POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCIV

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Generalization of the concept of variety and quasivariety to partial algebras through category theory

WARSZAWA 1983

PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-02220-5 ISSN 0012-3862

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

CONTENTS

Introduction

Our aim is to import "structure" (i.e. theory and methodology) into the area of partial algebras from that of total algebras. In our approach category theoretical abstraction serves to connect the two areas in a nontrivial manner.

Since previous experience in this endeavour shows that this importing job is somewhat elusive (it is full of deceptive appearences, which easily cause one to make premature decisions), we took every cost to avoid premature decision and ad-hocness. We did not want to develop any prejudices about the shape of the language (generalized identities) to be applied in partial algebras. We wanted the "concept of generalized identities" to pop up as a result of an unbiased investigation.

A large portion of our work has as its purpose an answer to the question: "what is the adequate language to think about partial algebras in". Therefore we wanted to have iff-style theorems on the adequateness of languages to some generalized algebraic purposes. To do this, "abstract model theory" (cf. Barwise [9]) seems to be the proper framework.

In our abstract-model-theoretical investigations (Section 2), the question "why exactly this concept of generalized identities" plays a central role. This is important in our field of applications, because in the theory of partial algebras there is still no satisfactory solution to the problem of finding the adequate generalization of the concept of identity (and related things): Various *ad hoc* concepts of generalized identity are running riot just as programming languages do in the Babel of computer science.

In order to avoid ad-hocness, we also tried to avoid unjustified decisions in the choice of the concept of "primitive class", i.e. in the choice of the closure operations H and S: using category theoretical terminology, we did not even require the pair $\langle H, S\rangle$ to be a factorization system (in similar investigations Banashewski–Herrlich [8] did require this). (Neither strongepi-strongmono nor isomorphism-anymono are factorization systems, in general.)

Keeping H and S to be variables, we stated a unique abstract theorem for HSP and, in applications, H and S can be rather freely and independently chosen. The extremal cases for the choice of H are SP (= Is SP) and $H_r SP$ (where H_r is the weakest manageable concept of homomorphic image). Between these extremes there is a continuity of possible choices of

H. In this way a unification of the theory of varieties and quasivarieties is obtained (as a by-product for the theory of total algebras).

In Banashewski–Herrlich [8], ***SP*** and ***HSP*** are separately treated (as a consequence of the restriction to factorization systems).

In our opinion, the following things strongly belong to the Birkhoff-style variety cult:

the identity concept (what can be said (E) and what does it mean ($\models$));

characterization of axiomatizable classes (the primitive classes $\boldsymbol{HSP}\mathscr{A}$ and the varieties $\mathrm{Md}\,E(\mathscr{A})$ coincide); HSP is a closure operator;

word-algebra concept, free extension of the word-algebra towards a class $\mathscr{A}$ of algebras or "the $\mathscr{A}$-free algebra" (factorization of the word-algebra by a specially obtained "fully invariant" congruence, which represents the identities);

the possibility to read the identities valid in $\mathscr{A}$ from the free extension of the word-algebra towards $\mathscr{A}$ (the word-algebra consists of "words", which are building blocks of the "texts" which are called identities, and which might hold in $\mathscr{A}$);

the calculus: an inference system (set of equation rewriting rules) designed specially for algebraic purposes, to obtain from any set of identities any identity which is a consequence of this set by the sole use of paper and pencil. These rules can be considered as the rules of the game of thinking in identities about algebras.

Our aim in this paper is to import all five of these into partial-algebra-theory. Because in the importation of the first four there is a grave danger of making unjustified decisions, we first investigated these concepts in an abstract category theoretical framework. (A similar investigation was carried independently by Banashewski–Herrlich [8] but with different applications.) The aim of this investigation is to destill the pure essence of these four items (i.e. of Birkhoff-style theory of varieties). The destilled and purified theory of varieties is then applied to the category of partial algebras (and also to that of models for fun) while anti-ad-hocness vigilance is not relaxed.

In Section 1 abstract category theoretical versions of the first four items (of the five listed above) are identified and investigated.

In Section 2 the categories are still abstract; a forgetful functor is attached to them and the concept of a language is introduced in abstract model theoretic style. The generalized identity concept is obtained as a result of investigations. The results obtained here (about the first four above-mentioned things) are the most important of all in this article: they are still completely general and at the same time they are specific enough to be applied mechanically in Section 3 to the categories of partial algebras, models and total algebras for any admissible choices of H and S. (But they have a much wider range of applicability, e.g. they can be applied to the category of structures having simultaneously relations, partial operations, and operations.)

In Section 3 the results of Section 2 are applied to (models and) partial algebras and to many different choices of the concepts of H and S. (As a by-product, a unification of the theory of varieties and quasivarieties is obtained.) The case of M in Theorem 2 and Lemma 3, both in Section 3, answer Problems 1 and 2 in Malcev [27], p. 328.

In Section 4 calculus is given and its completeness is proved.

In Section 5 there are some examples.

In Section 6 model theoretic consequences are obtained.

Relations to other Category Theoretical investigations of Universal Algebra: Monads (cf. MacLane [26]), "Arbib–Manes Functorial Algebras", and algebraic theories were compared with and related to the present approach in Németi–Sain [32], Section 6. Here we note that the category of all partial algebras of a fixed similarity type is not monadic. It is not even algebraic in the sense of Linton, or Herrlich–Ringel [20], or Herrlich–Strecker [21], Def. 32.1. The same holds for relational structures and algebraic systems in the sense of Malcev [27], cf. Sain [38]. Koubek and Adámek in Prague have recently generalized Arbib–Manes functorial algebras to cover partial algebras as well. Their approach would be a reasonable alternative to the present one. See Németi–Sain [32], Prop. 15. Connections with other approaches and detailed historical remarks can be found in Németi–Sain [32] between Definitions 4 and 5, at the end of Section 3, and at the end of Section 4.

A refinement of the present approach to go beyond quasivarieties can be found in Andréka–Németi [3], [4] and in Section 5 of Németi–Sain [32]. Andréka–Németi [5], [6] refine the present approach to treat all first order formulas. Heterogeneous algebras have been deeply investigated by similar means in Matthiessen [28].

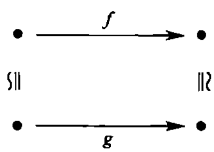
In the present exposition *only elementary category theory* is used; functors, natural transformations, adjointness, etc. are *not* used.

This work is a somewhat generalized and rearranged version of materials which we first formulated in theses submitted to the Eötvös Lóránd University in May 1974 (cf. also Pasztor [34], Sain [37]).

§ 1. The purely category theoretical version of Birkhoff's theorem

Throughout we work in an arbitrary but fixed category. Mor and Ob denote the classes of its morphisms and objects, respectively.

A class $\mathscr{H} \subseteq \mathrm{Mor}$ of arrows is *abstract* if it is closed w.r.t. isomorphisms, i.e. $f \in \mathscr{H}$ and commutativity of

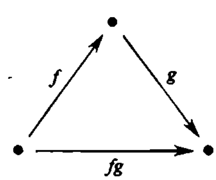


imply $g \in \mathscr{H}$.

A class of arrows is always supposed to be abstract.

Is, Epi, Mono denote the classes of all isomorphisms, all epimorphisms, and all monomorphisms, respectively.

Composition is written in the order



Let $\mathscr{H}$ and $\mathscr{G}$ be two classes of arrows. Then we define

$$\mathscr{H} \cdot \mathscr{G} \overset{\mathrm{d}}{=} \{hg\colon\ h \in \mathscr{H}, g \in \mathscr{G}\}.$$

$\xrightarrow{\mathscr{H}}$ denotes that the arrow is an element of $\mathscr{H}$. Thus, $\xrightarrow{f\ \mathrm{Epi}}$ is the same as $\overset{f}{\twoheadrightarrow}$.

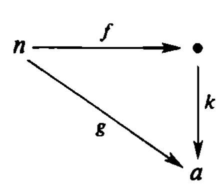
For an arrow $a \xrightarrow{f} b$ we write $\mathrm{so}\, f = a$ and $\mathrm{ta}\, f = b$ (for domain and codomain, respectively).

DEFINITION 1. Let $\mathscr{H} \subseteq \mathrm{Mor}$, $n \in \mathrm{Ob}$, and $\mathscr{A} \subseteq \mathrm{Ob}$. The *free $\mathscr{H}$-extension of n towards $\mathscr{A}$*, in symbols $\mathrm{fr}_n^{\mathscr{H}}\, \mathscr{A}$, is defined as follows

$$\mathrm{fr}_n^{\mathscr{H}}\, \mathscr{A} = f$$

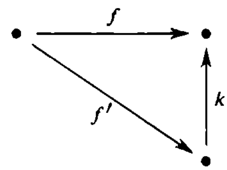
if (1) and (2) below hold:

(1) $f \in \mathscr{H}$ and for any $n \xrightarrow{g} a \in \mathscr{A}$ there exists an arrow k making the diagram



commutative (cf. Herrlich–Strecker [21], Def. 37.8);

(2) For any f' satisfying condition (1) there exists a unique k making the diagram



commutative.

We also introduce the abbreviation:

$$\mathrm{Fr}_n^{\mathcal{H}} \mathcal{A} \stackrel{\mathrm{d}}{=} \mathrm{ta}\,(\mathrm{fr}_n^{\mathcal{H}} \mathcal{A}). \quad \blacksquare$$

If f satisfies condition (1) of Definition 1, then we say that f is *free over* $\mathcal{A}$. This is called *f is $\mathcal{A}$-extendable* in Herrlich–Strecker [21], Def. 37.8. and *f is valid in $\mathcal{A}$*, in symbols $\mathcal{A} \models f$, in Németi–Sain [32], Def. 2–4.

Thus, intuitively speaking, the free extension $\mathrm{fr}_n^{\mathcal{H}} \mathcal{A}$ is the "largest" $\mathcal{H}$-arrow free over $\mathcal{A}$. Note that $\mathrm{fr}_n^{\mathcal{H}} \mathcal{A}$ is unique (up to isomorphism, of course).

The above definition is a generalization of reflection (cf. Mitchell [29]) or universal arrow (cf. Mac Lane [26]) and serves the purpose of representing the universal algebraic concept of *free algebra* over a class $\mathcal{A}$ of algebras in the sense of Henkin–Monk–Tarski [19], Def. 0.4.19 (where it is denoted by $\mathrm{Fr}_\alpha \mathcal{A}$). Note that we do *not* require $\mathrm{Fr}_n^{\mathcal{H}} \mathcal{A}$ to be an element of $\mathcal{A}$. (As opposed to $\mathrm{fr}_n^{\mathcal{H}} \mathcal{A}$, reflections correspond to the "Grätzer [18]-version" of free algebras.)

NOTATION. Let $N \subseteq \mathrm{Mor}$, $\mathcal{A} \subseteq \mathrm{Ob}$ be arbitrary. Now N can enlarge $\mathcal{A}$ in two ways:

$$\underleftarrow{N}\mathcal{A} \stackrel{\mathrm{d}}{=} \{\mathrm{ta}\, f \colon \xleftarrow{\;f\;} \cdot \in \mathcal{A}\},$$
$$\underrightarrow{N}\mathcal{A} \stackrel{\mathrm{d}}{=} \{\mathrm{so}\, f \colon \xrightarrow{\;f\;} \cdot \in \mathcal{A}\}.$$

$\underleftarrow{N}\mathcal{A}$ may be called the class of all N-quotients or N-images of elements of $\mathcal{A}$ while $\underrightarrow{N}\mathcal{A}$ is the class of all N-subobjects of elements of $\mathcal{A}$. Cf. Herrlich–Strecker [21], p. 165. ex. 23C, p. 235. ex. 31B, Prop. 36. 11!

CONVENTION. We use the letters H, S and their indexed versions H_i, S_i to denote:

$$H_i \mathcal{A} \stackrel{\mathrm{d}}{=} \underleftarrow{H}_i \mathcal{A} \quad \text{and} \quad S_i \mathcal{A} \stackrel{\mathrm{d}}{=} \underrightarrow{S}_i \mathcal{A}.$$

This convention is motivated by the generally accepted notations for the class of all subalgebras (S) and the class of all homomorphic images (H), cf. e.g. Grätzer [18]. Note that if $\mathrm{Is} \subseteq N$, then $\underleftarrow{N}\mathcal{A} \supseteq \mathcal{A} \subseteq \underrightarrow{N}\mathcal{A}$.

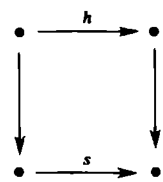
The class $\boldsymbol{P}\mathcal{A}$ is defined to be the class of all direct products (of sets) of elements of $\mathcal{A}$. Note that $\boldsymbol{P}\mathcal{A} \supseteq \mathcal{A}$, in general $\boldsymbol{P}$ is a closure operation, and the terminal object is in $\boldsymbol{P}\mathcal{A}$ (if it exists), cf. Herrlich–Strecker [21], ex. 31B.

Definition 2. A pair $\langle \mathcal{H}, S \rangle$ is a *factorization system* if (1)–(3) below hold:

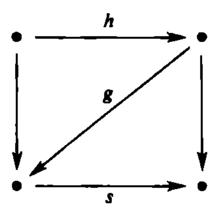
(1) $\mathcal{H} \cdot S = \mathrm{Mor}$;

(2) $\mathcal{H} \cap S = \mathrm{Is}$;

(3) For any diagram



where $h \in \mathcal{H}$, $s \in S$ there is a unique g making the diagram



commutative.

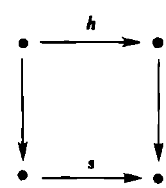
The unique arrow g is called the *diagonal fill-in.* ■

Cf. Németi–Sain [32], Def. 9, and Burmeister–John–Pasztor [14].

Recall the following properties of factorization systems (from Németi–Sain [32]):

(1) The factorizations are unique. Thus we can talk of the $\mathcal{H}$*-part* (or S-part) of any arrow.

(2) S can be defined from $\mathcal{H}$ (or vice versa) as: S is the class of all arrows s for which any commutative square



with $h \in \mathcal{H}$ has a diagonal fill-in as required in condition (3) of Definition 2.

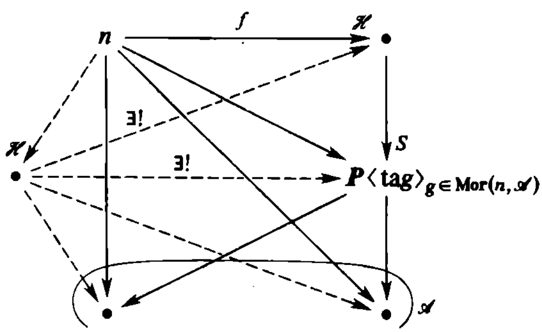
(3) $\mathcal{H} \cdot \mathcal{H} = \mathcal{H}$ and $S \cdot S = S$.

More on factorization systems can be found in Németi–Sain [32] between Definition 9 and Proposition 4. Note that the above Definition 2 is *different* from Herrlich–Strecker [21], Def. 33.1 ($\mathscr{H} \nsubseteq$ Epi, $S \nsubseteq$ Mono are allowed here).

PROPOSITION 1. *Let our category be $\mathscr{H}$-co-well-powered, have direct products, and let $\langle \mathscr{H}, S \rangle$ be a factorization system. Then the free extension $\mathrm{fr}_n^{\mathscr{H}} \mathscr{A}$ exists and $\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} \in SP\mathscr{A}$ for arbitrary $n \in \mathrm{Ob}$ and $\mathscr{A} \subseteq \mathrm{Ob}$.*

Proof. We give the proof only for the case when $\mathscr{A}$ is a set. If it is a proper class, then, by $\mathscr{H}$-co-well-poweredness and $\mathscr{H} \cdot S = \mathrm{Mor}$, the proof can be generalized (following the usual pattern).

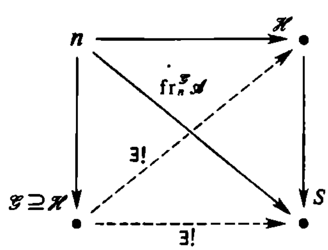
Consider the following diagram:



to see that $f = \mathrm{fr}_n^{\mathscr{H}} \mathscr{A}$. To check uniqueness, we can use the fact that, by Andréka–Németi [7], the existence of products and $\mathscr{H}$-co-well-poweredness imply $\mathscr{H} \subseteq$ Epi. ∎

Remark. Let $\langle \mathscr{H}, S \rangle$ and $\langle \mathscr{G}, M \rangle$ be factorization systems and let $\mathscr{H} \subseteq \mathscr{G}$. Then $\mathrm{fr}_n^{\mathscr{H}} \mathscr{A}$ is the $\mathscr{H}$-part of $\mathrm{fr}_n^{\mathscr{G}} \mathscr{A}$ if the latter exists.

Proof. Consider the diagram



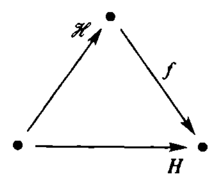
∎

From now on $S, \mathscr{H} \subseteq$ Mor, our category is $\mathscr{H}$-co-well-powered and has direct products. Furthermore, $\langle \mathscr{H}, S \rangle$ is always a factorization system; H is always a subset of Mor and $\mathscr{A} \subseteq$ Ob. About the above and following cf. Herrlich–Strecker [21], ex. 37G.

THEOREM 1. *Let* $\mathcal{N} \subseteq \mathrm{Ob}$ *be such that*

$$H\mathcal{N} = \mathrm{Ob}.$$

Suppose further that $H \cdot H = H$, *and that in any diagram*



commutativity implies $f \in H$. *Then*

$$HSP\mathcal{A} = H\{\mathrm{Fr}_n^{\mathscr{H}}\mathcal{A} : n \in \mathcal{N}\}.$$

Remark. We shall see later that this is a generalization of Birkhoff's theorem (on equational classes): $\mathcal{N}$ can be chosen to correspond to the absolutely free algebras (word-algebras) and $\mathrm{Fr}_n^{\mathscr{H}}\mathcal{A}$ to correspond to the free algebras ower $\mathcal{A}$, that is, to objects representing the identities valid in $\mathcal{A}$.

We can choose $H = \mathrm{Is}$ and thus the above theorem is also a generalization of the "quasivariety theorem", which means that here we have a unification of the variety and quasivariety theorems. This unified treatment of varieties and quasivarieties will be maintained throughout the paper.

Since H can be chosen freely, we also have a continuity between quasivariety and variety. All these will be seen more clearly later. ∎

To prove the theorem we need some lemmas which, however, are interesting in themselves. Therefore we state them separately.

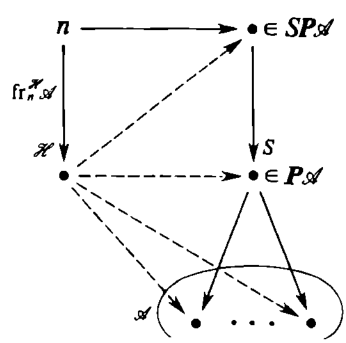
The following lemma holds without the hypotheses we imposed on the categories under investigation. The only one we need is that $\langle \mathscr{H}, S \rangle$ is a factorization system.

LEMMA 1. *We have*

$$\mathrm{fr}_n^{\mathscr{H}}\mathcal{A} = \mathrm{fr}_n^{\mathscr{H}} SP\mathcal{A}$$

if one of them exists.

Proof. Consider the diagram



to see that if an arrow is free over $\mathscr{A}$, then it is also free over $SP\mathscr{A}$. The converse is obvious since $SP\mathscr{A} \supseteq \mathscr{A}$. Therefore the same arrows are free over $\mathscr{A}$ and $SP\mathscr{A}$. ∎

LEMMA 2. *We have*

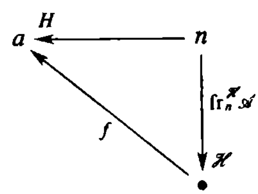
$$a \in SP\mathscr{A} \quad \textit{iff} \quad \mathrm{fr}_a^{\mathscr{H}} \mathscr{A} = 1_a.$$

Proof. (a) The "if" part follows from Prooosition 1.

(b) The "only if" part: Let f denote $\mathrm{fr}_a^{\mathscr{H}} \mathscr{A}$. $f = \mathrm{fr}_a^{\mathscr{H}}\, SP\mathscr{A}$ follows from Lemma 1. Since $a \in SP\mathscr{A}$, for some g, $fg = 1_a$. Therefore $f(gf) = f$, and by the uniqueness condition in the definition of $\mathrm{fr}_n^{\mathscr{H}} \mathscr{A}$ we have $gf = 1$. ∎

Proof of Theorem 1. (1) $HSP\mathscr{A} \supseteq H\{\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} : n \in \mathscr{N}\}$ by Proposition 1.

(2) By hypothesis $H \cdot H = H$; it is enough to see that $SP\mathscr{A} \subseteq H\{\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} : n \in \mathscr{N}\}$. Let $a \in SP\mathscr{A}$. Now there is an arrow $a \overset{H}{\leftarrow} n \in \mathscr{N}$, by the hypothesis: $H\mathscr{N} = \mathrm{Ob}$. Since $\mathrm{fr}_n^{\mathscr{H}} \mathscr{A}$ is free over $SP\mathscr{A}$, there is an arrow:



and by the hypothesis of the theorem, $f \in H$. ∎

COROLLARY 1. *We have*

$$SP\mathscr{A} = \{\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} : n \in \mathrm{Ob}\}.$$

Proof. By choosing $H = \mathrm{Is}$, the conditions of Theorem 1 are satisfied, see Andréka–Németi [7]. ∎

Note that in universal algebra this corollary implies Malcev's theorem on free classes cf. Malcev [27], p. 28.

To be able to appreciate Theorem 1 we want to see whether HSP is a closure operator or not. Only if it is a closure operator, can we regard the theorem as a characterization of the axiomatizable classes of some language.

Remark. SP is a closure operator.

Proof. The proof is a simplified version of the proof of the following theorem. ∎

Let $E \subseteq \mathrm{Epi}$ be arbitrary. Following the notations of Mitchell [29], p. 136, the class of *E-projective* objects is denoted by $\mathrm{Pj}(E)$, and for any $\mathscr{P} \subseteq \mathrm{Ob}$, the class of all epies f for which $\mathscr{P} \subseteq \mathrm{Pj}(\{f\})$ is denoted by $\mathrm{Ep}(\mathscr{P})$, cf. Herrlich–Strecker [21], Def. 31.6. The class E is said to be a *projective class* if $\mathrm{Ep}(\mathrm{Pj}(E)) = E$ and our category has enough E-projectives. (That is, $\underline{E}\mathrm{Pj}(E) = \mathrm{Ob}$.)

The following theorem implies that if H is a projective class, then HSP is a closure operation.

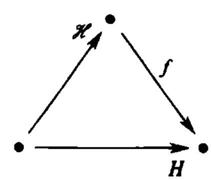
We say that a class H of arrows is *closed* w.r.t. $\boldsymbol{P}$ if the direct product of arrows of H is again in H.

THEOREM 2. *Suppose that*

(A) *Our category has enough H-projectives*; $H \supseteq \mathrm{Is}$;

(B) *H is closed* w.r.t. $\boldsymbol{P}$;

(C) *If*



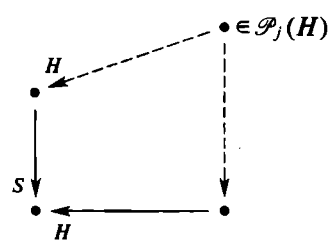
commutes, then $f \in H$.

Then HSP is a closure operator.

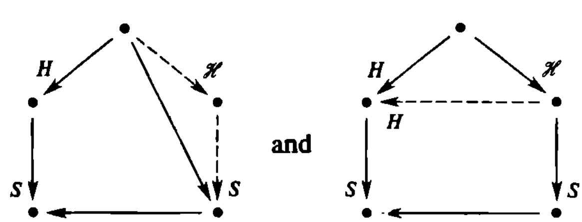
Proof. (1) Obviously $HSP\mathscr{A} \supseteq \mathscr{A}$ and $\mathscr{B} \subseteq \mathscr{A}$ implies $HSP\mathscr{B} \subseteq HSP\mathscr{A}$.

(2) We have to show that $HSPHSP\mathscr{A} = HSP\mathscr{A}$.

(a) To see that $SH\mathscr{A} \subseteq HS\mathscr{A}$, consider the diagrams:

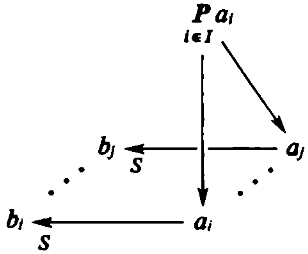


by condition (A), and

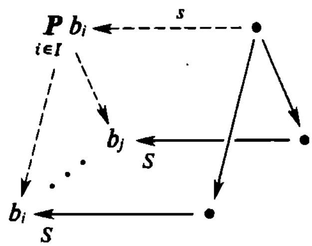


by Definition 2, and by condition (C).

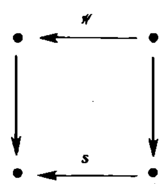
(b) $\boldsymbol{PS}\mathscr{A} \subseteq \boldsymbol{SP}\mathscr{A}$ because to any diagram



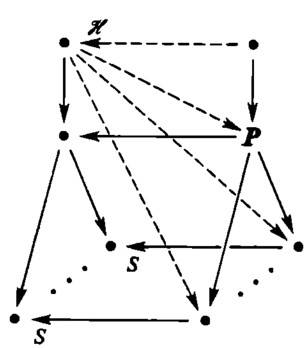
we can construct:



and $s \in S$, since for any square:



we have



And this is enough, by the remark following Definition 2.

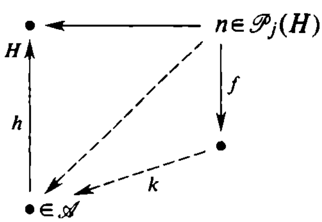
(c) $\boldsymbol{PH}\mathcal{A} \subseteq \boldsymbol{HP}\mathcal{A}$ because H is closed w.r.t. $\boldsymbol{P}$. ∎

Note that if H is a projective class, then the conditions of the theorem hold since any projective class is easily seen to be closed w.r.t. $\boldsymbol{P}$.

THEOREM 3. *For any H-projective object n,*

$$\mathrm{fr}_n^{\mathcal{K}}\,\mathcal{A} = \mathrm{fr}_n^{\mathcal{K}}\,HSP\mathcal{A}.$$

Proof. By Lemma 1 $\mathrm{fr}_n^{\mathcal{K}}\,\mathcal{A} = \mathrm{fr}_n^{\mathcal{K}}\,SP\mathcal{A}$. Now, if $n \xrightarrow{f}$ is free over $\mathcal{A}$, then it is also free over $H\mathcal{A}$ by the diagram:



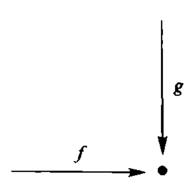
where kh completes the desired triangle. ∎

Some important classes of epies and monos. Now we define some important choices for the classes H and S. The results obtained so far will be applied to these classes which play an important role in the categories of models, partial algebras, and total algebras. (Especially, in the theory of partial algebras a greater differentiation in the definitions of special epi-classes is required than is usual in category theory.) We shall define these epi-classes in an abstract, purely category-theoretical manner and in the applications we shall state their "representation theorems". To prove the main results these abstract category theoretical definitions could be skipped and the representation theorems could be considered as definitions. We included them for sake of generality (and because of our conviction that they are important).

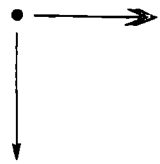
Is, H_w, and S_w stand for the classes of all isomorphisms, epimorphisms, and monomorphisms, respectively.

$\overset{f}{\twoheadrightarrow}$ denotes that $f \in \mathrm{Epi}$, and $\overset{f}{\rightarrowtail}$ denotes that $f \in \mathrm{Mono}$.

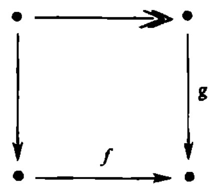
Recall from Pasztor [35], [36] the definition of the class H_r of "relative epies": $f \in H_r$ iff $f \in \mathrm{Epi}$ and for every $\xrightarrow{g}$ ta$f$ the diagram



has a lower bound



i.e. there is a commutative diagram



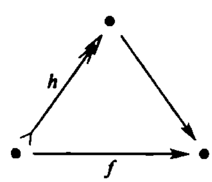
The classes H_s, S_s, S_r are defined by stipulating that the pairs

$$\langle H_s, S_w\rangle, \quad \langle H_w, S_s\rangle, \quad \langle H_r, S_r\rangle$$

be factorization systems.

In literature H_s and S_s are called *strong epi* and *strong mono* (practically they concide with "regular") cf. Herrlich–Strecker [21], p. 265, ex. 34K p. 103, Def. 16.13, Pasztor [35], [36], and Matthiessen [28].

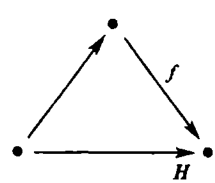
The class H_c of "closed epies" was defined by Pasztor as: $f \in H_c$ iff $f \in \mathrm{Epi}$ and



implies $h \in \mathrm{Is}$. (Cf. Burmeister–John–Pasztor [14], Pasztor [36].)

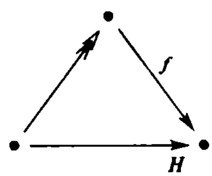
Note that $\mathrm{Is} \subseteq H_c \subseteq H_s \subseteq H_r \subseteq H_w \subseteq \mathrm{Mor}$.

PROPOSITION 2. *For any* $H \in \{H_s, H_r, H_w\}$ *we have* $H \cdot H = H$ *and*



implies $f \in H$.

For any $H \in \{H_c, \text{Is}\}$,



implies $f \in H$.

Proof. See Pasztor [35], [36], Burmeister–John–Pasztor [14]. ■

For more on H_c see Burmeister–John–Pasztor [14], Pasztor [36]. About H_r see Pasztor [36].

Note that under very mild hypotheses the following can be proved (Pasztor [35], [36]): $H_c \cdot H_c = H_c$ and every $H \in \{\text{Is}, H_c, H_s, H_r, H_w\}$ is preserved w.r.t. $\boldsymbol{P}$. Thus under very mild hypotheses, all the theorems stated so far hold for arbitrary $\mathscr{H}$, $H \in \{\text{Is}, H_c, H_s, H_r, H_w\}$. (Of course, the choice of S is determined by the choice of $\mathscr{H}$. The reason for omitting $\langle \text{Mor}, \text{Is} \rangle$ is that Mor-co-well-poweredness does not hold in many categories.)

Remark (Pasztor [35], [36], Burmeister–John–Pasztor [14]). In the categories of partial algebras, models, and total algebras all the pairs

$$\langle \text{Mor}, \text{Is} \rangle, \quad \langle H_w, S_s \rangle, \quad \langle H_r, S_r \rangle, \quad \langle H_s, S_w \rangle, \quad \langle \text{Is}, \text{Mor} \rangle$$

are factorization systems; and in the category of partial algebras there is no S for which $\langle H_c, S \rangle$ would be a factorization system. For any $H \in \{\text{Is}, H_c, H_s, H_r\}$ these categories have enough H-projectives; H_r, H_s are projective classes. (In contrast, these categories do not have "projectives in general", that is, H_w-projectives.) ■

More motivation to these ideas can be found in Sain [37], Pasztor [34] and in Gergely–Németi–Pasztor [17], where we published Definition 1, Proposition 1, Theorems 1–3 in a more detailed but less general form.

Remark. The categories to be considered as applications are regular in the sense of Matthiessen [28]. Though they are regular, the study of regular epies is not enough, cf. Pasztor [36], Burmeister–John–Pasztor [14] etc. (E.g. H_c is stronger than regular, H_r is weaker than regular.) We note that our concrete categories are *not algebraic* in sense of Herrlich–Strecker [21], Def. 32.1!! ■

§ 2. Category theoretical study of generalized identities

Throughout this section the category $\mathscr{C}$ under investigation together with the classes $\mathscr{H}$, S, and H of morphisms are supposed to satisfy the hypotheses of Theorems 1–3.

In this section we want to find a language $\mathscr{C}$ in which one can talk about the elements of Ob (i.e. the models of the language are to be the objects of the category) and we want to choose a sublanguage $E \subseteq \mathscr{L}$ such that the E-axiomatizable classes coincide with the HSP-closed classes. That is, the expressive power of the language E should be such that the most accurate description of $\mathscr{A}$ in E is the description of $HSP\mathscr{A}$.

The first language $\mathscr{L}$ should be very general, to provide sufficient freedom in the search for E, which is the real aim of this section.

Moreover, we need a "word-algebra-like" object $\mathfrak{M}$ together with its $\mathscr{A}$-relativized object $\mathfrak{W}_{\mathscr{A}}$ such that the elements of E valid in $\mathscr{A}$ could be "read-from" $\mathfrak{W}_{\mathscr{A}}$. (This "reading-from" $\mathfrak{W}_{\mathscr{A}}$ should be much more concrete than "finding-out" which equations are valid in $\mathfrak{W}_{\mathscr{A}}$; somehow the valid equations should be elements of the natural map from $\mathfrak{W}$ into $\mathfrak{W}_{\mathscr{A}}$.)

The generalized model-theoretic language concept (cf. Németi–Sain [33], Barwise [9] or Andréka–Gergely–Németi [1]) is a triple $\langle L, M, \models \rangle$ where $\models \subseteq M \times L$. L is the class of formulas, M is the class of models, and $\models$ is the relation of validity.

Here we consider languages of the form $\langle L, \mathrm{Ob}, \models \rangle$ and subclasses $E \subseteq L$. If $T \subseteq L$, then the class of models of T is denoted by $\mathrm{Md}(T) = \mathrm{Md}\, T$. For any $\mathscr{A} \subseteq \mathrm{Ob}$, the class of formulas (elements of L) valid in $\mathscr{A}$ is $\mathrm{Th}(\mathscr{A})$; the class of E-formulas (elements of E) valid in $\mathscr{A}$ is denoted by $E(\mathscr{A}) \stackrel{\mathrm{d}}{=} \mathrm{Th}(\mathscr{A}) \cap E$.

We are interested in the relationships between the closure operators $\mathrm{Md}\, E$ and HSP on the class Ob.

Remark. There is a certain amount of model theoretic intuition behind the present section. Neither Section 1 nor Sections 3–5 need any model theoretic intuition from the reader. Section 2 is a "bridge" between Section 1 and Sections 3–5. If the reader dislikes the flavour of model theory, then the question arises: "Is it possible to form this bridge without model theoretic arguments?". The answer is yes; a version of the present section is elaborated in Németi–Sain [32] within the frames of "pure Category Theory". Therefore the reader who finds this section hard to read is invited skip it, go straight to section 3, and fill in the missing details from Németi–Sain [32]. ■

We shall make a restriction on the language to the effect that the validity $\models$ be defined via the category $\mathscr{S}$ of sets and a forgetful functor U. Now we fix $\mathscr{S}$.

The objects of the category $\mathscr{S}$ are all the sets together with an additional object: the class $\mathrm{Ord} \overset{d}{=} V$ of all ordinals. The morphisms are all the mappings.

In the language L, the elements of V will be the variable symbols. (We chose $V = \mathrm{Ord}$ in order not to restrict the number of variables used in L(1).) The inclusion of Ord into $\mathscr{S}$ is not essential: it serves only convenience purposes: In order to make the exposition shorter, in certain applications we shall choose the language $L = L_{\infty,\omega}$ and this ∞ requires Ord (cf. Barwise [10], p. 9).

Anyway, the use of Ord and $L_{\infty,\omega}$ is only technical; they are used to obtain the final results which are formulated in $L_{\omega,\omega}$.

From now on, U is a "forgetful functor" on $\mathscr{C}$, that is $U: \mathscr{C} \to \mathscr{S}$ such that U preserves $\boldsymbol{P}$ (direct products) and for any $a, b \in \mathrm{Ob}\,\mathscr{C}$, U is one-one on $\mathrm{Mor}(a, b)$.

We shall use the shorthands: if $f \in \mathrm{Mor}$, $a \in \mathrm{Ob}$, then $f' \overset{d}{=} U(f)$ and $a' \overset{d}{=} U(a)$.

Now we make some restriction on the language L and validity $\models$. The basic idea is that the relation $\models$ should be defined by valuations of the variables V into the "universes of objects".

We shall suppose that there is a binary relation between L and subsets of V which is called "is defined on". (This relation is associated to $\models$ in a certain way.) I.e. for any $\varphi \in L$ and $\overset{r}{\rightarrowtail} V$ it is supposed to be meaningful to say that "φ is defined on r". The intuitive meaning of the latter is: "the free variables of φ are contained in the subset r of V".

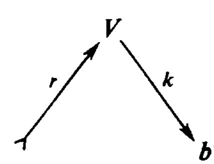
Further, we suppose that associated to $\models$ there is a ternary relation between Ob, L, and $\mathrm{Mor}\,\mathscr{S}$ denoted by "A" $\cdot \models \cdot [\cdot]$". I.e. for $b \in \mathrm{Ob}$, $\varphi \in L$, and $k \in \mathrm{Mor}\,\mathscr{S}$ the statement claiming that they are in this relation reads as: $b \models \varphi[k]$. The latter is pronounced as "the formula φ is true in the model b under the valuation k of the free variables occurring in φ".

For an object b, the identity morphism of b is denoted by 1_b. Recall that $b \overset{1_b}{\to} b$.

CONDITION 1. I. For any $\varphi \in L$:

(α) For any $\overset{r}{\rightarrowtail} V$ the formula φ is either defined on r or not, but φ is always defined on 1_V:

(β) Let φ be defined on r; then for any $b' \overset{k}{\leftarrow} \cdot \overset{r}{\rightarrowtail}$, the statement $b \models \varphi[k]$ is meaningful (and it is either true or false); and for any



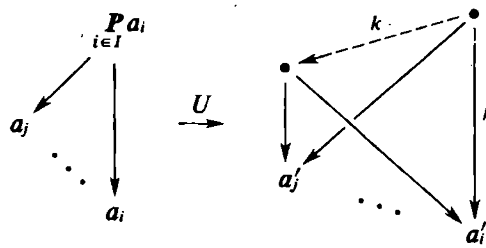
$b \models \varphi[rk]$ iff $b \models \varphi[k]$;

(γ) $b \models \varphi$ iff $b \models \varphi[k]$ for every $V \overset{k}{\to} b'$.

(1) To avoid such restriction Shafaat [41] was forced to make use of uniformity.

II. For any $\varphi, \psi \in L$ also $(\varphi \to \psi) \in L$ and for any $a \in \mathrm{Ob}\mathscr{C}$: $a \models (\varphi \to \psi)[k]$ iff ($a \models \varphi[k]$ implies $a \models \psi[k]$). ∎

Definition 1. (1) We say that φ is *preserved under* $\boldsymbol{P}$ (w.r.t. valuations) if for any diagram:



the hypothesis $(\forall i)$ $a_i \models \varphi[k_i]$ implies $\boldsymbol{P}a_i \models \varphi[k]$.

(2) For any $\mathscr{N} \subseteq$ Mor we say that φ is *preserved under* $\underleftarrow{\mathscr{N}}$ if for any diagram:

$$b \xleftarrow{\mathscr{N}\, f} \cdot \models \varphi[k]$$

also $b \models \varphi[kf']$.

(3) φ is *preserved under* $\underrightarrow{\mathscr{N}}$ if

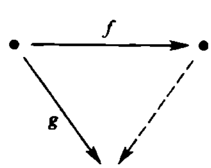
$$a \xrightarrow{f\ \mathscr{N}} \cdot \models \varphi[kf']$$

implies the existence of an l such that $a \models \varphi[l]$ and $lf' = kf'$. ∎

Note that if $(\forall h \in \mathscr{N})$ $h' \in$ Mono, then φ is preserved under $\underrightarrow{\mathscr{N}}$ iff for any $a \xrightarrow{f\ \mathscr{N}} \cdot \models \varphi[kf']$, also $a \models \varphi[k]$.

Definition 2. A formula $\varphi \in L$ is *constructive* if to any object there is a smallest arrow making φ true, and this arrow is a member of $\mathscr{H}$; that is:

For any $V \xleftarrow{r} \cdot \xrightarrow{k} a'$ such that φ is defined on r, there exists an $a \xrightarrow{f\mathscr{H}} \cdot \models \varphi[kf']$ such that for any $a \xrightarrow{g} \cdot \models \varphi[kg']$ there is a unique arrow making the diagram



commutative. ∎

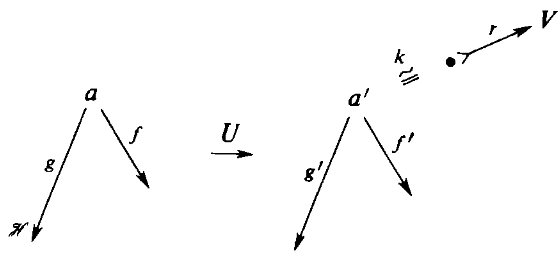
DEFINITION 3. We define the class Pr of *primitive formulas* to be a subclass of L which satisfies the following:

For any $p \in \mathrm{Pr}$, (1)–(3) below hold:

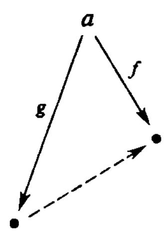
(1) p is preserved under $\underleftarrow{\mathrm{Mor}}$;

(2) p is constructive;

(3) If the situation



is such that for any primitive formula $p \in \mathrm{Pr}$ the hypothesis ta $g \models p[kg']$ implies ta$f \models p[kf']$, then there is an arrow making the diagram



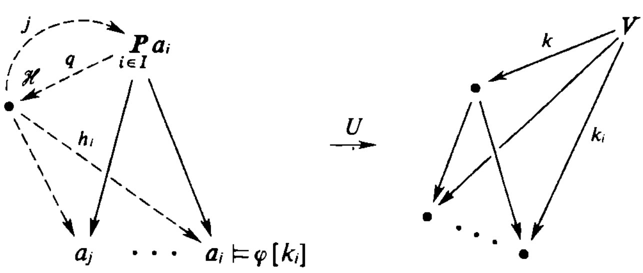
commutative. ∎

Note that the now defined class Pr of formulas does not necessarily exist, nor is necessarily unique.

LEMMA 1. *Let S be such that* $(\forall s \in S)\, s' \in \mathrm{Mono}$. *Then* $\varphi \in L$ *is constructive and is preserved under* $\underset{\sim}{\mathrm{Is}}$ *iff* φ *is preserved under* $\underset{\sim}{\boldsymbol{S}}\boldsymbol{P}$ *iff* φ *is preserved under* $\underset{\sim}{S}$ *and* $\boldsymbol{P}$.

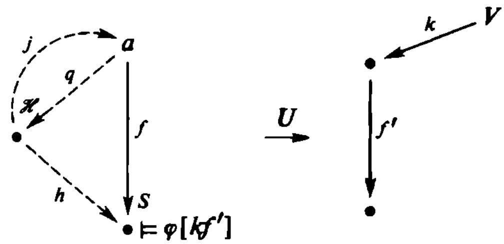
Proof. (1) Suppose φ is constructive and is preserved under $\underset{\sim}{\mathrm{Is}}$. Let $(\forall i \in I)\ a_i \models \varphi[k_i]$, and k be the direct product of $\langle k_i \rangle_{i \in I}$.

Consider the diagram

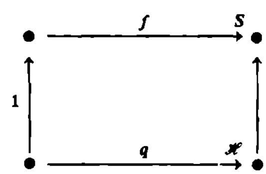


where q is that smallest arrow making $\varphi[kq']$ true the existence of which is postulated in condition (2) of Def. 2. Thus the h_i's in the diagram exist. By the properties of direct product, the j in the diagram also exists and $qj = 1$. By $qjq = q$ and the uniqueness condition in the definition of constructivity $jq = 1$; hence $j \in \mathrm{Is}$. Therefore $\underset{i \in I}{\boldsymbol{P}} a_i \models \varphi[k]$ since $\mathrm{ta}\, q \models \varphi[kq']$. We have proved that φ is preserved under $\boldsymbol{P}$. It remains to show that it is preserved under $\underset{\sim}{S}$.

Let $f \in S$ and $\mathrm{ta} f \models \varphi[kf']$. Now, consider the diagram



where q and h are the same as in the previous diagram. Now, j exists in the diagram because $\langle \mathscr{H}, S \rangle$ is a factorization system, and the diagram



commutes, and therefore the diagonal fill-in j exists. Now $a \models \varphi[k]$ is proved in the same way as in the case of direct products.

(2) Suppose φ is preserved under $\underset{\sim}{S}$ and $\boldsymbol{P}$. We want to show that φ is "constructive". (Evidently, φ is preserved under Is.)

Define $\mathscr{F} \overset{d}{=} \{f \in \mathrm{Mor}\colon a \overset{f}{\to} \cdot \models \varphi[kf']\}$. By $\mathscr{H}$-co-well-poweredness and $\mathscr{H} \cdot S = \mathrm{Mor}$ there is a *set* $E \subseteq \mathscr{H}$ such that for any $e \in E$ there is an $s \in S$ such that $es \in \mathscr{F}$, and for any $f \in \mathscr{F}$ there is an $e \in E$ and $g \in S$, such that $eg = f$. Thus $\mathrm{ta}\, e \models \varphi[ke']$ for any $e \in E$ and, since φ is preserved under $\underset{\sim}{S}\boldsymbol{P}$ if qs is the direct product of E and $q \in \mathscr{H}$, $s \in S$, then $\mathrm{ta}\, q \models \varphi[kq']$.

But also if $a \overset{f}{\to} \cdot \models \varphi[kf']$, then $f \in \mathscr{F}$ and hence there is an arrow $e \in E$ and $eg = f$. Denoting by π_e the projection (of the direct product) belonging to e, $q(s\pi_e g) = f$. This proves that q is the desired "smallest" arrow making φ true. ∎

By the above lemma we have obtained a (category oriented) characterization of universal Horn formulas:

Corollary 1. *In the categories of models, partial algebras, and total algebras the constructive formulas coincide with the universal Horn formulas, if* $\mathscr{H} = \mathrm{Epi}$.

Remark. In the case of partial algebras the language (the formulas of which we are talking about now) will be defined in Section 3. (In the other two cases the language is the usual first order one.)

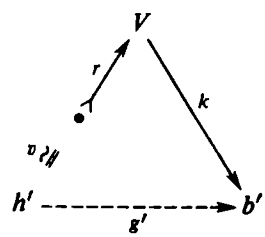
Proof. In the cases cf model and total algebras it is known that the universal Horn formulas are the $S_s P$-preserved ones (cf. Chang–Keisler [15]).

For partial algebras Theorem 1 in Section 3 implies the same. (Note that this theorem proves in a uniform manner for all three cases that the $S_s P$-preserved formulas coincide with the universal Horn ones.) ∎

DEFINITION 4. The formula φ *describes the object* n w.r.t. $\langle v, r\rangle$ if

$$n' \overset{v}{\cong} \cdot \overset{r}{\rightarrowtail} V,$$

and for any valuation $V \xrightarrow{k} b'$ $b \models \varphi[k]$ iff there is an $n \xrightarrow{g} b$ making the diagram



commutative. ∎

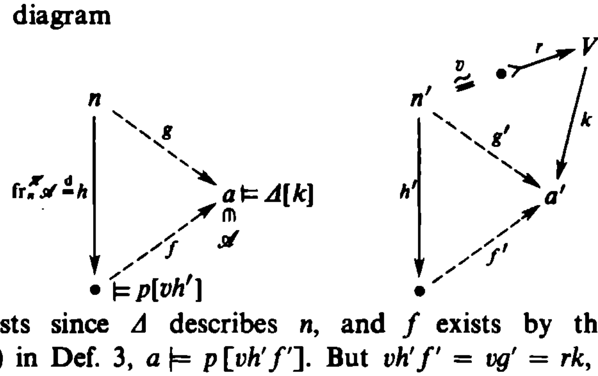
In the introduction to this section we formulated an aim to that the "generalized identities" valid in $\mathscr{A}$ should be "readable from" $\mathrm{fr}_n^{\mathscr{K}}\mathscr{A}$. The following theorem states that this aim has been attained.

THEOREM 1 ("read-from" theorem). *If* Δ *describes* n w.r.t. $\langle v, r\rangle$, *then for any primitive formula* $p \in \mathrm{Pr}$ *such that* p *is defined on* r:

$$\mathrm{Fr}_n^{\mathscr{K}}\mathscr{A} \models p[v(\mathrm{fr}_n^{\mathscr{K}}\mathscr{A})'] \quad \textit{iff} \quad \mathscr{A} \models (\Delta \to p).$$

Proof. Notation: $h \overset{\mathrm{d}}{=} \mathrm{fr}_n^{\mathscr{K}}\mathscr{A}$.

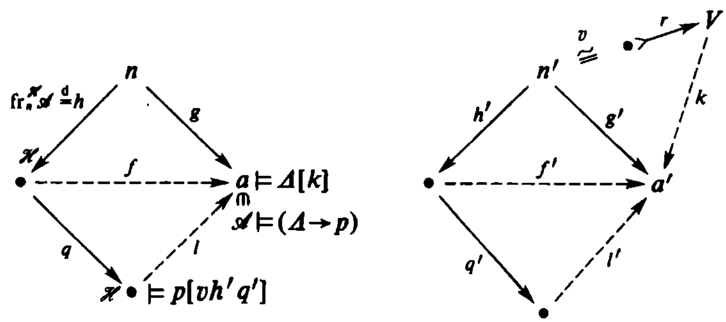
(a) The "only if" part: Let $a \in \mathscr{A}$ and $a \models \Delta[k]$, and let $\mathrm{Fr}_n^{\mathscr{K}}\mathscr{A} \models p[vh']$. Consider the diagram



where g exists since Δ describes n, and f exists by the definition of $\mathrm{fr}_n^{\mathscr{K}}\mathscr{A}$. By (1) in Def. 3, $a \models p[vh'f']$. But $vh'f' = vg' = rk$, whence by I(β) in Condition 1, $a \models p[k]$.

(b) The "if" part: Let $\mathscr{A} \models (\Delta \to p)$ and let $\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} \xrightarrow{q^{\mathscr{H}}}$ be the smallest arrow making p true, according to the definition of constructivity.

To see that hq is free over $\mathscr{A}$, consider the diagram for arbitrary $n \xrightarrow{g} \cdot \in \mathscr{A}$:



where k exists by the properties of $\mathscr{S}$. Also, f exists by the definition of $\mathrm{fr}_n^{\mathscr{H}} \mathscr{A}$; and l exists because $a \models p[vh'f']$ (since $vh'f' = rk$, $a \models p[k]$, which implies $a \models p[rk]$).

Now we want to show: $\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} \models p[vh']$. Since $hq \in \mathscr{H}$ is free over $\mathscr{A}$, there is an arrow $\xrightarrow{h} \cdot \underset{i}{\overset{q}{\rightleftarrows}} \cdot \models p[vh'q']$. Also $hqi = h$ and since p is preserved under $\underline{\mathrm{Mor}}$, $\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} \models p[vh']$. ∎

To state the generalized Birkhoff theorem, in addition to the hypotheses stated at the beginning of this section we impose the following:

Let $\mathscr{C}$ have enough H-projectives and to any H-projective object $n \in \mathrm{Pj}(H)$ let there be a fixed formula Δ_n which describes n w.r.t. $\langle v_n, r_n \rangle$. Also let the classes H, S be such that for all $h \in H$, $s \in S$ it holds that $h' \in \mathrm{Epi}$, $s' \in \mathrm{Mono}$ (in the category $\mathscr{S}$, of course).

In this section, the class E of generalized identities is fixed as:

$$E \stackrel{d}{=} \{(\Delta_n \to p) \colon n \in \mathrm{Pj}(H),\ p \in \mathrm{Pr},\ p \text{ is defined on } r_n\}.$$

THEOREM 2. *Under the above conditions, for any* $\mathscr{A} \subseteq \mathrm{Ob}\mathscr{C}$:

$$HSP\mathscr{A} = \mathrm{Md}\,E(\mathscr{A}).$$

Before proving the theorem we prove a lemma.

LEMMA 2. *Let* $\mathscr{M} \subseteq \mathrm{Mor}$ *be such that* $h' \in \mathrm{Epi}$ *for every* $h \in \mathscr{M}$. *Let* Δ *describe the object* n. *Then,* (a)–(c) *below hold.*

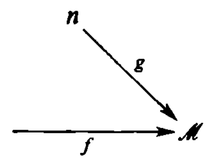
(a) Δ *is preserved under* $\boldsymbol{P}$;

(b) Δ *is preserved under* $\underline{\mathrm{Mor}}$;

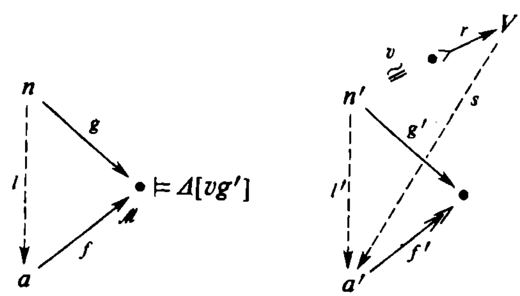
(c) Δ is *preserved under* $\underline{\mathscr{M}}$ iff $n \in \mathrm{Pj}(\mathscr{M})$.

Proof. (a) and (b) are obvious, only (c) should be proved.

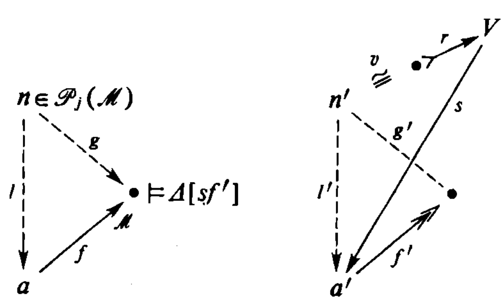
Suppose Δ is preserved under $\underset{\to}{\mathcal{M}}$. For any



consider the diagram:



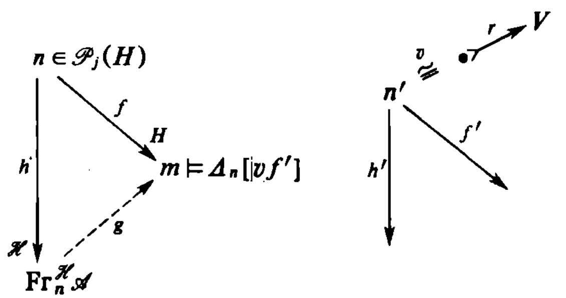
where the arrow s (such that $rsf' = vg'$) exists because $f' \in \mathrm{Epi}$ by the properties of the category $\mathcal{S}$ of sets. Since $\xrightarrow{f} \cdot \models \Delta[rsf']$, by I($\beta$) of condition 1 and the hypothesis that Δ is preserved under $\underset{\to}{\mathcal{M}}$, we have $a \models \Delta[s]$. Since Δ describes n, the arrow l exists and $vl' = rs$. Now, from $rsf' = vg'$ and $\overset{u}{\twoheadrightarrow}$ it follows that $l'f' = g'$. Since U is one-one on Mor $(n, \mathrm{ta}\,g)$, we have $lf = g$. Hence we see that $n \in \mathrm{Pj}(\mathcal{M})$. Suppose $n \in \mathrm{Pj}(\mathcal{M})$. For any $\xrightarrow[\mathcal{M}]{f} \cdot \models \Delta[sf']$ consider the diagram:



where g exists since Δ describes n, and l exists since n is $\mathcal{M}$-projective. By the properties of $\mathcal{S}$, there is an arrow z such that $rz = vl'$ and $sf' = zf'$. Since Δ describes n, $a \models \Delta[z]$. ∎

Proof of Theorem 2. (1) To prove $HSP\mathcal{A} \supseteq \mathrm{Md}\,E(\mathcal{A})$, let $m \models E(\mathcal{A})$. It is enough to show that $m \in H\{\mathrm{Fr}_n^{\mathcal{V}}\,\mathcal{A} \colon n \in \mathrm{Pj}(H)\}$. Since $\mathcal{C}$ has enough

H-projectives, $m \underset{H}{\leftarrow} n \in \mathrm{Pj}(H)$ exists and Δ_n describes n w.r.t. $\langle v, r \rangle$. In the diagram:



g exists because: Let $p \in \mathrm{Pr}$ be defined on r. By the "read-from" theorem: $\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} \models p[vh']$ implies $\mathscr{A} \models (\Delta_n \to p) \in E$, which further implies $m \models (\Delta_n \to p)$. Hence, by $m \models \Delta_n[vf']$, the hypothesis $\mathrm{Fr}_n^{\mathscr{H}} \mathscr{A} \models p[vh']$ implies $m \models p[vf']$. Now, Definition 3 yields the existence of g. By the relationship between $\mathscr{H}$ and H, also $g \in H$.

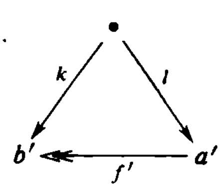
(2) To prove $HSP\mathscr{A} \subseteq \mathrm{Md}\,E(\mathscr{A})$, we show:

(a) $\mathscr{A} \models \psi \Rightarrow SP\mathscr{A} \models \psi$ if $\psi \in E$.

Let $(\Delta_n \to p) \in E$. The formula $\psi \overset{\mathrm{d}}{=} (\Delta_n \to p)$ is constructive, because for arbitrary $a' \overset{k}{\leftarrow}$ if $a \not\models \Delta_n[k]$, then 1_a is the smallest arrow making ψ true, else if $a \models \Delta_n[k]$, then the arrow making p true is the arrow belonging to ψ, as well. By Lemma 1, ψ is preserved under SP. By condition 1 this is enough.

(b) $\mathscr{A} \models \psi \Rightarrow H\mathscr{A} \models \psi$ if $\psi \in E$.

Let $b \underset{H}{\overset{f}{\leftarrow}} a \models (\Delta_n \to p)$ be arbitrary, where $n \in \mathrm{Pj}(H)$. Suppose $b \models \Delta_n[k]$. Since $\overset{f}{\twoheadrightarrow}$, there is an l making



commutative. Now by Lemma 2(c) $a \models \Delta_n[l_1]$ for some $l_1 f' = k$. Therefore $a \models p[l_1]$ and since p is preserved under $\overleftarrow{\mathrm{Mor}}$, $b \models p[l_1 f']$, i.e. $b \models p[k]$. ∎

Remark. In the categories of models, partial algebras (or total ones, of course) of a given type the hypotheses of the above theorem (Theorem 2) are satisfied for arbitrary $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$ and $S \in \{S_s, S_r, S_w\}$. This means that we have 12 different HSP-closure operators and the corresponding generalized identities. All 12 are different in partial algebras (only 6 are different in models and 2 are different in total algebras). H_w would be also interesting but it behaves in a rather pathological manner in partial algebras: the hypotheses of Theorem 2 are not met in the case of H_w, since $U(H_w) \nsubseteq$ Epi and in addition there are no H_w-projectives. (The category of partial algebras does not have projectives in general, but it has enough H_r-projectives, H_s-projectives etc.)

Accordingly, it is highly probable that there is no generalized identity concept for H_w (at least no one which would satisfy the rather mild hypotheses made in this paper. Certainly no universal formula would do, and if we allow $\forall\exists$-formulas, there are still problems we do not see how to solve.). ∎

§ 3. Generalized identities in partial algebras

Now we turn our attention to the following concrete categories:

Let t be an arbitrary finitary (²) similarity type (i.e. $t: I \to \omega$, where I is an arbitrary set and ω is the set of natural numbers); we shall consider:

the category T of t-type total (universal) algebras;

the category P of t-type partial algebras;

the category M of t-type models.

Observe that M and P are *not algebraic* in sense of Herrlich–Strecker [21], Def. 32.1 (!).

We choose our language L to be the usual infinitary language $L_{\infty\omega}$ (cf. Barwise [10]). The usual finitary first order language (with identity, and of type t) is denoted by $L_{\omega\omega}$. The infinitary language $L_{\infty\omega}$ has the same symbols as $L_{\omega\omega}$ except that we add variables v_α for all ordinals α and allow infinitely long conjunctions and disjunctions. More precisely: for any $Z \in \{M, P, T\}$ the language $L^Z_{\infty\omega}$ is the smallest class L for which

$$L \supseteq A^Z$$

(the atomic formulas) and

if $\varphi, \psi \in L$ and $v \in \text{Ord}$, then $\neg\varphi$, $\exists v\varphi$, $(\varphi \to \psi) \in L$;

if Φ is a subset of L, then $\bigwedge\Phi$, $\bigvee\Phi \in L$;

where the class A^Z of atomic formulas is defined as:

A^M: the usual atomic formulas, i.e. of the form $Rx_1 \ldots x_n$ or $x_1 = x_2$ where the x_i's are variables (elements of Ord) and R is a relation symbol;

A^P: formulas of the form $\exists!\tau$ and $\tau = \sigma$ where τ and σ are terms (of type t with variables from Ord);

(²) These same investigations can be carried through for arbitrary infinitary types (t: $I \to$ Ord), but to save space here we sacrifice generality.

A^T: the usual identities (of type t), i.e. formulas of the form $\tau = \sigma$ where τ and σ are terms.

The class of variables is $V = \mathrm{Ord}$.

The validity relation $\models$ is the usual, with the only additional definition for A^P: Let $\mathfrak{A}$ be a partial algebra and $k\colon V \to A$ be an arbitrary valuation of the variable symbols into the universe A of $\mathfrak{A}$.

$\mathfrak{A} \models \exists!\tau[k]$ iff in the partial algebra $\mathfrak{A}$ the term τ is defined at the evaluation k of the variables.

$\mathfrak{A} \models (\tau = \sigma)[k]$ iff ($\mathfrak{A} \models \exists!\tau[k]$, $\mathfrak{A} \models \exists!\sigma[k]$ and the values of τ and σ at valuation k coincide).

The formula $\varphi \in L_{\infty\omega}$ is defined on $\overset{r}{\rightarrowtail} V$ if the free variables of φ are elements of $\mathrm{Rg}(r)$.

Now, it is clear that $L^Z_{\infty\omega}$ satisfies condition 1. Now, we apply the results of Section 2 to the categories M, P, and T for the 12 possible choices:

$$H \in \{\mathrm{Is}, H_c, H_s, H_r\} \quad \text{and} \quad S \in \{S_s, S_r, S_w\}.$$

The following lemma is a representation theorem in M, P, and T for the above mentioned four H's and three S's which were defined in an abstract manner in Section 1.

Remark. As it was already noted in Section 1, the results of Section 2 can be applied in M, P, and T in such a manner that we ignore the category-theoretical definitions of the investigated epi-classes (H's) and mono-classes (S's). Thus the bored reader who is not interested in generalized (to abstract categories) results can consider the following lemma as a definition. ∎

LEMMA 1. (a) *In M, P, and T:*
$f \in \mathrm{Is}$ *iff f is an isomorphism;*
$f \in H_s$ *iff f is a factorization, that is* $\mathfrak{A} \xrightarrow{f} \mathfrak{B}$ *and* $\mathfrak{A}/\ker f \overset{\mathrm{nat}.f}{\cong} \mathfrak{B}$. *In other words: $\mathfrak{B}$ is the f-image of $\mathfrak{A}$. In Grätzer* [18], *p.* 81 *this is called a full onto homomorphism;*
$f \in H_r$ *iff f is onto.*

In P: $f \in H_c$ iff f is a closed homomorphism (see Höft [22], *or below. In Grätzer* [18], *p.* 81 *this is called a strong onto homomorphism.).*

In M: $f \in H_c$ iff f is a factorization and the kernel of f satisfies the axioms of equality (see below).

In T: $f \in H_c$ iff f is onto.

(b) *In P:*
$m \in S_s$ *iff m is a (strong) subalgebra embedding, cf. Grätzer* [18], *p.* 80;
$m \in S_r$ *iff m is a relative subalgebra embedding, cf. Grätzer* [18], *p.* 80;

$m \in S_w$ iff m is a weak subalgebra embedding, cf. Grätzer [18], *p.* 81; *that is, m is a one-one homomorphism* $\mathfrak{A} \xrightarrow{m} \mathfrak{B}$.

In M:

$m \in S_s = S_r$ iff m is a (strong) submodel embedding in the usual sense. Cf. Chang–Keisler [15], *p.* 21;

$m \in S_w$ iff m is a weak submodel embedding, i.e. a one-one homomorphism $\mathfrak{A} \xrightarrow{m} \mathfrak{B}$.

In T: *$S_s = S_r = S_w$ denotes the usual subalgebra embedding.*

(c) *In M, P, and T, **P** denotes the usual direct-product.*

Proof. To save space we omit the proof (see Pasztor [36]). ■

Remark (Pasztor [36]). In the categories P, M, and T the pairs

$$\langle H_s, S_w \rangle, \quad \langle H_r, S_r \rangle, \quad \langle H_w, S_s \rangle$$

are factorization systems. (To H_c there is no S to form a factorization system.)

Also for any $\mathscr{H} \in \{H_s, H_r, H_w\}$ and $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $\mathscr{H}$ and H are in the relationship required in Sections 1 and 2.

Now, it is easy to see that the 12 pairs $\langle H, S \rangle$ satisfy the hypotheses posed in Sections 1 and 2 and therefore the conclusions of the theorems stated so far hold for any of them. (E.g. $H_s S_s \boldsymbol{P}$ is a closure operator.)

As regards H_c, we would like to remark that this class is extensively used in the literature of partial algebras, e.g. Bauman & Pfanzagl [11], Grätzer [18], J. Schmidt [39], Höft [22], Burmeister [13], John [25]. Unfortunately, H_c is difficult to handle, e.g. is not a projective class. (On the other hand, H_c does have injectives in a weak sense as Pasztor [36] has pointed out.)

To save the reader the trouble of looking through the definitions in the literature, we quote them for the case when the type t has one unary function symbol. (The generalization is mechanical.)

(a) Let $h \in \mathrm{Mor}\, P$.

$h \in H_c$ iff $\langle A, f^{\mathfrak{A}} \rangle \xrightarrow{h} \langle B, f^{\mathfrak{B}} \rangle$ is an onto homomorphism and for any $a \in A$; $f^{\mathfrak{B}}$ is defined on $h(a)$ implies that $f^{\mathfrak{A}}$ is defined on a.

$h \in H_s$ iff $\langle A, f^{\mathfrak{A}} \rangle \xrightarrow{h} \langle B, f^{\mathfrak{B}} \rangle$ is an onto homomorphism and for any $b \in B$, if $f^{\mathfrak{B}}$ is defined on b, then there is an $a \in A$ such that $h(a) = b$ and $f^{\mathfrak{A}}$ is defined on a.

$h \in S_s$ iff $\langle A, f^{\mathfrak{A}} \rangle \xrightarrow{h} \langle B, f^{\mathfrak{B}} \rangle$ is a one-one homomorphism and for any $a \in A$; if $f^{\mathfrak{B}}$ is defined on $h(a)$, then $f^{\mathfrak{A}}$ is defined on a.

$h \in S_r$ iff h is one-one and for any $a, b \in A$ if $f^{\mathfrak{B}}(h(a)) = h(b)$, then $f^{\mathfrak{A}}$ is defined on a.

$h \in S_w$ iff h is one-one.

(b) Let $h \in \mathrm{Mor}\, M$.

$h \in H_s$ iff $\langle A, R^{\mathfrak{A}}\rangle \xrightarrow{h} \langle B, R^{\mathfrak{B}}\rangle$ is an onto homomorphism and for any $\langle c, d\rangle \in R^{\mathfrak{B}}$ there are $\langle a, b\rangle \in R^{\mathfrak{A}}$ such that $h(a) = c$, $h(b) = d$.

$h \in H_c$ iff h is as above, and for any $a, b \in A$ if $\langle h(a), h(b)\rangle \in R^{\mathfrak{B}}$, then also $\langle a, b\rangle \in R^{\mathfrak{A}}$.

The results at the end of this section show that H_c indeed has considerable significance in the theory of partial algebras. (All the same, we consider $H_s S_s P$ as the most important closure operator and the proper generalization of the notion of primitive class.)

Now we turn our attention to the H-projective objects.

LEMMA 2. (a) *In any category* Pj (Is) = Ob.

(b) *In the category T of total algebras*:

$$\mathrm{Pj}(H_c) = \mathrm{Pj}(H_s) = \mathrm{Pj}(H_r) = \text{"the class of word-algebras"}.$$

(c) *In M and P*:

$\langle A, R_i\rangle_{i\in I} \in \mathrm{Pj}(H_s)$ *iff for any $a \in A$ there are at most one $i \in I$, $b_1, \ldots, b_n \in A$, $j \leqslant n$ such that $a = b_j$ and $\langle b_1, \ldots, b_n\rangle \in R_i$. Note that, analogously to the case of word-algebras in T, here $\mathfrak{A} \in \mathrm{Pj}(H_s)$ if $\mathfrak{A}$ has as little "structure" as possible, as little is true in $\mathfrak{A}$ as possible*;

$\langle A, R_i\rangle_{i\in I} \in \mathrm{Pj}(H_r)$ *iff* $(\forall i \in I)$ $R_i = \emptyset$. *We call these "discrete" algebras (or "discrete" models), cf. e.g. Burmeister–John–Pasztor* [14].

(d) In P: $\mathfrak{A} \in \mathrm{Pj}(H_c)$ *iff $\mathfrak{A}$ is an initial segment of a (total) word-algebra (or Peano algebra), cf. Höft* [22], *p.* 203, *Burmeister* [13] *(i.e. $\mathfrak{A}$ is a relative subalgebra of a word-algebra and is generated by the generators of the word-algebra.)*

In M: $\mathrm{Pj}(H_c) = \mathrm{Ob}$.

(e) *The categories M, P, and T have enough H-projectives for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$. Also* Is, *$H_s$ and H_r are projective classes, while H_c is not.*

Proof. See Pasztor [36]. ∎

(Note that, obviously, $\mathrm{Pj}(H_r) \subseteq \mathrm{Pj}(H_s) \subseteq \mathrm{Pj}(H_c)$ and therefore in P all the projective objects are initial segments of a word-algebra.)

Remark. In the theory of total algebras, the *concept of word-algebra* (absolutely free algebra) is central. Therefore in starting to build up a theory for partial algebras the search for the proper (nontrivial) generalization (s) of this concept is vital. Our bet is *H-projectiveness*. One of the reasons is that we have *obtained* these as a result of our category theoretical study (which is not reported here in full detail) instead of choosing them by chance. And incidentally they happen to be analogous; e.g.: In T, in a word algebra as little is true as possible under the constraint that every algebra is a homomorphic image of a word algebra. Now, let us fix some H in P. The H-projectives are those partial algebras in which as little is true as possible under the constraint that every algebra should be the H-image of an

H-projective one. Consider as examples $\mathrm{Pj}(H_r)$ (where the constraint is weak) or $\mathrm{Pj}(H_s)$, where the constraint is still not too strong. $\mathrm{Pj}(H_s)$ is the free algebra (generalized word-algebra) concept in which partialness really gets through (comes into full display). In M we shall ignore H_c, because though $\mathrm{Is} \neq H_c$, still $\mathrm{Pj}(\mathrm{Is}) = \mathrm{Pj}(H_c)$ and therefore the obtained results would coincide. (Or approaching the question from a different angle, $H_c S_s \mathscr{A} = \mathrm{Is}\, S_s \mathscr{A} = S_s \mathscr{A}$ in M.)

In T the classes H_c, H_s, H_r coincide. We shall formulate our statements accordingly.

Note that the category P is the most interesting of the three investigated here. E.g. it is the only one in which H_c and H_r are nontrivial. ∎

Now, we turn our attention to the class Pr of primitive formulas. By the definition of Pr and Lemma 1 of Section 2, the class of primitive formulas depends only on S of our variables H, S, $\mathscr{K}$ etc. Therefore we denote it by $\mathrm{Pr}(S)$. Accordingly, in P there are 3 different classes of primitive formulas ($\mathrm{Pr}(S_s), \mathrm{Pr}(S_r), \mathrm{Pr}(S_w)$), in M there are two, and in T only one class.

Definition 1. (a) $\mathrm{Pr}(S_s) \overset{\mathrm{d}}{=} A^Z$ for $Z \in \{M, P, T\}$.

(b) $\mathrm{Pr}(S_w) \overset{\mathrm{d}}{=} \{x = y\colon x, y \in V\}$ in M and P.

(c) $\mathrm{Pr}(S_r) \overset{\mathrm{d}}{=} \{x = f(y_1, \ldots, y_n),\ x = y\colon x, y, y_i \in V\}$ in P. ∎

Remark. In the categories M, P, and T for any $S \in \{S_s, S_r, S_w\}$ the above defined $\mathrm{Pr}(S)$ satisfies Definition 3 of Section 2.

Proof. We leave the proof to the reader. ∎

Recall (cf. Chang–Keisler [15]) the definition of the *positive diagram* $D^+(\mathfrak{A})$ of a model (or algebra) $\mathfrak{A}$: A denotes the universe of $\mathfrak{A}$. Let in $\mathscr{S}$: $A \overset{v}{\cong} \cdot \rightarrowtail V$. Now, for any $Z \in \{M, P, T\}$ and $\mathfrak{A} \in \mathrm{Ob}Z$:

$$D^+(\mathfrak{A}) \overset{\mathrm{d}}{=} \bigwedge \{p \in A^Z\colon \mathfrak{A} \models p[v]\}.$$

Note that $D^+(\mathfrak{A}) \in L_{\infty\omega}$.

Remark. In any category $Z \in \{M, P, T\}$, the positive diagram $D^+(\mathfrak{A})$ describes $\mathfrak{A}$ v.r.t. $\langle v, r\rangle$. The proof is left to the reader. ∎

We have now reached the point where the generalized identities are falling into our hands. In M, P, and T for any H and S from the twelve we can mechanically substitute into the conditions of Theorem 2 of Section 2, obtaining:

$$\mathscr{E}_{HS} = \{(D^+(\mathfrak{A}) \to p)\colon \mathfrak{A} \in \mathrm{Pj}(H),\ p \in \mathrm{Pr}(S) \text{ and the variables occurring in } p \text{ also occur in } D^+(\mathfrak{A})\},$$

and the statement:

$$HSP\mathscr{A} = \mathrm{Md}\,\mathscr{E}_{HS}(\mathscr{A}) \quad \text{(for any class } \mathscr{A} \subseteq \mathrm{Ob}).$$

In principle, we have got our results in this way but we want a smooth syntactical characterization of the generalized identities belonging to H and S.

DEFINITION 2. For any $S \in \{S_s, S_r, S_w\}$ we define:

(a) In P, M, and T:

$$E_{H_r S} \overset{\mathrm{d}}{=} \Pr(S).$$

(b) In T:

$$E_{\mathrm{Is}\, S} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \tau_i = \sigma_i \to \tau = \sigma)\colon \tau,\ \sigma,\ \tau_i,\ \sigma_i \text{ are terms, } I \text{ is an arbitrary set}\}$$

(c) In M:

$$E_{\mathrm{Is}\, S} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} R_i x_{i1} \dots x_{in_i} \to p)\colon p \in \Pr(S),\ I \text{ is a set}\},$$

$$E_{H_s S} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} R_i x_{i1} \dots x_{in_i} \to p)\colon p \in \Pr(S),\ x_{ij} \neq x_{kl} \text{ if } ij \neq kl,\ I \text{ is a set}\}.$$

(d) In P:

$$E_{\mathrm{Is}\, S_s} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \tau_i = \sigma_i \to \tau = \sigma)\colon \tau, \sigma, \tau_i, \sigma_i \text{ are terms, } I \text{ is a set}\}$$

(the same as in T),

$$E_{\mathrm{Is}\, S_r} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \tau_i = \sigma_i \to x = f(y_1, \dots, y_n))\colon \tau_i, \sigma_i \text{ are terms; } x, y_i \in V;$$

and the function symbol f is optional, i.e. $x = y$ is also permitted on the right side, I is a set$\}$,

$$E_{\mathrm{Is}\, S_w} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \tau_i = \sigma_i \to x = y)\colon \tau_i, \sigma_i \text{ are terms; } x, y \in V;\ I \text{ is a set}\},$$

$$E_{H_c S_s} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \exists! \tau_i \to \tau = \sigma)\colon \tau_i, \tau, \sigma \text{ are terms, } I \text{ is a set}\},$$

$$E_{H_c S_r} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \exists! \tau_i \wedge \exists! \tau \wedge \exists! \sigma_1 \wedge \dots \wedge \exists! \sigma_n \to \tau = f(\sigma_1, \dots, \sigma_n))\colon \tau,\ \tau_i, \sigma_i \text{ are}$$

terms; and the function symbol f is optional, I is a set$\}$,

$$E_{H_c S_w} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \exists! \tau_i \wedge \exists! \tau \wedge \exists! \sigma \to \tau = \sigma\colon \tau, \sigma, \tau_i, \sigma_i \text{ are terms, } I \text{ is a set}\}.$$

The definitions of $E_{H_s S}$'s are the repetitions of the definitions of $E_{H_c S}$'s with the additional restriction that on the left of "$\to$" *all the variables be distinct*, and the restriction that the τ_i's are "simple".

(Notice that here we add the same restriction to $E_{H_c S}$ which in the case of M was added to the definition of $E_{H_c S} = E_{\mathrm{Is}\, S}$ to obtain $E_{H_s S}$ there.)

$$E_{H_s S_s} \overset{\mathrm{d}}{=} \{(\bigwedge_{i \in I} \exists! f_i(x_{i1} \dots x_{in_i}) \to \tau = \sigma)\colon \tau, \sigma \text{ are terms, } I \text{ is a set, and}$$

$$x_{ij} \neq x_{kl} \text{ if } ij \neq kl\},$$

$$E_{H_s S_r} \stackrel{\mathrm{d}}{=} \{(\bigwedge_{i\in I} \exists! \; f_i(x_{i1} \dots x_{in_i}) \to \tau = f(\sigma_1 \dots \sigma_n)) \colon \tau, \sigma_1, \dots, \sigma_n \in V \cup$$
$$\cup \{f_i(x_{i1} \dots x_{in_i})\}_{i\in I},\ x_{ij} \neq x_{kl} \text{ if } ij \neq kl,\ I \text{ is a set, and } f \text{ is optional}\},$$
$$E_{H_s S_w} \stackrel{\mathrm{d}}{=} \{(\bigwedge_{i\in I} \exists! f_i(x_{i1} \dots x_{in_i}) \to \tau = \sigma) \colon \tau, \sigma \in V \cup \{f_i(x_{i1} \dots x_{in_i})\}_{i\in I},$$
$$x_{ij} \neq x_{kl} \text{ if } ij \neq kl, \text{ and } I \text{ is a set}\}. \blacksquare$$

Of all these, we consider $E_{H_s S_s}$ (in P) to be the most important concept. Notice that in P, M, and T, $E_{\mathrm{Is}\, S_s}$ is the class of (infinitary) universal-Horn-formulas. In P, in $E_{H_r S_s}$ we have obtained a generalized version of the so called "strong identities"; cf. Höft [23], Burmeister [13], John [25]; and similarly in $E_{H_c S_r}$ and $E_{H_c S_w}$ we have obtained generalized versions of the "Evansian" and "week" identities, respectively, cf. John [25], Höft [23], Burmeister [13], Edgar [16].

Remark. $E_{\mathrm{Is}\, S_r}$ and $E_{\mathrm{Is}\, S_w}$ could have been defined in the style of $E_{H_c S_r}$ and $E_{H_c S_w}$. For example, the class $\{(\bigwedge_{i\in I} \tau_i = \sigma_i \to \tau = \sigma) \colon$ both τ and σ occur on the left side, too$\}$ is equivalent to $E_{\mathrm{Is}\, S_w}$. Therefore $E_{\mathrm{Is}\, S_r}$ and $E_{\mathrm{Is}\, S_w}$ could be called the quasi-Evansian-identities and the quasi-week-identities, respectively. ∎

Note that the expressive power of $E_{H_s S_s}$ is not much greater than that of the "strong" identities: if $\tau = \sigma$ is on the right side, in lucky cases we can say in the hypothesis "τ is defined" but we cannot say "τ and σ are defined", since if $\tau = \sigma$ is nontrivial, then they have common variables. But even, "if τ is defined, then $\tau = \sigma$" can be said only if τ is of the form $f(x_1 \dots x_n)$ and $x_1 \dots x_n$ is repetition-free.

Now, we state that these generalized identities (E_{HS}) are good (first only in the sense that generalized varieties coincide with generalized primitive classes). To see this, it remains to check that, loosely speaking, the E_{HS}'s are semantically equivalent to the $\mathscr{E}_{\mathrm{HS}}$'s. (After that we turn to make these generalized identities finitary.)

THEOREM 1. *In P, M, and T: for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$, and any class $\mathscr{A}$ of objects,*

$$HSP\mathscr{A} = \mathrm{Md}\, E_{HS}(\mathscr{A}).$$

Proof. (1) $HSP\mathscr{A} \subseteq \mathrm{Md}\, E_{HS}(\mathscr{A})$ since it is easy to check that E_{HS} is preserved under HSP.

(2) To see the converse inclusion we have to show that any $\varphi \in \mathscr{E}_{HS}$ is semantically equivalent to an element of E_{HS}.

In M this is obvious since there $\mathscr{E}_{HS} \subseteq E_{HS}$. (Disregarding the formulas of the form $x = x$.)

In P and T: $\mathscr{E}_{\mathrm{Is}\, S} \subseteq E_{\mathrm{Is}\, S}$ if we disregard the formulas $\exists! \tau$ which can be substituted by $\tau = y$ where y is an arbitrary, entirely new variable symbol.

Let $H \neq \mathrm{Is}$. Now, any H-projective object is an initial segment of a total word-algebra. (Cf. the remark following Lemma 2 of Section 3.) Any element of $\mathscr{E}_{HS}$ is of the form $(D^+(\mathfrak{A}) \rightarrow)$ where $p \in \mathrm{Pr}(S)$ and $\mathfrak{A} \in \mathrm{Pj}(H)$. Let the set $X \subseteq A$ be the free-generator of the word-algebra containing $\mathfrak{A}$. Now, we use the standard notation $\equiv$ of semantical equivalence of formulas, and also use the elements of X as variable symbols:

$$D^+(\mathfrak{A}) \equiv \big(\bigwedge_{i \in I} \exists! \, \tau_i \wedge \bigwedge_{i \in I} \tau_i = y_i\big),$$

where the τ_i's are terms with all of their variables in X and the y_i's are variable symbols different from each other and also from the elements of X. (More precisely: the variable y_i does not occur anywhere else in the formula.)

For formulas of this shape it is easy to see that

$$\big(\bigwedge_{i \in I} \exists! \, \tau_i \wedge \bigwedge_{i \in I} \tau_i = y_i \rightarrow p\big) \equiv \big(\bigwedge_{i \in I} \exists! \, \tau_i \rightarrow p[y_i/\tau_i]_{i \in I}\big),$$

where $p[y_i/\tau_i]_{i \in I}$ stands for the formula obtained from p by replacing every occurrence of y_i by τ_i for all $i \in I$. Considering again that $\exists! \, \tau \equiv \tau = y$ for arbitrary new variable y, the obtained formulas are elements of E_{HS} if $\mathfrak{A} \in \mathrm{Pj}(H)$ and $p \in \mathrm{Pr}(S)$. ■

By this theorem we have obtained twelve generalized Birkhoff theorems: at least that part which says that the varieties coincide with the primitive classes. Let us call these "axiomatizability" results.

Some of these have already been known: the axiomatizability result of Birkhoff for total varieties, that of Shafaat–Banashewski–Herrlich for total infinitary quasivarieties, and that of Burmeister-Höft for strong identities in partial algebras ($E_{H_r S_s}$).

A language L is *finitary* if every formula $\varphi \in L$ contains only finitely many symbols.

Now we turn our attention to the problem of obtaining finitary generalized identities.

Finitary generalized identities. Notice that in some cases we already have finitary generalized identities, e.g. $E_{H_r S}$ is already finitary. Also if $t = \emptyset$, then every E_{HS} is finitary: the generalized equations are $x = y$ and there are exactly two axiomatizable classes.

As already stated, $H_s S_s \boldsymbol{P}$ plays a distinguished role in our investigations because the formation of factor-structures, substructures and direct products are the most frequently used ways of obtaining new structures from old ones in the world of everyday life. And indeed, as stated in the following theorem, $E_{H_s S_s}$ can be replaced by $E_{H_s S_s} \cap L_{\omega\omega}$, that is, by the finitary elements of $E_{H_s S_s}$.

Definition 3. The *finitary generalized identities* (in P, T, and M) for H and S are defined as:

$$E^{\omega}_{HS} = E_{HS} \cap L_{\omega\omega}. \; ■$$

Recall that $L_{\omega\omega}$ is the ordinary finitary first order language. We shall see later that if $HSP\mathscr{A}$ is $L_{\omega\omega}$-axiomatizable, then it is also E^{ω}_{HS}-axiomatizable (more cannot be expected by any cost).

THEOREM 2. *In P, M, and T*:

1. (a) *If the type t is finite, then for any* $S \in \{S_s, S_r, S_w\}$ *and* $\mathscr{A}$,

$$H_s SP\mathscr{A} = \mathrm{Md}\, E^{\omega}_{H_s S}(\mathscr{A});$$

(b) *If the type t is degenerate (i.e. is finite and contains only constant symbols or unary relation symbols), then for any S and H the above finitary axiomatizability result holds.*

2. *In all the remaining cases there exists an* $\mathscr{A}$ *for which* $HSP\mathscr{A}$ *is not* $L_{\omega\omega}$*-axiomatizable.* (Note that $E_{H_r S}$ has already been dealt with in a remark and thus $H = H_r$ does not belong to the "remaining cases".)

Before proving the theorem we state a lemma.

DEFINITION 4. 1. Let $\mathscr{C}$ be a subcategory of M and let Op be an operator on Ob$\mathscr{C}$. Op is *nonexpanding* if

$$(\langle A, R_i\rangle_{i\in I} \in \mathrm{Op}\mathscr{A},\ J \subseteq I \text{ and } (\forall j \in J)\ R_j \neq \emptyset)$$

implies that

$$(\text{there is a } \langle B, F_i\rangle_{i\in I} \in \mathscr{A} \text{ such that } (\forall j \in J)\ F_j \neq \emptyset).$$

2. Let $N \subseteq M$. A language $\langle L, N, \models\rangle$ is called *referentially transparent* if the truthvalue of a formula $\varphi \in L$ in a model $\mathfrak{A} \in N$ depends only on those relation symbols which occur in φ. In other words: The language is referentially transparent if the implication: (the symbols occurring in φ are elements of $J \subseteq I$ and $(\forall j \in J)\ R_j = F_j$) implies

$$(\langle A, R_i\rangle_{i\in I} \models \varphi \quad \text{iff} \quad \langle A, F_i\rangle_{i\in I} \models \varphi)$$

holds for every $\varphi \in L$, $\mathfrak{A}, \mathfrak{B} \in N$. ∎

Notice that P and T are subcategories of M.

Remark. The formation of reduced products (P^r) is expanding iff the type t is infinite. H_r is also expanding in P. If $H \neq H_r$, then HSP is nonexpanding (for the H's and S's investigated so far). Every $L \subseteq L_{\infty\omega}$ is referentially transparent. ∎

Recall that a language L is finitary if every formula $\varphi \in L$ contains only a finite number of symbols.

LEMMA 3. *Let* $\langle L, N, \models\rangle$ *be a finitary referentially transparent language for some* $N \in \{M, P\}$ *and let Op be a nonexpanding operator on* N.

Now, if the type t is infinite, then there is an $\mathscr{A} \subseteq N$ *for which* $\mathrm{Op}\mathscr{A} \neq \mathrm{Md}\, L(\mathscr{A})$.

Proof. Let $\langle A, R_i\rangle_{i\in I} \in N$ be such that $(\forall i \in I)\ R_i \neq \emptyset$.

$\mathscr{A} \stackrel{d}{=} \{\langle A, F_i \rangle_{i \in I}\colon (\forall j \in J) F_j = R_j \text{ and } (\forall i \notin J) F_i = \emptyset,$

for some finite $J\}$.

Since Op is nonexpanding, $\langle A, R_i \rangle_{i \in I} \notin \mathrm{Op}\,\mathscr{A}$. Let $\mathscr{A} \models \varphi$ and J be a finite set such that all the relation symbols occurring in φ are in J. There is an $\langle A, F_i \rangle_{i \in I} \in \mathscr{A}$ such that $(\forall j \in J)\ F_j = R_j$ and by referential transparency $\langle A, R_i \rangle_{i \in I} \models \varphi$. Thus $\langle A, R_i \rangle_{i \in I} \in \mathrm{Md}\,L(\mathscr{A})$. ∎

The above lemma states that we cannot hope to get a better solution from abandoning classical first order language $\langle L_{\infty\omega}, M, \models \rangle$ and introducing some entirely new and funny language (or validity).

Proof of Theorem 2.

Ad 1. The basic idea of the proof is that only a finite part of the left side of the implication can influence the truthvalue of the right side, because in case (a) the distinctness of the variable symbols (repetition-free-ness), in case (b) the pathologicalness of the type prevents the atomic formulas (on the left) from forming "chain-like" infinite interconnected systems.

Ad (a). Any element of $E_{H_s S}$ is of the form:

$$\bigwedge_{i \in I} R_i x_{i1} \dots x_{in_i} \to p$$

in the case of M and

$$\bigwedge_{i \in I} \exists! f_i(x_{i1} \dots x_{in_i}) \to \tau = \sigma$$

in the case of P. The proof is the same for both cases; we write it down here for M. Notation: $\mathrm{Hyp} \stackrel{d}{=} \{R_i x_{i1} \dots x_{in_i}\}_{i \in I}$. Let $G \subseteq \mathrm{Hyp}$ be the set of all such formulas from Hyp which contain a variable symbol occurring in p. By the distinctness of variables G is finite, see the definition of $E_{H_s S}$.

Let $K \subseteq \mathrm{Hyp}$ be such that any relation symbol occurring in Hyp also occurs in K, and let K be finite. Such a K exists since the type t is finite.

Now, we claim that $(\bigwedge \mathrm{Hyp} \to p) \equiv (\bigwedge (G \cup K) \to p)$. It is enough to show that $(\bigwedge \mathrm{Hyp} \to p) \models (\bigwedge (G \cup K) \to p)$.

Let k be a valuation of the variables of G, K and p such that $\mathfrak{A} \models G \cup K[k]$ and $\mathfrak{A} \not\models p[k]$. Now, we can continue k to a valuation $k_1 \supseteq k$ such that $\mathfrak{A} \models \mathrm{Hyp}[k_1]$: It is enough to show that $\mathfrak{A} \models \mathrm{Hyp} \setminus (G \cup K)[k_1]$ and $k_1 \supseteq k$. By the construction of G the variables occurring in $\mathrm{Hyp} \setminus (G \cup K)$ do not occur in p; and by distinctness of variables they do not occur in $G \cup K$, either. Therefore in defining k_1 the values to the variables of $\mathrm{Hyp} \setminus (G \cup K)$ can be chosen independently of k. Since $\mathfrak{A} \models K[k]$, there is a valuation k_2 of the variables of $\mathrm{Hyp} \setminus (G \cup K)$ for which $\mathfrak{A} \models \mathrm{Hyp} \setminus (G \cup K)[k_2]$. Let $k_1 \stackrel{d}{=} k \cup k_2$. Clearly $\mathfrak{A} \not\models (\mathrm{Hyp} \to p)[k_1]$.

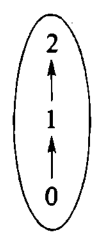
Ad (b). The proof is analogous to the above.

Ad 2. In the case of M and P, if the type t is infinite, then the statement follows from Lemma 3 of Section 3 (see the remark following the lemma).

In all the remaining cases the proof follows the following pattern: first we choose a structure $\mathfrak{C}$ unconstructible by HSP, i.e. if $\mathfrak{C} \in S_w HSP\mathscr{A}$, then already $\mathfrak{C} \in S_w\mathscr{A}$. Then we choose a class $\mathscr{A}$ such that there is an ultraproduct $\mathfrak{B}$ of elements from $\mathscr{A}$ and $\mathfrak{C} \in S_w\mathfrak{B}$ while $\mathfrak{C} \notin S_w\mathscr{A}$. By unconstructibility, $\mathfrak{B} \notin HSP\mathscr{A}$ and at the same time $\mathfrak{B} \in \mathrm{Md}\, L_{\omega\omega}(\mathscr{A})$. Note that the basic idea of the proof of Lemma 3 of Section 3 is similar.

The cases of P, M, and finite but not degenerate type t: We give the proof for P and a unary operation. Since P is a full subcategory of M and n-ary operations can be used in unary style, this proof applies also for M and arbitrary t.

ω is the set of natural numbers (i.e. ω is the smallest infinite cardinal); for every $n \in \omega$, n is the set of all natural numbers smaller than n. For every $i \leqslant \omega$ let $\mathfrak{B}_i \stackrel{d}{=} \langle i, \text{successor function}\rangle$, e.g.

$\mathfrak{B}_3 =$ 

$\mathscr{A} \stackrel{d}{=} \{\mathfrak{B}_i\}_{i<\omega}$ and $\mathfrak{C} \stackrel{d}{=} \mathfrak{B}_\omega$.

Notice that $\mathfrak{C}$ is unconstructible by $\mathrm{Is}SP$ and H_cSP (but it is constructible by H_sSP). Clearly $\mathfrak{C}$ is constructible from $\mathscr{A}$ by ultraproduct.

The case of T (and $H = \mathrm{Is}$):

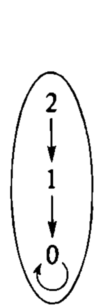
(α) Infinite type. Since arbitrary operations can be used as constants, the proof uses (infinitely many) constants: $t = \{\langle c_i, 0\rangle : i \in \beta\}$; $\beta \geqslant \omega$. Let $\mathfrak{B}_i$ be such that the first i constants coincide and all the others are distinct; (more precisely: the constant symbols c_j, $j < i$, have the same denotation in $\mathfrak{B}_i$ while the symbols c_k, $k \geqslant i$ have all different denotations). Again $\mathscr{A} \stackrel{d}{=} \{\mathrm{B}_i\}_{i<\omega}$ and $\mathfrak{C} \stackrel{d}{=} \mathfrak{B}_\omega$.

(β) The type t is finite but not degenerate: If the type t has an $1 < n$-ary function symbol, then the statement (nonaxiomatizability of SP) is known (cf. Malcev [27], p. 272).

Now, suppose t contains only unary function symbols.

$$\mathfrak{B}_i \stackrel{d}{=} \langle i, \{\langle j, j \dotplus 1\rangle : j \in i\}\rangle \quad \text{for} \quad i \leqslant \omega .$$

E.g.

$\mathfrak{B}_3 =$ 

Again $\mathscr{A} \stackrel{d}{=} \{\mathfrak{B}_i\}_{i<\omega}$ and $\mathfrak{C} \stackrel{d}{=} \mathfrak{B}_\omega$. ■

The above Theorem 2/1 states that $E^\omega_{H_sS}$ is a good generalization of the identity concept; it is a surprising consequence of Birkhoff's theorem that $HSP\mathscr{A}$ is $L_{\omega\omega}$-axiomatizable (of course in T, and for $H \neq \mathrm{Is}$). Here we have the same foi H_sSP.

The following theorem states that all the E^ω_{HS}'s are as good as possible: the only reason for $HSP\mathscr{A}$ to be not E^ω_{HS}-axiomatizable can be that it is not axiomatizable at all (in $L_{\omega\omega}$).

THEOREM 3. *In P, M, and T, for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$ and $\mathscr{A} \subseteq \mathrm{Ob}$,*

$$HSP\mathscr{A} \text{ is axiomatizable in } L_{\omega\omega}$$

iff

$$HSP\mathscr{A} = \mathrm{Md}\, E^\omega_{HS}(\mathscr{A}).$$

Proof. If $\mathscr{B} = HSP\mathscr{A}$ is axiomatizable in $L_{\omega\omega}$, then it is closed to ultraproducts. By Theorem 1 of Section 3, $\mathscr{B}$ is also axiomatizable by a set $\Sigma \subseteq E_{HS}$. Let $\sigma = (\bigwedge_{i\in I} p_i \to p) \in \Sigma$ be arbitrary. $\Psi \stackrel{d}{=} \{p_i \colon i \in I\} \cup \{\neg p\}$. Now, $\sigma \equiv \neg(\bigwedge \Psi)$.

Suppose that for all finite $\Phi \subseteq \Psi$ there is an $\mathfrak{A}_\Phi \models (\bigwedge \Phi)[k_\Phi]$, such that $\mathfrak{A}_\Phi \in \mathscr{B}$. By well known property of ultraproducts (cf. Chang–Keisler [15], Corollary 4.1.11) there exists an ultrafilter $\mathscr{D}$ such that

$$\mathop{\boldsymbol{P}}_{\Phi \in S_\omega(\Psi)} \mathfrak{A}_\Phi/\mathscr{D} \models \Psi[k]$$

where $S_\omega(\Psi)$ is the set all finite subsets of Ψ. But this is a contradiction, and thus there exists a finite $\Phi \subseteq \Psi$ such that $\mathscr{B} \models \neg(\bigwedge \Phi)$. Now, since the p_i's are prime formulas and $\mathop{\boldsymbol{P}}_{i\in\Phi} \mathfrak{A}_i \in \mathscr{B}$, we also have: $\neg p \in \Phi$. Thus to every $\sigma \in \Sigma$ there is a $\sigma_\omega \in E^\omega_{HS}$ such that $\mathscr{B} \models (\sigma_\omega \to \sigma)$. Thus $\mathscr{B} = \mathrm{Md}\{\sigma_\omega \colon \sigma \in \Sigma\}$. ■

Notation: $\boldsymbol{P}^r\mathscr{A}$ (Up $\mathscr{A}$) is the smallest class containing $\mathscr{A}$ and closed w.r.t. reduced products (ultraproducts).

$\underrightarrow{L}\mathscr{A}$ is the smallest class containing $\mathscr{A}$ and closed w.r.t. direct limits (cf. Grätzer [18], p. 152 or as "colomit" in Mac Lane [26], p. 57).

COROLLARY 1. *In P, M, and T for any* $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$ *and* $\mathscr{A} \subseteq \mathrm{Ob}$,

(a) $\mathrm{Md}E^{\omega}_{HS}(\mathscr{A}) = HSP^r\mathscr{A} = HSP\mathrm{Up}\mathscr{A}$;

(b) $\mathrm{Md}E^{\omega}_{HS}(\mathscr{A})$ *is the smallest HSP and* $\underrightarrow{L}$*-closed class containing* $\mathscr{A}$.

Proof. We prove only (a). HSP^r can be seen to be a closure operator. Therefore $HSP^r\mathscr{A}$ is closed w.r.t. ultraproducts, and "ultraroots" (because it is closed w.r.t. S_s). It is well known that a class is $L_{\omega\omega}$-axiomatizable iff it is closed w.r.t. ultraproducts and ultraroots. But now we can apply the above theorem since our class is closed w.r.t. HSP and is $L_{\omega\omega}$-axiomatizable. ∎

Note that if $H = \mathrm{Is}$, then this corollary states (and generalizes) the well known quasivariety theorem. If we apply the above theorem and corollary to P, we obtain a single theorem which states as extremal cases the quasivariety and the variety theorem. Also it can be stated in a more general form establishing a continuous hierarchy between quasivariety and variety: On the basis of Theorem 2 of Section 2 we can choose H, $\mathscr{H}$, and S rather freely and applying the remark following Def. 1 of Section 3 axiomatizability results can be obtained. To make the obtained identities finitary, the argument of Theorem 3 of Section 3 can be repeated since it does not rely on special properties of Is, H_c, H_s, etc.

Remark. In the case of H_s (for finite type) Theorem 2/1 of Section 3 is stronger than the above one!

Garrett Birkhoff has shown (cf. Birkhoff [12], p. 440) that the identities of an equationally defined class of (total) algebras are provable from the identities defining the class, using a very simple calculus all of whose proofs involve no linguistic expressions other than identities. (This kind of investigation *does belong* to the Birkhoff-style treatment of variety-like things.) ∎

§ 4. Calculus

Now we have languages E_{HS} and E^{ω}_{HS} for P, M, and T; $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$. We also have semantics $\models$ for them, and the next thing to do in such situations is to look for complete calculi (inference systems) to our languages (to our generalized indentities). A calculus for any generalized identity- concept E should be such that if we can infer from $\varphi \in E$ the formula ψ, then also $\psi \in E$. Thus in any derivation, the intermediate elements of the derivation are also elements of E. On the other hand, a calculus for E should not be more complicated than is made necessary by the structure of E (and the models of E). (In this sense, the calculus for $L_{\omega\omega}$ cannot replace Birkhoff's calculus for identities.) Actually, Birkhoff's calculus is algebraic (i) in form: the five rules are exactly the reformulation of "fully invariant congruence"; and (ii) in the proof of completeness.

Ad (i). We want to have a calculus, the set of rules of which is independent in such a manner that each has a well understood and easily explainable role. E.g. Birkhoff's calculus contains three rules for (equality's being) equivalence, one for congruence and one for fully-invariantness.

Now we introduce a tool (the concept of derivation rules) to define calculi.

Definition 1. Let $E \subseteq L_{\infty\omega}$ be arbitrary. The *calculus* for E is always defined by "abstract" derivation rules of the form:

$$A_1, \dots, A_n \vdash A_{n+1}.$$

Here A_i is not a formula in general but rather a *scheme for formulas*; for any formula $\varphi \in L_{\infty\omega}$, φ either has the form A_i or does not have the form A_i.

The above rule means that for any formulas $\varphi_1, \dots, \varphi_{n+1} \in E$, if φ_i has the form A_i (for all $i \leqslant n+1$), then we can infer φ_{n+1} from the formulas $\varphi_1, \dots, \varphi_n$. ■

The restriction $\varphi_i \in E$ in the above definition is important because we shall define different calculi for different E's by giving a unique calculus scheme from which the restriction $\varphi_i \in E$ will produce the special concrete calculi for the special E's. (E.g. given E_1 and E_2 and rule $A \vdash B$; the special rule for E_1 means that if φ, $\psi \in E_1$ and φ is of the form A and ψ is of the form B, then we can infer ψ from φ; to apply the special rule for E_2, φ, $\psi \in E_2$ is required which might yield quite different kinds of derivations.)

We also introduce a notation for substituting the variables of a formula by some other terms.

Let k: $V \to$ Terms be a function from the variable symbols into the class of terms. (Notice that Terms $\supseteq V$ and in M Terms $= V$.) Now, if $\varphi \in L_{\infty\omega}$, then $\varphi[\![k]\!]$ denotes the formula obtained from φ by substituting the variables everywhere and simultaneously according to k. E.g. if $k(x) = fy$ and $k(y) = y$, then

$$(x = y[\![k]\!]) = (fy = y)$$

or, to use simultaneousness, let $k(x) = fy$ and $k(y) = ffy$; now

$$(x = y[\![k]\!]) = (fy = ffy).$$

Another means of denoting substitution is the following. In a formula-scheme A_i the text "$\_\tau\_$" denotes a sequence of terms, say $\tau_1, \ldots, \tau_n$, such that $\tau = \tau_i$ for some i. Now, if in the same derivation rule occur the texts "$\_\tau\_$" and "$\_\sigma\_$", then these represent the same sequence of terms which differ only in one place (say the first is $\tau_1 \ldots \tau_{i-1}\, \tau\, \tau_{i+1} \ldots \tau_n$ and the second is $\tau_1 \ldots \tau_{i-1}\, \sigma\, \tau_{i+1} \ldots \tau_n$). In other words "$\_\sigma\_$" is obtained from "$\_\tau\_$" by replacing one occurrence of τ by σ. Similarly, $\_\tau\_$ might denote a unique term $f(\tau_1 \ldots \tau_n)$ in which $\tau = \tau_i$, e.t.c.

In this article we give calculi only for $S = S_s$. The reason for this is that S_s is the most important of the three kinds of subalgebra concepts, and we have to save space somehow. The calculi for the other cases can be obtained from these by not too sophisticated modifications.

The case of models (M).

DEFINITION 2. In the case of M, for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, the *calculi* for E_{HS_s} and $E^{\omega}_{HS_s}$ are defined by the following rules:

(A) 1. $\vdash (K \to x = x)$;

2. $(K \to x = y) \vdash (K \to y = x)$;

3. $(K \to x = y)$, $(K \to y = z) \vdash (K \to x = z)$;

4. $(K \to x = y)$, $(K \to R\_x\_) \vdash (K \to R\_y\_)$;

(B) $\vdash (K \wedge p \wedge L \to p)$;

(C) $(\bigwedge_{i \in I} p_i \to p)$, $\{(K \to p_i[\![k]\!])\}_{i \in I} \vdash (K \to p[\![k]\!])$.

In using the above rules, the formulas $(\bigwedge_{i \in \emptyset} p_i \to p)$ and p *should be treated as identical formulas.* (Thus e.g. as a special case of rule (C) we have $p \vdash (K \to p[\![k]\!]$.) ∎

Notice that in the calculus for $E_{H_r S_s}$ the formula K is the empty conjunction and thus e.g. rule (A)1 is $\vdash x = x$, rule (B) is automatically deleted e.t.c.

Recall that *strong completeness* means that, for any $\varphi \in E$, $\Sigma \subseteq E$, $\Sigma \models \varphi$ implies that $\Sigma \vdash \varphi$, and soundness means the opposite implication. (The symbol $\vdash$ denotes derivability.)

THEOREM 1. *For any* $E \in \{E_{HS_s}, E^{\omega}_{HS_s}: H \in \{\mathrm{Is}, H_c, H_s, H_r\}\}$ *the calculus for defined in Definition 4 of Section 4 is complete in the strong sense and sound.*

Proof. The proof is similar to Birkhoff's proof.

(a) To check soundness is a routine job.

(b) To prove completeness let $\Sigma \subseteq E$. To any $(K \to p) \in E$ such that

$\Sigma \not\vdash (K \to p)$, we construct a model $\mathfrak{M}$ such that $\mathfrak{M} \models \Sigma$ while $\mathfrak{M} \not\models (K \to p)$. First we treat equality as an ordinary relation.

$$\mathfrak{M} \stackrel{d}{=} \langle M, R_=, R_j \rangle_{j \in J}.$$

M is defined to be the set of variable symbols occurring in $(K \to p)$. For every relation symbol r (including "="):

$$R_r \stackrel{d}{=} \{\langle x_1, \ldots, x_n \rangle \in M^n \colon \Sigma \vdash (K \to r x_1 \ldots x_n)\}.$$

Now, it is easy to check that, for any valuation $k\colon V \to M$, and $q \in \Pr(S_s)$,

(1) $$\mathfrak{M} \models q[k] \quad \text{iff} \quad \Sigma \vdash (K \to q[\![k]\!]).$$

Now, $\mathfrak{M} \not\models (K \to p)$; to see this, let $i\colon V \to M$ be the identity on M. $\mathfrak{M} \models K[i]$ by inference rule (B) and at the same time $\mathfrak{M} \not\models p[i]$ by the hypothesis $\Sigma \not\vdash (K \to p)$, and by (1). (Note, that $p[\![i]\!] = p$.) However, $\mathfrak{M} \models \Sigma$ because:

Let $(\bigwedge_{i \in I} p_i \to q) \in \Sigma$, and $k\colon V \to M$ be such that $\mathfrak{M} \models \bigwedge_{i \in I} p_i[k]$. This means, by (1), that $\Sigma \vdash (K \to p_i[\![k]\!])$, for every $i \in I$. By inference rule (C) $\Sigma \vdash (K \to q[\![k]\!])$, and by (1) this means that $\mathfrak{M} \models q[k]$.

What remains to do is to help the situation that equality is an ordinary relation in $\mathfrak{M}$. Inference rule (A) serves exactly the purpose to ensure that the equality relation satisfies the first-order axioms of equality. (That is, the equality relation is a H_c-congruence.) This implies that the factormodel $\mathfrak{M}/R_=$ of $\mathfrak{M}$ by the equality relation $R_=$ is elementarily equivalent to $\mathfrak{M}$. And in $\mathfrak{M}/R_=$ the interpretation of equality is the usual. ∎

Remark. An important question concerning calculi is whether they are recursive or not. The calculi for $E^{\omega}_{HS_s}$ are recursive. ∎

The case of partial algebras (P). *For purely aesthetical reasons* we change the definition of $E^{\omega}_{HS_s}$ (of partial algebras): *In this section* if $(K \to \tau = \tau) \in E_{HS_s}$, then also $(K \to \exists! \tau) \in E_{HS_s}$.

If the reader does not like these new generalized identities, then he can eliminate them by replacing $\exists! \tau$ by $\tau = \tau$ in the calculus everywhere.

DEFINITION 3. In the case of P, for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$ the calculi for E_{HS_s} and $E^{\omega}_{HS_s}$ are defined by the following rules:

(0) $(K \to \exists! \_ \tau \_) \vdash (K \to \exists! \tau)$;

(A)
1. $\vdash (K \to x = x)$;
2. $(K \to \exists! \tau) \vdash (K \to \tau = \tau)$;
3. $(K \to \tau = \sigma) \vdash (K \to \sigma = \tau)$;
4. $(K \to \tau = \sigma)$, $(K \to \sigma = \nu) \vdash (K \to \tau = \nu)$;
5. $(K \to \tau = \sigma)$, $(K \to \exists! \_ \tau \_) \vdash (K \to \_ \tau \_ = \_ \sigma \_)$;
6. $(K \to \tau = \sigma) \vdash (K \to \exists! \tau)$;

(B) $\vdash (K \wedge p \wedge L \to p)$;

(C) $\dfrac{(\bigwedge_{i \in I} p_i \to p) \quad \{(K \to p_i[\![k]\!])\}_{i \in I}}{} \vdash (K \to [\![k]\!])$. ∎

THEOREM 2. *For any* $E \in \{E_{HS_s}, E^{\omega}_{HS_s}: H \in \{\mathrm{Is}, H_c, H_s, H_r\}\}$ *the calculus for E defined in Definition 3 of Section 4 is complete in the strong sense and sound.*

Proof. The proof proceeds analogously to the proof of Theorem 1 above. Let $\Sigma \subseteq E$, and $(K \to p) \in E$ be such that $\Sigma \nvdash (K \to p)$. $M \stackrel{d}{=} \{\tau: \Sigma \vdash (K \to \exists! \tau)$, and the variables occurring in τ also occur in $(K \to p)\}$.

The interpretation of the function symbols in $\mathfrak{M}$ is exactly that one which is used in the standard word-algebra constructions. The interpretation of the relation symbol "=":

$$R_= \stackrel{d}{=} \{\langle \tau, \sigma \rangle \in M^2: \ \Sigma \vdash \tau = \sigma\}.$$

Now, we claim that for any $k: V \to M$ and $q \in \mathrm{Pr}(S_s)$

$$\mathfrak{M} \models q[k] \quad \text{iff} \quad \Sigma \vdash (K \to q[\![k]\!]).$$

Indeed: $\mathfrak{M} \models \exists! \tau[k]$ iff (by rule (0)) $\tau[\![k]\!] \in M$ iff $\Sigma \vdash (K \to \exists! \tau[\![k]\!])$. $\mathfrak{M} \models \tau = \sigma[k]$ iff ($\mathfrak{M} \models \exists! \tau[k]$, $\mathfrak{M} \models \exists! \sigma[k]$ and $\Sigma \vdash (K \to \tau[\![k]\!] = \sigma[\![k]\!])$) iff (by rule (A) 6) $\Sigma \vdash (K \to \tau[\![k]\!] = \sigma[\![k]\!])$.

From this, by rules (B) and (C), $\mathfrak{M} \models \Sigma$ and $\mathfrak{M} \not\models (K \to p)$ is proved in the same manner as it was done in the proof of Theorem 1 above. Here again the problem of equality is dealt with in the same way as there. ∎

The case of total algebras (T). Now it is a trivial job to obtain the definition of the calculi for the E_{HS_s} and $E^{\omega}_{HS_s}$'s of T from the calculi of P. We could add the axiom $\exists! \tau$, for all terms τ, but then the formulas of the form "$\exists! \tau$" are meaningless and accordingly we delete them from the languages E_{HS}. Thus, since we known that "$\exists! \tau$", the only thing to do is to delete all formulas of the form $\exists! \tau$ from the calculi of P. The result can be seen to be complete.

DEFINITION 4. In the case of T, for any $H \in \{\mathrm{Is}, H_s\}$, the calculi for E_{HS_s} and $E^{\omega}_{HS_s}$ are defined by the following rules:

(A) 1. $\vdash (K \to x = x)$;

2. $(K \to \tau = \sigma) \vdash (K \to \sigma = \tau)$;

3. $(K \to \tau = \sigma)$, $(K \to \sigma = \nu) \vdash (K \to \tau = \nu)$;

4. $(K \to \tau = \sigma) \vdash (K \to \_\tau\_ = \_\sigma\_)$;

(B) $\vdash (K \wedge p \wedge L \to p)$;

(C) $(\bigwedge_{i \in I} p_i \to p)$, $\{(K \to p_i[\![k]\!])\}_{i \in I} \vdash (K \to p[\![k]\!])$. ∎

THEOREM 3. *For any* $E \in \{E_{HS_s}, E^{\omega}_{HS_s}: H \in \{\mathrm{Is}, H_s\}\}$, *the calculus for E defined in Definition 4 of Section 4 is complete in the strong sense and sound.*

Proof. There are two different ways of proving the theorem: one goes by an easy specialization of the calculus for P as outlined in the remark before the definition. The other way is to repeate proof of Theorem 2 above by slight modifications. ■

Remark. The calculus (and its completeness) for T can be obtained from the calculus (and completeness) for P by a mechanical specialization, because T is a closed subcategory of P. And indeed, since P is not a closed subcategory of M, the results obtained for M cannot be specialized to P in such a mechanical manner. ■

Note that the above defined calculus for $E^{\omega}_{H_s S_s}$ of T is exactly the famous *Birkhoff calculus for varieties* and the proof of Theorem 2 of Section 4 reduces exactly to Birkhoff's completeness proof, cf. Grätzer [18]. Selman [40], Th. 2, gave a complete calculus for quasivarieties, that is, for $E^{\omega}_{\mathrm{Is}\, S_s}$ of T. Thus his result is a special case of the above theorem. However, he has ten rules of inference while we have only six, which are in one-one correspondence with Birkhoff's rules. In the remaining part of his article (Th. 1) Selman gives a complete calculus for binary implication languages, that is for those formulas $(K \to p) \in E^{\omega}_{\mathrm{Is} S_s}$ in which $K \in A^T$. This is also a special case of the above theorem, if we observe the following.

All the calculi defined in this section have a high degree of independence (segmentability) in the following sense: We can chose $E \subseteq E_{\mathrm{Is}\, S_s}$ (cf. Definition 4.1) by making an arbitrary restriction on the hypothesis part of the formulas, and the calculus restricted to E will be complete. More precisely:

Let $\mathscr{K} \subseteq \{\bigwedge_{i \in I} p_i\colon p_i \in A^Z\}$ be arbitrary, and let $E \stackrel{d}{=} \{(K \to p)\colon K \in \mathscr{K}, p \in A^Z\}$. Now, the calculus restricted to E defined in Definition 3 of Section 4 is complete w.r.t. E. The proof of Theorem 2 of Section 4 gives a precise proof of this statement, too. Thus we have complete calculi for binary implication languages, not only for T, but for M and P, as well.

If we compare Selman's article and the completeness proofs of this section, we can observe an interesting confrontation of the so-called "logical" (or syntactical) completeness proofs and the algebraic (or purely semantical) completeness proofs. Selman has followed the logical approach while we here follow the algebraic. As it is usually the case, the algebraic proof is much shorter. (3)

(3) Concerning this confrontation in first order logic in general, we outlined our purely semantical approach in Andréka–Gergely–Németi [1], where we gave the simplest known (at least by us) proof of the Gödel–Henkin completeness theorem.

If we investigate a language, the investigation is incomplete without saying something about the expressive power of the language in question. A way of doing this is to characterize algebraically the axiomatizable classes; see e.g. Theorem 2 of Section 2, and 1, 2 of Section 2. Accordingly, it would be interesting to have a characterization of the classes axiomatizable by Selman's "binary implication formulas". It would be even more interesting to have such characterizations for different languages $E = \{(K \to p)\colon K \in \mathcal{K}, p \in A^Z\}$ where $\mathcal{K}$ is some fixed subset of $\{\bigwedge_{i\in I} p_i\colon p_i \in A^Z\}$.

§ 5. Examples

1. Small categories. Small categories are considered as partial semi-groups with two additional unary operations dom and cod. Consider the following set of $E^{\omega}_{H_c S_s}$-identities:

$$\exists! \, x\cdot y \wedge \exists! \, y\cdot z \to (x\cdot y)\cdot z = x\cdot(y\cdot z),$$
$$\operatorname{dom} x\cdot x = x,$$
$$x\cdot \operatorname{cod} x = x,$$
$$\exists! \, x\cdot y \to \operatorname{cod} x\cdot y = y,$$
$$\exists! \, x\cdot y \to x\cdot \operatorname{dom} y = x,$$
$$\exists! \, z\cdot \operatorname{cod} x \to z\cdot \operatorname{cod} x = z,$$
$$\exists! \operatorname{dom} x\cdot z \to \operatorname{dom} x\cdot z = z.$$

PROPOSITION. *A partial algebra $\langle A, \cdot, \operatorname{dom}, \operatorname{cod}\rangle$ is a small category iff it satisfies the above set of $E^{\omega}_{H_c S_s}$-identities.* ∎

By the above proposition, small categories form a $H_c S_s$-variety; but it is easy to check that they are not closed w.r.t. H_s.

2. Semigroups with partial invertation (and partial unit). Burmeister [13], p. IV. 4.6 describes an interesting class of partial algebras which, however, cannot be axiomatized even by a combination of the so-called strong and weak identities. Here is an $E^{\omega}_{H_s S_s}$-axiom system for that class (which thus proved to be a partial variety):

(1) $$x\cdot(y\cdot z) = (x\cdot y)\cdot z,$$

(2) $$\exists! \, h(x) \to x\cdot h(x) = 1,$$
$$\exists! \, (x) \to h(x)\cdot x = 1,$$

(3) $$\exists! \, 1 \to x\cdot 1 = x,$$
$$\exists! \, 1 \to 1\cdot x = x.$$

(The problem was caused by identities (2).)

Note that though this class is a variety, it is not an HS_r variety (for any H) and thus neither it is an S_r quasivariety.

3. Weakness. We must argue with a remark in Höft [23] which says that any notion of validity of equations is logically stronger than "weak validity". Consider the following notion of validity:

Let τ and σ be terms and let G be the set of all terms containing only such variable and function symbols which already occur in τ or σ. Now; define: $\tau = \sigma$ is *very weakly valid* in $\mathfrak{A}$ iff

$$\mathfrak{A} \models \big(\bigwedge_{\varrho \in G} \exists! \, \varrho \to \tau = \sigma \big).$$

Clearly "very weak" is weaker than "weak". (And at the same time "very weak" is referentially transparent, which wasn't even required.) By the way, $E^{\omega}_{H_c S_w}$ and $E_{H_c S_w}$ are also weaker than weak.

4. (Partial) relative subalgebras of a total variety. Let $\mathscr{V}$ be a variety in T. We want to characterize the class $S_r \mathscr{V}$, where S_r is taken in P (i.e. $\mathscr{V}$ is considered as $\mathscr{V} \subseteq \mathrm{Ob}\, P$). Let $E^{\omega}_{\mathrm{Is}\, S_r}$ be understood in P, i.e. be a proper subclass of all quasiidentities of P.

$$S_r \mathscr{V} = \mathrm{Md}\, E^{\omega}_{\mathrm{Is}\, S_r}(\mathscr{V}).$$

I.e., the quasiidentities $(\bigwedge_{i<n} \tau_i = \sigma_i \to x = f(y_1 \ldots y_n))$ valid in $\mathscr{V}$ define the partial algebras $S_r \mathscr{V}$ in P. Or, by an easy syntactical argument: the "Evansian quasiidentities" valid in $\mathscr{V}$ define exactly $S_r \mathscr{V}$.

Looking at the thing from a different angle: a partial algebra can be embedded into a variety $\mathscr{V}$ as relative-subalgebra if and only if it satisfies the generalized identities $E^{\omega}_{\mathrm{Is}\, S_r}(\mathscr{V})$.

§ 6. Some model-theoretic consequences

First of all, the results of Section 3 are model theoretic axiomatizability results. E.g.:

If a class (of models or partial algebras of finite type) is closed under $H_s S_s \boldsymbol{P}$ (i.e. under the formation of factormodels, submodels and direct products), then it is $L_{\omega\omega}$-axiomatizable.

The $H_s S_s \boldsymbol{P}$-closed classes (of finite type) are exactly the $E^{\omega}_{H_s S_s}$-axiomatizable ones; i.e. (in M) axiomatizable by formulas

$$\bigwedge_{i<n} R_i x_{i1} \ldots x_{in_i} \to x = y$$

where the x_{ij}'s are all distinct.

For any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$ the HSP-closed classes are exactly the E_{HS}-axiomatizable ones.

E.t.c.

In a similar fashion, all the results of Section 3 can be translated into model theoretic terms. Here model theory is understood in a widened sense, since partial algebras are also investigated as models.

Now, we turn our attention to preservation theorems. Recall that M, P, and T stand for "models", "partial algebras", and "algebras", respectively.

THEOREM 1. *For M, P, and T, for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$ and for any $\varphi \in L_{\omega\omega}$: φ is preserved under HSP iff $\varphi \equiv \bigwedge_{i<n} \psi_i$, where $\psi_i \in E^{\omega}_{HS}$.*

Proof. Suppose $\varphi \in L_{\omega\omega}$ is preserved under HSP. $\mathrm{Md}\{\varphi\}$ is $L_{\omega\omega}$-axiomatizable and closed w.r.t. HSP. By Theorem 1 of Section 3 it is E^{ω}_{HS}-axiomatizable. But since it is finitely axiomatizable, by compactness there is a finite subset $\{\psi_i\}_{i<n} \subseteq E^{\omega}_{HS}$ for which

$$\mathrm{Md}\{\varphi\} = \mathrm{Md}\{\psi_i\}_{i<n}. \ \blacksquare$$

By comparing this theorem and the nonaxiomatizability result of Section 3 (Theorem 2/2 of Section 3) we conclude that axiomatizability does not follow from preservation.

PROPOSITION. *For M, P, and T, for any $H \in \{\mathrm{Is}, H_c, H_s, H_r\}$, $S \in \{S_s, S_r, S_w\}$, and for any $\varphi \in L_{\infty\omega}$:*

φ is preserved under HSP iff $\varphi \equiv \bigwedge_{i\in I} \psi_i$, where $\psi_i \in E_{HS}$.

Proof. Suppose φ is preserved under HSP. By Theorem 1 of Section 3 there is a $\Sigma \subseteq E_{HS}$ such that $\mathrm{Md}\{\varphi\} = \mathrm{Md}\,\Sigma$. Now by using the form of E_{HS} it should be shown that the class Σ can be substituted by a set, and by making the variables distinct in distinct elements of Σ the quantification-problem can be avoided and the conjunction formed. $\blacksquare$

By the above theorem and proposition, we have obtained (in a single step) the S_sP, S_rP, S_wP-formulas, the H_cS_sP, H_cS_rP, H_cS_wP, the H_sS_sP e.t.c.-formulas for M, P, and T (as well for the finitary as the infinitary case).

Of these, the finitary S_sP and H_rS_sP-formulas are already known for M and T (cf. Chang–Keisler [15]).

Now, we mention some of the obtained characterizations explicitly:

In M, the finitary H_sS_sP formulas are finite conjunctions of formulas of the form:

$$\Big(\bigwedge_{i<n} R_i x_{i1} \dots x_{in_i} \to p\Big)$$

where the x_{ij}'s are all distinct, and p is an atomic formula. (I.e. the H_sS_sP

formulas are the universal Horn ones with distinct variables on the left.) The same in P:

$$\Big(\bigwedge_{i<n} \exists! f_i(x_{i1} \dots x_{in_i}) \to \tau = \sigma\Big)$$

where the x_{ij}'s are distinct.

In M, P, and T for $S \in \{S_s, S_r, S_w\}$ the SP-formulas are the universal Horn formulas with only elements of $\mathrm{Pr}(S)$ on the right. E.g. in P and for S_r:

$$\Big(\bigwedge_{i<n} p_i \to x = f(y_1 \dots y_m)\Big),$$

where p_i is atomic formula; (or the $S_w \boldsymbol{P}$ formulas are the universal Horn ones with the restriction that if $\tau = \sigma$ is on the right side, then τ and σ should occur on the left).

In M the $H_s S_w \boldsymbol{P}$-formulas are the universal Horn ones with distinct variables on the left and $x = y$ on the right side.

Note that the model theory of partial operations is entirely underdeveloped; e.g. it would be interesting to develop the theory of models of the form:

$$\langle A, R_i, P_j, F_k\rangle_{i\in I, j\in J, k\in K},$$

where

R_i is a relation,
P_j is a partial operation,
F_k is an operation.

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