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Abstract logics

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Classical abstract logics

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Preface

Both papers in this volume are concerned with abstract logics. The first one is an introduction to the general theory of abstract logics and the second deals with a particular kind of logics, called classical logics.

An abstract logic is a pair $\langle \mathscr{A}, \mathscr{C} \rangle$ where $\mathscr{A}$ is an algebra and $\mathscr{C}$ is a closure system on the carrier of $\mathscr{A}$. Obviously, $\mathscr{C}$ may be replaced by the closure operator Cn corresponding to $\mathscr{C}$. Abstract logics are obtained as generalization of various logical notions. If $\mathscr{C} = \{A, B\}$ where $B \subseteq A$ $=$ the carrier of $\mathscr{A}$ then $\mathscr{C}$ is called *elementary closure system* and the abstract logic $\langle \mathscr{A}, \mathscr{C} \rangle$ forms a logical matrix, a concept introduced by Łukasiewicz and Tarski in 1930. At about the same time, Tarski introduced the consequence operation (or entailment relation) as the fundamental logical notion. It is a special case of a closure operator. Also, as observed first by Lindenbaum, formalized languages are algebraic systems, i.e., sets supplied with (free) operations determined by formation rules. Hence, a formalized language conceived as an algebra of sentential formulas together with a consequence operation generated by logical axioms and rules of inference or, perhaps defined otherwise, is an abstract logic.

Since the 1930's several abstract generalizations, either algebraic or closure theoretic, arose from the logical field. I only mention cylindric algebras of Tarski and polyadic algebras of Halmos and, on the other hand, *Théorie metamathematique des ideaux* of A. Robinson (1955).

Note that Boole's mathematical thought already reached the level of abstract logics. Any Boolean algebra with the family of all its filters is a paradigm of abstract logics. Similarly, a Boolean algebra together with a single ultrafilter may be taken as paradigm of a logical matrix (elementary abstract logic).

The class of abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$, with algebras $\mathscr{A}$ of a fixed similarity type, constitute a category when supplied with suitably defined morphisms. Here, the analogy with general topology is the guiding idea so that the morphisms of abstract logics are defined as "continuous" homomorphisms. It is natural to continue the said analogy and, to introduce projective and inductive generation of abstract logics as well as the notion of a dual space. Thus, we construct a general framework of the theory of abstract logics.

Certainly, like the category of all topological spaces, categories of abstract logics are too large to be interesting or important when considered as a whole. One may prefer to study certain classes of abstract logics possessing certain properties, distinguished from this or that point of view. For example, topological separation properties, which originated in geometry, are of no use from the purely logical point of view adopted here. We are concerned with other properties: some of these are set-theoretical properties of closure systems (finiteness, logical compactness, regularity), the others involve the closure system and the underlying algebra, simultaneously (structurality, invariance, negativity, disjunctivity).

Properties of abstract logics which are distinguished from the logical point of view are those of "good" logics. These are either

(I) logics we actually use (!) or, at least, are able to use or

(II) logics which appear as semantical interpretations of the former ones.

The primary logics (I) seem to be necessarily free in the general algebraic sense. On the other hand, the most natural secondary logics (II) are logical quotients of primary logics modulo logical congruences. They are, in several particular cases, known as Lindenbaum-Tarski algebras. Therefore the theory of abstract logics cannot overlook free logics and must consider logical congruences in a general setting.

The theory of abstract logics constitutes a framework for semantical investigations. Here, we only consider the problem of the so called completeness theorem, which is the generally accepted minimal requirement on what semantics provide. Given a logic $\langle \mathscr{A}, \mathrm{Cn} \rangle$, in the semantics we look for a class $\mathscr{K}$ of interpretations (logical matrices) with the following property (completeness of the logic with respect to $\mathscr{K}$): $p \notin \mathrm{Cn}(X)$ iff there is an interpretation in $\mathscr{K}$ which makes p invalid (false, unsatisfied) while making all elements of X valid (true, satisfied). Our idea, partly due to R. Wójcicki is to make the completeness problem trivial. Indeed, if one views the logic $\langle \mathscr{A}, \mathscr{C} \rangle$ as a bundle of logical matrices, one may generalize the old Lindenbaum construction to show that abstract logics, if structural or invariant *are* complete with respect to themselves and, hence, with respect to their logical quotients.

However undesirable it might seem, the trivial completeness theorem underlies the investigations of all the diverse (sentential) logics by H. Rasiowa and R. Sikorski. On the other hand, their exemplary studies may suggest that the framework of abstract logics is the result of an over-eagerness to generalize beyond reasonable understanding. Indeed, all the logical quotients obtained by Rasiowa and Sikorski are "nice" algebras, e.g. lattices with unit, together with a closure system of filters (or, in particular, I-filters). One may argue that this closure system is inherent in the corresponding algebra and what seemingly matters here is the *unit* which again is inseparable from the algebra (lattice with unit). If the free logics considered by Rasiowa and Sikorski are typical for primary logics then as secondary logics one may only expect algebras involving an "internal" closure system. Therefore, it is a useless luxury to climb to the level of abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$ where the closure system $\mathscr{C}$ cannot be easily manufactured from the algebra $\mathscr{A}$.

There is indeed some point in this reasoning. However, what matters here is the question whether there exists a primary logic such that in logical quotients the algebra and the closure system are independent, so to speak.

The answer is an emphatic YES and constitutes the ultimate justification for the theory of abstract logics. This answer comes from the non-Fregean logic (NFL), and in particular, its basic level called the sentential calculus with identity (SCI). The SCI does not fit into the lattice-theoretic framework and requires a general theory of logical matrices and abstract logics. You will find some details at the end of the first paper.

The theory of abstract logics is kept here on the level which corresponds to sentential calculi. However, it easily generalizes to the level which corresponds to open (quantifier-free) logics with both sentential and nominal variables. Again, the open non-Fregean logic calls necessarily for a general theory of algebraic structures together with closure systems, possibly elementary. It pleases me very much to acknowledge that J. Łoś was the first (1949) to introduce such structures.

The roots of the second paper may again be found in non-Fregean logic. NFL is extremely weak (as a closure operator) and extremely rich (as closure system). Hence, the non-Fregean logic (or SCI, in particular) forces one to consider an uncountable lattice of logics, that is, the complete lattice of all its extensions (stronger closure operator and smaller closure system, the language being fixed). We face a genuine embarrassment of riches. The first attempt to deal with it consists in dividing all the extensions of NFL (or SCI) into (1) elementary ones, obtained by adding new axioms and (2) the non-elementary ones. All the elementary extensions have very good properties and have been labelled classical logics. The Fregean logic, known from textbooks of mathematical logic, *is* an elementary extension of NFL and in a sense a maximal one. Furthermore, logical quotients of sufficiently strong elementary extensions of SCI appear as nice algebras, e.g., Boolean algebras. On the other hand, all non-elementary extensions possess rather bad properties and are, at least, very strange. The question arises whether we can formulate in general terms the deep difference between elementary and non-elementary extensions of NFL (or SCI, at least). An attempt to do that is the content of the paper on classical logics. A very simple definition of classical abstract logics is assumed and a pleasant result is obtained: an abstract logic is classical iff it is equivalent in a sense to a Boolean logic, i.e., a Boolean algebra together with all (!) filters.

Roman Suszko

CLASSICAL ABSTRACT LOGICS

S. L. Bloom and D. J. Brown

1. Introduction

An *abstract logic* $\langle \mathscr{A}, \mathscr{C} \rangle$ consists of an algebra $\mathscr{A}$ and a closure system $\mathscr{C}$ on $\mathscr{A}$ (the same letter denotes both an algebra and its underlying set). A classical abstract logic (henceforth, "classical logic") is an abstract logic whose closure system is finite, negative and disjunctive (these terms will be defined later). If $\mathscr{A}$ is the algebra of formulas of the standard two-valued propositional calculus (or a Boolean algebra), and $\mathscr{C}$ is the collection of theories (or the collection of all filters) then $\langle \mathscr{A}, \mathscr{C} \rangle$ is a classical logic.

In this paper, classical logics are characterized in several ways, and the category of classical logics is compared with the category of Boolean spaces. As a result, one sees that many of the properties enjoyed by the standard propositional and predicate calculi are consequences of rather simple properties of their respective consequence operations. An interesting corollary to the main theorem is the following: a closure system $\mathscr{C}$ is classical iff $\mathscr{C}$ is isomorphic to the collection of all filters of a Boolean algebra (see Corollary 1).

2. Preliminaries

A closure system $\mathscr{C}$ on an algebra $\mathscr{A}$ is a collection of subsets of $\mathscr{A}$ closed under arbitrary intersection. A closure operation Cn on $\mathscr{A}$ is a function from the power set of $\mathscr{A}$ into itself such that, for all $X, Y \subseteq \mathscr{A}$

$$X \subseteq \mathrm{Cn}(X) = \mathrm{Cn}(\mathrm{Cn}(X)), \qquad X \subseteq Y \Rightarrow \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y).$$

Every closure system $\mathscr{C}$ defines a closure operation Cn by:

$$\mathrm{Cn}(X) = \bigcap (T \in \mathscr{C}\colon X \subseteq T),$$

and every closure operation Cn defines a closure system $\mathscr{C}$:

$$\mathscr{C} = \{X\colon X = \mathrm{Cn}(X)\}.$$

(see [2], pp. 41-42). The letter $\mathscr{C}$ (sometimes with subscripts) will always denote a closure system, and the letters Cn will denote a closure operation. With no further explanation, we will now consider an abstract logic as either a pair $\langle \mathscr{A}, \mathscr{C}\rangle$ or as a pair $\langle \mathscr{A}, \mathrm{Cn}\rangle$, whichever is more convenient.

A *Boolean logic*(1) $\langle \mathscr{A}, \mathscr{C}\rangle$ consists of a Boolean algebra $\mathscr{A}$ and the closure system of all filters in $\mathscr{A}$. If $\mathscr{A}$ is a Boolean algebra, the "corresponding Boolean logic" is the pair $\langle \mathscr{A}, \mathscr{C}\rangle$ where $\mathscr{C}$ is the collection of all filters in $\mathscr{A}$.

For the remainder of the paper, we assume that all algebras have at least a unary function (which is always denoted $-a$) and a binary function (which is always denoted $a \vee b$).

A closure operation Cn on the algebra $\mathscr{A}$ is:

i) *finite* if, for every $X \subseteq A$, $\mathrm{Cn}(X) = \bigcup(\mathrm{Cn}(Y)\colon\ Y \subset X,\ Y \text{ finite})$;

ii) *negative* if, for every $X \subseteq A$, $a \in A$:

$$a \in \mathrm{Cn}(X) \Leftrightarrow \mathrm{Cn}(X, -a) = A;$$

iii) *disjunctive* if, for every $X \subseteq A$, $a, b \in A$:

$$\mathrm{Cn}(a, X) \cap \mathrm{Cn}(b, X) = \mathrm{Cn}(a \vee b, X);$$

iv) *classical* if Cn is finite, negative and disjunctive. $\mathscr{C}$ and $\langle \mathscr{A}, \mathscr{C}\rangle$ are classical if the corresponding Cn is.

PROPOSITION 1. *Let $\mathscr{A}$ be the algebra of formulas of the standard (two-valued) propositional calculus, and let* Cn *be the closure operation defined on $\mathscr{A}$ via the usual truth-functional tautologies and the rule modus ponens. Then (with respect to the obvious connectives)* Cn *is a classical closure operation.*

PROPOSITION 2. *Let $\mathscr{A}$ be a Boolean algebra and let $\mathscr{C}$ be the closure system of all filters in $\mathscr{A}$. Then (with respect to the complement and join operations) $\mathscr{C}$ is a classical.*

The proofs of the above propositions are straight-forward and are omitted. In the next section we obtain partial converses to these propositions using the concept of a dual space of an abstract logic (introduced by Brown in [1]). Before making the necessary definitions, a preliminary result is needed.

PROPOSITION 3. *If $\langle \mathscr{A}, \mathrm{Cn}\rangle$ is classical, then for any $X \subseteq \mathscr{A}$*

$$\mathrm{Cn}(X) = \bigcap(T \in \mathscr{M}\colon\ X \subseteq T)$$

where $\mathscr{M}$ is the collection of maximal Cn-*theories.*

(A Cn-*theory* if a subset T such that $T = \mathrm{Cn}(T)$. T is *maximal* if the only theories over T are T and $\mathscr{A}$ ($\neq T$).) A proof of this proposition

(1) Martin Davis uses the term "Boolean logic" differently. It can be seen from our Theorem 2 that Davis' "Boolean Logics" are our classical logics.

may be given by imitating the proof of Tarski's theorem for the propositional calculus: A theory is the intersection of all maximal theories containing it.

Let $L = \langle \mathscr{A}, \mathrm{Cn}\rangle$ be a classical logic, and let $\mathscr{M}$ be the collection of maximal Cn-theories.

DEFINITION D. Let $s\colon P(A) \to P(\mathscr{M})$ be the function defined by

$$s(X) = \{T \in \mathscr{M} :\cdot X \subseteq T\}.$$

The *dual space* of L is the pair $\langle \mathscr{M}, \mathscr{C}\rangle$ where $\mathscr{C}$ is the collection of all subsets of $\mathscr{M}$ of the form $s(X)$.

$\mathscr{C}$ is a closure system on $\mathscr{M}$ since $\bigcap s(X_i) = s(\bigcup X_i)$.

3. The category of classical logics

Let $L_i = \langle \mathscr{A}_i, \mathscr{C}_i\rangle$ $(i = 1, 2)$ be classical logics. A *morphism* h: $\mathscr{A}_1 \to \mathscr{A}_2$ is a function that preserves at least $-$ and $\vee$. (Note that $\mathscr{A}_1$ and $\mathscr{A}_2$ need not have the same type.) A *logical morphism* from L_1 to L_2 is a morphism h from A_1 to A_2 such that for each set C in $\mathscr{C}_2$, $\check{h}(C)$, the inverse image of C under h, belongs to $\mathscr{C}_1$. It is clear that the collection of classical logics and logical morphisms form a category. Let $\boldsymbol{L}$ denote this category.

Let $\boldsymbol{B}$ be the category of Boolean spaces and continuous maps. (A Boolean space is a compact Hausdorff space having a basis of clopen sets.) Our first theorem concerns the connection between the categories $\boldsymbol{L}$ and $\boldsymbol{B}$.

THEOREM 1. *There are contravarient functors* $F\colon \boldsymbol{L} \to \boldsymbol{B}$ *and* $G\colon \boldsymbol{B} \to \boldsymbol{L}$ *such that* (i) $\boldsymbol{B} \xrightarrow{G} \boldsymbol{L} \xrightarrow{F} \boldsymbol{B}$ *is identity* (*modulo homeomorphism*) *and* (ii) *there is a natural transformation of the identity functor on* $\boldsymbol{L}$ *to* $\boldsymbol{L} \xrightarrow{F} \boldsymbol{B} \xrightarrow{G} \boldsymbol{L}$.

Proof. The functor F assigns to each classical logic its dual space. The functor G assigns to each Boolean space the Boolean logic corresponding to the Boolean algebra of its clopen sets. The value F gives to a logical morphism will be explained below.

Let $\langle \mathscr{M}, \mathscr{C}\rangle$ be the dual space of the classical logic $\langle \mathscr{A}, \mathrm{Cn}\rangle$. First, a topology is put on $\mathscr{M}$ using the function s of definition D. Then it is shown that in this topology $\mathscr{C}$ is the collection of closed sets, so that s and $\mathscr{C}$ define the same topology.

Since $s(a) \cap s(b) = s(a, b)$ for any a, b in $\mathscr{A}$, the collection of sets $s(a_1, \ldots, a_n)$ form a basis for a topology on $\mathscr{M}$ [2]. Notice that for any a in $\mathscr{A}$,

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[2] We are writing $s(a)$ for $s(\{a\})$, $s(a, b)$ for $s(\{a, b\})$, etc.

(1) $$s(a) \cap s(-a) = \emptyset, \quad s(a) \cup s(-a) = \mathscr{M}.$$

Indeed, if $a \in M$, $M \in \mathscr{M}$ then $\mathrm{Cn}(-a, M) = \mathscr{A}$, since Cn is negative, and hence $-a \notin M$. Furthermore, if $a \notin M$, then $M \subseteq \mathrm{Cn}(-a, M) \neq \mathscr{A}$, so that $-a \in \mathscr{M}$. (1) shows that each set $s(a)$ is clopen.

We now show that for any a, b in $\mathscr{A}$,

(2) $$s(a) \cup s(b) = s(a \vee b).$$

Suppose that $a \in M$, $M \in \mathscr{M}$. Then $\mathrm{Cn}(a, M) \cap \mathrm{Cn}(b, M) = M$ since $M \subseteq \mathrm{Cn}(b, M)$. But then $a \vee b \in M$, since Cn is disjunctive. Similarly it follows that $s(b) \subseteq s(a \vee b)$. Hence $s(a) \cup s(b) \subseteq s(a \vee b)$. Conversely, if $a \vee b \in M$, then not both of the sets $\mathrm{Cn}(a, M)$, $\mathrm{Cn}(b, M)$ are proper supersets of M. Hence either a or b belongs to M, proving (2). Notice that it follows from (1) and (2) that the sets $s(a)$, $a \in \mathscr{A}$, are closed under complementation, finite union *and* intersection: $s(a) \cap s(b) = s\big(-(-a \vee -b)\big)$. The topology defined from the clopen basis $\{s(a)\colon a \in \mathscr{A}\}$ is Hausdorff: if $a \in M_1$, $a \notin M_2$, then $M_1 \in s(a)$, $M_2 \in s(-a)$ and the sets $s(a)$, $s(-a)$ are disjoint, by (1).

In order to show that $\mathscr{M}$ is compact, suppose $\mathscr{M} = \bigcup \big(s(X_i)\colon i \in I\big)$ where each X_i is a finite subset of $\mathscr{A}$. By (1) and (2), we may assume that X_i is a unit set, say $X_i = \{a_i\}$. If for each finite set $i_1, \ldots, i_n$ there is some element of $\mathscr{M}$ not in $s(a_{i_1}) \cup \ldots \cup s(a_{i_n})$, then $\mathrm{Cn}(a_{i_1}, \ldots, a_{i_n}) \neq \mathscr{A}$. Since Cn is finite and negative, $\mathrm{Cn}(\{-a_i\colon i \in I\}) \neq \mathscr{A}$. But then, by Proposition 3, some M_0 in $\mathscr{M}$ contains $-a_i$, all $i \in I$, and thus $M_0 \notin \bigcup \big(s(a_i)\colon i \in I\big)$, a contradiction. This proves: the dual space of a classical logic is a Boolean space.

If $O \subseteq \mathscr{M}$ is open, then O is a union of basis elements, say $O = \bigcup s(a_i)$. But then the complement of O is $\bigcap s(-a_i) = s(\bigcup \{-a_i\})$. This shows that every closed set belongs to the closure system $\mathscr{C}$. Conversely, if $C \in \mathscr{C}$, $C = s(X)$ for some $X \subseteq \mathscr{A}$. But then $C = \bigcap \big(s(a)\colon a \in X\big)$, and since each set $s(a)$ is closed, C is closed as well. Thus, $\mathscr{C}$ is the collection of closed sets.

Suppose now that $L_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ are classical logics with dual spaces $\mathscr{M}_i$ $(i = 1, 2)$. If $h\colon L_1 \to L_2$ is a logical morphism, then for each M in $\mathscr{M}_2$, $\check{h}(\mathscr{M})$ is in $\mathscr{M}_1$. Indeed, one can show that the maximal theories in any classical logic are precisely those theories T which satisfy: for any $a \in \mathscr{A}$, either $a \in T$ or $-a \in T$, not both. Now if $a \notin \check{h}(M)$, then $h(a) \notin M$, so $-h(a) = h(-a) \in M$: hence $-a \in \check{h}(M)$. This shows that $\check{h}$ maps elements of $\mathscr{M}_2$ into $\mathscr{M}_1$. In fact, $\check{h}$ is continuous (with respect to the Boolean space topologies on $\mathscr{M}_1$ and $\mathscr{M}_2$). Indeed, writing g for $\check{h}$, it is easily seen that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{h} & \mathscr{A}_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ P(\breve{M}_1) & \xrightarrow{\breve{g}} & P(\breve{M}_2) \end{array}$$

where $s_i(a) = \{M \in \mathscr{M}_i\colon a \in \mathscr{M}\}$ $(i = 1, 2)$. Thus $\breve{g}\big(s_1(a)\big) = s_2\big(h(a)\big)$, so that $g(=\breve{h})$ is continuous.

The functor F assigns to each logical morphism h from L_1 to L_2 the continuous map $F(h) = \breve{h}\colon M_2 \to M_1$. Since $\overbrace{(hg)} = \breve{g}\breve{h}$, the above argument shows that F is a contravariant functor from $\boldsymbol{L}$ to $\boldsymbol{B}$.

If X is a Boolean space, $G(X)$ is the classical logic corresponding to the dual algebra of X. The theory of Boolean algebras shows that any continuous map between Boolean spaces is induced by a homomorphism between the dual algebras. It only needs to be pointed out that all such homomorphisms are logical morphisms, since the inverse image of a filter is a filter. Thus G is a contravariant functor from $\boldsymbol{B}$ to $\boldsymbol{L}$. It is clear that $\boldsymbol{B} \overset{G}{\to} \boldsymbol{L} \overset{F}{\to} \boldsymbol{B}$ is the identity (modulo homomorphism).

Finally, it remains to prove that there is a natural transformation from the identity functor on $\boldsymbol{L}$ to the functor $\boldsymbol{L} \overset{F}{\to} \boldsymbol{B} \overset{G}{\to} \boldsymbol{L}$. This transformation assigns to the classical logic $L = \langle \mathscr{A}, \mathscr{C}\rangle$ the morphism $s\colon \mathscr{A} \to \mathscr{A}^*$, where $\mathscr{A}^*$ is the algebra of clopen subsets of the dual space of L and s is the function defined in definition D. (It follows from (1) and (2) that s *is* a morphism.) In Corollary 4 below it is shown that s is logical. It is not difficult to verify that if h is a logical morpism from $\langle \mathscr{A}_1, \mathscr{C}_1\rangle$ to $\langle \mathscr{A}_2, \mathscr{C}_2\rangle$, then the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{h} & \mathscr{A}_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ \mathscr{A}_1^* & \xrightarrow{G(F(h))} & \mathscr{A}_2^* \end{array}$$

This completes the proof of Theorem 1.

Remark. Using the notation of the preceding paragraph, we have established that $\mathscr{A}^*$ is logically isomorphic to $\mathscr{A}/\!\sim$, where $\sim$ is the partial congruence on $\mathscr{A}$ induced by the morphism s. Indeed, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{s} & \mathscr{A}^* \\ {\scriptstyle \mathrm{nat}}\downarrow & \nearrow{\scriptstyle \bar{s}} & \\ \mathscr{A}/\!\sim & & \end{array}$$

where $\text{nat}(a) = \{b: s(a) = s(b)\}$, and $\bar{s}$ is the induced isomorphism. This is a generalization of the well-known fact that a Boolean algebra $\boldsymbol{B}$ is isomorphic to the Stone field $P(\mathscr{B})$ (see, e. g. H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, Theorem II, 8.1).

The reader will have noticed that it is possible to define classical closure operations (and systems) on *sets* $\mathscr{A}$, as well as algebras. One says that: Cn is *negative* if for each $\mathscr{A}$ there is $a, b \in \mathscr{A}$ such that for all $X \subseteq \mathscr{A}$: $a \in \text{Cn}(X)$ iff $\text{Cn}(X, b) = \mathscr{A}$. A similar definition is given for Cn is *disjunctive*. For the moment, then, consider classical closure systems *on sets*. From Proposition 2 and the proof of Theorem 1, one obtains the following representation for the closure system of filters of a Boolean algebra.

COROLLARY 1. *$\mathscr{C}$ is a classical closure system iff $\mathscr{C}$ is isomorphic* (*as a lattice*[3])) *to the closure system of all filters of a Boolean algebra.*

4. The characterization theorems

In this section, classical logics are characterized in two ways. Theorem 2 and Corollary 2 are rather concrete descriptions. First, several definitions are required.

For any algebra $\mathscr{A}$ (of the kind being considered), we let $\text{Taut}(\mathscr{A})$ be the set of those elements $a \in \mathscr{A}$ such that $h(a) = 1$ for every morphism h from $\mathscr{A}$ into the two element Boolean algebra.

We define the function $a \to b$ as usual:

$$a \to b = -a \vee b.$$

A subset T of $\mathscr{A}$ is *closed* if b belongs to T whenever a and $a \to b$ belong to T.

THEOREM 2. *The logic $\langle \mathscr{A}, \text{Cn} \rangle$ is classical iff*

1) $\text{Taut}(\mathscr{A}) \subseteq \text{Cn}(\emptyset)$, *and*

2) Cn *is finite, and*

3) *for each $X \subseteq \mathscr{A}$, $a, b \in \mathscr{A}$, the "deduction property" holds:*

$$b \in \text{Cn}(X, a) \quad \textit{iff} \quad a \to b \in \text{Cn}(X).$$

Before presenting the proof of Theorem 2, we note that it implies a seemingly stronger result. For any abstract logic $\langle \mathscr{A}, \text{Cn} \rangle$ define the closure operation Pr on $\mathscr{A}$ by:

$a \in \text{Pr}(X)$ iff there is a finite sequence $b_1, \ldots, b_n$ such that $b_n = a$ and for each $i \leqslant n$, $b_i \in X$ or $b_i \in \text{Cn}(\emptyset)$ or b_i follows from earlier b_j, b_k by "modus ponens".

COROLLARY 2. *$\langle \mathscr{A}, \text{Cn} \rangle$ is classical iff* $\text{Taut}(\mathscr{A}) \subseteq \text{Cn}(\emptyset)$ *and* $\text{Cn} = \text{Pr}$.

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(3) If $\mathscr{C}$ is a closure system, then $\langle \mathscr{C}, \subseteq \rangle$ is a complete lattice.

Proof of Theorem 2. Suppose that $L = \langle \mathscr{A}, \mathrm{Cn} \rangle$ is classical and that $a \notin \mathrm{Cn}(\emptyset)$. Then, by Proposition 3, there is an element M in the dual space $\mathscr{M}$ of L such that $a \notin M$. Now M may be considered a homomorphism from the algebra $\mathscr{A}^*$ of clopen subsets of M into $\mathbf{2}$, the two element Boolean algebra. Thus the composition

$$\mathscr{A} \xrightarrow{s} \mathscr{A}^* \xrightarrow{M} \mathbf{2}$$

is a morphism of $\mathscr{A}$ into $\mathbf{2}$ taking a into 0, so that $a \notin \mathrm{Taut}(\mathscr{A})$. Since Cn is finite by definition, it must only be shown that the deduction property holds. Suppose that $b \in \mathrm{Cn}(X, a)$ and M is any maximal theory over X. If $a \in M$, then $b \in M$, so that $-a \vee b \in M$. If $a \notin M$, then $-a \in M$, so again $-a \vee b \in M$. From Proposition 3, it follows that $a \to b$ is in $\mathrm{Cn}(X)$. The other direction is easier.

Now suppose that Cn is a closure operation satisfying Conditions 1), 2) and 3). It must be shown that Cn is negative and disjunctive. As for negativity, suppose that $a \in \mathrm{Cn}(X)$ and b is arbitrary. From 3) it follows that $\mathrm{Cn}(X)$ is closed. Also $a \to (-a \to b) \in \mathrm{Taut}(\mathscr{A}) \subseteq \mathrm{Cn}(X)$. Thus $-a \to b \in \mathrm{Cn}(X)$, so $b \in \mathrm{Cn}(X, -a)$. Since b was arbitrary, we have shown that $\mathrm{Cn}(X, -a) = \mathscr{A}$. To complete the demonstration of negativity, suppose that $\mathrm{Cn}(X, -a) = \mathscr{A}$. In particular, $a \in \mathrm{Cn}(X, -a)$, so $-a \to a \in \mathrm{Cn}(X)$. But $(-a \to a) \to a \in \mathrm{Taut}(\mathscr{A})$, so that $a \in \mathrm{Cn}(X)$.

The proof that Cn is disjunctive is similar and is omitted.

COROLLARY 3. *Suppose that* $\langle \mathscr{A}, \mathrm{Cn} \rangle$ *is classical. Then* $T = \mathrm{Cn}(T)$ *iff* $\mathrm{Cn}(\emptyset) \subseteq T$ *and* T *is closed.*

Proof. The necessity of these conditions is clear. Suppose then that T is a subset of $\mathscr{A}$ satisfying both conditions. If $a \in \mathrm{Cn}(T)$, then $a \in \mathrm{Cn}(t_1, \ldots, t_n)$, for some t_i in T. By Theorem 2,

$$\big(t_1 \to \big(t_2 \to \ldots \to (t_n \to a)\big) \ldots\big) \in \mathrm{Cn}(\emptyset) \subseteq T.$$

Using induction and the fact that T is closed, we see that $a \in T$, so $\mathrm{Cn}(T) = T$.

COROLLARY 4. *Let* $L = \langle \mathscr{A}, \mathrm{Cn} \rangle$ *be classical. The function* s *(of definition* D) *is a logical morphism from* L *to the Boolean logic corresponding to the Boolean algebra* $\mathscr{A}^*$ *of clopen subsets of the dual space of* L.

Proof. It was shown in the proof of Theorem 1 that s is a morphism. We need only show that if F is a filter in $\mathscr{A}^*$, $\breve{s}(F) = \mathrm{Cn}(\breve{s}(F))$. Using the preceding corollary, it is sufficient to show $\mathrm{Cn}(\emptyset) \subseteq \breve{s}(F)$ and that $\breve{s}(F)$ is closed. If $a \in \mathrm{Cn}(\emptyset)$, $s(a)$ is the maximum element in $\mathscr{A}^*$ and hence belongs to F. If a and $a \to b \in \breve{s}(F)$, then $s(a)$ and $-s(a) \cup s(b)$ are in F. Thus $s(b) \in F$, since F is a filter. This completes the proof.

Our last theorem concerns the concept of "projective generation" of closure systems. This notion (as well as its dual, "inductive generation") was studied by Brown in [1].

DEFINITION. Let $\langle \mathscr{A}, \mathscr{C} \rangle$ and $\langle \mathscr{B}_i, \mathscr{C}_i \rangle$ $(i \in I)$ be abstract logics. Let H_i be a subset of the morphisms from $\mathscr{A}$ to $\mathscr{B}_i$. $\mathscr{C}$ is *projectively generated* by H_i and $\mathscr{C}_i$ $(i \in I)$ if $\mathscr{C}$ is the coarsest closure system on $\mathscr{A}$ such that every member of H_i is a logical morphism.

LEMMA (Brown [1]). *Suppose that* Cn *is projectively generated by* $\mathrm{Cn}_{\mathscr{B}}$ *and a single morphism* $h\colon \mathscr{A} \to \mathscr{B}$. *Then for any* $X \subseteq \mathscr{A}$, $\mathrm{Cn}(X) = \breve{h}\big(\mathrm{Cn}_{\mathscr{B}}(h(X))\big)$.

THEOREM 3. *$\langle \mathscr{A}, \mathscr{C} \rangle$ is classical iff there is a Boolean logic $\langle \mathscr{B}, \mathscr{C}_{\mathscr{B}} \rangle$ and a surjective morphism $h\colon \mathscr{A} \to \mathscr{B}$ such that $\mathscr{C}$ is projectively generated by h and $\mathscr{C}_{\mathscr{B}}$.*

Proof. (Sufficiency.) Suppose $h\colon \mathscr{A} \to \mathscr{B}$, is a surjective morphism from $\mathscr{A}$ onto the Boolean algebra $\mathscr{B}$. We use Theorem 2 to show that $\mathscr{C}$ is classical. First, if $a \in \mathrm{Taut}(\mathscr{A})$, clearly $h(a) \in \mathrm{Taut}(\mathscr{B}) \subseteq \mathrm{Cn}_{\mathscr{B}}(\emptyset)$. Hence $a \in \breve{h}\,\mathrm{Cn}_{\mathscr{B}}(\emptyset) = \mathrm{Cn}(\emptyset)$. In order to prove that Cn is finite, we show the union of a directed family of sets in $\mathscr{C}$ is also in $\mathscr{C}$ (see [2], p. 45). So suppose that $T_i = \mathrm{Cn}(T_i)$ $(i \in I)$ is a directed family. Then, by the lemma, $\mathrm{Cn}(\bigcup T_i) = \breve{h}\,\mathrm{Cn}_{\mathscr{B}}\big(h(\bigcup T_i)\big)$. Since

$$T_i = \breve{h}\,\mathrm{Cn}_{\mathscr{B}}\big(h(T_i)\big), \qquad h(\bigcup T_i) \subseteq \bigcup \mathrm{Cn}_{\mathscr{B}}\big(h(T_i)\big).$$

Now since $\mathrm{Cn}_{\mathscr{B}}$ is finite,

$$\mathrm{Cn}_{\mathscr{B}}\Big(\bigcup \mathrm{Cn}_{\mathscr{B}}\big(h(T_i)\big)\Big) = \bigcup \mathrm{Cn}_{\mathscr{B}}\big(h(T_i)\big).$$

Hence,

$$\mathrm{Cn}(\bigcup T_i) \subseteq \breve{h} \bigcup \mathrm{Cn}_{\mathscr{B}}\big(h(T_i)\big) = \bigcup \breve{h}\,\mathrm{Cn}_{\mathscr{B}}\big(h(T_i)\big) = \bigcup T_i.$$

The proof that Cn satisfies the deduction property is straightforward and is omitted.

(Necessity.) If $\langle \mathscr{A}, \mathscr{C} \rangle$ is classical, let $h = s$, the surjection of $\mathscr{A}$ onto the dual algebra $\mathscr{B}$ of the dual space of $\langle \mathscr{A}, \mathscr{C} \rangle$. We have already shown that s is a logical morphism. It must be shown that

$$a \in \mathrm{Cn}(X) \quad \text{iff} \quad s(a) \in \mathrm{Cn}_{\mathscr{B}}\big(\{s(x)\colon x \in X\}\big).$$

First suppose that $a \in \mathrm{Cn}(X)$ and that F is a filter in $\mathscr{B}$ containing $\{s(x)\colon x \in X\}$. Then $\breve{s}(F)$ is a Cn-theory containing X, and hence a. Thus $s(a) \in F$, so $s(a) \in \mathrm{Cn}_{\mathscr{B}}\big(\{s(x)\colon x \in X\}\big)$.

Now suppose that $a \notin \mathrm{Cn}(X)$. Let M be a maximal Cn-theory containing X and $-a$. The set $\{s(m)\colon m \in M\}$ $\big(= s^*(M)\big)$ has the finite intersection property and can thus be extended to a maximal filter F in $\mathscr{B}$. Since clearly $M \subseteq \breve{s}(F) \neq \mathscr{A}$, M must be equal to $\breve{s}(F)$. But then $\{s(x)\colon x \in X\} \subseteq s^*(M) \subseteq F$ and $s(a) \notin F$. Hence $s(a) \notin \mathrm{Cn}_{\mathscr{B}}\big(\{s(x)\colon x \in X\}\big)$.

Remark. In general, classical logics are not closed under projective generation. Indeed, if $\mathscr{B}_n$ $(n = 1, 2, \ldots)$ is the two element Boolean algebra **2**, let $\mathscr{A} = \mathscr{B}_1 \times \mathscr{B}_2 \times \ldots$ Let p_n be the projection of $\mathscr{A}$ onto its nth coordinate. The closure system $\mathscr{C}$ which is projectively generated by p_n and $\mathscr{C}_n$ ($\mathscr{C}_n$ = all filters in **2**) consists of all filters F in $\mathscr{A}$ of the form $F_1 \times F_2 \times \ldots$ where F_n is a filter in **2**. This closure system is *not* finite: if $a_k \in \mathscr{A}$ is the function defined by $a_k(n) = 0$ if $n \leqslant k$; 1 otherwise, and a is the zero function, then $a \in \mathrm{Cn}(a_1, a_2, \ldots)$ but $a \notin \mathrm{Cn}(a_1, \ldots, a_n)$.

References

[1] D. J. Brown, *Abstract Logics*, Ph. D. Thesis, Stevens Institute of Technology 1969.

[2] P. M. Cohn, *Universal Algebra*, Harper and Row 1965.