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SEMIGROUP OF A QUASIORDINARY SINGULARITY

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Introduction

In a series of papers (cf. [2] and its bibliography), Abhyankar and Moh have studied plane algebroid curve singularities, mainly over an algebraically closed field k of characteristic zero. Starting with the characteristic pairs of such a curve singularity $A = k[[X, Z]]/(F) = k[[X]][\zeta]$ where $F \in k[[X]][Z]$ is monic and irreducible, they construct a special basis of the free $k[[X]]$-module A, which in turn gives a minimal set of generators of the value-semigroup of A. M. Micus in his thesis [7] has generalized this result to e-dimensional quasiordinary singularities ($e \geqslant 2$). He used this result to associate a semigroup $\subset \frac{1}{n}\mathbf{N}_0^e$ for some n with such singularities, and gave the connection with the distinguished exponents introduced by Lipman [4]. In this paper we give a short valuationtheoretic proof of the fact that this semigroup is defined intrinsically by A at least in the case of a surface singularity, a fact contained also in Lipman's thesis (proven there for the distinguished exponents) and characterize for $e \geqslant 1$ those semigroups of $\frac{1}{n}\mathbf{N}_0^e$ which occur as the semigroup of a quasiordinary singularity.

2. Strict linear combinations

(2.1) Notation. Let $e \in \mathbf{N}$; for $r_i = (\varrho_{i1}, \ldots, \varrho_{ie}) \in \mathbf{Q}^e$, $i = 1, 2$, define $r_1 \leqslant r_2$ iff $\varrho_{1j} \leqslant \varrho_{2j}$ for $j = 1, \ldots, e$. For $r := (\varrho_1, \ldots, \varrho_e) \in \mathbf{Q}^e$, set $\lfloor r \rfloor := (\lfloor \varrho_1 \rfloor, \ldots, \lfloor \varrho_e \rfloor)$.

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Let $n \in \mathbf{N}$, and put $\Delta_n := \frac{1}{n}\mathbf{Z}$. Then Δ_n^e is an (if $e \geqslant 2$ not linearly) ordered $\mathbf{Z}$-submodule of $\mathbf{Q}^e$.

Let $h \in \mathbf{N}$ and $r = (r_1, \ldots, r_h)$ be an h-tuple of elements of Δ_n^e. Let $r_i = (\varrho_{i1}, \ldots, \varrho_{ie})$ for $i = 1, \ldots, h$ and for $i = 0, \ldots, h$ let $\delta_i(r) = \delta_i$ be the positive gcd of minors of order e of the $(e, e+1)$-matrix $\left(nE_e, {}^t((n\varrho_{lj})_{1 \leqslant l \leqslant i, 1 \leqslant j \leqslant e})\right)$; here E_e denotes the (e, e)-unit matrix. Define $\theta_{i+1}(r) = \theta_{i+1} := \delta_i(r)/\delta_h(r)$ for $i = 0, \ldots, h$; then $\theta_{i+1} | \theta_i$ for $1 \leqslant i \leqslant h$ and $\theta_{h+1} = 1$. Define $v_i(r) = v_i := \theta_i/\theta_{i+1}$ for $i = 1, \ldots, h$; thus $v_i = \delta_{i-1}/\delta_i$ for $i = 1, \ldots, h$. We call the sequences $\theta(r) = (\theta_1, \ldots, \theta_h)$ and $v(r) = (v_1, \ldots, v_n)$ the divisor sequence and the v-sequence associated with the sequence r, respectively. Given $i \in \{1, \ldots, h\}$ the elements $v_1, \ldots, v_i$ may be calculated knowing only $r_1, \ldots, r_i$.

(2.2) LEMMA. *For $i \in \{0, \ldots, h\}$ let N_i be the $\mathbf{Z}$-submodule of Δ_n^e generated by $\mathbf{Z}^e$ and the elements $r_1, \ldots, r_i$. Let $\gamma \in \mathbf{Z}$ and $i \in \{1, \ldots, h\}$. The element γr_i is contained in N_{i-1} iff $v_i | \gamma$.*

Proof. The theory of elementary divisors shows that $\operatorname{Card}(\Delta_n^e/N_i) = \delta_i$ for $i = 0, \ldots, h$. Choose $i \in \{1, \ldots, h\}$ and let N_i' be the $\mathbf{Z}$-submodule of Δ_n^e generated by N_{i-1} and γr_i. Then $\operatorname{Card}(\Delta_n^e/N_i') = |\gcd(\delta_{i-1}, \gamma\delta_i)|$; thus $N_i' = N_{i-1}$ iff $|\gcd(v_i, \gamma)| = v_i$.

(2.3) PROPOSITION. *Let $a = a_0 + \sum_{i=1}^h \alpha_i r_i$, $b = b_0 + \sum_{i=1}^h \beta_i r_i$ with a_0, b_0 in $\mathbf{Z}^e$ and integers α_i, β_i such that $0 \leqslant \alpha_i, \beta_i < v_i$ for $i = 1, \ldots, h$. Then $a = b$ iff $a_0 = b_0$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, h$.*

Proof. Suppose $a = b$. If $\alpha_i = \beta_i$ for $i = 1, \ldots, h$ then $a_0 = b_0$. Suppose there exists $j \in \{1, \ldots, h\}$ such that $\alpha_j \neq \beta_j$; choose j maximal with this property. We may assume that $\alpha_j > \beta_j$. Then $0 = a - b = c + \sum_{i=1}^j \gamma_i r_i$, $c \in \mathbf{Z}^e$, $\gamma_1, \ldots, \gamma_{i-1} \in \mathbf{Z}$, $0 < \gamma_j < v_j$; so $\gamma_j r_j$ is contained in the $\mathbf{Z}$-module N_{j-1} which contradicts (2.2).

(2.4) PROPOSITION. *Every element a of the $\mathbf{Z}$-module N_h has a unique representation of the form*

$$a = a_0 + \sum_{i=1}^{h} \alpha_i r_i, \quad a_0 \in \mathbf{Z}^e,$$

$\alpha_i \in \mathbf{N}_0$ and $0 < \alpha_i < v_i$ for $i = 1, \ldots, h$.

Proof. Uniqueness follows from (2.3). Lemma (2.2) shows that $v_i r_i$ is contained in the $\mathbf{Z}$-module N_{i-1} for $i = 1, \ldots, h$. Let $a = a_0 + \sum_{i=1}^h \alpha_i r_i$, $a_0 \in \mathbf{Z}^e$, $\alpha_i \in \mathbf{Z}$ for $i = 1, \ldots, h$. Starting with $j = h$ we write $\alpha_j = \beta_j \cdot v_j + \gamma_j$ with $\beta_j, \gamma_j \in \mathbf{Z}$ and $0 \leqslant \gamma_j < v_j$ and get the claimed representation by recursion.

(2.5) We define $r_0 = q_0 = m_0 = (0, \ldots, 0) \in \mathbf{Z}^e$ and by recursion for $i = 1, \ldots, h$

$$q_i := -\sum_{j=1}^{i-1} \frac{\theta_j}{\theta_i} q_j + r_i, \qquad m_i := m_{i-1} + q_i,$$

and we set $\nu_0 := 1$. Then $q := (q_1, \ldots, q_h)$, $m := (m_1, \ldots, m_h)$ are sequences in Δ_n^e and so have an associated divisor sequence and ν-sequence.

(2.6) PROPOSITION. *The following holds true:*

(i) *Each of the three sequences* $(m_1, \ldots, m_h)$, $(q_1, \ldots, q_h)$, $(r_1, \ldots, r_h)$ *in* Δ_n^e *determines the other two and the associated sequences of divisors and* ν*-sequences are the same.*

(ii) *For every* $i = 1, \ldots, h$ *the* **Z**-*submodules of* Δ_n^e *generated by* $\mathbf{Z}^e$ *and* $m_1, \ldots, m_i$, *respectively* $q_1, \ldots, q_i$, *respectively* $r_1, \ldots, r_i$, *are the same.*

(iii) *The following are equivalent:*

(1) *The sequence* $(m_1, \ldots, m_h)$ *is linearly ordered:* $m_0 < m_1 < m_2 < \ldots < m_h$.

(2) $q_i > 0$ *for* $i = 1, \ldots, h$.

(3) *For* $i = 1, \ldots, h$ *we have* $r_i > \nu_{i-1} r_{i-1}$.

If these conditions are satisfied then for $i = 1, \ldots, h$

$$(*) \qquad r_i > \sum_{j=1}^{i-1} (\nu_j - 1) r_j.$$

Proof. It is easy to see that each of the sequences $(m_1, \ldots, m_h)$, $(q_1, \ldots, q_h)$ and $(r_1, \ldots, r_h)$ defines the same sequence of divisors and that (i) and (ii) hold true.

With regard to (iii), it is clear that conditions in (1) and (2) are equivalent. Now for $i \geqslant 2$

$$r_i = \sum_{j=1}^{i-1} \frac{\theta_j}{\theta_i} q_j + q_i = \sum_{j=1}^{i-1} \frac{\theta_j}{\theta_{i-1}} \frac{\theta_{i-1}}{\theta_i} q_j + q_i = \nu_{i-1} r_{i-1} + q_i$$

which shows that (2) and (3) are equivalent.

If these conditions are satisfied then $r_1 > (\nu_0 - 1) r_0$ and it is easy to prove $(*)$ by induction using the condition in (3).

(2.7) DEFINITION. An element $a \in \Delta_n^e$ is called a *strict linear combination of* $r_1, \ldots, r_h$ if a has a representation $a = a_0 + \sum_{i=1}^h \alpha_i r_i$ where $a_0 \in \mathbf{N}_0^e$ and for $i = 1, \ldots, h$ the coefficients α_i are integers such that $0 \leqslant \alpha_i < \nu_i$.

(2.8) COROLLARY. *Let the conditions of* (2.6)(iii) *be satisfied.*

(a) *The set*

$$\sum_{i=1}^{h} \alpha_i r_i, \qquad \alpha_i \in N_0, \qquad 0 \leqslant \alpha_i < \nu_i \textit{ for } i = 1, \ldots, h,$$

is linearly ordered.

(b) *The set of strict linear combinations of* $r_1, \ldots, r_h$ *is a subsemigroup of* Δ_n^e *consisting of nonnegative elements.*

Proof. (a) Let

$$a = \sum_{i=1}^{h} \alpha_i r_i \neq \sum_{i=1}^{h} \beta_i r_i = b$$

be strict linear combinations of $r_1, \ldots, r_h$ and choose $j \in \{1, \ldots, h\}$ maximal such that $\alpha_j \neq \beta_j$; we may assume that $\alpha_j > \beta_j$. According to (2.6) (∗) we have

$$r_j > \sum_{i=1}^{j-1} (v_i - 1) r_i \geqslant \sum_{i=1}^{j-1} \beta_i r_i$$

and this implies that $a > b$.

(b) Let a and b be strict linear combinations of $r_1, \ldots, r_h$,

$$a = a_0 + \sum_{i=1}^{h} \alpha_i r_i, \qquad b = b_0 + \sum_{i=1}^{h} \beta_i r_i,$$

$a_0, b_0 \in \mathbf{N}_0^e$, $\alpha_i, \beta_i \in \mathbf{N}_0$, $0 \leqslant \alpha_i, \beta_i < v_i$ for $i = 1, \ldots, h$. Then $a + b = a_0 + b_0 + c + \sum_{i=1}^{h} \gamma_i r_i$ with $c \in \mathbf{Z}^e$, $\gamma_i \in \mathbf{N}_0$, $0 \leqslant \gamma_i < v_i$ for $i = 1, \ldots, h$

According to (2.4) for every $i \in \{1, \ldots, h\}$ we may write

$$v_i r_i = \bar{a}_0 + \sum_{j=1}^{i-1} \bar{\alpha}_j r_j \quad \text{with } \bar{a}_0 \in \mathbf{Z}^e, \bar{\alpha}_j \in \mathbf{N}_0, 0 \leqslant \bar{\alpha}_j < v_j \text{ for } j = 1, \ldots, i.$$

From (2.6) (∗) we get

$$\bar{a}_0 \geqslant v_i r_i - \sum_{j=1}^{i-1} (v_j - 1) r_j > (v_i - 1) r_i \geqslant 0.$$

3. Quasiordinary singularities

(3.1) Notations. Let k be an algebraically closed field of characteristic 0, put $R_1 = R = k[[X_1, \ldots, X_e]]$, the formal power series ring over k in e variables. Let n be a natural number and define $R_n := k[[X_1^{1/n}, \ldots, X_e^{1/n}]]$. Let K_n be the quotient field of R_n and denote K_1 by K. Then K_n/K is a Galois extension and the Galois group is isomorphic to $(\mathbf{Z}_n)^e$ where $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$. Let Γ_n be those nonnegative rationals with denominator n; for $m \in \Gamma_n^e$, $m = (\mu_1, \ldots, \mu_e)$ let $\underline{X}^m = X_1^{\mu_1} \cdot \ldots \cdot X_e^{\mu_e}$.

Consider the natural pairing

$$\Gamma_n^e \times \mathbf{Z}_n^e \to \mathbf{Z}_n, \qquad (m, g) \mapsto \sum_{i=1}^{e} n\mu_i \cdot \gamma_i$$

where $m = (\mu_1, \ldots, \mu_e)$, $g = (\gamma_1, \ldots, \gamma_e)$. Let $m := (m_1, \ldots, m_h)$ be a sequence of elements in Γ_n^e; define the divisor sequence $\theta(m)$ and the sequences $r_1, \ldots, r_h$,

$q_1, \ldots, q_h$ as in (2.5) and put $M_i := \underline{X}^{m_i}$, $Q_i := \underline{X}^{q_i}$, $P_i := \underline{X}^{r_i}$ for $i = 1, \ldots, h$. For every $i \in \{0, \ldots, h\}$ the fields $K(P_1, \ldots, P_i)$, $K(Q_1, \ldots, Q_i)$, $K(M_1, \ldots, M_i)$ coincide and the just mentioned pairing and Galois theory show that each of these fields has degree θ_1/θ_{i+1} over K.

(3.2) Let $F \in R[Z]$ be an irreducible unitary polynomial whose discriminant has the form $\underline{X}^m E$ where $m \in \mathbb{N}_0^e$ and E is a unit in R. In the case $e = 1$ this is always true. Such polynomials will be called *quasiordinary*. Abhyankar [1] and more recently Luengo [6] have shown that there exist $n \in \mathbb{N}$ and ζ in R_n such that $F(\zeta) = 0$ so that $R[Z]/(F) = R[\zeta]$. A ring A having a representation of this form will also be called *quasiordinary*. If $e = 1$ then A is the ring of a plane irreducible algebroid curve. In his thesis Lipman [4] has studied these singularities for the case $e = 2$ — they play an important role in the resolution of surface singularities, cf. Zariski [8] — and has shown the following (which holds true for arbitrary e). Let $\zeta = \sum_{r \in \Gamma_n^e} c_r \underline{X}^r$. Then there are $h \in \mathbb{N}_0$ and $m_1 < \ldots < m_h$ in Γ_n^e such that [using the notation in (2.2)]

(i) $m_i \in \operatorname{Supp}(\zeta)$ for $i = 1, \ldots, h$.

(ii) $\operatorname{Supp}(\zeta) \subset N_h$.

(iii) For $r \in \operatorname{Supp}(\zeta)$ let $\tau(r)$ be the least $i \in \{0, \ldots, h\}$ such that $r \in N_i$. Then $r \geqslant m_{\tau(r)}$ and $\tau(m_i) = i$ for $i = 1, \ldots, h$.

The elements $m_1, \ldots, m_h$ will be called the *distinguished exponents of* ζ.

Let v and r be the v-sequence and r-sequence associated to $(m_1, \ldots, m_h)$. It is easy to see by Lemma (2.2): The condition $\tau(m_i) = i$ for $i = 1, \ldots, h$ is equivalent to the condition $v_i > 1$ for $i = 1, \ldots, h$.

Furthermore, Lipman [4] has shown: one may normalize ζ in such a way that $\operatorname{Supp}(\zeta) \cap \mathbb{N}_0^e = \emptyset$ and if $m_i = (\mu_{i1}, \ldots, \mu_{ie})$ for $i = 1, \ldots, h$ are the distinguished exponents belonging to ζ then $\mu_{11} > 1$ if $\mu_{1j} = 0$ for $j = 2, \ldots, e$ and the h-tuples $(\mu_{11}, \ldots, \mu_{h1}), \ldots, (\mu_{1e}, \ldots, \mu_{he})$ are decreasing with respect to the lexicographic order.

(3.3) It is clear that $A = R[\zeta]$ is a free R-module. In his thesis, M. Micus [6] has constructed — following the methods developped by Abhyankar [2] in the case $e = 1$, especially using approximate roots — a special basis of A as R-module, thus generalizing the Abhyankar–Moh-epimorphism theorem from the case of a plane irredubible curve singularity to the case of quasiordinary singularities.

Let us say that an element $g \in A$ has order $r \in \Gamma_n^e$ if $g = \underline{X}^r E$ where E is a unit in R_n, and write $\operatorname{ord}(g) = r$. Define

$$G_j := \begin{cases} Z & \text{for } j = 1, \\ \operatorname{App}_Z^{\theta_j}(F) & \text{for } j = 2, \ldots, h. \end{cases}$$

($\operatorname{App}_Z^{\theta_j}(F)$ is the θ_jth approximate root of F.) Let $g_j = G_j(\zeta)$ for $j = 1, \ldots, h$. For $b := (\beta_1, \ldots, \beta_h) \in \mathbb{N}_0^h$ put $g^b := g_1^{\beta_1} \ldots g_h^{\beta_h}$. Let $B := \{(\beta_1, \ldots, \beta_h) | 0 \leqslant \beta_j < v_j$ for $1 \leqslant j \leqslant h\}$. With these notations M. Micus has shown:

(3.4) THEOREM. *For* $i = 1, \ldots, h$ *the element* g_i *has order* r_i. *The set* $\{g^b | b \in B\}$ *is an R-basis of A. Let* $g = \sum_{b \in B} F_b g^b$ *with* $F_b \in R$ *be an element of A of order r. There is a unique* $b = (\beta_1, \ldots, \beta_h) \in B$ *such that* $r = \operatorname{ord}(F_b) + \sum_{i=1}^{h} \beta_i r_i$.

(3.5) COROLLARY. *Let* $\Gamma(\zeta) := \{r \in \Gamma_n^e |$ *there exist* $g \in A$ *with* $\operatorname{ord}(g) = r\}$. *Then* $\Gamma(\zeta)$ *is the set of strict linear combinations of* $r_1, \ldots, r_h$.

(3.6) COROLLARY. *Let* $\bar{A}$ *be the integral closure of A and* $\bar{\Gamma} := N_h \cap \Gamma_n^e$. *Then an element* $g \in R_n$ *is in* $\bar{A}$ *iff* $r \in \bar{\Gamma}$ *for each* $r \in \operatorname{Supp}(g)$.

(3.7) COROLLARY. *Let* $T' := \{\sum_{i=1}^{h} \beta_i r_i | (\beta_1, \ldots, \beta_h) \in B\}$ *and set* $T := \{t' - \lfloor t' \rfloor \, | t' \in T'\}$. *Then* $\bar{A}$ *is a free R-module having R-basis* $\{\underline{X}^t | t \in T\}$. *In particular,* $\bar{A}$ *is a Cohen–Macaulay ring.*

(3.8) Consider the semigroup $\Gamma(\zeta)$; it will be called the *semigroup belonging to* ζ. Then we find h and the elements $r_1, \ldots, r_h$, in the following way. The set $\{r - a \geqslant 0 | r \in \Gamma(\zeta), a \in \mathbf{N}_0^e\}$ has a smallest element, namely r_1; so we may calculate v_1. By recursion, we find r_i as the smallest element in the set

$$\{r - a - \sum_{j=1}^{i-1} \alpha_j r_j | r \in \Gamma(\zeta),\ a \in \mathbf{N}_0^e,\ 0 \leqslant \alpha_j < v_j \text{ for } 0 \leqslant j < i\}$$

and we may calculate v_i.

4. The invariance of the semigroup

(4.1) If $e = 1$ the integral closure $\bar{A} = k[[t]]$ of A is a discrete valuation ring; let v be the normalized valuation defined by $\bar{A}$. Then $\Gamma(A) := \{v(g) | g \in A \setminus \{0\}\}$ is an invariant of A. Let n be the smallest nonzero value of v on A. As ζ is normalized there are elements $r_1, \ldots, r_h \in \Gamma_n$ such that $\{n, nr_1, \ldots, nr_h\}$ is a minimal set of generators of $\Gamma(A)$ and the sequence $(r_1, \ldots, r_h)$ is uniquely determined by A; thus $\Gamma(\zeta) = \frac{1}{n}\Gamma(A)$.

(4.2) PROPOSITION. *Let* $r_1, \ldots, r_h$ *be elements in* Γ_n^e, *let* $v = (v_1, \ldots, v_h)$ *be the corresponding v-sequence. Define* $v_0 = 0$. *Then the conditions*

$$v_i > 1, \quad r_i > v_{i-1} r_{i-1} \quad \text{for } i = 1, \ldots, h$$

are necessary and sufficient for the existence of a normalized quasiordinary ζ *such that* $\Gamma(\zeta)$ *is the set of strict linear combinations of* $r_1, \ldots, r_h$.

Proof. We calculate the m-sequence belonging to r and use the construction given by Lipman for the existence of a quasiordinary ζ having this m-sequence as sequence of distinguished exponents.

(4.3) In the case $e = 1$ the conditions of the proposition are the well-known conditions that must hold for the minimal set of generators of a subsemigroup Γ of Γ_n in order that Γ is the semigroup of a plane irreducible algebroid curve singularity, cf. e.g. Angermüller [4] or W. Micus [8].

(4.4) Lipman [4] has shown that in the case $e = 2$ the sequence $(m_1, \ldots, m_h)$ is an invariant of A by studying very carefully the resolution of A by special formal monoidal and quadratic transforms of A. We give a new proof of this result by showing that the sequence $(r_1, \ldots, r_h)$ is an invariant of A.

Consider the representation $R \subset A = R[\zeta] \subset \bar{A} \subset R_n$. Let $X := X_1$, $Y := X_2$ and $m_i = (\lambda_i, \mu_i)$ for $i = 1, \ldots, h$. Let $L = \mathrm{Quot}(A)$ and $d := [L:K]$. The following results are contained in Lipman [4]: If $\lambda_1 > 0$ then $\mathfrak{p} := (X, \zeta)$ is a prime ideal of height 1 and its multiplicity in A is $\min(d, d\lambda)$. If $\mu_1 > 0$ then there is a corresponding result for the ideal $\mathfrak{q} := (Y, \zeta)$. Furthermore, d is an invariant of A and if $\lambda_1 \leqslant 1$ then $0 < \mu_1 \leqslant 1$ and $\mathfrak{p}$ and $\mathfrak{q}$ are the only prime ideals of height 1 which may be singular.

If the singular locus $\mathrm{Sing}(A)$ of A has two components then these are defined by prime ideals of the form $\mathfrak{p} = (X, \zeta)$ and $\mathfrak{q} = (Y, g(\zeta))$ with an element $g(\zeta) \in A$.

If A is regular then $\zeta = 0$ and $\Gamma(\zeta) = \mathbf{N}_0^2$. Now let A be not regular. Consider the case where the tangent cone of A is reducible. Let π: $\mathrm{Bl}_{\mathfrak{m}}(A) \to \mathrm{Spec}(A)$ be the canonical map of $\mathrm{Bl}_{\mathfrak{m}}(A) = \mathrm{Proj}(\bigoplus_h \mathfrak{m}^h)$; the exceptional divisor $\pi^{-1}(\mathfrak{m})$ has two components V and W, defined by $X = 0$ and $Y = 0$ in the projective plane which is the exceptional divisor of the blow-up of $k[[X, Y, Z]]$ with respect to the maximal ideal. Let $\mathfrak{p}$ and $\mathfrak{q}$ be those prime ideals of height 1 of A which contain respectively $\pi(V)$ and $\pi(W)$.

Now we have the following situation. If the tangent cone of A is reducible or if $\mathrm{Sing}(A)$ has two components there are two intrinsically defined prime ideals of height 1; in the representation of A considered above these are the prime ideals containing X respectively Y.

Let v, w be the normalized valuations of L corresponding to the restriction of L of the valuations of K_n defined by the ideals $(X^{1/n})$, $(Y^{1/n})$. An element $g \in \bar{A}$ has an order iff $\omega(g) = 0$ for each essential valuation ω of the Krull ring $\bar{A}$ which is not equivalent to v and w.

If all first components in the elements of $t \in T$ are zero put $\varepsilon = n$; otherwise let ε/n ($\varepsilon \in \mathbf{N}$) be the smallest nonzero first component of the elements of T. Then n/ε is the ramification index of v over K and is defined intrinsically by $\min\{v(g) | g \in A \text{ has positive order}\}$ if the multiplicity of $\mathfrak{p}$ is d; in the other case λ_1 is defined intrinsically and the ramification index is given by $\min\{v(g)/\lambda_1 | g \in A \text{ has positive order}\}$. In the same way we define σ/n for the valuation w using the second component of the elements of T. Now

$$\left(\frac{\varepsilon}{n} v(g_i), \frac{\sigma}{n} w(g_i)\right) = r_i, \qquad 1 \leqslant i \leqslant h,$$

and this implies that $\Gamma(\zeta)$ is defined intrinsically.

In the remaining case where Sing(A) has only one component one has to consider a strict resolution of A in the sense of Lipman and use induction with respect to the length of this resolution. But in this case the calculations given by Lipman are quite simple and short.

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