COLLOQUIUM MATHEMATICUM

VOL. XXXII 1974 FASC. 1

HOMOGENEOUS LATTICES AND LATTICE-ORDERED GROUPS

BY

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A lattice L is *homogeneous* if, for any $a, b \in L$, $a \leqslant b$, there exists an automorphism f of L such that $f(a) = b$ and $x \leqslant f(x)$ for all $x \in L$. Every lattice that admits a lattice-ordered group structure is homogeneous. In this paper structural properties of homogeneous lattices are developed. In particular, homogeneous lattices that are linearly ordered or that have a finite number of meet disjoint elements are investigated. In general, this paper shows that many properties of lattice-ordered groups also hold for homogeneous lattices. As a consequence, this gives purely lattice-theoretic proofs for several lattice-ordered group results.

Let L be an arbitrary lattice. If $a \leqslant b$ in L, then the *interval* from a to b is the convex sublattice $\{x \mid x \in L \text{ and } a \leqslant x \leqslant b\}$, and is denoted by $[a, b]$. If $[a, b] = \{a, b\}$, then b *covers* a and this is written $a \prec b$. If $a, b \in L$, and a and b are not comparable, then write $a \| b$. A lattice L is *non-trivial* if $|L| > 1$. The automorphism group of L is denoted by $A(L)$. Let $A^*(L)$ be the set of all automorphisms of L such that $f \in A^*(L)$ if and only if, for all $x \in L$, $x \leqslant f(x)$ or, for all $x \in L$, $f(x) \leqslant x$.

In [6], Dwinger* defines homogeneous lattices in a slightly different way: for every a and b in L, there exists a homomorphism f of L such that $f(a) = b$ and such that if $a < b$ or $a \| b$, then f can be chosen so that, for all $x \in L$, $x < f(x)$ or $x \| f(x)$, respectively. Example 1.6 in Section 1 shows that the class of homogeneous lattices discussed in [6] is properly contained in the class of homogeneous lattices as defined in this paper. However, the results obtained in [6] for homogeneous lattices also apply to the class of homogeneous lattices defined in this paper.

1. General results on homogeneous lattices and homogeneous chains.

THEOREM 1.1. *If L is a homogeneous lattice, then L has a transitive group of automorphisms.*

Proof. Let $x, y \in L$. Then there exist automorphisms f and g of L such that $f(x) = x \vee y$ and $g(y) = x \vee y$. Hence $g^{-1} f(x) = y$.

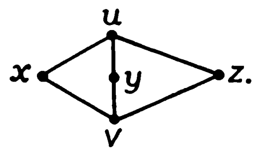
* The author wishes to thank Ph. Dwinger for many helpful discussions concerning homogeneous lattices.

THEOREM 1.2. *If L is a non-trivial homogeneous lattice, then L has neither first nor last element.*

Proof. Suppose $1 \in L$ is the last element, $a \in L$, $a \neq 1$. Let $f \in A(L)$ be such that $f(z) = 1$. Then $1 < f(1)$ which is impossible.

THEOREM 1.3. *If L is homogeneous, then L is distributive.*

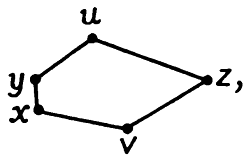
Proof. Suppose L has a sublattice



Let $f \in A^*(L)$ be such that $f(v) = x$. Then $f(y \wedge z) = f(y) \wedge f(z) = f(v) = x$, so $f(y) \geqslant x$ and $f(z) \geqslant x$. Since $f \in A^*(L)$, $f(y) \geqslant y$ and $f(z) \geqslant z$. Thus

$$u = x \vee y \leqslant f(y) \quad \text{and} \quad u = x \vee z \leqslant f(z).$$

So $u \leqslant f(y) \wedge f(z) = f(y \wedge z) = f(v) = x$ which is impossible. Suppose L has a sublattice



and thus let $f \in A^*(L)$ be such that $f(v) = x$. Then

$$f(y \wedge z) = f(y) \wedge f(z) = f(v) = x,$$

so $f(z) \geqslant x$. But again $f \in A^*(L)$, so $f(y) \geqslant y$ and $f(z) \geqslant z$. Hence $y \leqslant u = x \vee z \leqslant f(z)$. But

$$y \leqslant f(y) \wedge f(z) = f(y \wedge z) = x$$

which is impossible.

Definition 1.4. A lattice C is a *chain* if it is a linearly ordered set. A chain C is *dense* if it is non-trivial and if, for any $a, b \in C$, $a < b$, there exists a $c \in C$ such that $a < c < b$. A chain C is *scattered* if it has no subchain which is dense. If M is an arbitrary lattice and C is a chain, the lattice $C \circ M$ will denote the *lexicographic product* of C and M (see [1], p. 199). If a is an ordinal, Z^a denotes the set of all sequences of integers of type a, with finitely many non-zeros, ordered antilexicographically [10]. (See also [14].)

LEMMA 1.5. *Let L be a chain with a transitive group of automorphisms. Then L is homogeneous.*

Proof. Let $a, b \in L$ and $h \in A(L)$, $h(a) = b$. Without loss of generality, let $a < b$. Let $L = X \cup Y$, where

$$X = \{x \in L \mid x \leqslant h(x)\} \quad \text{and} \quad Y = \{x \in L \mid x > h(x)\}.$$

Then $X \cap Y = \emptyset$. Define $g: L \to L$ by $g(x) = h(x)$ for $x \in X$, and by $g(x) = h^{-1}(x)$ for $x \in Y$. Since $X \cap Y = \emptyset$, g is well defined. Moreover, $x \leqslant g(x)$ for all $x \in L$ and $g(a) = b$. A straightforward argument shows g is an automorphism of L.

Example 1.6. Let C denote the chain of all ordinals less than the first uncountable ordinal. Let I denote the half-open unit interval of real numbers $[0, 1)$. Then $C \circ I$ is a linearly ordered set called the *long line*. Deleting the initial element from $C \circ I$ gives a chain L. It is easily seen that the chain L has a transitive group of automorphisms and thus, by Lemma 1.5, is homogeneous. However, any automorphism of L has a fixed point, so L does not admit a lattice-ordered group structure. This shows that the class of homogeneous lattices properly contains the class of lattices that admit a lattice-ordered group structure.

The following two theorems are due to Morel ([10], p. 213, and [11], p. 200). (See also Sankaran [13], p. 18.)

THEOREM 1.7. *If C is a scattered chain with a transitive group of automorphisms, then $C \cong Z^a$ for some ordinal a.*

THEOREM 1.8. *If C is a chain that admits a linearly-ordered group structure, then C satisfies exactly one of the following conditions:*

(i) *$C \cong Z^a$ for some ordinal a;*

(ii) *C is dense;*

(iii) *$C \cong D \circ Z^a$, where a is a non-zero ordinal and D is a dense chain which admits a linearly-ordered group structure.*

Now, using Lemma 1.5 and Theorem 1.7, a result for homogeneous chains, similar to Theorem 1.8, is obtained. (Also see [12].)

THEOREM 1.9. *Let C be a homogeneous chain. Then exactly one of the following conditions holds:*

(i) *$C \cong Z^a$ for some ordinal a;*

(ii) *C is dense;*

(iii) *$C \cong D \circ Z^a$, where a is a non-zero ordinal and D is a homogeneous chain.*

Proof. If C is a one-element chain, then $C \cong Z^0$. If (ii) does not hold and C is non-trivial, then there exist $a, b \in C$ such that $a \prec b$. Hence the family of convex scattered subchains of C that contain a is non-void. Take the union of all such convex scattered subchains to obtain a maximal convex scattered subchain of C containing a. Call this subchain $M(a)$. By homogeneity, for any $x \in C$, such a maximal subchain $M(x)$

can be constructed. Also, by homogeneity, convexity and maximality, $M(x) \cong M(y)$ for all $x, y \in C$. Maximality and convexity imply that if $y \in M(x)$, then $M(y) = M(x)$. The collection of all such $M(x)$ for $x \in C$ forms a partition of C into convex isomorphic subchains. This partition, therefore, induces a congruence relation δ on C. If f is an automorphism of C, then $f(M(x)) = M(f(x))$. Since C has a transitive group of automorphisms, so does $D = L/\delta$. By Lemma 1.5, D is homogeneous. Also, since the $M(x)$ are maximal with respect to being scattered, if D is non-trivial, then it is dense. It is easily seen that $M(x)$ has a transitive group of automorphisms. Thus, by Theorem 1.7, $M(x) \cong Z^a$ for some non-zero ordinal a.

Note that the congruence relation δ in the proof of Theorem 1.9 is a *convex congruence* on C. See, for example, [8] for a discussion of such congruence relations and their use in lattice-ordered groups.

Theorem 1.9 gives, as a corollary, the following parallel of Mal'cev's result for countable linearly-ordered groups [9]. Q will denote the rational numbers under the usual ordering.

COROLLARY 1.10. *Let C be a countable homogeneous chain. Then exactly one of the following conditions holds:*

(i) $C \cong Z^a$ *for some countable ordinal* a;

(ii) $C \cong Q$;

(iii) $C \cong Q \circ Z^a$ *for some non-zero countable ordinal* a.

Proof. This follows from Theorems 1.9 and 1.2 and the fact that, up to isomorphism, Q is the only countable chain that is dense with neither first nor last element.

2. Disjoint elements in homogeneous lattices. In this section results similar to those of Conrad and Clifford [5] and Conrad [3] and [4] are obtained. See also Fuchs [7], p. 82.

Let L be a homogeneous lattice. A subset $S = \{s_i\}_{i \in I}$ of incomparable elements is called *$\wedge$-disjoint* if $s_i \wedge s_j = s_i \wedge s_k$ for any $i, j, k \in I$, $i \neq j$, $i \neq k$. If S is $\wedge$-disjoint and $x = s_i \wedge s_j$ for $i \neq j$, then S is said to be *$\wedge$-disjoint relative to x*. The notion of $\vee$-disjoint is defined dually. If L is homogeneous and S is $\wedge$-disjoint relative to x, then for $y \in L$ such that $y \neq x$ there exists a set T which is $\wedge$-disjoint relative to y, and $|S| = |T|$.

In the remainder of this paper, every homogeneous lattice will have $\wedge$-disjoint sets of cardinality at most n, where n is a finite integer.

LEMMA 2.1. *Let L be a homogeneous lattice. If n is the maximum cardinality of a $\wedge$-disjoint set, then n is the maximum cardinality of a $\vee$-disjoint set.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a $\wedge$-disjoint set relative to $e \in L$. By Theorem 1.3, L is distributive. Hence the sublattice of L generated by $\{a_1, \ldots, a_n\}$

is 2^n. Let $d = a_1 \vee \ldots \vee a_n$. Then the co-atoms of the sublattice generated by $\{a_1, \ldots, a_n\}$ form a $\vee$-disjoint set relative to d. A dual argument now proves the lemma.

Choose and fix an $e \in L$. Let $\{a_1, \ldots, a_n\}$, where n is maximal, be $\wedge$-disjoint relative to e. Select $\{b_1, \ldots, b_n\}$ $\vee$-disjoint relative to e, where, by virtue of Lemma 2.1, n is also maximal.

LEMMA 2.2. *The intervals* $[e, a_i]$ *and* $[b_i, e]$, $1 \leqslant i \leqslant n$, *are both chains.*

Proof. Suppose $[e, a_1]$ is not a chain. Then there exist $x, y \in [e, a_1]$, $x \| y$. If $x \wedge y = e$, then the set $\{x, y, a_2, \ldots, a_n\}$ is $\wedge$-disjoint which contradicts the maximality of n. For $x \wedge y > e$, let $f \in A^*(L)$ be such that $f(x \wedge y) = e$. Then $a_1 \geqslant x \geqslant f(x) \geqslant e$ and $a_1 \geqslant y \geqslant f(y) \geqslant e$ and $f(x) \wedge f(y) = e$. Since $x \| y$, $f(x) \| f(y)$, and hence $f(x) \neq e$ and $f(y) \neq e$. But then $\{f(x), f(y), a_2, \ldots, a_n\}$ is a $\wedge$-disjoint set.

Definition 2.3. Let L be homogeneous and fix an $e \in L$. For $a > e$, let

$$a\hat{\ } = \{b \geqslant e \mid b \wedge x = e \text{ if and only if } a \wedge x = e\}.$$

Similarly, for $a < e$, let

$$a\check{\ } = \{b \leqslant e \mid b \vee x = e \text{ if and only if } b \vee x = e\}.$$

In the language of lattice-ordered groups, $a\hat{\ }$ is called the *carrier* of a (see [7], p. 72). Using distributivity, it is easily shown that $a\hat{\ }$ is a convex sublattice of L for any $a > e$.

Suppose the maximal cardinality of a $\wedge$-disjoint subset of L is n. If $S = \{a_1, \ldots, a_n\}$ is $\wedge$-disjoint relative to e, then $\hat{a_i}$ is a chain for each i. For if not, let $x, y \in \hat{a_i}$, $x \| y$. Then the set

$$\{x, y, a_1 \vee (x \wedge y), \ldots, a_n \vee (x \wedge y)\}$$

is of cardinality $n+1$ and $\wedge$-disjoint relative to $x \wedge y$.

LEMMA 2.4. *The interval* $[b_1, a_i]$, $1 \leqslant i \leqslant n$, *is a chain for exactly one* i.

Proof. First, suppose both $[b_1, a_1]$ and $[b_1, a_2]$ are chains. Let $f \in A^*(L)$, $f(e) = b_1$. Then

$$f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2) = f(e) = b_1.$$

But $f(a_1) \neq b_1$ and $f(a_2) \neq b_1$ since $a_1 \neq e$ and $a_2 \neq e$. Also $f(a_1) \in [b_1, a_1]$ and $f(a_2) \in [b_1, a_2]$ which, by the hypothesis, are chains. Thus $f(a_1) \wedge f(a_2)$ is either e or the minimum of $f(a_1)$ and $f(a_2)$. Thus $f(a_1) \wedge f(a_2) = b_1$ is impossible.

Now it will be shown that $[b_1, a_i]$ is a chain for some i. Let $f \in A^*(L)$, $f(b_1) = e$. By Lemma 2.2, $[b_1, e]$ is a chain, and so is $[f(b_1), f(e)] = [e, f(e)]$. Moreover, $f(e) > e$. Hence $a_j \wedge f(e) = t > e$ for some j, and $a_k \wedge f(e) = e$ for $k \neq j$. Thus $f(e) \in \hat{a_j}$. Now consider the set $\check{b_1}$. Since it is a chain, $f(\check{b_1})$ is also a chain. Note $[e, f(e)] \subseteq f(\check{b_1})$.

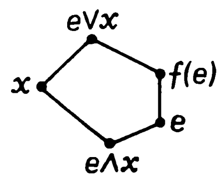
Next, claim $b_1 \epsilon f(b_1^{\check{}})$. It is enough to show that $f^{-1}(b_1) \epsilon b_1^{\check{}}$ or, equivalently, $f^{-1}(b_1) \vee b_j = e$ for $j = 2, \ldots, n$. But this is the same as showing $b_1 \vee f(b_j) = f(e)$. Since $b_1 \vee b_j = e$, it follows $f(b_1) \vee f(b_j) = f(e)$. Since $f(b_1) = e$, this gives $e \vee f(b_j) = f(e)$. But

$$e = b_1 \vee b_j \leqslant b_1 \vee f(b_j),$$

so $e \vee f(b_j) \leqslant b_1 \vee f(b_j)$. But also

$$f(e) = e \vee f(b_j) \geqslant b_1 \vee f(b_j).$$

Thus $f(e) = b_1 \vee f(b_j)$ as desired, so $b_1 \epsilon f(b_1^{\check{}})$. At this stage we now have the intervals $[e, a_i]$ and $[b_1, f(e)]$ with the common intersection $[e, f(e)]$. Suppose there exists an $x \epsilon [b_1, a_i]$ such that $x \notin [e, a_i]$ and $x \notin [b_1, f(e)]$. Then $x \| e$. Form $x \vee e$ and $x \wedge e$. Since $[b_1, f(e)]$ is an interval, $x \vee e > f(e)$. Thus $x \vee e = x \vee f(e)$. Similarly, $x \wedge e = x \wedge f(e)$. Since $e \neq f(e)$, it follows that



is a sublattice, which is impossible. Thus $[b_1, a_i]$ is indeed a chain.

Note 2.5. It follows from Lemma 2.4 that the set $b_1^{\check{}} \cup a_i^{\hat{}} \cup \{e\}$ is a convex subchain for exactly one i. This argument can be given for each $b_i^{\check{}}$, $1 \leqslant i \leqslant n$, renumbering the a_i, if necessary, to give

$$C_i = a_i^{\hat{}} \cup b_i^{\check{}} \cup \{e\}.$$

The C_i, $1 \leqslant i \leqslant n$, are convex subchains.

LEMMA 2.6. *Let L be homogeneous with $C_1, C_2, \ldots, C_n$ as in Note 2.5. Let D be any convex subchain, $e \epsilon D$. Then $D \subseteq C_i$ for some i.*

Proof. Let $d \epsilon D$, $d > e$. Form $\{d \wedge a_1, \ldots, d \wedge a_n\}$ with $a_i > e$, $a_i \epsilon C_i$. At least one of these meets is not e, for otherwise $\{d, a_1, \ldots, a_n\}$ would be $\wedge$-disjoint relative to e. So assume $d \wedge a_1 > e$. It follows that $d \wedge a_i = e$ for $2 \leqslant i \leqslant n$. For if not, say $d \wedge a_2 > e$, then the chain $[e, d]$ would contain the two non-comparable elements $d \wedge a_1$ and $d \wedge a_2$ which is impossible. We claim $d \epsilon a_1^{\hat{}}$. For let $a_1 \wedge x = e$. If $d \wedge x > e$, then the set $\{x \wedge d, a_1, a_2, \ldots, a_n\}$ is $\wedge$-disjoint which is impossible. If, on the other hand, $d \wedge y = e$ and $a_1 \wedge y > e$, the set $\{d, a_1 \wedge y, a_2, \ldots, a_n\}$ is $\wedge$-disjoint relative to e which is also impossible. Thus $\{d \mid d \epsilon D, d \geqslant e\}$ is in $a_1^{\hat{}}$. Similarly, $\{d \mid d \epsilon D, d \leqslant e\}$ is in $b_j^{\check{}}$ for some j. But since D is a chain, Lemma 2.4 guarantees that $j = 1$ as desired.

Note 2.7. We infer directly from Lemmas 2.6 and 2.2 that if the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are both $\wedge$-disjoint relative to e, and n is maximal, then there is some permutation ν of $\{1, \ldots, n\}$ such that $a_1^{\wedge} = b_{\nu(1)}^{\wedge}, \ldots, a_n^{\wedge} = b_{\nu(n)}^{\wedge}$.

LEMMA 2.8. *Let L be a homogeneous lattice, C_i as in Note 2.5 and $c \in C_i$. Let $f \in A^*(L)$ be such that either $f(c) = e$ or $f(e) = c$. Then $f(C_i) = C_i$, and thus $f|_{C_i}$ is an automorphism of C_i.*

Proof. Let $c \in C_1$, $c < e$, and $f \in A^*(L)$, $f(c) = e$. In the proof of Lemma 2.4 it was shown that $f(c^{\vee}) \subseteq C_1$. For any $a \in C_1$ with $e < f(e) < a$, since $[c, a]$ is a chain, so is $[e, f(a)]$. So, by Lemma 2.6, $f(a) \in C_1$. Thus $f(C_1) \subseteq C_1$. Note $e \in f(C_1)$ and $e < f(e)$. Now consider f^{-1}. Observe that $f^{-1} \in A^*(L)$ and $f^{-1}(f(e)) = e$. Applying f gives

$$ff^{-1}(C_1) = C_1 \subseteq f(C_1) \subseteq C_1,$$

so $f(C_1) = C_1$. A dual argument applies for $c > e$. Finally, if $f(e) = c$, just go through the above argument with f^{-1}.

THEOREM 2.9. *Let L be a homogeneous lattice such that n is the maximum cardinality of a $\wedge$-disjoint set. Let C_i, $1 \leqslant i \leqslant n$, be as in Note 2.5. Then the C_i are homogeneous chains.*

Proof. Let $x, y \in C_i$. Consider $f \in A^*(L)$ such that $f(x) = e$. Let $g \in A^*(L)$ be such that $g(e) = y$. Then gf is an automorphism of L, $gf(x) = y$. By Lemma 2.8, $gf(C_i) = C_i$, so $gf|_{C_i}$ is an automorphism of C_i. Hence C_i has a transitive group of automorphisms, and so, by Lemma 1.5, C_i is homogeneous.

Thus when viewed in terms of lattice-ordered groups, the C_i correspond to shifting subgroups. See, for example, Byrd, Conrad and Lloyd [2].

LEMMA 2.10. *Let P be the sublattice of L generated by the $a_i^{\wedge}$, $1 \leqslant i \leqslant n$. Then*

$$P \cong \prod_{i=1}^{n} (a_i^{\wedge} \cup \{e\}).$$

Dually, if Q is the sublattice of L generated by the $b_i^{\vee}$, then

$$Q \cong \prod_{i=1}^{n} (b_i^{\vee} \cup \{e\}).$$

Proof. Let $p \in P$, $p > e$. Then, by distributivity,

$$p = \bigvee_i \big(\bigwedge_j x_{i_j}\big), \qquad \text{where } x_{i_j} \in a_{i_j}.$$

By disjointness, this simplifies to $x_{i_1} \vee \ldots \vee x_{i_k}$ with $x_{i_j} \in a_{i_j}^{\wedge}$; $i_j \neq i_l$ implies $a_{i_j}^{\wedge} \neq a_{i_l}^{\wedge}$, and $x_{i_j} > e$. This representation is also unique, for if $p = y_{i_1} \vee \ldots \vee y_{i_m}$, then $y_{i_1} = p \wedge y_i = y_{i_1} \wedge x_{i_j}$, and hence $y_{i_1} \leqslant x_{i_j}$. But a similar argument shows that $x_{i_j} \leqslant y_{i_1}$. Define

$$h\colon P \to \prod_{i=1}^{n} (a_i \cup \{e\})$$

by $(h(p))_i = x_i$ if x_i occurs in the above representation, and by $(h(p))_i = e$ otherwise. Then h is easily seen to be an isomorphism.

Definition 2.11. Let $S(e)$ denote the *convex hull* of P and Q. That is,

$$S(e) = \{y \in L \mid q \leqslant y \leqslant p \text{ with } p \in P \text{ and } q \in Q\}.$$

Then $S(e)$ is a convex sublattice of L, $e \in S(e)$. This construction of the sublattice $S(e)$ can be done for any element $x \in L$, with x in place of e, to give a convex sublattice $S(x)$. All such sublattices will be isomorphic since L is homogeneous.

Define a relation σ on L by $\langle x, y\rangle \in \sigma$ if and only if $S(x) = S(y)$.

LEMMA 2.12. *The relation σ is a congruence relation for L. Moreover, $[y]_\sigma = S(y)$ for all $y \in L$.*

Proof. Clearly, σ is an equivalence relation. To show that $[y]_\sigma = S(y)$, it is sufficient to show that if $x \in S(y)$, then $S(x) = S(y)$. Since all the $S(y)$ are isomorphic and L is homogeneous, it is enough to show that if $x \in S(e)$, then $e \in S(x) \subseteq S(e)$. First, suppose $x > e$ and $x = a_1$, i.e. x is in a set of n $\wedge$-disjoint elements, which is $\wedge$-disjoint relative to e. Thus

$$\{x \vee a_2, x \vee a_3, \ldots, x \vee a_n, a_1'\}$$

are $\wedge$-disjoint relative to x, where $a_1' \in \hat{a_1}$, $a_1' > x$. But these elements are also in $S(e)$. Since $[e, a_1] = [e, x]$ is a chain, by Lemma 2.6, $e \in S(x)$. Let $c_1, \ldots, c_n$ be $\vee$-disjoint relative to x. Then, since $[c_i, a_1]$ is a chain, so is $[e \wedge c_i, e \wedge a_1]$, which is $[e \wedge c_i, e]$. By Lemma 2.6, $e \wedge c_i \in S(e)$. Hence the c_i are in $S(e)$ since $S(e)$ is convex, i.e. $e \wedge c_i < c_i < x$ and $e \wedge c_i$ and x are in $S(e)$. Hence all the $\wedge$-disjoint and $\vee$-disjoint elements of $x = a_1$ are in $S(e)$, so $e \in S(x) \subseteq S(e)$, so $S(x) = S(e)$. If $x = a_1 \vee \ldots \vee a_k$ say, then x is one of n elements pairwise disjoint to $a_1 \vee \ldots \vee a_{k-1}$, the others being

$$a_1 \vee \ldots \vee a_{k-1} \vee a_{k+1}, \quad a_1 \vee \ldots \vee a_{k-1} \vee a_{k+2}, \quad \ldots, \quad a_1 \vee \ldots \vee a_{k-1} \vee a_n,$$

together with

$$a_1' \vee a_2 \vee \ldots \vee a_{k-1}, \quad \ldots, \quad a_1 \vee \ldots \vee a_{k-1}', \quad \text{where } a_i' \in \hat{a_i}, \ a_i' > a_i.$$

Now apply the previous argument to show $S(x) = S(a_1 \vee \ldots \vee a_{k-1})$ and induct on k. A dual argument shows that if $x < e$ holds, then $x \in S(e)$ implies $S(x) = S(e)$. Finally, if $x \in S(e)$, $x \| e$, then $x \wedge e$ and $x \vee e$ are in $S(e)$, since $S(e)$ is a sublattice. But $x \vee e \geqslant e \geqslant x \wedge e$, and hence $S(x \vee e) = S(e) = S(x \wedge e)$. But also $x \vee e \geqslant x \geqslant x \wedge e$, so $x \in S(x \wedge e)$. Thus $S(x) = S(x \wedge e) = S(e)$ as desired.

Now it will be shown that σ is a lattice congruence relation. Let $\langle x, y\rangle \in \sigma$. Without loss of generality, let $x = e$ so that, say, $y = a_1 \vee \ldots \vee a_k$ with $a_i \wedge a_j = e$. To see that σ is a congruence relation it is sufficient to show that, for an arbitrary $z \in L$, $\langle x\vee z, y\vee z\rangle \in \sigma$ and $\langle x\wedge z, y\wedge z\rangle \in \sigma$. But

$$y\vee z = (a_1\vee z)\vee(a_2\vee z)\vee \ldots \vee(a_k\vee z).$$

The interval $[e\vee z, a_1\vee z]$ is a homomorphic image of the interval $[e, a_1]$, and thus it is either a non-trivial chain or a single element. If it is a non-trivial chain, then, by Lemma 2.6, it is in $\hat{d_i}$, where $\{d_1, \ldots, d_n\}$ is the set of n $\wedge$-disjoint elements relative to $e\vee z$. Thus $y\vee z \in S(e\vee z)$ and, therefore, $(y\vee z, e\vee z)\in\sigma$. Similarly,

$$y\wedge z = (a_1\vee \ldots \vee a_k)\wedge z = (a_1\wedge z)\vee \ldots \vee (a_k\wedge z),$$

and $[e\wedge z, a_1\wedge z]$ is a non-trivial chain or a single element, so $y\wedge z\in S(e\wedge z)$.

THEOREM 2.13. *We have*

$$S(e) \cong \prod_{i=1}^{n} C_i,$$

where the C_i *are as in Note* 2.5.

Proof. Let $y\in S(e)$, $y \neq e$. Let $y\vee e = x_1\vee \ldots \vee x_k$, $x_i\in \hat{a_i}$, $x_i > e$ and $y\wedge e = z_{i_1}\wedge \ldots \wedge z_{i_m}$, $z_{i_j} < e$, $z_{i_j}\in \check{b}_{i_j}$. First, we claim

$$\{1, \ldots, k\}\cap\{i_1, \ldots, i_m\} = \emptyset.$$

For suppose $y\vee e \geqslant x_1 > z_1 \geqslant y\wedge e$. Let $t = (x_1\wedge y)\vee z_1$. The interval $[z_1, x_1]$ is linearly ordered and both t and e are in $[z_1, x_1]$. Hence either $t\vee e = e$ or $t\wedge e = e$. But

$$t\wedge e = [(x_1\wedge y)\vee z_1]\wedge e = (e\wedge x_1\wedge y)\vee(z_1\wedge e) = y_1 \neq e$$

while

$$t\vee e = [(x_1\wedge y)\vee z_1]\vee e = (x_1\wedge y)\vee e = (x_1\vee e)\wedge(y\vee e) = x_1 \neq e.$$

Let $y\in S(e)$. Let $P(y) = \{i \mid e < x_i \in \hat{a_i}$ and x_i occurs in the representation of $y\vee e\}$. Analogously define $Q(y)$ for $y\wedge e$. Then we have shown that $Q(y)\cap P(y) = \emptyset$.

Define

$$\alpha\colon S(e) \to \prod_{i=1}^{n} C_i$$

by $(\alpha(y))_i = x_i$ if $i\in P(y)$, where x_i occurs in the representation in P of $y\vee e$; by $(\alpha(y))_i = z_i$ if $i\in Q(y)$, where z_i occurs in the representation in Q of $y\wedge e$; and by $(\alpha(y))_i = e$ otherwise. Then α is well defined since the representations are unique and $P(y)\cap Q(y) = \emptyset$. Also α is 1-1, for if $y\vee e = y'\vee e$ and $y\wedge e = y'\wedge e$, then, by distributivity, $y = y'$. To show that α is a homomorphism, observe that

$$(y\vee y')\vee e = (y\vee e)\vee(y'\vee e) \quad \text{and} \quad (y\vee y')\wedge e = (y\wedge e)\vee(y'\wedge e).$$

Thus $(a(y\vee y'))_i = (a(y))_i \vee (a(y'))_i$. Similarly for $y\wedge y'$.

It remains to show that a is onto $\prod_{i=1}^{n} C_i$. From the definition of a it will be sufficient to show that if

$$q = z_1\wedge \ldots \wedge z_k \in Q, \qquad z_i \in C_i,\, z_i < e$$

and if

$$p = x_{k+1}\vee \ldots \vee x_{k+m} \in P, \qquad x_i \in C_i,\, x_k > e, \qquad \text{with } k+m \leqslant n,$$

then there exists a $y\in S(e)$ such that $y\vee e = p$ and $y\wedge e = q$. Let $\{\bar{a}_1, \ldots, \bar{a}_n\}$ be n $\wedge$-disjoint elements relative to q. Let $f\in A^*(L)$ be such that $f(e) = q$. (Without loss of generality, assume $f(a_i) = \bar{a}_i$.) Since $q\in S(e)$, e and p are in the sublattice generated by $\bar{a}_1^{\wedge} \cup \ldots \cup \bar{a}_n^{\wedge}$, i.e. $S(q) = S(e)$. Let

$$e = e_1\vee \ldots \vee e_r \quad \text{and} \quad p = p_1\vee \ldots \vee p_r \vee p_{r+1}\vee \ldots \vee p_{r+t}$$

with e_i and $p_i \in \bar{a}_i^{\wedge}$, and $p_i > q$, $e_i > q$, and $r+t < n$. Since $q < e < p$, such a unique representation is possible. We claim $e_i = p_i$ for $1 \leqslant i \leqslant r$. For if not, suppose $e_1 < p_1$. Then the interval $e_2\vee \ldots \vee e_r$, $p_1\vee e_2\vee \ldots \vee e_r$ would be a chain containing e. Hence it would be in some C_i. But

$$q \leqslant z_i < e_2\vee \ldots \vee e_r < e < p_1\vee e_2\vee \ldots \vee e_r < x_i \leqslant p$$

which violates the original choice of p and q, i.e. $P(p)\cap Q(q) = \emptyset$. Now let $y > q$ be given by $y = p_{r+1}\vee \ldots \vee p_{r+t}$. Then

$$e\wedge y = q$$

and

$$e\vee y = e_1\vee \ldots \vee e_r\vee p_{r+1}\vee \ldots \vee p_{r+t} = p_1\vee \ldots \vee p_r\vee p_{r+1}\vee \ldots \vee p_{r+t} = p,$$

as desired.

COROLLARY 2.14. *Let L be a homogeneous lattice, and e an arbitrary fixed element of L. Suppose L satisfies the following property:*

(1) *every $x > e$ is the join of at most n elements $\wedge$-disjoint relative to e.*

Then L is the product of homogeneous chains.

Proof. For $e\in L$ and any set A $\wedge$-disjoint relative to e, $|A| \leqslant n$. Thus one can form $S(e)$ as in Theorem 2.13. If $x > e$, then $x\in S(e)$. If $x < e$, then $x\in S(e)$ also; for if not, $S(x)\cap S(e) = \emptyset$, and then e cannot be expressed as join of at most n $\wedge$-disjoint elements relative to x. This violates the assumptions that L is homogeneous and that (1) holds for L. Thus $S(e) = L$ and the result follows from Theorems 2.13 and 2.9.

COROLLARY 2.15. *Let L be a homogeneous lattice with n the maximal cardinality of a $\wedge$-disjoint set. If every interval in L is of finite length, then*

$$L \cong \prod_{i=1}^{n} Z_i,$$

where Z_i is the chain of all integers.

Proof. By Theorem 2.9, the C_i are homogeneous chains. Since all intervals are finite, Theorem 1.9 implies $C_i \cong Z$ for all i. Moreover, if $x > e$, then $x \in S(e)$ since the interval $[e, x]$ has a finite length. Thus $L = S(e)$ and the corollary follows from Theorem 2.13.

Let L be homogeneous with n the maximal cardinality of a $\wedge$-disjoint set. Let σ be as in Lemma 2.12. The following is analogous to the result number 8 in [3]:

THEOREM 2.16. *The lattice L/σ is homogeneous and any set of $\wedge$-disjoint elements has cardinality less than n.*

Proof. Let $S(a)$ and $S(b) \in L/\sigma$. Let $f \in A^*(L)$ be such that $f(a) = b$. Define $F\colon L/\sigma \to L/\sigma$ by $F(S(x)) = S(f(x))$. Note $f(S(x)) = S(f(x))$. Thus $S(x) = S(y)$ implies $S(f(x)) = S(f(y))$. It follows that F is well defined, 1-1 and onto. Since σ is a congruence relation, $S(x) \vee S(y) = S(x \vee y)$. Thus

$$F(S(x) \vee S(y)) = S(f(x \vee y)) = S(f(x) \vee f(y)) = S(f(x)) \vee S(f(y))$$
$$= F(S(x)) \vee F(S(y)).$$

Similarly for meets. Thus F is an automorphism of L/σ. It remains to show some such automorphism satisfies the monotonicity condition. If $S(a) \leqslant S(b)$ in L/σ, then $a \wedge b \leqslant b$ and $a \wedge b \in S(a)$ since $S(a \wedge b) = S(a) \wedge \wedge S(b) = S(a)$. Let $f \in A^*(L)$ be such that $f(a \wedge b) = b$. Then $x \leqslant f(x)$ for all $x \in L$. Thus $S(x) \leqslant S(f(x))$. Therefore, let $F(S(x)) = S(f(x))$ be the desired automorphism of L/σ.

The second claim of the theorem will now be demonstrated. Let $d \in L \setminus S$, $d > e$. Let $r(d) = \{i \mid d > \hat{a_i}\}$, i.e. $i \in r(d)$ and $x \in \hat{a_i}$ imply $d > x$. Claim $|r(d)| \geqslant 2$. For suppose $|r(d)| = 1$. So $d > \hat{a_1}$ say. For each i, $2 \leqslant i \leqslant n$, there exists an $x_i \in \hat{a_i}$ such that $x_i \| d$. Let $t_i = x_i \wedge d$. Since $\hat{a_i}$ is a chain, for all $x \in \hat{a_i}$, $d \wedge x = t_i$ or $d \wedge x = x$. Form $t = t_2 \vee \ldots \vee t_n$. Let $\{s_1, s_2, \ldots, s_n\}$ be n $\wedge$-disjoint elements relative to t. Note

$$\{t, s_1, s_2, \ldots, s_n\} \subseteq S(e).$$

If $s_i \wedge a_1 > e$ and $s_j \wedge a_1 > e$, then $a_1 \wedge (s_i \wedge s_j) > e$. But then $a_1 \wedge t > e$ which is impossible. So for, say, $s_2, \ldots, s_n$, we have $s_i \wedge s_j = t$ and $a_1 \wedge s_i = e$, $2 \leqslant i, j \leqslant n$. But then $d > t$ and $s_i > t$ imply $d \wedge s_i \geqslant t$. Let

$$s_i = s_2^i \vee \ldots \vee s_n^i, \qquad \text{where } s_j^i \in \hat{a_j}.$$

Then

$$d \wedge s_i = (d \wedge s_2^i) \vee \ldots \vee (d \wedge s_n^i) \leqslant t_2 \vee \ldots \vee t_n = t.$$

Thus $d \wedge s_i = t$. So $\{d, s_2, \ldots, s_n\}$ are $\wedge$-disjoint to t which implies, by Note 2.7, that $d \in S$ which is impossible.

The case $|r(d)| = 0$ is handled analogously.

Thus $|r(d)| \geqslant 2$ for all $d \in L \setminus S$, $d > e$. Moreover, if $\{d_1, \ldots, d_m\}$ are such that $d_i \in L \setminus S$, $d_i > e$ and $d_i \wedge d_j \in S$, then $r(d_i) \cap r(d_j) = \emptyset$ for all $1 \leqslant i < j \leqslant m$. Therefore, $|r(d_i)| \geqslant 2$ implies $m < n$, as desired.

In the special case where $n = 2$ we get the following result which is similar to that of Conrad and Clifford for lattice-ordered groups [5]:

THEOREM 2.17. *Let L be a homogeneous lattice such that the maximal cardinality of any $\wedge$-disjoint set is 2. Then $L \cong C_3 \circ (C_2 \times C_1)$, where the C_i, $i = 1, 2, 3$, are homogeneous chains.*

Proof. By Theorem 2.16, L/σ is homogeneous. Moreover, if $x \not\equiv y(\sigma)$, then $x > y$ or $y < x$ in L. Letting $C_3 = L/\sigma$, the conclusion follows.

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Reçu par la Rédaction le 12. 1. 1973