *for every $\varepsilon > 0$ and $j \in \mathbb{N}$ there exists a constant $C_{\varepsilon,j}$ (depending on δ as $\delta \to 0$ and on $\varrho - r$ as $r \to \varrho$) such that for $z \notin \mathbf{L}_\varepsilon$ we have*

$$|\mathcal{M}(\chi_{\tilde{r},r}u)(z)| \le \begin{cases} C_{\varepsilon,j}\langle \ln r\rangle^m r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \textit{for } \operatorname{Re} z \le 0, \\ C_{\varepsilon,j}\langle \ln r\rangle^m \widetilde{r}^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \textit{for } \operatorname{Re} z > 0, \end{cases}$$

(iii) *for every $\varepsilon > 0$ and $z \in \mathbf{L}_\varepsilon$,*

$$|\mathcal{M}(\chi_{\tilde{r},r}u)(z)| \le C_\varepsilon \langle \ln r\rangle^m \frac{\widetilde{r}^{-\operatorname{Re} z}}{\operatorname{dist}(\operatorname{Im} z, \{\operatorname{Im} \zeta_0, \ldots, \operatorname{Im} \zeta_b\})^{m+1}}$$

(with C_ε independent of r as $r \to 0_+$).

Similarly for every function χ_r with $r < \varrho$,

(i′) *$\mathcal{M}(\chi_r u)$ extends to a holomorphic function on $\mathbb{C} \setminus Z$,*

(ii′) *for every $\varepsilon > 0$ there exists a constant C_ε (independent of r as $r \to 0_+$) such that for $z \notin \mathbf{L}_\varepsilon$ we have*

$$|\mathcal{M}(\chi_r u)(z) \le C_\varepsilon \langle \ln r\rangle^m r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle,$$

(iii′) *for every $\varepsilon > 0$ and $z \in \mathbf{L}_\varepsilon$,*

$$|\mathcal{M}(\chi_r u)(z)| \le C_\varepsilon \langle \ln r\rangle^m \frac{r^{-\operatorname{Re} z}}{\operatorname{dist}(\operatorname{Im} z, \{\operatorname{Im} \zeta_0, \ldots, \operatorname{Im} \zeta_b\})^{m+1}}.$$

Moreover, for $j = 0, \ldots, b$ the Laplace distributions T_j coincide with the difference of the boundary values of $\frac{1}{2\pi i}\mathcal{M}(\chi_{\tilde{r},r}u)$ (and of $\frac{1}{2\pi i}\mathcal{M}(\chi_r u)$) accross the line L_j.

Proof. We may assume that u is in the standard form $u(x) = T[x^\alpha]$ for $0 < x < \varrho$, where $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ with $\omega = \ln\varrho$. Define $G_{\tilde r,r} = \mathcal{M}\chi_{\tilde r,r}$, where $\chi_{\tilde r,r}$ is the cut-off function as in Proposition 7.1. Let $r < \varrho = e^\omega$ and choose κ so that $r < \varrho e^{-2\kappa} = \mathring{r}$. Since

$$(d/dz)^\lambda G_{\tilde r,r}(z-\alpha) = \mathcal{M}((-\ln x)^\lambda \chi_{\tilde r,r})(z-\alpha),$$

it follows from (9.3) and (9.4), after entering with the Mellin transformation under the integral sign, that

$$\mathcal{M}(\chi_{\tilde r,r}u)(z) = \sum_{\lambda=0}^{m_k}(-1)^\lambda \int_0^\infty \left(\frac{d}{dz}\right)^\lambda G_{\tilde r,r}(z-\alpha)T_{\lambda,\kappa}(\alpha)\,d\alpha \quad \text{for } \operatorname{Re} z < 0. \tag{10.1}$$

By (9.3) and Proposition 7.1, for every $\varepsilon > 0$ and $p \in \mathbb{N}$ there exists a constant $C_{\varepsilon,p,r} = C_{\varepsilon,p}|\ln r|^\lambda$ such that for $0 < \alpha < \infty$,

$$|(d/dz)^\lambda G_{\tilde r,r}(z-\alpha)T_{\lambda,\kappa}(\alpha)| \le \begin{cases} C_{\varepsilon,p,r}\dfrac{r^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle^p}e^{-\kappa\alpha} & \text{for } \operatorname{Re} z \le 0,\\ C_{\varepsilon,p,r}\dfrac{\widetilde{r}^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle^p}e^{-\kappa\alpha} & \text{for } \operatorname{Re} z > 0,\ |z-\alpha| \ge \varepsilon \end{cases}$$

(since $\alpha(\ln r - \omega + \kappa) < -\kappa\alpha$). Hence, for a new constant $C = C_{\varepsilon,p}|\ln r|^m$,

$$|\mathcal{M}(\chi_{\tilde r,r}u)(z)| \le \begin{cases} C\dfrac{r^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle^p} & \text{for } \operatorname{Re} z \le 0,\\ C\dfrac{\widetilde{r}^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle^p} & \text{for } \operatorname{Re} z > 0,\ z \notin (\overline{\mathbb{R}}_+)_\varepsilon. \end{cases}$$

Thus $\mathcal{M}(\chi_{\tilde r,r}u)$ satisfies the conditions (i) and (ii) with $Z = L = \overline{\mathbb{R}}_+$. To prove (iii) observe that by Proposition 7.1,

$$G_{\tilde r,r}(z) = -\frac{r^{-z}}{z} + \widetilde{G}_{\tilde r,r}(z) \tag{10.2}$$

with $\widetilde{G}_{\tilde r,r}$ entire on $\mathbb{C}$. Hence in a similar way to the above we get for $p = 1$ the following estimate with $C = C_\varepsilon|\ln r|^\lambda$:

$$|(d/dz)^\lambda \widetilde{G}_{\tilde r,r}(z-\alpha)T_{\lambda,k}(\alpha)| \le \begin{cases} C\dfrac{r^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle}e^{-\kappa\alpha} & \text{for } \operatorname{Re} z \le 0,\\ C\dfrac{\widetilde{r}^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle}e^{-\kappa\alpha} & \text{for } \operatorname{Re} z > 0. \end{cases}$$

Observe that the function

$$H^{\lambda,\kappa}_{\tilde r,r}(z) := \int_0^\infty \left(\frac{d}{dz}\right)^\lambda \widetilde{G}_{\tilde r,r}(z-\alpha)T_{\lambda,\kappa}(\alpha)\,d\alpha$$

is entire and for $\operatorname{Re} z > 0$,

$$|H^{\lambda,\kappa}_{\tilde r,r}(z)| \le C\widetilde{r}^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle$$

with a new constant C.

Thus in the study of the behaviour of $\mathcal{M}(\chi_{\tilde r,r}u)$ near $\overline{\mathbb{R}}_+$ we are reduced, by (10.1) and (10.2), to the integrals of the form

$$H_r^{\lambda,\kappa}(z)=\int_0^\infty\left(\frac{d}{dz}\right)^\lambda\left(\frac{r^{\alpha-z}}{\alpha-z}\right)T_{\lambda,\kappa}(\alpha)\,d\alpha$$

for which condition (iii) (with $m=\lambda$) is easily verified since by (9.3),

$$|H_r^{\lambda,\kappa}(z)|\le Cr^{-\operatorname{Re}z}/|\operatorname{Im}z|^{\lambda+1}\quad\text{ for } z\in(\overline{\mathbb{R}}_+)_\varepsilon.$$

The last assertion follows easily from the expression for $H_r^{\lambda,\kappa}$.

In the case where u is an analytic function (in a neighbourhood of zero) we have slightly more information on the Mellin transforms of $\chi_{\tilde r,r}u$:

THEOREM 10.2. *Let u be a function analytic at zero of convergence radius not less than $0<\varrho$. Then for any cut-off function $\chi_{\tilde r,r}$ with $r<\varrho$,*

(i) *$\mathcal{M}(\chi_{\tilde r,r}u)$ is meromorphic on $\mathbb{C}$ with simple poles at the points of $\mathbb{N}_0$,*

(ii) *for any $0<\varepsilon<1/2$ and any $j\in\mathbb{N}$ there exists a constant $C_{\varepsilon,j}$ (independent of r as $r\to 0$) such that for $z\notin(\mathbb{N}_0)_\varepsilon:=\{z\in\mathbb{C}:\operatorname{dist}(z,\mathbb{N}_0)<\varepsilon\}$,*

$$|\mathcal{M}(\chi_{\tilde r,r}u)(z)|\le\begin{cases}C_{\varepsilon,j}r^{-\operatorname{Re}z}/\langle\operatorname{Im}z\rangle^j & \text{for }\operatorname{Re}z\le 0,\\ C_{\varepsilon,j}\widetilde{r}^{-\operatorname{Re}z}/\langle\operatorname{Im}z\rangle^j & \text{for }\operatorname{Re}z\ge 0.\end{cases}\tag{10.3}$$

An analogous assertion (with $r=\widetilde r$) is valid for the cut-off function χ_r.

Proof. Write

$$\chi_{\tilde r,r}(x)f(x)=\sum_{j=0}^\infty a_jx^j\chi_{\tilde r,r}(x)\quad\text{ for }0\le x\le r$$

with the series being absolutely convergent. Then

$$\mathcal{M}(\chi_{\tilde r,r}f)(z)=\sum_{j=0}^\infty a_j(\mathcal{M}\chi_{\tilde r,r})(z-j).$$

Fix $r<\mathring{r}<\varrho$. By Proposition 7.1 we get

$$|\mathcal{M}(\chi_{\tilde r,r}f)(\alpha+i\beta)|\le\begin{cases}\dfrac{C_{\varepsilon p}}{\langle\beta\rangle^p}\displaystyle\sum_{j=0}^\infty|a_j|r^{-\alpha+j} & \text{for }\alpha\le -1,\\ \dfrac{C_{\varepsilon p}}{\langle\beta\rangle^p}\Big(\displaystyle\sum_{j=0}^k|a_j|\widetilde{r}^{-\alpha+j}+\sum_{j=k+1}^\infty|a_j|r^{-\alpha+j}\Big) & \\ \qquad\text{for }\alpha+i\beta\notin(\mathbb{N}_0)_\varepsilon,\ k-1<\alpha\le k\ (k=0,1,\ldots). &\end{cases}$$

Put $c=\sum_{j=0}^\infty|a_j|\mathring{r}^j$ and note that $\sum_{j=0}^\infty|a_j|\widetilde{r}^j<c$ and $\sum_{j=0}^\infty|a_j|r^j<c$. Then estimate (10.3) follows immediately from the above formulas with $C^*_{\varepsilon p}=cC_{\varepsilon p}(1+\delta^\alpha)$, since $\widetilde{r}^{-\alpha}+r^{-\alpha}=\widetilde{r}^{-\alpha}(1+\delta^\alpha)$.

11. Characterization of GAFs in terms of Mellin transforms. In this section we prove results converse to those of Section 10. We decided to present them separately since some of the assumptions made below are much weaker than the corresponding assertions of Section 10. These results find important application to singular partial differential equations in Chapter III.

We retain the notation of Section 9.

THEOREM 11.1. *A distribution $u \in D'((0,\varrho))$ extendable over zero is a generalized analytic function of type Z of convergence radius not less than ϱ if for any cut-off function $\chi_{\tilde r,r}$ with $r<\varrho$,*

(i) *$\mathcal{M}(\chi_{\tilde r,r}u)$ holomorphic in $\{\operatorname{Re} z<\omega\}$ (for some $\omega\in\mathbb{R}$) extends to a holomorphic function on $\mathbb{C}\setminus Z$,*

(ii) *there exist $s\in\mathbb{R}$ and $\theta\in\mathbb{R}_+$ such that for any $\varepsilon>0$ and any $\kappa>0$ there is a constant $C_{\varepsilon,\kappa}$ (depending on r) such that for $z=\alpha+i\beta\notin\mathbf{L}_\varepsilon$,*

$$|\mathcal{M}(\chi_{\tilde r,r}u)(\alpha+i\beta)|\le\begin{cases}C_{\varepsilon,\kappa}\langle\beta\rangle^s(re^\kappa)^{-\alpha} & \text{for } \alpha\le 0,\\ C_{\varepsilon,\kappa}e^{(\theta+\kappa)|\beta|}(\widetilde{r}e^{-\kappa})^{-\alpha} & \text{for } \alpha\ge 0,\end{cases}$$

(iii) *for some $\varepsilon>0$ there exists $p\in\mathbb{N}_0$ such that for any $\kappa>0$,*

$$|\mathcal{M}(\chi_{\tilde r,r}u)(\alpha+i\beta)|\le C_\kappa\frac{(\widetilde{r}e^{-\kappa})^{-\alpha}}{(\operatorname{dist}(\beta,\{\operatorname{Im}\zeta_0,\ldots,\operatorname{Im}\zeta_b\}))^p}\quad \text{for } \alpha+i\beta\in\mathbf{L}_\varepsilon.$$

Proof. First observe that by the Phragmén–Lindelöf type Theorem 6.6 the condition (ii) may be replaced by the following stronger one:

(ii′) there exists $s\in\mathbb{R}$ such that for any $\varepsilon>0$ and $\kappa>0$ there is a constant $C_{\varepsilon,\kappa}$ such that for $z=(\alpha+i\beta)\notin\mathbf{L}_\varepsilon$,

$$|\mathcal{M}(\chi_{\tilde r,r}u)(\alpha+i\beta)|\le\begin{cases}C_{\varepsilon,\kappa}\langle\beta\rangle^s(re^\kappa)^{-\alpha} & \text{for } \alpha\le 0,\\ C_{\varepsilon,\kappa}\langle\beta\rangle^s(\widetilde{r}e^{-\kappa})^{-\alpha} & \text{for } \alpha> 0.\end{cases}$$

For simplicity we consider only the standard case where $\mathbf{L}=Z=\overline{\mathbb{R}}_+$. The general case is treated by a simple extension of the argument given below. Let

$$F(z)=\frac{1}{2\pi i}\mathcal{M}(\chi_{\tilde r,r}u)(z)$$

with $r<\varrho$. By (i), F is holomorphic on $\mathbb{C}\setminus\overline{\mathbb{R}}_+$ and since (iii) holds we deduce from Theorem 4.2 that the difference of the boundary values of F is a Laplace distribution in $L'_{(\ln\tilde r)}(\overline{\mathbb{R}}_+)$. Denote that distribution by T. Since $\widetilde{r}$ can be chosen arbitrarily close to ϱ and T is independent of the choice of the cut-off function, it follows that $T\in L'_{(\ln\varrho)}(\overline{\mathbb{R}}_+)$.

Define $\widetilde{u}(x)=T[x^\alpha]$ for $0<x<\varrho$. Then $\widetilde{u}(x)$ is a GAF of convergence radius $\ge\varrho$. Hence it follows from Theorem 10.1 that $\mathcal{M}(\chi_{\tilde r,r}\widetilde{u})\in\mathcal{O}(\mathbb{C}\setminus\overline{\mathbb{R}}_+)$ and satisfies the estimates (ii) and (iii). Since again by Theorem 10.1, $\frac{1}{2\pi i}\mathcal{M}(\chi_{\tilde r,r}\widetilde{u})$ is a defining function for T, it follows from Theorem 4.2 that the function

$$H(z)=\mathcal{M}(\chi_{\tilde r,r}u)-\mathcal{M}(\chi_{\tilde r,r}\widetilde{u})$$

is entire and for every $\kappa>0$ satisfies the estimate

$$|H(z)|\le C_\kappa(\widetilde{r}e^{-\kappa})^{-\operatorname{Re} z}\quad\text{for } z\in(\overline{\mathbb{R}}_+)_{\varepsilon_0}\tag{11.1}$$

for some fixed ε_0. Since H also satisfies the estimate

$$|H(z)|\le\begin{cases}C_{\varepsilon,\kappa}\langle\operatorname{Im} z\rangle^s(re^\kappa)^{-\operatorname{Re} z} & \text{for } \operatorname{Re} z\le 0,\\ C_{\varepsilon,\kappa}\langle\operatorname{Im} z\rangle^s(\widetilde{r}e^{-\kappa})^{-\operatorname{Re} z} & \text{for } \operatorname{Re} z\ge 0,\end{cases}\tag{11.2}$$

for any $\varepsilon, \kappa > 0$ and $z \notin (\overline{\mathbb{R}}_+)_\varepsilon$, it follows from (11.1) that (11.2) holds in fact for all $z \in \mathbb{C}$. Thus it follows from Theorem 6.4 that H is the Mellin transform of a Mellin distribution with support in $[\widetilde{r}, r]$. Consequently, $u = \widetilde{u}$ on $(0, \widetilde{r})$ for any $\widetilde{r} < \varrho$ and hence $u = \widetilde{u}$ on $(0, \varrho)$, which was to be proved.

R e m a r k 11.1. It should be underlined that in order to prove that a distribution u is a GAF it is not enough to show that for every $j = 0, \ldots, b$ the difference T_j of the boundary values of $\frac{1}{2\pi i}\mathcal{M}(\chi_{\tilde{r},r}u)$ across the lines L_j is a Laplace distribution. Indeed, u may then differ from the GAF

$$u_0(x) = \sum_{j=0}^{b} T_j[x^j]$$

by a function flat of infinite order at zero.

R e m a r k 11.2. Setting formally $\widetilde{r} = r$ in Theorem 11.1 and recalling the convention $\chi_{r,r} = \chi_r$ we obtain a version of that theorem for the cut-off functions χ_r. We must, however, impose some a priori conditions on u so that the product $\chi_r u$ makes sense. For instance, one can suppose that u is locally bounded on $(0, \varrho)$ and $O(x^\omega)$ at zero for some $\omega \in \mathbb{R}$. The proof of such a variant follows the lines of the proof of Theorem 11.1 and is slightly simpler than that proof.

R e m a r k 11.3. If the assumptions of Theorem 11.1 hold for a fixed cut-off function $\chi_{\tilde{r},r}$ then u is a GAF of convergence radius $\geq \widetilde{r}$. An analogous statement applies to Remark 11.2.

R e m a r k 11.4. The choice of the point $\alpha = 0$ in the estimate in (ii) is completely irrelevant. Zero can be replaced by any $\mathring{\alpha} \in \mathbb{R}$.

Next we present a "converse" of Theorem 10.2 describing analytic functions in terms of their Mellin transforms.

THEOREM 11.2. *Suppose $u \in D'((0, \varrho))$ satisfies the assumptions of Theorem* 11.1 *with $Z = \mathbb{N}_0$, $\mathbf{L} = \overline{\mathbb{R}}_+$ and $p = 1$. Then u is an analytic function at zero of convergence radius not less than ϱ.*

P r o o f. It follows from Theorem 11.1 that $u(x) = T[x^\alpha]$ for $0 < x < \varrho$, where

$$T = \frac{1}{2\pi i}\big(\lim_{\beta\to 0_+} \mathcal{M}\chi_r u(\cdot + i\beta) - \lim_{\beta\to 0_+} \mathcal{M}\chi_r u(\cdot - i\beta)\big)$$

and the limits are taken in the weak topology in $L'_{(\ln\varrho)}(\overline{\mathbb{R}}_+)$. Moreover, T must be supported by the set $\mathbb{Z} = \mathbb{N}_0$. Thus in a neighbourhood of every point $j \in \mathbb{N}_0$, T is a finite combination of the Dirac delta $\delta_{(j)}$ and its derivatives. Since $p = 1$, no derivatives of $\delta_{(j)}$ are allowed so that $T = \sum_{j=0}^{\infty} a_j \delta_{(j)}$ and hence

$$u(x) = T[x^\alpha] = \sum_{j=0}^{\infty} a_j x^j \quad \text{for } 0 < x < \varrho,$$

which proves that u is analytic at zero with convergence radius not less than ϱ.

12. The Borel and Taylor transformations in the class of GAFs. Let

$$u(x) = \sum_{j=0}^{b} T_j[x^z] \quad \text{for } 0 < x < \varrho$$

with $T_j \in L'_{(\ln \varrho)}(L_j)$, $j = 0, \ldots b$, be a GAF. Define

$$\mathbf{T} = \{T_0, \ldots, T_b\}.$$

The operation $\mathcal{B}$ which to u assigns the set $\mathbf{T}$ is called the *Borel transform* of u (see Remark 14.1). Thus $\mathbf{T} = \mathcal{B}u$. Denote by $\mathfrak{L}$ the space of all finite subsets $\mathbf{T} = \{T_0, \ldots, T_b\}$ with $T_j \in L'_{(-\infty)}(L_j)$ for some different (1) half-lines $L_j = \zeta_j + \overline{\mathbb{R}}_+$ with $\zeta_j \in \mathbb{C}$, $j = 0, \ldots, b$. The elements of $\mathfrak{L}$ can be added: if $\mathbf{T}_1 = \{T_0^1, \ldots, T_{b_1}^1\}$ and $\mathbf{T}_2 = \{T_0^2, \ldots, T_{b_2}^2\}$ with $T_j^l \in L'_{(-\infty)}(L_j^l)$, $l = 1, 2$, $j = 0, 1, \ldots, b_l$, then $\mathbf{T}_1 + \mathbf{T}_2 = \{T_0, \ldots, T_b\}$ with $T_k \in L'_{(-\infty)}(L_k)$, $k = 0, \ldots, b$, where L_k are all different half-lines among the lines L_j^l and

$$T_k = \sum_{L_j^l \subset L_k} T_j^l.$$

Further, we equip $\mathfrak{L}$ with the sequence convergence topology: we say that $\mathbf{T}_n = \{T_0^n, \ldots, T_{b_n}^n\}$, $n \in \mathbb{N}_0$, tends to zero if there exists $b \in \mathbb{N}_0$ such that $b_n \le b$ for $n \in \mathbb{N}_0$ and $T_j^n \to 0$ in the sense of the weak topology in the space of Laplace distributions. The space $\mathfrak{L}$ endowed with the above structures becomes a linear topological space and it follows easily from the definition of GAF that the Borel transform $\mathcal{B} : \mathrm{GAF} \to \mathfrak{L}$ is linear (cf. 13.1 below) and onto. Next, we equip the space GAF with the topology induced from $\mathfrak{L}$ and call it the *spectral topology* in GAF. Clearly $\mathcal{B}$ is then a continuous mapping. We now prove that $\mathcal{B}$ is injective. Due to the linearity of $\mathcal{B}$ it is enough to show that if $u \in$ GAF and $\mathcal{B}u = 0$ then $u \equiv 0$. Suppose that u has a convergence radius $\ge \varrho$ and take the cut-off function $\chi_{\tilde{r},r}$ with $r < \varrho$. By Theorem 10.1, $\mathcal{M}(\chi_{\tilde{r},r}u)$ is entire and satisfies (i)–(iii) of Theorem 11.1. Hence by that theorem, $u \equiv 0$.

The transformation inverse to $\mathcal{B}$ is called the *Taylor transformation* and is denoted by $\mathcal{T}$. The transformation $\mathcal{T}$ can be regarded as a variant of the right sided Laplace transformation.

13. Operations on generalized analytic functions. The space of generalized analytic functions is closed with respect to fundamental algebraic and differential operations. Below we investigate those operations in detail. Since, as was shown in Section 12, a generalized analytic function is completely described by its Borel transform we describe the effect of those operations on the Borel transforms.

13.1. *Addition.* If $u_1, u_2 \in$ GAF and $\mathcal{B}u_l = \mathbf{T}_l = \{T_0^l, \ldots, T_{b_l}^l\}$, $l = 1, 2$, where $T_j^l \in L'_{(\omega)}(L_j^l)$, $l = 1, 2$, $j = 0, \ldots, b_l$, then $u_1 + u_2 \in$ GAF and

$$\mathcal{B}(u_1 + u_2) = \mathbf{T}_1 + \mathbf{T}_2$$

with the addition of elements of $\mathfrak{L}$ defined in Section 12.

(1) This means that $L_j \cap L_k = \emptyset$ for $j, k = 0, \ldots, b$, $j \ne k$.

13.2. *Multiplication.* Let u_1, u_2 be as above. Then $u_1 \cdot u_2 \in \mathrm{GAF}$ and

$$\mathcal{B}(u_1 \cdot u_2) = \mathbf{T}_1 * \mathbf{T}_2, \tag{13.1}$$

where the convolution in $\mathfrak{L}$ is defined as follows: $\mathbf{T}_1 * \mathbf{T}_2 = \{T_0, \ldots, T_b\}$, where $T_k \in L'_{(\omega)}(L_k)$, $k = 0, \ldots, b$, with L_k being all different half-lines among the lines $L^1_j + L^2_{j'}$, $j = 0, \ldots, b_1$, $j' = 0, \ldots, b_2$, and

$$T_k = \sum_{L^1_j + L^2_{j'} \subset L_k} T^1_j * T^2_{j'};$$

here $T^1_j * T^2_j$ denotes the convolution of Laplace distributions as defined in Section 4.1.

In particular, if $u \in \mathrm{GAF}$, $\mathcal{B}u = \mathbf{T} \in \mathfrak{L}$ and f is an analytic function in a neighbourhood of zero, $f(x) = \sum_{j=0}^{\infty} a_j x^j$, then

$$\mathcal{B}(fu) = \sum_{j=0}^{\infty} a_j T * \delta_j.$$

P r o o f. It is enough to consider the standard case $u_j(x) = T_j[x^\alpha]$, $j = 1, 2$, with $T_j \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$. Then it follows from the results of Section 4.1 that $T_1 * T_2 \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$. Moreover, we have

$$u_1(x) \cdot u_2(x) = T_1[x^\alpha] \cdot T_2[x^\gamma] = T_1[T_2[x^{\alpha+\gamma}]] = T_1 * T_2[x^\alpha]$$

by the definition of convolution of Laplace distributions.

It is worthwhile to note that the space $\mathcal{L}$ equipped with the operation of convolution is a convolution algebra and formula (13.1) means that the Borel transformation is a homomorphism of the multiplicative algebra of GAFs onto the convolution algebra $\mathfrak{L}$.

13.3. *Differentiation.* Let $u \in \mathrm{GAF}$ and $\mathcal{B}u = T \in L'_{(\omega)}(L)$. Then $\left(x\frac{d}{dx}\right)u \in \mathrm{GAF}$ and

$$\mathcal{B}\left(\frac{d}{dx}u\right) = \alpha(T * \delta_{(-1)}) \in L'_{(\omega-1)}(L).$$

More generally, it follows from this formula and 13.2 that

$$\mathcal{B}\left(P\left(x\frac{d}{dx}\right)u\right) = P(\alpha) \cdot Bu,$$

which shows that GAFs are specially well adapted to the study of operators of the form $P\left(x\frac{d}{dx}\right)$, where $P(z)$ is a complex polynomial in z.

13.4. *Primitivization.* We begin with a lemma.

LEMMA 13.1. *Let $T \in D'(\Omega)$, where $0 \in \Omega$, an open set in $\mathbb{R}$. Then there exists a distribution $\widetilde{T} \in D'(\Omega)$ such that $\alpha \cdot \widetilde{T} = T$. Any two such distributions differ by a constant multiple of δ_0.*

P r o o f. Represent T as the difference of the boundary values of a holomorphic function $F \in \mathcal{O}(V \setminus \Omega)$, where V is a complex neighbourhood of Ω. Then $\widetilde{T}$ is defined as the difference of the boundary values of the function $F(z)/z \in \mathcal{O}(V \setminus \Omega)$. The second statement follows from the fact that all distributional solutions of the equation $\alpha \cdot S = 0$ are of the form $S = c \cdot \delta_0$ with some $c \in \mathbb{C}$.

The distribution $\widetilde{T}$ which exists by the lemma is denoted by $\mathrm{reg}(T/\alpha)$.

Let $u\in$GAF and $\mathcal{B}u = T\in L'_{(\omega)}(L)$. Then there exists a GAF $\int u$, called the *primitive* of u, such that $\left(x\frac{d}{dx}\right)\int u = u$ and

$$\mathcal{B}\Big(\int u\Big) = \mathbf{T} \in \mathfrak{L},$$

where

$$\mathbf{T} = \operatorname{reg}\left(\frac{T * \delta_{(1)}}{\alpha}\right) \in L'_{(\omega+1)}(L \cup \overline{\mathbb{R}}_+) \quad \text{if } L \subset \mathbb{R},$$

and $\mathbf{T} = (T_1, T_2)$ with $T_1 \in L'_{(\omega+1)}(L)$, $T_2 \in L'_{(0)}(\overline{\mathbb{R}}_+)$ with

$$T_1 = \frac{1}{\alpha}(T * \delta_{(1)}), \quad T_2 = c\delta_{(0)} \quad (c \in \mathbb{C}) \quad \text{if } L \not\subset \mathbb{R}.$$

13.5. *Analytic change of variables.* Let $u \in$ GAF and

$$\mathcal{B}u = \mathbf{T} = \{T_0, \dots, T_b\} \quad \text{with } T_p \in L'_{(-\infty)}(L_p), \quad p = 0, \dots, b.$$

Let $y = g(x)$ be an analytic function in a neighbourhood of zero with $g(0) = 0$ and such that $\left(x\frac{d}{dx}\right)g(0) > 0$. Then $u \circ g \in GAF$ and

$$\mathcal{B}(u \circ g) = \{\widetilde{T}_0, \dots, \widetilde{T}_b\}, \quad \widetilde{T}_p \in L'_{(-\infty)}(L_j), \quad p = 0, \dots, b,$$

with

$$\widetilde{T}_p = \sum_{j=1}^{\infty} \sigma_j(z)\Big(\sum_{l=1}^{\infty} c_l T_p * \delta_{(j+l-1)}\Big), \quad p = 0, \dots, b, \tag{13.2}$$

where

$$\sigma_j(z) = \sum\nolimits^{j} \frac{(-z-1)(-z-2)\dots(-z-j)}{i_2!\dots i_k!} c_1^{-z-1-i_0} c_2^{i_2} \dots c_k^{i_k} \tag{13.3}$$

with $\sum^j$ denoting summation over $i_0, i_2, \dots, i_k \in \mathbb{N}_0$ such that (1)

$$-i_0 + 2i_2 + \dots + ki_k = j, \quad -i_0 + i_2 + \dots + i_k = 0,$$

and

$$c_j = \frac{1}{j!}\frac{d^j}{dy^j} g^{-1}(0).$$

P r o o f. Without affecting generality we may assume that u is in the standard form, i.e. $u(y) = T[y^\alpha]$ for $0 < y < \varrho$, where $T \in L'_{(\ln \varrho)}(\overline{\mathbb{R}}_+)$ for some $0 < \varrho$. Further, we may assume that g^{-1} maps $(0, r]$ diffeomorphically onto $(0, t]$ (for some $0 < r < \varrho$ and $t > 0$) and is of convergence radius (at zero) greater than r.

As in the proof of Theorem 9 in [Zie1] we have

$$\int_0^t u(g(x)) x^{-z-1}\, dx = \int_0^r \frac{u(y)}{g'(g^{-1}(y))} (g^{-1}(y))^{-z-1}\, dy. \tag{13.4}$$

Set $b(y) = u(y)/g'(g^{-1}(y))$. We are interested in the holomorphic extension of the function

$$\Phi(z) = \int_0^r b(y)(g^{-1}(y))^{-z-1}\, dy.$$

(1) Observe that for every j, there are only finitely many such nonzero indices.

By assumption $g^{-1}(y) > 0$ on $(0, r]$, hence it is enough to consider the function

$$\widetilde{\Phi}(z) = \int_0^{r_0} b(y)(g^{-1}(y))^{-z-1}\,dy \tag{13.5}$$

for r_0 sufficiently small. Indeed, the function

$$\theta(z) = \int_{r_0}^{r} b(y)(g^{-1}(y))^{-z-1}\,dy$$

is entire and satisfies the estimate

$$|\theta(z)| \le \begin{cases} Ct_1^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle & \text{for } \operatorname{Re} z \le 0, \\ Ct_2^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle & \text{for } \operatorname{Re} z > 0, \end{cases} \tag{13.6}$$

where $t_1 = \sup_{y\in(r_0,r)} |g^{-1}(y)|$ and $t_2 = \inf_{y\in(r_0,r)} |g^{-1}(y)|$. To study the function $\widetilde{\Phi}$ we first observe that

$$g^{-1}(y) = \sum_{j=1}^{\infty} c_j y^j \qquad \text{for } |y| \le r \tag{13.7}$$

and $c_1 > 0$. We shall make use of the following power series expansion at a point $v > 0$ for $0 < w < v$:

$$(v+w)^{\zeta} = v^{\zeta} + \sum_{j=1}^{\infty} \frac{\zeta \ldots (\zeta - j + 1)}{j!} v^{\zeta - j} w^j, \tag{13.8}$$

valid for any $\zeta \in \mathbb{C}$. Applying (13.8) to $v = c_1$ and $w = \sum_{j=2}^{\infty} c_j y^{j-1}$ and multiplying the power series w^j we get

$$\Big(\sum_{j=1}^{\infty} c_j y^j\Big)^{-z-1} = \sum_{j=0}^{\infty} \sigma_j(z) y^{-z-1+j}, \tag{13.9}$$

where σ_j are defined by (13.3). Hence we get

$$\widetilde{\Phi}(z) = \sum_{j=0}^{\infty} \sigma_j(z) \Psi_b(z-j), \tag{13.10}$$

where

$$\Psi_b(z) = \int_0^{r_0} b(y) y^{-z-1}\,dy.$$

The function b is the product of a GAF u of convergence radius $> \varrho$ and of an analytic function $(g^{-1})'$ of convergence radius $> r$. Hence it follows from 13.2 that b is a GAF of convergence radius $\ge r$ and by Theorem 10.1 the Mellin transform $\Psi_b(z) = \mathcal{M}(\chi_{r_0} b)(z)$ is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{R}}_+$ and there exists $p > 0$ such that for every $\varepsilon > 0$,

(i) $\quad |\Psi_b(z)| \le C_\varepsilon r_0^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle \qquad$ for $z \notin (\overline{\mathbb{R}}_+)_\varepsilon$,

(ii) $\quad |\Psi_b(z)| \le C_\varepsilon r_0^{-\operatorname{Re} z}/|\operatorname{Im} z|^p \qquad$ for $z \in (\overline{\mathbb{R}}_+)_\varepsilon$.

Moreover, since $(g^{-1})'(y) = \sum_{j=1}^{\infty} j c_j y^{j-1}$, it follows from formula (13.1) that

$$\Psi_b(z) = \sum_{j=1}^{\infty} j c_j \Psi(z - j + 1), \tag{13.11}$$

where $\Psi(z) = \mathcal{M}(\chi_{r_0} u)$. Set

$$A = \sum_{j=2}^{\infty} |c_j| r_0^j.$$

Then from (13.3), (13.10) and (i) we get, for $z \notin (\overline{\mathbb{R}}_+)_\varepsilon$,

$$|\widetilde{\Phi}(z)| \le C_\varepsilon \Big(\sum_{j=1}^{\infty} \frac{(|z|+1)\ldots(|z|+j)}{j!} c_1^{-\operatorname{Re} z - 1 - j} A^j \Big) \frac{r_0^{-\operatorname{Re} z}}{\langle \operatorname{Im} z \rangle} \le \widetilde{C}_\varepsilon (1 - A/c_1)^{-|z|} r_0^{-\operatorname{Re}} c_1^{-\operatorname{Re} z} / \langle \operatorname{Im} z \rangle \tag{13.12}$$

in view of (13.8) (here we assume $A < c_1$, which can always be done provided r_0 is sufficiently small). In a completely analogous way, by using the estimate (ii) we obtain

$$|\widetilde{\Phi}(z)| \le \widetilde{C}_\varepsilon (1 - A/c_1)^{-|z|-1} r_0^{-\operatorname{Re} z} / |\operatorname{Im} z|^p \quad \text{for } z \in (\overline{\mathbb{R}}_+)_\varepsilon. \tag{13.13}$$

Since $u \circ g$ is a Mellin distribution (see [Zie1]) it follows from Theorem 6.4 that for every $\kappa > 0$,

$$|\mathcal{M}(\chi_t \cdot u \circ g)(z)| \le C_\kappa (t e^\kappa)^{-\operatorname{Re} z} / \langle \operatorname{Im} z \rangle^s \quad \text{for } \operatorname{Re} z < \omega$$

for some $\omega \in \mathbb{R}$ and $s > 0$. This combined with the estimates (13.12), (13.13) and (13.6) shows that the assumptions of Theorem 11.1 are satisfied (cf. Remark 11.4). It follows from that theorem (see also Remarks 11.2 and 11.3) that $u \circ g$ is a GAF of convergence radius $\ge r_0$. Formula (13.2) follows from (13.10) and (13.11) by computing the boundary values.

14. Resurgent functions. Resurgent functions constitute an important subclass of GAFs, and occur as solutions to singular ordinary and partial differential equations (see Chapter III and Appendix II).

Definition 14.1. Let Σ be a discrete subset in $\mathbb{C}$. A generalized analytic function

$$u(x) = \sum_{j=0}^{b} T_j[x^\zeta], \quad 0 < x < e^\omega,$$

with $T_j \in L'_{(\omega)}(L_j)$ is called a *resurgent function* of type Σ if $T_j \in \mathcal{R}_{(\omega)}^{(-\infty)}(L_j, \Sigma)$, $j = 0, \ldots, b$.

The following classical theorem provides us with most elementary examples of resurgent functions.

Theorem 14.1 ([Tou]). *Let $\omega > 0$. The function*

$$u(x) = T[x^\alpha] \quad \textit{for } 0 < x < e^{-\omega}$$

is a resurgent function with $T = Y \cdot G$, where $G \in \widetilde{\mathcal{O}}_{(-\omega)}(\mathbb{C})$ (1), if and only if

(1) Recall that (in accordance with the notation of Section 4) $\widetilde{\mathcal{O}}_{(-\omega)}(\mathbb{C}) = \{\Phi \in \mathcal{O}(\mathbb{C})$: $|\Phi(z)| \le C_\kappa e^{(\omega+\kappa)|z|}$ for any $\kappa > 0$ and $z \in \mathbb{C}\}$.

$$u(x) = \sum_{n=0}^{\infty} \frac{a_n}{(-\ln x)^{n+1}} \qquad \textit{for } 0 < x < e^{-\omega},$$

where the series $\sum_{n=0}^{\infty} a_n \zeta^n$ *has convergence radius not less than* $1/\omega$.

Proof. Suppose $G \in \widetilde{\mathcal{O}}_{(-\omega)}(\mathbb{C})$. Then $|G(z)| < C_\kappa e^{(\omega+\kappa)|z|}$ for every $\kappa > 0$. Writing $G(z) = \sum_{j=0}^{\infty} b_j z^j$ we get, for $\varrho > 0$ ($\varrho = |z|$),

$$|b_j| \le C_\kappa e^{(\omega+\kappa)\varrho} \varrho^{-j}.$$

The minimum of the function $\mathbb{R}^+ \ni \varrho \mapsto e^{(\omega+\kappa)\varrho}\varrho^{-j}$ is attained for $\varrho = j/(\omega+\kappa)$ and equals $(\omega+\kappa)^j (j/e)^{-j}$. Hence

$$|b_j|(j/e)^j \le C_\kappa (\omega+\kappa)^j.$$

By the Stirling formula

$$\Gamma(1+x) \sim (x/e)^x \sqrt{2\pi x} \qquad \text{as } x \to \infty$$

we get $(j/e)^j \sim \Gamma(1+j)(2\pi j)^{-1/2}$. Thus if $\kappa' > \kappa$ we find

$$|b_j| \Gamma(1+j) \le C_{\kappa'} (\omega+\kappa')^j,$$

which proves that the series $\sum_{j=0}^{\infty} b_j \Gamma(1+j) z^j$ has convergence radius not less than $1/\omega$. By (1.2) we get, for $0 < x < e^{-\omega}$,

$$u(x) = YG(\alpha)[x^\alpha] = \int_0^\infty \Big(\sum_{j=0}^{\infty} b_j \alpha^j x^\alpha \Big)\, d\alpha$$

$$= \sum_{j=0}^{\infty} b_j \int_0^\infty \alpha^j x^\alpha \, d\alpha = \sum_{j=0}^{\infty} \frac{a_j}{(-\ln x)^{j+1}}$$

with $a_j = b_j \cdot j!$. Conversely, if $\sum_{j=0}^{\infty} b_j \Gamma(j+1) z^j$ is holomorphic in the disc $|z| < 1/\omega$ then for every $\kappa > 0$ there exists $C_\kappa > 0$ such that for all $j \in \mathbb{N}_0$,

$$|b_j| \le C_\kappa (\omega+\kappa)^j (j/e)^{-j}.$$

The function $G(z) = \sum_{j=0}^{\infty} b_j z^j$ is entire. Write

$$\mu(\varrho) = \sup_{|z| \le \varrho} |G(z)|, \qquad \widetilde{\mu}(\varrho) = \sup_j |b_j| \varrho^j.$$

Then

$$\widetilde{\mu}(\varrho) < \mu(\varrho) \le \frac{1+\kappa}{\kappa} \widetilde{\mu}(\varrho(1+\kappa)). \tag{14.1}$$

We have

$$\widetilde{\mu}(\varrho) \le C_\kappa ((\omega+\kappa)\varrho)^j (j/e)^{-j}.$$

The maximum of the function $\mathbb{R}_+ \ni t \mapsto A^t (t/e)^{-t}$ equals e^{At}, hence $\widetilde{\mu}(\varrho) \le C_\kappa e^{(\omega+\kappa)\varrho}$ and the desired estimate follows in view of (14.1).

Remark 14.1. We have shown in Theorem 14.1 that the Borel transform of $u = \sum_{j=0}^{\infty} a_j/(-\ln x)^{j+1}$ equals $Y(\alpha)(\sum_{j=0}^{\infty} a_j \alpha^j / j!)$, i.e. coincides with the classical Borel transformation, which to a (divergent) series $\sum_{j=0}^{\infty} a_j / s^{j+1}$ assigns the series $\sum_{j=0}^{\infty} a_j \alpha^j / j!$ which stands a better chance of being convergent.

14.1. *Alien derivatives of resurgent functions.* Let $u(x) = T[x^z]$, $0 < x < e^\omega$, where $T \in \mathcal{R}^k_{(\omega)}(L, \Sigma)$, be a resurgent function of type Σ. We introduce the operations

$$\delta^L_\sigma u(x) = (\Delta^L_\sigma T)[x^\alpha], \quad 0 < x < e^\omega,$$

where Δ^L_σ are defined by (5.2) for $\sigma \in L \cap \Sigma$.

It follows from (5.3) that if

$$u_j(x) = T_j[x^\zeta]$$

with $T_j \in \mathcal{R}^k_{(\omega)}(L_j, \Sigma_j)$, $L_j = \zeta_j + \overline{\mathbb{R}}_+$, $j = 1, 2$, then for $\sigma \in (L_1 + L_2) \cap (\Sigma_1 + \Sigma_2)$ with $\sigma \neq \zeta_1 + \zeta_2$ we have the following Leibniz rule:

$$\delta^{L_1+L_2}_\sigma (u_1 \cdot u_2) = (\delta^{L_1}_{\sigma-\zeta_2} u_1) \cdot u_2 + u_1(\delta^{L_2}_{\sigma-\zeta_1} u_2).$$

In particular, in the standard situation of resurgent functions

$$u_j(x) = T_j[x^\alpha],$$

where $T_j \in \mathcal{R}^{-\infty}_{(-\infty)}(\overline{\mathbb{R}}_+, \mathbb{N})$, we get under the notation $\delta^{\mathbb{R}}_n = \delta_n$ for $n \in \mathbb{N}$:

$$\delta_n(u_1 \cdot u_2) = (\delta_n u_1) \cdot u_2 + u_1(\delta_n u_2). \tag{14.2}$$

Thus δ_n are derivations in the class of (standard) resurgent functions. We call them (following J. Ecalle) *alien derivations* to distinguish them from the standard differentiation. The kernel of δ_n, $n \in \mathbb{N}_0$, is the following:

LEMMA 14.1. *Let u be a standard resurgent function*

$$u(x) = T[x^\alpha], \quad 0 < x < e^{-\omega},$$

where $T \in \mathcal{R}^k_{(-\omega)}(\overline{\mathbb{R}}_+, \mathbb{N}_0)$ with $k \geq 0$, $\omega > 0$. Then u is of the form

$$u(x) = \sum_{l=0}^{k-1} c_l (\ln x)^l + \sum_{j=0}^{\infty} \frac{a_j}{(-\ln x)^{j+1}} \quad \textit{for } 0 < x < e^{-\omega}, \tag{14.3}$$

where the series $\sum_{j=0}^\infty a_j \zeta^j$ has convergence radius not less than $1/\omega$, if and only if $\delta_n u = 0$ for $n \in \mathbb{N}_0$.

Proof. It follows from the definition of the operations Δ_n that $\operatorname{var} T \in \widetilde{\mathcal{O}}^k_{(-\omega)}(\widetilde{\mathbb{C} \setminus \mathbb{N}_0})$ is in $\widetilde{\mathcal{O}}^k_{(-\omega)}(\mathbb{C} \setminus \{0\})$ if and only if $\Delta_n T = 0$ for $n \in \mathbb{N}_0$. On the other hand, $\operatorname{var} T \in \widetilde{\mathcal{O}}^k_{(-\omega)}(\mathbb{C} \setminus \{0\})$ if and only if T can be represented as

$$T = \sum_{j=0}^{k-1} \widetilde{c}_j \delta^{(j)}_{(0)} + Y \cdot H,$$

where H extends to a function in $\widetilde{\mathcal{O}}_{(-\omega)}(\mathbb{C})$. The statement now follows from Theorem 14.1.

Thus, one can say that alien derivatives measure how far a given resurgent function is from functions of the form (14.3). Alien derivations are useful in a whole range of classification problems modulo functions (14.3) (cf. [M], [E2], [E3], [PCN]). Moreover, due to their derivation property (14.2) they apply well to nonlinear singular ODEs (see Appendix II.1).

14.2. *Taylor–Fourier representation of resurgent functions.* Recall that in the definition of a GAF u the singularities of the Mellin transforms of $\chi_r u$ are chosen to be half-lines parallel to the real axis and directed along $\mathbb{R}_+$. This corresponds to a generalized Taylor expansion of u at zero. In the case of a resurgent function we can rotate those half-lines by any angle θ to obtain the so-called Taylor–Fourier representation. The case of $\theta = \pm\pi/2$ corresponds to the Fourier representation of u, as explained below.

We start with the definition of the space of functions of exponential growth in a conical domain V. We say that V is a *conical domain at* ∞ if V is open in $\mathbb{C}$ and there exists an open cone $\Gamma \subset \mathbb{C}$ and $\zeta \in \mathbb{C}$ such that $\zeta + \Gamma \subset V$. For any such V and $\omega \in \mathbb{R}$ we define

$$\widetilde{\mathcal{O}}_{(\omega)}(V) = \{\Phi \in \mathcal{O}(V) : \sup_{z\in\widetilde{V}} |e^{(\omega-\kappa)|z|}\Phi(z)| < \infty \text{ for any } \kappa > 0 \text{ and every proper conical subset } \widetilde{V} = \widetilde{b} + \widetilde{\Gamma} \subset V\}.$$

THEOREM 14.2. *Let*

$$u(x) = T[x^\alpha], \quad 0 < x < e^\omega, \tag{14.4}$$

where $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$. Suppose $T = Y \cdot \Phi$, where Φ extends to a function in $\widetilde{\mathcal{O}}_{(\omega)}(V)$ with V a conical domain containing $\overline{\mathbb{R}}_+$. Then for every angle θ such that the line $L_\theta = e^{i\theta}\overline{\mathbb{R}}_+$ is contained in V, we have

$$u(x) = \int_{L_\theta} \Phi(z)x^z\,dz \quad \text{for } 0 < x < e^\omega. \tag{14.5}$$

Proof. Let $\Delta(\theta, R)$ be the sector on the sphere $|z| = R$ between 0 and θ. Clearly $\Delta(\theta, R) = R \cdot \Delta(\theta, 1)$, and for every fixed $x < e^\omega$ and $\kappa > 0$ we have

$$\Big|\int_{\Delta(\theta,R)} \Phi(z)x^z\,dz\Big| \le R \int_{\Delta(\theta,1)} |\Phi(R\gamma)|x^R\,d\gamma \le C \cdot Re^{(-\omega+\kappa)R}x^R. \tag{14.6}$$

Now the right hand side of (14.6) tends to zero as $R \to \infty$ if κ is chosen so small that $e^\kappa x < e^\omega$. This proves that the integral over $\mathbb{R}_+$ in (14.4) can be replaced by the integral over L_θ as in (14.5).

Remark 14.2. If V is large enough so that $i\overline{\mathbb{R}}_+ \subset V$ then (14.5) assumes the form

$$u(x) = i\int_0^\infty x^{iR}\Phi(iR)\,dR \quad \text{for } 0 < x < e^\omega,$$

which in the variable $s = -\ln x$ becomes the Fourier transform of the function $R \mapsto Y \cdot \Phi(iR)$. This motivates the term "Taylor–Fourier representation" for (14.5).

Remark 14.3. Observe that in the case of a standard resurgent function $u(x) = T[x^\alpha]$ with $T \in \mathcal{R}^{-\infty}_{(-\infty)}(\mathbb{R}_+, \mathbb{N}_0)$ the rotation of $\mathbb{R}_+$ by an angle θ leads to the occurrence of infinitely many half-lines $L^n_\theta = n + L_\theta$, $n \in \mathbb{N}_0$, and (14.5) assumes the form

$$u(x) = \sum_{n=0}^\infty \int_{L^n_\theta} x^z\Phi(z)\,dz.$$

III. Applications to singular linear differential equations

15. Special functions as generalized analytic functions. We now show that all important special functions (except the elliptic functions) are generalized analytic functions.

We start by recalling the definition of some special functions and transforming them to the form suitable for our purposes:

• the *error function*

$$\mathrm{Erf}(y) = \int_0^y e^{-t^2}\,dt = \frac{y}{2}\int_0^1 e^{-y^2\alpha}\,\frac{d\alpha}{\sqrt{\alpha}} \qquad \text{for } y>0;$$

• the *complementary error function*

$$\mathrm{Erfc}(y) = \int_y^\infty e^{-t^2}\,dt = \frac{y}{2}\int_1^\infty e^{-y^2\alpha}\,\frac{d\alpha}{\sqrt{\alpha}} \qquad \text{for } y>0;$$

• the *incomplete gamma functions*

$$\gamma(a,y) = \int_0^y e^{-t}t^{a-1}\,dt = y^a\int_0^1 e^{-y\alpha}\alpha^{a-1}\,d\alpha \qquad \text{for } y>0,$$

$$\Gamma(a,y) = \int_y^\infty e^{-t}t^{a-1}\,dt = y^a\int_1^\infty e^{-y\alpha}\alpha^{a-1}\,d\alpha \qquad \text{for } y>0$$

(clearly $\gamma(a,y)+\Gamma(a,y)=\Gamma(a)$, the "usual" Euler Γ-function);

• the *exponential integral*

$$-\mathrm{Ei}(-y) = \int_y^\infty e^{-t}\,\frac{dt}{t} = \int_1^\infty e^{-y\alpha}\,\frac{d\alpha}{\alpha} \qquad \text{for } y>0;$$

• the *logarithmic integral*

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln t} = -\int_1^\infty x^\alpha\,\frac{d\alpha}{\alpha} \qquad \text{for } 0<x<1.$$

Further examples of special functions are listed at the end of the section.

Observe that the function Li is an example of a GAF of the form

$$u(x) = \int_0^\infty x^\alpha T(\alpha)\,d\alpha, \tag{15.1}$$

where

$$T = \frac{-Y(\alpha-1)}{\alpha}.$$

Recall the most elementary example of a GAF which is not analytic.

Example 15.1. For $0<x<1$ we have with $T=Y$ (= the Heaviside function)

$$\frac{1}{-\ln x} = \int_0^\infty x^\alpha\,d\alpha.$$

It can be generalized as

EXAMPLE 15.2. If $a \in \mathbb{C} \setminus \mathbb{N}_0$ then

$$(-\ln x)^a = \frac{1}{\Gamma(-a)} \alpha_+^{-a-1}[x^\alpha] \quad \text{for } 0 < x < 1$$

(α_+^{-a-1} denotes the homogeneous distribution of order $-a-1$ supported by $\overline{\mathbb{R}}_+$ (cf. [GS])). For $j \in \mathbb{N}_0$ we have, for $0 < x$,

$$(-\ln x)^j = \delta^{(j)}[x^\alpha].$$

Proof. Let $\operatorname{Re} a < 0$. Substituting $t = \alpha y$ (for a fixed $y > 0$) in

$$\Gamma(-a) = \int_0^\infty e^{-t} t^{-a-1}\, dt$$

we get

$$y^a = \frac{1}{\Gamma(-a)} \int_0^\infty e^{-y\alpha} \alpha^{-a-1}\, d\alpha,$$

which gives the desired result in the variable $y = -\ln x$. For the remaining $a \in \mathbb{C} \setminus \mathbb{N}_0$ the result follows by analytic continuation. The second statement of the example is clear.

We can now show that all the special functions listed at the beginning are generalized analytic functions (in the variables $x = e^{-y^2}$ or $x = e^{-y}$). Indeed, in view of Example 15.1 and the convolution formula (13.1) we have

$$\operatorname{Erf}(\sqrt{-\ln x}) = \frac{1}{2\Gamma(-1/2)} \alpha_+^{-3/2} * \alpha_+^{-1/2} Y(1-\alpha)[x^\alpha] \quad \text{for } 0 < x < 1,$$

$$\operatorname{Erfc}(\sqrt{-\ln x}) = \frac{1}{2\Gamma(-1/2)} \alpha_+^{-3/2} * \alpha^{-1/2} Y(\alpha-1)[x^\alpha] \quad \text{for } 0 < x < 1.$$

Further,

$$\gamma(a, -\ln x) = T[x^\alpha] \quad \text{for } 0 < x < 1,$$

where

$$T = \begin{cases} \dfrac{1}{\Gamma(-a)} \alpha_+^{-a-1} * \alpha_+^{a-1} Y(1-\alpha) & \text{for } a \in \mathbb{C} \setminus \mathbb{N}_0, \\ \delta^{(j)} * \alpha_+^{j-1} Y(1-\alpha) & \text{for } j \in \mathbb{N}_0; \end{cases}$$

and

$$\Gamma(a, -\ln x) = T[x^\alpha] \quad \text{for } 0 < x < 1,$$

where

$$T = \begin{cases} \dfrac{1}{\Gamma(-a)} \alpha_+^{-a-1} * \alpha^{a-1} Y(\alpha-1) & \text{for } a \in \mathbb{C} \setminus \mathbb{N}_0, \\ \delta^{(j)} * \alpha_+^{j-1} Y(\alpha-1) & \text{for } j \in \mathbb{N}_0. \end{cases}$$

Finally,

$$\operatorname{Ei}(\ln x) = -\frac{1}{\alpha} Y(\alpha-1)[x^\alpha] \quad \text{for } 0 < x < 1.$$

All the special functions considered at the beginning of this section are closely related to the *confluent hypergeometric function* traditionally denoted by ${}_1F_1(a, c; z)$ and given by the series (see [Sl])

$$ {}_1F_1(a, c; z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \dots $$

of convergence radius ∞. The function ${}_1F_1(a, c; z)$ is a solution of the confluent hypergeometric equation also called the *Kummer equation*:

$$ y\frac{d^2 f}{dy^2} + (c - y)\frac{df}{dy} - af = 0. \tag{15.2} $$

The Kummer equation has a regular singular point at zero and an irregular point at ∞. The latter statement means that the equation (15.2) in the variable $s = 1/y$ has a singular point at zero which is not regular. However, considered in the logarithmic variable $y = -\ln x$ the Kummer equation assumes the form

$$ -\ln x\left(x\frac{d}{dx}\right)^2 u - (c + \ln x)\left(x\frac{d}{dx}\right)u - au = 0, \tag{15.3} $$

which, after dividing both sides by $-\ln x$ and grouping the terms, becomes

$$ \left(x\frac{d}{dx}\right)^2 u + \left(x\frac{d}{dx}\right)u + \frac{c}{\ln x}\left(x\frac{d}{dx}\right)u + \frac{a}{\ln x}u = 0. \tag{15.3$'$} $$

Note that the irregular singular point at ∞ for the equation (15.2) got transformed to a regular singular point at zero for the equation (15.3) and the price we paid was the replacement of the coefficients of (15.2) by functions of the form $v(x) = (-\ln x)^{-1}$. Since $v(0) = 0$, the operator $\big(x\big(x\frac{d}{dx}\big)\big)^2 + x\big(x\frac{d}{dx}\big)$ can be regarded as the principal part of the equation (15.3$'$) Thus (15.3) is a Fuchsian type equation whose coefficients are generalized analytic functions. Leaving the delicate general problem of solvability of such an equation to the following section, we solve (15.3) directly. We seek solutions of (15.3) in the form $u(x) = T[x^\alpha]$. From Example 15.1, formula (13.1) and the operational properties of the Borel transformation we get the equation

$$ (\alpha^2 + \alpha)T = Y * ((c\alpha + a)T) \tag{15.4} $$

for the Laplace distribution T. Differentiating (15.4) with respect to α we arrive at the equation (called by Malgrange the Laplace equation; cf. [Tou])

$$ (\alpha^2 + \alpha)\frac{dT}{d\alpha} = ((c - 2)\alpha + a - 1)T. \tag{15.4$'$} $$

Outside the points $\alpha = 0$, $\alpha = -1$ the equation (15.4$'$) is solved by the functions

$$ T(\alpha) = C|\alpha|^{a-1}|\alpha + 1|^{c-a-1}. $$

It is easy to check that if $a - 1 \notin -\mathbb{N}_0$ and $c - a - 1 \notin -\mathbb{N}_0$ then (15.4$'$) has the following two distributional solutions supported by $[-1, 0]$ and $[0, \infty)$, respectively:

$$ T_1 = \alpha_-^{a-1}(\alpha + 1)_+^{c-a-1}, \qquad T_2 = (\alpha + 1)^{c-a-1}\alpha_+^{a-1} $$

($(\alpha+1)_+^b$ denotes the translate by -1 of the homogeneous distribution α_+^b). Thus equation (15.3$'$) and consequently the equivalent equation (15.3) has two solutions which are GAFs:

$$ u_1(x) = \alpha_-^{a-1}(\alpha + 1)_+^{c-a-1}[x^\alpha], \qquad u_2(x) = (\alpha + 1)^{c-a-1}\alpha_+^{a-1}[x^\alpha]. $$

This result is compatible with the so-called Barens formulas for the function ${}_1F_1(a,c;y)$ (which satisfies the Kummer equation),

$$ {}_1F_1(a,c;-\ln x) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_{-1}^{0} x^{\alpha}(-\alpha)^{a-1}(\alpha+1)^{c-a-1}\,d\alpha $$

for $a > 0$ and $c - a > 0$, and for the function $U(a,c;y)$ which is another solution of the Kummer equation,

$$ U(a,c;-\ln x) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} x^{\alpha}\alpha^{a-1}(\alpha+1)^{c-a-1}\,d\alpha $$

for $a > 0$.

Next we prove that u_2 is a resurgent function. Indeed, it is easy to observe that the function $T_2(\alpha)$ for $\alpha > 0$ extends to a function $\Psi(z)$ having logarithmic singularities at 0 and -1. Thus we can compute the difference of the boundary values of Ψ across $\mathbb{R}$ (according to Section 5.1 we denote it by $\Delta_0 T_2$):

$$ \Delta_0 T_2 = \lim_{\beta\to 0_+} \Psi(\cdot + i\beta) - \lim_{\beta\to 0_+} \Psi(\cdot - i\beta). $$

It follows easily from the formula $z^{a-1} = |z|^{a-1}e^{i(a-1)\arg z}$ that

$$ \Delta_0 T_2 = (1 - e^{2\pi i(a-1)})T_2. \tag{15.5} $$

From (15.5) we get, in view of the notation of Section 14,

$$ \delta_0 u_2(x) = (1 - e^{2\pi i(a-1)})u_2(x). $$

This means that the jump at zero of the Borel transform of u_2 $(= T_2)$ recovers u_2 up to a constant. In other words, u_2 resurges from the singularity of T_2 at zero. The same is also true for the other singular point -1. This justifies the name "resurgent function" introduced in Section 14.

The other solution u_1 is clearly not resurgent but $u_1 + u_2$ is, and one can carry out analogous computations for it.

Summing up, the Kummer equation admits a basis of resurgent solutions. The reader interested in the "alien" analysis of other Kummer type equations is referred to [Maj].

To end this section we complete the list of special functions with those expressible by the values of the functions ${}_1F_1$ and U on the imaginary axis (see [Sl]):

- the *integral sine*

$$ \operatorname{Si} y = \int_0^y \frac{\sin t}{t}\,dt = \frac{\pi}{2} - \frac{1}{2} i(e^{-iy}U(1,1;iy) + e^{iy}U(1,1;-iy)), $$

- the *integral cosine*

$$ \operatorname{Ci} y = -\int_y^\infty \frac{\cos t}{t}\,dt = \frac{\pi}{2} - \frac{1}{2} i(e^{-iy}U(1,1;iy) - e^{iy}U(1,1;-iy)), $$

- the *Bessel functions*

$$ J_\nu(y) = \frac{y^\nu e^{-iy}}{2^\nu \Gamma(\nu+1)}\, {}_1F_1\left(\frac{1}{2} + \nu, 1 + 2\nu; 2iy\right), $$

- the *Fresnel integrals*

$$\begin{aligned}\mathrm{S}(y) &= \frac{1}{\sqrt{2\pi}}\int_0^y \frac{\sin t}{\sqrt{t}}\,dt \\ &= \sqrt{\frac{y}{2\pi}}\,i\Big({}_1F_1\Big(\frac12,\frac32;-ye^{\frac12 i\pi}\Big) - {}_1F_1\Big(\frac12,\frac32;-ye^{-\frac12 i\pi}\Big)\Big),\\ \mathrm{C}(y) &= \frac{1}{\sqrt{2\pi}}\int_0^y \frac{\cos t}{\sqrt{t}}\,dt \\ &= \sqrt{\frac{y}{2\pi}}\Big({}_1F_1\Big(\frac12,\frac32;-ye^{\frac12 i\pi}\Big) + {}_1F_1\Big(\frac12,\frac32;-ye^{-\frac12 i\pi}\Big)\Big).\end{aligned}$$

16. Fuchsian type ODEs with generalized analytic coefficients. Consider an operator R of the form

$$R\Big(x, x\frac{d}{dx}\Big) = P\Big(x\frac{d}{dx}\Big) - Q\Big(x, x\frac{d}{dx}\Big), \tag{16.1}$$

where $P(z)$ is a polynomial of degree $m \in \mathbb{N}_0$ and

$$Q\Big(x, x\frac{d}{dx}\Big) = \sum_{\nu=0}^{m} a_\nu(x)\Big(x\frac{d}{dx}\Big)^\nu, \tag{16.2}$$

where a_ν is a GAF of convergence radius $\ge \varrho > 0$ of the form

$$a_\nu(x) = \int_0^\infty x^\alpha \mu_\nu(\alpha), \qquad 0 < x < \varrho, \tag{16.3}$$

where $\mu_\nu \in L'_{(\ln\varrho)}(\overline{\mathbb{R}}_+)$ is a Borel measure such that for $\nu = 0, \ldots, m$,

$$\lim_{r\to 0_+} \int_0^\infty r^\alpha |\mu_\nu(\alpha)| = 0. \tag{16.4}$$

The condition (16.4) means that

$$\lim_{x\to 0_+} a_\nu(x) = 0 \qquad (\nu = 0, \ldots, m), \tag{16.5}$$

hence the operator Q can be regarded as a "perturbation" of the principal part $P\big(x\frac{d}{dx}\big)$. The corresponding operator R is then called a *Fuchsian type operator with generalized analytic coefficients*. Moreover, it follows from (16.5) that zero is an isolated singular point of R. Therefore we may assume that R has no singular points on $(0, \varrho)$. We shall consider an equation of the form

$$Ru = f \qquad \text{on } (0, \varrho), \tag{16.6}$$

where f is a generalized analytic function of type Z (with arbitrary m),

$$f(x) = \sum_{k=0}^{b} T_k[x^z] \qquad \text{for } 0 < x < \varrho, \tag{16.7}$$

where $T_k \in L'_{(\ln\varrho)}(L_k)$, $L_k = \zeta_\kappa + \overline{\mathbb{R}}_+$, $\zeta_k \in \mathbb{C}$, $k = 0, \ldots, b$. Clearly (16.6) has solutions u on $(0, \varrho)$ which are analytic functions. We shall study the behaviour of those solutions at zero.

Define

$$\operatorname{Char} P = \{z : P(z) = 0\},$$

$$Z^P = (Z \cup \operatorname{Char} P) + \overline{\mathbb{R}}_+, \qquad \mathbf{L}^P = \bigcup_{k=0}^{b} L_k \cup \{\operatorname{Char} P + \overline{\mathbb{R}}_+\}.$$

THEOREM 16.1 (existence). *Let R and f be as above. Let u be any solution of* (16.6) *on $(0,\varrho)$. Then for all t with $0<t<\varrho$, the function $u_t = \chi_t u$ (where χ_t is the characteristic function of $(0,t]$) extends to a Mellin distribution and*

(i) $\mathcal{M}u_t \in \mathcal{O}(\mathbb{C} \setminus Z^P)$,

(ii) *for any $\varepsilon > 0$ there exists $r(\varepsilon)$ with $0 < r(\varepsilon) \le t$ and $C_\varepsilon > 0$ such that for $\alpha + i\beta \in \mathbb{C} \setminus \overline{\mathbf{L}^P_\varepsilon}$,*

$$|\mathcal{M}u_t(\alpha+i\beta)| \le \begin{cases} C_\varepsilon (r(\varepsilon))^{-\alpha} & \text{for } \alpha \ge 0, \\ C_\varepsilon t^{-\alpha} & \text{for } \alpha < 0. \end{cases}$$

Proof. Fix $r < \varrho$. Suppose u is a solution of (16.6) on $(0,\varrho)$ and $u_r = \chi_r u \in \mathcal{M}'_{(a)}((0,r])$ for some $a \in \mathbb{R}$. Then

$$Ru_r = f_r + \sum_{j=0}^{m-1} c_j \delta_{(r)}^{(j)} \quad \text{in } D'(\overline{\mathbb{R}}_+), \tag{16.8}$$

where $f_r = \chi_r f$ and c_j are suitable complex constants. Computing the Mellin transform of (16.8) we get, in view of (16.1)–(16.3),

$$P(z) g_r(z) = \sum_{\nu=0}^{m} \int_0^\infty (z-\alpha)^\nu g_r(z-\alpha)\mu_\nu(\alpha) + h_r(z), \tag{16.9}$$

where

$$g_r(z) = \mathcal{M}u_r(z), \qquad h_r(z) = \mathcal{M}f_r(z) + r^{-z} \sum_{j=0}^{m-1} c_j (z+1)\dots(z+j) r^{-1-j}.$$

Dividing both sides by P we get the following convolution type equation for $g_r(z)$:

$$g_r(z) = \sum_{\nu=0}^{m} \int_0^\infty \frac{(z-\alpha)^\nu}{P(z)} g_r(z-\alpha)\mu_\nu(\alpha) + \frac{h_r(z)}{P(z)}.$$

We shall solve it by the method of successive approximations according to the scheme

$$\begin{aligned} g_r^0 &= \frac{h_r(z)}{P(z)}, \\ g_r^n(z) &= \sum_{\nu=0}^{m} \int_0^\infty \frac{(z-\alpha)^\nu}{P(z)} g_r^{n-1}(z-\alpha)\mu_\nu(\alpha) + \frac{h_r(z)}{P(z)}. \end{aligned} \tag{16.10}$$

Next to compute the Mellin transform of f_r we write f in the form (16.7) and by the results of Section 10 we get

$$\mathcal{M}f_r(z) = \sum_{k=0}^{b} T_k \left[\frac{r^{\zeta - z}}{\zeta - z} \right] \quad \text{for } z \notin Z.$$

Moreover, it follows from Theorem 10.1(ii′) for f_r that the function g_r^0 satisfies

(i_0) $g_r^0 \in \mathcal{O}(\mathbb{C} \setminus (Z \cup \operatorname{Char} P))$,

(ii_0) for every $\varepsilon > 0$ there exists $C_{\varepsilon,r} > 0$ such that for $\alpha + i\beta \notin (\mathbf{L} \cup \operatorname{Char} P)_\varepsilon$,

$$|g_r^0(\alpha + i\beta)| \le C_{\varepsilon,r} r^{-\alpha}/\langle\beta\rangle.$$

It is clear from (10) that

$$(\mathrm{i}_n) \qquad g_r^n \in \mathcal{O}(\mathbb{C} \setminus Z^P).$$

Now, take $z \notin \mathbf{L}_\varepsilon^P$. Then there exists $K_\varepsilon > 0$ such that

$$\left|\frac{(z-\alpha)^\nu}{P(z)\alpha^\nu}\right| < K_\varepsilon \quad \text{for } z \notin (\operatorname{Char} P + \overline{\mathbb{R}}_+)_\varepsilon,\ \alpha \in \overline{\mathbb{R}}_+,\ \nu = 0, \ldots, m.$$

Further, set

$$d(r) = \max_{0 \le \nu \le m} \Big(\int_0^\infty r^\gamma |\mu_\nu(\gamma)| \Big).$$

Then we get

$$(\mathrm{ii}_n) \qquad |g_r^n(\alpha + i\beta)| \le \frac{\widetilde{C}_\varepsilon}{\langle\beta\rangle} \Big(\sum_{j=0}^n (K_\varepsilon(m+1)d(r))^j \Big) r^{-\alpha} \quad \text{for } \alpha + i\beta \notin \mathbf{L}_\varepsilon^P.$$

Fix $\varepsilon > 0$ and choose $r = r(\varepsilon)$ such that

$$K_\varepsilon(m+1)d(r) < 1$$

(which is possible in view of (4)). Then it follows by a standard reasoning that the sequence $g_{r(\varepsilon)}^n$ converges locally uniformly to a function $g_{r(\varepsilon)}$ holomorphic on $\mathbb{C} \setminus \overline{\mathbf{L}_\varepsilon^P}$ and such that

$$(16.11) \qquad |g_{r(\varepsilon)}(\alpha + i\beta)| \le C_\varepsilon (r(\varepsilon))^{-\alpha}/\langle\beta\rangle \quad \text{for } \alpha + i\beta \in \mathbb{C} \setminus \overline{\mathbf{L}_\varepsilon^P}.$$

Clearly the function $g_{r(\varepsilon)}$ solves the equation (16.9). Moreover, $g_{r(\varepsilon)}$ is a unique such function holomorphic in $\mathbb{C} \setminus \overline{\mathbf{L}_\varepsilon^P}$ and satisfying (16.11). By computing the inverse Mellin transform we deduce by Theorem 6.3 that

$$\widetilde{u}_{r(\varepsilon)} = \mathcal{M}^{-1} g_{r(\varepsilon)}$$

is a unique Mellin distribution ($\widetilde{u}_{r(\varepsilon)} \in M'_{(a)}((0, r(\varepsilon)])$ with $a = \min(\min_{0 \le k \le b} \operatorname{Re} \zeta_k$, $\min(\operatorname{Re} z : P(z) = 0)))$ which solves the equation (16.8).

Now let v be an arbitrary (analytic) solution of (16.6) on $(0, \varrho)$. Fix $r = r(\varepsilon) < \varrho$ for some ε and consider $v_r = \chi_r v$. Then v_r satisfies the equation (16.8). By the part that we have just proved there exists a Mellin distribution $\widetilde{u}_r \in M'_{(a)}((0, r])$ satisfying (16.8) and such that

$$(16.12) \qquad \mathcal{M}\widetilde{u}_r \in \mathcal{O}(\mathbb{C} \setminus \overline{\mathbf{L}_\varepsilon^P}),$$

$$(16.13) \qquad |\mathcal{M}\widetilde{u}_r(\alpha + i\beta)| \le C_\varepsilon r^{-\alpha}/\langle\beta\rangle \quad \text{for } \alpha + i\beta \in \mathbb{C} \setminus \mathbf{L}_\varepsilon^P.$$

By the uniqueness of solution of the Cauchy problem for the operator R at the point $x = r = r(\varepsilon)$ we infer that $v_r = \widetilde{u}_r$. Thus (16.12) and (16.13) hold for v_r.

Now take any $t < \varrho$ and consider the function $v_{r,t} = v_t - v_r$. Clearly $v_{r,t}$ is a bounded function supported by $[r,t]$. Thus $\mathcal{M}v_{r,t}$ is entire (see Theorem 6.4) and

$$|\mathcal{M}v_{r,t}(\alpha+i\beta)| \le C\int_r^t x^{-\alpha-1}\,dx = C\frac{r^{-\alpha}-t^{-\alpha}}{\alpha} \quad \text{for } \alpha+i\beta\in\mathbb{C}. \tag{16.14}$$

Since $r = r(\varepsilon)$ and ε can be taken arbitrarily small, the assertions of the theorem follow from (16.12)–(16.14).

Theorem 16.1 does not assert that the solutions of (16.6) are GAFs. To ensure this, more restrictive assumptions on the coefficients of the operator R must be imposed.

Remark 16.1. If $\mu_\nu(\alpha) \equiv 0$ for $0 \le \alpha \le \varrho_0$ for some fixed $\varrho_0 > 0$ and all $\nu = 0,\ldots,m$ (e.g. if the operator R has analytic coefficients at zero) then by extending the argument used in the proof of Theorem 13.3 of [SZ] one can prove that there exists $0 < \mathring{r} < \varrho$ such that all solutions of (6) are GAFs of convergence radius not less than $\mathring{r}$.

THEOREM 16.2 (regularity). *Under the notation of this section, suppose that there exists a tubular neighbourhood V of $\overline{\mathbb{R}}_+$ such that for $\nu = 0,\ldots,m$,*

$$\mu_\nu = Y(\alpha)\alpha^k q_\nu(\alpha)d\alpha, \tag{16.15}$$

where the q_ν extend to functions in $\widetilde{\mathcal{O}}_{(\ln\varrho)}(V)$ and $k+1$ is the maximal multiplicity of the characteristic roots of P. Then there exists $0 < \mathring{r}$ such that any solution of (16.6) on $(0,\varrho)$ is a GAF of type Z^P and convergence radius not less than $\mathring{r}$.

The proof of Theorem 16.2 will be based on Theorem 11.1. In view of Theorem 16.1 it is enough to verify condition (iii′) of Theorem 10.1 for $z \in \mathbf{L}^P_\varepsilon$ and hence for $z \in (L_k)_\varepsilon$ $(k = 0,\ldots,b)$ and for $z \in (c_j + \mathbb{R}_+)_\varepsilon$ with $c_j \in \operatorname{Char} P$. By means of suitable translations the proof can be reduced to the standard case of $(\mathbb{R}_+)_\varepsilon$. The proof of the latter fact is preceded by the following

LEMMA 16.1. *Let $Q \in \widetilde{\mathcal{O}}_{(\omega)}(V)$, where $\omega \in \mathbb{R}$ and $V = \{z \in \mathbb{C} : \operatorname{dist}(z,\overline{\mathbb{R}}_+) < \delta\}$ for some $\delta > 0$. Let $G \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+)$, where $W = \{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}$, be such that*

$$|G(\alpha+i\beta)| \le Ct^{-\alpha} \quad \textit{for } \alpha \le -1$$

with $0 < t < e^\omega$ and C independent of α and of β. Then for $z \in V$ with $\operatorname{Im} z > 0$ and $\operatorname{Im} z$ close enough to zero, we have

$$\int_{\mathbb{R}_+} Q(w)G(z-w)\,dw = \int_{\Gamma_z} Q(z-w)G(w)\,dw,$$

where Γ_z is the path consisting of the intervals $(-\infty+i\varepsilon, \operatorname{Re} z + i\varepsilon]$ and $[\operatorname{Re} z + i\varepsilon, z]$, where $0 < \varepsilon < \delta$ is such that $-\Gamma_z + z \subset V$. An analogous statement is valid for $z \in V$ with $\operatorname{Im} z < 0$.

Proof. For z such that $0 < \operatorname{Im} z < \varepsilon$ we have by the change of variable

$$\int_{\mathbb{R}_+} Q(w)G(z-w)\,dw = -\int_{z-\mathbb{R}_+} Q(z-w)G(w)\,dw.$$

Thus we need only prove that

$$(16.16)\qquad \int_{z-\mathbb{R}_+} Q(z-w)G(w)\,dw = \int_{\Gamma_z} Q(z-w)G(w)\,dw.$$

To this end we note that for fixed z the integral in (16.16) is holomorphic on the set bounded by the contour consisting of Γ_z and $z-\mathbb{R}_+$. Write $w = s+i\gamma$, $z=\alpha+i\beta$ with $\beta>0$. Then $z-w = \alpha - s + i(\beta-\gamma)$ and for any fixed $\kappa>0$,

$$\Big|\int_{s+i\beta}^{s+i\varepsilon} Q(z-w)G(w)\,dw\Big| \le C_\kappa \int_\beta^\varepsilon e^{(-\omega+\kappa)(\alpha-s)}C(\gamma)t^{-s}\,d\gamma$$
$$\le Ce^{(-\omega+\kappa)|\alpha-s|}t^{-s} \qquad \text{for } s\le -1.$$

Now we observe that the last expression tends to zero as $s\to-\infty$. Indeed, for s small negative we have $|\alpha-s| = \alpha - s$ and we may always assume that $t<e^{\omega-\kappa}$. This proves the lemma.

Proof of Theorem 16.2. Let u be a solution of (16.6) on $(0,\varrho)$. Fix $\varepsilon_0>0$ sufficiently small so that $z-\Gamma_z\subset V$ for $|\operatorname{Im} z|\le\varepsilon_0$. Let $\mathring{r} = r(\varepsilon_0)$. Setting

$$g_{\mathring{r}} = \mathcal{M}(\chi_{\mathring{r}}u),\qquad h_{\mathring{r}} = \mathcal{M}(\chi_{\mathring{r}}f) + \mathring{r}^{-z}\sum_{j=1}^{m-1} c_j(z+1)\dots(z+j)\mathring{r}^{-1-j}$$

we see as in the proof of Theorem 16.1 that $g_{\mathring{r}}$ satisfies the equation

$$(16.17)\qquad g_{\mathring{r}}(z) = \frac{1}{P(z)}\sum_{\nu=0}^{m}\int_0^\infty (z-\alpha)^\nu g_{\mathring{r}}(z-\alpha)\alpha^k q_\nu(\alpha)\,d\alpha + \frac{h_{\mathring{r}}(z)}{P(z)}.$$

By Theorem 16.1(ii) we have

$$|g_{\mathring{r}}(\alpha+i\beta)|\le C_0\mathring{r}^{-\alpha}\qquad\text{for } \alpha+i\beta\in\mathbb{C}\setminus\overline{\mathbf{L}^P_{\varepsilon_0}}.$$

By Lemma 16.1 we may replace the integral over $\mathbb{R}_+$ in (16.17) by integration over the curve Γ_z. Then (16.17) can be written as

$$(16.18)\qquad g_{\mathring{r}}(z) - \frac{1}{P(z)}\sum_{\nu=0}^{m}\int_z^{\operatorname{Re} z+i\varepsilon_0} w^\nu g_{\mathring{r}}(w)(z-w)^k q_\nu(z-w)\,dw$$
$$= \frac{1}{P(z)}\sum_{\nu=0}^{m}\int_{\operatorname{Re} z+i\varepsilon_0}^{-\infty+i\varepsilon_0} w^\nu g_{\mathring{r}}(w)(z-w)^k q_\nu(z-w)\,dw + \frac{h_{\mathring{r}}(z)}{P(z)}.$$

Let z be close to a characteristic root c_j with $\operatorname{Re}(z-c_j)>0$. Subject to a translation of the variable z, we may assume that $c_j=0$. If $2\varepsilon_0$ is less than the minimum of $|\operatorname{Im}(c_j-c_k)|$ over all j,k such that $\operatorname{Im} c_j\ne\operatorname{Im} c_k$, then the expression on the right hand side of (16.18) makes sense for $0<|\operatorname{Im} z|\le\varepsilon_0$. By the estimates given above and the assumption on q_ν we get for $z=\alpha+i\beta$, $\nu=0,\dots,m$,

$$(16.19)\quad \Big|\int_{\operatorname{Re} z+i\varepsilon_0}^{-\infty+i\varepsilon_0} w^\nu g_{\mathring{r}}(w)(z-w)^k q_\nu(z-w)\,dw\Big|$$

$$\le C_\kappa \int_{-\infty}^{\alpha} (\varepsilon_0+|\gamma|)^\nu |z-w|^k (e^\kappa/\varrho)^{\alpha-\gamma}\mathring{r}^{-\gamma}\,d\gamma$$

$$\le C'_\kappa(\varepsilon_0+|\alpha|)^\nu \mathring{r}^{-\alpha}\int_{-\infty}^{\alpha}(\varepsilon_0+|\alpha-\gamma|)^\nu|\alpha-\gamma+i(\beta-\varepsilon_0)|^k(e^\kappa\mathring{r}/\varrho)^{\alpha-\gamma}\,d\gamma$$

$$\le \widetilde{C}_\kappa(\varepsilon_0+|\alpha|)^\nu\mathring{r}^{-\alpha}$$

since $\int_0^\infty(\varepsilon_0+\theta)^\nu(|\beta-\varepsilon_0|+\theta)^k(e^\kappa\mathring{r}/\varrho)^\theta\,d\theta<\infty$ because $e^\kappa\mathring{r}<\varrho$ if κ is small. By assumption on the multiplicity of the characteristic roots we get

$$(16.20)\qquad |1/P(\alpha+i\beta)|<C/|\beta|^{k+1}\quad \text{for } \alpha\in\mathbb{R} \text{ and } |\beta|\le\varepsilon_0.$$

Further, from Theorem 10.1(iii) and (16.20) we get, for some $l'\in\mathbb{N}_0$,

$$(16.21)\qquad \left|\frac{h_{\mathring{r}}(\alpha+i\beta)}{P(a+i\beta)}\right|<C\mathring{r}^{-\alpha}/|\beta|^{l'}\quad \text{for } 0<|\beta|\le\varepsilon_0.$$

Denote by $H_{\mathring{r}}$ the expression on the right hand side of (16.18). Then it follows from the above estimates that for some $l\in\mathbb{N}_0$,

$$(16.22)\qquad |H_{\mathring{r}}(\alpha+i\beta)|\le C\mathring{r}^{-\alpha}/|\beta|^{l}\quad \text{for } 0<|\beta|\le\varepsilon_0.$$

Denote by R_ν the operator

$$R_\nu H(z)=\int_z^{\operatorname{Re} z+i\varepsilon_0} w^\nu q_\nu(z-w)(z-w)^k H(w)\,dw$$

and put

$$\widetilde{R}H(z)=-\frac{1}{P(z)}\sum_{\nu=0}^{m}R_\nu H(z).$$

Then (16.18) assumes the form $(I-\widetilde{R})g_{\mathring{r}}(z)=H_{\mathring{r}}(z)$. Hence

$$g_{\mathring{r}}(z)=\sum_{j=0}^{\infty}\widetilde{R}^j H_{\mathring{r}}(z)$$

provided the series is convergent. To prove the convergence, set

$$C_1=\max_{0\le\nu\le m}\sup_{[-i\varepsilon_0,0]}|q_\nu(w)|.$$

Then from (16.22) we get for $z=\alpha+i\beta$ with $0<\beta\le\varepsilon_0$ and $w=\alpha+i\gamma$,

$$|R_\nu H_{\mathring{r}}(\alpha+i\beta)|\le C\mathring{r}^{-\alpha}\int_\beta^{\varepsilon_0}(|\alpha|+\gamma)^\nu q_\nu(i(\beta-\gamma))(\gamma-\beta)^k\frac{1}{|\gamma|^l}\,d\gamma$$

$$\le C\cdot C_1(|\alpha|+\varepsilon_0)^\nu\mathring{r}^{-\alpha}\int_\beta^{\varepsilon_0}\frac{(\gamma-\beta)^k}{\gamma^l}\,d\gamma$$

$$\le C\cdot C_1(|\alpha|+\varepsilon_0)^\nu \mathring{r}^{-\alpha} 2\sum_{\mu=0}^{k}\binom{k}{\mu}\frac{1}{l-k-1}\frac{1}{\beta^{l-k-1}}.$$

Since $c_j = 0$ has multiplicity not higher than $k+1$ we hence get, for $0<\beta\le\varepsilon_0$ and $l\ge k+2$,

$$|\widetilde{R}H_{\mathring{r}}(\alpha+i\beta)|\le C\frac{\widetilde{C}}{l-k-1}\frac{\mathring{r}^{-\alpha}}{\beta^l}.$$

By induction,

$$|\widetilde{R}^j H_{\mathring{r}}(\alpha+i\beta)|\le C\frac{\mathring{r}^{-\alpha}}{\beta^l}\left(\frac{\widetilde{C}}{l-k-1}\right)^j.$$

Thus if l is large enough so that $\widetilde{C}<l-k-1$ we get

$$|g_{\mathring{r}}(\alpha+i\beta)|\le CA\mathring{r}^{-\alpha}/\beta^l \quad \text{for } \alpha>-1,\ 0<\beta<\varepsilon_0,$$

where $A=\sum_{j=0}^\infty(\widetilde{C}/(l-k-1))^j$.

Now, by similar considerations for $-\varepsilon_0<\beta<0$ relating to the path symmetric to Γ_z with respect to $\mathbb{R}$ we obtain the desired result by Theorems 16.1 and 11.1.

17. Fuchsian type PDEs with "constant" coefficients. The simplest example of an elliptic partial differential operator with irregular singularities at infinity is the Laplace operator

$$\Delta_2=(\partial/\partial s_1)^2+(\partial/\partial s_2)^2.$$

In logarithmic coordinates $s_1=-\ln x_1,\ s_2=-\ln x_2$ the operator Δ_2 assumes the form

$$\widetilde{\Delta}_2=(x_1\partial/\partial x_1)^2+(x_2c\partial/\partial x_2)^2.$$

Recall that the locally integrable function

$$E(s_1,s_2)=\frac{1}{4\pi}\ln(s_1^2+s_2^2)\quad \text{for } s\ne 0$$

is a fundamental solution of Δ_2 (i.e. $\Delta_2E=\delta_{(0)}$). In logarithmic variables E becomes the function

$$u(x_1,x_2)=\frac{1}{4\pi}\ln((\ln x_1)^2+(\ln x_2)^2)$$

and satisfies the equation $\widetilde{\Delta}_2u=\delta_{(1,1)}$, where $\delta_{(1,1)}$ is the Dirac delta at $(1,1)\in\mathbb{R}^2$. We shall prove below that u is a GAF in the "radial" variable y_1, where $x_1=y_1$, $x_2=y_1\cdot y_2$.

Example 17.1. For every fixed $0<y_2<\infty$ the function $y_1\mapsto u(y_1,y_1\cdot y_2)$ is a generalized function of type $(\overline{\mathbb{R}}_+,1)$ and convergence radius ≥ 1. Moreover, its Borel transform is the distribution

$$T_0[\varphi]= -\frac{2\gamma-\ln 2}{2\pi}\varphi(0)+\frac{1}{2\pi}\int_0^\infty \ln\varrho\frac{d}{d\varrho}(y_2^{\varrho/2}\varphi(\varrho))\,d\varrho + \frac{1}{2\pi}\int_0^\infty\frac{1-\cos(\frac{\varrho}{2}\ln y_2)}{\varrho}y_2^{\varrho/2}\varphi(\varrho)\,d\varrho$$

(here γ is the Euler constant). Hence for $a_1 > 0$,

$$T_0(a_1) = -\frac{1}{2\pi} y_2^{a_1/2} \frac{\cos(\frac{a_1}{2}\ln y_2)}{a_1}.$$

Thus $y_1 \mapsto u(y_1, y_1 \cdot y_2)$ is a resurgent function.

Proof. Denote by χ_t the characteristic function of the interval $(0,t]$ for $0 < t < 1$. We shall compute the Mellin transform $\mathcal{M}^1$ (with respect to y_1) of the function $\chi_t(y_1)u(y_1, y_1 \cdot y_2)$ for a fixed $y_2 > 0$. We have

$$\mathcal{M}^1(\chi_t u(y_1, y_1 \cdot y_2))(\zeta_1) = \frac{1}{4\pi}\int_0^t \ln\left\{2\left(\left(-\ln y_1 - \frac{1}{2}\ln y_2\right)^2 + \left(-\frac{1}{2}\ln y_2\right)^2\right)\right\} y_1^{-\zeta_1-1}\, dy_1.$$

Thus it is enough to compute the one-dimensional Mellin transform

$$\mathcal{M}^1(\chi_t \ln\{2((-\ln y_1 + C)^2 + C^2)\})(\zeta_1) = \int_0^t \ln((-\ln y_1 + C)^2 + C^2) y_1^{-\zeta_1-1}\, dy_1 + \ln 2 \cdot \int_0^t y_1^{-\zeta_1-1}\, dy_1,$$

where $C = C(y_2) = -\frac{1}{2}\ln y_2$. Since

$$\int_0^t y_1^{-\zeta_1-1}\, dy_1 = \frac{t^{-\zeta_1}}{-\zeta_1}$$

we have to compute the first integral. We know that C varies in a bounded set thus for y_1 sufficiently small we have

$$\ln((-\ln y_1 + C)^2 + C^2) = 2\ln(-\ln y_1 + C) + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} C^{2j}(-\ln y_1 + C)^{-2j}$$

and the series is absolutely convergent. By Example 15.2 we have

$$\mathcal{M}^1(\chi_t \ln(-\ln y_1 + C))(\zeta_1) = \frac{\gamma}{\zeta_1} t^{-\zeta_1} + \int_0^{\infty} \ln\varrho \frac{d}{d\varrho}\left(\frac{e^{-C\varrho} t^{\varrho-\zeta_1}}{\varrho-\zeta_1}\right) d\varrho,$$

$$\mathcal{M}^1(\chi_t(-\ln y_1 + C)^{-2j})(\zeta_1) = \frac{1}{\Gamma(2j)} \int_0^{\infty} \frac{\varrho^{2j-1} e^{-C\varrho} t^{\varrho-\zeta_1}}{\varrho-\zeta_1}\, d\varrho \quad \text{for } \zeta_1 \in \mathbb{C}\setminus[0,\infty),$$

where Γ is the Euler gamma function and γ is the Euler constant. Since

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}(C\varrho)^{2j} = \cos(C\varrho)$$

we hence get

$$\mathcal{M}^1(\chi_t \ln((-\ln y_1 + C)^2 + C^2))(\zeta_1) = t^{-\zeta_1}\frac{2\gamma}{\zeta_1} + 2\int_0^{\infty} \ln\varrho \frac{d}{d\varrho}\left(\frac{e^{-C\varrho} t^{\varrho-\zeta_1}}{\varrho-\zeta_1}\right) d\varrho + 2\int_0^{\infty} \frac{1-\cos(C\varrho)}{\varrho} \frac{e^{-C\varrho} t^{\varrho-\zeta_1}}{\varrho-\zeta_1}\, d\varrho.$$

We have thus found that

$$\mathcal{M}^1(\chi_t u(y_1, y_1\cdot y_2))(\zeta_1) = \frac{1}{4\pi}\bigg(\frac{2\gamma-\ln 2}{\zeta_1} + 2\int_0^\infty \ln\varrho \frac{d}{d\varrho}\bigg(\frac{y_2^{\varrho/2}t^\varrho}{\varrho-\zeta_1}\bigg)\,d\varrho + 2\int_0^\infty \frac{1-\cos\left(\frac{\varrho}{2}\ln y_2\right)}{\varrho}\frac{y_2^{\varrho/2}t^\varrho}{\varrho-\zeta_1}\,d\varrho\bigg)t^{-\zeta_1}, \tag{17.1}$$

which easily shows that the assumptions of Theorem 11.1 (complemented by Remark 11.2) are satisfied. This proves that $y_1 \mapsto u(y_1, y_1\cdot y_2)$ is a GAF. Concerning its Borel transform (in variable y_1), let

$$T_0 = \frac{1}{2\pi i}\big(\lim_{b_1\to 0_+}\mathcal{M}^1(\chi_t u(y_1,y_1\cdot y_2))(\cdot + ib_1) - \lim_{b_1\to 0_-}\mathcal{M}^1(\chi_t u(y_1,y_1\cdot y_2))(\cdot + ib_1)\big),$$

where the limit is taken in the sense of distributions. Then from (17.1) we easily get the desired explicit expression for T_0.

Returning to the original variables $y_1 = x_1$, $y_2 = x_2/x_1$ in the formula $u(y_1, y_1\cdot y_2) = T_0[y_1^\varrho]$ and observing that

$$\frac{1}{\varrho}(y_2^{(1+i)\varrho/2} + y_2^{(1-i)\varrho/2}) = \frac{2}{\varrho}y_2^{\varrho/2}\cos\left(\frac{\varrho}{2}\ln y_2\right)$$

we obtain the following expression for u:

$$u(x_1,x_2) = \frac{-2\gamma-\ln 2}{4\pi} + \frac{1}{2\pi}\int_0^\infty x_1^\varrho\frac{d}{d\varrho}(x_1\cdot x_2)^{\varrho/2}\ln\varrho\,d\varrho - \frac{1}{2\pi}\int_0^\infty \frac{x_1^\varrho(x_1\cdot x_2)^{\varrho/2} - \frac{1}{2}(x_1^{(1-i)\varrho/2}x_2^{(1+i)\varrho/2} + x_1^{(1+i)\varrho/2}x_2^{(1-i)\varrho/2})}{\varrho}\,d\varrho,$$

which shows that u is in fact a GAF in the two variables x_1, x_2 (cf. Section 18).

We now pass to the general case of the "fundamental solution" u:

$$\mathcal{P}\bigg(y_1\frac{\partial}{\partial y_1},\ldots,y_n\frac{\partial}{\partial y_n}\bigg)u = \delta_{(\mathbf{1})} \qquad (\mathbf{1} = (1,\ldots,1)\in\mathbb{R}^n) \tag{17.2}$$

to a singular PDO $\mathcal{P}$ in n variables. We first recall some pertinent notions and fix notation. Let $\mathcal{P}$ be a polynomial in the complex variable $\zeta = (\zeta_1,\ldots,\zeta_n)\in\mathbb{C}^n$, $n>1$. To the polynomial $\mathcal{P}$ we relate the operator $\mathcal{P}(y\partial/\partial y)$ on $\mathbb{R}^n_+ = \mathbb{R}_+\times \overset{(n)}{\cdots} \times\mathbb{R}_+$, where $y\partial/\partial y = (y_1\partial/\partial y_1,\ldots,y_n\partial/\partial y_n)$. Then the polynomial $\mathcal{P}$ is called the *Mellin symbol* of the operator $P(y\partial/\partial y)$. The solutions of the equation (17.2) lie in the scale of spaces $\mathfrak{M}'_a$, $a\in\mathbb{R}^n$, defined as follows:

Denote by $\mu:\mathbb{R}^n\to\mathbb{R}^n_+$ the diffeomorphism

$$\mu(s) = e^{-s} := (e^{-s_1},\ldots,e^{-s_n}).$$

Let $a\in\mathbb{R}^n$ and define (for details see [SZ], §4)

$$\mathfrak{M}_a = \mathfrak{M}_a(\mathbb{R}^n_+) = \{\sigma\in C^\infty(\mathbb{R}^n_+) : (y^{a+\mathbf{1}}\sigma)\circ\mu\in S(\mathbb{R}^n)\},$$

where we write $y^z = y_1^{z_1}\ldots y_n^{z_n}$ for $y\in\mathbb{R}^n$ and $z\in\mathbb{C}^n$, and $S(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions. We equip $\mathfrak{M}_a$ with a natural topology induced from $S(\mathbb{R}^n)$, and define $\mathfrak{M}'_a$ as the topological dual of $\mathfrak{M}_a$. It follows from the S'-version of the Schwartz kernel theorem ([SZ], Th. 4.3) that $\mathfrak{M}'_a$ is isomorphic in a natural way to $\mathfrak{M}'_{a_1}(\mathbb{R}_+;\mathfrak{M}'_{a'}(\mathbb{R}^{n-1}_+))$ (here $a=(a_1,a')$), the space of linear continuous functionals on $\mathfrak{M}_{a_1}$ with values in $\mathfrak{M}'_{a'}(\mathbb{R}^{n-1}_+)$.

Observing that $u\in\mathfrak{M}'_a(\mathbb{R}^n_+)$ if and only if $e^{as}(u\circ\mu)\in S'(\mathbb{R}^n)$ we can define the $\mathcal{M}_a$-*Mellin transform* of u by the formula

$$\mathcal{M}_a u=(2\pi)^{n/2}\mathcal{F}^{-1}(e^{as}(u\circ\mu)), \tag{17.3}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transformation in $S'(\mathbb{R}^n)$ defined on test functions $\sigma\in S(\mathbb{R}^n)$ by the formula

$$\mathcal{F}^{-1}\sigma(y)=(2\pi)^{-n/2}\int\limits_{\mathbb{R}^n}e^{i\xi y}\sigma(\xi)\,d\xi.$$

If u has support contained in a bounded set (in $\mathbb{R}^n$) then

$$\mathcal{M}_a u(b)=u[y^{-a-ib-\mathbf{1}}], \tag{17.4}$$

i.e. it coincides with the n-dimensional Mellin transform of u:

$$\mathcal{M}u(z)=u[y^{-z-\mathbf{1}}]. \tag{17.5}$$

We shall also need the following parameter version of Theorem 8.2 on the modified Cauchy transformation:

THEOREM 17.1. *Let $T\in S'(\mathbb{R}^n)$ and fix $\overset{*}{a}\in\mathbb{R}^n$. Fix cut-off functions $\chi\in C^\infty_{(0)}(\mathbb{R}_+)$ with $\chi\equiv1$ in a neighbourhood of zero and $\sigma\in C^\infty_0(\mathbb{R}^{n-1}_+)$, and define*

$$K(\zeta)=\mathcal{M}(\chi(x_1)\sigma(x'))(\zeta)\quad \textit{for } \zeta\in(\mathbb{C}\setminus\{0\})\times\mathbb{C}^{n-1},$$
$$K'(\zeta')=\mathcal{M}'\sigma(\zeta')\quad \textit{for } \zeta'\in\mathbb{C}^{n-1}.$$

Set

$$\widetilde{\mathcal{C}}^\pm(\zeta)=\widetilde{\mathcal{C}}^\pm T(\zeta)=\frac{1}{(2\pi)^n}T[K(\zeta-\overset{*}{a}-i\gamma)]\quad \textit{for } \pm\operatorname{Re}\zeta_1>\pm\overset{*}{a}_1,\ \zeta'\in\mathbb{C}^{n-1}$$

and

$$\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta')=(\widetilde{\mathcal{C}}'T)_{\overset{*}{a}_1}(\zeta')=\frac{1}{(2\pi)^{n-1}}T[K'(\zeta'-\overset{*}{a}'-i\gamma')]\in S'(\mathbb{R})\quad \textit{for } \zeta'\in\mathbb{C}^{n-1} \tag{17.6}$$

(in (17.6), *T is regarded as an element of $S'(\mathbb{R}^{n-1};S'(\mathbb{R}))$ under a canonical isomorphism $S'(\mathbb{R}^n)\simeq S'(\mathbb{R}^{n-1};S'(\mathbb{R}))$; see* [SZ], *Th.* 4.3*). Then*

$$\widetilde{\mathcal{C}}^\pm T\in\mathcal{O}(\{\pm\operatorname{Re}\zeta_1>\pm\overset{*}{a}_1\}\times\mathbb{C}^{n-1}),\qquad (\widetilde{\mathcal{C}}'T)_{\overset{*}{a}_1}\in\mathcal{O}(\mathbb{C}^{n-1};S'(\mathbb{R})),$$

and in the sense of convergence in $S'(\mathbb{R}^n)$,

$$\lim_{\substack{a\to\overset{*}{a}\\ a_1<\overset{*}{a}_1}}\widetilde{\mathcal{C}}^-T(a+i\cdot)-\lim_{\substack{a\to\overset{*}{a}\\ a_1>\overset{*}{a}_1}}\widetilde{\mathcal{C}}^+T(a+i\cdot)=(\widetilde{\mathcal{C}}'T)_{\overset{*}{a}_1}(\overset{*}{a}'+i\cdot) \tag{17.7}$$

(here $(\widetilde{\mathcal{C}}'T)_{\overset{*}{a}_1}(\overset{*}{a}'+i\cdot)\in S'(\mathbb{R}^{n-1};S'(\mathbb{R}))$ is regarded as an element of $S'(\mathbb{R}^n)$).

Proof. By translation of the variable ζ we may assume $\overset{*}{a}=0$. In view of formula (17.3) we have, with $\omega(s)=\chi(e^{-s_1})\sigma(e^{-s'})$ and $\omega'(s')=\sigma(e^{-s'})$,

$$K(a+ib)=\begin{cases}(2\pi)^{n/2}\mathcal{F}^{-1}(e^{as}\omega(s))(b) & \text{for } a_1<0\\ (2\pi)^{n/2}\mathcal{F}^{-1}(e^{as}(\omega(s)-\omega'(s')))(b) & \text{for } a_1>0.\end{cases}$$

Now by the formula for the Fourier transform of convolution

$$\widetilde{\mathcal{C}}^{-}T(a+i\,\cdot) = \mathcal{F}^{-1}(e^{as}\omega(s)\mathcal{F}T) \qquad \text{for } a_1 < 0,$$
$$\widetilde{\mathcal{C}}^{+}T(a+i\,\cdot) = \mathcal{F}^{-1}(e^{as}(\omega(s)-\omega'(s'))\mathcal{F}T) \qquad \text{for } a_1 > 0.$$

First we prove that the limits in (17.7) exist. To this end, since $\mathcal{F}$ and $\mathcal{F}^{-1}$ are topological isomorphisms of S onto S, it is enough to show that for every $\psi \in S(\mathbb{R}^n)$,

$$(17.8)\qquad \begin{aligned} e^{as}\omega(s)\psi(s) &\to \omega(s)\psi(s) && \text{in } S(\mathbb{R}^n) \text{ as } a \to 0 \text{ with } a_1 < 0,\\ e^{as}(\omega(s)-\omega'(s'))\psi(s) &\to (\omega(s)-\omega'(s'))\psi(s) && \text{in } S(\mathbb{R}^n) \text{ as } a \to 0 \text{ with } a_1 > 0. \end{aligned}$$

But this is simple in view of the properties of the support of $\omega(s)$ and $\omega(s) - \omega'(s')$ and the fact that all derivatives of ω and ω' are bounded on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively.

From (17.8) we get

$$\lim_{\substack{a'\to 0\\ a_1\to 0_-}} \widetilde{\mathcal{C}}^{-}T(a+i\,\cdot) - \lim_{\substack{a'\to 0\\ a_1\to 0_+}} \widetilde{\mathcal{C}}^{+}T(a+i\,\cdot) = \mathcal{F}^{-1}\big(\omega'(s')\mathcal{F}T\big) = T * (\delta_{(0)} \otimes \mathcal{F}'^{-1}\omega'),$$

where $\delta_{(0)}$ is the Dirac delta at zero in variable s_1 and $\mathcal{F}'^{-1}$ denotes the inverse Fourier transformation in the variables $s' = (s_2, \ldots, s_n)$. Now from (17.3) we see that

$$K'(ib') = (2\pi)^{(n-1)/2}\mathcal{F}'^{-1}\omega'(b')$$

and therefore

$$T * \big(\delta_{(0)} \otimes \mathcal{F}'^{-1}\omega'\big) = (\widetilde{\mathcal{C}}'T)_0(i\,\cdot),$$

which proves (17.7). The proof of the holomorphy of $\widetilde{\mathcal{C}}^{\pm}T$ and $(\widetilde{\mathcal{C}}'T)_{\mathring{a}_1}$ is left to the reader.

Corollary 17.1. *Let H be a function holomorphic on an open set $U \subset \mathbb{C}^n$. Fix $\mathring{a} \in \mathbb{R}^n$ and suppose that the function $b \mapsto H(\mathring{a}+ib)$, defined for $b \in \mathbb{R}^n$ such that $\mathring{a}+ib \in U$, extends to a distribution in $S'(\mathbb{R}^n)$ which we denote by $H_{\mathring{a}}$. Further, suppose that there exists an open set $U^1 \subset \mathbb{C}$ such that for every $\zeta_1 \in U^1$ the function $b' \mapsto H_{\zeta_1}(\mathring{a}'+ib')$, defined for $b' \in \mathbb{R}^{n-1}$ such that $(\zeta_1, \mathring{a}'+ib') \in U$, extends to a distribution $H_{\zeta_1,\mathring{a}'}$ in $S'(\mathbb{R}^{n-1})$ and the distribution-valued function*

$$(17.9)\qquad U^1 \ni \zeta_1 \mapsto H_{\zeta_1,\mathring{a}'} \in S'(\mathbb{R}^{n-1})$$

is holomorphic on U^1. Finally, assume that there exists a regularization $\widetilde{H}_{\mathring{a}_1,\mathring{a}'} \in S'(\mathbb{R}; S'(\mathbb{R}^{n-1}))$ of the function $b_1 \mapsto H_{\mathring{a}_1+ib_1,\mathring{a}'} \in S'(\mathbb{R}^{n-1})$, defined for $b_1 \in \mathbb{R}$ with $\mathring{a}_1+ib_1 \in U^1$, such that $\widetilde{H}_{\mathring{a}_1,\mathring{a}'} = H_{\mathring{a}}$ under the canonical isomorphism $S'(\mathbb{R}; S'(\mathbb{R}^{n-1})) \simeq S'(\mathbb{R}^n)$. Then the function

$$\widetilde{\mathcal{C}}'_{\zeta_1}(\zeta') = \frac{1}{(2\pi)^{n-1}} H_{\zeta_1,\mathring{a}'}[K'(\zeta'-\mathring{a}'-i\gamma')], \qquad (\zeta_1,\zeta') \in U^1 \times \mathbb{C}^{n-1},$$

is holomorphic on $U^1 \times \mathbb{C}^{n-1}$, and for every fixed $\zeta' \in \mathbb{C}^{n-1}$ the distribution $\widetilde{\mathcal{C}}'_{\mathring{a}_1}(\zeta') \in S'(\mathbb{R})$ is a regularization of the function

$$b_1 \mapsto \widetilde{\mathcal{C}}'_{\mathring{a}_1+ib_1}(\zeta')$$

defined for $b_1 \in \mathbb{R}$ such that $\mathring{a}_1 + ib_1 \in U^1$. Moreover, the function

$$\widetilde{\psi}(\zeta) = \begin{cases} \widetilde{\mathcal{C}}^{-}(\zeta) & \textit{for } \operatorname{Re}\zeta_1 < \mathring{a}_1,\ \zeta' \in \mathbb{C}^{n-1},\\ \widetilde{\mathcal{C}}^{+}(\zeta) + \widetilde{\mathcal{C}}'_{\zeta_1}(\zeta') & \textit{for } \operatorname{Re}\zeta_1 > \mathring{a}_1,\ \zeta_1 \in U^1,\ \zeta' \in \mathbb{C}^{n-1}, \end{cases}$$

extends to a holomorphic function on $(\{\operatorname{Re}\zeta_1 < \overset{*}{a}_1\} \cup U^1) \times \mathbb{C}^{n-1}$ (*here* $\widetilde{\mathcal{C}}^{\pm}(\zeta) = (\widetilde{\mathcal{C}}^{\pm} H_{\overset{*}{a}})(\zeta)$ *as in Theorem* 17.1).

P r o o f. Since the function (17.9) is holomorphic on U^1 it is immediate that $\widetilde{\mathcal{C}}'_{\zeta_1}(\zeta')$ is holomorphic on $U^1 \times \mathbb{C}^{n-1}$. The second assertion is also clear because $\widetilde{H}_{\overset{*}{a}_1,\overset{*}{a}'} = H_{\overset{*}{a}}$. Finally, the extendibility of the function $\widetilde{\psi}$ follows from (17.7) and a version of the Painlevé Theorem 2.1 with a holomorphic parameter.

We shall also need the following parameter version of Proposition 8.1:

Corollary 17.2. *Let* $\mathbb{C}^{n-1} \ni \zeta' \mapsto T_{\zeta'} \in E'(\mathbb{R})$ (= *the space of compactly supported distributions*) *be a distribution-valued holomorphic function which is rapidly decreasing as a function of* $\operatorname{Im}\zeta'$, *locally uniformly in* $\operatorname{Re}\zeta'$, *and such that the orders of* $T_{\zeta'}$ *are uniformly bounded. Suppose that* $T_{\zeta'}$ *restricted to an interval* $(0, \mathring{b})$ *with* $\mathring{b} > 0$ *is a function* $T_{\zeta'}(\gamma_1)$ *for* $\zeta' \in \mathbb{C}^{n-1}$, *and for* $j = 0, 1$ *and some* $p, l \in \mathbb{N}_0$,

$$\left|\left|\left| \frac{\partial^j}{\partial \gamma_1^j} T_{a'+i\,\cdot}(\gamma_1) \right|\right|\right|_{S;l} \le \frac{C_l}{\gamma_1^p}, \qquad \gamma_1 \in (0, \mathring{b}),$$

locally uniformly with respect to $a' \in \mathbb{R}^{n-1}$, *where*

$$|||\sigma|||_{S;l} = \sup_{x\in\mathbb{R}^{n-1}} \langle x\rangle^l \left(\sum_{|\alpha|\le l} \left| \left(\frac{\partial}{\partial x}\right)^{\alpha} \sigma(x) \right| \right)$$

for $\sigma \in S(\mathbb{R}^{n-1})$. *Then for* $\mathring{a}_1 \le a_1 < 0$ *and small* $b_1 > 0$,

$$|||C^- T_{a'+i\,\cdot}(a_1 + ib_1)|||_{S;l} \le \widetilde{C}_l / b_1^{\hat{p}}$$

locally uniformly in $a' \in \mathbb{R}^{n-1}$, *where* $\widehat{p} = \max(p, 1 + \widetilde{p})$, $\widetilde{p} = \sup_{\zeta'\in\mathbb{C}^{n-1}} \operatorname{order} T_{\zeta'}$ *and*

$$C^- T_{\zeta'}(\zeta_1) = -\frac{1}{2\pi} T_{\zeta'}\left[\frac{1}{\zeta_1 - i\gamma_1}\right] \qquad \text{for } \zeta' \in \mathbb{C}^{n-1},\ \operatorname{Re}\zeta_1 < 0.$$

The corollary follows easily from Proposition 8.1 since C^- commutes with differentiations in variable b'.

Let us return to the general equation (17.2) in two variables y_1, y_2. As in Example 17.1 we are interested in establishing generalized analyticity of the solution u in y_1 with y_2 fixed. This is connected with the following geometric condition.

Definition 17.1. We say that a complex polynomial $\mathcal{P}(\zeta_1, \zeta_2)$ satisfies *condition* (A_1) if there exists a convex cone Γ_- in the plane (b_1, a_2) around the direction $(0, -1)$ with vertex at $(0, -\delta)$ (for some $\delta > 0$) such that

(A_1) $\operatorname{Re}\operatorname{Char}\mathcal{P} := \{(a_1, b_1, a_2) \in \mathbb{R}^3 : \exists b_2 \in \mathbb{R} : P(a_1 + ib_1, a_2 + ib_2) = 0\}$ has empty intersection with the wedge $W = \mathbb{R} \times \Gamma_- = \{(a_1, b_1, a_2) : a_1 \in \mathbb{R}, (b_1, a_2) \in \Gamma_-\}$

for $|b_1|$ large (see Figure 1).

The geometric condition (A_1) can be reformulated in an analytic way as follows. Let $c_1(\zeta_1), \ldots, c_m(\zeta_1)$ be functions such that

$$\mathcal{P}(\zeta_1, \zeta_2) = a_m \prod_{j=1}^{m} (\zeta_2 - c_j(\zeta_1)). \tag{17.10}$$

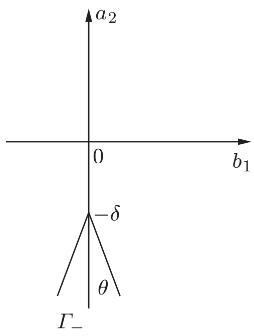

Fig. 1

It turns out that c_j are holomorphic functions on $\mathbb{C}$ with ramification points at the discriminant set $\{\Delta(\zeta_1) = 0\}$, where

$$\Delta(\zeta_1) = \prod_{j<k} (c_j(\zeta_1) - c_k(\zeta_1)).$$

Definition 17.1′. We say that $\mathcal{P}$ satisfies *condition* (A_1') if there exist constants $\theta > 0$ and $\delta > 0$ such that for $j = 1, \ldots, m$,

$$(A_1') \qquad \operatorname{Re} c_j(a_1 + ib_1) > -\theta|b_1| - \delta$$

for large $|b_1|$.

Clearly if we assume (A_1') we may take for Γ_- in (A_1) any cone around $(0, -1)$ of opening angle not greater than θ. Conversely, for θ in (A_1') we can take the opening angle of the cone Γ_- satisfying (A_1).

To formulate the theorem we introduce a final bunch of notation.

Fix $\overset{*}{a} \in \mathbb{R}^2$. We shall consider the functions c_j, $j = 1, \ldots, m$, defined as follows. For $\operatorname{Re} \zeta_1 < \overset{*}{a}_1$ close to $\overset{*}{a}_1$, $c_j(\zeta_1)$ are the holomorphic functions satisfying (17.10) and we choose the following extension of c_j to $\{\operatorname{Re} \zeta_1 > \overset{*}{a}_1\}$. Let θ_ν $(\nu = 1, \ldots, N)$ be all points in $\mathbb{C}$ such that $\Delta(\theta_\nu) = 0$ and for some j, $1 \le j \le m, c_j$ has Puiseux expansion at θ_ν with minimal $p > 1$ and $\operatorname{Re} \theta_\nu > \overset{*}{a}_1$.

At θ_ν, c_j has value $c_j(\theta_\nu)$. For $\overset{\circ}{\zeta}_1 \in \mathbb{R}_+ + \theta_\nu$, we define

$$c_j(\overset{\circ}{\zeta}_1) := c_j^+(\overset{\circ}{\zeta}_1) = \lim_{\zeta_1 \to \overset{\circ}{\zeta}_1,\, \operatorname{Im} \zeta_1 > \operatorname{Im} \overset{\circ}{\zeta}_1} c_j(\zeta_1).$$

We also define

$$c_j^-(\overset{\circ}{\zeta}_1) = \lim_{\zeta_1 \to \overset{\circ}{\zeta}_1,\, \operatorname{Im} \zeta_1 < \operatorname{Im} \overset{\circ}{\zeta}_1} c_j(\zeta_1).$$

Denote by B_μ $(\mu = 1, \ldots, M)$ all points in $\mathbb{R}$ such that for some j with $1 \le j \le m$,

$$(17.11) \qquad \operatorname{Re} c_j(\overset{*}{a}_1 + iB_\mu) = \overset{*}{a}_2.$$

For j satisfying (17.11) we define $\operatorname{sgn}(j; \mu) = \operatorname{sgn}(c_j; B_\mu) = +$ if for $a_1 > \overset{*}{a}_1$ close to $\overset{*}{a}_1$, $b_1 \mapsto \operatorname{Re} c_j(a_1 + ib_1)$ is an increasing function in a neighbourhood of B_μ. Otherwise we put $\operatorname{sgn}(j;\mu) = -$. Finally, we set

$$I^0(B_\mu) = \{j : \text{formula (17.11) holds}\}$$

$$I^+(\theta_\nu) = \{j : c_j \text{ has a Puiseux expansion at } \theta_\nu \text{ with } p > 1 \text{ and } \operatorname{Re} c_j(\mathring{a}_1 + ib_1) > \mathring{a}_2 \text{ for } b_1 > \operatorname{Im}\theta_\nu \text{ close to} \operatorname{Im}\theta_\nu\}.$$

Finally, let

$$L_\mu = \mathbb{R} + iB_\mu \quad \text{for } \mu = 1,\dots,M,$$
$$\widetilde{L}_\nu = \overline{\mathbb{R}}_+ + \theta_\nu \quad \text{for } \nu = 1,\dots,N, \quad Z = \{\zeta_1 : \Delta(\zeta_1) = 0\}.$$

THEOREM 17.2. *Fix* $\mathring{a} \in \mathbb{R}^2$. *Let* $\mathcal{P}(\zeta_1,\zeta_2)$ *be a polynomial satisfying condition* (A_1) *and such that for* $j = 1,\dots,m$, $\operatorname{Re} c_j(\mathring{a}_1 + ib_1) \not\equiv \mathring{a}_2$ *for* $b_1 \in \mathbb{R}$. *Consider a solution* $v \in \mathfrak{M}'_{\mathring{a}}(\mathbb{R}^2_+)$ *of the equation*

$$\mathcal{P}\left(y_1\frac{\partial}{\partial y_1}, y_2\frac{\partial}{\partial y_2}\right)v = f \quad \text{in } \mathbb{R}^2_+,$$

where f *is a Mellin distribution such that* $\mathcal{M}f \in \mathcal{O}(\mathbb{C}^2)$ *and for some* $t = (t_1,t_2)$ *and* $s > 0$,

$$|\mathcal{M}f(\zeta)| \le C\langle\zeta\rangle^s t^{-\operatorname{Re}\zeta} \quad \text{for } \operatorname{Re}\zeta \ge \mathring{a}. \tag{17.12}$$

Then for every $\sigma \in C_0^\infty((0,t_2))$, $v[\cdot\sigma(y_2)] \in \mathfrak{M}'_{\mathring{a}_1}(\mathbb{R}_+)$ (*here* $v \in \mathfrak{M}'_{\mathring{a}_1}(\mathbb{R}_+;\ \mathfrak{M}'_{\mathring{a}_2}(\mathbb{R}_+))$) *is a generalized function in variable* y_1 *of convergence radius not less than* t_1.

Proof. Let $\chi_{\tilde{r},r}$ be the cut-off function as in Section 7 with $r < t_1$ and let $\sigma \in C_0^\infty((0,t_2))$ have support in $[\widetilde{\delta},\delta]$ with some $0 < \widetilde{\delta} < \delta < t_2$. Define

$$\begin{aligned} G(\zeta_1) &= \mathcal{M}\chi_{\tilde{r},r}(\zeta_1) && \text{for } \zeta_1 \in \mathbb{C}\setminus\{0\},\\ K'(\zeta_2) &= \mathcal{M}\sigma(\zeta_2) && \text{for } \zeta_2 \in \mathbb{C},\\ K(\zeta_1,\zeta_2) &= G(\zeta_1)K'(\zeta_2) && \text{for } \zeta_1 \in \mathbb{C}\setminus\{0\}, \quad \zeta_2 \in \mathbb{C},\\ F(\zeta) &= \mathcal{M}f(\zeta) && \text{for } \zeta \in \mathbb{C}^2. \end{aligned} \tag{17.13}$$

Let $(1/\mathcal{P})_{\mathring{a}}$ be a distribution in $S'(\mathbb{R}^2)$ extending the function $b \mapsto 1/\mathcal{P}(\mathring{a} + ib)$ for b such that $\mathcal{P}(\mathring{a} + ib) \ne 0$, and such that

$$\mathcal{M}_{\mathring{a}}v = F(\mathring{a} + i\,\cdot)(1/P)_{\mathring{a}}. \tag{17.14}$$

We have

$$\mathcal{M}(\chi_{\tilde{r},r}(y_1)\sigma(y_2)v)(\zeta) = \widetilde{\mathcal{C}}^-(\zeta), \tag{17.15}$$

where

$$\widetilde{\mathcal{C}}^\pm(\zeta) = \frac{1}{(2\pi)^2}\left(\frac{1}{\mathcal{P}}\right)_{\mathring{a}}[F(\mathring{a} + i\gamma)K(\zeta - \mathring{a} - i\gamma)] \quad \text{for } \pm\operatorname{Re}\zeta_1 > \pm\mathring{a}_1,\ \zeta_2 \in \mathbb{C}. \tag{17.16}$$

Further, it follows from Corollary 17.1 that for fixed $\zeta_2 \in \mathbb{C}$ the holomorphic extension of $\widetilde{\mathcal{C}}^-(\cdot,\zeta_2)$ to (a subset of) $\mathbb{C}$ coincides for $\operatorname{Re}\zeta_1 > \mathring{a}_1$ with the function $\widetilde{\mathcal{C}}^+(\zeta) + \widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2)$, where $\widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2)$ is the holomorphic extension of the function

$$\widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) = \frac{1}{2\pi i}\int\limits_{\operatorname{Re}\theta = \mathring{a}_2} \frac{K'(\zeta_2 - \theta)F(\zeta_1,\theta)}{\mathcal{P}(\zeta_1,\theta)}\,d\theta$$

defined for $\zeta_1 = \mathring{a}_1 + ib_1$ with $b_1 \neq B_\mu$ for $\mu = 1, \ldots, M$. Since the function $\mathbb{C} \ni \theta \mapsto K'(\zeta_2 - \theta)F(\zeta_1, \theta)$ is rapidly decreasing along the imaginary axis locally uniformly in ζ_1 and ζ_2, it follows that the integral over the line $\operatorname{Re}\theta = \mathring{a}_2$ may be replaced by an integral over $\operatorname{Re}\theta = r$ (for large $r > 0$) if we add suitable residue terms. To this end define, for $\zeta_1 \in \mathbb{C}$,

$$I_r^+(\zeta_1) = \{j : r > \operatorname{Re} c_j(\mathring{a}_1 + ib_1) > \mathring{a}_2 \text{ for } b_1 > \operatorname{Im}\zeta_1, \text{ close to} \operatorname{Im}\zeta_1\}.$$

In view of (17.10) we have

$$\widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) = \frac{-1}{a_m} \sum_{j \in I_r^+(\zeta_1)} \frac{K'(\zeta_2 - c_j(\zeta_1))F(\zeta_1, c_j(\zeta_1))}{\prod_{q=1,\, q\neq j}^m (c_j(\zeta_1) - c_q(\zeta_1))} + \frac{1}{2\pi i a_m} \int_{\operatorname{Re}\theta = r} \frac{K'(\zeta_2 - \theta)F(\zeta_1, \theta)}{\prod_{j=1}^m (\theta - c_j(\zeta_1))}\, d\theta. \tag{17.17}$$

We shall prove that the integral

$$I(r) = \int_{\operatorname{Re}\theta = r} \frac{K'(\zeta_2 - \theta)F(\zeta_1, \theta)}{\mathcal{P}(\zeta_1, \theta)}\, d\theta$$

tends to zero for any fixed $\zeta_1 \in \mathbb{C}$ and $\zeta_2 \in \mathbb{C}$. Indeed, it follows from (17.12) and (6.7) in Theorem 6.4 with $j = |s| + 2$ that for large r,

$$\begin{aligned} I(r) &\le C(\zeta_1, \zeta_2)\left(\frac{\delta}{t_2}\right)^r \int_{\operatorname{Re}\theta = r} \frac{\langle(\zeta_1, \theta)\rangle^s}{\langle \zeta_2 - \theta\rangle^{|s|+2} |P(\zeta_1, \theta)|}\, d\theta \\ &\le C'(\zeta_1, \zeta_2)\left(\frac{\delta}{t_2}\right)^r \int_{\operatorname{Re}\theta = r} \frac{d\theta}{\langle\theta\rangle^2} \end{aligned}$$

and since $\delta < t_2$, $I(r) \to \infty$ as $r \to \infty$, as claimed. Consequently, it follows from (17.17) that for any fixed ζ_1 with Re ζ_1 close to $\mathring{a}_1$ (so that $\operatorname{Re} c_j(\zeta_1) > \mathring{a}_2$ for $j \in I^+(\zeta_1) := I_\infty^+(\zeta_1)$) and $\zeta_2 \in \mathbb{C}$ we have

$$\widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) = \frac{-1}{a_m} \sum_{j \in I^+(\zeta_1)} \frac{K'(\zeta_2 - c_j(\zeta_1))F(\zeta_1, c_j(\zeta_1))}{\prod_{q=1,\, q\neq j}^m (c_j(\zeta_1) - c_q(\zeta_1))}. \tag{17.18}$$

Computing the inverse Mellin transform $(\mathcal{M}^2)^{-1}$ with respect to ζ_2 we find

$$(\mathcal{M}^2)^{-1}\widetilde{\mathcal{C}}'_{\zeta_1}(y_1) = \sigma(y_2)A(\zeta_1, y_2), \tag{17.19}$$

where

$$A(\zeta_1, y_2) = \sum_{j \in I^+(\zeta_1)} \frac{y_2^{c_j(\zeta_1)} F(\zeta_1, c_j(\zeta_1))}{\prod_{q=1,\, q\neq j}^m (c_j(\zeta_1) - c_q(\zeta_1))}. \tag{17.20}$$

Note that in formula (17.20) we may drop the restriction that $\operatorname{Re}\zeta_1$ is close to $\mathring{a}_1$. We shall study the behaviour of $A(\zeta_1, y_2)$ as $\operatorname{Re}\zeta_1 \to \infty$ for a fixed $0 < y_2 < t_2$. To this end observe that all denominators in (17.20) are separated from zero for $|\zeta|$ large (they can only vanish at a finite number of the points θ_ν). Consequently, from (17.12) we get for

$\operatorname{Re}\zeta_1 \ge \mathring{a}_1$, $|\zeta_1|$ large,

$$|A(\zeta_1, y_2)| \le C \sum_{j\in I^+(\zeta_1)} \langle(\zeta_1, c_j(\zeta_1))\rangle^s \left(\frac{t_2}{y_2}\right)^{-\operatorname{Re} c_j(\zeta_1)} t_1^{-\operatorname{Re}\zeta_1} \le C \sum_{j\in I^+(\zeta_1)} \langle(\zeta_1, c_j(\zeta_1))\rangle^s \left(\frac{t_2}{y_2}\right)^{\theta|\operatorname{Im}\zeta_1|+\delta} t_1^{-\operatorname{Re}\zeta_1} \tag{17.21}$$

in view of condition (A_1'). Being an algebraic function each c_j is of polynomial growth:

$$|c_j(\zeta_1)| \le C^*|\zeta_1|^{s^*} \quad \text{for } |\zeta_1| \to \infty,\ j = 1, \dots, m,$$

for some constants $C^*, s^* > 0$. Thus for every $\varepsilon > 0$ and $\operatorname{Re}\zeta_1 \ge \mathring{a}_1$,

$$\sum_{j\in I^+(\zeta_1)} \langle(\zeta_1, c_j(\zeta_1))\rangle^s \le C(\varepsilon) e^{\varepsilon(\operatorname{Re}\zeta_1 + |\operatorname{Im}\zeta_1|)}.$$

Hence we get from (17.21),

$$|A(\zeta_1, y_2)| \le \widetilde{C}(t_2/y_2)^\delta e^{\theta(\ln(t_2/y_2)+\varepsilon)|\operatorname{Im}\zeta_1|}(t_1 e^{-\varepsilon})^{-\operatorname{Re}\zeta_1} \quad \text{for } |\zeta_1| \text{ large and } \operatorname{Re}\zeta_1 \ge \mathring{a}_1.$$

Next, we consider the behaviour of $\mathcal{C}^+(\zeta)$ as $\operatorname{Re}\zeta_1 \to \infty$ for a fixed $\zeta_2 \in \mathbb{C}$. Since $(1/\mathcal{P})_{\mathring{a}}$ is a tempered distribution of order $\le m$ we see in view of (17.12) ([1]) and Proposition 7.1 that

$$|\widetilde{\mathcal{C}}^+(\zeta_1, \zeta_2)| \le C \sup_{\gamma\in\mathbb{R}^2} \langle\gamma\rangle^m \Big| \sum_{|\alpha|\le m} \frac{\partial^\alpha}{\partial\gamma^\alpha} K(\zeta - \mathring{a} - i\gamma)\Big| \le \widetilde{C}\widetilde{r}^{-\operatorname{Re}\zeta_1} \sup_{\gamma_2\in\mathbb{R}} \langle\gamma_2\rangle^m \Big| \sum_{l=0}^m \frac{\partial^l}{\partial\gamma_2^l} \mathcal{M}\sigma(\zeta_2 - \mathring{a}_2 - i\gamma_2)\Big| \le C'\widetilde{r}^{-\operatorname{Re}\zeta_1} \quad \text{for } \operatorname{Re}\zeta_1 > \mathring{a}_1. \tag{17.22}$$

Now observe that by (17.15) the Mellin transform $\mathcal{M}^1$ (with respect to y_1) of the Mellin distribution $\chi_{\tilde{r},r}(y_1)v$ evaluated at the function σ of y_2 equals

$$\mathcal{M}^1(\chi_{\tilde{r},r}(y_1)v[\cdot\,\sigma(y_2)])(\zeta_1) = \widetilde{\mathcal{C}}^-(\zeta_1, -1) \quad \text{for } \operatorname{Re}\zeta_1 < \mathring{a}_1.$$

Consequently, from (17.19), (17.20) we get the following expression for the holomorphic extension Ψ of $\widetilde{\mathcal{C}}(\zeta_1, -1)$:

$$\Psi(\zeta_1) = \begin{cases} \widetilde{\mathcal{C}}^-(\zeta_1, -1) & \text{for } \operatorname{Re}\zeta_1 < \mathring{a}_1, \\ \widetilde{\mathcal{C}}^+(\zeta_1, -1) + \int_{\tilde{\delta}}^{t_2} \sigma(y_2) A(\zeta_1, y_2)\, dy_2 & \text{for } \operatorname{Re}\zeta_1 > \mathring{a}_1. \end{cases}$$

Combining (17.21) and (17.22) we observe that the estimates (ii) and (iii) of Theorem 11.1 are satisfied for $|\zeta_1|$ large. We now prove that (iii) holds for ζ in a bounded set. The only doubtful points are those on the line $\mathring{a}_1 + i\mathbb{R}$. Since by definition $K(\zeta) = G(\zeta_1)K'(\zeta_2)$ we have

$$\widetilde{\mathcal{C}}^\pm = \frac{1}{2\pi}\widetilde{\mathcal{C}}'_{\mathring{a}_1}(\zeta_2)[G(\zeta_1 - \mathring{a}_1 - i\,\cdot)] \quad \text{for } \operatorname{Re}\zeta_1 \ne \mathring{a}_1,$$

([1]) Since F is holomorphic the estimate (17.12) also holds for the derivatives of F.

where for $\zeta_2 \in \mathbb{C}$, $\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2) \in S'(\mathbb{R})$ is a suitable regularization (see Corollary 17.1) of the function

$$b_1 \mapsto \widetilde{\mathcal{C}}'_{\overset{*}{a}_1+ib_1}(\zeta_2) \quad \text{for } b_1 \neq B_\mu \ (\mu = 1, \ldots, M). \tag{17.23}$$

Let κ be a $C_0^\infty(\mathbb{R})$ function which is 1 in a neighbourhood of the points B_μ ($\mu = 1, \ldots, M$) and $\operatorname{Im}\theta_\nu$ ($\nu = 1, \ldots, N$). Write, for $\operatorname{Re}\zeta_1 \neq \overset{*}{a}_1$,

$$\Psi_1(\zeta) = \frac{1}{2\pi}\kappa\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2)[G(\zeta_1 - \overset{*}{a}_1 - i\,\cdot)],$$
$$\Psi_2(\zeta) = \frac{1}{2\pi}(1-\kappa)\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2)[G(\zeta_1 - \overset{*}{a}_1 - i\,\cdot)].$$

It follows that $\Psi_2(\cdot, \zeta_2)$ is holomorphic in a complex neighbourhood of the points $\overset{*}{a}_1 + iB_\mu$ ($\mu = 1, \ldots, M$) and $\overset{*}{a}_1 + i\operatorname{Im}\theta_\nu$ ($\nu = 1, \ldots, N$), so we are interested in Ψ_1. By Proposition 7.1,

$$G(\zeta_1) = \frac{-1}{\zeta_1} + \overline{G}(\zeta_1)$$

with $\overline{G} \in \mathcal{O}(\mathbb{C})$. Inserting this in the definition of Ψ_1 we find that modulo a holomorphic factor we are led to consider the function

$$\Psi_3(\zeta) = \frac{1}{2\pi}\kappa\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2)\Big[\frac{-1}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}\Big] \quad \text{for } \operatorname{Re}\zeta_1 \neq \overset{*}{a}_1.$$

Summing up we conclude (by Corollary 17.1) that in a neighbourhood of the points $\overset{*}{a}_1 + iB_\mu$, $\overset{*}{a}_1 + i\operatorname{Im}\theta_\nu$ the holomorphic continuation of the function $\widetilde{\mathcal{C}}^-$ has the same jumps as the function

$$\Phi(\zeta_1, \zeta_2) = \begin{cases} \dfrac{-1}{2\pi}\kappa\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}\Big[\dfrac{1}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}\Big] & \text{for } \operatorname{Re}\zeta_1 < \overset{*}{a}_1,\ \zeta_2 \in \mathbb{C}, \\ \dfrac{-1}{2\pi}\kappa\widetilde{\mathcal{C}}'_{\overset{*}{a}_1}\Big[\dfrac{1}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}\Big] + \widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) & \text{for } \operatorname{Re}\zeta_1 > \overset{*}{a}_1,\ \zeta_2 \in \mathbb{C}. \end{cases} \tag{17.24}$$

Now, in view of (17.18) the assertion follows from Corollary 17.2.

Before establishing explicit formulas for the Borel transform of v in y_1 we prove the following.

Lemma 17.1. *Let F be holomorphic in a neighbourhood of the quarter disc $\{0 < |\zeta| \leq \varepsilon,\ 0 \leq \operatorname{Re}\zeta \leq \varepsilon,\ 0 \leq \operatorname{Im}\zeta \leq \varepsilon\}$ for some $\varepsilon > 0$ and integrable along the radii at zero. Denote by χ_ε the characteristic function of the interval $(0, \varepsilon) \subset \mathbb{R}$. Define*

$$\Psi(\zeta) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_0^\varepsilon \frac{F(i\gamma)}{\zeta - i\gamma}\,d\gamma & \text{for } \operatorname{Re}\zeta < 0, \\ \dfrac{1}{2\pi i}\displaystyle\int_0^\varepsilon \frac{F(i\gamma)}{\zeta - i\gamma}\,d\gamma + F(\zeta) & \text{for } \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta > 0, \\ \dfrac{1}{2\pi i}\displaystyle\int_0^\varepsilon \frac{F(i\gamma)}{\zeta - i\gamma}\,d\gamma & \text{for } \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta < 0. \end{cases}$$

Then $\Psi \in \mathcal{O}(\Delta_\varepsilon)$, where $\Delta_\varepsilon = \{|\zeta| < \varepsilon\} \setminus [0, \varepsilon)$, and for $|\alpha| < \varepsilon$,

$$\lim_{b \to 0_+}(\Psi(\alpha + ib) - \Psi(\alpha - ib)) = \begin{cases} 0 & \text{for } \alpha < 0, \\ F(\alpha) & \text{for } \alpha > 0, \end{cases}$$

where the limit is taken in the sense of distributions.

P r o o f. The fact that Ψ is holomorphic on Δ_ε is clear from the properties of the Cauchy transformation (cf. §7). Let

$$\mathcal{C}(\zeta) = \frac{1}{2\pi}\int_0^\varepsilon \frac{F(\alpha)}{\zeta-\alpha}\,d\alpha.$$

Then $\mathcal{C}(\zeta) \in \mathcal{O}(\Delta_\varepsilon)$ and it is a defining function for $\chi_\varepsilon(\alpha)F(\alpha)$. Observe that for $\operatorname{Re}\zeta < 0$,

$$\Psi(\zeta) = \mathcal{C}(\zeta) + \frac{1}{2\pi}\int_{S_\varepsilon^+} \frac{F(w)}{\zeta-w}\,dw, \tag{17.25}$$

where S_ε^+ is the arc on the sphere $\{|w| = \varepsilon\}$ joining $i\varepsilon$ to ε. By analytic continuity (17.25) holds for $\zeta \in \Delta_\varepsilon$. Finally, note that the function $\frac{1}{2\pi}\int_{S_\varepsilon^+}\frac{F(w)}{\zeta-w}\,dw$ is holomorphic in $\{|\zeta| < \varepsilon\}$ and the assertion of the lemma follows in view of the properties of $\mathcal{C}(\zeta)$.

Now, we are in a position to prove the following important corollary to Theorem 17.2:

THEOREM 17.3. *Retain the assumptions and notation of Theorem* 17.2 *and suppose that* $\Delta(\zeta_1) \neq 0$ *for* $\zeta_1 \in L_\mu$, $\mu = 1,\ldots,M$ (*then* $M = m$) *and* $\Delta(\zeta_1) \neq 0$ *for* $\zeta_1 \in \mathring{a}_1 + i\mathbb{R}$. *Then there exists a distinguished solution* $\mathring{v} \in \mathfrak{M}'_{\mathring{a}}(\mathbb{R}^2_+)$ *of the equation* $\mathcal{P}(y\partial/\partial y)\mathring{v} = f$ *of the form*

$$\mathring{v}(y_1,y_2) = \sum_{\mu=1}^{M} T^\mu_{y_2}[y_1^{\zeta_1}] + \sum_{\mu=1}^{\widetilde{N}} \widetilde{T}^\nu_{y_2}[y_1^{\zeta_1}] \quad \textit{for } 0 < y_1 < t_1,\ 0 < y_2 < t_2,$$

where for $\mu = 1,\ldots,M$,

$$T^\mu_{y_2}(\zeta_1) = \chi^{\mathring{a}_1}_{L_\mu}\cdot\frac{(-1)}{a_m}\sum_{j\in I^0(B_\mu)}\operatorname{sgn}(j;\mu)\frac{y_2^{c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1)}F(\zeta_1, c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1))}{\prod_{q=1,\,q\neq j}^m (c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1) - c_q^{\operatorname{sgn}(j;\mu)}(\zeta_1))},$$

and for $\nu = 1,\ldots,\widetilde{N}$,

$$\widetilde{T}^\nu_{y_2}(\zeta_1) = \chi^{\mathring{a}_1}_{\tilde{L}_\nu}\frac{(-1)}{a_m}\sum_{j\in I^+(\theta_\nu)}\left(\frac{y_2^{c_j^+(\zeta_1)}F(\zeta_1,c_j^+(\zeta_1))}{\prod_{q=1,\,q\neq j}^m(c_j^+(\zeta_1)-c_q^+(\zeta_1))} - \frac{y_2^{c_2^-(\zeta_1)}F(\zeta_1,c_j^-(\zeta_1))}{\prod_{q=1,\,q\neq j}^m(c_j^-(\zeta_1)-c_q^-(\zeta_1))}\right),$$

where

$$\{\theta_{N+1},\ldots,\theta_{\widetilde{N}}\} = (Z\setminus\{\theta_1,\ldots,\theta_N\}) \cap \{\operatorname{Re} z_1 \geq \mathring{a}_1\}$$

and for $\nu = N+1,\ldots,\widetilde{N}$,

$$I^+(\theta_\nu) = \{j : \operatorname{Re} c_j(\mathring{a}_1 + ib_1) > \mathring{a}_2 \textit{ for } b_1 > \operatorname{Im}\theta_\nu \textit{ close to } \operatorname{Im}\theta_\nu\}.$$

Finally, $\chi^{\mathring{a}_1}_{L_\mu}$ *is the characteristic function of the half-line* $L_\mu \cap \{\operatorname{Re}\zeta_1 \geq \mathring{a}_1\}$; *similarly* $\chi^{\mathring{a}_1}_{\tilde{L}_\nu}$ *is the characteristic function of the half-line* $\widetilde{L}_\nu$. *Any other solution* $v \in \mathfrak{M}'_{\mathring{a}}(\mathbb{R}^2_+)$ *of the above equation differs from* $\mathring{v}$ *by a function*

$$\sum_{\mu=1}^m D_\mu y_1^{\mathring{a}_1+iB_\mu} y_2^{c_\mu(\mathring{a}_1+iB_\mu)}, \tag{17.26}$$

where D_μ ($\mu = 1,\ldots,m$) *are arbitrary complex constants and* c_μ *is such that* $\operatorname{Re} c_\mu(\mathring{a}_1 + iB_\mu) = \mathring{a}_2$.

Proof. Retain the notation from the proof of Theorem 17.2. Since $\Delta(\zeta_1) \neq 0$ for $\zeta_1 \in \mathring{a}_1 + i\mathbb{R}$ it follows that $\mathcal{P}$ has simple roots only on $\mathring{a} + i\mathbb{R}$ (so $M = m$) and hence $\mathcal{M}_{\mathring{a}} v$ is a distribution of order at most one. Thus $\widetilde{\mathcal{C}}'_{\mathring{a}_1}(\zeta_2)$ is a distribution of order at most 1. On the other hand, the function given by (17.23) is locally integrable and polynomially bounded at infinity. Thus it defines a regular distribution which differs from (17.23) by a linear combination of Dirac deltas at the points $\mathring{a}_1 + iB_\mu$, $\mu = 1, \ldots, m$. Since any such combination leads to a solution of the homogeneous equation, we are reduced, by (17.24), to the study of the jumps of the function

$$\widetilde{\Phi}(\zeta_1, \zeta_2) = \begin{cases} \dfrac{-1}{2\pi} \displaystyle\int_{\mathbb{R}} \frac{\kappa(\gamma_1) \widetilde{\mathcal{C}}'_{\mathring{a}_1 + i\gamma_1}(\zeta_1)}{\zeta_1 - \mathring{a}_1 - i\gamma_1} \, d\gamma_1 & \text{for } \operatorname{Re} \zeta_1 < \mathring{a}_1, \\ \dfrac{-1}{2\pi} \displaystyle\int_{\mathbb{R}} \frac{\kappa(\gamma_1) \widetilde{\mathcal{C}}'_{\mathring{a}_1 + i\gamma_1}(\zeta_1)}{\zeta_1 - \mathring{a}_1 - i\gamma_1} \, d\gamma_1 + \widetilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) & \text{for } \operatorname{Re} \zeta_1 > \mathring{a}_1. \end{cases}$$

The jumps are now easily computed from (17.18) in view of Lemma 17.1.

Remark 17.1. Clearly, even if $\Delta(\zeta_1)$ vanishes at some points in L_μ, one gets similar explicit formulas for T^μ, $\mu = 1, \ldots, M$, and $\widetilde{T}^\nu$, $\nu = 1, \ldots, \widetilde{N}$, as in Theorem 17.3 with the characteristic functions $\chi^{\mathring{a}_1}_{L_\mu}$ and $\chi^{\mathring{a}_1}_{L_\nu}$ replaced by suitable homogeneous distributions.

EXAMPLE 17.2. Let $0 \neq \mathring{\alpha} \in \mathbb{R}^2$ and denote by $u_{\mathring{\alpha}} \in \mathfrak{M}'_{\mathring{\alpha}}(\mathbb{R}^2)$ the solution of the equation $Pu_{\mathring{\alpha}} = \delta_{(1)}$ on $\mathbb{R}^2_+$, where $P(x\partial/\partial x) = (x_1 \partial/\partial x_1)^2 + (x_2 \partial/\partial x_2)^2$, such that

$$\mathcal{M}_{\mathring{\alpha}} u_{\mathring{\alpha}}(\beta) = \frac{1}{(\mathring{\alpha}_1 + i\beta_1)^2 + (\mathring{\alpha}_2 + i\beta_2)^2} \in L^1(\mathbb{R}^2).$$

In the radial variables $x_1 = y_1,\ x_2 = y_1 y_2,\ u(y_1, y_1 y_2)$ satisfies the equation $\mathcal{P}u = \delta_{(1)}$, where

$$\mathcal{P}\left(y \frac{\partial}{\partial y}\right) = \left(y_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{y_2}\right)^2 + \left(y_2 \frac{\partial}{\partial y_2}\right)^2.$$

We have

$$P(\zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)^2 + \zeta_2^2 = 2(\zeta_2 - c_1(\zeta_1))(\zeta_2 - c_2(\zeta_1)),$$

where $c_1(\zeta_1) = \frac{1+i}{2}\zeta_1$, $c_2(\zeta_1) = \frac{1-i}{2}\zeta_1$ and $\Delta = i\zeta_1$. Further, we find

$$B_1 = \alpha_1 - 2\alpha_2, \quad B_2 = 2\alpha_2 - \alpha_1$$

and since c_1 and c_2 are regular there are no points θ_ν.

Define

$$Z = \{0\},$$
$$L_1 = \{\zeta_1 \in \mathbb{C} : \zeta_1 = a_1 + i(\alpha_1 - 2\alpha_2)\},$$
$$L_2 = \{\zeta_1 \in \mathbb{C} : \zeta_1 = a_1 + i(2\alpha_2 - \alpha_1)\},$$
$$\mathbf{L} = L_1 \cup L_2 \cup Z, \quad \mathbf{L}_{\mathring{a}_1} = \mathbf{L} \cap \{\operatorname{Re} \zeta_1 \geq \mathring{a}_1\}.$$

Clearly, the polynomial $\mathcal{P}(\zeta_1, \zeta_2)$ satisfies condition (A_1).

Thus it follows from Theorem 17.3 that

$$u_{\mathring{\alpha}}(y_1, y_1 y_2) = -\frac{1}{4\pi}\left(\int_{L_1} \frac{y_2^{(1+i)\zeta_1/2} y_1^{\zeta_1}}{\zeta_1} \, d\zeta_1 + \int_{L_2} \frac{y_2^{(1-i)\zeta_1/2} y_1^{\zeta_1}}{\zeta_1} \, d\zeta_1 \right).$$

Returning to the original variables $x_1 = y_1$, $x_2 = y_1y_2$ we see that the function $u_{\overset{*}{\alpha}}(x_1, x_2)$ has the form

$$(17.27) \qquad -\frac{1}{4\pi}\bigg(\int\limits_{L_1} \frac{x_1^{(1-i)\zeta_1/2} x_2^{(1+i)\zeta_1/2}}{\zeta_1}\, d\zeta_1 + \int\limits_{L_2} \frac{x_1^{(1+i)\zeta_1/2} x_2^{(1-i)\zeta_1/2}}{\zeta_1}\, d\zeta_1\bigg),$$

which makes sense for $0 < x_1x_2 < 1$. Hence we see that $u_{\overset{*}{\alpha}}(x_1, x_2)$ for $0 < x_1 < 1$ and $0 < x_2 < 1$ is a generalized analytic function in both variables (see Section 18). Since $\big(\frac{1-i}{2}\zeta_1\big)^2 + \big(\frac{1+i}{2}\zeta_1\big)^2 = 0$ we observe that in (17.27) we integrate over two half-lines

$$\widetilde{L}_1 = \bigg\{\bigg(\frac{1-i}{2}\zeta_1, \frac{1+i}{2}\zeta_1\bigg) : \zeta_1 \in L_1\bigg\}, \qquad \widetilde{L}_2 = \bigg\{\bigg(\frac{1+i}{2}\zeta_1, \frac{1-i}{2}\zeta_1\bigg) : \zeta_1 \in L_2\bigg\}$$

issuing from the (two) intersection points of the complex characteristic set $\{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 + z_2^2 = 0\}$ with the imaginary plane $\overset{*}{\alpha} + i\mathbb{R}^2$ (for a fixed $\overset{*}{\alpha} \in \mathbb{R}^2$), and contained in the characteristic set. The integration over $\widetilde{L}_1, \widetilde{L}_2$ is performed with respect to the measure induced from that on the characteristic set. Further, observe that it follows from Theorem 17.3 that the density $1/\zeta_1$ is the inverse of the difference over the point ζ_1 of the two sheets of the characteristic set $\{z_1^2 + z_2^2 = 0\}$ (in our case $c_1(\zeta_1) - c_2(\zeta_1) = -2i\zeta_1$). This geometric description of formula (17.27) carries over to all (second order) operators considered in this section.

Example 17.3. Consider the function

$$u(x) = \frac{-1}{(n-2)|S^{n-1}|}((\ln x_1)^2 + \ldots + (\ln x_n)^2)^{(2-n)/2}, \qquad x \in \mathbb{R}_+^n,$$

where $|S^{n-1}|$ is the measure of the unit sphere S^{n-1} in $\mathbb{R}^n$ and $n > 2$, which satisfies the equation

$$(17.28) \qquad \bigg(\bigg(x_1\frac{\partial}{\partial x_1}\bigg)^2 + \ldots + \bigg(x_n\frac{\partial}{\partial x_n}\bigg)^2\bigg)u = \delta_{(1)} \quad \text{in } \mathbb{R}_+^n.$$

Clearly, u is the well known fundamental solution of the Laplace operator in $\mathbb{R}^n$ in logarithmic variables. We shall prove that as an iterated integral

$$(17.29) \qquad u(x) = C \int\limits_{\mathbb{R}_+} \bigg(\int\limits_{S^{n-2}} x_1^{(1-i)r/2} x'^{(1+i)\omega' r/2} d\omega'\, r^{n-3}\, dr\bigg),$$

where C is a suitable constant, $x' = (x_2, \ldots, x_n)$, $\omega' = (\omega_2, \ldots, \omega_n) \in \mathbb{R}^{n-1}$ and $S^{n-2} = \{\omega' \in \mathbb{R}^{n-1} : \|\omega'\| = 1\}$.

Proof. Computing the Mellin transform of both sides of (17.28) we find

$$\mathcal{M}_0 u(\beta) = -1/\|\beta\|^2$$

as a locally integrable function on $\mathbb{R}^n$. Take a cut-off function κ of the form

$$\kappa(x) = \chi_r(x_1)\sigma'(x_2, \ldots, x_n),$$

where $\sigma' \in C_0^\infty((0,1)^{n-1})$ and χ_r is the characteristic function of $(0, r]$, $r < 1$. Then

$$\mathcal{M}(\kappa u)(z) = \widetilde{\mathcal{C}}^-(z) = \frac{1}{(2\pi)^n}\bigg(\frac{1}{\|\gamma\|^2}\bigg)[K(z - i\gamma)],$$

where

$$K(z) = \mathcal{M}\kappa(z) = G(z_1)K'(z'), \quad G(z_1) = \mathcal{M}^1\chi_r(z_1), \quad K'(z') = \mathcal{M}'\sigma'(z').$$

According to the general theory presented for $n = 2$ in the proof of Theorem 17.2, we are interested in the set of holomorphy in z_1 of the function

$$\widetilde{\mathcal{C}}'_{z_1}(z') = \frac{1}{(2\pi)^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \frac{K'(z' - i\theta')}{z_1^2 + \|i\theta'\|^2}\, d\theta'.$$

Introducing radial coordinates in $\mathbb{R}^n$: $\theta' = \varrho\omega'$ with $\omega' \in S^{n-2}$, we get

$$\widetilde{\mathcal{C}}'_{z_1}(z') = \frac{1}{(2\pi)^{n-1}} \int\limits_0^\infty \frac{\varrho^{n-2}}{z_1^2 + (i\varrho)^2} \widetilde{K}(z', i\varrho)\, d\varrho = \frac{(-i)^{n-2}}{2(2\pi)^{n-1}} \int\limits_{-\infty}^\infty \frac{(i\varrho)^{n-2}}{z_1^2 + (i\varrho)^2} \widetilde{K}(z', i\varrho)\, d\varrho,$$

where for $\tau \in \mathbb{C}$,

$$\widetilde{K}(z', \tau) = \int\limits_{S^{n-2}} K'(z' - \tau\omega')\, d\omega'$$

is an even (i.e. $\widetilde{K}(z', \tau) = \widetilde{K}(z', -\tau)$) holomorphic function on $\mathbb{C}^n$. Since $n > 2$ it follows that for fixed z' the function $z_1 \mapsto \widetilde{\mathcal{C}}'_{z_1}(z')$ satisfies the assumptions of Lemma 17.1. Thus it follows from Theorem 17.2 and Theorem 17.3 for $\mathcal{P} = z_1^2 + z_2^2$ and $\overset{*}{a}_1 = 0$ that the Borel transform of κu in variable x_1 equals

$$\Xi_{z'}(\alpha_1) = CY(\alpha_1)\alpha_1^{n-3}\widetilde{K}(z', i\alpha_1),$$

where C is a suitable numerical constant. Since

$$((\mathcal{M}')^{-1}\widetilde{K})(x', \tau) = \int\limits_{S^{n-2}} ((\mathcal{M}')^{-1}K')(x')x'^{\tau\omega'}\, d\omega' = K'(x') \int\limits_{S^{n-2}} x'^{\tau\omega}\, d\omega'$$

we get the following integral representation for u:

$$u(x) = C \int\limits_{\mathbb{R}_+} \Big(\int\limits_{S^{n-2}} x_1^\tau x_2^{i\tau\omega_2} \ldots x_n^{i\tau\omega_n} \tau^{n-3} d\tau \Big)\, d\omega'.$$

By Theorem 14.2 we can replace integration over $\mathbb{R}_+$ by that over the line $\frac{1-i}{2}\mathbb{R}_+$, giving us finally formula (17.29).

Remark 17.2. Observe that as in dimension 2,

$$\Big(\frac{1-i}{2}\tau\Big)^2 + \Big\|\frac{1+i}{2}\tau\omega'\Big\|^2 = 0,$$

hence

$$\mathbb{R}_+ \times S^{n-2} \ni (\tau, \omega) \mapsto \Big(\frac{1-i}{2}\tau, \frac{1+i}{2}\tau\omega'\Big) \in \mathbb{C}^n$$

describes a subset S_0 of real dimension $n-1$ of the characteristic set $\{z_1^2 + \ldots + z_n^2 = 0\}$.

Since the dimension of the characteristic set equals $2n - 2$ there is much room in the characteristic set to vary the set $S_{\overset{*}{\alpha}}$. This possibility is reflected in the holomorphy properties of the spectral density resembling those of the resurgent functions. Since the resurgence phenomenon observed here originates in the characteristic set we call it the *characteristic resurgence*.

18. GAFs in several variables. Formula (17.27) in Example 17.2 clearly demonstrates that generalized analytic functions of two variables occur naturally as solutions to elliptic Fuchsian PDEs. Actually, we could have developed from the beginning the theory of GAFs in several variables. For simplicity we have chosen to start with the case of a single variable. In fact, the theory of GAFs in several variables mimics, with no essential changes, that in one variable. This becomes specially evident if we adopt suitable vector notation.

Thus if $a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ then we write $a < b$ (resp. $a \le b$) if $a_j < b_j$ (resp. $a_j \le b_j$) for $j = 1, \ldots, n$. If $x \in \mathbb{R}^n$, $0 < x$ and $z \in \mathbb{C}^n$ we write $x^z = x_1^{z_1} \cdot \ldots \cdot x_n^{z_n}$. Extending this notation to complex variables we shall write $z^{-\mathbf{1}} = \frac{1}{z_1} \ldots \frac{1}{z_n}$ for $z \in \mathbb{C}$ with $z_j \neq 0$ $(j = 1, \ldots, n)$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. More generally, for $w \in (\widetilde{\mathbb{C} \setminus \{0\}})^n := \widetilde{\mathbb{C} \setminus \{0\}} \times \ldots \times \widetilde{\mathbb{C} \setminus \{0\}}$ we shall consider the exponential function

$$\mathbb{C}^n \ni z \mapsto w^z = w_1^{z_1} \cdot \ldots \cdot w_n^{z_n} \in \mathbb{C}.$$

It will also be convenient to define the real and imaginary part of a $z \in \mathbb{C}^n$ as the vectors $\operatorname{Re} z = (\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$, $\operatorname{Im} z = (\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n)$.

Before passing to the definition of a Laplace distribution in n variables we recall the already classical notion of a hyperfunction in several variables (cf. e.g. [Ka]).

Let Ω be an open set in $\mathbb{R}^n$. The space $B(\Omega)$ of *hyperfunctions* on Ω is defined in an intrinsic way as $B(\Omega) := H^n_\Omega(W, \mathcal{O})$, the nth local cohomology space of the sheaf $\mathcal{O}$ of germs of holomorphic functions on W, where W is a complex neighbourhood of Ω, i.e. an open set in $\mathbb{C}^n$ such that $\Omega \subset W$ as a closed subset. We shall need the following explicit description of $B(\Omega)$:

Let A be a nonsingular $n \times n$ real matrix (i.e. $A \in GL(n, \mathbb{R})$). Let W be an open subset of $\mathbb{C}^n$ such that $\Omega = W \cap \mathbb{R}^n$. Then $B(\Omega)$ is canonically isomorphic to the quotient space

$$B(\Omega) = \frac{\mathcal{O}(\{\alpha + i\beta \in W : \forall j, (A\beta)_j \neq 0\})}{\sum_{j=1}^n \mathcal{O}(\{\alpha + i\beta \in W : \forall k \neq j, (A\beta)_k \neq 0\}} \tag{18.1}$$

(note that $B(\Omega)$ is independent of the choice of W and $A \in GL(n, \mathbb{R})$ since it can be defined intrinsically by local sheaf cohomologies).

The set $\{z \in W : \forall j,\ (A\beta)_j \neq 0\}$ splits into $\bigcup_{\sigma_j = \pm 1} W^\sigma_A$ $(\sigma = (\sigma_1, \ldots, \sigma_n))$, where $W^\sigma_A = \Omega + i\Gamma^\sigma_A$ and

$$\Gamma^\sigma_A = \{\beta \in \mathbb{R}^n : \forall j,\ \sigma_j (A\beta)_j > 0\}. \tag{18.2}$$

Let $F \in \mathcal{O}(\bigcup_\sigma W^\sigma_A)$ and denote by $f = [F]_A$ the hyperfunction determined by F (as an element of the quotient space in (18.1)). If $A = \mathrm{Id}$ we write $[F]$ instead of $[F]_{\mathrm{Id}}$. Observe that $F \circ A^{-1} \in \mathcal{O}(\bigcup_\sigma W^\sigma_{\mathrm{Id}})$ and define the composite

$$f \circ A^{-1} := [F \circ A^{-1}] \tag{18.3}$$

as an element of the quotient space in (18.1) with $A = \mathrm{Id}$.

It is possible to compose hyperfunctions with analytic mappings (see [Ka]) but we shall need only the case described above.

Let Γ be an arbitrary convex cone and suppose $H \in \mathcal{O}(\Omega + i\Gamma)$. We define $b_\Gamma(H)$, the boundary value of H, as an element of $B(\Omega)$ as follows: Choose A such that for some

σ, $\Gamma_A^\sigma \subset \Gamma$. Then $b_\Gamma(H)$ is defined as the element of (18.1) whose defining function Ψ is given by

$$\Psi(z) = \begin{cases} \operatorname{sgn}\sigma\, H(z) & \text{for } z \in W_A^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

It can be proved that the definition of $b_\Gamma(H)$ is independent of the choice of A and of the cone Γ in the sense that if $\Gamma' \subset \Gamma$ is another convex cone then $b_{\Gamma'}(H) = b_\Gamma(H)$.

If $F \in \mathcal{O}(\bigcup_\sigma W_A^\sigma)$ then

$$[F] = \sum_\sigma \operatorname{sgn}\sigma\, b_{\Gamma_A^\sigma}(F^\sigma), \tag{18.4}$$

where $F^\sigma = F|_{W_A^\sigma}$. Also

$$b_\Gamma(H) \circ A = b_{A^{-1}(\Gamma)}(H \circ A)$$

for $H \in \mathcal{O}(\Omega + i\Gamma)$.

The representation (18.4) of a hyperfunction as a sum of boundary values is very convenient because under some growth conditions (given below), boundary values are distributions and we have the following n-dimensional variant of Theorem 2.1:

THEOREM 18.1 ([Ko]). *Let $H \in \mathcal{O}(\Omega + i\Gamma)$ be of polynomial growth* (*i.e. for any proper subset $\Omega_0 \subset \Omega$ and proper subcone $\Gamma_0 \subset \Gamma$, and for $z \in \Omega_0 + i\Gamma$ with $|\operatorname{Im} z|$ close to zero, we have $|H(z)| < C/|\operatorname{Im} z|^N$ for some positive constants C and N*). *Then in the sense of distributional convergence*

$$b_\Gamma(H) = \lim_{\beta\to 0,\, \beta\in\Gamma} H(\alpha + i\beta) \in D'(\Omega).$$

Let $K = K_1 \times \ldots \times K_n$ be a compact subset of $\mathbb{R}^n$. We shall denote by $B_K(\mathbb{R}^n)$ the space of hyperfunctions with support contained in K. Hyperfunctions with compact support can also be defined in a similar way to distributions; namely, as analytic functionals, i.e. functionals on the space of analytic functions on K:

Denote by $\mathcal{A}(K)$ the inductive limit $\mathcal{A}(K) = \varinjlim_{U\supset K} \mathcal{O}(U)$, where U ranges over all complex neighbourhoods of K. Then we have the following isomorphism.

THEOREM 18.2 (Martineau, Harvey [Ko]).

$$B_K(\mathbb{R}^n) \simeq \mathcal{A}'(K).$$

The isomorphism is given by the duality

$$\langle [\Psi], \phi\rangle = (-1)^n \int\limits_{\mathcal{C}_1\times\ldots\times\mathcal{C}_n} \Psi(z)\phi(z)dz \quad \textit{for } \phi \in A(K), \tag{18.5}$$

where $[\Psi] \in B_K(\mathbb{R}^n)$ is the hyperfunction having $\Psi \in \mathcal{O}(\prod_{j=1}^n (\mathbb{C} \setminus K_j))$ as a defining function and for $j = 1, \ldots, n$, $\mathcal{C}_j$ is a closed curve (*corresponding to ϕ*) *encircling K_j once anticlockwise and contained in the set where ϕ is holomorphic.*

Let $M \in B_K(\mathbb{R}^n)$. Then by the isomorphism (18.5) the function

$$\Psi^{\mathrm{Id}}(z) = \frac{1}{(2\pi i)^n} M_{(\xi)}[(\xi - z)^{-1}] \in \mathcal{O}\Big(\prod_{j=1}^n (\mathbb{C} \setminus K_j)\Big)$$

is a defining function of M. It is called the *standard defining function.*

EXAMPLE 18.1. $[z^{-1}] = (-2\pi i)^n \delta_{(0)}$, where $[z^{-1}]$ is regarded as an element of the space in (18.1) with $A = \mathrm{Id}$ and $\delta_{(0)}$ is the Dirac delta at $0 \in \mathbb{R}^n$. More generally, if A is a nonsingular real $n \times n$ matrix we have

$$[(Az)^{-1}]_A = \frac{(-2\pi i)^n}{|\det A|}\delta_{(0)}, \tag{18.6}$$

where $[(Az)^{-1}]_A$ is an element of the quotient space in (18.1).

We may now proceed to the definition of a Laplace distribution in the hyperfunction setting. Clearly the half-line $\overline{\mathbb{R}}_+$ should now be replaced by the positive orthant $\overline{\mathbb{R}}^n_+ = \overline{\mathbb{R}}_+ \times \overset{(n)}{\cdots} \times \overline{\mathbb{R}}_+$, and a tubular neighbourhood of $\overline{\mathbb{R}}_+$ by a polytubular neighbourhood W of $\overline{\mathbb{R}}^n_+$ defined as the Cartesian product $W = W_1 \times \ldots \times W_n$, where W_j $(j = 1, \ldots, n)$ is an open tubular neighbourhood of $\overline{\mathbb{R}}_+$.

Let $\omega \in \mathbb{R}^n$ and let V be an open polytubular subset of $\mathbb{C}^n$. The space $\widetilde{\mathcal{O}}_{(\omega)}(V)$ of holomorphic functions on V of *exponential growth* (*of type* ω) is defined as

$$\widetilde{\mathcal{O}}_{(\omega)}(V) = \{H \in \mathcal{O}(V) : \sup_{\zeta \in K} |\mathbf{e}^{(\omega-\delta)\zeta} H(\zeta)| < \infty \text{ for every } 0 < \delta \in \mathbb{R}^n \text{ and every proper polytubular subset } K \text{ of } V\}. \tag{18.7}$$

Note that $\mathbf{e} = (e, \ldots, e) \in \mathbb{R}^n$ and due to the vector notation (introduced above) the definition of $\widetilde{\mathcal{O}}_{(\omega)}$ looks practically the same as in dimension 1.

Now let $A \in GL(n, \mathbb{R})$, and let W be a polytubular neighbourhood of $\overline{\mathbb{R}}^n_+$. We define

$$\begin{aligned} W \#^A \overline{\mathbb{R}}^n_+ &= \{z \in W : (A(z - x))_j \neq 0 \text{ for } x \in \overline{\mathbb{R}}^n_+ \text{ and } j = 1, \ldots, n\}, \\ W \#^A_j \overline{\mathbb{R}}^n_+ &= \{z \in W : (A(z - x))_p \neq 0 \text{ for } \overline{\mathbb{R}}^n_+ \text{ and } p \neq j\} \quad (j = 1, \ldots, n). \end{aligned} \tag{18.8}$$

Observe that for $A = \mathrm{Id}$ we have (omitting the superscript Id)

$$\begin{aligned} W \# \overline{\mathbb{R}}^n_+ &= (W_1 \setminus \overline{\mathbb{R}}_+) \times \ldots \times (W_n \setminus \overline{\mathbb{R}}_+), \\ W \#_j \overline{\mathbb{R}}^n_+ &= (W_1 \setminus \overline{\mathbb{R}}_+) \times \ldots \times W_j \times \ldots \times (W_n \setminus \overline{\mathbb{R}}_+). \end{aligned}$$

In analogy with (18.7) we define for fixed $a \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$,

$$\widetilde{\mathcal{O}}^k_a(W \#^A \overline{\mathbb{R}}^n_+) = \{\psi \in \mathcal{O}(W \#^A \overline{\mathbb{R}}^n_+) : \sup_{\alpha + i\beta \in K} e^{a\alpha} (\mathrm{dist}(\beta, \varDelta))^k |\psi(\alpha + i\beta)| < \infty \text{ for } 0 < \delta\}.$$

Here K ranges over all proper tubular subsets of W and $\varDelta$ is the set $\{\beta \in \mathbb{R}^n : (A\beta)_j = 0$ for some $j = 1, \ldots, n\}$. With analogously defined spaces $\widetilde{\mathcal{O}}^k_a(W \#^A_j \overline{\mathbb{R}}^n_+)$ we consider for fixed $\omega \in \mathbb{R}$ the spaces:

$$\widetilde{\mathcal{O}}^\infty_{(w)}(W \#^A \overline{\mathbb{R}}^n_+ = \varprojlim_{a<w} \varinjlim_{k \in \mathbb{N}_0} \widetilde{\mathcal{O}}^k_a(W \#^A \overline{\mathbb{R}}^n_+), \qquad \widetilde{\mathcal{O}}^\infty_{(w)}(W \#^A_j \overline{\mathbb{R}}^n_+) = \varinjlim_{a<w} \varinjlim_{k \in \mathbb{N}_0} \widetilde{\mathcal{O}}^k_a(W \#^A_j \overline{\mathbb{R}}^n_+).$$

The space of *Laplace distributions* (*of type* $\omega \in \mathbb{R}^n$) *supported by* $\overline{\mathbb{R}}^n_+$ is defined to be

$$L'_{(\omega)}(\overline{\mathbb{R}}^n_+) = \widetilde{\mathcal{O}}^k_{(w)}(W \#^A \overline{\mathbb{R}}^n_+) \Big/ \sum_{j=1}^n \widetilde{\mathcal{O}}^k_{(w)}(W \#^A_j \overline{\mathbb{R}}^n_+). \tag{18.9}$$

As in the case of hyperfunctions, it can be proved by introducing suitable local cohomologies with growth that the space of Laplace distributions is independent of the choice of the polytubular neighbourhood W of $\overline{\mathbb{R}}^n_+$ and of A. One may also proceed differently by showing that the space on the right hand side of (18.9) is canonically isomorphic to the dual space of the space $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ of Laplace test functions defined by means of a natural extension of the definition in dimension 1:

$$L_{(\omega)}(\overline{\mathbb{R}}^n_+) = \{\phi \in C^\infty(\overline{\mathbb{R}}^n_+) : \sup_{y\in\overline{\mathbb{R}}^n_+} |e^{-ay}(\partial/\partial y)^\alpha \phi(y)| < \infty \quad \text{for some } a < \omega \text{ and any } \alpha \in \mathbb{N}_0^n\}.$$

With the space $L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ defined as the dual of $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$, the identity in (18.9) is an isomorphism resembling that in Theorem 4.2 and proved as in dimension 1.

In particular, given a $T \in L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ we assign to it a defining function $\Psi \in \lim_{k\in\mathbb{R},\, a<\omega} \mathcal{O}^k_a(\mathbb{C} \# \overline{\mathbb{R}}^n_+)$ by the formula

$$\Psi(z) = \frac{1}{(2\pi i)^n} T\left[\frac{\mathbf{e}^{-(z-w)^2}}{(z-w)^{\mathbf{1}}}\right] \quad \text{for } z \in \mathbb{C}^n \# \overline{\mathbb{R}}^n_+.$$

In applications, however, it will be convenient to take instead of Ψ the defining function

$$\Psi_r(z) = \frac{1}{(2\pi i)^n} T\left[\frac{r^{z-w}}{(z-w)^{\mathbf{1}}}\right] \quad \text{for } z \in \mathbb{C}^n \# \overline{\mathbb{R}}^n_+, \tag{18.10}$$

where $r \in \mathbb{R}^n_+$, $r < \ln\omega$ (vector notation!). Then by the n-dimensional variant of Theorem 4.1 (see [SZ], Th. 8.2) we get the estimate

$$|\Psi_r(\alpha + i\beta)| \le C_k r^{-\alpha}/\|\beta\|^k \quad \text{for } \alpha + i\beta \in K, \tag{18.11}$$

where K is any set the form $\widetilde{W} \cap \{\mathbb{R}^n + i\Delta\}$ with Δ a proper subcone of the set $\{\beta \in \mathbb{R}^n : \beta_j \ne 0,\ j = 1,\dots,n\}$ and $\widetilde{W}$ a polytubular neighbourhood of $\overline{\mathbb{R}}^n_+$. To get the versions of the above statements corresponding to a matrix $A \in GL(n,\mathbb{R})$ with $\det A = 1$, we modify (18.10) to

$$\Psi^A_r(z) = \frac{1}{(2\pi i)^n} T\left[\frac{r^{z-w}}{(A(z-w))^{\mathbf{1}}}\right] \quad \text{for } z \in \mathbb{C}^n \#^A \overline{\mathbb{R}}^n_+.$$

Clearly Ψ^A_r satisfies (18.11) with the only difference that Δ in K is a proper subcone of $\{\beta \in \mathbb{R}^n : (A\beta)_j \ne 0, j = 1,\dots,n\}$. Then, using the notation introduced at the beginning of the section we have

$$T = \sum_{\sigma\in\{-1,1\}^n} \operatorname{sgn}\sigma\, b_{\Gamma^\sigma_A}(\Psi^{A,\sigma}_r),$$

where $\Psi^{A,\sigma}_r = \Psi^A_r|_{W^\sigma_A}$.

Now, given a Laplace distribution $T \in L'_{(\ln\varrho)}(\overline{\mathbb{R}}^n_+)$ for some $\varrho \in \mathbb{R}^n_+$, we assign to it (as in §9) the function

$$f(w) = T[w^z] \quad \text{for } w \in (\widetilde{\mathbb{C}\setminus\{0\}})^n,\ \|w\| < \varrho, \tag{18.12}$$

called a *generalized analytic function*. For a given f of the form (18.12) the distribution T is called the (multidimensional) *Borel transform* of f.

The most fundamental examples of GAFs in several variables are functions of the form

$$u(x) = \sum_{|\gamma|=0}^{\infty} \frac{a_\gamma}{(-\ln x)^{\gamma+\mathbf{1}}} \tag{18.13}$$

(here, for $x \in \mathbb{R}_+^n$, $\ln x = (\ln x_1, \ldots, \ln x_n)$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^n$). Let $\omega \in \mathbb{R}_+^n$. The n-dimensional variant of Theorem 14.1 states that u is of the form (18.13) for $0 < x < \exp(-\omega) := (e^{-\omega_1}, \ldots, e^{-\omega_n})$, where the series $\sum_{|\gamma|=0}^{\infty} a_\gamma \zeta^\gamma$ has polyradius of convergence not less than $\omega^{-\mathbf{1}}$, if and only if u is a generalized analytic function of the form

$$u(x) = T[x^\alpha] \quad \text{for } 0 < x < \exp(-\omega)$$

with $T = Y(\alpha_1) \cdot \ldots \cdot Y(\alpha_n) \cdot G$, where $G \in \widetilde{\mathcal{O}}_{(-\omega)}(\mathbb{C}^n)$, $G(\zeta) = \sum_{|\gamma|=0}^{\infty} \frac{a_\gamma}{\gamma!} \zeta^\gamma$.

More generally, one may need to consider GAFs whose Borel transforms are supported by $\mathbb{R}_+^n$-type sets L, i.e. subsets of $\mathbb{C}^n$ such that $L = \Phi(\overline{\mathbb{R}}_+^n)$, where Φ is a holomorphic mapping. In some cases Φ may have singularities (e.g. ramification points or poles). However, as long as we are dealing with solutions to linear Fuchsian type PDEs the Borel transforms of solutions turn out to be supported by small sets. We have already observed in Example 17.2 that the particular solution $u_{\overset{*}{\alpha}}$ of the equation

$$\left(\left(x_1 \frac{\partial}{\partial x_1}\right)^2 + \left(x_2 \frac{\partial}{\partial x_2}\right)^2\right) u_{\overset{*}{\alpha}} = \delta_{(1,1)}$$

was a GAF whose two-dimensional Borel transform is supported by two half-lines $\widetilde{L}_1$ and $\widetilde{L}_2$. More generally, we have the following.

Corollary 18.1. *Retain the notation of Theorem* 17.3. *In addition to the assumptions of Theorem* 17.3 *suppose that for* $j = 1, \ldots, m$, $c_j(\zeta_1) \sim (A_j + iB_j)\zeta_1^{\delta_j}$ *as* $|\zeta_1| \to \infty$ *uniformly on any closed subcone in* $\operatorname{Re} \zeta_1 > \overset{*}{a}_1$, *where* $A_j, B_j, \delta_j \in \mathbb{R}$. *Then the distinguished solution v is a GAF in two variables and can be written in the form*

$$\begin{aligned} v(y_1, y_2) = &- \frac{1}{a_m} \sum_{\mu=1}^{n} \sum_{j \in I^0(B_\mu)} \operatorname{sgn}(j;\mu) \\ &\times \int\limits_{L^{\varrho_j}_{\mu,\overset{*}{a}_1}} y_1^{\zeta_1} y_2^{c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1)} \frac{F(\zeta_1, c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1))}{\prod_{q=1,\, q\neq j}^{m} (c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1) - c_q^{\operatorname{sgn}(j;\mu)}(\zeta_1))}\, d\zeta_1 \\ &- \frac{1}{a_m} \sum_{\nu=1}^{\widetilde{N}} \sum_{j \in I^+(\theta_\nu)} \Bigg(\int\limits_{\tilde{L}^{\varrho_j}_\nu} y_1^{\zeta_1} y_2^{c_j^+(\zeta_1)} \frac{F(\zeta_1, c_j^+(\zeta_1))}{\prod_{q=1,\, q\neq j}^{m} (c_j^+(\zeta_1) - c_q^+(\zeta_1))}\, d\zeta_1 \\ &- \int\limits_{\tilde{L}^{\varrho_j}_\nu} y_1^{\zeta_1} y_2^{c_2^-(\zeta_1)} \frac{F(\zeta_1, c_j^-(\zeta_1))}{\prod_{q=1,\, q\neq j}^{m} (c_j^-(\zeta_1) - c_q^-(\zeta_1))}\, d\zeta_1 \Bigg), \end{aligned}$$

where $\widetilde{L}^{\varrho_j}_\nu = e^{i\varrho_j}\overline{\mathbb{R}}_+ + \theta_\nu$ *and* $L^{\varrho_j}_{\mu,\overset{*}{a}_1} = e^{i\varrho_j}\overline{\mathbb{R}}_+ + \overset{*}{a}_1 + iB_\mu$ *with arbitrary* $\varrho_j \in \mathbb{R}$ *such that* $|\varrho_j| < \pi/2$ *and* $A_j \cos \varrho_j \delta_j - B_j \sin \varrho_j \delta_j > 0$. *The integrals are absolutely convergent for* $0 < y_1 < t_1$, $0 < y_2 < t_2$.

Proof. This follows from Theorems 17.2, 17.3 and the estimate

$$\begin{aligned} \left| \frac{y_1^{\zeta_1} y_2^{c_j(\zeta_1)} F(\zeta_1, c_j(\zeta_1))}{\prod_{q=1,\, q\neq j}^{m} (c_j(\zeta_1) - c_q(\zeta_1))} \right| &\le C \langle (\zeta_1, c_j(\zeta_1)) \rangle^s \left(\frac{t_2}{y_2}\right)^{\operatorname{Re} c_j(\zeta_1)} \left(\frac{t_1}{y_1}\right)^{\operatorname{Re} \zeta_1} \\ &\le C^* |\zeta_1|^{\overset{*}{s}} \left(\frac{t_2}{y_2}\right)^{-r^{\delta_j}(A_j \cos \varrho_j \delta_j - B_j \sin \varrho_j \delta_j)} \left(\frac{t_1}{y_1}\right)^{-r \cos \varrho_j} \end{aligned}$$

for ζ_1 of the form $\overset{*}{a}_1 + iB_\mu + e^{i\varrho_j} r$, or $\theta_\nu + e^{i\varrho_j} r$, where $r \in \mathbb{R}_+$.

Observe that we may vary arbitrarily the parameters ϱ_j as long as for the curves $\{(\zeta_1, c_j^{\pm}(\zeta_1)) : \zeta_1 \in L^{\varrho_j}_{\mu,\mathring{a}_1}\}$ and $\{(\zeta_1, c_j^{\pm}(\zeta_1)) : \zeta_1 \in \widetilde{L}^{\varrho_j}_{\nu}\}$ we have $\operatorname{Re} c_j^{\pm}(\zeta_1) \to \infty$ as $\operatorname{Re}\zeta_1 \to \infty$. Thus in contrast to dimension 1 there is no "privileged" position for those curves. The general requirement is that they should be as close as possible to the real plane (cf. Section 14.2). For Example 17.2 this is achieved by choosing the angles $\varrho_j = \pm\pi/4$. However, the situation changes drastically when we pass to solutions to nonlinear equations (see Appendix II.2).

19. Linear elliptic Fuchsian type PDEs with generalized analytic coefficients. By a *Fuchsian type partial differential operator* with coefficients being generalized analytic functions we understand the following n-dimensional analog of the operator (16.1):

$$R(x, x\partial/\partial x) = P(x\partial/\partial x) - Q(x, x\partial/\partial x), \tag{19.1}$$

where in accordance with the notation of Section 18, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $x\partial/\partial x$ is the vector $x\partial/\partial x = (x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n)$. We assume that R is a linear operator of order m, with generalized analytic coefficients in a neighbourhood of zero. Thus

$$Q(x, x\partial/\partial x) = \sum_{|\nu|\le m} a_\nu(x)(x\partial/\partial x)^\nu,$$

where the a_ν are GAFs of convergence radius $\ge \varrho > 0$ of the form

$$a_\nu(x) = \int_{\mathbb{R}^n_+} x^\alpha \alpha^{\mathbf{k}} q_\nu(\alpha)\, d\alpha, \tag{19.2}$$

where q_ν extends to a function in $\widetilde{\mathcal{O}}_{(\ln\varrho)}(V)$ with V a polytubular neighbourhood of $\mathbb{R}^n_+$ and $\mathbf{k} = (k, \ldots, k) \in \mathbb{N}_0^n$. We easily observe that $\lim_{x\to 0,\, 0<x} a_\nu(x) = 0$, hence the decomposition (19.1) is unique. In particular, replacing the vector $x\partial/\partial x$ in $P(x\partial/\partial x)$ by $z \in \mathbb{C}^n$ we obtain a polynomial $P(z)$ on $\mathbb{C}^n$, which we call the *principal Mellin symbol* of R at zero. We shall assume the following "ellipticity" condition: for every $\mathring{\alpha}\in\mathbb{R}^n$ there exist $A < \infty$ and $C_0 > 0$ such that

$$|P(\mathring{\alpha} + i\beta)| > C_0(1 + \|\beta\|)^m \quad \text{for } \|\beta\| > A. \tag{19.3}$$

If the operator R given by (19.1) is such that its principal Mellin symbol P satisfies (19.3) we say that R is an *elliptic Fuchsian type PDO*. We shall consider the equation

$$R(x, x\partial/\partial x)u = f, \tag{19.4}$$

where $f(x) = T_0[x^\alpha]$ for $0 < x < t$ with $T_0 \in L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$. Our intention is to prove that there exists a solution u to (19.4) in a neighbourhood of zero which is a GAF. Clearly the solution u (if it exists) is not unique.

The main hypothesis that ensures the solvability of (19.4) in GAFs is the so-called cone condition for the polynomial $P(z)$ (= the principal Mellin symbol of R) introduced below:

Definition 19.1. Let $P(z_1, \ldots, z_n)$ be a complex polynomial in $\mathbb{C}^n$. We say that P satisfies the *cone condition* on $\mathbb{R}^n$ if there exists a complex polytubular neighbourhood of $\mathbb{R}^n$ (i.e. a neighbourhood of the form $V = V_1 \times \ldots \times V_n$, where V_j are complex tubular neighbourhoods of $\mathbb{R}$) and an open convex cone $\Gamma \subset \mathbb{R}^n = \operatorname{Im} \mathbb{C}^n$ such that

$$P(z) \neq 0 \quad \text{on } (\mathbb{R}^n + i\Gamma) \cap V.$$

R e m a r k 19.1. Observe that the cone condition is satisfied if $\{y : \operatorname{Im}(d_z P)(y) = 0\} \cap \Gamma = \emptyset$ as z runs through $\{\alpha \in \mathbb{R}^n : P(\alpha) = 0\}$. Thus, in a way, it measures how the zero sets of the imaginary part of the differential of P change as we move along the zero set of P in $\mathbb{R}^n$.

R e m a r k 19.2. The cone condition is clearly satisfied if the set $\{x \in \mathbb{R}^n : P(x) = 0\}$ is discrete and $P(x)$ does not approach zero as $x \to \infty$, $x \in \mathbb{R}^n$.

Suppose P satisfies the cone condition on $\mathbb{R}^n$. Then there exists $p \in \mathbb{N}_0$ such that

$$\frac{1}{|P(z)|} \leq \frac{C}{\|\operatorname{Im} z\|^p} \quad \text{for } \operatorname{Im} z \in \Gamma,\ \|\operatorname{Im} z\| \text{ small.}$$

The smallest such p is called the *multiplicity* of the set $\{\alpha \in \mathbb{R}^n : P(\alpha) = 0\}$ (with respect to Γ).

We next formulate a multidimensional variant of Lemma 16.1:

LEMMA 19.1. *Let $Q \in \widetilde{\mathcal{O}}_{(\omega)}(V)$, where V is a polytubular neighbourhood of $\overline{\mathbb{R}}^n_+$. Let $G \in \mathcal{O}(W)$, where W is a convex local wedge over $\mathbb{R}^n$, be such that*

$$|G(\alpha + i\beta)| \leq Ct^{-\alpha} \quad \text{for } \alpha \leq -1 \text{ and } \|\beta\| \text{ small}$$

with $0 < t < \exp w$ and C independent of α and β. Then for $z \in W$ with $\|\operatorname{Im} z\|$ small enough we have

$$\int_{\mathbb{R}^n_+} Q(w)G(z-w)\,dw = \int_{\Delta_z} Q(z-w)G(w)\,dw, \tag{19.5}$$

where $\Delta_z = \operatorname{Re} z + (-\overline{\mathbb{R}}^n_+ + i\mathring{\varepsilon}) \cup (-\operatorname{bd} \mathbb{R}^n_+ + i[\operatorname{Im} z, \mathring{\varepsilon}])$ and $\mathring{\varepsilon} \in \mathbb{R}^n$ is a fixed point such that $\mathbb{R}^n + i\mathring{\varepsilon} \subset W$ (here $\operatorname{bd} \mathbb{R}^n_+$ denotes the boundary of $\mathbb{R}^n_+$ in $\mathbb{R}^n$ and $[\operatorname{Im} z, \mathring{\varepsilon}]$ is the line segment in $\mathbb{R}^n$ with end points $\operatorname{Im} z$ and $\mathring{\varepsilon}$).

P r o o f. We have

$$\int_{\mathbb{R}^n_+} Q(w)G(z-w)\,dw = \int_{z-\mathbb{R}^n_+} Q(z-w)G(w)\,dw$$

and the assertion will follow if we show that

$$\Big(\int_{z-\mathbb{R}^n_+} - \int_{\Delta_z}\Big) Q(z-w)G(w)\,dw = 0.$$

Since $(z - \mathbb{R}^n_+) \cup \Delta_z$ is the boundary in $\mathbb{C}^n$ of the set $\operatorname{Re} z + (-\mathbb{R}^n_+ + i(\operatorname{Im} z, \mathring{\varepsilon}))$ of real dimension $n+1$, the result follows from the Cauchy–Poincaré theorem due to the estimates at infinity similar to those in dimension 1.

THEOREM 19.1. *Assume the notation and assumptions introduced above. Further, suppose that the polynomial P satisfies the cone condition on $\mathbb{R}^n$ with respect to a convex open cone Γ and $-\Gamma$. Let k in* (19.2) *be such that $k+1$ is the multiplicity of $\{\alpha \in \mathbb{R}^n : P(\alpha) = 0\}$. Then there exists $\mathring{r} > 0$ and a GAF u in n variables of convergence* *radius not less than* $\mathring{\mathbf{r}} = (\mathring{r}, \ldots, \mathring{r})$ *which solves the equation* (19.4) *in the polyinterval* $\{0 < x < \mathring{\mathbf{r}}\}$.

Proof. We seek a solution u in the form $u(x) = T[x^\alpha]$, where $T \in L'_{(\omega)}$ for some $\omega \in \mathbb{R}^n$. Denote by μ_ν the measure $\alpha^{\mathbf{k}} q_\nu(\alpha) d\alpha$. Since $f = T_0[x^\alpha]$ we obtain from (19.1) and (19.2) the following identity for the Laplace distribution T:

$$P(\alpha)T = \sum_{|\nu|=0}^{m} \int_{\mathbb{R}^n_+} (\alpha-\gamma)^\nu T(\cdot - \gamma)\mu_\nu(\gamma) + T_0. \tag{19.6}$$

Dividing formally both sides by $P(\alpha)$ we get

$$T = \sum_{|\nu|=0}^{m} \int_{\mathbb{R}^n_+} (\alpha-\gamma)^\nu \frac{T(\cdot - \gamma)}{P(\alpha)}\mu_\nu(\gamma) + \frac{T_0}{P(\alpha)}. \tag{19.7}$$

The definition of the quotients $T(\cdot - \gamma)/P(\alpha)$ and $T_0/P(\alpha)$ requires explanation. We concentrate on the case of T_0. Choose a defining function for T_0 in the form

$$h_r^A(z) = T_0\Big[\frac{\mathbf{r}^{\zeta - z}}{A(\zeta - z)}\Big] \quad \text{for } z \in \mathbb{C} \#^A \overline{\mathbb{R}}^n_+,$$

where $\mathbf{r} = (r, \ldots, r)$ for some $r > 0$ and A is a nonsingular real $n \times n$ matrix of determinant 1 such that $A(\mathbb{R}^n_+)$ is a proper subcone of Γ (then $A(-\mathbb{R}^n_+)$ is a proper subcone of $-\Gamma$). Thus according to (18.4), T_0 can be represented as

$$T_0 = \sum_{\sigma \in \{-1,1\}^n} \operatorname{sgn}\sigma\, b_{\Gamma_A^\sigma}(h_r^{A,\sigma}),$$

where for $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_j \in \{-1, 1\}$, $j = 1, \ldots, n$, we write as in (18.2),

$$\Gamma_A^\sigma = \{\beta \in \mathbb{R}^n : \forall j,\ \sigma_j(A\beta)_j > 0\}$$

and $h_r^{A,\sigma}$ is the restriction of $\operatorname{sgn}\sigma\, h_r^A$ to the wedge $W_A^\sigma = \mathbb{R}^n + i\Gamma_A^\sigma$. Recall that in order to compute the boundary values $b_{\Gamma_A^\sigma}(h_r^{A,\sigma})$ we do not need the whole cones Γ_A^σ but only their portions contained in the cones Γ and $-\Gamma$. Thus by the cone condition we may define

$$\frac{T_0}{P(\alpha)} := \sum_{\sigma \in \{-1,1\}^n} \operatorname{sgn}\sigma\, b_{\Gamma_A^\sigma}\bigg(\frac{h_r^{A,\sigma}(z)}{P(z)}\bigg), \tag{19.8}$$

where $h_r^{A,\sigma}(z)/P(z)$ are considered as functions on $W_{A,\Gamma}^\sigma = \mathbb{R}^n + i(\Gamma \cap \Gamma_A^\sigma)$ for $\sigma \neq -\mathbf{1}$ and $h_r^{A,-\mathbf{1}}(z)/P(z)$ is regarded as a function on $W_{A,\Gamma}^{-\mathbf{1}} = \mathbb{R}^n + i(-\Gamma \cap \Gamma_A^{-\mathbf{1}})$. Clearly $T_0/P(\alpha)$ is a distribution on $\mathbb{R}^n$ with support contained in $\{\alpha \in \mathbb{R}^n : P(\alpha) = 0\} \cup \operatorname{supp} T_0$. Since by ellipticity the set $\{\alpha \in \mathbb{R}^n : P(\alpha) = 0\}$ is bounded it follows that the support of $T_0/P(\alpha)$ is contained in a positive quadrant $L_0 = \mathring{\alpha} + \overline{\mathbb{R}}^n_+$ for some fixed $\mathring{\alpha} \in \mathbb{R}^n$. To see that it is a Laplace distribution we estimate its defining functions in $W_{A,\Gamma}^\sigma$:

$$\left|\frac{h_r^{A,\sigma}(\alpha + i\beta)}{P(\alpha + i\beta)}\right| \le \widetilde{C}\frac{\mathbf{r}^{-\alpha}}{\prod_{j=1}^n (\sigma_j(A\beta)_j)^{k_j}|P(\alpha + i\beta)|} \le C\frac{\mathbf{r}^{-\alpha}}{\prod_{j=1}^n (\sigma_j(A\beta)_j)^{k_j+m}}.$$

In a similar way, we put

$$g_r^A(z) = T\Big[\frac{\mathbf{r}^{\zeta - z}}{A(\zeta - z)}\Big] \quad \text{for } z \in \mathbb{C}^n \#^A \overline{\mathbb{R}}^n_+$$

and observe that for $\gamma \in \mathbb{R}^n$,

$$T(\cdot - \gamma)\left[\frac{\mathbf{r}^{\zeta - z}}{A(\zeta - z)}\right] = T\left[\frac{\mathbf{r}^{\zeta+\gamma - z}}{A(\zeta + \gamma - z)}\right] = g_r^A(z-\gamma).$$

Thus instead of solving equation (19.4) we shall look for a function $g_r^A(z)$ such that

$$g_r^A(z) = \sum_{|\nu|=0}^{m} \frac{1}{P(z)} \int_{\mathbb{R}^n_+} (z-\gamma)^\nu g_r^A(z-\gamma)\mu_\nu(\gamma) + g_r^{0,A}(z), \tag{19.9}$$

where

$$g_r^{0,A}(z) = h_r^A(z)/P(z).$$

Actually the equation (19.9) should be understood as a family of equations on the local wedges $W^\sigma_{A,\Gamma}$ ($\sigma \in \{-1,1\}^n$) for the functions $g_r^{0,A,\sigma} = g_r^{0,A}|_{W^\sigma_{A,\Gamma}}$.

Now by using Lemma 19.1 we are led to the equation

$$\begin{aligned} g_r^A(z) - \sum_{|\nu|=0}^{m} \frac{1}{P(z)} \int_{z}^{\operatorname{Re} z + i\mathring{\varepsilon}} \int_{\operatorname{Re} z - \operatorname{bd} \mathbb{R}^n_+} w^\nu g_r^A(w)(z-w)^k q_\nu(z-w)\,dw \\ = \sum_{|\nu|=0}^{m} \frac{1}{P(z)} \int_{\operatorname{Re} z + i\mathring{\varepsilon} - \mathbb{R}^n_+} w^\nu g_r^A(w)(z-w)^k q_\nu(z-w)\,dw + \frac{h_r^A(z)}{P(z)}. \end{aligned} \tag{19.10}$$

Again (19.10) is understood as a family of equations for $g_\nu^{A,\sigma}$ on the local wedges $W^\sigma_{A,\Gamma}$. We solve those equations by the method of successive approximations based on a combination of the methods of the proofs of Theorems 16.1 and 16.2 in dimension 1. First we invert the operator on the left hand side of (19.10). To fix ideas take $\sigma = \mathbf{1}$ and consider the space

$$\begin{aligned} \mathcal{H}_{\mathring{r},l} = \{H \in \mathcal{O}(W^{\mathbf{1}}_{A,\Gamma}) : |H(\alpha + i\beta)| \le C\mathring{r}^{-\alpha}/\|\beta\|^l \\ \text{for } \beta \in \Delta, \text{ any bounded proper subcone of } \Gamma^{\mathbf{1}}_A\} \end{aligned}$$

for some fixed $\mathring{r} \in \mathbb{R}^n_+$ and $l \in \mathbb{Z}$. Denote by $\mathcal{R}$ the operator

$$\mathcal{R}H(z) = \sum_{|\nu|=0}^{m} \frac{1}{P(z)} \int_{z}^{\operatorname{Re} z + i\mathring{\varepsilon}} \int_{\operatorname{Re} z - \operatorname{bd} \mathbb{R}^n_+} H(w) w^\nu (z-w)^k q_\nu(z-w)\,dw$$

for $H \in \mathcal{H}_{\mathring{r},l}$, where $\mathring{\varepsilon}$ is sufficiently small.

We shall prove that if l is large enough then the operator $\mathcal{R}$ maps $\mathcal{H}_{\mathring{r},l}$ into $\mathcal{H}_{\mathring{r},l}$ and $\operatorname{Id} - \mathcal{R}$ is invertible. As in the proof of Theorem 16.2 we arrive at the following estimates:

$$|\mathcal{R}H(\alpha + i\beta)| \le \frac{C \cdot \widetilde{C}}{l-k-1} \frac{\mathring{r}^{-\alpha}}{\|\beta\|^l} \quad \text{for } \beta \in \Delta,$$

where C is the constant corresponding to $H \in \mathcal{H}_{\mathring{r},l}$ and $\widetilde{C}$ is independent of $H \in \mathcal{H}_{\mathring{r},l}$.

By iterating the above estimate we get

$$|\mathcal{R}^j H(\alpha + i\beta)| \le C \frac{\widetilde{C}^j}{(l-k-1)^j} \frac{\mathring{r}^{-\alpha}}{\|\beta\|^l}.$$

Hence for the inverse $(\mathrm{Id}-\mathcal{R})^{-1}$ of $\mathrm{Id}-\mathcal{R}$ we find
$$|(\mathrm{Id}-\mathcal{R})^{-1}H(\alpha+i\beta)| \le CC^* \mathring{r}^{-\alpha}/\|\beta\|^l,$$
where $C^* = \sum_{j=0}^{\infty} \widetilde{C}^j/(l-k-1)^j$ is convergent provided l is large enough.

Now, in order to solve (19.10) it remains to solve the equation

$$(19.11)\quad g_r^A(z) = (\mathrm{Id}-\mathcal{R})^{-1}\Big(\sum_{|\nu|=0}^{m} \frac{1}{P(z)} \int\limits_{\operatorname{Re} z + i\mathring{\varepsilon} - \mathbb{R}_+^n} w^\nu g_r^A(w)(z-w)^k q_\nu(z-w)\, dw + \frac{h_r^A(z)}{P(z)}\Big)$$

in the space $\mathcal{H}_{r,l}$ for suitable r and l. Then for $H \in \mathcal{H}_{r,l}$ we find, as in (16.19),

$$\begin{aligned}
\Big|\sum_{|\nu|=0}^{m} &\frac{1}{P(\alpha+i\beta)} \int\limits_{\alpha+i\mathring{\varepsilon}-\mathbb{R}_+^n} w^\nu H(w)(\alpha+i\beta-w)^k q_\nu(\alpha+i\beta-w)dw\Big| \\
&\le \sum_{|\nu|=0}^{m} \frac{1}{|P(\alpha+i\beta)|} C \frac{C_\kappa}{\mathring{\varepsilon}^l} \prod_{j=1}^{n} \int\limits_{-\infty}^{\alpha_j} (|\mathring{\varepsilon}_j| + |\gamma_j|)^{\nu_j} r_j^{-\gamma_j} \\
&\quad \times |(\alpha+i\beta)_j - \gamma_j - i\mathring{\varepsilon}_j|^{k_j} (e^{\kappa/n}/\varrho)^{\alpha_j-\gamma_j}\, d\gamma_j \\
&\le \sum_{|\nu|=0}^{m} C \frac{C_1}{\widetilde{C}} \prod_{j=1}^{n} \frac{(|\mathring{\varepsilon}_j| + |\alpha_j|)^{\nu_j}}{(1+|\alpha|)^m} r_j^{-\alpha_j} \int\limits_{-\infty}^{\alpha_j} (\mathring{\varepsilon}_j + |(\alpha-\gamma)_j|)^{\nu_j} \\
&\quad \times |(\alpha-\gamma)_j + i(\beta_j - \mathring{\varepsilon}_j)|^{k_j} (e^{\kappa/n} r_j/\varrho)^{(\alpha-\gamma)_j}\, d\gamma_j \\
&\le C\cdot C_3 \cdot d(r) r^{-\alpha}
\end{aligned}$$

with
$$d(r) = \prod_{j=1}^{n} \int\limits_{0}^{\infty} (\mathring{\varepsilon}_j + |\theta_j|)^{\nu_j} |\theta_j + |\beta_j - \mathring{\varepsilon}_j||^{k_j} (e^{\kappa/n} r_j/\varrho)^{\theta}\, d\theta,$$
since by the cone condition for P we have
$$|P(z)| \ge \widetilde{C}(1+\|\operatorname{Re} z\|)^m \quad \text{for } \operatorname{Im} z = \mathring{\varepsilon}.$$
Observe that $d(r) \to 0$ as $r \to 0$, thus by the standard iteration procedure we find that for r sufficiently small the equation (19.11) has a unique holomorphic solution $g_r^{A,\mathbf{1}}$ on the wedge $W_{A,\Gamma}^{\mathbf{1}}$ belonging to the space $\mathcal{H}_{r,l}$. In a similar way we find solutions $g_r^{A,\sigma}$ on the remaining wedges $W_{A,\Gamma}^{\sigma}$. Then the distribution
$$T = \sum_\sigma \operatorname{sgn}\sigma\, b_{\Gamma_A^\sigma}(g_r^{A,\sigma})$$
is a solution of the equation (19.4). To see that T is a Laplace distribution it is enough to check that the support of T is contained in a certain positive quadrant. But it is easy to see (by inspecting the iteration procedure) that outside the set $\{\alpha\in\mathbb{R}^n : P(\alpha)=0\}+\overline{\mathbb{R}}_+^n$ the sum of the boundary values of g_r^A is zero at those points at which it is zero for h_r^A. Consequently,
$$\operatorname{supp} T \subset \operatorname{supp} T_0 \cup \{\alpha\in\mathbb{R}^n : P(\alpha)=0\} + \overline{\mathbb{R}}_+^n.$$

R e m a r k 19.3. The choice of $\mathbb{R}^n$ in Theorem 19.1 is completely insignificant. Instead one may take any real type plane of the form $\mathring{\zeta} + \mathbb{R}^n$ for some $\mathring{\zeta} \in \mathbb{C}^n$ and consider the equation $R(x, x\partial/\partial x)u = f$, where $f(x) = T[x^\zeta]$ for $T \in L'_{(\omega)}(\mathring{\zeta} + \overline{\mathbb{R}}^n_+)$. Clearly in that case we have to assume that P satisfies the cone condition on $\mathring{\zeta} + \mathbb{R}^n$.

Example 19.1. The polynomial $P(z_1, z_2) = z_1^2 + z_2^2$ satisfies the cone condition on a plane $\mathring{\zeta} + \mathbb{R}^2$ ($\mathring{\zeta} \in \mathbb{C}^2 \setminus \{0\}$) with respect to any proper subcone $\Gamma \subset \mathbb{R}^2 \setminus \mathbb{R} \cdot (\operatorname{Im} \mathring{\zeta})$.

Indeed, let $\mathring{\zeta} = (\mathring{a}_1, \mathring{a}_2) + i(\mathring{b}_1, \mathring{b}_2)$. Let $\mathring{b} = (\mathring{b}_1, \mathring{b}_2) \neq 0$. Then the set $\{z \in \mathring{\zeta} + \mathbb{R}^2 : P(z) = 0\}$ consists of two points $\zeta^1 = (-\mathring{b}_2, \mathring{b}_1) + i(\mathring{b}_1, \mathring{b}_2)$ and $\zeta^2 = (\mathring{b}_2, -\mathring{b}_1) + i(\mathring{b}_1, \mathring{b}_2)$. Next observe that $\operatorname{Im} P(z) = 0$ over the points $(-\mathring{b}_2, \mathring{b}_1)$ and $(\mathring{b}_2, -\mathring{b}_1)$ if and only if the vector $\beta = \operatorname{Im} z$ is orthogonal to $(-\mathring{b}_2, \mathring{b}_1)$ and to $(\mathring{b}_2, -\mathring{b}_1)$.

It follows from the above result that $z_1^2 + z_2^2$ does not satisfy the cone condition on $\mathbb{R}^2$.

Actually, the result of Example 19.1 is valid in any dimension. Namely, the polynomial $P(z) = z_1^2 + \ldots + z_n^2$ satisfies the cone condition on $\mathbb{R}^n + i\mathring{b}$ (for $\mathring{b} \neq 0$) with respect to any proper subcone Γ contained in the complement of the set of all vectors normal to

$$\{\alpha \in \mathbb{R}^n : \|\alpha\| = \|\mathring{b}\|,\ (\alpha, \mathring{b}) = 0\}.$$

Observe that for $\mathring{b} = 0$ the set of normal vectors to $\{0\}$ is the whole $\mathbb{R}^n$.

It follows from the results of Section 18 that Theorem 19.1 applies to operators R whose coefficients $a_\nu(x)$ are functions of the form

$$a_\nu(x) = \sum_{|\gamma|=0}^{\infty} \frac{a_\gamma}{(-\ln x)^{\gamma+\mathbf{k}+\mathbf{1}}} \quad \text{with} \quad \overline{\lim_{|\gamma|\to\infty}} \sqrt[|\gamma|]{|a_\gamma|} < \infty. \tag{19.12}$$

In particular, it follows from Example 19.1 that Theorem 19.1 applies to equations of the form

$$\left(\sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial x_j} \right)^2 + \sum_{|\nu|\le 2} a_\nu(x) \left(x \frac{\partial}{\partial x} \right)^\nu \right) u = f, \tag{19.13}$$

where a_ν are of the form (19.12) with $\mathbf{k} = 0$, and $f \in L'_{(c)}(\mathring{\zeta} + \overline{\mathbb{R}}^n_+)$ for some $\mathring{\zeta} \in \mathbb{C}^n \setminus \{0\}$. In the variables $y_1 = -\ln x_1, \ldots, y_n = -\ln x_n$ the equation (19.13) becomes

$$\left(\sum_{j=1}^{n} \left(\frac{\partial}{\partial y_j} \right)^2 + \sum_{|\nu|\le 2} b_\nu(y) \left(\frac{\partial}{\partial y} \right)^\nu \right) v = g,$$

where

$$b_\nu(y) = \sum_{|\gamma|=0}^{\infty} a_\gamma / y^{\gamma+\mathbf{1}},$$

i.e. the Laplace equation with a perturbation which is analytic at infinity. The study of the behaviour of solutions to this equation as $y \to \infty$ is one of main applications of the results of the paper.

R e m a r k 19.4. If we assume that the coefficients a_ν are GAFs whose Borel transforms are arbitrary Laplace distributions on $\overline{\mathbb{R}}^n_+$ which vanish in a neighbourhood of zero then the result of Theorem 19.1 remains valid and its proof substantially simplifies. In particular, one gets the result for operators with analytic coefficients at zero. Since the operator R is linear it follows that Theorem 19.1 holds for operators whose coefficients are sums of GAFs of the types considered in Theorem 19.1 and described above.

Concerning the uniqueness of solution to equations of type (19.4) note the following simple

PROPOSITION. *Let* $f \in \mathfrak{M}'_{\mathring{a}}$ *for some* $\mathring{a} \in \mathbb{R}^n$. *Then there exists a solution* $u \in \mathfrak{M}'_{\mathring{a}}$ *of the equation*

$$P(x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n)u = f. \tag{19.14}$$

The solution u *is unique in the space* $\mathfrak{M}'_{\mathring{a}}$ *modulo distributions* $v \in \mathfrak{M}'_{\mathring{a}}$ *such that* $\mathcal{M}_{\mathring{a}}v$ *is supported by the set* $\{\beta \in \mathbb{R}^n : P(\mathring{a} + i\beta) = 0\}$, *and* $P(\mathring{a} + i\beta)\mathcal{M}_{\mathring{a}}v = 0$.

Proof. By applying the $\mathcal{M}_{\mathring{a}}$-Mellin transformation, (19.14) is reduced to a division problem in $S'(\mathbb{R}^n)$: find $V \in S'(\mathbb{R}^n)$ such that

$$P(\mathring{a} + i\,\cdot)V = \mathcal{M}_{\mathring{a}}f. \tag{19.15}$$

By the Łojasiewicz–Hörmander theorem the division problem (19.15) is solvable in $S'(\mathbb{R}^n)$ and the uniqueness of V is as asserted by the proposition.

Appendix I. The symbol of a distribution in the sense of A. Weinstein. Conormal distributions

The concept of the symbol of a distribution at a point (say zero) is based on the idea of expanding the distribution into (poly) homogeneous terms. Clearly this is not always possible even for conormal distributions (recalled below). However, if one abandons the idea that the expansions should involve only discrete sets of homogeneous distributions one is naturally led to consider distributions of the form $u = T[x_+^\alpha]$, where T is a Laplace distribution and x_+^α is homogeneous of order α (see [Zie1], §8, for details). Distributions of that form constitute a subclass of conormal distributions (see the definition below), and in the opinion of the author—a fundamental one, not only in the study of singular linear equations but also in nonlinear ordinary and partial differential equations (see Appendices II.1 and II.2).

The basic idea in microlocal analysis of singularities of distributions is to study the growth properties of their Fourier transforms localized to a point (say zero) and to a (cotangent) direction $\xi \in \mathbb{R}^n$. According to these ideas A. Weinstein introduces in [We] distribution-valued functions $[1,\infty) \ni \tau \mapsto g^\tau \in D'(\mathbb{R}^n)$, where for given $g \in D'(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$g_\xi^\tau[\phi] = g[\tau^{n/2}e^{-i+x\xi}\phi(\sqrt{\tau}, x)] \quad \text{for } \phi \in C_0^\infty(\mathbb{R}^n). \tag{I.1}$$

For ϕ in a bounded set (in $C_0^\infty\mathbb{R}^n$) the function $g^\tau[\phi]$ is bounded as $\tau \to \infty$ by τ^N for some $N \in \mathbb{R}$. Therefore it is convenient to introduce the set $S^N(D'(\mathbb{R}^n))$ consisting of those functions (I.1) for which $g^\tau[\phi]$ is bounded by τ^N as $\tau \to \infty$ (one then writes $g^\tau = O(\tau^N)$). Let $\xi \in \mathbb{R}^n$ and g be a distribution such that $g_\xi^\tau = O(\tau^N)$ for some $N \in \mathbb{R}$. Following [We] we define the *principal symbol* of order N of g at the point $(0,\xi)$ (denoted by $\sigma^N_{(0,\xi)}(g)$) as the element of the quotient space $S^N(D'(\mathbb{R}^n))/S^{N-1/2}(D'(\mathbb{R}^n))$ given by g_ξ^τ. Intuitively speaking, $\sigma^N_{(0,0)}(g)$ measures the order of flatness of g at zero, and $\sigma^N_{(0,\xi)}(g)$, for $\xi \neq 0$, the degree of singularity of g at zero as observed from the (cotangent) direction ξ.

In the case of a homogeneous distribution one should be able to compute the symbol explicitly. This is very simple for δ-distributions (and their derivatives as well as primitives). For instance (see [We]),

$$\sigma^{m/2+1/2}_{(0,0)}(\delta^{(m)}) = \tau^{m/2+1/2}\delta^{(m)}, \qquad \sigma^{m+1/2}_{(0,\xi)}(\delta^{(m)}) = \tau^{m+1/2}(i\xi)^m\delta^{(0)}.$$

We shall concentrate on the case of homogeneous functions

$$\mathbb{R} \ni x \mapsto (x^2)^\lambda \quad \text{for } \lambda > 0,\ \lambda \notin \mathbb{N}.$$

Clearly the (complete) symbol at $(x,\xi) = (0,0)$ of $(x^2)^\lambda$ equals $\frac{1}{\tau^\lambda}(x^2)^\lambda$. The computation of the symbol at a point $(0,\xi)$, $\xi \neq 0$, is more delicate (due to the oscillatory term $e^{-i\sqrt{\tau}\xi x}$). Let $\phi \in C_0^\infty(\mathbb{R})$ with $\phi \equiv 1$ in a neighbourhood of zero. Consider the product

$$u(x) = \phi(x)(x^2)^\lambda \quad \text{for } \lambda \text{ as above.}$$

Then u is a compactly supported function which is stable under the operator xd/dx, i.e. there exists $s \in \mathbb{R}$ such that

$$(xd/dx)^j u \in {}^\infty H_{(s)}(\mathbb{R}) \quad \text{for } j \in \mathbb{N}_0, \tag{I.2}$$

where ${}^\infty H_{(s)}(\mathbb{R})$ is the Besov space ([Hö]) (instead of ${}^\infty H_{(s)}$ we could have taken any other suitable space). By using the Fourier inversion formula we may write u in the form

$$u(x) = \int_{\mathbb{R}} e^{ix\xi} a_\lambda(\xi)\, d\xi,$$

where a_λ is the Fourier transform of u:

$$a_\lambda(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}u(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}u(x)\, dx.$$

Now, from (I.2) we get (see [Hö], Lemma 18.2.4)

$$|(d/d\xi)^j a_\lambda(\xi)| \le C_j(1+|\xi|)^{\mu-j}, \qquad j \in \mathbb{N}_0,$$

for $\mu = -s-1/2$. More generally, we may replace $a_\lambda(\xi)$ by a function $a(x,\xi) \in C^\infty(\mathbb{R}\times\mathbb{R})$ satisfying the estimate

$$|(d/dx)^k(d/d\xi)^j a(x,\xi)| \le C_{jk}(1+|\xi|)^{\mu-j} \quad \text{for all } j,k \in \mathbb{N}_0. \tag{I.3}$$

The collection of such functions $a(x,\xi)$ is called the *space of symbols of order* μ and is denoted by $S^\mu(\mathbb{R}\times\mathbb{R})$ (see [Hö]). The corresponding distribution

$$u = \int e^{ix\xi} a(x,\xi)\, d\xi \tag{I.4}$$

((I.4) should be understood as the Fourier transform of $a(x,\xi)$ in the sense of tempered distributions in $S'(\mathbb{R})$) is called a *conormal distribution* with respect to zero.

For a conormal distribution (I.4) with $a \in S^\mu$, Hörmander [Hö] defines the (principal) symbol at $(0,\xi)$ as an element of the quotient space $S^\mu(\mathbb{R}\times\mathbb{R})/S^{\mu-1}(\mathbb{R}\times\mathbb{R})$ determined by $a(x,\xi)$. The fundamental result of [We] (Th. 3.2.5) states that, in the case of a conormal distribution, both the Hörmander symbol and the symbol defined by Weinstein coincide, i.e. up to a constant factor the Weinstein symbol at $(0,\xi)$ as a function of τ is equal to the Hörmander symbol at $(0,\tau\xi)$.

Returning to the case of the distribution $u = \phi(x)(x^2)^\lambda$, we observe that $a_\lambda(\xi)$ is asymptotically homogeneous of order $-2\lambda-1$. More precisely, we have for $\lambda \notin \mathbb{Z}$,

$$a_\lambda(\xi) = \frac{-2}{\sqrt{2\pi}} \sin(\lambda\pi)\Gamma(2\lambda+1)|\xi|^{-2\lambda-1} + \sigma(\xi), \tag{I.5}$$

where $\sigma(\xi)/|\xi|^j \to 0$ as $|\xi| \to \infty$ for any $j \in \mathbb{N}$. This follows easily (cf. [Ły2]) from the fact that

$$\mathcal{F}((x^2)^\lambda) = \frac{-1}{\pi} \sin(\lambda\pi)\Gamma(2\lambda+1)|\xi|^{-2\lambda-1}.$$

Thus up to a constant factor (independent of λ) the symbol of $(x^2)^\lambda$ at $(0,\xi)$ equals

$$\sin(\lambda\pi)\Gamma(2\lambda+1)|\tau\xi|^{-2\lambda-1}.$$

A general symbol $a \in S^\mu(\mathbb{R}\times\mathbb{R})$ need not, however, be asymptotically homogeneous in the sense (I.5) nor expand in an asymptotic series of homogeneous distributions of decreasing orders. Therefore in the general case of a conormal distribution the explicit computation of the symbol becomes complicated (if at all possible). To grasp the idea of the difficulties consider the example of the function

$$g(x) = \frac{1}{x^2 - \ln(x^2)} \tag{I.6}$$

given by A. Weinstein in [We]. Direct computations presented in [We] do not lead to the explicit expression for the symbol. Luckily, the function g is a generalized analytic function:

For $t > 0$, let

$$h(t) = \frac{1}{t - \ln t}.$$

For $0 < t < -\ln t$ we have

$$h(t) = \frac{1}{-\ln t}\left(\frac{1}{1 + \frac{t}{-\ln t}}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j t^j}{(-\ln t)^{j+1}},$$

which in view of the formula

$$\frac{1}{(-\ln t)^{j+1}} = \int_0^\infty t^\alpha \frac{\alpha^j}{j!}\, d\alpha \quad \text{for } 0 < t < 1,\ j \in \mathbb{N}_0$$

leads to the expression

$$h(t) = \int_0^\infty t^\alpha T(\alpha)\, d\alpha, \tag{I.7}$$

where for $k \in \mathbb{N}_0$,

$$T(\alpha) = \sum_{j=0}^{k} \frac{(-1)^j(\alpha-j)^j}{j!} \quad \text{for } k < \alpha \le k+1.$$

Clearly $|T(\alpha)| \le e^\alpha$ so h is a GAF of convergence radius $\ge 1/e$. Replacing x by t^2 in (I.7) we find that

$$g(x) = \int_0^\infty (x^2)^\alpha T(\alpha) d\alpha \quad \text{for } |x| \le 1/\sqrt{e}, \tag{I.8}$$

with T given above. Combining (I.8) with the formulas for the symbol of $(x^2)^\lambda$ we see that the "complete" symbol of g at $(0,0)$ is given by the assignment

$$\tau \mapsto \int_0^\infty (x^2)^\alpha \frac{1}{\tau^\alpha} T(\alpha)\, d\alpha,$$

hence the principal symbol of order zero (modulo $S^{-1/2}(D'(\mathbb{R}))$) equals

$$\sigma^0_{(0,0)}(g) = \int_0^{1/2} (x^2)^\alpha \frac{1}{\tau^\alpha} T(\alpha)\, d\alpha.$$

Similarly, to compute the symbol at $(0,\xi)$ for $\xi \neq 0$, we first note that for the GAF

$$r(x) = \int_{1/4}^\infty (x^2)^\alpha T(\alpha)\, d\alpha \quad \text{for } |x| \le 1/\sqrt{e},$$

r^τ_ξ is bounded by $\tau^{-3/2}$. Hence up to a constant multiplier the principal symbol of order -1 (modulo $S^{-3/2}(D'(\mathbb{R}))$) is the function

$$\sigma^{-1}_{(0,\xi)}(g) = \int_0^{1/4} \tau^{-2\alpha-1}|\xi|^{-2\alpha-1} \sin(\alpha\pi) \Gamma(2\alpha+1) T(\alpha)\, d\alpha,$$

which can be regarded as a GAF at the point $\tau=\infty$ of convergence radius ∞. Analogously one may write down the symbol modulo $S^\mu(D'(\mathbb{R}))$ for any $\mu < -1$. However, unlike the situation in the case of $(0,0)$ the complete symbol at $(0,\xi)$ cannot be written in the form of a GAF due to the divergence effect caused by the factor $\Gamma(2\alpha+1)$. At this point we remark that the heuristic formula

$$\sigma^{N_1+N_2}_{(0,\xi)}(u_1 u_2) = \int \sigma^{N_1}_{(0,\eta)}(u_1) \cdot \sigma^{N_2}_{(0,\xi-\eta)}(u_2)\, d\eta$$

for the symbol of the product of distributions (formulated in [We]) is valid (and meaningful) in the case of GAFs. It is in fact a direct consequence of the formula (13.1) which states that the Borel transform of the product of GAFs is the convolution of the Borel transforms of the factors.

Finally, a few words about the case of dimension $n > 1$. Distributions conormal with respect to zero are replaced by distributions conormal with respect to the walls $\{x_1 = 0\}, \ldots, \{x_n = 0\}$. It is easy to see that the algebra of smooth vector fields tangent simultaneously to all the hyperplanes $\{x_1 = 0\}, \ldots, \{x_n = 0\}$ is spanned by the vector fields $x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n$. Thus the conormality condition (2) for a distribution u (with compact support) assumes for instance the form

$$\text{(I.9)} \qquad (x_1\partial/\partial x_1)^{\gamma_1} \ldots (x_n\partial/\partial x_n)^{\gamma_n} u \in L^2(\mathbb{R}^n) \quad \text{for any } \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^u.$$

For u restricted to $\mathbb{R}^n_+$ the condition (I.9) implies (cf. [SZ]) that u is a Mellin distribution $u \in \mathcal{M}'_{(-1/2,\ldots-1/2)}$ and for every $\gamma \in \mathbb{N}_0^n$ the Mellin transform $\mathcal{M}u$ satisfies the condition

$$(\alpha + i\beta)^\gamma \mathcal{M}u(\alpha + i\beta) \in L^2(\mathbb{R}^n) \quad \text{for } \gamma \in \mathbb{N}_0^n$$

as a function of β for any fixed $\alpha < -\mathbf{1/2} = (-1/2, \ldots, 1/2)$. Thus u is a Mellin distribution of Mellin order $-\infty$ or equivalently $x^{\mathbf{1/2}}u$ is a Mellin multiplier: $x^{\mathbf{1/2}}u \in \mathcal{M}_{[-\mathbf{1}]}$ (see [SZ] for the definitions and notation). Summing up, from the point of view of the Mellin analysis of singularities distributions conormal with respect to the walls $\{x_1 = 0\}, \ldots, \{x_n = 0\}$ are smooth functions, i.e. the Mellin transformation is well adapted to this geometric situation. Moreover, given an arbitrary distribution (in a neighbourhood of zero) one may describe how close it is to being conormal (say, with respect to zero) by studying the growth order (in the β-direction) of suitably restricted Mellin transforms of u. This leads to a natural definition of the second wave front set with respect to (the conormal bundle at) zero (see [ZieK] for details and [B] for applications to nonlinear equations).

The case of the Mellin transformation, described above, illustrates the general idea that the study of distributions whose singularities are related to a certain (algebraic) set should be carried out by means of a suitable integral transformation related to the geometry of that set.

Appendix II. Nonlinear singular differential equations

1. The case of ordinary differential equations. We consider the Euler equation

$$\frac{d}{dy}\phi - \phi = g(y), \tag{II.1}$$

where $g(y) = \sum_{j=1}^{\infty} b_j y^{-j}$ is an analytic function of variable y^{-1} for large y. We are interested in solutions of (II.1) as $y \to \infty$. Recall that the point ∞ is an irregular singular point. Consequently, (II.1) is solved in hyperfunctions or formal power series in y^{-1}.

We shall investigate solutions of (II.1) by adapting the ideas of J. Ecalle ([E1], [PCN]). First, by the change of variables ($x = e^{-y}$), we transform (II.1) to the equivalent form

$$-x\frac{d}{dx}\psi(x) - \psi(x) = f(x), \tag{II.2}$$

where

$$f(x) = g(-\ln x) = \sum_{j=1}^{\infty} b_j(-\ln x)^{-j}.$$

We compute the Borel transform of (II.2). We find that

$$-(\alpha+1)B(\psi) = B(f) \quad \text{in } D'(\mathbb{R}), \tag{II.3}$$

where

$$B(f)(\alpha) = Y(\alpha)\sum_{j=1}^{\infty} b_j \frac{\alpha^{j-1}}{(j-1)!}, \tag{II.4}$$

according to Section 14.

Define

$$A(\alpha) = \sum_{j=1}^{\infty} b_j \frac{\alpha^{j-1}}{(j-1)!} \quad \text{for } \alpha \in \mathbb{R}$$

and observe that, since the series $\sum b_j s^j$ is convergent, the function A extends to an entire function $A(z)$ on $\mathbb{C}$ of exponential growth (Theorem 14.1).

By solving (II.3) in the space of distributions we get

$$B(\psi) = -Y(\alpha)\frac{A(\alpha)}{\alpha+1}. \tag{II.5}$$

It is seen from (II.5) that ψ is a resurgent function and we shall compute the "alien derivative" $\Delta_{-1}(B(\psi))$.

Note that $B(\psi)$ restricted to $\mathbb{R}_+$ equals $-A(\alpha)/(\alpha+1)$ and extends to a meromorphic function $-A(z)/(z+1)$ whose difference of boundary values on $\mathbb{R}$ is

$$b\left(-\frac{A(z)}{z+1}\right) = 2\pi i A(-1)\delta_{(-1)},$$

in view of Example 18.1. Thus $\Delta_{-1}B(\psi) = 2\pi i A(-1)\delta_{(0)}$. Passing to the corresponding generalized analytic functions we see that the general solution $\psi(x,u) = \psi(x) + ux^{-1}$ of (II.2) can now be written as

$$\psi(x,u) = \psi(x) + ux^{-1}(\Delta_{-1}/(2\pi i A(-1)))B(\psi)[x^\alpha], \tag{II.6}$$

where

$$\psi(x) = -\int_0^\infty \frac{A(\alpha)}{\alpha+1}x^\alpha\, d\alpha.$$

Now we proceed to the main topic of this Appendix:

"*Alien analysis*" *of a nonlinear Euler equation.* We consider the nonlinear counterpart of (II.1), i.e. an equation of the form

$$\frac{d}{dy}\phi - \phi = b(y,\phi), \tag{II.7}$$

where $b(y,\phi)$ is a convergent power series in y^{-1} for y large and in ϕ for ϕ small, i.e.

$$b(y,\phi) = \sum_{r=0}^{\infty} b_r(y^{-1})\phi^r$$

with b_r analytic in y^{-1}. In logarithmic variables $x = e^{-y}$, (II.7) becomes

$$-x\frac{d}{dx}\psi(x) - \psi(x) = \sum_{r=0}^{\infty} b_r\left(\frac{1}{-\ln x}\right)\psi^r, \tag{II.8}$$

We are seeking a general solution $\psi(x,u)$ of (II.8). It should depend on one complex parameter u in a nonlinear way. We can write $\psi(x,u)$ as a formal series

$$\psi(x,u) = \sum_{n=0}^{\infty} u^n x^{-n}\psi^{(n)}(x), \tag{II.9}$$

where $\psi^n(x)$ are certain functions (later on they will turn out to be generalized analytic functions). Note that (II.9) is a generalization of the corresponding formula given in the linear case. Inserting (II.9) in (II.8) we find by equating the coefficients of the same powers

$u^n x^{-n}$ that

$$
\text{(II.10)}\quad
\begin{aligned}
&-x\frac{d}{dx}\psi^0-\psi^0=\sum_{r=0}^{\infty}b_r\cdot(\psi^0)^r \quad \text{for} \quad n=0,\\
&-x\frac{d}{dx}\psi^n+(n-1)\psi^{(n)}=\sum_{r=1}^{\infty}b_r\Big(\sum_{n_1+\ldots+n_r=n}\psi^{(n_1)}\cdot\ldots\cdot\psi^{(n_r)}\Big)\\
&\hfill \text{for } n\ge 1.
\end{aligned}
$$

Next we compute formally the Mellin transform of (II.10). Under the notation

$$\Psi^{(n)}=B\psi^{(n)},\quad B_r=Bb_r,$$

we easily deduce, from the operational rules for the Borel transformation, that

$$
\text{(II.11)}\quad
\begin{aligned}
&(-\alpha-1)\Psi^{(n)}=B_0+\sum_{r=1}^{\infty}B_r*(\Psi^{(0)})^r, \qquad n=0,\\
&(-\alpha+n-1)\Psi^{(n)}=\sum_{r=1}^{\infty}B_r*\Big(\sum_{n_1+\ldots+n_r=n}\Psi^{(n_1)}*\ldots*\Psi^{(n_r)}\Big), \qquad n\ge 1.
\end{aligned}
$$

The convolution equations (II.11) are not easy to solve since the right hand sides of (II.11) contain the unknown $\Psi^{(n)}$. In order to separate them we consider a parameter version of (II.8):

$$-x\frac{d}{dx}\psi(x)-\psi(x)=\varepsilon b(-\ln x,\psi).$$

with ε close to 1. Now we seek the functions $\psi^{(n)}$ in the formal form

$$\text{(II.12)}\qquad \psi^{(n)}(x)=\sum_{k=0}^{\infty}\varepsilon^k\psi^{(n,k)}(x).$$

Observe that then

$$\sum_{n=0}^{\infty}u^n x^{-n}\psi^{(n,0)}(x)$$

is a solution of the homogeneous equation

$$-x\frac{d}{dx}\psi-\psi=0.$$

Thus

$$-x\frac{d}{dx}\psi^{(n,0)}+(n-1)\psi^{(n,0)}=0$$

and hence $\psi^{(n,0)}(x)=C_n x^{n-1}$, where C_n is a complex constant. Consequently,

$$\text{(II.13)}\qquad \Psi^{(n,0)}=B\Psi^{(n,0)}=C_n\delta_{(n-1)},$$

where $\delta_{(n-1)}$ is the Dirac delta at $n-1$. Since we want $\Psi^{(n,0)}$ to be resurgent functions we put $C_0=0$, $C_1=C\neq 0$ and $C_j=0$ for $j=2,3,\ldots$ Now multiply the right hand side of (II.11) by ε.

Inserting the Borel transform of (II.12) we get (under the notation $\Psi^{(n,k)} = B\psi^{(n,k)}$) by equating the coefficients of the powers of ε:

$$
(-\alpha-1)\Psi^{(0,1)} = B_0
$$

$$
(-\alpha-1)\Psi^{(0,k)} = \sum_{r=1}^{\infty} B_r * \sum_{k_1+\ldots+k_r=k-1} \Psi^{(0,k_1)} * \ldots * \Psi^{(0,k_r)}, \qquad k = 2,3,\ldots \ (n=0),
$$

(II.14)

$$
(-\alpha+n-1)\Psi^{(n,k)} = \sum_{r=1}^{\infty} B_r * \sum_{\substack{n_1+\ldots+n_r=n \\ k_1+\ldots+k_r=k-1}} \Psi^{(n_1,k_1)} * \ldots * \Psi^{(n_r,k_r)}, \qquad k = 1,2,\ldots \ (n \ge 1).
$$

In particular for $n=0$, $k=1$ we have, by (II.13),

$$
\Psi^{(0,1)} = \frac{B_0}{-\alpha-1}
$$

and the equations (II.14) are solved recurrently. One can prove that the functions $\Psi^{(n,k)}$ and then $\Psi^{(n)}$ for $\varepsilon = 1$ are resurgent functions with singular set $\Omega = -\mathbb{Z}$. The precise location of the singularities can easily be derived by applying the "alien" calculus:

Lemma. *$\Delta_\omega \Psi^{(n)} \neq 0$ at most for $\omega = -1,0,1,\ldots,n-1$. Thus if we set $\dot{\Psi}^{(n)}(\alpha) = \Psi^{(n)}(\alpha+n)$ we have for the Borel transform $\Psi(\alpha,u)$ of $\psi(x,u)$,*

$$
\Psi(\alpha,u) = \sum_{n=0}^{\infty} u^n \dot{\Psi}^{(n)},
$$

and since $\dot{\Psi}^{(n)}$ has $\{-1-n,-n,1-n,\ldots,-1\}$ as singular set it follows that the singular set for Ψ is $-\mathbb{N}$.

Proof. Fix (n,k) and suppose that $\omega \in \mathbb{C}$ is such that

$$
\Delta_\omega \Psi^{(n,k_j)} = 0 \quad \text{for } k_j = 0,1,\ldots,k_1.
$$

Applying Δ_ω to (II.14) we get (in view of the algebraic properties of Δ_ω) for $n > 0$,

$$
(-\alpha+n-1-\omega)\Delta_\omega \Psi^{(n,k)} = \sum_{\mathbf{n},\mathbf{k}} B_{\mathbf{n}} * \Delta_\omega(\Phi^{(n_1,k_1)} * \ldots * \Phi^{(n_r,k_r)}) = 0,
$$

where $\mathbf{n} = (n_1,\ldots,n_r) \in \mathbb{N}_0^r$ and $\mathbf{k} = (k_1,\ldots,k_r) \in \mathbb{N}_0^r$ with r ranging from 0 to ∞ if $n=1$ and from 1 to ∞ otherwise (here $B_{\mathbf{n}}$ are certain entire functions). Hence

$$
\Delta_\omega \Psi^{(n,k)} = c\delta_{(n-1-\omega)},
$$

but by the definition of Δ_ω, $\Delta_\omega \Psi^{(n,k)}$ can be a singular hyperfunction (distribution) only at zero. Thus we must have $\omega = n-1$. The proof now goes by induction.

Now we generalize the expression (II.6) to the nonlinear case. To this end we apply Δ_{-1} to both sides of (II.11) for $n=0$. Since $\Delta_{-1} B_r \equiv 0$ (B_r being entire) we get by the Leibniz rule

$$
-\alpha\Delta_{-1}\Psi^{(0)} = \sum_{r=1}^{\infty} B_r * \Delta_{-1}(\Psi^{(0)} * \overset{(r)}{\ldots} * \Psi^{(0)}).
$$

By applying again the Leibniz rule to the convolutions we observe that $\Delta_{-1}\Psi^{(0)}$ satisfies (II.11) for $n = 1$. Regarding $\Psi^{(0)}$ as a fixed function we see that it is a linear homogeneous equation and hence by the uniqueness of solutions we get

$$\Delta_{-1}\Psi^{(0)} = A_{-1}\Psi^{(1)}$$

for some constant $A_{-1} \in \mathbb{C}$. More generally, one proves that

$$\Delta_{-1}\Psi^{(n-1)} = nA_{-1}\Psi^{(n)} \quad \text{for } n \geq 0$$

and hence

$$\Psi^{(n)} = \frac{1}{n!}(\Delta_{-1}/A_{-1})n\Psi_{(0)}.$$

Consequently (recalling the definition of $\dot{\Delta}_{-1}$), we get

$$\Psi(\alpha, u) = \exp(u\dot{\Delta}_{-1}/A_{-1})\Psi^{(0)},$$

which is the desired generalization.

Next we pass to the so-called "bridge equation" which establishes a connection between the alien and the classical derivatives of the general solution $\Psi(\alpha, u)$.

PROPOSITION. *For* $\omega \in -\mathbb{N}$ *we have*

$$\dot{\Delta}_\omega\Psi(\alpha, u) = \mathbf{A}_\omega\Psi(\alpha, u),$$

where

$$\mathbf{A}_\omega = u^\omega A_\omega u\frac{d}{du}$$

with some constants $A_\omega \in \mathbb{C}$.

Proof. As in the reasoning above we apply $\dot{\Delta}_\omega$ to the equation

$$(-\alpha - 1)\Psi = \sum_{r=0}^{\infty} B_r * \Psi * \overset{(r)}{\ldots} * \Psi.$$

We get

$$(-\alpha - 1)\dot{\Delta}_\omega\Psi = \sum_{r=0}^{\infty} B_r * \dot{\Delta}_\omega(\Psi * \ldots * \Psi)$$

and note that $\dot{\Delta}_\omega\Psi$ satisfies a linear homogeneous equation. The same equation is satisfied by $u\frac{d}{du}\Psi$ and hence we get

$$\dot{\Delta}_\omega\Psi = a_\omega(u)u\frac{d}{du}\Psi$$

for a function $a_\omega(u)$ which we compute below. Since

$$\Psi(\alpha, u) = \sum_{n=0}^{\infty} u^n\delta_{(-n)} * \Psi^{(n)}$$

it follows that for $\omega = -k \in -\mathbb{N}$,

$$\Delta_{-k}\Psi(\alpha, u) = \sum_{n=0}^{\infty} u^n\delta_{(k-n)} * \Delta_{-k}\Psi^{(n)} = a_{-k}(u)\sum_{n=1}^{\infty} nu^{n-1}\delta_{(-n)} * \Psi^{(n)}.$$

By equating the coefficients of $\delta_{(k-n)}$ we get

$$\Delta_{-k}\Psi^{(n)} = 0 \quad \text{for } k \geq n$$

and

$$u^n \Delta_{-k}\Psi^{(n)} = a_{-k}(u)(n+k)u^{n-k-1}\Psi^{(n)} \quad \text{for } k < n.$$

Hence $a_{-k}(u) = A_{-k}u^{-k+1}$ for some constant $A_{-k} \in \mathbb{C}$, as desired.

We end this appendix by noting the importance of the sequence $\{A_{-k}\}_{k\in\mathbb{N}}$ thus obtained. It is unique up to a rescaling $A_{-k} \mapsto \lambda^{-k}A_{-k}$ for some complex λ. Next, it is not difficult to note that $\{A_{-k}\}_{k\in\mathbb{N}}$ is invariant (up to rescaling) under the analytic changes (at infinity) of the variable y and of the unknown function ϕ (in a neighbourhood of zero). In other words, the $\{A_{-k}\}_{k\in\mathbb{N}}$ form a system of analytic invariants of the equation (II.7). It turns out that the system is complete in the sense that the equality of the A_{-k}'s is a necessary and sufficient condition for the equations of type (II.7) to be analytically conjugate. Finally, the system is free in the sense that every sequence $\{A_{-k}\}_{k\in\mathbb{N}}$ corresponds to an equation of type (II.7).

2. The case of partial differential equations. The results of Section 17, where multidimensional resurgence effect have been established for solutions to singular Fuchsian PDEs, open way to the expectations that the program outlined above in the case of ordinary differential equations can be carried over to nonlinear singular partial differential equations. Below we present an example in the (model) case of the nonlinear Laplace equation

$$-(\partial^2/\partial y_1^2 + \partial^2/\partial y_2^2)v = |v|^{q-1}v \quad \text{for } q > 1. \tag{II.8}$$

The Laplace operator in the logarithmic variables $y_1 = -\ln x_1$, $y_2 = -\ln x_2$ assumes the form (cf. Section 17)

$$\widetilde{\Delta} = (x_1\partial/\partial x_1)^2 + (x_2\partial/\partial x_2)^2,$$

i.e. becomes an operator with regular singularities along the walls $\{x_1 = 0\}$, $\{x_2 = 0\}$. By adopting the terminology from the ordinary differential equations, the above property means that infinity is an irregular singular point for Δ which is responsible for the fact that harmonic functions are not analytic at infinity. Nevertheless, it follows from the results of Section 17 that they are generalized analytic functions (in 2 variables) and it is hoped that this property will be valid for solutions to nonlinear Laplace equations. As an example consider the radial solution v of (II.8)

$$v(y_1, y_2) = C(y_1^2 + y_2^2)^{-1/(q-1)}$$

for a suitable numerical constant C. We shall prove that the function v in the logarithmic variables $y_1 = -\ln x_1$, $y_2 = -\ln x_2$ is a GAF. More generally, we shall prove this for the functions

$$u_\lambda(x_1, x_2) = ((\ln x_1)^2 + (\ln x_2)^2)^{\lambda-1} \quad \text{for } \lambda \in \mathbb{C} \setminus \mathbb{Z}.$$

To this end we first observe that with a numerical constant A,

$$u_\lambda(x_1, x_2) = A \int_{\mathbb{R}^2} \frac{x^{i\gamma}}{P(i\gamma)^\lambda}\, d\gamma,$$

where $P(z_1, z_2) = z_1^2 + z_2^2$ and $1/P^\lambda(i\gamma)$ is the distribution obtained by analytic continuation in λ of the function $1/P^\lambda(i\gamma)$ for $\lambda < 0$.

Let $\kappa \in C_0^\infty(\mathbb{R}_+)$ and $K'(z_2) = \mathcal{M}\kappa(z_2)$. According to the general theory we are interested in the analytic continuation in z_1 of the partial Cauchy transform

$$\widetilde{\mathcal{C}}'_{z_1}(\kappa u_\lambda)(z_2) = \int\limits_{\operatorname{Re}\theta=0} \frac{K'(z_2-\theta)}{P(z_1,\theta)^\lambda}\,d\theta.$$

The continuation is done by replacing the integral over $\operatorname{Re}\theta = 0$ by an integral over $\operatorname{Re}\theta = r$ and passing with r to $+\infty$. It remains to compute the "residue" terms. However (unlike the case of $\lambda = 1$), the residues of the function

$$\theta \mapsto \frac{1}{P(z_1,\theta)^\lambda} = \frac{1}{(\theta+iz_1)^\lambda(\theta-iz_1)^\lambda}$$

for $\lambda \in \mathbb{C}\setminus\mathbb{Z}$ lie on the half-lines $-iz_1 + \mathbb{R}_+$ and $iz_1 + \mathbb{R}_+$. Thus, when computing the Borel transformation we have to take into account the jumps of $1/P(z_1,\theta)^\lambda$ across these lines. This leads to the expression of u_λ as a GAF (in 2 variables) of the form

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} \frac{x_1^{z_1}x_2^{-iz_1+\theta}}{(\theta+iz_1)^\lambda(\theta-2iz_1)^\lambda}\,d\theta\,dz_1 + \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} \frac{x_1^{z_1}x_2^{iz_1+\theta}}{(\theta-iz_1)^\lambda(\theta-2iz_1)^\lambda}\,d\theta\,dz_1.$$

After rotation $z_1 = (1+i)a/2$ in the first term and $z_1 = (1-i)a/2$ in the second we arrive at the Taylor type expansion

$$\int\limits_{\mathbb{R}_+^2} \frac{x_1^{(1+i)/2}ax_2^{(1-i)a/2+\theta}}{(\theta+(1-i)a/2)^\lambda(\theta+(1-i)a)^\lambda}\,da\,d\theta + \int\limits_{\mathbb{R}_+^2} \frac{x_1^{(1-i)/2}ax_2^{(1+i)a/2+\theta}}{(\theta+(1-i)a/2)^\lambda(\theta+(1+i)a)^\lambda}\,da\,d\theta.$$

Note that the Borel transforms are now supported by $\mathbb{R}_+^2$-type sets and not by half-lines as in the case of $\lambda = 1$. Also note the 2-dimensional resurgence effect (in variables a and θ).

References

[A] E. Andronikoff, *Intégrales de Nilsson et faisceaux constructibles*, Bull. Soc. Math. France 120 (1992), 51–85.

[B] J. M. Bony, *Second microlocalization and propagation of singularities for semilinear hyperbolic equations*, Taniguchi Symp. H.E.R.T. Kataka, 1984, 11–49.

[Br] H. Bremermann, *Distributions*, *Complex Variables and Fourier Transform*, Addison-Wesley, 1965.

[C] J. B. Conway, *Functions of One Complex Variable*, Grad. Texts in Math. 11, Springer, 1973.

[E1] J. Ecalle, *Les fonctions résurgentes*, Publ. Math. Université de Paris-Sud, several volumes.

[E2] —, *Singularités irrégulières et résurgence multiple*, in: Cinq applications des fonctions résurgentes, preprint 84 T 62, Orsay 2-42.

[E3] —, *Finitude des cycle-limites et accéléro-sommation de l'application de retour*, preprint 90–36, Université de Paris-Sud.

[GS] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Academic Press, New York, 1964.

[Ka] A. Kaneko, *Introduction to Hyperfunctions*, Math. Appl., Kluwer, Dordrecht, 1988.

[Ko] H. Komatsu, *An introduction to the theory of hyperfunctions*, in: Lecture Notes in Math. 287, Springer, 1973, 1–43.

[L] J. Leray, *Le calcul différentiel et intégral sur une variété analytique complexe*, Bull. Soc. Math. France 87 (1959), 81–180.

[Ły1] G. Łysik, *On the structure of Mellin distributions*, Ann. Polon. Math. 51 (1990), 219–228.

[Ły2] —, *The Taylor transformation of analytic functionals with non-bounded carrier*, Studia Math. 108 (1994), 159–176.

[Maj] H. Majima, *Resurgent equations and Stokes multipliers for the generalized hypergeometric differential equation*, preprint, Ochanomizu University, 1990.

[M] B. Malgrange, *Introduction aux travaux de J. Ecalle*, prépublication de l'Institut de Fourier, Université de Grénoble, 20 (1984).

[Mo] M. Morimoto, *Analytic functionals with non-compact carrier*, Tokyo J. Math. 1 (1978), 77–103.

[N] N. Nilsson, *Some growth and ramification properties of certain multiple integrals*, Ark. Mat. 5 (1965), 463–476.

[Pl] M. Pliś, *The Mellin analytic functionals and the Laplace–Beltrami operator on the Minkowski half-plane*, Studia Math. 99 (1991), 263–276.

[PCN] E. Pham, B. Candelpergher et C. Nosmos, *Visite aux sources*, prépublication de l'Université de Nice, 1989.

[Schl] H. Schlichtkrull, *Hyperfunctions and Harmonic Analysis on Symmetric Spaces*, Progr. in Math. 49, Birkhäuser, Boston, 1984.

[Sl] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, 1960.

[Sz] Z. Szmydt, *Fourier Transformation and Linear Differential Equations*, PWN–Polish Sci. Publ., Warszawa, and Reidel, Dordrecht, 1977.

[SZ] Z. Szmydt and B. Ziemian, *The Mellin Transformation and Fuchsian Type PDEs*, Math. Appl., Kluwer Acad. Publ., 1992.

[T] E. C. Titchmarsh, *The Theory of Functions*, Oxford Univ. Press, London, 1949.

[Tou] J. C. Tougeron, *Gevrey expansions and applications*, preprint, University of Toronto, 1991.

[VDZ] V. S. Vladimirov, Yu. N. Drozhzhinov and B. I. Zav'yalov, *Multidimensional Tauberian Theorems for Generalized Functions*, Nauka, Moscow, 1986 (in Russian).

[We] A. Weinstein, *The order and symbol of a distribution*, Trans. Amer. Math. Soc. 241 (1978), 1–54.

[Wi] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, N.J., 1946.

[Zie1] B. Ziemian, *Taylor formula for distributions*, Dissertationes Math. 264 (1988).

[Zie2] —, *Elliptic corner operators in spaces with continuous radial asymptotics II*, in: Partial Differential Equations, Banach Center Publ. 27, part 2, Inst. Math., Polish Acad. Sci., Warszawa, 1992, 555–580.

[Zie3] —, *Continuous radial asymptotics for solutions to elliptic Fuchsian equations in 2-dimensions*, in: Proc. Sympos. Microlocal Analysis and its Applications, Publ. RIMS, Kyoto Kokyuroku 750, 1991.

[ZieK] B. Ziemian and H. Kołakowski, *Second microlocalization and the Mellin transformation*, Publ. RIMS Kyoto Univ. 26 (1990), 785–801.

Symbol index

Subject index