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Holomorphic Sobolev spaces on the ball

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Introduction

In this work we derive inequalities among various weighted Sobolev norms for holomorphic functions on the unit ball B in C^n. To be precise, we introduce a family dv_q, $q \geqslant 0$, of probability measures on $\bar{B}$ by letting $dv_q(z) = (\Gamma(n+q)/\pi^n\Gamma(q))(1-\|z\|^2)^{q-1}dv(z)$ for $q > 0$ and letting dv_0 be the normalized surface measure on ∂B which is the weak* limit of dv_q as $q \to {}^{+}0^{+}$. We define $\|f\|_{p,q;s}$ to be the supremum of the $L^p(dv_q)$, $0 < p < \infty$, norms of derivatives of f up to order s. (In Section 5 we extend this definition to allow s to be an arbitrary real number when f is holomorphic.) We will show (Theorems 5.3 and 6.13) that, roughly speaking, to estimate the norm $\|f\|_{p,q;s}$ it suffices to consider derivatives of f in the radial direction. (See also Boas [7] for an L^2-version of this result.) This observation leads to a number of interesting inequalities involving the norms $\|\cdot\|_{p,q;s}$. Thus, for example, we obtain a Lipschitz estimate for holomorphic functions f with $\|f\|_{p,q;s} < \infty$ which is sharper than the classical Sobolev estimate in the continuous category (Corollary 5.5). We also obtain BMO (bounded mean oscillation) and Hardy–Littlewood type estimates for such functions, which extend those given by Graham [15] and Krantz [17] (Theorem 5.14 and Proposition 5.15). In addition, we obtain sharp inequalities between the norms $\|\cdot\|_{p_j,q_j;s_j}$ $(j = 1, 2)$ on holomorphic functions which imply, in particular, that $\|\cdot\|_{p,q_1;s_1}$ is equivalent to $\|\cdot\|_{p,q_2;s_2}$ whenever both q_j are positive and $q_1 - q_2 = p(s_1 - s_2)$ (Theorem 5.12(i)). Except in the trivial case $q_1 = q_2 = 0$, this result fails if $p \neq 2$ and $q_1 \cdot q_2 = 0$, in which case sharp inclusions are given (Theorems 5.12(ii), 5.12(iii) and 5.13).

We also prove a sequence of results concerning the fractional powers of the Bergman and the Poisson–Szegö kernels which are of interest on their own right, and useful in establishing the above mentioned estimates (Theorems 1.10, 3.3, 3.4 and their corollaries). In addition, we will prove projection theorems, of the type considered by Forelli and Rudin [14] for the norms $\|\cdot\|_{p,q;s}$. The estimates are a rather straightforward extension of the Forelli–Rudin result in

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the case $s \geqslant 0$, but are more delicate in the case $s < 0$ (Theorems 3.9, 6.14 and Corollary 6.3).

All of the above results are based on reproducing formulas for holomorphic functions which are derived in Section 1 from functional Hilbert space considerations. Since the kernels in these formulas are produced by rather abstract methods, it is necessary to analyze their singularities asymptotically in order to obtain the L^p-estimates discussed above (Theorem 2.1, 2.2, and Corollaries 2.3 and 2.4).

The paper is organized as follows: In Section 0, we give the preliminaries and notation of this work. It also contains some integral representations for certain special functions which are relevant to our reproducing kernels. Section 1 deals with functional Hilbert spaces of holomorphic functions and the equivalence of various, rather abstract, Sobolev spaces. In Section 2 we describe the asymptotic behavior of the relevant reproducing kernels near ∂B. The spaces $L^p_q = L^p(dv_q)$ and their holomorphic subspaces A^p_q are introduced and studied in Section 3. In Section 4, we prepare some norm estimates for certain integral operators on Sobolev spaces. The above mentioned results concerning Sobolev norms are in Section 5. Finally, in Section 6 we give various projection theorems, especially for Sobolev spaces with a negative number of derivatives.

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0. Preliminaries and notation

Throughout this work, we use the following notation:

0.1. The vector space C^n. We fix a positive integer n, and denote by C^n the vector space of ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers, with inner product and norm, given by

$$\langle z, \zeta \rangle = z_1 \bar{\zeta}_1 + \ldots + z_n \bar{\zeta}_n, \qquad \|z\| = \langle z, z \rangle^{1/2}.$$

For $r > 0$, $B(r) \equiv B_n(r) = \{z \in C^n \colon \|z\| < r\}$ denotes the ball of radius r, centered at the origin, in C^n. The *unit ball* $B \equiv B_n$ in C^n is then $B = B_n(1)$. In the one-dimensional case, we let $\Delta(r)$ denote $B_1(r)$ and we let Δ denote the *unit disk* $B_1(1)$ in C.

0.2. Multinomials. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in Z^n_+$ and $z = (z_1, \ldots, z_n) \in C^n$, we use

$$\alpha! = \alpha_1! \cdots \alpha_n!, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

We also use the partial differential operators $\partial_j = \partial/\partial z_j$ $(j = 1, \dots, n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. The *radial derivative operator* is then

$$\mathcal{R} \equiv \sum_{j=1}^{n} z_j \partial_j,$$

and we set

$$D_l \equiv \mathcal{R} + l \qquad (l \in \boldsymbol{C}),$$

with $D = D_1$.

For a domain Ω in $\boldsymbol{C}^n$, $\mathcal{O}(\Omega)$ denotes the class of all holomorphic functions on Ω. Thus, any $f \in \mathcal{O}(B)$ can be represented as

$$f(z) = \sum a_\alpha z^\alpha \qquad (z \in B),$$

and hence

$$\{D_l^s f\}(z) = \sum (|\alpha| + l)^s a_\alpha z^\alpha \qquad (l \in \boldsymbol{C}),$$

where $a_\alpha = a_\alpha(f) \in \boldsymbol{C}$, $\alpha \in \boldsymbol{Z}_+^n$ and $s \in \boldsymbol{Z}_+$. We may, by virtue of the last formula, define $D_l^s f$ for any $s \in \boldsymbol{R}$ and any $l > 0$. In particular, $D_l^s f$ is in $\mathcal{O}(B)$ for any $s \in \boldsymbol{R}$ and any $f \in \mathcal{O}(B)$.

0.3. Measures. The letter v stands for the Lebesgue measure on $\boldsymbol{C}^n$ while σ is the surface measure on the boundary ∂B of B, normalized so that $\sigma(\partial B) = 1$. For $q \geqslant 0$, dv_q is the probability measure on $\bar{B}$, defined by $dv_0 = d\sigma$ when $q = 0$ and by

$$dv_q(z) = \frac{1}{\pi^n} \frac{\Gamma(n+q)}{\Gamma(q)} (1 - \|z\|^2)^{q-1} dv(z)$$

when $q > 0$. It should be noted that as a measure on $\bar{B}$, $dv_q \to dv_0$ in the weak* sense as $q \to 0^+$. This can be verified easily by a calculation based on polar coordinates. In particular, if f is a continuous function on $\bar{B}$, then

$$\int f\, dv_0 = \int_{\partial B} f\, d\sigma = \lim_{q \to 0^+} \int_B f\, dv_q.$$

On the other hand,

$$\int f\, dv_q = \int_B f\, dv_q \qquad (q > 0),$$

if f is integrable with respect to dv_q.

0.4. Combinatorial notation. For m, r, $s \in \boldsymbol{C}$, we use

$$(r)_m = \frac{\Gamma(r+m)}{\Gamma(r)}, \qquad \binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}$$

and the familiar *beta-function* $B(r, s) = \Gamma(r)\Gamma(s)/\Gamma(r+s)$. In particular, $(r)_m = r(r+1) \cdots (r+m-1)$ if $m \in \boldsymbol{Z}_+$. For $m \in \boldsymbol{Z}_+$ and $r, s \in \boldsymbol{C}$, we also define

$$C^m(r, s) = \Gamma(r+1) \sum_{j=0}^{m} \frac{1}{(1)_{r-j}(s)_{r-j}},$$

and thus

$$C^0(r, s) = 1/(s)_r, \qquad C^1(r, s) = \{r^2+(s-1)r+1\}/(s)_r.$$

In particular, $C^m(r, s) = C^m(m, s)$ for $m,\ r \in \boldsymbol{Z}_+$ with $r \leqslant m$, and $s \in \boldsymbol{C}$.

0.5. Special functions. For $a,\ b,\ c \in \boldsymbol{R}$ and $z \in \Delta$, we consider the *hypergeometric* function

$$F(a, b; c{:}z) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{m!(c)_m} z^m,$$

and hence $F(a, b; b{:}z) = (1-z)^{-a}$. We note the identity

$$F(a, b; c{:}\ z) = (1-z)^{c-a-b} F(c-a, c-b; c{:}z),$$

and the integral representation, for $c > b > 0$,

$$F(a, b; c{:}z) = \frac{1}{B(b, c-b)} \int_0^1 (1-tz)^{-a} t^{b-1} (1-t)^{c-b-1} dt.$$

We also consider the related function

$$G_{a,b}(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(a)_m}{(m+1)^b} z^m \qquad (z \in \Delta),$$

and thus

$$G_{a,0}(z) = (1-z)^{-a},$$

$$G_{a,-1}(z) = F(a, 2; 1{:}z) = (1-z)^{-a-1}[1+(a-1)z],$$

$$G_{a,1}(z) = F(a, 1; 2{:}z) = \frac{1}{z}\frac{1}{a-1}\{(1-z)^{-a+1}-1\} \qquad (a \neq 1),$$

$$G_{1,1}(z) = G_{2,2}(z) = F(1, 1; 2{:}z) = -\frac{1}{z}\log(1-z),$$

$$G_{3,2}(z) = \tfrac{1}{2}(1-z)^{-1}\{1-(1-z)\frac{1}{z}\log(1-z)\},$$

$$G_{4,2}(z) = \tfrac{1}{6}(1-z)^{-2}\{1+3(1-z)-2(1-z)^2\frac{1}{z}\log(1-z)\}.$$

Also

$$G_{a,b}(z) = \frac{1}{\Gamma(b)}\int_0^\infty (1-e^{-t}z)^{-a}t^{b-1}e^{-t}dt \qquad (b>0),$$

$$\left(z\frac{d}{dz}+1\right)^s G_{a,b}(z) = G_{a,b-s}(z) \qquad (a,\, b,\, s\in \boldsymbol{R}).$$

In particular, for z, $\zeta\in B$ and any a, b, $s\in\boldsymbol{R}$,

(0.1) $$D^s|_z G_{a,b}(\langle z,\zeta\rangle) = G_{a,b-s}(\langle z,\zeta\rangle).$$

Another special function that we consider is

$$H_{a,s}(z) = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{C^s(m,a)}z^m \qquad (z\in\Delta)$$

where $s\in\boldsymbol{Z}_+$ and $a\in\boldsymbol{R}$. In view of 0.4, $H_{a,0}(z)=(1-z)^{-a}$ and

$$H_{a,1}(z) = \int_0^\infty (1-e^{-t}z)^{-a}e^{-(a-1)t/2}\frac{\sinh(\gamma(a)t)}{\gamma(a)}dt$$

where

$$\gamma(a) = \tfrac{1}{2}\{(a+1)(a-3)\}^{1/2}$$

and $a>1$. Thus

$$H_{3,1}(z) = \int_0^\infty (1-e^{-t}z)^{-3}te^{-t}dt$$

and hence

$$H_{3,1}(z) = G_{3,2}(z) = \tfrac{1}{2}(1-z)^{-1}\left\{1-(1-z)\frac{1}{z}\log(1-z)\right\},$$

Moreover,

$$H_{1,1}(z) = \sum_{m=0}^{\infty}\frac{z^m}{m^2+1} = 1+z\int_0^\infty (1-e^{-t}z)^{-1}e^{-t}\sin t\,dt.$$

0.6. Smoothness of functions. Let Ω be a domain or a closure of a domain in $\boldsymbol{C}^n$, and consider the classes $C^m(\Omega)$, $m\in\boldsymbol{Z}_+$ and $C^\infty(\Omega)$. Thus $f\in C^m(\Omega)$ if and only if for any $\alpha\in\boldsymbol{Z}_+^n$ with $|\alpha|\leqslant m$, $\partial^\alpha f$ exists and belongs to $C(\Omega)\equiv C^0(\Omega)$. For $0<\varepsilon\leqslant 1$, we define two Lipschitz classes of order ε, $\mathrm{Lip}_\varepsilon(\Omega)$ and $\Lambda_\varepsilon(\Omega)$, as follows: $f\in C(\Omega)$ is said to belong to $\mathrm{Lip}_\varepsilon(\Omega)$ or to $\Lambda_\varepsilon(\Omega)$ if there exists a constant $A>0$ such that $|f(z\pm\zeta)-f(z)|\leqslant A\|\zeta\|^\varepsilon$ or $|f(z+\zeta)+f(z-\zeta)-2f(z)|\leqslant A\|\zeta\|^\varepsilon$, respectively, for all $z\in\Omega$ and all $\zeta\in\boldsymbol{C}^n$ such that $z\pm\zeta\in\Omega$. Evidently, for every $0<\varepsilon<1$, $\mathrm{Lip}_1(\Omega)$

$\subset \Lambda_1(\Omega) \subset \Lambda_\varepsilon(\Omega) = \mathrm{Lip}_\varepsilon(\Omega)$. We now define $\Lambda_a(\Omega)$ for every $a > 0$ as follows: If $a = m+\varepsilon$ with $m \in \boldsymbol{Z}_+$ and $0 < \varepsilon \leqslant 1$, then $f \in \Lambda_a(\Omega)$ if and only if $f \in C^m(\Omega)$ and $\partial^\alpha f \in \Lambda_\varepsilon(\Omega)$ for every $\alpha \in \boldsymbol{Z}_+^n$ with $|\alpha| = m$. For future reference we recall two classical results due to Hardy–Littlewood and Zygmund (see [11, pp. 74–76]): Let $f \in \mathcal{O}(\Delta)$. Then $f \in \Lambda_\varepsilon(\bar{\Delta})$, $0 < \varepsilon < 1$, if and only if $\sup\{(1-|z|)^{1-\varepsilon}|f'(z)|\colon z \in \Delta\} < \infty$ and $f \in \Lambda_1(\bar{\Delta})$ if and only if $\sup\{(1-|z|)|f''(z)|\colon z \in \Delta\} < \infty$.

We also recall the result of Tricomi and Erdelyi [22] concerning the asymptotic expansion of a ratio of two gamma functions. Let a and b be any complex numbers. Then in the sector $\{z \in \boldsymbol{C} \setminus \{0\}\colon |\arg z| < \pi\}$, we have

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{m=0}^{\infty} \gamma_m(a, b) z^{-m}$$

as $z \to \infty$ within the sector, where

$$\gamma_m(a, b) = (b-a)_m q_m(a, b)$$

and where the coefficients $q_m(a, b)$ $(m = 0, 1, \ldots)$ are determined from the expansion

$$e^{-at}(1-e^{-t})^{b-a-1} = \sum_{m=0}^{\infty} q_m(a, b) t^{m+b-a-1} \qquad (0 < |t| < 2\pi).$$

Here the symbol $\sim$ in the above asymptotic expansion means that for any $k \in \boldsymbol{Z}_+$,

$$z^k \left\{ z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} - \sum_{m=0}^{\infty} \gamma_m(a, b) z^{-m} \right\} \to \gamma_k(a, b)$$

as $|z| \to \infty$ within the sector. The first three coefficients of $\gamma_m(a, b)$ are easily shown to be

$$\gamma_0(a, b) = 1,$$

$$\gamma_1(a, b) = \tfrac{1}{2}(a-b)(a+b-1),$$

$$\gamma_3(a, b) = \tfrac{1}{24}(a-b)(a-b-1)\{3(a+b-1)^2-(a-b+1)\}.$$

Given two-complex-valued functions f and g on a non-void set X, we write $f \lesssim g$ on X if there exists a positive constant A such that $|f(x)| \leqslant A|g(x)|$ for every $x \in X$. We also write $f \simeq g$ on X if $f \lesssim g$ and $g \lesssim f$ on X.

As an example, we record the following simple consequence of Stirling's formula, namely that for fixed $a, b \in \boldsymbol{R}$ such that $-a$ and $-b$ are not in $\boldsymbol{Z}_+$, we have

$$(a)_m/(b)_m \simeq (m+1)^{a-b} \qquad (m \in \boldsymbol{Z}_+).$$

In a similar fashion for $a \in \boldsymbol{R}$ with $-a \notin \boldsymbol{Z}_+$ and $s \in \boldsymbol{Z}_+$, we have

$$C^s(m, a) \simeq \frac{1}{(a)_m}(m+1)^{2s} \quad (m \in \boldsymbol{Z}_+).$$

1. Hilbert spaces of holomorphic functions

A Hilbert space $\mathscr{H}$ of complex-valued functions on a non-void set X under pointwise operations is called a *functional Hilbert space* on X, if for every $x \in X$, the point evaluation functional $f \mapsto f(x)$ from $\mathscr{H}$ to $\boldsymbol{C}$ is continuous. In this case, there is for each $x \in X$ a unique function $k_x \in \mathscr{H}$ such that $f(x) = \langle f, k_x \rangle$ for every $f \in \mathscr{H}$. The function $K(x, y) = k_y(x)$ on $X \times X$ is then the *reproducing kernel* for $\mathscr{H}$. One verifies easily that K is *positive-definite*, in short $K \gg 0$, on X, in the sense that for any finite sequence $x_1, \ldots, x_N \in X$, the matrix $[K(x_i, x_j)]$ is positive-definite. Moreover, if X is a complex manifold Ω and if $\mathscr{H} \subset \mathcal{O}(\Omega)$, then the reproducing kernel K of $\mathscr{H}$ is *sesqui-holomorphic* on Ω, i.e. for any $z \in \Omega$, $K(\cdot, z)$ and $\overline{K(z, \cdot)}$ are in $\mathcal{O}(\Omega)$.

The following result is classical (see [2], [3], [8]):

THEOREM 1.1. *Assume that* $K \gg 0$ *on* X. *Then there is a unique functional Hilbert space* $\mathscr{H}$ *on* X *with* K *as its reproducing kernel. Moreover, if* X *is a complex manifold and* K *is sesqui-holomorphic on* X, *then* $\mathscr{H} \subset \mathcal{O}(X)$.

Thus it follows that there is a one-to-one correspondence between positive-definite kernels K on X and functional Hilbert spaces $\mathscr{H} \equiv \mathscr{H}(K{:}X)$ on X. One easily verifies the following assertion:

PROPOSITION 1.2. *Let* $c_j \geqslant 0$ *and let* $K_j \gg 0$ $(j = 1, 2)$ *on* X. *Then*:

(i) $c_1K_1 + c_2K_2 \gg 0$ *on* X;
(ii) $K_1 \cdot K_2 \gg 0$ *on* X;
(iii) $\phi(K_1) \gg 0$ *on* X *whenever* ϕ *is an entire function on* $\boldsymbol{C}$ *with non-negative Taylor coefficients. In particular,* $e^{K_1} \gg 0$ *on* X.

For kernels K_1 and K_2 on X, we write $K_1 \gg K_2$ on X if $K_1 - K_2 \gg 0$ on X. We have:

PROPOSITION 1.3. *Let* $K_j \gg 0$ *on* X *and let* $\mathscr{H}_j = \mathscr{H}(K_j{:}X)$ *be its associated functional Hilbert space on* X $(j = 1, 2)$. *Let* $c > 0$. *Then the following statements are equivalent*:

(i) $c^2K_1 \gg K_2$ *on* X;
(ii) $\mathscr{H}_2 \subsetneq \mathscr{H}_1$ *and* $\|f\|_1 \leqslant c\|f\|_2$ *for every* $f \in \mathscr{H}_2$;

In particular, $\mathscr{H}_1 \simeq \mathscr{H}_2$ *(i.e.* $\mathscr{H}_1 \hookrightarrow \mathscr{H}_2$ *and* $\mathscr{H}_2 \hookrightarrow \mathscr{H}_1$*) if and only if* $(c_1c_2)^2K_1 \gg c_2^2K_2 \gg K_1$ *on* X *for some* $c_j > 0$ $(j = 1, 2)$.

Many interesting examples of sesqui-holomorphic kernels on the ball B of C^n are of the form

$$K(z, \zeta) = \sum c_\alpha z^\alpha \bar{\zeta}^\alpha \qquad (z, \zeta \in B)$$

where $c_\alpha \equiv c_\alpha(K) \in C$, $\alpha \in Z^n_+$. The family of such kernels is denoted by $\mathscr{K}(B)$. For any $K \in \mathscr{K}(B)$ we associate the index set

$$\Gamma(K) = \{\alpha \in Z^n_+ \colon\ c_\alpha(K) \neq 0\}.$$

By $\mathscr{K}_+(B)$ we denote the positive cone in $\mathscr{K}(B)$ which consists of all $K \in \mathscr{K}(B)$ such that $K \gg 0$ on B. We then have:

THEOREM 1.4. *For* $K \in \mathscr{K}(B)$, *the following statements are equivalent:*

(i) $K \in \mathscr{K}_+(B)$;
(ii) $c_\alpha(K) > 0$ *for every* $\alpha \in \Gamma(K)$.

Moreover, in both cases the associated functional Hilbert space $\mathscr{H} = \mathscr{H}(K\colon B)$ *consists of all functions* f *in* $\mathscr{O}(B)$ *of the form*

$$f(z) = \sum_{\alpha \in \Gamma(K)} a_\alpha z^\alpha \qquad (z \in B)$$

such that

$$\|f\|_K^2 \equiv \sum_{\alpha \in \Gamma(K)} c_\alpha^{-1} |a_\alpha|^2 < \infty.$$

In particular, $\{\sqrt{c_\alpha} z^\alpha \colon \alpha \in \Gamma(K)\}$ *forms an orthonormal basis for* $\mathscr{H}$.

COROLLARY 1.5. *Let* $K_j \in \mathscr{K}_+(B)$ *with* $\mathscr{H}_j = \mathscr{H}(K_j\colon B)$ *and* $c_\alpha^{(j)} \equiv c_\alpha(K_j)$ $(j = 1, 2)$ *for* $\alpha \in Z^n_+$, *and let* $a > 0$. *Then the following are equivalent:*

(i) $a^2K_1 - K_2 \in \mathscr{K}_+(B)$;
(ii) $\mathscr{H}_2 \hookrightarrow \mathscr{H}_1$ *and* $\|f\|_{K_1} \leqslant a\|f\|_{K_2}$ *for every* $f \in \mathscr{H}_2$;
(iii) $c_\alpha^{(2)} \leqslant a^2 c_\alpha^{(1)}$ *for every* $\alpha \in Z^n_+$, *and thus* $\Gamma(K_2) \subset \Gamma(K_1)$.

In particular, $\mathscr{H}_1 \simeq \mathscr{H}_2$ *if and only if* $c_\alpha^{(1)} \simeq c_\alpha^{(2)}$ *on* Z^n_+ *(and hence* $\Gamma(K_1) = \Gamma(K_2)$*).*

As an application we consider a holomorphic function ϕ on the unit disk Δ of the form

$$\phi(\lambda) = \sum_{m=0}^{\infty} \frac{1}{m!} b_m \lambda^m \qquad (\lambda \in \Delta)$$

with $b_m > 0$ for $m = 0, 1, \ldots$ It follows that

$$K_\phi(z, \zeta) = \phi(\langle z, \zeta \rangle) = \sum \frac{1}{\alpha!} b_{|\alpha|} z^\alpha \bar{\zeta}^\alpha$$

is a kernel in $\mathscr{K}_+(B)$ with $c_\alpha(K_\phi) = b_{|\alpha|}/\alpha!$ for $\alpha \in Z^n_+ = \Gamma(K_\phi)$. Moreover, $\mathscr{H} = \mathscr{H}(K_\phi{:}B)$ possesses the characteristic property of invariance of the norm $\|\cdot\| \equiv \|\cdot\|_{K_\phi}$ under unitary compositions, i.e. $\|f \circ U\| = \|f\|$ for every $f \in \mathscr{H}$ and every unitary transformation U of C^n.

As our first example, we introduce a family of unitarily invariant kernels K_q, $q \in R$, in $K_+(B)$ by setting $K_q(z, \zeta) = h_q(\langle z, \zeta\rangle)$ where $h_q \in \mathcal{O}(\Delta)$, is defined by

$$h_q(\lambda) = \begin{cases} (1-\lambda)^{-q}, & q > 0, \\ (1-q)^{-1}F(1, 1; 2-q{::}\lambda), & q \leqslant 0. \end{cases}$$

We let $\mathscr{H}_q = \mathscr{H}(K_q{:}B)$ be the associated functional Hilbert space on B with the induced norm $\|\cdot\|_q \equiv \|\cdot\|_{K_q}$. We also let $c_\alpha(q) \equiv c_\alpha(K_q)$, and note that

$$c_\alpha(q) = \begin{cases} (q)_{|\alpha|}/\alpha!, & q > 0, \\ (|\alpha|!)^2/((1-q)_{|\alpha|+1}\alpha!), & q \leqslant 0, \end{cases}$$

for $\alpha \in Z^n_+ = \Gamma(K_q)$. Note also that for $q \leqslant 0$ we have (see 0.5)

$$h_q(\lambda) = (1-\lambda)^{-q} \int_0^1 t^{-q}(1-t\lambda)^{q-1}\, dt \qquad (\lambda \in \Delta).$$

In particular,

$$h_0(\lambda) = -\frac{1}{\lambda}\log(1-\lambda).$$

The norm $\|\cdot\|_{n+q}$ in $\mathscr{H}_{n+q}$ for $q \geqslant 0$ admits a concrete realization in the form (see 0.3) of

$$\|f\|^2_{n+q} = \int |f|^2 dv_q \qquad (f \in \mathscr{H}_{n+q},\ q \geqslant 0).$$

Here, for $q = 0$, f in the above integral is replaced by its boundary function $f_* \in L^2(\partial B)$, i.e.

$$\|f\|^2_n = \sup_{0<r<1} \int_{\partial B} |f_r|^2 dv_0 = \int_{\partial B} |f_*|^2 dv_0 \qquad (f \in \mathscr{H}_n),$$

where f_r denotes the *dilation* of f, defined by $f_r(z) = f(rz)$. It follows that $\mathscr{H}_n$ is the *Hardy-space* $H^2 = H^2(B)$, while $\mathscr{H}_{n+q}$ for $q > 0$ is the *weighted Bergman space* $A^2_q = A^2_q(B)$, and thus, by continuity, $A^2_0 = H^2 = \mathscr{H}_n$. The reproducing kernel for $A^2_q = \mathscr{H}_{n+q}$, $q \geqslant 0$, is, of course, $K_{n+q}(z, \zeta) = (1-\langle z, \zeta\rangle)^{-(n+q)}$ with K_n and K_{n+1} being the ordinary *Szegö* and *Bergman kernels*, respectively, on B (see also [14]).

We shall show below that for $q < 0$, the space $\mathscr{H}_{n+q}$ can be viewed as a Sobolev space of holomorphic functions on B. To this end, for q, $s \in R$ and $f \in \mathcal{O}(B)$, we define (see 0.2)

$$\|f\|_{q,s} = \|D^s f\|_q,$$

and thus, for $q \geqslant 0$

$$\|f\|^2_{n+q,s} = \int |D^s f|^2 dv_q \qquad (q \geqslant 0).$$

We denote by $\mathscr{H}_{q,s}$ the functional Hilbert space of all functions f in $\mathscr{O}(B)$ satisfying $\|f\|_{q,s} < \infty$. It then follows by a straightforward calculation that the reproducing kernel $K_{q,s}$ of $\mathscr{H}_{q,s}$ is a unitarily invariant kernel in $\mathscr{K}_+(B)$ with

$$c_\alpha(q, s) = c_\alpha(q)(|\alpha|+1)^{-2s} \qquad (\alpha \in \boldsymbol{Z}^n_+ = \Gamma(K_{q,s})).$$

In particular (see 0.5), for $q > 0$ and $s \in \boldsymbol{R}$,

$$(1.1) \qquad K_{q,s}(z, \zeta) = G_{q,2s}(\langle z, \zeta\rangle) \qquad (z, \zeta \in B).$$

When $s \in \boldsymbol{Z}_+$ and $q \in \boldsymbol{R}$, we also define (see 0.2)

$$|||f|||^2_{q,s} = \sum_{|\alpha| \leqslant s} \frac{|\alpha|!}{\alpha!} \|\partial^\alpha f\|^2_q \qquad (f \in \mathscr{O}(B)),$$

and hence, for $q \geqslant 0$,

$$|||f|||^2_{n+q,s} = \sum_{|\alpha| \leqslant s} \frac{|\alpha|!}{\alpha!} \int \|\partial^\alpha f\|^2 dv_q \qquad (q \geqslant 0, s \in \boldsymbol{Z}_+).$$

The weight $|\alpha|!/\alpha!$ is included to make the norm invariant under unitary coordinate changes. We let $\tilde{\mathscr{H}}_{q,s}$ denote the functional Hilbert space of all functions f in $\mathscr{O}(B)$ satisfying $|||f|||_{q,s} < \infty$. It follows, again, that the reproducing kernel $\tilde{K}_{q,s}$ of $\tilde{\mathscr{H}}_{q,s}$ is a unitarily invariant kernel in $\mathscr{K}_+(B)$ with $\Gamma(\tilde{K}_{q,s}) = \boldsymbol{Z}^n_+$ and with $\tilde{c}_\alpha(q, s) = c_\alpha(\tilde{K}_{q,s})$ given by

$$\tilde{c}_\alpha(q, s) = \begin{cases} 1/C^s(|\alpha|, q)\alpha!, & q > 0, \\ 1/A^s(|\alpha|, q)\alpha!, & q > 0, \end{cases}$$

where $C^s(|\alpha|, q)$ is as in 0.4 and where

$$A^s(m, q) = m! \sum_{j=0}^{s} \frac{(1-q)_{m-j+1}}{\{(m-j)!\}^3}.$$

In particular (see 0.5), for $q > 0$ and $s \in \boldsymbol{Z}_+$,

$$\tilde{K}_{q,s}(z, \zeta) = H_{q,s}(\langle z, \zeta\rangle) \qquad (z, \zeta \in B).$$

Evidently, $\tilde{K}_{q,0} = K_{q,0} = K_q$ for any $q \in \boldsymbol{R}$ and, moreover, by using some of the identities found in 0.5 we find that $\tilde{K}_{3,1} = K_{3,1}$ and that $K_{2,1} = K_{1,1/2} = K_0$. This indicates that the spaces $\mathscr{H}_{q,s}$ and $\mathscr{H}_{q-2s}$ (for q, $s \in \boldsymbol{R}$), and $\tilde{\mathscr{H}}_{q,s}$ and $\mathscr{H}_{q,s}$ (for $q \in \boldsymbol{R}$, $s \in \boldsymbol{Z}_+$) are intimately connected. Indeed, our next result asserts that the spaces $\mathscr{H}_{q,s}$, $\tilde{\mathscr{H}}_{q,s}$ and $\mathscr{H}_{q-2s}$ are equivalent. In Section 5 we will give an L^p-version of this result (see Theorem 5.3 and 5.12).

THEOREM 1.6. *For any* $q \in \boldsymbol{R}$, $\mathscr{H}_{q,s} \simeq \mathscr{H}_{q-2s}$ *for every* $s \in \boldsymbol{R}$, *and* $\tilde{\mathscr{H}}_{q,s} \simeq \mathscr{H}_{q-2s}$ *for every* $s \in \boldsymbol{Z}_+$.

Proof. From the definitions of $c_\alpha(q)$, $c_\alpha(q, s)$ and $\tilde{c}_\alpha(q, s)$, and by using Stirling's formula (see 0.6) we deduce that

$$c_\alpha(q) \simeq |\alpha|!(|\alpha|+1)^{q-1}/\alpha! \qquad (q \in \boldsymbol{R}, \alpha \in \boldsymbol{Z}^n_+),$$
$$\tilde{c}_\alpha(q, s) \simeq c_\alpha(q)(|\alpha|+1)^{-2s} \qquad (q \in \boldsymbol{R}, s \in \boldsymbol{Z}_+, \alpha \in \boldsymbol{Z}^n_+),$$
$$c_\alpha(q, s) = c_\alpha(q)(|\alpha|+1)^{-2s} \qquad (q, s \in \boldsymbol{R}, \alpha \in \boldsymbol{Z}^n_+).$$

It follows that $c_\alpha(q, s) \simeq c_\alpha(q-2s) \simeq \tilde{c}_\alpha(q, s)$ for $\alpha \in \boldsymbol{Z}^n_+$. The proof now follows immediately from Corollary 1.5. ∎

As an immediate corollary we obtain a concrete realization of the norm $\|\cdot\|_{n+q}$ in $\mathscr{H}_{n+q}$ for $q < 0$.

COROLLARY 1.7. *Let* $q \in \boldsymbol{R}$ *and* $f \in \mathcal{O}(B)$. *Then* $\mathscr{H}_{n+q} \simeq \mathscr{H}_{n+1,(1-q)/2}$. *In particular,* $f \in \mathscr{H}_{n+q}$ *if and only if*

$$\|f\|^2_{n+1,(1-q)/2} = \int |D^{(1-q)/2} f|^2 dv_1 < \infty.$$

We will need an elementary "integration by parts" formula for the operator D^s. In this direction we remark that by virtue of Theorem 1.6, we have, for $q, s \in \boldsymbol{R}$, that $\mathscr{H}_q \cap \mathscr{H}_{q,s} \simeq \mathscr{H}_q \cap \mathscr{H}_{q-2s}$, and the latter is $\mathscr{H}_q$ if $s \leqslant 0$ and is $\mathscr{H}_{q-2s}$ if $s \geqslant 0$.

LEMMA 1.8. *Let* q, $s \in \boldsymbol{R}$ *and assume that* f, $g \in \mathscr{H}_q \cap \mathscr{H}_{q,s}$. *Then*

$$\langle D^s f, g\rangle_q = \langle f, D^s g\rangle_q.$$

Proof. Using the previous notation, we have, for

$$f = \sum a_\alpha z^\alpha, \qquad g = \sum b_\alpha z^\alpha,$$

that

$$\langle D^s f, g\rangle_q = \sum \{c_\alpha(q)\}^{-1}(|\alpha|+1)^s a_\alpha \bar{b}_\alpha = \langle f, D^s g\rangle_q,$$

and the proof is complete. ∎

We also record for future reference the following elementary consequence of the reproducing formula for $\mathscr{H}_{q,s}$, $q > 0$, (1.1), and (0.1) of 0.5.

THEOREM 1.9. *Let* $s \in \boldsymbol{R}$ *and* $z \in B$. *Then for any* $f \in \mathscr{H}_{q,s}$, $q > 0$,

$$f(z) = \langle D^s f, G_{q,s}(\langle\cdot, z\rangle)\rangle_q \qquad (f \in \mathscr{H}_{q,s}, q > 0).$$

In particular, for any $f \in \mathscr{H}_{n+q,s}$, $q \geqslant 0$,

$$f(z) = \int \{D^s f\}(\zeta) G_{n+q,s}(\langle z, \zeta\rangle) dv_q(\zeta) \qquad (f \in \mathscr{H}_{n+q,s}, q \geqslant 0).$$

The space $\mathscr{H}_q$, $q > 0$, admits some remarkable geometrical properties due to the fact that its reproducing kernel

$$K_q(z, \zeta) = (1-\langle z, \zeta\rangle)^{-q} \qquad (q > 0, z, \zeta \in B)$$

is a power of the Bergman kernel $K_{n+1}(z, \zeta) = (1-\langle z, \zeta\rangle)^{-(n+1)}$. Let

$\phi \in \operatorname{Aut}(B)$, i.e. ϕ is a biholomorphic mapping of B onto itself with (non-vanishing) Jacobian J_ϕ. Evidently, for any $z, \zeta \in B$,

$$(1-\langle z, \zeta\rangle)^{-(n+1)} = (1-\langle \phi(z), \phi(\zeta)\rangle)^{-(n+1)} J_\phi(z)\overline{J_\phi(\zeta)},$$

and thus

$$K_q(z, \zeta) = K_q(\phi(z), \phi(\zeta))\{J_\phi(z)\}^{q/(n+1)}\{\overline{J_\phi(\zeta)}\}^{q/(n+1)} \quad (q > 0). \tag{1.2}$$

THEOREM 1.10. *Let* $\phi \in \operatorname{Aut}(B)$. *Then the mapping* $T_\phi f = (f \circ \phi)\cdot[J_\phi]^{q/(n+1)}$, $f \in \mathscr{H}_q$, $q > 0$, *is a unitary transformation of* $\mathscr{H}_q$ *onto* $\mathscr{H}_q$. *In particular,* $\{\sqrt{\alpha!/(q)_{|\alpha|}}\phi^\alpha[J_\phi]^{q/(n+1)}\colon \alpha \in Z_+^n\}$ *is an orthonormal basis for* $\mathscr{H}_q$, $q > 0$.

Proof. Let $f \in \mathscr{H}_q$ $(q > 0)$. We have to show that $T_\phi f \in \mathscr{H}_q$. To this end we may assume that $c_f \equiv \|f\|_q > 0$. It follows (see, for example, [3]) that

$$c_f^2 K_q(\phi(z), \phi(\zeta)) - f(\phi(z))\overline{f(\phi(\zeta))} \gg 0 \quad \text{on } B. \tag{1.3}$$

On the other hand, by (1.2)

$$\begin{aligned} &c_f^2 K_q(z, \zeta) - \{T_\phi f\}(z)\overline{\{T_\phi f\}(\zeta)} \\ &\quad = \{J_\phi(z)\}^{q/(n+1)}\{\overline{J_\phi(\zeta)}\}^{q/(n+1)}\{c_f^2 K_q(\phi(z), \phi(\zeta)) - f(\phi(z))\overline{f(\phi(\zeta))}\}, \end{aligned}$$

and this kernel is positive-definite on B by (1.3). It follows, once again, from [3] that $T_\phi f \in \mathscr{H}_q$ and $\|T_\phi f\|_q \leqslant c_f = \|f\|_q$. The above argument may be repeated with the automorphism $\psi = \phi^{-1} \in \operatorname{Aut}(B)$. Then $T_\psi f \in \mathscr{H}_q$ with $\|T_\psi f\|_q \leqslant \|f\|_q$ for every $f \in \mathscr{H}_q$. It follows that $f = T_\phi(T_\psi f)$ and $f = T_\psi(T_\phi f)$ and $\|T_\phi f\|_q = \|T_\psi f\|_q = \|f\|_q$ for any f in $\mathscr{H}_q$, $q > 0$. This concludes the proof. ∎

When $q \geqslant n$, an alternative proof, which is simpler and more concrete, of the above theorem is available. This proof is based on the invariance property of the measure dv_{q-n} under $\operatorname{Aut}(B)$, and may be extended to an L^p-setting (see Theorem 3.3).

2. Some estimates

From Theorem 1.6 we deduce that the kernels $K_{q,s}$ and K_{q-2s} $(q, s \in \boldsymbol{R})$ and the kernel $\tilde{K}_{q,s}$ $(q \in \boldsymbol{R}, s \in \boldsymbol{Z}_+)$ are all equivalent in the positive-definiteness sense. In fact, by the considerations of 0.5 we find that these kernels are closely related to one another and in many instances they are even identical to one another. The leading term of these kernels is found to be $(1-\langle z, \zeta\rangle)^{-(q-2s)}$ for $q > 2s$ and $[1-q+2s]^{-1}F(1, 1; 2-q+2s\colon\langle z, \zeta\rangle)$ for $q \leqslant 2s$. A more precise formulation of this statement, for the generic functions $G_{a,b}$ and $H_{a,s}$ (see 0.5) with $a, b \in \boldsymbol{R}$ and $s \in \boldsymbol{Z}_+$, is given in Theorems 2.1, 2.2 and their corollaries below.

We begin by estimating the coefficients $C^s(m, a)$ (see 0.4–0.6) for $m, s \in \mathbf{Z}_+$ and $a \in \mathbf{R}$. By definition and by the Tricomi–Erdelyi asymptotic expansion (see 0.6) we have, for $m \to \infty$,

$$C^s(m, a) = \frac{1}{(a)_m} \sum_{j=0}^{s} \frac{\Gamma(m+1)}{\Gamma(m+1-j)} \cdot \frac{\Gamma(m+a)}{\Gamma(m+a-j)}$$
$$\sim \frac{1}{(a)_m} \sum_{j=0}^{s} (m+1)^{2j} \sum_{k=0}^{\infty} \gamma_k(j)(m+1)^{-k}$$
$$\sim \frac{1}{(a)_m}(m+1)^{2s} \sum_{k=0}^{\infty} \delta_k (m+1)^{-k},$$

where $\gamma_k(j) = \gamma_k(j; a)$, $\delta_k = \gamma_k(s; a)$ and $\gamma_0(j) = \delta_0 = 1$ $(0 \leqslant j \leqslant s)$. It follows that

$$\frac{1}{C^s(m, a)} \sim \frac{(a)_m}{(m+1)^{2s}} \sum_{k=0}^{\infty} r_k (m+1)^{-k} \qquad (r_0 = 1,\ m \to \infty).$$

From this, and using the special functions and smoothness notation of 0.5 and 0.6, we obtain, by employing the above mentioned results of Hardy–Littlewood and Zygmund, the following theorem:

THEOREM 2.1. *Let $a \in \mathbf{R}$ and $s \in \mathbf{Z}_+$. Then for any $N \in \mathbf{Z}_+$ such that $N \geqslant \max(0, [a-2s])$, where $[b]$ denotes the integer-value of $b \in \mathbf{R}$, we have*

$$H_{a,s}(z) = \sum_{k=0}^{N} r_k G_{a,2s+k}(z) + f_N(z) \qquad (z \in \Delta,\ r_0 = 1),$$

where $f_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+2s-a+1}(\bar{\Delta})$.

We now establish an asymptotic expansion for the function $G_{a,b}$.

THEOREM 2.2. *For $a, b \in \mathbf{R}$, the function $G_{a,b}$ admits the following properties:*

(i) *When $-a = k \in \mathbf{Z}_+$, $G_{a,b}$ is a polynomial of degree $k = -a$ of the form*

$$G_{-k,b}(z) = \sum_{m=0}^{k} (-1)^m (m+1)^{-b} \binom{k}{m} z^m;$$

(ii) *When $-b = k \in \mathbf{Z}_+$, $G_{a,b}$ is of the form*

$$G_{a,-k}(z) = (1-z)^{-(a+k)} P_k(z)$$

with $P_0(z) \equiv 1$, $P_1(z) = 1 + (a-1)z$ and where for $a \neq 1$, P_k is a polynomial of degree k with $P_k(0) = 1$ and $P_k^{(k)}(0) = k!(a-1)^k$. For $a = 1$, $k \geqslant 1$, P_k is a polynomial of degree $k-1$ with $P_k(0) = 1$ and $P_k^{(k-1)}(0) = (k-1)!$;

(iii) *When $-a \notin \mathbf{Z}_+$ and $a - b \notin \mathbf{Z}$, $G_{a,b}$ is of the form*

$$G_{a,b}(z) = \frac{\Gamma(a-b)}{\Gamma(a)} (1-z)^{-(a-b)} \sum_{k=0}^{N} \gamma_k (1-z)^k + f_N(z) \qquad (z \in \Delta,\ \gamma_0 = 1)$$

with $f_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+b-a+1}(\bar{\Delta})$ *for any* $N \in \boldsymbol{Z}_+$ *such that* $N \geqslant \max(0, [a-b])$;

(iv) *When* $-a \notin \boldsymbol{Z}_+$ *and* $b-a \in \boldsymbol{Z}_+$, $G_{a,b}$ *is of the form*

$$G_{a,b}(z) = \frac{(-1)^{b-a+1}}{\Gamma(a)\cdot(b-a)!}(1-z)^{(b-a)}\frac{1}{z}\log(1-z)\sum_{k=0}^{N}\gamma_k(1-z)^k + f_N(z) \qquad (z \in \Delta,\ \gamma_0 = 1)$$

with $f_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+b-a+1}(\bar{\Delta})$ *for any* $N \in \boldsymbol{Z}_+$;

(v) *When* $-a \notin \boldsymbol{Z}_+$ *and* $a-b \in \boldsymbol{Z}_+ \setminus \{0\}$, $G_{a,b}$ *is of the form*

$$G_{a,b}(z) = \frac{\Gamma(a-b)}{\Gamma(a)}(1-z)^{(b-a)}\sum_{k=0}^{a-b-1}\gamma_k(1-z)^k$$

$$+\frac{1}{z}\log(1-z)\sum_{k=0}^{N}\gamma_{a-b+k}(1-z)^k + f_N(z) \qquad (z \in \Delta,\ \gamma_0 = 1)$$

with $f_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+1}(\bar{\Delta})$ *for every* $N \in \boldsymbol{Z}_+$.

Proof. Item (i) is straightforward while item (ii) follows directly from

$$G_{a,-k}(z) = \left(z\frac{d}{dz}+1\right)^k (1-z)^{-a} \qquad (k = 0, 1, \ldots).$$

We now prove the remaining items. For $-a \notin \boldsymbol{Z}_+$ and $b-a \notin \boldsymbol{Z}_+$ we have, by virtue of the Tricomi–Erdelyi asymptotic expansion,

$$\Gamma(m+a)(m+1)^{-b} = \Gamma(m+a-b)\frac{\Gamma(m+a)}{\Gamma(m+a-b)}(m+1)^{-b}$$

$$\sim \Gamma(m+a-b)\sum_{k=0}^{\infty}\gamma_k(m+1)^{-k} \qquad (\gamma_0 = 1).$$

In particular, if also $a-b \notin \boldsymbol{Z}_+$, we obtain (see 0.6),

$$G_{a,b}(z) = \frac{\Gamma(a-b)}{\Gamma(a)}\sum_{k=0}^{N}\gamma_k G_{a-b,k}(z)^k + g_N(z) \qquad (z \in \Delta,\ \gamma_0 = 1) \tag{2.1}$$

with $g_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+b-a+1}(\bar{\Delta})$ for any $N \in \boldsymbol{Z}_+$ such that $N \geqslant \max(0, [a-b])$. Since $G_{a-b,0}(z) = (1-z)^{-(a-b)}$, item (iii) follows by iteration.

In a similar manner, for $-a \notin \boldsymbol{Z}_+$ and $b-a \in \boldsymbol{Z}_+$,

$$G_{a,b}(z) = \sum_{m=b-a}^{\infty}\frac{(a)_m}{m!}\frac{1}{(m+1)^b}z^m + P_{b-a-1}(z)$$

$$= \frac{1}{\Gamma(a)}\frac{1}{z}\sum_{m=b-a+1}^{\infty}\frac{\Gamma(m-1+a)}{\Gamma(m)}m^{-b}z^m + P_{b-a-1}(z)$$

with

$$P_{b-a-1}(z) = \sum_{m=0}^{b-a-1} \frac{(a)_m}{m!} \frac{1}{(m+1)^b} z^m .$$

It follows that

$$G_{a,b}(z) = \frac{1}{\Gamma(a)} \frac{1}{z} \sum_{k=0}^{N} \gamma_k \sum_{m=b-a+1}^{\infty} \frac{\Gamma(m+a-b)}{\Gamma(m+1)} m^{-k} z^m + g_N(z) \qquad (\gamma_0 = 1)$$

with $g_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+b-a+1}(\bar{\Delta})$ for every $N \in \boldsymbol{Z}_+$. By iteration we also obtain

$$G_{a,b}(z) = \frac{1}{\Gamma(a)} \frac{1}{z} \sum_{k=0}^{N} \gamma_k \sum_{m=b-a+1+k}^{\infty} \frac{\Gamma(m+a-b-k)}{\Gamma(m+1)} z^m + f_N(z) \qquad (\gamma_0 = 1)$$

with $f_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+b-a+1}(\bar{\Delta})$ for every $N \in \boldsymbol{Z}_+$. Item (iv) now follows by observing that

$$\sum_{m=j+1}^{\infty} \frac{\Gamma(m-j)}{\Gamma(m+1)} z^m = \frac{(-1)^{j+1}}{j!} (1-z)^j \log(1-z) + q_j(z) \qquad (j = 0, 1, \ldots)$$

where q_j is a polynomial of degree j.

We now assume $-a \notin \boldsymbol{Z}_+$ and $a-b \in \boldsymbol{Z}_+ \backslash \{0\}$. In this case we may use (2.1), and thus, by iteration

$$G_{a,b}(z) = \frac{\Gamma(a-b)}{\Gamma(a)} (1-z)^{-(a-b)} \sum_{k=0}^{a-b-1} \gamma_k (1-z)^k$$

$$+ \frac{\Gamma(a-b)}{\Gamma(a)} \sum_{k=a-b}^{N} \gamma_k G_{a-b,k}(z) + g_N(z) \qquad (\gamma_0 = 1)$$

with $g_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+b-a+1}(\bar{\Delta})$ for every $N \in \boldsymbol{Z}_+$ such that $N \geqslant a-b$. We now use item (iv) and replace the above N by $N+a-b$. This gives (v) and the proof is complete. ■

As an immediaty corollary of this theorem we obtain, with the aid of Theorem 2.1, the following result:

COROLLARY 2.3. *Let* $s \in \boldsymbol{Z}_+$ *and* $a \in \boldsymbol{R}$ *such that* $-a \notin \boldsymbol{Z}_+$. *Then the function* $H_{a,s}$ *admits the following properties:*

(i) *When* $a-2s \notin \boldsymbol{Z}$, $H_{a,2s}$ *is of the form*

$$H_{a,s}(z) = \frac{\Gamma(a-2s)}{\Gamma(a)} (1-z)^{-(a-2s)} \sum_{k=0}^{N} \gamma_k (1-z)^k + f_N(z) \qquad (z \in \Delta,\ \gamma_0 = 1)$$

with $f_N \in \mathcal{O}(\Delta) \cap \Lambda_{N+2s-a+1}(\bar{\Delta})$ *for any* $N \in \boldsymbol{Z}_+$ *such that* $N \geqslant \max(0, [a-2s])$;

(ii) *When* $2s-a\in Z_+$, $H_{a,s}$ *is of the form*

$$H_{a,s}(z)=\frac{(-1)^{a-1}}{\Gamma(a)(2s-a)!}(1-z)^{2s-a}\frac{1}{z}\log(1-z)\sum_{k=0}^{N}\gamma_k(1-z)^k+f_N(z)$$

$$(z\in\Delta,\ \gamma_0=1)$$

with $f_N\in\mathcal{O}(\Delta)\cap\Lambda_{N+2s-a+1}(\bar\Delta)$ *for any* $N\in\boldsymbol{Z}_+$;

(iii) *When* $a-2s\in Z_+\setminus\{0\}$, $H_{a,s}$ *is of the form*

$$H_{a,s}(z)=\frac{\Gamma(a-2s)}{\Gamma(a)}(1-z)^{-(a-2s)}\sum_{k=0}^{a-2s-1}\gamma_k(1-z)^k$$

$$+\frac{1}{z}\log(1-z)\sum_{k=0}^{N}\gamma_{a-2s+k}(1-z)^k+f_N(z)\qquad(z\in\Delta,\ \gamma_0=1)$$

with $f_N\in\mathcal{O}(\Delta)\cap\Lambda_{N+1}(\bar\Delta)$ *for every* $N\in\boldsymbol{Z}_+$.

The function $H_{a,s}$ was also studied by Boas [6]* for the special case of $a=n+1$. In this special case Boas, by using different methods, has obtained a result similar to, but somewhat less precise than, the above corollary. He also obtained in [6] the integral representation, appearing in 0.5, of $H_{a,1}$ when $a=n+1$.

Another corollary of Theorem 2.2 is the following result:

COROLLARY 2.4. *For* a, $b\in\boldsymbol{R}$, *the function* $G_{a,b}$ *admits the following properties*:

(i) *When* $a>b$, $G_{a,b}$ *is of the form*

$$G_{a,b}(z)=(1-z)^{-(a-b)}F(z)\qquad(z\in\Delta)$$

where $F\in\mathcal{O}(\Delta)\cap\Lambda_{a,b}(\bar\Delta)$. If, *in addition*, $-b=k\in\boldsymbol{Z}_+$ *then* F *is a polynomial of degree* k *if* $a\neq 1$ *or* $k=0$, *and of degree* $k-1$ *otherwise*;

(ii) *When* $a=b$ *and* $-a\notin\boldsymbol{Z}_+$ *then* $G_{a,b}$ *is of the form*

$$G_{a,b}(z)=-\frac{1}{\Gamma(a)}\frac{1}{z}\log(1-z)+F(z)\qquad(z\in\Delta)$$

where $F\in\mathcal{O}(\Delta)\cap\Lambda_1(\bar\Delta)$. *If, in addition*, $a=1$ *then* $F\equiv 0$;

(iii) *When* $a=b=-k$, $k\in\boldsymbol{Z}_+$, $G_{a,b}$ *is the polynomial*

$$G_{a,b}(z)=\sum_{m=0}^{k}(-1)^m(m+1)^k\binom{k}{m}z^m;$$

---

* Note that the weight $|\alpha|!/\alpha!$ should be included in the scalar product appearing in p. 275 of [6].

(iv) *When* $a < b$, $G_{a,b}$ *is in* $\mathcal{O}(\Delta) \cap \Lambda_{b-a}(\bar{\Delta})$. *If, in addition,* $-a = k \in \mathbf{Z}_+$ *then* $G_{a,b}$ *is the polynomial*

$$G_{a,b}(z) = \sum_{m=0}^{k} (-1)^m (m+1)^{-b} \binom{k}{m} z^m.$$

Proof. Items (ii) and (iii), and the additional parts of items (i) and (iv) can be read off directly from Theorem 2.2. We therefore prove only the main parts of (i) and (iv), and thus we may assume that $-a \notin \mathbf{Z}_+$.

Assume first that $b > a$. By letting $N = 0$ in items (iii) and (iv) of Theorem 2.2, we obtain, using the classical results of Hardy–Littlewood and Zygmund (see 0.6), that $G_{a,b} \in \mathcal{O}(\Delta) \cap \Lambda_{b-a}(\bar{\Delta})$ and the present (iv) follows. We now assume that $a > b$. In this case we choose the N in items (iii) and (v) of Theorem 2.2 to be very large so that $f_N \in \mathcal{O}(\Delta) \cap \Lambda_M(\bar{\Delta})$ where $M > a-b$. By the above mentioned results of Hardy–Littlewood and Zygmund, $(1-z)^{a-b} f_N \in \mathcal{O}(\Delta) \cap \Lambda_{a-b}(\bar{\Delta})$, and $(1-z)^{a-b} z^{-1} \log(1-z) \in \mathcal{O}(\Delta) \cap \Lambda_{a-b}(\bar{\Delta})$ for $a-b \in \mathbf{Z}_+ \setminus \{0\}$. This gives the present (i), and the proof is complete. ∎

We will also need some well-known estimates for the growth at the boundary of the integral

$$I_{s,q}(z) = \int |1 - \langle z, \zeta \rangle|^{-s} dv_q(\zeta) \qquad (z \in B),$$

with $s \in \mathbf{R}$ and $q \geqslant 0$ (see 0.3). This integral may be evaluated by means of hypergeometric functions (see 0.5), resulting in

$$I_{s,q}(z) = F(s/2, s/2; n+q : \|z\|^2)$$

or

$$I_{s,q}(z) = (1 - \|z\|^2)^{n+q-s} F(n+q-s/2, n+q-s/2; n+q : \|z\|^2).$$

We also note that in the notation of Section 1,

$$I_{s,q}(z) = \|(1 - \langle \cdot, z \rangle)^{-s/2}\|_{n+q}^2$$

and that

$$I_{2(n+q),q}(z) = K_{n+q}(z, z) = (1 - \|z\|^2)^{-(n+q)}.$$

By using Stirling's formula and the notation of 0.6 we obtain:

THEOREM 2.5. *For any* $s \in \mathbf{R}$, $q \geqslant 0$ *and* $z \in \mathbf{B}$,

$$I_{s,q}(z) \simeq \begin{cases} (1 - \|z\|^2)^{-(s-n-q)}, & s > n+q, \\ \|z\|^{-2} \log(1 - \|z\|^2)^{-1}, & s = n+q, \\ 1, & s < n+q. \end{cases}$$

One also verifies easily the following identities:

$$I_{0,q}(z) = 1, \quad I_{2,2n}(z) = \|z\|^{-2}\log(1-\|z\|^2)^{-1},$$
$$I_{2(n+q),q}(z) = (1-\|z\|^2)^{-(n+q)},$$
$$I_{s,q}(z) = (1-\|z\|^2)^{n+q-s} I_{2(n+q)-s,q}(z).$$

3. The space L^p_q

For $0 < p \leqslant \infty$ we denote by L^p_q the L^p-space with respect to the measure dv_q (see 0.3), $q \geqslant 0$, and we let $\|\cdot\|_{p,q}$ denote the associated norm. (Note that this is at odds with the notation for the Sobolev norm of the Hilbert space $\mathscr{H}_{q,s}$ in Section 1.) Thus

$$\|f\|_{p,q} = \{\int |f|^p dv_q\}^{1/p}.$$

Of course, we are using the term "norm" rather loosely since $\|\cdot\|_{p,q}$ does not satisfy the triangle inequality for $0 < p < 1$, but in this case $\varrho(f, g) = \|f-g\|^p_{p,q}$ defines a metric on L^p_q which turns L^p_q into a complete topological vector space.

The next result is essentially due to Forelli and Rudin [14].

THEOREM 3.1. *Let $r > 0$ and let K be a measurable kernel on $B \times B$ satisfying*

$$\int |K(z, \zeta)| dv_s(\zeta) \lesssim I_{n+r,s}(z), \qquad \int |K(\zeta, z)| dv_s(\zeta) \lesssim I_{n+r,s}(z)$$

for any $s > 0$ and $z \in B$, and let

$$\{Pf\}(z) = \int f(\zeta) K(z, \zeta) dv_r(\zeta) \quad (z \in B).$$

For $1 \leqslant p < \infty$ and $0 < q < rp$, the operator P is continuous on L^p_q.

Proof. We first consider the case $p = 1$. By Fubini's theorem and Theorem 2.5 we have for $0 < q < r$

$$\|Pf\|_{1,q} \leqslant \int |f(\zeta)| \cdot \{\int |K(z, \zeta)| dv_q(z)\} dv_r(\zeta)$$
$$\leqslant c_1 \int |f(\zeta)| \cdot I_{n+r,q}(\zeta) dv_r(\zeta) \leqslant c \|f\|_{1,q},$$

where c and c_1 are positive constants.

For $1 < p < \infty$ and $0 < q < rp$ we choose d with

$$\max((q-r)/(p-1), 0) < d < \min(q/(p-1), r)$$

and we let $\lambda(z) = 1 - \|z\|^2$ $(z \in B)$. Then by assumption and Hölder's inequality, with $p' = p/(p-1)$, we have

$$|\{Pf\}(z)|$$
$$\leqslant \{\int |f(\zeta)|^p |K(z, \zeta)| [\lambda(\zeta)]^{d(p-1)} dv_r(\zeta)\}^{1/p} \{\int |K(z, \zeta)| [\lambda(\zeta)]^{-d} dv_r(\zeta)\}^{1/p'}$$
$$\leqslant \{c_1 I_{n+r,r-d}(z)\}^{1/p'} \{\int |f(\zeta)|^p |K(z, \zeta)| [\lambda(\zeta)]^{d(p-1)} dv_r(\zeta)\}^{1/p}$$
$$\leqslant c_2^{1/p'} [\lambda(z)]^{-d/p'} \{\int |f(\zeta)|^p |K(z, \zeta)| [\lambda(\zeta)]^{d(p-1)} dv_r(\zeta)\}^{1/p},$$

where Theorem 2.5, with the observation that $0 < d < r$, has been used. It follows from Fubini's theorem and the assumption, as $q-d(p-1) > 0$, that

$$\|Pf\|^p_{p,q} \leqslant c^p_3 \int |f(\zeta)|^p [\lambda(\zeta)]^{d(p-1)} \{\int |K(z, \zeta)| [\lambda(z)]^{d(1-p)} dv_q(z)\} dv_r(\zeta)$$
$$\leqslant c^p_4 \int |f(\zeta)|^p [\lambda(\zeta)]^{d(p-1)} I_{n+r,q-d(p-1)}(\zeta) dv_r(\zeta).$$

Since also $r > q-d(p-1)$, a second application of Theorem 2.5 gives

$$\|Pf\|^p_{p,q} \leqslant c^p_5 \int |f(\zeta)|^p [\lambda(\zeta)]^{q-1} dv(\zeta)$$

where c_j ($j = 1, 2, 3, 4, 5$) are positive constants. It follows that $\|Pf\|_{p,q} \leqslant c\|f\|_{p,q}$, where $c > 0$ is a constant, and the proof is complete. ■

The next result is similar to Theorem 3.1, but more delicate. To prove it we use an argument of Flett [12] suitably adopted to the present setting. For a continuous function f on B, the *means* of f are defined by

$$M_p(f:\varrho) = \{\int_{\partial B} |f(\varrho z)|^p dv_0(z)\}^{1/p} \quad (0 < \varrho < 1,\ 0 < p \leqslant \infty).$$

Theorem 3.2. *Let $q \geqslant 0$ and let K be a measurable kernel on $\bar{B}\times\bar{B}$ satisfying $|K(z, \zeta)| \lesssim |1-\langle z, \zeta\rangle|^{-(n+q)}$. Define*

$$\{Pf\}(z) = \int f(\zeta) K(z, \zeta) dv_q(\zeta).$$

Then:

(i) *For $1 \leqslant p \leqslant r \leqslant \infty$ and $f \in L^p_q$, we have*

$$M_r(Pf:\varrho) \lesssim (1-\varrho)^{-[(n+q)/p-n/r]} \|f\|_{p,q} \quad (0 < \varrho < 1);$$

(ii) *For $1 \leqslant p \leqslant r \leqslant \infty$, $np < (n+q)r$ and $f \in L^p_q$, we have*

$$M_r(Pf:\varrho) = o\big((1-\varrho)^{-[(n+q)/p-n/r]}\big) \quad \text{as } \varrho \to 1^-;$$

(iii) *For $1 < p < r \leqslant \infty$, $p \leqslant k < \infty$ and $f \in L^p_q$, we have*

$$\{\int_0^1 M^k_r(Pf:\varrho)(1-\varrho)^{k[(n+q)/p-n/r]-1} d\varrho\}^{1/k} \leqslant c\|f\|_{p,q},$$

with a constant $c = c(k, p, q, r) > 0$.

Proof. For $0 < \varrho < 1$, set $K_\varrho(z, \zeta) = K(\varrho z, \zeta)$ ($z, \zeta \in \bar{B}$), and define an operator P_ϱ by

$$\{P_\varrho f\}(z) = \int f(\zeta) K_\varrho(z, \zeta) dv_q(\zeta) = \{Pf\}(\varrho z).$$

It follows from Theorem 2.5 that for any $l \geqslant 0$ and any $p > (n+l)/(n+q)$,

$$\|K_\varrho(z, \cdot)\|_{p,l},\ \|K_\varrho(\cdot, z)\|_{p,l} \lesssim (1-\varrho)^{-[(n+q)-(n+l)/p]} \quad (z \in \bar{B},\ 0 < \varrho < 1). \tag{3.1}$$

To prove (i), we choose $s \geqslant 1$ such that $1-s^{-1} = p^{-1}-r^{-1}$ and define a, b, $c > 0$ with $a^{-1}+b^{-1}+c^{-1} = 1$ by letting $a^{-1} = r^{-1}$, $b^{-1} = p^{-1}-r^{-1}$ and

$c^{-1} = s^{-1} - r^{-1}$. By Hölder's inequality

$$\begin{aligned}|\{P_\varrho f\}(z)| &= \left|\int K_\varrho(z, \zeta) f(\zeta) dv_q(\zeta)\right| \\ &\leqslant \int |K_\varrho(z, \zeta)| \cdot |f(\zeta)| dv_q(\zeta) \\ &= \int (|K_\varrho(z, \zeta)|^{s/a} |f(\zeta)|^{p/a} (|K_\varrho(z, \zeta)|^{s/c})(|f(\zeta)|^{p/b}) dv_q(\zeta) \\ &\leqslant \|f\|_{p,q}^{p/b} \|K_\varrho(z, \cdot)\|_{s,q}^{s/c} \cdot \left\{\int |K_\varrho(z, \zeta)|^s |f(\zeta)|^p dv_q(\zeta)\right\}^{1/r},\end{aligned}$$

so by Fubini's theorem and (3.1)

$$M_r(Pf:\varrho) = \|P_\varrho f\|_{r,0} \lesssim (1-\varrho)^{-[(n+q)/p - n/r]} \|f\|_{p,q}$$

and (i) follows. To prove (ii), we assume that $\varepsilon > 0$ is given. Then we can write $f = f_1 + f_2$ where f_1 is a bounded function with compact support on $\bar{B}$, and $\|f_2\|_{p,q} < \varepsilon$. Moreover, since $(n+q)r > np$, we have for all $0 < \varrho < 1$ sufficiently close to 1, $(1-\varrho)^{(n+q)/p - n/r} < \varepsilon/(\|f_1\|_{\infty,q} + 1)$. It follows that

$$M_r(Pf_1:\varrho) \leqslant \|f_1\|_{\infty,q} \leqslant \varepsilon(1-\varrho)^{-[(n+q)/p - n/r]},$$

and, thus, by (i)

$$\begin{aligned}M_r(Pf:\varrho) &\leqslant M_r(Pf_1:\varrho) + M_r(Pf_2:\varrho) \\ &\leqslant M_r(Pf_1:\varrho) + c(1-\varrho)^{-[(n+q)/p - n/r]} \|f_2\|_{p,q} \\ &< \varepsilon(1+c)(1-\varrho)^{-[(n+q)/p - n/r]}\end{aligned}$$

where $c > 0$ is a constant independent of ϱ and ε. This proves (ii).

To prove (iii), we fix r with $1 < r \leqslant \infty$ and define, for each $f \in L_q^p$ with $1 \leqslant p \leqslant r$

$$\{Tf\}(\varrho) = (1-\varrho)^{-n/r} M_r(Pf:\varrho) \qquad (0 < \varrho < 1).$$

By (i), Tf is a well-defined function from $(0, 1)$ to $\boldsymbol{R}_+$ with

$$\{Tf\}(\varrho) \leqslant c_1(1-\varrho)^{-(n+q)/p} \|f\|_{p,q} \qquad (0 < \varrho < 1)$$

where $c_1 = c_1(p, q, r) > 0$ is a constant. Moreover, T is sublinear on L_q^p, that is for $f_1, f_2 \in L_q^p$, $T(f_1 + f_2) \leqslant Tf_1 + Tf_2$ on (0,1). It also follows that T is of strong type (∞, ∞). When, on the other hand, $p < \infty$ we have

$$\begin{aligned}\{\varrho \in (0, 1)\colon \{Tf\}(\varrho) > s\} &\subset \{\varrho \in (0, 1)\colon c_1(1-\varrho)^{-(n+q)/p} \|f\|_{p,q} > s\} \\ &= \{\varrho \in (0, 1)\colon (1-\varrho)^{n+q} < (c_1 s^{-1} \|f\|_{p,q})^p\}.\end{aligned}$$

Let ν_q be the measure $d\nu_q(\varrho) = -d(1-\varrho)^{n+q}$ on $(0, 1)$. It follows that

$$\nu_q\{\varrho \in (0, 1)\colon \{Tf\}(\varrho) > s\} \leqslant (c_1 s^{-1} \|f\|_{p,q})^p,$$

i.e. as a mapping from the measure space $(\bar{B}, dv_q)$ to the measure space $((0, 1), d\nu_q)$, T is of weak type (p, p) for $1 \leqslant p \leqslant r$ $(1 \leqslant r \leqslant \infty)$, and its weak

(p, p)-norm is c_1. Thus, by the Marcinkiewicz interpolation theorem, T is of strong type (p, p) for $1 < p < r$, i.e.

$$\Big\{\int_0^1 [\{Tf\}(\varrho)]^p\,dv_q(\varrho)\Big\}^{1/p} \leqslant c_2 c_1 \|f\|_{p,q}$$

where $c_2 = c_2(p, r) \equiv 2\{(1-p/r)^{-1}+(p-1)^{-1}\}^{1/p}$ (and thus $c_2(p, \infty) = 2(p/(p-1))^{1/p}$. It follows that

$$\int_0^1 M_r^p(Pf:\varrho)(1-\varrho)^{n+q-pn/r-1}\,d\varrho \leqslant [(n+q)^{-1/p}c_1c_2]^p\|f\|^p_{p,q} \tag{3.2}$$

which is the desired result for $k = p$. For $p < k < \infty$, we have by (i)

$$\begin{aligned}
&\int_0^1 M_r^k(Pf:\varrho)(1-\varrho)^{k[(n+q)/p-n/r]-1}\,d\varrho \\
&\quad = \int_0^1 M_r^p(Pf:\varrho)(1-\varrho)^{n+q-pn/r-1}\{(1-\varrho)^{(n+q)/p-n/r}M_r(Pf:\varrho)\}^{k-p}\,d\varrho \\
&\quad \leqslant (c_1\|f\|_{p,q})^{k-p}\int_0^1 M_r^p(Pf:\varrho)(1-\varrho)^{n+q-pn/r-1}\,d\varrho,
\end{aligned}$$

and the desired estimate follows from (3.2). ∎

For $0 < p \leqslant \infty$ and $q \geqslant 0$, we let $A^p_q = A^p_q(B)$ denote the subspace of L^p_q consisting of holomorphic functions on B. In particular, when $q = 0$ we obtain the Hardy-class $H^p = A^p_0$, which we identify in the usual way as a subspace of $L^p_0 = L^p(\partial B)$ (see [21]). Moreover, using the notation of Section 1, we see that A^2_q is the functional Hilbert space $\mathscr{H}_{n+q}$ $(q \geqslant 0)$ with the reproducing kernel

$$k_q(z, \zeta) \equiv K_{n+q}(z, \zeta) = (1-\langle z, \zeta\rangle)^{-(n+q)} \qquad (z, \zeta \in B). \tag{3.3}$$

Let $\phi \in \mathrm{Aut}(B)$. Then, by (1.2), for any $z, \zeta \in B$

$$k_q(z, \zeta) = k_q(\phi(z), \phi(\zeta))\{J_\phi(z)\overline{J_\phi(\zeta)}\}^{(n+q)/(n+1)} \qquad (q \geqslant 0). \tag{3.4}$$

In particular, for any $z \in \bar{B}$

$$dv_q(z) = |J_\phi(z)|^{-2(n+q)/(n+1)}dv_q(\phi(z)) \qquad (q \geqslant 0). \tag{3.5}$$

By H and SH we denote the classes of all (real-valued) harmonic and subharmonic, respectively, functions on B. Thus $u \in H$ if and only if $\pm u \in SH$. For $q > 0$, we define $(SH)^1_q = SH \cap L^1_q$, while for $q = 0$ we let $(SH)^1_0$ denote the class of all $u \in SH$ such that $\sup_{0<r<1}\int_{\partial B}|u_r|\,dv_0 < \infty$. In this case we note that $u \in (SH)^1_0$ if and only if there exists a finite Borel measure $\hat{u}$ on ∂B such that its Poisson integral is the least harmonic majorant of u on B (see, for example, Stein [21, p. 8]).

We also consider the usual classes PH and PSH of pluri-harmonic and pluri-subharmonic, respectively functions on B. Evidently, $H \subset SH$, $PSH \subset SH$ and $PH = PSH \cap H$. Finally, parallel to the class $(SH)_q^1$ $(q \geqslant 0)$, we let $H_q^1 = H \cap (SH)_q^1$, $(PSH)_q^1 = PSH \cap (SH)_q^1$ and $(PH)_q^1 = PH \cap (SH)_q^1$.

We now prove a sequence of results which are of interest in their own right. The first result below is the L^p-version of Theorem 1.10.

THEOREM 3.3. *Let* $0 < p \leqslant \infty$ *and* $q \geqslant 0$, *and let* $\phi \in \mathrm{Aut}(B)$. *Then the mapping* T_ϕ, *given by* $T_\phi f = (f \circ \phi)\cdot[J_\phi]^{2(n+q)/p(n+1)}$ *is a linear isometry of* L_q^p *onto* L_q^p *and of* A_q^p *onto* A_q^p.

Proof. The theorem is trivial for $p = \infty$. For $0 < p < \infty$, we use (3.5) with $w = \phi(z)$, $z \in \bar{B}$. Thus

$$\|T_\phi f\|_{p,q}^p = \int |f(\phi(z))|^p \cdot |J_\phi(z)|^{2(n+q)/(n+1)} dv_q(z)$$
$$= \int |f(w)|^p dv_q(w) = \|f\|_{p,q}^p,$$

concluding the proof. ∎

THEOREM 3.4. *Let* $q \geqslant 0$ *and let* $u \in (PSH)_q^1$. *Then for any* $z \in B$,

$$u(z) \leqslant \int u(\zeta)\frac{|k_q(z,\zeta)|^2}{k_q(z,z)} dv_q(\zeta). \tag{3.6}$$

Equality, at some point $z \in B$, *holds if and only if* $u \in (PH)_q^1$. *Here, for* $q = 0$, $u\,dv_q$ *in the above integral has to be replaced by the finite Borel measure* $d\hat{u}$ *on* ∂B.

Proof. By the submean-value property of subharmonic functions we have

$$\psi(0) \leqslant \int_{\partial B} \psi(r\eta)dv_0(\eta) \qquad (0 \leqslant r < 1)$$

for any $\psi \in SH$. In particular, if also $\psi \in (SH)_q^1$, $q > 0$, then

$$\psi(0)\int_0^1 r^{2n-1}(1-r^2)^{q-1}dr \leqslant \frac{\Gamma(n)}{2\pi^n}\int_B \psi(\eta)(1-\|\eta\|^2)^{q-1}dv(\eta)$$

or

$$\psi(0) \leqslant \int \psi(\eta)dv_q(\eta).$$

This inequality is also true for $q = 0$, provided that $\psi\,dv_q$ is replaced by $d\hat{\psi}$. It is also clear that we have equality in (3.7) if and only if $\psi \in H_q^1$.

Let $\phi \in \mathrm{Aut}(B)$ with $\phi(0) = z$. By assumption, $u \circ \phi \in (SH)_q^1$ and hence by (3.7), using the $q = 0$ convention,

$$u(z) = u \circ \phi(0) \leqslant \int u \circ \phi(\eta)dv_q(\eta). \tag{3.8}$$

On the other hand, by (3.5)

$$\int u(\phi(\eta))dv_q(\eta) = \int u(\phi(\eta))|J_\phi(\eta)|^{-2(n+q)/(n+1)}dv(\phi(\eta)).$$

Let $\zeta = \phi(\eta)$. Then by (3.3) and (3.4),

$$k_q(z,\zeta) = k_q(\phi(0), \phi(\eta)) = \{J_\phi(0)\overline{J_\phi(\eta)}\}^{-(n+q)/(n+1)}.$$

In particular,

$$|J_\phi(\eta)|^{-2(n+q)/(n+1)} = \frac{|k_q(z,\zeta)|^2}{k_q(z,z)},$$

and hence

$$\int u(\phi(\eta))dv_q(\eta) = \int u(\zeta)\frac{|k_q(z,\zeta)|^2}{k_q(z,z)}dv_q(\zeta). \tag{3.9}$$

Note that by passing to the limit as $q \to 0^+$, (3.9) remains valid for $q = 0$ provided udv_q is replaced by $d\hat{u}$. Combining (3.8) and (3.9) yields (3.6).

If $u \in (PH)^1_q$, then (3.6) must also hold with u replaced by $-u$, so in fact we have equality in (3.6). Conversely, if equality holds for some point $z \in B$, then it follows from (3.8) that for some $\phi \in \mathrm{Aut}(B)$ with $\phi(0) = z$,

$$u(\phi(0)) = \int u(\phi(\eta)dv_q(\eta)),$$

and hence $u \circ \phi \in H^1_q$. This implies, since $u \in PSH$, that $u \circ \phi \in H \cap PSH = PH$, and thus $u \circ \phi \in (PH)^1_q$. It follows that $u = (u \circ \phi) \circ \phi^{-1} \in (PH)^1_q$, and the proof is complete. ■

As a corollary of this theorem, we have:

COROLLARY 3.5. *Let* $0 < p < \infty$, $q \geqslant 0$ *and let* $f \in A^p_q$. *Then:*

(i) *For any* $\zeta \in B$

$$|f(\zeta)|^p \leqslant \int |f(z)|^p \frac{|k_q(\zeta, z)|^2}{k_q(\zeta,\zeta)} dv_q(z)$$

with equality at some point $\zeta \in B$ *if and only if* f *is constant on* B. *Here for* $q = 0$, f *in the above integral is understood as the boundary value function* f_* *of* $f \in A^p_0 = H^p$;

(ii) *For any* $\zeta \in B$

$$|f(\zeta)| \leqslant \{k_q(\zeta,\zeta)\}^{1/p} \|f\|_{p,q}$$

with equality at some point $\zeta \in B$ *if and only if* $f(z) \equiv \lambda(1 - \langle z, \zeta\rangle)^{-2(n+q)/p}$ *for some constant* $\lambda \in C$ *and every* $z \in B$.

Proof. Since $|f|^p \in (PSH)^1_q$, the inequality in (i) follows directly from Theorem 3.4. Moreover, by the same theorem, equality in (i) holds for some point $\zeta \in B$ if and only if $|f|^p$ is also pluri-harmonic on B. This, since $f \in \mathcal{O}(B)$, is equivalent to f being a constant on B, and (i) follows.

To prove item (ii), we fix a point $\zeta \in B$ and define a function g in A_q^p by letting $g(z) \equiv f(z)\cdot(1-\langle z, \zeta\rangle^{2(n+q)/p}$ $(z \in B)$. Item (ii) now follows by using item (i) with g in place of f. ∎

Item (ii) of the above corollary admits a pluri-subharmonic version which is not as sharp:

COROLLARY 3.6. *Let $q \geqslant 0$ and let $u \in (PSH)_q^1$ be non-negative on B, Then for any $\zeta \in B$,*

$$u(\zeta) \leqslant 2^{2(n+q)} k_q(\zeta, \zeta) \int u dv_q,$$

where for $q = 0$, udv_q is replaced by $d\hat{u}$.

Proof. This follows from Theorem 3.4 and by observing that for $z \in \bar{B}$ and $\zeta \in B$, $|k_q(z, \zeta)| \leqslant 2^{n+q} k_q(\zeta, \zeta)$. ∎

For $q \geqslant 0$ and $p > 1$, we consider the class $(SH)_q^p$ consisting of all $u \in SH$ such that $|u|^p \in (SH)_q^1$. Evidently, $(SH)_q^p$ is a subclass of $(SH)_q^1$, and for $q = 0$, $u \in (SH)_0^p$ if and only if there exists a boundary function u_* in L_0^p such that its Poisson integral is the least harmonic majorant of u on B (see Stein [21], p. 8), i.e. $d\hat{u} = u_* dv_0$ on ∂B with $u_* \in L_0^p$. For $u \in (SH)_0^p$ we identify $\|u\|_{p,0}$ with $\|u_*\|_{p,0} = \sup_{0<r<1} \|u_r\|_{p,0} < \infty$. We also let $(PSH)_q^p = PSH \cap (SH)_q^p$.

For z, $\zeta \in \bar{B}$ and $q \geqslant 0$, we consider the kernel

$$\mathscr{K}_q(z, \zeta) = \frac{|k_q(z, \zeta)|^2}{k_q(z, z)} = \left(\frac{1-\|z\|^2}{|1-\langle z, \zeta\rangle|^2}\right)^{n+q} \tag{3.10}$$

and the operator

$$\{\mathscr{P}_q f\}(z) = \int f(\zeta) \mathscr{K}_q(z, \zeta) dv_q(\zeta). \tag{3.11}$$

Note that $\mathscr{K}_0$ is the familiar *Poisson–Szegö kernel* and that $\mathscr{P}_0$ reduces to the identity operator on L_0^1 (see Rudin [19, p. 40]). In our case, $\mathscr{K}_q$ is measurable on $\bar{B} \times \bar{B}$ with

$$|\mathscr{K}_q(z, \zeta)| \leqslant 2^{n+q} |1-\langle z, \zeta\rangle|^{-(n+q)}.$$

It follows from Theorem 3.2, that the operator $\mathscr{P}_q$ shares with the operator P there the properties (i)–(iii) listed in the theorem. These properties when combined with Theorem 3.4 yield the following result:

THEOREM 3.7. *Let $q \geqslant 0$. Then:*

(i) *For $1 < p < r \leqslant \infty$, $p \leqslant k < \infty$ and $u \in (PSH)_q^p$;* we have

$$\left\{\int_0^1 M_r^k(u:\varrho)(1-\varrho)^{k[(n+q)/p-n/r]-1} d\varrho\right\}^{1/k} \leqslant c\|u\|_{p,q}$$

with a constant $c = c(k, p, r) > 0$;

(ii) *For* $0 < p < r \leqslant \infty$, $p \leqslant k < \infty$ *and* $f \in A^p_q$, *we have*

$$\{\int_0^1 M_r^k(f:\varrho)(1-\varrho)^{k[n+q)/p-n/r]-1}\, d\varrho\}^{1/k} \leqslant c\,\|f\|_{p,q}$$

with a constant $c = c(k,\ p,\ q,\ r) > 0$.

Proof. As $|u| \in (PSH)^p_q$, we may assume in (i) that $u \geqslant 0$ on B, and since $(PSH)^p_q \subset (PSH)^1_q$, we have by Theorem 3.4 and (3.10)-(3.11), that $u \leqslant \mathscr{P}_q u$. Since also u and the kernel $\mathscr{K}_q$ are non-negative, we deduce that $M_r(u:\varrho) \leqslant M_r(\mathscr{P}_q u:\varrho)$ for any $0 \leqslant \varrho < 1$. Here for $q = 0$, $\mathscr{P}_q u$ is understood as $\mathscr{P}_0 u_*$ ($u_* \in L^p_0$). Item (i) for $q > 0$ now follows immediately from Theorem 3.2 (iii) with $\mathscr{P}_q$ in place of P. If, on the other hand, $q = 0$ then since $u \in (PSH)^p_0$, there exists a sequence u_{r_j} of dilations by r_j ($0 < r_j < 1$) of u such that $u_{r_j} \in L^p_0 \cap PSH$ ($j = 1, 2, \ldots$) and

$$\lim_{j \to \infty} \|u_{r_j}\|_{p,0} = \|u_*\|_{p,0} \equiv \|u\|_{p,0}.$$

We now apply again Theorem 3.2 (iii) with u_{r_j} in place of f and $\mathscr{P}_0$ in place of P, and note that $M_r(u_{r_j}:\varrho) \leqslant M_r(\mathscr{P}_0 u_{r_j}:\varrho)$ for any $0 \leqslant \varrho < 1$. Letting $j \to \infty$ gives item (i) for $q = 0$, and hence (i) follows.

Item (ii) follows by applying (i) to the pluri-subharmonic function $|f|^{p/2}$, and replacing p, r, k by 2, $2r/p$, $2k/p$, respectively. This concludes the proof. ∎

COROLLARY 3.8 *Let* $q_1, q_2 \in [0, \infty)$ *and* $p_1, p_2 \in (0, \infty]$ *be related by* $(n+q_1)/p_1 = (n+q_2)/p_2$. *Then*:

(i) *If also* $1 < p_1 \leqslant p_2 < \infty$, *then* $\|\mathscr{P}_{q_1} f\|_{p_2,q_2} \leqslant c_1 \|f\|_{p_1,q_1}$ *for every* f *in* $L^{p_1}_{q_1}$;

(ii) *If also* $1 < p_1 \leqslant p_2 < \infty$, *then* $\|u\|_{p_2,q_2} \leqslant c_2 \|u\|_{p_1,q_1}$ *for every* u *in* $(PHS)^{p_1}_{q_1}$;

(iii) *If also* $0 < p_1 \leqslant p_2 < \infty$, *then* $\|f\|_{p_2,q_2} \leqslant c_3 \|f\|_{p_1,q_1}$ *for every* f *in* $A^{p_1}_{q_1}$. *Here* $c_j = c_j(p_1, q_1, p_2, q_2) > 0$ *is a constant for* $j = 1, 2, 3$.

Proof. To prove item (i), we consider first the case $p_1 = p_2$. In this case $q_1 = q_2$. If $q_1 = 0$ there is nothing to prove, for $\mathscr{P}_0$ is the identity operator on L^1_0. If $q_1 > 0$, then (i) is a special case of Theorem 3.1 with q_1 in place of $r = q$, p_1 in place of p and $\mathscr{K}_{q_1}$ in place of K. We now consider the case $p_1 < p_2$ (and hence $q_1 < q_2$). In this case, (i) is a special case of Theorem 3.2 (iii) with p_1 in place of p, q_1 in place of q, p_2 in place of $r = k$, q_2 in place of $k(n+q)/p - n$ and $\mathscr{K}_{q_1}$ in place of K.

We now prove item (ii). If $p_2 = \infty$ then $p_1 = \infty$, and since $u \in SH$ we have that $\|u\|_{\infty,q_2} = \|u\|_{\infty,q_1}$ for any $q_1, q_2 \geqslant 0$. Thus (ii) is valid when $p_2 = \infty$. When $p_1 = p_2 < \infty$ we find that $q_1 = q_2$, and thus (ii) holds trivially. We may therefore assume that $1 < p_1 < p_2 < \infty$. In this case, (ii) is a special case of Theorem 3.7 (i) with p_1 in place of p, q_1 in place of q, p_2 in place of $r = k$ and q_2 in place of $k(n+q)/p-n$.

Item (iii) follows by applying (ii) to the pluri-subharmonic function $|f|^{p_1/2}$, and replacing p_1, p_2 by 2, $2p_2/p_1$ respectively. Alternatively, (iii) also follows from Theorem 3.7 (ii), as in the above proof of (iii). This concludes the proof. ∎

We conclude this section with the following well-known projection theorem. It is due to Korányi and Vági [16] when $q = 0$ and to Forelli and Rudin [14] when $q > 0$.

THEOREM 3.9. *Let $q \geqslant 0$ and $1 < p < \infty$. Then the operator*

$$\{P_q\}(z) = \int f(\zeta)k_q(z, \zeta)dv_q(\zeta)$$

is a continuous projection of L^p_q onto A^p_q with norm $m_q(p)$ satisfying $m_q(2) = 1$ and $m_q(p) = m_q(p')$ where $p' = p/(p-1)$. Moreover, P_q is the orthogonal projection of L^2_q onto A^2_q. In particular, the dual space of A^p_q is isomporphic to $A^{p'}_q$, with the duality given by the L^2_q-pairing.

Proof. The continuity of P_q is a special case of Theorem 3.1 in the case $q > 0$. The case $q = 0$ is contained in [16]. The remaining properties follow from the reproducing property of k_q. ∎

4. Norm estimates

In Section 5 we will need estimates for operators of the form

$$\{P_{t,r}f\}(z) = \int f(\zeta)L_t(z, \zeta)dv_r(\zeta) \qquad (r \geqslant 0,\ t \in \mathbf{R})$$

where L_t is a measurable kernel on $\bar{B} \times \bar{B}$ satisfying

$$|L_t(z, \zeta)| \lesssim |1-\langle z, \zeta\rangle|^{-(n+t)}.$$

For $s \in \mathbf{R}$, we let $\{M_s f\}(z) = (1-\|z\|^2)^s f(z)$. It follows that if $r > 0$ and $r-s > 0$, then $P_{r-s,r} = cP_{r-s,r-s}M_s$ where $c = c(r, s) > 0$ is a constant.

LEMMA 4.1. *Let $0 < p < \infty$, $q \geqslant 0$ and $s \in \mathbf{R}$ such that $q+ps > 0$. Then:*

(i) *If also $1 \leqslant p < \infty$ and $q > 0$, then, for $r > (q+ps)/p$, $P_{r-s,r}$ maps L^p_{q+ps} continuously into L^p_q;*

(ii) *If also $0 < p \leqslant 1$ and $q \geqslant 0$, then, for $r > \{n(1-p)+q+ps\}/p$, $P_{r-s,r}$ maps A^p_{q+ps} continuously into L^p_q;*

Proof. To prove (i), we observe that, since $0 < q < (r-s)p$, Theorem 3.1 applies with $r-s$ in place of r and $P_{r-s,r-s}$ in place of P. Thus, for $f \in L^p_{q+ps}$,

$$\|P_{r-s,r}f\|_{p,q} = c_1\|P_{r-s,r-s}M_s f\|_{p,q} \leqslant c_2\|M_s f\|_{p,q} = c_3\|f\|_{p,q+ps}$$

where $c_j = c_j(q, p, r, s) > 0$ is a constant for $j = 1, 2, 3$, and (i) follows.

To prove (ii), we use Corollary 3.8 (iii) to deduce that, for f in $\mathcal{O}(B)$,

$$\begin{aligned} |\{P_{r-s,r}f\}(z)| &\leqslant c_1 \int |f(\zeta)(1-\langle\zeta, z\rangle)^{-(n+r-s)}|\, dv_r(\zeta) \\ &\leqslant c_2\{\int |f(\zeta)(1-\langle\zeta, z\rangle)^{-(n+r-s)}|^p\, dv_{p(n+r)-n}(\zeta)\}^{1/p} \end{aligned}$$

where $c_j = c_j(p, r, s) > 0$ is a constant ($j = 1, 2$). Thus by Fubini's theorem and Theorem 2.5, since $p(n+r-s) > n+q$, we have

$$\begin{aligned} \|P_{r-s,r}f\|_{p,q}^p &\leqslant c_3 \int |f(\zeta)|^p I_{p(n+r-s),q}(\zeta)\, dv_{p(n+r)-n}(\zeta) \\ &\leqslant c_4 \|f\|_{p,q+ps}^p, \end{aligned}$$

where $c_j = c_j(q, p, r, s) > 0$ is a constant ($j = 3, 4$), and the proof is complete. ∎

Unfortunately, the preceding result fails when $q = 0$ and $p > 1$. However, we do have:

Lemma 4.2. *Let* $1 \leqslant p \leqslant \infty$, $r \geqslant 0$ *and* $s > 0$. *Then* $P_{r-s,r}$ *is a continuous operator on* L_0^p.

Proof. We consider first the case $r > 0$. In this case we may assume without loss of generality that $0 < s < r$. Indeed, if $0 < r \leqslant s$ we choose t with $0 < t < r \leqslant s$ and thus, since $|1-\langle z, \zeta\rangle|^{s-t} \leqslant 2^{s-t}$, $P_{r-s,r}$ is *a fortiori* of type $P_{r-t,r}$. Now let $f \in L_0^p$ and consider its Poisson integral $\tilde{f}$. Thus $\tilde{f}$ is a (complex-valued) harmonic function on B whose restriction to ∂B is f. In particular, for any $0 < \varrho < 1$ we have $M_p(\tilde{f}:\varrho) \leqslant \|f\|_{p,0}$, and thus for any $q > 0$

$$\|\tilde{f}\|_{p,q} \leqslant \|f\|_{p,0} \qquad (q > 0). \tag{4.1}$$

Indeed, (4.1) is trivial for $p = \infty$, and to prove it for $p < \infty$ we observe that

$$\begin{aligned} \|\tilde{f}\|_{p,q}^p &= \frac{\Gamma(n+q)}{\Gamma(n)\Gamma(q)} \int_0^1 M_p^p(\tilde{f}:\sqrt{\varrho})(1-\varrho)^{q-1}\varrho^{n-1}\, d\varrho \\ &\leqslant \|f\|_{p,0}^p \frac{\Gamma(n+q)}{\Gamma(n)\Gamma(q)} \int_0^1 (1-\varrho)^{q-1}\varrho^{n-1}\, d\varrho = \|f\|_{p,0}^p. \end{aligned}$$

We assume now that $0 < s < r$. We have

$$|\{P_{r-s,r}\tilde{f}\}(z)| \leqslant c_0 \int |\tilde{f}(\zeta)||1-\langle\zeta, z\rangle|^{-(n+r-s)}\, dv_r(\zeta)$$

where $c_0 > 0$ is a constant. For the case $p = 1$, we obtain from Fubini's theorem, Theorem 2.5, and (4.1)

$$\begin{aligned} \|P_{r-s,r}f\|_{1,0} &= \int_{\partial B} |\{P_{r-s,r}\tilde{f}\}(z)|\, dv_0(z) \leqslant c_0 \int |\tilde{f}(\zeta)| \cdot I_{n+r-s,0}(\zeta)\, dv_r(\zeta) \\ &\leqslant c_0 c \int |\tilde{f}(\zeta)|\, dv_s(\zeta) = c_0 c \|\tilde{f}\|_{1,s} \leqslant c_0 c \|f\|_{1,0}, \end{aligned}$$

where $c = c(r, s) > 0$ is a constant. Similarly, for the case $p = \infty$

$$\|P_{r-s,r}f\|_{\infty,0} \leqslant c_0 \|\tilde{f}\|_{\infty,r} (\sup_{z \in \partial B} I_{n+r-s,r}(z))$$

$$\leqslant c_0 d \|\tilde{f}\|_{\infty,r} \leqslant c_0 d \|f\|_{\infty,0},$$

where $d = d(r, s) > 0$ is a constant. Thus it follows from the Riesz–Thorin interpolation theorem that $P_{r-s,r}$ is continuous on L_0^p for $1 \leqslant p \leqslant \infty$ (with norm $\leqslant c_0 c^{1/p} d^{1/p'}$ where $1/p + 1/p' = 1$).

This proves the lemma for the case $r > 0$. The case $r = 0$ is similar but simpler since harmonic extensions to B of functions in L_0^p are not needed here in the proof. The lemma is now complete. ■

The case $q = 0$ of the following result can be found in a slightly different form in Folland and Stein [13].

Proposition 4.3. *Let* $q \geqslant 0$, $1 < p_1 < p_2 < \infty$, *and* $s = (n+q)/p_1 - n/p_2$. *Then* $P_{q-s,q}$ *maps* $L_q^{p_1}$ *continuously into* $L_0^{p_2}$.

The proof of this proposition will be facilitated by a few preliminary lemmas. For $z \in \bar{B}$, $\delta \geqslant 0$, let

$$Q_\delta(z) = \{\zeta \in \bar{B}\colon |1 - \langle z, \zeta \rangle| < \delta\}, \qquad \tilde{Q}_\delta(z) = \bar{B} \backslash Q_\delta(z).$$

Note that $U(Q_\delta(z)) = Q_\delta(Uz)$ for any unitary transformation U of C^n, and thus $v_q(Q_\delta(z))$ is a non-negative radially symmetric function on $\bar{B}$, for $q \geqslant 0$. Note also that $Q_\delta(z) = \bar{B}$ for $\delta > 2$ and tat $v_q(Q_\delta(z)) = 0$ for $\delta \leqslant 1 - \|z\|$.

We shall need the elementary estimate $v_0(Q_\delta(z)) \leqslant c(0)\delta^n$, $c(0) = \Gamma(n+1)/(4\Gamma^2(n/2+1))$ (see Rudin [19, p. 67]). From this is follows easily that

$$v_q(Q_\delta(z)) \leqslant c(q)\delta^{n+q} \tag{4.2}$$

where

$$c(q) = 2^{q-2} n\Gamma(n+q)/(\Gamma(q+1)\Gamma^2(n/2+1)) \qquad (q \geqslant 0).$$

Lemma 4.4. *Let* $q \geqslant 0$ *and* $\delta \geqslant 0$; *and* $z \in \bar{B}$. *Then for* $0 < \varepsilon < n+q$

$$\int_{Q_\delta(z)} |1 - \langle z, \zeta \rangle|^{-(n+q-\varepsilon)} dv_q(\zeta) \leqslant \frac{n+q-\varepsilon}{\varepsilon} c(q)\delta^\varepsilon,$$

and for $\varepsilon > 0$

$$\int_{\tilde{Q}_\delta(z)} |1 - \langle z, \zeta \rangle|^{-(n+q+\varepsilon)} dv_q(\zeta) \leqslant \frac{n+q+\varepsilon}{\varepsilon} c(q)\delta^{-\varepsilon}.$$

Proof. By (4.2), we have

$$\int_{Q_\delta(z)} |1-\langle z, \zeta\rangle|^{-(n+q-\varepsilon)} dv_q(\zeta) = \int_{\delta^{-(n+q-\varepsilon)}}^{\infty} v_q\big(Q_{t^{-1/(n+q-\varepsilon)}}(z)\big)dt$$

$$\leqslant c(q) \int_{\delta^{-(n+q-\varepsilon)}}^{\infty} t^{-(n+q)/(n+q-\varepsilon)}dt$$

$$= \frac{n+q-\varepsilon}{\varepsilon} c(q)\delta^{\varepsilon}.$$

Similarly

$$\int_{\tilde{Q}_\delta(z)} |1-\langle z, \zeta\rangle|^{-(n+q+\varepsilon)} dv_q(\zeta) = \int_{0}^{\delta^{-(n+q+\varepsilon)}} v_q\big(Q_{t^{-1/(n+q+\varepsilon)}}(z)\big)dt$$

$$\leqslant c(q) \int_{0}^{\delta^{-(n+q+\varepsilon)}} t^{-(n+q)/(n+q+\varepsilon)}dt$$

$$= \frac{n+q+\varepsilon}{\varepsilon} c(q)\delta^{-\varepsilon},$$

and the lemma follows. ■

We denote by χ_A the characteristic function of a set A. For $\delta \geqslant 0$ and $z, \zeta \in \bar{B}$ it is clear that

$$\chi_{Q_\delta(z)}(\zeta) = \chi_{Q_\delta(\zeta)}(z). \tag{4.3}$$

LEMMA 4.5. *Let* $1 \leqslant p \leqslant \infty$, $q \geqslant 0$, *and* $q/p < s < n+q$. *For* $\delta \geqslant 0$, *consider the integral operator*

$$\{T_\delta f\}(z) = \int_{Q_\delta(z)} f(\zeta)|1-\langle z, \zeta\rangle|^{-(n+q-s)} dv_q(\zeta).$$

Then

$$\|T_\delta f\|_{p,0} \leqslant c\delta^{s-q/p}\|f\|_{p,q} \qquad (f \in L^p_q)$$

where $c = c(q, p, s) > 0$ *is a constant.*

Proof. For the case $p = \infty$, we have by Lemma 4.4

$$\|T_\delta f\|_{\infty,0} \leqslant \frac{n+q-s}{s} c(q)\delta^{s}\|f\|_{\infty,q}.$$

Similarly, for the case $p = 1$ we have by virtue of (4.2) and Lemma 4.4

$$\|T_\delta f\|_{1,0} \leqslant \frac{n+q-s}{s-q} c(0)\delta^{s-q}\|f\|_{1,q}.$$

For $1 < p < \infty$ we choose a number t with

$$\max(q/p,\, s-(n+q)/p') < t < \min(s,\, (n+q)/p) \qquad (1/p+1/p' = 1),$$

and use Hölder's inequality. This, together with Lemma 4.4, gives

$$|\{T_\delta f\}(z)| \leqslant \left(\int_{Q_\delta(z)} \frac{|f(\zeta)|^p dv_q(\zeta)}{|1-\langle z, \zeta\rangle|^{n+q-pt}}\right)^{1/p} \left(\int_{Q_\delta(z)} \frac{dv_q(\zeta)}{|1-\langle z, \zeta\rangle|^{n+q-p'(s-t)}}\right)^{1/p'}$$
$$\leqslant \left\{c(q)\frac{n+q-p'(s-t)}{p'(s-t)}\right\}^{1/p'} \delta^{s-t} \left(\int_{Q_\delta(z)} \frac{|f(\zeta)|^p dv_q(\zeta)}{|1-\langle z, \zeta\rangle|^{n+q-pt}}\right)^{1/p}.$$

Using Fubini's theorem, (4.3), and a second application of Lemma 4.4 gives

$$\|T_\delta f\|_{p,0} \leqslant \left\{c(q)\frac{n+q-p'(s-t)}{p'(s-t)}\right\}^{1/p'} \left\{c(0)\frac{n+q-pt}{pt-q}\right\}^{1/p} \delta^{s-q/p} \|f\|_{p,q},$$

which is the desired result. ∎

LEMMA 4.6. *Let* $1 \leqslant p < \infty$, $q \geqslant 0$, *and* $s > 0$ *with* $0 < ps < n+q$. *For* $\delta > 0$, *consider the integral operator*

$$\{\tilde{T}_\delta f\}(z) = \int_{\tilde{Q}_\delta(z)} f(\zeta)|1-\langle z, \zeta\rangle|^{-(n+q-s)} dv_q(\zeta).$$

Then

$$|\{\tilde{T}_\delta f\}(z)| \leqslant d\delta^{s-(n+q)/p} \|f\|_{p,q}$$

where $d = d(q,\, p,\, s) > 0$ *is a constant, given by*

$$d = \left\{c(q)\frac{p(n+q-s)}{n+q-s}\right\}^{1/p'} \qquad (1/p+1/p' = 1).$$

Proof. The case $p = 1$ follows from the definition of the set $\tilde{Q}_\delta(z)$. For $1 < p < \infty$ we have by Hölder's inequality

$$|\{\tilde{T}_\delta f\}(z)| \leqslant \|f\|_{p,q}\{\int_{\tilde{Q}_\delta(z)} |1-\langle z, \zeta\rangle|^{-p'(n+q-s)} dv_q(\zeta)\}^{1/p'},$$

and the result follows from the second part of Lemma 4.4 with $\varepsilon = [n+q-ps]/(p-1)$. This concludes the proof. ∎

We can now easily complete the proof of Proposition 4.3, using an argument from [13, Lemma 15.3]. We may assume that the operator $T = P_{q-s,s}$ is of the form

$$\{Tf\}(z) = \int f(\zeta)|1-\langle z, \zeta\rangle|^{-(n+q-s)} dv_q(\zeta)$$

with $q \geqslant 0$, $s = (n+q)/p_1 - n/p_2$, and $1 < p_1 < p_2 < \infty$. To prove that T maps $L_q^{p_1}$ continuously into $L_q^{p_2}$, it is sufficient to show, by the Marcinkiewicz theorem, that T, as an operator from the measure space $(\bar{B},\, v_q)$ to $(\partial B,\, v_0)$, is of weak type $(p_1,\, p_2)$ whenever $s = (n+q)/p_1 - n/p_2$.

For $\delta > 0$, we write, in the notation of Lemmas 4.5 and 4.6, $T = T_\delta + \tilde{T}_\delta$. Fix $t > 0$ and $f \in L_q^{p_1}$ and let $d = d(q, p_1, s)$ be the constant of Lemma 4.6. We set $\delta = (2d\|f\|_{p_1,q}t^{-1})^{p_2/n}$. Thus, by Lemma 4.6 we have $|\{\tilde{T}_\delta f\}(z)| \leqslant t/2$ for all z, and so

$$\text{(4.4)} \qquad v_0\{z \in \partial B\colon |\{Tf\}(z)| > t\} \leqslant v_0\{z \in \partial B\colon |\{T_\delta f\}(z)| > t/2\}.$$

But it follows from Lemma 4.5 with $c = c(q, p_1, s)$ that

$$v_0\{z \in \partial B\colon |\{T_\delta f\}(z)| > t/2\} \leqslant (2c\delta^{s-q/p_1}\|f\|_{p_1,q}t^{-1})^{p_1},$$

so by our choice of δ, we have

$$v_0\{z \in \partial B\colon |\{T_\delta f\}(z)| > t/2\} \leqslant \left(\frac{c}{d}\right)^{p_1}(2d\|f\|_{p_1,q}t^{-1})^{p_2}.$$

Thus by (4.4), T is of weak type (p_1, p_2) and the proof is complete. ∎

5. Sobolev norms

For $s \in \mathbf{Z}_+$, $q > 0$, and $0 < p \leqslant \infty$, the (weighted) Sobolev space $W_{q,s}^p$ is the completion of $C^\infty(\bar{B})$ with respect to the "norm"

$$|||f|||_{p,q;s} = \left\{\sum_{|\alpha|+|\beta| \leqslant s} \frac{|\alpha|!}{\alpha!}\|\partial^\alpha\bar{\partial}^\beta f\|_{p,q}^p\right\}^{1/p}.$$

The space $W_{0,s}^p$ can be defined in a similar manner except that the derivatives are taken with respect to local coordinates in ∂B. It follows that $W_{q,s}^p$ is contained in $L_q^p \equiv W_{q,0}^p$ with a continuous inclusion, and when $1 \leqslant p \leqslant \infty$, $W_{q,s}^p$ is a Banach space normed by $|||\cdot|||_{p,q;s}$. As in L_q^p, $|||\cdot|||_{p,q;s}$ is not a proper norm for $0 < p < 1$, but in this case $\varrho(f, g) = |||f-g|||_{p,q;s}^p$ defines a metric on $W_{q,s}^p$, rendering it into a complete topological vector space.

We let $\mathcal{W}_{q,s}^p = \mathcal{W}_{q,s}^p(B)$ denote the subspace of $W_{q,s}^p$ consisting of holomorphic functions on B. Here, when $q = 0$, we adopt the usual convention of identifying Hardy classes with subspaces of $L_0^p = L_0^p(\partial B)$ (see also 0.3). It follows that $\mathcal{W}_{q,s}^p$ is a subspace of $A_q^p \equiv \mathcal{W}_{q,0}^p$, and, moreover, by Corollary 3.5 (ii) we also deduce that point evaluations on $\mathcal{W}_{q,s}^p$ are bounded and that $\mathcal{W}_{q,s}^p$ is a closed subspace of $W_{q,s}^p$. Note also that for $f \in \mathcal{W}_{q,s}^p$,

$$|||f|||_{p,q;s} = \left\{\sum_{|\alpha| \leqslant s} \frac{|\alpha|!}{\alpha!}\|\partial^\alpha f\|_{p,q}^p\right\}^{1/p}.$$

When $p = 2$, $W_{q,s}^2$ and $\mathcal{W}_{q,s}^2$ are both Hilbert spaces, and the latter coincides with the functional Hilbert space $\tilde{\mathcal{H}}_{n+q,s}$ discussed in Section 1. It follows that

$$\tilde{K}_{n+q,s}(z, \zeta) = H_{n+q,s}(\langle z, \zeta\rangle) \qquad (z, \zeta \in B)$$

is the reproducing kernel of $\mathcal{W}_{q,s}^2$ (see also 0.5 and Corollary 2.3).

We next introduce a different weighted Sobolev norm on holomorphic functions as follows: For $q \geqslant 0$, $0 < p \leqslant \infty$, and $s \in \boldsymbol{R}$ we set

$$\|f\|_{p,q;s} := \|D^s f\|_{p,q}$$

whenever $f \in \mathcal{O}(B)$ (see also 0.2), and we let $A^p_{q,s} = \{f \in \mathcal{O}(B)\colon \|f\|_{p,q;s} < \infty\}$. Thus it follows that $A^p_{q;0} = A^p_q$ and that $A^p_{q;s}$ coincides with the functional Hilbert space $\mathscr{H}_{n+q,s}$ of Section 1. In particular,

$$K_{n+q,s}(z, \zeta) = G_{n+q,2s}(\langle z, \zeta\rangle) \quad (z, \zeta \in B)$$

is the reproducing kernel of $A^2_{q;s}$ (see also 0.5, Theorem 1.9, and Theorem 2.2).

We shall need the following lemma:

LEMMA 5.1. *Let $q \geqslant 0$, $s \in \boldsymbol{R}$ and $\alpha \in \boldsymbol{Z}^n_+$. For $z, \zeta \in B$:*

(i) $\partial^\alpha|_z G_{n+q,s}(\langle z, \zeta\rangle) = (n+q)_{|\alpha|}\bar{\zeta}^\alpha h(n+q+|\alpha|;\, s;\, 1+|\alpha|\colon\langle z, \zeta\rangle)$,

where for $a, b \in \boldsymbol{R}$ and $c > 0$,

$$h(a; b; c\colon\lambda) = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{(a)_m}{(m+c)^b}\lambda^m \quad (\lambda \in \Delta);$$

(ii) *If, in addition, $s < n+q+|\alpha|$, then*

$$h(n+q+|\alpha|;\, s;\, 1+|\alpha|\colon\langle z, \zeta\rangle) = (1-\langle z, \zeta\rangle)^{-(n+q+|\alpha|-s)}F(\langle z, \zeta\rangle),$$

where $F \in \mathcal{O}(\Delta) \cap A_{n+q+|\alpha|-s}(\bar{\Delta})$.

Proof. The first part follows by a direct computation. As for the second part, we note that for $a, b \in \boldsymbol{R}$ and $c > 0$, the two holomorphic functions $h(a; b; c\colon\cdot)$ and $h(a; b; 1\colon\cdot) \equiv G_{a,b}$ exhibit the same behavior on the unit disk Δ. Thus, the second part follows from the first part and Corollary 2.4 (i) ∎

We shall also need the following elementary lemma:

LEMMA 5.2. *Let $q \geqslant 0$, $0 < p < \infty$, and $s \in \boldsymbol{R}$. Then $\mathcal{O}(\bar{B})$ is dense in $A^p_{q,s}$.*

Proof. For $f \in A^p_{q,s}$, and $0 \leqslant \varrho < 1$ the dilation f_ϱ defined by $f_\varrho(z) = f(\varrho z)$ is holomorphic on the ball $B(1/\varrho)$ and moreover $(D^s f)_\varrho = D^s(f_\varrho)$. Thus, it follows that $D^s(f_\varrho)$ converges pointwise (almost everywhere in the case $q = 0$) to $D^s f$. Moreover, by subharmonicity of $|D^s f|^p$, the norms $\|f_\varrho\|_{p,q;s} = \|(D^s f)_\varrho\|_{p,q}$ increase to $\|f\|_{p,q;s} = \|D^s f\|_{p,q}$ as $\varrho \to 1^-$. From this it follows that $D^s(f_\varrho) = (D^s f)_\varrho$ converges to $D^s f$ in the norm of L^p_q as $\varrho \to 1^-$, and the lemma is proved. ∎

A part of the L^p-version of Theorem 1.6 is contained in the next theorem. When $q = 1$ and $p = 2$ this result was also obtained by Boas [7] with B replaced by a domain with a smooth boundary.

THEOREM 5.3. *Let $q \geqslant 0$, $0 < p \leqslant \infty$, and $s \in \boldsymbol{Z}_+$. Then $A^p_{q,s} = \mathscr{W}^p_{q,s}$ and their norms are equivalent.*

Proof. Since D^s is a differential operator of order s, it is clear that

$$\|f\|_{p,q;s} \leqslant c|||f|||_{p,q;s} \qquad (f \in \mathscr{W}^{p}_{q,s})$$

for some constant $c = c(p, s) > 0$. For the reverse inequality, we may in view of Lemma 5.2, assume that $f \in \mathcal{O}(\bar{B})$. In this case, we have by Theorem 1.9, that for any $r \geqslant 0$

$$f(z) = \int \{D^s f\}(\zeta) G_{n+r,s}(\langle z, \zeta\rangle) dv_r(\zeta).$$

It follows from Lemma 5.1 that for any multi-index α with $|\alpha| \leqslant s < n+r+|\alpha|$,

$$\{\partial^\alpha f\}(z) = \int \{D^s f\}(\zeta) K(z, \zeta)\, dv_r(\zeta) \tag{5.1}$$

where

$$K(z, \zeta) = \bar{\zeta}^\alpha (1-\langle z, \zeta\rangle)^{-(n+r+|\alpha|-s)} F_{\alpha,r,s}(\langle z, \zeta\rangle) \tag{5.2}$$

with

$$F_{\alpha,r,s} \in \mathcal{O}(\Delta) \cap \Lambda_{n+r+|\alpha|-s}(\bar{\Delta}).$$

In particular,

$$|K(z, \zeta)| \lesssim |1-\langle z, \zeta\rangle|^{-(n+r)} \qquad (|\alpha| \leqslant s < n+r+|\alpha|,\ r \geqslant 0).$$

Choosing r such that $r > \max\bigl(q/p, (q+n(1-p))/p, s-|\alpha|-n\bigr)$, we obtain the desired result, for $q > 0$ and $0 < p < \infty$, from Lemma 4.1 (with $s = 0$).

The case $q = 0$ and the case $p = \infty$ are more delicate, and so a modification of the above proof is required. From (5.2) we obtain, for $\varepsilon = s-|\alpha|$,

$$K(z, \zeta) = N(z, \zeta)(1-\langle z, \zeta\rangle)^{-(n+r-\varepsilon)} \tag{5.3}$$

with $N \in \Lambda_{n+r-\varepsilon}(\bar{B} \times \bar{B})$ (see 0.6 for notation) where $0 \leqslant \varepsilon < n+r$, $r \geqslant 0$ ($\varepsilon = s-|\alpha|$).

In particular, $|N|$ is bounded by, say, c_0 on $\bar{B} \times \bar{B}$. We assume first that $\varepsilon > 0$ (i.e. $|\alpha| < s$). Then, it follows from Theorem 2.5, that for $p = \infty$ and $q \geqslant 0$,

$$\|\partial^\alpha f\|_{\infty,q} \leqslant c_0 \|D^s f\|_{\infty,q} \cdot \sup_{z \in B} I_{n+r-\varepsilon,r}(z) \leqslant c_0 c_1 \|D^s f\|_{\infty,q}$$

where $c_1 = c_1(r, \varepsilon) > 0$ is a constant, and the desired result follows. For $q = 0$ and $1 \leqslant p \leqslant \infty$, the desired result follows from Lemma 4.2 (with $s = \varepsilon > 0$). Moreover, since the inclusion $A^p_0 \subset A^p_{p\varepsilon}$ is continuous, the desired result for $q = 0$ and $0 < p \leqslant 1$ follows from Lemma 4.1 (ii) (with $q = 0$, $s = \varepsilon > 0$) by choosing $r > \max\bigl([n(1-p)+p\varepsilon]/p, n-\varepsilon\bigr)$. The proof of the theorem is now complete except for the case $\varepsilon = 0$, and $p = \infty$, $q \geqslant 0$ or $0 < p < \infty$, $q = 0$.

In the case $\varepsilon = 0$ (i.e. $|\alpha| = s$), the function N is in $\Lambda_{n+r}(\bar{B} \times \bar{B})$ with $r \geqslant 0$. Thus, it follows from (5.1), (5.3) and (3.3), and from the reproducing property of the kernel $k_r(z, \zeta) = (1-\langle z, \zeta\rangle)^{-(n+r)}$ that

$$\{\partial^\alpha f\}(z) = N(z, z) g(z) + \{Tg\}(z)$$

where

$$\{Tg\}(z) = \int g(\zeta)[N(z, \zeta) - N(z, z)]k_r(z, \zeta)dv_r(\zeta),$$

with $g = D^s f$. Letting $\{Sg\}(z) = N(z, z)g(z)$, it is clear, since N is bounded, that $\|Sg\|_{p,q} \leqslant c_0 \|g\|_{p,q}$. To estimate $\|Tg\|_{p,q}$, we choose $r > 0$, and so $N \in \Lambda_{n+r}(\bar{B} \times \bar{B}) \subset C^n(\bar{B} \times \bar{B})$. In particular,

$$|N(z, \zeta) - N(z, z)| \leqslant c_2 \|z - \zeta\| \leqslant \sqrt{2} c_2 |1 - \langle z, \zeta\rangle|^{1/2},$$

where $c_2 > 0$ is a constant. Thus

$$|N(z, \zeta) - N(z, z)| \cdot |k_r(z, \zeta)| \leqslant \sqrt{2} c_2 |1 - \langle z, \zeta\rangle|^{-(n+r-1/2)},$$

and hence, in the notation of Section 4, the operator T is of the form $T = P_{r-1/2,r}$, $r > 0$. For $p = \infty$, and $q \geqslant 0$, we have from Theorem 2.5 that $\|Tg\|_{\infty,q} \leqslant c_3 \|g\|_{\infty,q}$ where $c_3 = c_3(r) > 0$ is a constant. For $1 \leqslant p \leqslant \infty$, we have by Lemma 4.2 (with $s = 1/2$) that $\|Tg\|_{p,0} \leqslant c_4 \|g\|_{p,0}$ where $c_4 = c_4(r, p) > 0$ is a constant. Similarly, since the inclusion $A^p_{p/2} \supset A^p_0$ is continuous, we have by Lemma 4.1 (ii) (with $q = 0, s = 1/2$) that for $0 < p \leqslant 1$, $\|Tg\|_{p,0} \leqslant c_5 \|g\|_{p,0}$ with the choice of $r > \{n(1-p)+p/2\}/p$, where $c_5 = c_5(r, p) > 0$ is a constant. The proof is now complete. ∎

In view of this theorem we may regard $A^p_{q,s}$ a subspace of $W^p_{q,s}$ when $q \geqslant 0$, $0 < p \leqslant \infty$, and $s \in \mathbf{Z}_+$. Although not relevant to our work here, we note that for $q > 0$ we could define the space $W^p_{q,s}$ for any $s \in \mathbf{R}_+$ by using the complex method to interpolate between integer values of s. In particular, $W^p_{q,s}$ is the restriction to B of the Bessel potential space $\mathscr{L}^p_s(C^n)$ when $1 < p < \infty$ (see, for example, Adams [1, pp. 219–221]). With this definition, we also find that $A^p_{q,s} = W^p_{q,s} \cap \mathcal{O}(B)$ provided $0 < p < \infty$. We will omit the proof, and only mention that the coincidence of these spaces can be deduced from the fact that the spaces $A^p_{q,s}$ interpolate properly, between values of s, with the complex method.

LEMMA 5.4. *Let $f \in A^p_{q,s}$ with $0 < p \leqslant \infty$, $q \geqslant 0$, and $s \in \mathbf{R}$. Then for any $\alpha \in \mathbf{Z}^n_+$ and any $z \in B$*

$$|\{\partial^\alpha f\}(z)| \lesssim \begin{cases} \|f\|_{p,q;s}(1 - \|z\|^2)^{-[(n+q)/p+|\alpha|-s]}, & s < (n+q)/p + |\alpha|, \\ \|f\|_{p,q;s}\|z\|^{-2}\log(1 - \|z\|^2)^{-1}, & s = (n+q)/p + |\alpha|, \\ \|f\|_{p,q;s}, & s > (n+q)/p + |\alpha|. \end{cases}$$

Proof. By Lemma 5.2, we may assume that $f \in \mathcal{O}(\bar{B})$. To prove the lemma, we consider first the case $1 < p < \infty$. In this case we have, by (5.1) and (5.2).

$$\{\partial^\alpha f\}(z) = \int \{D^s f\}(\zeta) K(z, \zeta) dv_r(\zeta) \quad (r \geqslant 0),$$

where

$$|K(z, \zeta)| \leqslant c_0 |1 - \langle z, \zeta\rangle|^{-(n+r+|\alpha|-s)},$$

for some constant $c_0 > 0$, if also $r > s-n-|\alpha|$. For $q > 0$, we choose $r > \max(q/p, s-n-|\alpha|)$ and apply Hölder's inequality with $1/p+1/p' = 1$. This gives

$$|\{\partial^\alpha f\}(z)| \leqslant c_0 c_1 \|D^s f\|_{p,q} \{I_{p'(n+r+|\alpha|-s),(r-q)p'+q}(z)\}^{1/p'}$$

where $c_1 = c_1(r, q, p)$ is given (see 0.4) by

$$c_1(r, q, p) = \frac{1}{B(n, r)} \{B(n, q)\}^{1/p} \{B(n, (r-q)p'+q)\}^{1/p'},$$

and the desired result follows from Theorem 2.5. When $q = 0$, we choose $r > \max(0, s-|\alpha|-n/p)$ and express the above integral in polar coordinates. Apply now Hölder's inequality on the boundary integral of this expression, and use Theorem 2.5. This, using the notation of 0.5, and with $\|f\|_{p,0;s} = 1$, gives

$$\begin{aligned}|\{\partial^\alpha f\}(z)| &\leqslant \frac{c_0}{B(n, r)} \int_0^1 \varrho^{n-1}(1-\varrho)^{r-1} I^{1/p'}_{(n+r+|\alpha|-s)p',0}(\sqrt{\varrho}\, z)\, d\varrho \\ &\leqslant \frac{c_0 c_2}{B(n, r)} \int_0^1 \varrho^{n-1}(1-\varrho)^{r-1} (1-\varrho\|z\|^2)^{-[n/p+r+|\alpha|-s]} d\varrho \\ &= c_0 c_2 F(n/p+r+|\alpha|-s, n; n+r: \|z\|^2) \\ &= c_0 c_2 (1-\|z\|^2)^{-[n/p+|\alpha|-s]} F(n/p'+s-|\alpha|, r; n+r: \|z\|^2),\end{aligned}$$

where $c_2 = c_2(|\alpha|, r, p, s) > 0$ is a constant. The desired result now follows from Stirling's formula and the very definition of hypergeometric functions (see 0.5 and 0.6). This proves the lemma for the case $1 < p < \infty$. The proof for the case $p = \infty$ is very similar, and it could have also been obtained by letting $p \to \infty$ in each step of the above proof for $1 < p < \infty$.

For $0 < p \leqslant 1$, we use Corollary 3.8 (iii) to obtain

$$\|f\|_{p,q;s} = \|D^s f\|_{p,q} \geqslant c\|D^s f\|_{2,t} = c\|f\|_{2,t;s}$$

where $t = 2(n+q)/p-n$ and $c = c(p, q) > 0$ is a constant. The desired estimate now follows from the case $p = 2$, and the proof is complete. ∎

From the well-known characterization of Lipschitz functions in terms of the growth of their derivatives (see [19] and [20], and 0.6), we obtain the following smoothness result for the space $A^p_{q,s}$, which for $q = 0$, $s = 1$ and $n < p < \infty$ is due to Graham [15] and Krantz [17].

COROLLARY 5.5. *Let* $0 < p \leqslant \infty$ *and* $q \geqslant 0$. *Then, for* $s > (n+q)/p$, $A^p_{q,s} \subset \Lambda_{s-(n+q)/p}(\bar{B})$, *and the inclusion is continuous.*

Roughly speaking, the corollary asserts that the number of derivatives lost in the transition from the L^p-category to the continuous-category is $(n+q)/p$. In the case $q = 1$ and $1 \leqslant p < \infty$, this should be compared with the classical

Sobolev imbedding theorem for $W^p_{\cdot,s}$ which gives a loss of $2n/p$ derivatives (see, for example, [1, p.98]). It should also be noted that holomorphic functions in $\Lambda_{s-(n+q)/p}(\bar{B})$ are actually smoother than advertised in "complex tangential" directions. We refer to Rudin [19, pp. 101–109] for more details.

The next lemma is similar to Lemma 5.4.

LEMMA 5.6. *Let $f \in A^p_{q,s}$ with $0 < p \leqslant \infty$, $q \geqslant 0$ and $s \in \boldsymbol{R}$. Then for any $t \in \boldsymbol{R}$ and any $z \in B$*

$$|\{D^t f\}(z)| \lesssim \begin{cases} \|f\|_{p,q;s}(1-\|z\|^2)^{-[(n+q)/p+t-s]}, & s < (n+q)/p+t, \\ \|f\|_{p,q;s}\|z\|^{-2}\log(1-\|z\|^2)^{-1}, & s = (n+q)/p+t, \\ \|f\|_{p,q;s}, & s > (n+q)/p+t. \end{cases}$$

Proof. Again, by Lemma 5.2, we may assume that $f \in \mathcal{O}(\bar{B})$. It follows from (0.1) and Theorem 1.9 that for $r \geqslant 0$

$$\{D^t f\}(z) = \int \{D^s f\}(\zeta) G_{n+r,s-t}(\langle z, \zeta\rangle) dv_r(\zeta) \qquad (r \geqslant 0).$$

Moreover, by Corollary 2.4 (i) we find that

$$|G_{n+r,s-t}(\langle z, \zeta\rangle)| \lesssim |1-\langle z, \zeta\rangle|^{-(n+r+t-s)},$$

if also $r > s-n-t$. The proof now proceeds exactly as in the proof of Lemma 5.4 with t in place of $|\alpha|$. This concludes the proof. ■

We list some additional elementary properties of the spaces $A^p_{q,s}$.

PROPOSITION 5.7. *Let $0 < p \leqslant \infty$, $q \geqslant 0$ and $s \in \boldsymbol{R}$. Then:*

(i) *For any $\alpha \in \boldsymbol{Z}^n_+$ and any $z \in B$, the functional $f \mapsto \{\partial^\alpha f\}(z)$ is continuous on $A^p_{q,s}$;*

(ii) *For any $t \in \boldsymbol{R}$ and any $z \in B$, the functional $f \mapsto \{D^\alpha f\}(z)$ is continuous on $A^p_{q,s}$;*

(iii) *$A^p_{q,s}$ is a complete topological vector space, and is a Banach space for $1 \leqslant p \leqslant \infty$;*

(iv) *For $1 < p < \infty$ and $1/p+1/p' = 1$, the Banach spaces $A^p_{q,s}$ and $A^{p'}_{q,-s}$ are isomorphically dual to each other with respect to the L^2_q-pairing.*

Proof. Items (i) and (ii) follow from Lemmas 5.4 and 5.6, respectively. Item (iii) follows from either (i) or (ii) by a standard normal family argument. Item (iv) follows from Theorem 3.9 and Lemma 1.8. Note that in the notation of Lemma 1.8, the holomorphic L^2_q-pairing is the $A^2_q = \mathcal{H}_{n+q}$-pairing $\langle,\rangle_{n+q}$. Thus, by Lemma 1.8, for $f \in A^p_{q,s}$ and $g \in A^{p'}_{q,-s}$ we have $\langle f, g\rangle_{n+q} = \langle D^s f, D^{-s} g\rangle_{n+q}$, and hence, by Hölder's inequality, $|\langle f, g\rangle_{n+q}| \leqslant \|f\|_{p,q;s}\|g\|_{p',q;-s}$. This concludes the proof. ■

At this stage it is perhaps proper to remark that the norm of $A^p_{q,s}$ was based on the differential operator $D = D_1 = \mathcal{R}+1$. It is easily seen however that it could as well be based on $D_l = \mathscr{R}+l$ for any $l>0$ (see 0.1), i.e. for any $f\in\mathcal{O}(B)$ one has

$$\|D_l^s f\|_{p,q} \simeq \|D^s f\|_{p,q} \qquad (0<p\leqslant\infty,\ q\geqslant 0,\ s\in\boldsymbol{R}). \tag{5.4}$$

For $s\in\boldsymbol{Z}_+$, this result follows directly from Theorem 5.3 while, in general, for any $s\in\boldsymbol{R}$ the result follows by employing the same arguments as in the proof of Theorem 5.3, and by using the identity

$$\{D_l^r f\}(z) = \int\{D^s f\}(\zeta)h(\langle z,\zeta\rangle)dv_q(z)$$

where

$$h(\lambda) = \sum_{m=0}^{\infty}\frac{(n+q)_m}{m!}\frac{(m+l)^r}{(m+1)^s}\lambda^m \qquad (r\in\boldsymbol{R},\ \lambda\in\Delta).$$

With this remark we prove the following lemma:

LEMMA 5.8. *Let* $q\geqslant 0$, *and* $k,\ l>0$ *such that* $k+l=q+2$. *Then*:
(i) *If* $f,\ g\in A_q^2$, *then* $D_{n+q}f$ *and* $D_{n+q+1}g$ *are in* A_{q+2}^2, *and*

$$(n+q)(n+q+1)\int f\bar{g}\,dv_q = \int(D_{n+q}f)\overline{(D_{n+q+1}g)}\,dv_{q+2};$$

(ii) *If* $f\in A_q^p$ *and* $g\in A_q^{p'}$ *with* $1<p<\infty$ *and* $1/p+1/p'=1$, *then*

$$\left|\int f\bar{g}\,dv_q\right| \lesssim \|f\|_{p,kp;1}\|g\|_{p',lp';1}.$$

Proof. Since $A_q^2=\mathscr{H}_{n+q}$ $(q\geqslant 0)$, the first part of item (i) follows from Theorem 1.6 and (5.4). As for the identity of item (i), we let

$$f(z)=\sum_\alpha a_\alpha z^\alpha, \qquad g(z)=\sum_\alpha b_\alpha z^\alpha, \qquad (z\in B).$$

Then

$$\begin{aligned}(n+q)(n+q+1)\int f\bar{g}\,dv_q &= (n+q)(n+q+1)\sum_\alpha\frac{\alpha!}{(n+q)_{|\alpha|}}a_\alpha\bar{b}_\alpha\\ &= \sum_\alpha\frac{\alpha!}{(n+q+2)_{|\alpha|}}(|\alpha|+n+q)a_\alpha\cdot(|\alpha|+n+q+1)\bar{b}_\alpha\\ &= \int(D_{n+q}f)\overline{(D_{n+q+1}g)}\,dv_{q+2},\end{aligned}$$

and the identity follows. To prove item (ii), we assume first that $f,\ g\in\mathcal{O}(\bar{B})$ and use (i). The result then follows from Hölder's inequality and (5.4), and Lemma 5.2. ■

Before proceeding with the next theorem, we shall prepare some basic facts. For $q\geqslant 0$, the measure dv_q when $n=1$ is denoted by dA_q. Thus, dA_1 is the normalized area Lebesgue measure while dA_0 is the normalized arc-length measure on the boundary $\partial\Delta$ of the unit disk Δ. Starting with the "slice integration formula" (see, for example, [19, p. 15])

$$\int_{\partial B} F(\zeta)dv_0(\zeta) = \int_{\partial B} \Big(\int_{\partial\Delta} F(\lambda\zeta)dA_0(\lambda)\Big)dv_0(\zeta)$$

one shows easily that, for $q > 0$,

$$\int_{B} F(\zeta)dv_q(\zeta) = \frac{\Gamma(n+q)}{\Gamma(n)\Gamma(1+q)}\int_{\partial B}\Big(\int_{\Delta} F(\lambda\zeta)dA_q(\lambda)\Big)dv_q(\zeta) \tag{5.5}$$

When $q \to 0^+$, this formula reduces to the previous formula, and hence it is also correct for $q = 0$ (see also 0.3).

A well-known result of Littlewood–Paley [18] states that for any $p \geqslant 2$ there exists a constant $c = c(p) > 0$ such that for $\phi \in A_0^p(\Delta)$

$$\int_{\Delta} |\phi'|^p dA_p \leqslant c \int_{\partial\Delta} |\phi|^p dA_0. \tag{5.6}$$

This leads to the following "Littlewood–Paley inequality in the ball". Recall that $\mathscr{R}$ (see 0.2) is the radial derivative operator.

Proposition 5.9. *Let $f \in A_0^p$ with $p \geqslant 2$. Then*

$$\int_{B} |\mathscr{R}f|^p dv_p \leqslant c\frac{\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)}\int_{\partial B} |f|^p dv_0.$$

In particular,

$$\|Df\|_{p,p} \leqslant d\|f\|_{p,0}$$

where $d = d(p) = 1+\{c(p)/(pB(n, p))\}^{1/p}$.

Proof. For $\zeta \in \bar{B}$, we consider the slice function $f_\zeta(\lambda) = f(\lambda\zeta)$, $\lambda \in \Delta$. Thus $\lambda f_\zeta'(\lambda) = \{\mathscr{R}f\}(\lambda\zeta)$. It follows from (5.5) and (5.6) that

$$\begin{aligned}\int_{B} |\mathscr{R}f|^p dv_p &= \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)}\int_{\partial B}\int_{\Delta} |\lambda|^p|f_\zeta'(\lambda)|^p dA_p(\lambda)dv_0(\zeta)\\ &\leqslant c\frac{\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)}\int_{\partial B}\int_{\partial\Delta} |f_\zeta(\lambda)|^p dA_0(\lambda)dv_0(\zeta)\\ &= c\frac{\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)}\int_{\partial B} |f(\zeta)|^p dv_0(\zeta),\end{aligned}$$

and the result follows. ∎

Another fact that we need concerns the space of holomorphic functions of *bounded mean oscillation* in the ball which is denoted by $BMOA = BMOA(B)$ (see [10]). This space consists of all $f \in A_0^2$ such that

$$\|f\|_{BMOA} \equiv \sup\Big\{\Big|\int_{\partial B} f\bar{g}dv_0\Big|:\ g \in A_0^2,\ \|g\|_{1,0} = 1\Big\} < \infty.$$

Evidently, $BMOA$ is a Banach space with the norm $\|\cdot\|_{BMOA}$ and it serves as a dual of A_0^1. Moreover, the orthogonal projector P_0 (see Theorem 3.9) of L_0^2 onto A_0^2 maps L_0^∞ continuously onto $BMOA$. We also have $A_0^\infty \subset BMOA \subset A_0^p$, $0 < p < \infty$, and the injections are continuous.

LEMMA 5.10. *Let μ be the measure $d\mu(\zeta) = (1 - \|\zeta\|^2)^{-1} dv_1(\zeta)$ on B, and let T be the operator $\{Tf\}(z) = (1 - \|z\|^2)\{Df\}(z)$ on $\mathcal{O}(B)$. Then T maps A_0^2 and $BMOA$ continuously into $L^2(d\mu)$ and $L^\infty(d\mu)$, respectively.*

Proof. For any $f \in \mathcal{O}(B)$ we have, by Theorem 1.6

$$\|Tf\|_{L^2(d\mu)} = \frac{1}{(n+1)!}\|f\|_{2,2;1} \simeq \|f\|_{2,0},$$

and the first part of the lemma follows. In addition, for $z \in B$ and $f \in BMOA$, we have by Theorem 1.9, Corollary 2.4 (i) and Theorem 2.5,

$$\begin{aligned} |\{Tf\}(z)| &= (1 - \|z\|^2)|\{Df\}(z)| \\ &= (1 - \|z\|^2)\left|\int f(\zeta) G_{n,-1}(\langle z, \zeta\rangle)\, dv_0(\zeta)\right| \\ &\leqslant n(1 - \|z\|^2)\|f\|_{BMOA} \sup_{z \in \partial B} I_{n+1,0}(z) \\ &\leqslant nc\|f\|_{BMOA}, \end{aligned}$$

where $c > 0$ is a (universal) constant. It follows that $\|Tf\|_{L^\infty(d\mu)} \leqslant nc\|f\|_{BMOA}$, and the proof is complete. ■

This lemma provides an alternative proof of Proposition 5.9, in the form of the next corollary, which does not rely on the Littlewood–Paley inequality in the disk.

COROLLARY 5.11. *Let $f \in A_0^p$ with $p \geqslant 2$. Then $\|Df\|_{p,p} \lesssim \|f\|_{p,0}$. In particular, $\|\mathcal{R}f\|_{p,p} \lesssim \|f\|_{p,0}$.*

Proof. We consider the measure μ and the operator T of the lemma. In addition, we also consider the orthogonal projector P_0 of L_0^2 onto A_0^2. It follows from the lemma that the operator TP_0 maps L_0^2 and L_0^∞ continuously into $L^2(d\mu)$ and $L^\infty(d\mu)$, respectively. Thus the map TP_0, as a mapping from the measure space $(\partial B, dv_0)$ to the measure space $(B, d\mu)$ is of types $(2, 2)$ and (∞, ∞). It follows from the Riesz–Thorin interpolation theorem that TP_0 is of type (p, p) for $2 \leqslant p \leqslant \infty$. Since P_0 is the identity operator, it also follows that T maps A_0^p continuously into $L^p(d\mu)$ for $2 \leqslant p \leqslant \infty$, which is a reformulation of the corollary. This concludes the proof. ■

The next theorem together with Theorem 5.3 gives the L^p-version of Theorem 1.6.

THEOREM 5.12. *Let* $0 < p \leqslant \infty$, $q_j \geqslant 0$ *and* $s_j \in \mathbf{R}$ $(j = 1, 2)$ *such that* $(q_1 - q_2)/p = s_1 - s_2$.

(i) *If* $0 < p \leqslant \infty$ *and* $q_j > 0$ $(j = 1, 2)$, *then* $A^p_{q_1,s_1} \simeq A^p_{q_2,s_2}$;

(ii) *If* $2 \leqslant p \leqslant \infty$ *and* $q_2 = 0$, *then* $A^p_{0,s_2} \subset A^p_{q_1,s_1}$ *and the inclusion is continuous*;

(iii) *If* $0 < p \leqslant 2$ *and* $q_2 = 0$, *then* $A^p_{q_1,s_1} \subset A^p_{0,s_2}$ *and the inclusion is continuous.*

Proof. The case $p = 2$ of this theorem is contained in Theorem 1.6, while the case $p = \infty$ is completely trivial. In the latter case $s_1 = s_2$, and thus $A^\infty_{q_1,s_1} = A^\infty_{q_2,s_1}$ and the norms are identical. For item (i) it suffices by symmetry to show that for $f \in \mathcal{O}(\bar{B})$ we have $\|D^{s_2}f\|_{p,q_2} \lesssim \|D^{s_1}f\|_{p,q_1}$. By replacing f with $D^{s_2}f$ and s_1 with $s_1 - s_2$ we may assume without loss of generality that $s_2 = 0$. By Theorem 1.9 we have, for $r \geqslant 0$

$$f(z) = \int \{D^{s_1}f\}(\zeta) G_{n+r,s_1}(\langle z, \zeta\rangle)\, dv_r(\zeta) \quad (r \geqslant 0), \tag{5.7}$$

where by Corollary 2.4 (i), if also $r > s_1 - n$

$$|G_{n+r,s_1}(\langle z, \zeta\rangle)| \lesssim |1 - \langle z, \zeta\rangle|^{-(n+r-s_1)}. \tag{5.8}$$

We choose $r > \max(s_1 - n,\ q_1/p,\ q_1/p + n(1-p)/p)$ and apply Lemma 4.1 to obtain

$$\|f\|_{p,q_2} = \|P_{r-s_1,r} D^{s_1} f\|_{p,q_2} \lesssim \|D^{s_1}f\|_{p,q_2+ps} = \|D^{s_1}f\|_{p,q_1},$$

and the desired estimate follows.

To prove (ii), we have to show that, for $f \in \mathcal{O}(\bar{B})$, $\|D^{s_1}f\|_{p,q_1} \lesssim \|D^{s_2}f\|_{p,0}$ with $2 \leqslant p < \infty$. As before, it is sufficient to consider the case $s_2 = 0$. Moreover, since in this case $s_1 = q_1/p$, we have by (i) that $\|D^{s_1}f\|_{p,q_1} \simeq \|Df\|_{p,p}$, i.e. we may also assume that $s_1 = 1$. Hence we only have to show that $\|Df\|_{p,p} \lesssim \|f\|_{p,0}$. But this is exactly Proposition 5.9 or Corollary 5.11.

For item (iii), it suffices once again to only consider the case $s_2 = 0$ and $s_1 = 1$. Thus, for $f \in \mathcal{O}(\bar{B})$ we have to show that $\|f\|_{p,0} \lesssim \|Df\|_{p,p} = \|f\|_{p,p;1}$. For this purpose we must consider separatly the case $0 < p \leqslant 1$ and $1 < p \leqslant 2$.

The case $1 < p \leqslant 2$ can be deduced from item (ii) by a duality argument. Indeed, by Theorem 3.9

$$\|f\|_{p,0} \lesssim \sup\{|\textstyle\int f\bar{g}\, dv_0| : g \in A^{p'}_0,\ \|g\|_{p',0} = 1\} \quad (1/p + 1/p' = 1).$$

Thus, it follows from Lemma 5.8 (ii) (with $q = 0$ and $k = l = 1$) that

$$\|f\|_{p,0} \lesssim \|f\|_{p,p;1} \cdot \sup\{\|g\|_{p',p';1} : g \in A^{p'}_0,\ \|g\|_{p',0} = 1\}.$$

But since $p' \geqslant 2$, it follows from (ii) that, with g as above, $\|g\|_{p',p';1} = \|Dg\|_{p',p'} \lesssim \|g\|_{p',0} = 1$, and thus (iii) is proved in the case $1 < p \leqslant 2$.

For the case $0 < p \leqslant 1$ we choose $r > n(1-p)/p+1$ and apply (5.7)–(5.8) (with $s_1 = 1$) and Corollary 3.8 (iii) (with $p_1 = p$, $q_1 = p(n+r)-n$, $p_2 = 1$ and $q_2 = r$) to obtain

$$|f(z)|^p \lesssim \int |\{Df\}(\zeta)|^p |1-\langle z, \zeta\rangle|^{-p(n+r-1)} dv_{p(n+r)-n}(\zeta).$$

Thus, by Fubini's theorem and Theorem 2.5

$$\begin{aligned}\|f\|_{p,0}^p &\lesssim \int |\{Df\}(\zeta)|^p \cdot I_{p(n+r-1),0}(\zeta) dv_{p(n+r)-n}(\zeta)\\ &\simeq \int |\{Df\}(\zeta)|^p dv_{p-1}(\zeta) = \|Df\|_{p,p}^p = \|f\|_{p,p;1}^p,\end{aligned}$$

and the theorem is proved. ■

We also obtain inclusion relations amongst the spaces $A_{q,s}^p$ for different values of p. Special cases of the following result have been also obtained by Coifman and Rochberg [9], Graham [15] and Krantz [17].

THEOREM 5.13. *Let* $0 < p_1 < p_2 < \infty$, $q_j \geqslant 0$ *and* $s_j \in \boldsymbol{R}$ $(j = 1, 2)$ *such that* $(n+q_1)/p_1-(n+q_2)/p_2 = s_1-s_2$. *Then* $A_{q_1,s_1}^{p_1} \subset A_{q_2,s_2}^{p_2}$ *and the inclusion is continuous.*

Proof. The conditions of the theorem entail the condition $s_1-s_2 > \max((q_1-q_2)/p_1, (q_1-q_2)/p_2)$. Moreover, it is sufficient to show that for $f \in \mathcal{O}(\bar{B})$ we have

$$\|D^{s_2}f\|_{p_2,q_2} \lesssim \|D^{s_1}f\|_{p_1,q_1}.$$

For the case $q_2 > 0$ we have by Theorem 5.12 (i), since $p_2(s_1-s_2)+q_2 > q_1 \geqslant 0$, that

$$\|D^{s_2}f\|_{p_2,q_2} \simeq \|D^{s_1}f\|_{p_2,p_2(s_1-s_2)+q_2}.$$

Since $(n+q_1)/p_1 = [n+p_2(s_1-s_2)+q_2]/p_2$, it follows from Corollary 3.8 (iii) that

$$\|D^{s_1}f\|_{p_2,p_2(s_1-s_2)+q_2} \lesssim \|D^{s_1}f\|_{p_1,q_1}$$

and the desired estimate follows.

In the case $q_2 = 0$, we have $s_1-s_2 = (n+q_1)/p_1-n/p_2 > q_1/p_1 > 0$, and we distinguish two subcases. If also $p_2 \leqslant 2$, then by Theorem 5.12 (iii)

$$\|D^{s_2}f\|_{p_2,0} \lesssim \|D^{s_1}f\|_{p_2,p_2(s_1-s_2)}$$

and again, by Corollary 3.8 (iii), the desired estimate follows.

In the remaining subcase we have $q_2 = 0$ and $p_2 > p$ where $p = \max(p_1, 2)$. Let r be defined by $(n+r)/p = (n+q_1)/p_1$, and thus $r \geqslant q_1 \geqslant 0$. It follows from Theorem 1.9 and Corollary 2.4 (i) that

$$\{D^{s_2}f\}(z) = \int \{D^{s_1}f\}(\zeta) G_{n+r,s_1-s_2}(\langle z, \zeta\rangle) dv_r(\zeta)$$

with

$$|G_{n+r,s_1-s_2}(\langle z, \zeta\rangle)| \lesssim |1-\langle z, \zeta\rangle|^{-(n+r-s_1+s_2)},$$

and hence

$$D^{s_2}f = P_{r-(s_1-s_2),r}D^{s_1}f.$$

Since $s_1-s_2 = (n+r)/p-n/p_2$, it follows from Proposition 4.3 that

$$\|D^{s_2}f\|_{p_2,0} = \|P_{r-(s_1-s_2),r}D^{s_1}f\|_{p_2,0} \lesssim \|D^{s_1}f\|_{p,r}.$$

Once again, by Corollary 3.8 (iii),

$$\|D^{s_1}f\|_{p,r} \lesssim \|D^{s_1}f\|_{p_1,q_1}$$

and the theorem is proved. ■

The next theorem provides additional information to that given by Corollary 5.5 and Theorem 5.13. Before stating the theorem, however, we recall some definitions. The space of holomorphic functions of *vanishing mean oscilation* in the ball, denoted by $VMOA = VMOA(B)$, is a subspace of $BMOA$ consisting of all $f \in BMOA$ so that $f = P_0 g$ for some $g \in C(\partial B)$. Alternatively, $VMOA$ can be characterized as the closure in $BMOA$ *of* $\mathcal{O}(\bar{B})$ (see [10]).

THEOREM 5.14. *Let* $0 < p < \infty$, $q \geqslant 0$ *and* $s \geqslant 0$.

(i) *If* $s > (n+q)/p$ *then* $A^p_{q,s} \hookrightarrow \Lambda_{s-(n+q)/p}(\bar{B})$;

(ii) *If* $0 \leqslant s < (n+q)/p$ *then* $A^p_{q,s} \hookrightarrow A_q^{p(n+q)/[(n+q)-ps]}$;

(iii) *If* $s = (n+q)/p$ *then* $A^p_{q,s} \hookrightarrow VMOA$.

Proof. Assertion (i) is Corollary 5.5. Assertion (ii) is trivial for the case $s = 0$, and it is an immediate consequence of Theorem 5.13 for $0 < s < (n+q)/p$ and $0 < p < \infty$.

To prove assertion (iii), we first show that $A^p_{q,s} \hookrightarrow BMOA$. To this end we choose p_1 with $\max(p, 1) < p_1 < \infty$ and use Theorem 5.13 to conclude that $A^p_{q,s} \hookrightarrow A^{p_1}_{0,n/p_1}$. Since $1 < p_1 < \infty$, we deduce from Proposition 5.7 (v) that $A^{p_1}_{0,n/p_1} = (A^{p_1'}_{0,-n/p_1})^*$ where $1/p_1+1/p_1' = 1$, and the duality is with respect to the L^2_0-pairing. However, once again, by Theorem 5.13, $A^1_0 \hookrightarrow A^{p_1'}_{0,-n/p_1}$ and so $A^p_{q,s} \hookrightarrow (A^{p_1'}_{0,-n/p_1})^* \hookrightarrow (A^1_0)^* = BMOA$. Now, since by Lemma 5.2, $\mathcal{O}(\bar{B})$ is dense in $A^p_{q,s}$ and since $VMOA$ is a closed subspace of $BMOA$ which contains $\mathcal{O}(\bar{B})$ we also conclude that $A^p_{q,s} \hookrightarrow VMOA$, and the proof is complete. ■

For $q = 0$ and $s = 1$, assertion (i) and (ii) of the above theorem are due to Graham [15] and Krantz [17]. Krantz has also proved the inclusion $A^n_{0,1} \hookrightarrow BMOA$ which is part of assertion (iii) of the theorem.*)

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* This assertion has been recently refined and extended in several interesting directions. For details, we refer to Beatrous (Proc. Math. Soc. 97 (1986), 23–29), Burbea (Pacific J. Math. 127 (1987), 1–17), and Ahern and Bruna (Duke Math. J., 56 (1988), 129–142).

6. Projections in Sobolev spaces

It will be convenient to extend the norm $\|\cdot\|_{p,q;s}$ of $A^p_{q,s}$ to non-holomorphic functions. We introduce a differential operator $\mathscr{D}$ in C^n by $\mathscr{D} = D-\bar{\mathscr{R}}$, and thus $\mathscr{D} = \mathscr{R}-\bar{\mathscr{R}}+1$ where (see 0.2) $\mathscr{R}$ is the radial derivative operator. Of course, $\mathscr{D}$ agrees with D on $\mathscr{O}(B)$. For $s \in Z_+$, $q \geqslant 0$ and $0 < p \leqslant \infty$ we define

$$\|f\|_{p,q;s} = \|\mathscr{D}^s f\|_{p,q}.$$

This is meaningful even when $q = 0$ because the operator $\mathscr{R}-\bar{\mathscr{R}}$ is tangent to ∂B. It is also clear that for any $f \in W^p_{q,s}$ we have $\|f\|_{p,q;s} < \infty$. On the other hand, $\|\cdot\|_{p,q;s}$ is not a norm on $W^p_{q,s}$ for $s \in Z_+ \backslash \{0\}$, even when $1 \leqslant p \leqslant \infty$, since the equation $\mathscr{D}f = 0$ has non-trivial solutions (for example, $f(z) = \bar{z}_1$ is such a solution).

The fact that $\mathscr{R}-\bar{\mathscr{R}}$ is tangential along ∂B allows us to integrate by parts without introducing boundary terms. In fact, we have:

LEMMA 6.1. *Let $s \in Z_+$ and $q \geqslant 0$, and assume that f, $\mathscr{D}^s f \in L^1_q$ and that $g \in C^s(\bar{B})$. Then*

$$\int \mathscr{D}^s f \bar{g}\, dv_q = \int f \overline{\mathscr{D}^s g}\, dv_q.$$

Proof. Assume first that $q > 0$. If f has compact support and $s = 1$, then the result follows from a straightforward computation, and the case $s > 1$ then follows by iteration. For the general case, note that since $\mathscr{R}-\bar{\mathscr{R}}$ is tangent to any sphere centered at the origin, $\mathscr{D}$ commutes with multiplication by any radially symmetric function. Thus, letting χ_r be the characteristic function of the ball $B(r) = \{z \in C^n\colon \|z\| < r\}$, we have

$$\begin{aligned}\int \mathscr{D}^s f \bar{g}\, dv_q &= \lim_{r\to 1^-} \int \chi_r \mathscr{D}^s f \bar{g}\, dv_q = \lim_{r\to 1^-} \int \mathscr{D}^s(\chi_r f)\bar{g}\, dv_q \\ &= \lim_{r\to 1^-} \int \chi_r f \overline{\mathscr{D}^s g}\, dv_q = \int f \overline{\mathscr{D}^s g}\, dv_q,\end{aligned}$$

and the result follows for $q > 0$. The proof for the case $q = 0$ is similar, and could have been also obtained by letting $q \to 0^+$ in the previous proof. ■

Our next result concerns the orthogonal projection P_q of L^2_q onto A^2_q, $q \geqslant 0$, as given in Theorem 3.9.

THEOREM 6.2. *For $q \geqslant 0$, $\mathscr{D}$ commutes with P_q. More precisely, for any $s \in Z_+$ and any $f \in L^1_q$ such that $\mathscr{D}^s f \in L^1_q$, we have $D^s P_q f = P_q \mathscr{D}^s f$.*

Proof. For any $h \in \mathscr{O}(\Delta)$ such that $\overline{h(\lambda)} = h(\bar{\lambda})$ for all $\lambda \in \Delta$, we have

$$D|_z h(\langle z, \zeta\rangle) = \overline{D|_\zeta h(\langle \zeta, z\rangle)} = \langle z, \zeta\rangle h'(\langle z, \zeta\rangle)$$

for every $z, \zeta \in \bar{B}$ with $|\langle z, \zeta\rangle| < 1$. In particular, since $k_q(z, \zeta) = h_{n+q}(\langle z, \zeta\rangle)$ with $h_r(\lambda) = (1-\lambda)^{-r}$, $r > 0$, we obtain from Lemma 6.1,

$$\begin{aligned}\{D^s P_q f\}(z) &= \int f(\zeta) D^s|_z k_q(z, \zeta) dv_q(\zeta) = \int f(\zeta) D^s|_z h_{n+q}(\langle z, \zeta\rangle) dv_q(\zeta)\\ &= \int f(\zeta) \overline{D^s|_\zeta h_{n+q}(\langle \zeta, z\rangle)} dv_q(\zeta) = \int \{\mathscr{D}^s f\}(\zeta) h_{n+q}(\langle \zeta, z\rangle) dv_q(\zeta)\\ &= \int \{\mathscr{D}^s f\}(\zeta) k_q(z, \zeta) dv_q(\zeta) = \{P_q \mathscr{D}^s f\}(z),\end{aligned}$$

and the theorem is proved. ∎

As an immediate corollary, we obtain the following generalization to Sobolev spaces of the projection theorem for L^p_q onto A^p_q $(1 < p < \infty,\ q \geqslant 0)$ which appears in Theorem 3.9.

COROLLARY 6.3. *Let* $s \in \boldsymbol{Z}_+$, $q \geqslant 0$ *and* $1 \leqslant p < \infty$. *Then:*

(i) *For* $1 < p < \infty$ *and* $q \geqslant 0$, *we have* $\|P_q f\|_{p,q;s} \lesssim \|f\|_{p,q;s}$ *for every* $f \in W^p_{q,s}$. *In particular,* P_q *projects* $W^p_{q,s}$ *continuously into its holomorphic subspace* $A^p_{q,s}$;

(ii) *For* $1 \leqslant p < \infty$, $q > 0$ *and any* $0 < r < pq$ *we have* $\|P_q f\|_{p,r;s} \lesssim \|f\|_{p,r;s}$ *for every* $f \in W^p_{r,s}$ *and thus* P_q *projects* $W^p_{r,s}$ *continuously onto* $A^p_{r,s}$.

Proof. To prove (i), we use Theorems 3.9 and 6.2. Then

$$\|P_q f\|_{p,q;s} = \|D^s P_q f\|_{p,q} = \|P_q \mathscr{D}^s f\|_{p,q} \lesssim \|\mathscr{D}_s f\|_{p,q} = \|f\|_{p,q;s}.$$

The second assertion of (i) follows from Theorem 5.3. To prove (ii), we use Theorems 3.1 and 6.2. Then

$$\|P_q f\|_{p,r;s} = \|D^s P_q f\|_{p,r} = \|P_q \mathscr{D}^s f\|_{p,r} \lesssim \|\mathscr{D}_s f\|_{p,r} = \|f\|_{p,r;s},$$

and again, the second assertion of (ii) follows from Theorem 5.3. Note also that we have used the fact that $\|\cdot\|_{p,q;s} \lesssim |||\cdot|||_{p,q;s}$ in the proofs of (i) and (ii). The proof is now complete. ∎

In order to extend the preceding result to Sobolev norms $\|\cdot\|_{p,q;s}$ with negative integer s we will need to develop some ideas of Bell [4]. For $t \in \boldsymbol{R}$ we consider the differential operator $D_t = \mathscr{R} + t$ (see 0.2), and for each $k \in \boldsymbol{Z}_+$ we let

$$\Phi^k = \frac{1}{k!}(1-\|z\|^2)^k D_{n+1} \cdots D_{n+k} \qquad (z \in \boldsymbol{C}^n) \tag{6.1}$$

with $\Phi^0 = 1$. Then for any $s \in \boldsymbol{Z}_+$ we have the identity

$$\Phi^s = 1 + D_n\Big\{(1-\|z\|^2) \sum_{k=1}^{s} \frac{1}{k} \Phi^{k-1}\Big\} \qquad (z \in \boldsymbol{C}^n) \tag{6.2}$$

which can be readily established by induction on s.

LEMMA 6.4. *Let* $f \in C^1(\bar{B})$ *with* $f|_{\partial B} = 0$ *and let* $h \in A^1_1$. *Then* $\int h\overline{D_n f}\,dv = 0$.

Proof. Since $h \in A^1_1$ can be approximated by dilations, we may assume that $h \in \mathcal{O}(\bar{B})$. Integration by parts yields

$$\int h\overline{D_n f}\,dv = n\int h\bar{f}\,dv + \int h\overline{\mathcal{R} f}\,dv = n\int h\bar{f}\,dv - \Big(\sum_{j=1}^{n} \bar{\partial}_j \bar{z}_j h\Big)\bar{f}\,dv = 0,$$

and the result follows. ■

COROLLARY 6.5. *Let* $s \in \boldsymbol{Z}_+$ *and consider the orthogonal projection* $P_1\colon L^2_1 \to A^2_1$. *Then* $P_1 \circ \Phi^s = P_1$.

Proof. There is nothing to prove if $s = 0$, so we let $s \in \boldsymbol{Z}_+ \setminus \{0\}$. In this case, we have to show that $P_1(\Phi^s g) = P_1 g$ for any $g \in C^s(\bar{B})$. (Note that $C^s(\bar{B})$ is dense in L^1_2.) Using identity (6.2), we may write $\Phi^s g = g + D_n f$ where $f \in C^1(\bar{B})$ with $f|_{\partial B} = 0$. Thus, by Lemma 6.4, $P_1(D_n f) = 0$, and the result follows. ■

LEMMA 6.6. *Let* $0 < p \leqslant \infty$, $q > 0$, $s \in \boldsymbol{R}$ *and* $t \in \boldsymbol{Z}_+$. *If* $t - s < q/p$ *or if* $s \in \boldsymbol{Z}_+$, *then for any* $f \in \mathcal{O}(B)$ *we have*

$$|||\lambda^s f|||_{p,q;t} \lesssim \|f\|_{p,q;t-s},$$

where $\lambda(z) = 1 - \|z\|^2$ $(z \in B)$.

Proof. By definition

$$|||\lambda^s f|||_{p,q;t} \lesssim \sum_{|\alpha|+|\beta| \leqslant t} \|\partial^\alpha \bar{\partial}^\beta (\lambda^s f)\|_{p,q}. \tag{6.3}$$

We will show that each term in the sum on the right is dominated by $\|f\|_{p,q;t-s}$. For $z \in B$, we have

$$|\{\partial^\alpha \bar{\partial}^\beta (\lambda^s f)\}(z)| \lesssim \sum_{\gamma+\delta=\alpha} |\{\partial^\gamma \bar{\partial}^\beta \lambda^s\}(z)| \cdot |\{\partial^\delta f\}(z)|$$

and thus

$$|\{\partial^\alpha \bar{\partial}^\beta (\lambda^s f)\}(z)| \lesssim \sum_{|\delta| \leqslant t} [\lambda(z)]^{k(\delta)} \cdot |\{\partial^\delta f\}(z)| \tag{6.4}$$

where

$$k(\delta) = \begin{cases} \max(0, |\delta| - t + s), & s \in \boldsymbol{Z}_+, \\ |\delta| - t + s, & s \in \boldsymbol{R} \setminus \boldsymbol{Z}_+, \end{cases} \tag{6.5}$$

We consider first the case $p = \infty$. In this case, by assumption, s is in $\boldsymbol{Z}_+$ or $s > t$. It follows from (6.5) that $k(\delta) \geqslant 0$. If $k(\delta) = 0$, then, by Theorem 5.3,

$$\|\delta^\delta f\|_{\infty,q} \lesssim |||f|||_{\infty,q;|\delta|} \simeq \|f\|_{\infty,q;|\delta|}.$$

If, on the other hand, $k(\delta) > 0$ then by Lemma 5.4, for any $z \in B$ we have

$$|\{\partial^\delta f\}(z)| \lesssim \|f\|_{\infty,q;|\delta|-k(\delta)}[\lambda(z)]^{-k(\delta)}.$$

It follows that for all δ in Z^n_+,

$$\|\lambda^{k(\delta)}\partial^\delta f\|_{\infty,q} \leqslant \|f\|_{\infty,q;|\delta|-k(\delta)}.$$

Since, by (6.5), $|\delta|-k(\delta) \leqslant t-s$, we also have $\|\lambda^{k(\delta)}\partial^\delta f\|_{\infty,q} \lesssim \|f\|_{\infty,q;t-s}$ and the result for $p = \infty$ follows from (6.3) and (6.4). ■

For the case $0 < p < \infty$ we have, by assumption and (6.5), that $q+pk(\delta) > 0$ and that $|\delta|-k(\delta) \leqslant t-s$, $\delta \in Z^n_+$. Thus, by Theorems 5.3 and 5.12 (i),

$$\|\lambda^{k(\delta)}\partial^\delta f\|_{p,q} = \left\{\frac{B(n, q+pk(\delta))}{B(n, q)}\right\}^{1/p} \|\partial^\delta f\|_{p,q+pk(\delta)}$$

$$\lesssim |||f|||_{p,q+pk(\delta);|\delta|} \simeq \|f\|_{p,q+pk(\delta);|\delta|} \simeq \|f\|_{p,q;|\delta|-k(\delta)} \lesssim \|f\|_{p,q;t-s}$$

and the result follows from (6.3) and (6.4). This concludes the proof. ■

COROLLARY 6.7. *Let* $0 < p \leqslant \infty$, $q > 0$ *and* $s, t \in Z_+$. *Then the operator* Φ^s *maps* $A^p_{q,t}$ *continuously into* $W^p_{q,t}$.

Proof. For $f \in A^p_{q,t}$ and $\lambda(z) = 1-\|z^2\|$ $(z \in B)$ we have by (6.1) and Lemma 6.6,

$$|||\Phi^s f|||_{p,q;t} = (s!)^{-1}|||\lambda^s D_{n+1}\cdots D_{n+s}f|||_{p,q;t}$$

$$\lesssim \|D_{n+1}\cdots D_{n+s}f\|_{p,q;t-s} = \|D^{t-s}D_{n+1}\cdots D_{n+s}f\|_{p,q}.$$

Since, however, the above differential operators commute, we conclude from (5.4) that

$$\|D^{t-s}D_{n+1}\cdots D_{n+s}f\|_{p,q} \simeq \|D^t f\|_{p,q} = \|f\|_{p,q;t},$$

and the result follows.

It is convenient here to denote by $\langle\ ,\ \rangle_q$ the inner-product of L^2_q, $q \geqslant 0$. We note, however, that this notation is at odds with that used in Lemma 1.8. When $q > 0$, $1 < p < \infty$ and $s \in Z_+$, we also let $\mathring{W}^p_{q,s}$ stand for the $W^p_{q,s}$-closure of $C^\infty_0(B)$, which consists of all those functions in $C^\infty(B)$ which have compact support in B. Evidently, $\mathring{W}^p_{q,s}$ is a closed subspace of $W^p_{q,s}$ with $\mathring{W}^p_{q,0} = W^p_{q,0} = L^p_q$. We also recall that the Sobolev space $W^p_{q,-s}$ is defined as the space of all distributions f on B with $1/p+1/p' = 1$ and

$$(6.6) \qquad |||f|||_{p,q;-s} \equiv \sup\{|\langle f, \phi\rangle_q| : \phi \in C^\infty_0(B) \cap S^{p'}_{q,s}\} < \infty,$$

where

$$S^p_{q,s} = \{g \in W^p_{q,s}: |||g|||_{p,q;s} \leqslant 1\}.$$

Thus $W^p_{q,-s}$ is isometrically isomorphic to the dual $(\mathring{W}^{p'}_{q,s})^*$, with respect to the L^2_q-pairing, of $\mathring{W}^{p'}_{q,s}$, and thus it is itself a separable and a reflexive Banach space with the norm $|||\cdot|||_{p,q;-s}$. By analogy with the notation of Section 5, we let $\mathscr{W}^p_{q,-s}$ be the subspace of $W^p_{q,-s}$ consisting of holomorphic functions on B. We also let $\mathscr{V}^p_{q,-s}$ be the space of all $f \in \mathcal{O}(B)$ such that

$$|||f|||'_{p,q;-s} \equiv \sup\{\langle f_r, \phi\rangle_q|:\ 0 < r < 1,\ \phi \in C^\infty(\bar{B}) \cap S^{p'}_{q,s}\} < \infty.$$

We shall need the following refinement of Proposition 5.7 (iv). Here, for $q \geqslant 0$ and $1 < p < \infty$, $m_q(p)$ is the norm of the projection P_q, as given in Theorem 3.9, of L^p_q onto A^p_q, and thus $m_q(p) = m_q(p') \geqslant 1$ with $m_q(2) = 1$.

LEMMA 6.8. *Let* $q \geqslant 0$, $1 < p < \infty$ *and* $s \in \boldsymbol{R}$. *Then, for* $f \in A^p_q \cap A^p_{q,-s}$ *we have*

$$[m_q(p)]^{-1}\|f\|_{p,q;-s} \leqslant \sup\{|\langle f, g\rangle_q|:\ g \in \mathcal{O}(\bar{B}),\ \|g\|_{p',q;s} \leqslant 1\} \leqslant \|f\|_{p,q;-s}.$$

Proof. By Theorem 3.9 and Proposition 5.7 (iv) we have

$$[m_q(p)]^{-1}\|f\|_{p,q;-s} \leqslant \sup\{|\langle D^{-s}f, D^s g\rangle_q|:\ g \in A^{p'}_{q,s},\ \|g\|_{p',q;s} \leqslant 1\} \leqslant \|f\|_{p,q;-s}.$$

The result now follows from Lemma 1.8 and Lemma 5.2. ∎

LEMMA 6.9. *Let* $1 < p < \infty$, $s \in \boldsymbol{Z}_+$ *and* $0 < r_0 \leqslant r < 1$. *For* $f \in \mathcal{O}(B)$ *we have*

$$|||f_r|||_{p,1;-s} \leqslant c_{p,s}(r_0)|||f|||_{p,1;-s}$$

where $c_{p,s}(r_0) \equiv r_0^{-(2n/p+s)}$.

Proof. For $\phi \in C^\infty_0(B)$, the function $\phi^r \equiv r^{-2n}\phi_{1/r}$ is in $C^\infty_0(B(r))$. In particular, $\phi^r \in C^\infty_0(B)$ and $\langle f_r, \phi\rangle_1 = \langle f, \phi^r\rangle_1$, and thus, by definition,

$$|\langle f_r, \phi\rangle_1| = |\langle f, \phi^r\rangle_1| \leqslant |||f|||_{p,1;-s}|||\phi^r|||_{p,1;s}.$$

But,

$$|||\phi^r|||_{p',1;s} = \left\{\sum_{|\alpha|+|\beta| \leqslant s} \frac{|\alpha|!}{\alpha!} r^{2n(1-p')-p'(|\alpha|+|\beta|)} \|\partial^\alpha \bar{\partial}^\beta \phi\|^{p'}_{p',1}\right\}^{1/p'}$$

$$\leqslant r^{-(2n/p+s)}|||\phi|||_{p',1;s} \leqslant r_0^{-(2n/p+s)}|||\phi|||_{p',1;s},$$

and thus

$$|\langle f_r, \phi\rangle_1| \leqslant c_{p,s}(r_0)|||f|||_{p,1;-s}|||\phi|||_{p',1;s} \qquad (\phi \in C^\infty_0(B)).$$

Taking the supremum over all $\phi \in C^\infty_0(B)$ gives the desired result. ∎

LEMMA 6.10. *Let* $1 < p < \infty$ *and* $s \in \boldsymbol{Z}_+$. *For* $f \in \mathcal{O}(B)$ *we have* $|||f|||_{p,1;-s} \leqslant |||f|||'_{p,1;-s}$.

Proof. Fix a $\phi \in C_0^\infty(B)$ and choose $0 < r < 1$ sufficiently near 1 so that $\phi_r \in C_0^\infty(B)$. Then

$$\langle f, \phi\rangle_1 = r^{2n}\langle f_r, \phi_r\rangle_1$$

and thus, by definition,

$$|\langle f, \phi\rangle_1| \leqslant r^{2n}|||f|||'_{p,1;-s}|||\phi_r|||_{p',1;s}.$$

But

$$|||\phi_r|||_{p',1;s} = \left\{\sum_{|\alpha|+|\beta|\leqslant s} \frac{|\alpha|!}{\alpha!} r^{-2n} r^{p'(|\alpha|+|\beta|)} \|\partial^\alpha \bar{\partial}^\beta \phi\|_{p',1}^{p'}\right\}^{1/p'}$$
$$\leqslant r^{-2n/p'}|||\phi|||_{p',1;s}$$

and hence

$$|\langle f, \phi\rangle_1| \leqslant r^{2n/p}|||f|||'_{p,1;-s}|||\phi|||_{p',1;s}.$$

Thus

$$|\langle f, \phi\rangle_1| \leqslant |||f|||'_{p,1;-s}|||\phi|||_{p',1;s} \qquad (\phi \in C_0^\infty(B)).$$

The result now follows by taking the supremum over all $C_0^\infty(B)$. ■

LEMMA 6.11. *Let* $1 < p < \infty$ *and* $s \in \mathbf{Z}_+$. *For* $f \in \mathcal{O}(B)$ *we have* $\|f\|_{p,1;-s} \lesssim |||f|||_{p,1;-s}$.

Proof. For any $0 < r < 1$, we have by Lemma 6.8, Theorems 3.9 and 5.3, and Corollary 6.5

$$\begin{aligned}\|f_r\|_{p,1;-s} &\simeq \sup\{|\langle f_r, g\rangle_1|:\ g \in \mathcal{O}(\bar{B}),\ \|g\|_{p',1;s} \leqslant 1\}\\ &\simeq \sup\{|\langle f_r, P_1 g\rangle_1|:\ g \in \mathcal{O}(\bar{B}) \cap S^{p'}_{1,s}\}\\ &= \sup\{|\langle f_r, P_1 \Phi^s g\rangle_1|:\ g \in \mathcal{O}(\bar{B}) \cap S^{p'}_{1,s}\}\\ &= \sup\{|\langle f_r, \Phi^s g\rangle_1|:\ g \in \mathcal{O}(\bar{B}) \cap S^{p'}_{1,s}\}\end{aligned}$$

Now, by (6.1), the function $\Phi^s g$, for $g \in \mathcal{O}(\bar{B}) \cap S^{p'}_{1,s}$, vanishes to order s on ∂B, and, therefore, is a limit in $W^{p'}_{1,s}$ of functions in $C_0^\infty(B)$. It follows from (6.6) and Corollary 6.7 that

$$\begin{aligned}\|f_r\|_{p,1;-s} &\lesssim |||f_r|||_{p,1;-s}\cdot\sup\{|||\Phi^s g|||_{p',1;-s}:\ g \in \mathcal{O}(\bar{B}) \cap S^{p'}_{1,s}\}\\ &\lesssim \|f_r\|_{p,1;-s}.\end{aligned}$$

In particular, taking $1/2 \leqslant r < 1$, we deduce from Lemma 6.9 that

$$\|f_r\|_{p,1;-s} \lesssim |||f_r|||_{p,1;-s} \leqslant 2^{2n/p+s}|||f_r|||_{p,1;-s}.$$

The desired result follows now by observing that $\|f\|_{p,1;-s} = \sup\{\|f_r\|_{p,1;-s}$: $1/2 < r < 1\}$. ■

We now prove:

THEOREM 6.12. *Let* $q > 0$, $1 < p < \infty$ *and* $s \in \mathbf{Z}_+$. *Then* $\mathscr{V}^p_{q,-s} \simeq A^p_{q,-s} \simeq A^p_{q+sp}$, *i.e. on* $\mathcal{O}(B)$ *the norms* $|||\cdot|||'_{p,q;-s}$, $\|\cdot\|_{p,q;-s}$ *and* $\|\cdot\|_{p,q;+sp}$ *are equivalent to one another. In particular* $\mathscr{V}^p_{q,-s} \simeq \mathscr{V}^p_{q+mp,m-s}$ *for any* $m \in \mathbf{Z}$ *with* $-q/p < m \leqslant s$.

Proof. The equivalence of the norms $\|\cdot\|_{p,q;-s}$ and $\|\cdot\|_{p,q+sp}$ is a special case of Theorem 5.12 (i). Now, let $f \in \mathcal{O}(B)$ and $0 < r < 1$. Then, by Lemma 6.8 and Theorem 5.3,

$$\begin{aligned}\|f_r\|_{p,q;-s} &\simeq \sup\{|\langle f_r, g\rangle_q|:\ g \in \mathcal{O}(\bar{B}),\ \|g\|_{p',q;s} \leqslant 1\}\\ &\simeq \sup\{|\langle f_r, g\rangle_q|:\ g \in \mathcal{O}(\bar{B}) \cap S^{p'}_{q,s}\}\\ &\leqslant \sup\{|\langle f_r, \phi\rangle_q|:\ \phi \in C^\infty(\bar{B}) \cap S^{p'}_{q,s}\},\end{aligned}$$

and thus

$$\|f\|_{p,q;-s} = \sup_{0<r<1} \|f_r\|_{p,q;-s} \lesssim |||f|||'_{p,q;-s}.$$

Conversely, by Theorem 3.9, Lemma 6.8 and Corollary 6.3 (i),

$$\begin{aligned}\sup\{|\langle f_r, \phi\rangle_q|:\ \phi \in C^\infty(\bar{B}) \cap S^{p'}_{q,s}\} &\simeq \sup\{|\langle f_r, P_q\phi\rangle_q|:\ \phi \in C^\infty(\bar{B}) \cap S^{p'}_{q,s}\}\\ &\simeq \|f_r\|_{p,q;-s}\cdot\sup\{\|P_q\phi\|_{p',q;s}:\ \phi \in C^\infty(\bar{B}) \cap S^{p}_{q,s}\}\\ &\lesssim \|f_r\|_{p,q;-s}.\end{aligned}$$

It follows that

$$|||f|||'_{p,q;-s} \lesssim \sup_{0<r<1} \|f_r\|_{p,q;-s} = \|f\|_{p,q;-s},$$

proving the first part of the theorem. The second part follows from the first part and Theorem 5.12 (i) ■

For $p = 2$, the equivalence of $\mathscr{W}^2_{1,-s}$ and $\mathscr{V}^2_{1,-s}$ in the next theorem was also obtained by Bell and Boas [5] when B is replaced by a smooth pseudoconvex domain.

THEOREM 6.13. *Let* $1 < p < \infty$ *and* $s \in \mathbf{Z}_+$. *Then* $\mathscr{W}^p_{1,-s} \simeq \mathscr{V}^p_{1,-s} \simeq A^p_{1,-s} \simeq A^p_{1+sp}$, *i.e. on* $\mathcal{O}(B)$ *the norms* $|||\cdot|||_{p,1;-s}$, $|||\cdot|||'_{p,1;-s}$, $\|\cdot\|_{p,1;-s}$ *and* $\|\cdot\|_{p,1+sp}$ *are equivalent to the another.*

Proof. The equivalence of the norms $|||\cdot|||'_{p,1;-s}$, $\|\cdot\|_{p,1;-s}$ and $\|\cdot\|_{p,1+sp}$ is a special case of the previous theorem. Let $f \in \mathcal{O}(B)$. Then by Lemma 6.10, $|||f|||_{p,1;-s} \leqslant |||f|||'_{p,1;-s}$. On the other hand, by the previous equivalence and Lemma 6.11,

$$|||f|||'_{p,1;-s} \simeq |||f|||_{p,1;-s} \lesssim |||f|||_{p,1;-s},$$

and the desired result follows. ■

We can now formulate a projection theorem for the negative Sobolev spaces $W^p_{1,-s}$ onto their holomorphic subspaces $\mathscr{W}^p_{1,-s} \simeq A^p_{1,-s}$ $(1 < p < \infty$, $s \in \mathbf{Z}_+)$. As before, P_q denotes the orthogonal projection of L_q onto A^p_q $(q > 0)$.

THEOREM 6.14. *Let* $1 < p < \infty$ *and let* $s \in \mathbf{Z}_+$. *Then for* $q \geqslant s+1$, *the projection* P_q *extends to a continuous projection of* $W^p_{1,-s}$ *onto its holomorphic subspace* $A^p_{1,-s}$.

Proof. Let P^*_q denotes the adjoint of P_q with respect to the L^2_1-pairing. We have to show that P_q is a continuous operator from $W^p_{1,-s}$ into itself. This, since $\mathring{W}^{p'}_{1,s} = (W^p_{1,-s})^*$ in the L^2_1-pairing, is equivalent in showing that P^*_q is a continuous operator from $\mathring{W}^{p'}_{1,s}$ into itself. From the definition of P_q follows that

$$P^*_q = \lambda^{q-1} Q_q \qquad (\lambda(z) = 1 - \|z\|^2),$$

where

$$\{Q_q f\}(z) = \frac{1}{nB(n, q)} \int f(\zeta) k_q(z, \zeta) dv_1(\zeta) \qquad (f \in L^1_1).$$

with

$$k_q(z, \zeta) = (1 - \langle z, \zeta\rangle)^{-(n+q)} \qquad (z, \zeta \in B).$$

Let $f \in \mathring{W}^{p'}_{1,s}$, and thus f, $\mathscr{D}^s f \in L^{p'}_1$. It follows from Lemma 6.1 that

$$\begin{aligned} nB(n, q)\{D^s Q_q f\}(z) &= \int f(\zeta) D^s|_z k_q(z, \zeta) dv_1(\zeta) \\ &= \int f(\zeta) \overline{D^s|_\zeta k_q(\zeta, z)} dv_1(\zeta) \\ &= \int \{\mathscr{D}^s f\}(\zeta) k_q(z, \zeta) dv_1(\zeta), \end{aligned}$$

and hence

$$D^s Q_q f = Q_q \mathscr{D}^s f \qquad (f \in \mathring{W}^{p'}_{1,s}). \tag{6.7}$$

By Lemma 6.6, as $q \geqslant s+1 > s+1/p$, and by Theorem 5.12 (i) we have

$$\begin{aligned} |||P^*_q f|||_{p',1;s-q+1} = \|Q_q f\|_{p',1;s} &\lesssim \|Q_q f\|_{p',1;s-q+1} \\ &\simeq \|Q_q f\|_{p',1+p'(q-1);s} = \|D^s Q_q f\|_{p',1+p'(q-1)}. \end{aligned}$$

It follows from (6.7) that

$$\begin{aligned} |||P^*_q f|||_{p',1;s} &\lesssim \|Q_q \mathscr{D}^s f\|_{p',1+p'(q-1)} \\ &= \{nB(n, 1+p'(q-1))\}^{-1/p'} \|\lambda^{q-1} Q_q \mathscr{D}^s f\|_{p',1}. \end{aligned}$$

On the other hand, for $q \geqslant 1$ the operator $\lambda^{q-1} Q_q$ is of the form

$$\{\lambda^{q-1} Q_q g\}(z) = \int g(\zeta) K(z, \zeta) dv_1(\zeta)$$

where

$$|K(z,\zeta)| \leqslant 2^{q-1}|1-\langle z,\zeta\rangle|^{-(n+1)},$$

and thus, by Theorem 3.1,

$$\|\lambda^{q-1}Q_q\mathscr{D}^s f\|_{p',1} \lesssim \|\mathscr{D}^s f\|_{p',1} = \|f\|_{p',1;s} \lesssim |||f|||_{p',1;s}.$$

Consequently,

$$|||P_q^* f|||_{p',1;s} = |||f|||_{p',1;s},$$

and hence P_q^* is a continuous operator from $W_{1,s}^{p'}$ into itself. Since $\mathring{W}_{1,s}^{p'}$ is the $W_{1,s}^{p'}$-closure of $C_0^\infty(B)$, the proof will be completed if we can show that P_q^* maps $C_0^\infty(B)$ into $\mathring{W}_{1,s}^{p'}$. But if $\phi \in C_0^\infty(B)$ then $Q_q\phi \in C^\infty(\bar{B})$ and $P_q^*\phi$ vanishes to order $q-1$ on ∂B. Thus, as $r-1 \geqslant s$, $P_q^*\phi \in \mathring{W}_{1,s}^{p'}$, and the theorem is proved. ■

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