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D. J. BROWN and R. SUSZKO

Abstract logics

S. L. BLOOM and D. J. BROWN

Classical abstract logics

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Preface

Both papers in this volume are concerned with abstract logics. The first one is an introduction to the general theory of abstract logics and the second deals with a particular kind of logics, called classical logics.

An abstract logic is a pair $\langle \mathscr{A}, \mathscr{C} \rangle$ where $\mathscr{A}$ is an algebra and $\mathscr{C}$ is a closure system on the carrier of $\mathscr{A}$. Obviously, $\mathscr{C}$ may be replaced by the closure operator Cn corresponding to $\mathscr{C}$. Abstract logics are obtained as generalization of various logical notions. If $\mathscr{C} = \{A, B\}$ where $B \subseteq A$ = the carrier of $\mathscr{A}$ then $\mathscr{C}$ is called *elementary closure system* and the abstract logic $\langle \mathscr{A}, \mathscr{C} \rangle$ forms a logical matrix, a concept introduced by Łukasiewicz and Tarski in 1930. At about the same time, Tarski introduced the consequence operation (or entailment relation) as the fundamental logical notion. It is a special case of a closure operator. Also, as observed first by Lindenbaum, formalized languages are algebraic systems, i.e., sets supplied with (free) operations determined by formation rules. Hence, a formalized language conceived as an algebra of sentential formulas together with a consequence operation generated by logical axioms and rules of inference or, perhaps defined otherwise, is an abstract logic.

Since the 1930's several abstract generalizations, either algebraic or closure theoretic, arose from the logical field. I only mention cylindric algebras of Tarski and polyadic algebras of Halmos and, on the other hand, *Théorie metamathematique des ideaux* of A. Robinson (1955).

Note that Boole's mathematical thought already reached the level of abstract logics. Any Boolean algebra with the family of all its filters is a paradigm of abstract logics. Similarly, a Boolean algebra together with a single ultrafilter may be taken as paradigm of a logical matrix (elementary abstract logic).

The class of abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$, with algebras $\mathscr{A}$ of a fixed similarity type, constitute a category when supplied with suitably defined morphisms. Here, the analogy with general topology is the guiding idea so that the morphisms of abstract logics are defined as "continuous" homomorphisms. It is natural to continue the said analogy and, to introduce projective and inductive generation of abstract logics as well as the notion of a dual space. Thus, we construct a general framework of the theory of abstract logics.

Certainly, like the category of all topological spaces, categories of abstract logics are too large to be interesting or important when considered as a whole. One may prefer to study certain classes of abstract logics possessing certain properties, distinguished from this or that point of view. For example, topological separation properties, which originated in geometry, are of no use from the purely logical point of view adopted here. We are concerned with other properties: some of these are set-theoretical properties of closure systems (finiteness, logical compactness, regularity), the others involve the closure system and the underlying algebra, simultaneously (structurality, invariance, negativity, disjunctivity).

Properties of abstract logics which are distinguished from the logical point of view are those of "good" logics. These are either

(I) logics we actually use (!) or, at least, are able to use or

(II) logics which appear as semantical interpretations of the former ones.

The primary logics (I) seem to be necessarily free in the general algebraic sense. On the other hand, the most natural secondary logics (II) are logical quotients of primary logics modulo logical congruences. They are, in several particular cases, known as Lindenbaum-Tarski algebras. Therefore the theory of abstract logics cannot overlook free logics and must consider logical congruences in a general setting.

The theory of abstract logics constitutes a framework for semantical investigations. Here, we only consider the problem of the so called completeness theorem, which is the generally accepted minimal requirement on what semantics provide. Given a logic $\langle \mathscr{A}, \mathrm{Cn} \rangle$, in the semantics we look for a class $\mathscr{K}$ of interpretations (logical matrices) with the following property (completeness of the logic with respect to $\mathscr{K}$): $p \notin \mathrm{Cn}(X)$ iff there is an interpretation in $\mathscr{K}$ which makes p invalid (false, unsatisfied) while making all elements of X valid (true, satisfied). Our idea, partly due to R. Wójcicki is to make the completeness problem trivial. Indeed, if one views the logic $\langle \mathscr{A}, \mathscr{C} \rangle$ as a bundle of logical matrices, one may generalize the old Lindenbaum construction to show that abstract logics, if structural or invariant *are* complete with respect to themselves and, hence, with respect to their logical quotients.

However undesirable it might seem, the trivial completeness theorem underlies the investigations of all the diverse (sentential) logics by H. Rasiowa and R. Sikorski. On the other hand, their exemplary studies may suggest that the framework of abstract logics is the result of an over-eagerness to generalize beyond reasonable understanding. Indeed, all the logical quotients obtained by Rasiowa and Sikorski are "nice" algebras, e.g. lattices with unit, together with a closure system of filters (or, in particular, I-filters). One may argue that this closure system is inherent in the corresponding algebra and what seemingly matters here is the *unit* which again is inseparable from the algebra (lattice with unit). If the free logics considered by Rasiowa and Sikorski are typical for primary logics then as secondary logics one may only expect algebras involving an "internal" closure system. Therefore, it is a useless luxury to climb to the level of abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$ where the closure system $\mathscr{C}$ cannot be easily manufactured from the algebra $\mathscr{A}$.

There is indeed some point in this reasoning. However, what matters here is the question whether there exists a primary logic such that in logical quotients the algebra and the closure system are independent, so to speak.

The answer is an emphatic YES and constitutes the ultimate justification for the theory of abstract logics. This answer comes from the non-Fregean logic (NFL), and in particular, its basic level called the sentential calculus with identity (SCI). The SCI does not fit into the lattice-theoretic framework and requires a general theory of logical matrices and abstract logics. You will find some details at the end of the first paper.

The theory of abstract logics is kept here on the level which corresponds to sentential calculi. However, it easily generalizes to the level which corresponds to open (quantifier-free) logics with both sentential and nominal variables. Again, the open non-Fregean logic calls necessarily for a general theory of algebraic structures together with closure systems, possibly elementary. It pleases me very much to acknowledge that J. Łoś was the first (1949) to introduce such structures.

The roots of the second paper may again be found in non-Fregean logic. NFL is extremely weak (as a closure operator) and extremely rich (as closure system). Hence, the non-Fregean logic (or SCI, in particular) forces one to consider an uncountable lattice of logics, that is, the complete lattice of all its extensions (stronger closure operator and smaller closure system, the language being fixed). We face a genuine embarrassment of riches. The first attempt to deal with it consists in dividing all the extensions of NFL (or SCI) into (1) elementary ones, obtained by adding new axioms and (2) the non-elementary ones. All the elementary extensions have very good properties and have been labelled classical logics. The Fregean logic, known from textbooks of mathematical logic, *is* an elementary extension of NFL and in a sense a maximal one. Furthermore, logical quotients of sufficiently strong elementary extensions of SCI appear as nice algebras, e.g., Boolean algebras. On the other hand, all non-elementary extensions possess rather bad properties and are, at least, very strange. The question arises whether we can formulate in general terms the deep difference between elementary and non-elementary extensions of NFL (or SCI, at least). An attempt to do that is the content of the paper on classical logics. A very simple definition of classical abstract logics is assumed and a pleasant result is obtained: an abstract logic is classical iff it is equivalent in a sense to a Boolean logic, i.e., a Boolean algebra together with all (!) filters.

Roman Suszko

ABSTRACT LOGICS

D. J. BROWN and R. SUSZKO

Introduction(*)

The most important instance of an abstract logic is the ordered pair $\langle \mathscr{B}, \mathscr{F} \rangle$, where $\mathscr{B}$ is a Boolean algebra and $\mathscr{F}$ is the family of all filters of $\mathscr{B}$. The objects of the category of abstract logics will be ordered pairs $\langle \mathscr{A}, \mathscr{C} \rangle$, where $\mathscr{A}$ is an abstract algebra and $\mathscr{C}$ is a closure system on $|\mathscr{A}|$, the carrier of $\mathscr{A}$. To complete our definition of the category of abstract logics, we must define the morphisms, or structure preserving maps. We note that the family of homomorphisms between two Boolean algebras not only preserve the algebraic structure, but also the closure structure in the sense that homomorphisms pull filters back into filters. Logical morphisms are then defined as homomorphisms which pull closed sets back into closed sets.

Another category, closely associated with the category of abstract logics is the category of closure spaces. The prime example of a closure space is a topological space. The objects in this category are $\langle S, C \rangle$ where S is a nonempty set and C is a closure system on S. The morphisms are defined in the same manner as continuous maps are in the category of topological spaces, i.e. morphisms pull closed sets back into closed sets. Obviously the forgetful functor sends abstract logics into closure spaces, but a more important functor is that which sends abstract logics into their natural dual space. This construction is motivated by the relation between a Boolean algebra and its associated Stone space or dual space. As in the case of the classical propositional calculus where the formalized deductive system is isomorphic, modulo logical equivalence, to a Boolean algebra and the semantics of the classical propositional calculus, or model space is isomorphic to the Stone space of the Boolean algebra, one should interpret an abstract logic as a formalized deductive system and its natural dual space as its semantics or model space. An abstract logic may have many duals, hence a result of some interest is that an abstract logic is logically compact if and only if every one of its dual spaces is topologically compact.

(*) Donald J. Brown — Cowles Foundation for Research in Economics at Yale University. Roman Suszko — Mathematics Department, Stevens Institute of Technology. This research (on the side of the first author) was supported in part by a Bell Telephone Laboratories Doctoral Fellowship and the Office of Naval Research.

The work of Lindenbaum, Henkin, Rasiowa, and Sikorski show in many cases of interest that an adequate semantics in the sense of completeness theorems can be obtained from the language. For the propositional calculi this is usually done by treating the language as an abstract algebra defining an appropriate congruence and factoring the algebra. The models are then defined as these quotient algebras with a distinguished subset. Such structures have been termed logical matrices in the literature. Usually the quotient algebra is a lattice and the distinguished subset consists of only one element, the unit element in the lattice. An excellent summary of this algebraic approach to semantics is in "The Mathematics of Metamathematics" by Rasiowa and Sikorski. We show that this construction is a special case of projective generation in the category of abstract logics.

Although the applications of the abstract theory is restricted to propositional calculi in this paper, similar results have been obtained for open first order logics. For these results and a brief history of the ideas which motivated this research, we suggest that the reader consult Brown's 1969 dissertation, Abstract Logics.

This work has been taken mainly from the above mentioned dissertation, where the principal advisor was Roman Suszko, but we would like to express our gratitude to the participants of the seminar in mathematical logic held at Stevens in 1968/1969. In particular Dr. Stephen Bloom and Dr. Robert Quackenbush both of whom have been kind enough to allow us to include several of their results in this paper.

I. Elementary properties of closure systems and closure operations

DEFINITION 1. A family $\mathscr{C}$ of subsets of a non-empty set S is said to be a *closure system* on S if $\mathscr{C}$ is closed under arbitrary intersections, i.e. $\bigcap\mathscr{F}$ is in $\mathscr{C}$ if $\mathscr{F} \subseteq \mathscr{C}$. Clearly $S \in \mathscr{C}$. Define $\overline{\mathscr{C}}$ to be $\mathscr{C} - \{S\}$.

DEFINITION 2. A mapping Cn of $P(S)$, the power set of S, into itself is called a *closure operator* or *consequence operation* on S if for all $X, Y \subseteq S$

$$X \subseteq \mathrm{Cn}(X) = \mathrm{Cn}(\mathrm{Cn}(X)), \tag{1}$$

$$\mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y) \quad \text{if} \quad X \subseteq Y. \tag{2}$$

THEOREM 1. *Every closure system $\mathscr{C}$ on S constitutes a complete lattice under set inclusion where* $\operatorname{Inf}_i X_i = \bigcap_i X_i$ *and* $\operatorname{Sup}_i X_i = \operatorname{Inf}\{Z \in \mathscr{C} \mid \bigcup_i X_i \subseteq Z\}$.

THEOREM 2. *Every closure operator* Cn *on* S *defines the closure system* $\mathscr{C}(\mathrm{Cn}) = \{X \subseteq S \mid \mathrm{Cn}(X) = X\}$ = *the family of all* Cn-*closed subsets of* S. *Clearly,* $\bigcap \mathscr{C}(\mathrm{Cn}) = \mathrm{Cn}(\emptyset)$ *where* $\emptyset$ *is the empty set and* $S \in \mathscr{C}(\mathrm{Cn})$.

THEOREM 3. *The mapping* $\mathrm{Cn} \to \mathscr{C}(\mathrm{Cn})$ *is a bijective correspondence between closure operators on* S *and closure systems on* S.

Proof. Obviously, $\mathscr{C}(\mathrm{Cn}_1) \neq \mathscr{C}(\mathrm{Cn}_2)$ if $\mathrm{Cn}_1 \neq \mathrm{Cn}_2$. Given a closure system on S, set $\mathrm{Cn}(X) = \bigcap \{Z \in \mathscr{C} \mid X \subseteq Z\}$. Then, Cn is a closure operator on S such that $\mathscr{C}(\mathrm{Cn}) = \mathscr{C}$.

THEOREM 4. *The family of all closure operators on* S *is a complete lattice under the ordering*

$$\text{(3)} \qquad \mathrm{Cn}_1 \leqslant \mathrm{Cn}_2 \quad \text{iff} \quad \mathrm{Cn}_1(X) \subseteq \mathrm{Cn}_2(X), \text{ all } X \subseteq S.$$

Here, $\mathrm{Cn} = \operatorname*{Inf}_i \mathrm{Cn}_i$ *iff* $\mathrm{Cn}(X) = \bigcap_i \mathrm{Cn}_i(X)$, *all* $X \subseteq S$, *and* $\operatorname*{Sup}_i \mathrm{Cn}_i = \mathrm{Inf}\{\mathrm{Cn} \mid \mathrm{Cn}_i \leqslant \mathrm{Cn}, \text{ all } i\}$.

THEOREM 5. *The family of all closure systems on* S *is a complete lattice under set inclusion where*

$$\operatorname*{Inf}_i \mathscr{C}_i = \bigcap_i \mathscr{C}_i = \{X \subset S \mid X \in \mathscr{C}_i, \text{ all } i\},$$

$$\operatorname*{Sup}_i \mathscr{C}_i = \mathrm{Inf}\{\mathscr{C} \mid \mathscr{C}_i \subset \mathscr{C}, \text{ all } i\}.$$

THEOREM 6. *The complete lattice of all closure operations on* S *and the complete lattice of all closure systems on* S *are dually isomorphic under the correspondence* $\mathrm{Cn} \to \mathscr{C}(\mathrm{Cn})$.

Th above theorems are due to Moore [1], Birkhoff [1], [2], Tarski [1] and Ore [1]. Examples, of closure systems are the following:

(i) The family of all filters in a Boolean algebra.

(ii) The family of all closed sets in a topological space. Here, Cn and $\mathscr{C}$ have the additional properties: $\mathrm{Cn}(\emptyset) = \emptyset \in \mathscr{C}$ and $\mathrm{Cn}(X \cup Y) = \mathrm{Cn}(X) \cup \mathrm{Cn}(Y)$, that is, $X \cup Y$ is in $\mathscr{C}$ if both X and Y are in $\mathscr{C}$.

DEFINITION 3. If $\mathscr{B}$ is a non-empty family of subsets of S then $[\mathscr{B}]$ = the collection of all arbitrary intersections of members of $\mathscr{B}$, and $\mathscr{B}$ is said to be a *basis* of $[\mathscr{B}]$. Obviously, $\mathscr{C}$ is a basis of $\mathscr{C}$.

THEOREM 7. $[\mathscr{B}]$ *is the smallest closure system on* S *containing* $\mathscr{B}$.

Proof. $[\mathscr{B}]$ is a closure system such that $\mathscr{B} \subseteq [\mathscr{B}]$. If $\mathscr{C}$ is any closure system on S, containing $\mathscr{B}$ then clearly $[\mathscr{B}] \subseteq \mathscr{C}$.

DEFINITION 4. Let $\mathrm{Cn}_1, \mathrm{Cn}_2$ be closure operators on S. Then, the operation Cn, defined as $\mathrm{Cn}(X) = \mathrm{Cn}_1(\mathrm{Cn}_2(X))$ for $X \subseteq S$, is called the *composition* of $\mathrm{Cn}_1, \mathrm{Cn}_2$ and we write $\mathrm{Cn} = \mathrm{Cn}_1 \circ \mathrm{Cn}_2$. In general, $\mathrm{Cn}_1 \circ \mathrm{Cn}_2$ will not be a closure operator on S.

THEOREM 8. *Ore* [1]. *The following conditions are equivalent:*

(4) $\mathrm{Cn}_1\circ\mathrm{Cn}_2$ *is a closure operator on* S,

(5) $\mathrm{Cn}_1\circ\mathrm{Cn}_2 = \mathrm{Sup}(\mathrm{Cn}_1, \mathrm{Cn}_2)$, *i.e.*,

$$\mathscr{C}(\mathrm{Cn}_1)\cap\mathscr{C}(\mathrm{Cn}_2) = \{X \subseteq S \mid X = \mathrm{Cn}_1(\mathrm{Cn}_2(X))\},$$

(6) $\mathrm{Cn}_1\circ\mathrm{Cn}_2$ *is idempotent, i.e.*,

$$\mathrm{Cn}_1\Big(\mathrm{Cn}_2\big(\mathrm{Cn}_1(\mathrm{Cn}_2(X))\big)\Big) = \mathrm{Cn}_1(\mathrm{Cn}_2(X)), \quad \textit{all } X \subseteq S.$$

THEOREM 9. *The composition* $\mathrm{Cn} = \mathrm{Cn}_1\circ\mathrm{Cn}_2$ *is idempotent if and only if* $\mathrm{Cn}_2\circ\mathrm{Cn}_1 \leqslant \mathrm{Cn}$, *i.e.*,

$$\mathrm{Cn}_2(\mathrm{Cn}_1(X)) \subseteq \mathrm{Cn}_1(\mathrm{Cn}_2(X)), \quad \textit{all } X \subseteq S.$$

Proof. Suppose $\mathrm{Cn}_2\circ\mathrm{Cn}_1 \leqslant \mathrm{Cn}$. It follows that $\mathrm{Cn}_2\big(\mathrm{Cn}_1(\mathrm{Cn}_2(X))\big) \subseteq \mathrm{Cn}_1\big(\mathrm{Cn}_2(\mathrm{Cn}_2(X))\big)$. Hence, *i. e.*

$$\mathrm{Cn}(\mathrm{Cn}(X)) = \mathrm{Cn}_1\Big(\mathrm{Cn}_2\big(\mathrm{Cn}_1(\mathrm{Cn}_2(X))\big)\Big) \subseteq \mathrm{Cn}_1\Big(\mathrm{Cn}_1\big(\mathrm{Cn}_2(\mathrm{Cn}_2(X))\big)\Big) = \mathrm{Cn}(X).$$

Therefore, $\mathrm{Cn}\circ\mathrm{Cn} \leqslant \mathrm{Cn}$. On the other hand, $\mathrm{Cn}(X) \subseteq \mathrm{Cn}_2(\mathrm{Cn}(X))$. Hence, $\mathrm{Cn}(X) \subseteq \mathrm{Cn}_1\big(\mathrm{Cn}_2(\mathrm{Cn}(X))\big) = \mathrm{Cn}(\mathrm{Cn}(X))$, i.e., $\mathrm{Cn} \leqslant \mathrm{Cn}\circ\mathrm{Cn}$. Thus, Cn is idempotent. Suppose conversely, that $\mathrm{Cn}\circ\mathrm{Cn} = \mathrm{Cn}$. Then, by Theorem 8, Cn is a closure operator. Hence,

$$X \subseteq \mathrm{Cn}_1(\mathrm{Cn}_2(X)),$$

$$\mathrm{Cn}_1(X) \subseteq \mathrm{Cn}_1(\mathrm{Cn}_2(X)), \quad \mathrm{Cn}_2(\mathrm{Cn}_1(X)) \subseteq \mathrm{Cn}_2\big(\mathrm{Cn}_1(\mathrm{Cn}_2(X))\big)$$

and, finally,

$$\mathrm{Cn}_1\big(\mathrm{Cn}_2(\mathrm{Cn}_1(X))\big) \subseteq \mathrm{Cn}_1\Big(\mathrm{Cn}_2\big(\mathrm{Cn}_1(\mathrm{Cn}_2(X))\big)\Big) = \mathrm{Cn}(\mathrm{Cn}(X)) = \mathrm{Cn}(X).$$

On the other hand,

$$\mathrm{Cn}_2(\mathrm{Cn}_1(X)) \subseteq \mathrm{Cn}_1\big(\mathrm{Cn}_2(\mathrm{Cn}_1(X))\big).$$

Therefore, $\mathrm{Cn}_2(\mathrm{Cn}_1(X)) \subseteq \mathrm{Cn}(X)$. Thus, $\mathrm{Cn}_2\circ\mathrm{Cn}_1 \leqslant \mathrm{Cn}$.

II. Some properties of closure operators and closure systems

When speaking of closure operators and systems we take the logical point of view and use the corresponding terminology. Given a closure operator Cn on S, a set $X \subseteq S$ is said to be Cn-*consistent* if $\mathrm{Cn}(X) \neq S$ and Cn-*inconsistent* if $\mathrm{Cn}(X) = S$. The collection of all Cn-consistent closed sets, $\mathscr{C}(\mathrm{Cn})-\{S\}$, will be denoted $\overline{\mathscr{C}}(\mathrm{Cn})$. Maximal Cn-consistent subsets of S are called Cn-*complete.* Since every Cn-complete set is Cn-closed, it follows that Cn-complete sets are maximal in $\overline{\mathscr{C}}(\mathrm{Cn})$ and, conversely.

DEFINITION 1. Cn is said to be *finite* (or algebraic) if for each $X \subseteq S$, $\mathrm{Cn}(X) = \bigcup\{\mathrm{Cn}(Y) \mid \text{finite } Y \subseteq X\}$. Cn is said to be *logically compact*, *l*-compact if every Cn-inconsistent set includes some finite Cn-inconsistent subset.

In order to describe directly finite and compact closure operators we need the following:

DEFINITION 2. A non-empty family $\mathscr{F}$ of subsets of S is called *inductive* if for every chain $\mathscr{G}$ in $\mathscr{F}$ (or every upward directed subfamily $\mathscr{G}$ of $\mathscr{F}$) where the ordering is set inclusion, $\mathrm{Sup}\,\mathscr{G} = \bigcup \mathscr{G}$ belongs to $\mathscr{F}$.

THEOREM 1. (Schmidt [1]). *A closure operator* Cn *is finite iff the closure system* $\mathscr{C}(\mathrm{Cn})$ *is inductive.*

THEOREM 2. *A closure operator* Cn *is l-compact iff the family of all* Cn-*consistent sets is inductive.*

Proof. Suppose Cn l-compact and let $\mathscr{G} = \{X_i\}$ be a chain of consistent sets. If $X = \bigcup \mathscr{G}$ is inconsistent then some finite subset Y of X is inconsistent. Consequently, there exists $X_j \in \mathscr{G}$ such that $Y \subseteq X_j$. Hence, X_j is inconsistent, a contradiction. If, on the other hand, Cn is not l-compact then there exists an inconsistent set X such that every finite subset of X is consistent. But the family $\mathscr{F}$ of all those subsets is directed and $\bigcup \mathscr{F} = X$ is inconsistent. Hence, the family of all consistent sets is not inductive.

THEOREM 3. *If* $\mathscr{F}$ *is an inductive family then to every* $X \in \mathscr{F}$ *there exists in* $\mathscr{F}$ *a maximal set which includes* X.

COROLLARY 3.1. (Lindenbaum). *If* Cn *is l-compact then every* Cn-*consistent set is contained in a* Cn-*complete set.*

THEOREM 4. *If the family* $\overline{\mathscr{C}}(\mathrm{Cn})$ *is inductive then the closure system* $\mathscr{C}(\mathrm{Cn})$ *and the collection of all* Cn-*consistent sets are inductive.*

Proof. Inductivity of $\overline{\mathscr{C}}(\mathrm{Cn})$ obviously implies that of $\mathscr{C}(\mathrm{Cn})$. Suppose $\overline{\mathscr{C}}(\mathrm{Cn})$ is inductive and let $\{X_i\}$ be a chain of Cn-consistent sets. Then, $\{\mathrm{Cn}(X_i)\}$ is a chain in $\overline{\mathscr{C}}(\mathrm{Cn})$. By hypothesis, the set $\bigcup_i \mathrm{Cn}(X_i)$ belongs to $\overline{\mathscr{C}}(\mathrm{Cn})$. Hence, $\bigcup_i X_i$ is Cn-consistent.

THEOREM 5. (Birkhoff [1]). *Let* $\{\mathscr{F}_i\}$ *be a collection of inductive families on* S. *Then,* $\mathscr{F} = \bigcap_i \mathscr{F}_i$ *is an inductive family on* S.

Proof. Let $\mathscr{D}$ be a directed subfamily of $\mathscr{F}$. Then $\bigcup \mathscr{D}$ is in $\mathscr{F}_i$ for all i, and hence, $\bigcup \mathscr{D}$ belongs to $\mathscr{F}$.

THEOREM 6. *A closure operator* Cn *is finite and l-compact if and only if the family* $\overline{\mathscr{C}}(\mathrm{Cn})$ *is inductive.*

Proof. By Theorems 4 and 2.

COROLLARY 6.1. *Let* $\{Cn_i\}$ *be a family of finite* (*and* 1-*compact*) *closure operators on* S. *Then,* $\operatorname{Sup}_i \mathrm{Cn}_i$ *is a finite* (*and* 1-*compact*) *closure operator on* S.

Proof. By Theorem I.6 and Theorem 5.

DEFINITION 3. A closure operator Cn on S is said to have the NEG-*property* (or to be negative) if for every $a \in S$ there exists $b \in S$ such that the equivalence

$$(*) \qquad a \in \mathrm{Cn}(X) \quad \text{iff} \quad \mathrm{Cn}(X \cup \{b\}) = S$$

holds for all $X \subseteq S$.

THEOREM 7. *If the closure operator* Cn *on* S *has the* NEG-*property then for all* $a, b \in S$,

$$(**) \qquad \mathrm{Cn}(a) \cap \mathrm{Cn}(b) = \mathrm{Cn}(\emptyset) \quad \textit{and} \quad \mathrm{Cn}(a, b) = S$$

whenever $(*)$ *holds for all* $X \subseteq S$.

Proof. Suppose, Cn has the NEG-property and $(*)$ holds for all $X \subseteq S$. If $x \in \mathrm{Cn}(a) \cap \mathrm{Cn}(b)$ then there exists $y \in S$ such that for all $Y \subseteq S$, $x \in \mathrm{Cn}(Y)$ iff $\mathrm{Cn}(Y \cup \{y\}) = S$. Clearly, $\mathrm{Cn}(b, y) = S$ since $x \in \mathrm{Cn}(b)$. Therefore, $a \in \mathrm{Cn}(y)$, by $(*)$. But $x \in \mathrm{Cn}(a)$. Hence, $x \in \mathrm{Cn}(y)$. Consequently, $\mathrm{Cn}(y, y) = \mathrm{Cn}(y) = S$ and $x \in \mathrm{Cn}(\emptyset)$. On the other hand, $\mathrm{Cn}(a, b) = S$ because $a \in \mathrm{Cn}(a)$.

DEFINITION 4. A Closure operator Cn on S is called *proper* if $\mathrm{Cn}(a) \neq \mathrm{Cn}(b)$ for some $a, b \in S$ or, equivalently, $\emptyset \neq X \neq S$ for some $X \in \mathscr{C}(\mathrm{Cn})$. Cn is said to be *regular* if the Cn-complete sets constitute a basis of $\mathscr{C}(\mathrm{Cn})$.

THEOREM 8. (Żandarowska [1]). *A proper closure operator* Cn *on* S *is regular if and only if* $\mathscr{C}(\mathrm{Cn})$ *has a basis* $\mathscr{B}$ *such that* (i) $\emptyset, S \notin \mathscr{B}$ *and* (ii) *if* $X \subseteq Y$ *then* $X = Y$, *for all* $X, Y \in B$.

Proof. If Cn is proper then the family of Cn-complete sets has both properties (i) and (ii). If, on the other hand, a non-empty family $\mathscr{B}$ of subsets of S has both properties (i) and (ii) and, $[\mathscr{B}] = \mathscr{C}(\mathrm{Cn})$ then $\mathscr{B}$ is the family of all Cn-complete sets.

DEFINITION 5. A Cn-closed set X is called *meet-irreducible* if $X \neq \bigcap \mathscr{F}$ for every family of Cn-closed sets $\mathscr{F}$ such that $X \subseteq \bigcap \mathscr{F}$ and $X \notin \mathscr{F}$. Obviously, every Cn-complete set is Cn-irreducible.

THEOREM 9. (Pierce [1], pp. 49–51). *If* Cn *is finite, i.e.,* $\mathscr{C}(\mathrm{Cn})$ *is inductive then the family of all* Cn-*irreducible* Cn-*closed sets constitute the smallest basis of* $\mathscr{C}(\mathrm{Cn})$.

III. Basic concepts of closure spaces

DEFINITION 1. If S is a non-empty set, Cn a closure operator on S and $\mathscr{C}$ a closure system on S then the pairs $\langle S, \mathrm{Cn}\rangle$ and $\langle S, \mathscr{C}\rangle$ are called *closure spaces*. Because of the dual correspondence between closure operators and closure systems, we may identify $\langle S, \mathrm{Cn}\rangle$ and $\langle S, \mathscr{C}\rangle$ as the same closure space, where $\mathscr{C} = \mathscr{C}(\mathrm{Cn})$.

Obviously, every topological space is a closure space. Hence, the theory of closure spaces is a generalization of topological spaces. In particular, we make use of continuous maps in a generalized sense. (However, the general topological principles like separation axioms do not play any direct role in our theory since it aims at logical applications.)

DEFINITION 2. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two closure spaces, $\mathscr{S}_i = \langle S_i, \mathrm{Cn}_i\rangle = \langle S_i, \mathscr{C}_i\rangle$ for $i = 1, 2$. A mapping $f\colon S_1 \to S_2$ is said to be *continuous* if $\check{f}(Z) \epsilon \mathscr{C}_1$ for all $Z \epsilon \mathscr{C}_2$. Here $\check{f}(Z)$ = the counter image of Z under $f = \{a \epsilon S_1 \mid f(a) \epsilon Z \subseteq S_2\}$. The set of all continuous maps of $\mathscr{S}_1$ into $\mathscr{S}_2$ will be denoted by $\mathrm{Hom}(\mathscr{S}_1, \mathscr{S}_2)$. A Bijective map $f\colon S_1 \to S_2$ is called a *homeomorphism* between $\mathscr{S}_1$ and $\mathscr{S}_2$ if both f and its inverse $\check{f}$ are continuous, i.e., $f \epsilon \mathrm{Hom}(\mathscr{S}_1, \mathscr{S}_2)$ and $\check{f} \epsilon \mathrm{Hom}(\mathscr{S}_2, \mathscr{S}_1)$. Then, the closure spaces $\mathscr{S}_1$, $\mathscr{S}_2$ are said to be *homeomorphic*.

THEOREM 1. *The following conditions are equivalent:*

(1) f *is continuous,*

(2) $f(\mathrm{Cn}_1(X)) \subseteq \mathrm{Cn}_2(f(X))$, *all* $X \subseteq S_1$,

(3) $\mathrm{Cn}_1(\check{f}(Y)) \subseteq \check{f}(\mathrm{Cn}_2(Y))$, *all* $Y \subseteq S_2$,

(4) $\mathrm{Cn}_1(X) \subseteq (\check{f}\mathrm{Cn}_2(f(X)))$, *all* $X \subseteq S_1$.

Proof. Equivalence of (2) and (4) is obvious. Suppose (1). If $X \subseteq S_1$ then $f(X) \subseteq \mathrm{Cn}_2(f(X))$, $X \subseteq \check{f}(\mathrm{Cn}_2(f(X)))$ and, $\mathrm{Cn}_1(X) \subseteq \mathrm{Cn}_1(\check{f}(\mathrm{Cn}_2(f(X))))$. But $\check{f}(\mathrm{Cn}_2(f(X)) \epsilon \mathscr{C}_1$ by (1). Consequently, $\mathrm{Cn}_1(\check{f}(\mathrm{Cn}_2(f(X)))) = \check{f}(\mathrm{Cn}_2(f(X)))$. Therefore, $\mathrm{Cn}_1(X) \subseteq \check{f}(\mathrm{Cn}_2(f(X)))$. Thus, (1) implies (4). Suppose (2). Then,

$$f(\mathrm{Cn}_1(\check{f}(Y))) \subseteq \mathrm{Cn}_2(\check{f}(f(Y))) \subseteq \mathrm{Cn}_2(Y) \qquad \text{for all } Y \subseteq S_2.$$

Hence, (2) implies (3). Suppose (3). If $Y \epsilon \mathscr{C}_2$ then $\mathrm{Cn}_1(\check{f}(Y)) \subseteq \check{f}(\mathrm{Cn}_2(Y)) = \check{f}(Y)$, i.e., $\check{f}(Y) \epsilon \mathscr{C}_1$. Thus (3) implies (1).

THEOREM 2. *The class of all closure spaces defines a category, whose objects are closure spaces and whose morphisms are continuous maps.*

Proof. If $f \epsilon \mathrm{Hom}(\mathscr{S}_1, \mathscr{S}_2)$ and $g \epsilon \mathrm{Hom}(\mathscr{S}_2, \mathscr{S}_3)$ then the composition $(g \circ f) \epsilon \mathrm{Hom}(\mathscr{S}_1, \mathscr{S}_3)$. Let $E_s\colon S \to S$ be the identity map on S, $E_s(a) = a$ for every $a \epsilon S$. Then, $E_s \epsilon \mathrm{Hom}(\mathscr{S}, \mathscr{S})$ and for all closure spaces S_1 and S_2, if $f \epsilon \mathrm{Hom}(\mathscr{S}, \mathscr{S}_2)$ and $g \epsilon \mathrm{Hom}(\mathscr{S}_1, \mathscr{S})$ then $f \circ E_s = f$ and $E_s \circ g = g$. Finally, if $f_i \epsilon \mathrm{Hom}(\mathscr{S}_i, \mathscr{S}_{i+1})$ for $i = 1, 2, 3$ then $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$.

The properties of a closure space are essentially those of the underlying closure operator, i.e., closure system. Hence, correspondingly, we distinguish closure spaces which are proper, finite (inductive), l-compact, regular and those which have the NEG-property. Similarly, we will talk about bases of closure spaces.

IV. Galois connections and dual spaces

Given two non-empty sets S, T and a mapping $R\colon T \to P(S)$ we define two mappings $\overset{*}{R}\colon P(S) \to P(T)$ and $\overset{+}{R}\colon P(T) \to P(S)$ such that for $X \subseteq S$, $U \subseteq T$ and $u \in T$:

$$u \in \overset{*}{R}(X) \quad \text{iff} \quad X \subseteq R(u), \tag{1}$$

$$\overset{+}{R}(U) = \bigcap \{R(u) \mid u \in U\}. \tag{2}$$

They have the following properties where $X, X_1, X_2 \subseteq S$ and U, U_1, $U_2 \subseteq T$:

$$\text{if} \quad U_1 \subseteq U_2 \quad \text{then} \quad \overset{+}{R}(U_2) \subseteq \overset{+}{R}(U_1), \tag{3}$$

$$\text{if} \quad X_1 \subseteq X_2 \quad \text{then} \quad \overset{*}{R}(X_2) \subset \overset{*}{R}(X_1), \tag{4}$$

$$X \subseteq \overset{+}{R}(U) \quad \text{iff} \quad U \subseteq \overset{*}{R}(X), \tag{5}$$

$$\overset{*}{R}\big(\overset{+}{R}(\overset{*}{R}(X))\big) = \overset{*}{R}(X), \tag{6}$$

$$\overset{+}{R}\big(\overset{*}{R}(\overset{+}{R}(U))\big) = \overset{+}{R}(U). \tag{7}$$

Consequently, the mappings $\mathrm{Cn}_S^R\colon P(S) \to P(S)$ and $\mathrm{Cn}_T^R\colon P(T) \to P(S)$ defined as follows

$$\mathrm{Cn}_S^R(X) = \overset{+}{R}(\overset{*}{R}(X)), \qquad \mathrm{Cn}_T^R(U) = \overset{*}{R}\big(\overset{+}{R}(U)\big) \tag{8}$$

are closure operators on S and T, respectively, called Galois closure operators defined by R.

THEOREM 1. (Ore [2]). *The closure systems* $\mathscr{C}(\mathrm{Cn}_S^R)$ *and* $\mathscr{C}(\mathrm{Cn}_T^R)$ *are dually isomorphic where* $\overset{*}{R}\colon \mathscr{C}(\mathrm{Cn}_S^R) \to \mathscr{C}(\mathrm{Cn}_T^R)$ *is the dual isomorphism and* $\overset{+}{R}$ *is inverse its.*

DEFINITION 1. Let $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ be a closure space. A Galois representation for $\mathscr{S}$ is a pair $\langle B, R\rangle$ where

$$B \neq \emptyset, \quad \text{and} \quad R\colon B \to P(S), \tag{9}$$

$$\mathrm{Cn} = \mathrm{Cn}_S^R. \tag{10}$$

Then, the closure space $D = \langle B, \mathrm{Cn}_B^R\rangle$ will be called a *dual space* for S.

Everett [1] has shown that every closure space $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ has a Galois representation $\langle \mathscr{B}, E_{\mathscr{B}}\rangle$ where $\mathscr{B} = \mathscr{C}(\mathrm{Cn})$ and $E_{\mathscr{B}} =$ the identity map on $\mathscr{B}$. We present a slight modification of Everett's theorem.

THEOREM 2. *If* $\mathscr{B} \subseteq \mathscr{C}(\mathrm{Cn})$ *is a basis of the closure space* $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ *then* $\langle \mathscr{B}, E_{\mathscr{B}}\rangle$ *is a Galois representation for* S.

Proof. If $X \subseteq S$ then $\overset{*}{E}(X) = \{Y \in \mathscr{B} \mid X \subseteq Y\}$ and since $[\mathscr{B}] = \mathscr{C}(\mathrm{Cn})$, $\overset{+}{E}\big(\overset{*}{E}(X)\big) = \bigcap \overset{*}{E}(X)$. Hence, $\mathrm{Cn} = \mathrm{Cn}_{\mathscr{B}}^E$.

DEFINITION 2. If $\mathscr{B} \subseteq \mathscr{C}(\mathrm{Cn})$ is a basis of the closure space $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ then the $\mathscr{B}$-*natural* dual space for $\mathscr{S}$ is $\mathscr{S}^* = \langle \mathscr{B}, \mathrm{Cn}_{\mathscr{B}}^E\rangle$ where E is the identity map on $\mathscr{B}$. The space $\mathscr{S}^*$ is called the *natural* or *proper natural* dual space of $\mathscr{S}$ if correspondingly, $\mathscr{B} = \mathscr{C}(\mathrm{Cn})$ or $\mathscr{B} = \overline{\mathscr{C}}(\mathrm{Cn})$.

THEOREM 3. *Let* $\mathscr{S}^* = \langle \mathscr{B}, \mathrm{Cn}^*\rangle$ *be the* $\mathscr{B}$-*natural dual for the closure space* $\mathscr{S} = \langle S, \mathrm{Cn}^*\rangle$ *and* $\mathscr{C} = \mathscr{C}(\mathrm{Cn})$. *Then*

(11) $\mathrm{Cn}^*(\emptyset) = \emptyset$ *and, for all* $\mathscr{F}_1, \mathscr{F}_2 \subseteq \mathscr{B}$,

(12) $\mathrm{Cn}^*(\mathscr{F}_1 \cup \mathscr{F}_2) = \mathrm{Cn}^*(\mathscr{F}_1) \cup \mathrm{Cn}^*(\mathscr{F}_2)$ *if and only if*

(13) $S \notin \mathscr{B}$ *and, for all* $A \in \mathscr{B}$ *and* $X, Y \in \mathscr{C}(\mathrm{Cn})$,

(14) *if* $X \cap Y \subseteq A$ *then either* $X \subseteq A$ *or* $Y \subseteq A$.

Proof. Since, $A \in \mathrm{Cn}^*(\mathscr{F})$ if $\bigcap \mathscr{F} \subseteq A$, we easily infer that $\mathrm{Cn}^*(\emptyset) = \emptyset$ iff $S \notin \mathscr{B}$. On the other hand, (12) and (14) are equivalent because for all $A \in \mathscr{B}$ and all $\mathscr{F} \subseteq \mathscr{B}$, $A \in \mathrm{Cn}^*(\mathscr{F})$ iff $\bigcap \mathscr{F} \subseteq A$.

THEOREM 4. *Let* $\mathscr{B}_1, \mathscr{B}_2$ *be two bases of the closure space* $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ *and* $\mathscr{S}_1^*, \mathscr{S}_2^*$ *be the corresponding* $\mathscr{B}_i$-*natural dual spaces,* $i = 1, 2$. *If* $\mathscr{B}_1 \subseteq \mathscr{B}_2$ *then the identity map* $E\colon \mathscr{B}_1 \to \mathscr{B}_2$ *is in* $\mathrm{Hom}(S_1, S_2)$.

Proof. If $\mathscr{F} \subseteq \mathscr{B}_1$ then $\mathscr{F}$ is Cn_1^*-closed iff $\mathscr{F} = \mathscr{G} \cap \mathscr{B}_2$ for some Cn_2^*-closed $\mathscr{G}$. Hence, if $\mathscr{G}$ is Cn_2^*-closed then $\breve{E}(\mathscr{G})$ is Cn_1^*-closed.

DEFINITION 3. A closure space $S = \langle S, \mathrm{Cn}\rangle$ is *topologically compact*, t-compact if every family $\mathscr{G} \subseteq \mathscr{C}(\mathrm{Cn})$ such that $\bigcap \mathscr{G} = \mathrm{Cn}(\emptyset)$ has a finite subfamily $\mathscr{G}_f$ such that $\bigcap \mathscr{G}_f = \mathrm{Cn}(\emptyset)$.

THEOREM 5. *A closure space* $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ *is* l-*compact if and only if some* (*or, equivalently, every*) *dual space for* S *is* t-*compact.*

Proof. Let $\mathscr{S}^* = \langle S^*, \mathrm{Cn}^*\rangle$ be a t-compact dual space for $\mathscr{S}$. Suppose ϱ be the dual isomorphism between $\mathscr{C}(\mathrm{Cn})$ and $\mathscr{C}(\mathrm{Cn}^*)$. If $\{X_i\}$ is a chain of Cn-consistent sets and $X = \bigcup_i X_i$ then X is Cn-consistent. For, if $\sup_i X_i = \sup_i \mathrm{Cn}(X_i) = \mathrm{Cn}(X) = S$ then

$$\bigcap_i \varrho(\mathrm{Cn}(X_i)) = \inf_i \varrho(\mathrm{Cn}(X_i)) = \varrho(\mathrm{Cn}(X)) = \varrho(S) = \mathrm{Cn}^*(\emptyset).$$

By t-compactness of S^*,

$$\mathrm{Inf}\{\varrho(\mathrm{Cn}(X_{i_1})), \ldots, \varrho(\mathrm{Cn}(X_{i_k}))\} = \varrho(\mathrm{Cn}(X_{i_1}) \cap \ldots \cap \varrho(\mathrm{Cn}(X_{i_k})) = \mathrm{Cn}^*(\emptyset) = \varrho(S).$$

Hence, by duality,

$$\mathrm{Sup}\{\mathrm{Cn}(X_{i_1}), \ldots, \mathrm{Cn}(X_{i_k})\} = \mathrm{Cn}(X_{i_j}) = S \quad \text{where} \quad 1 \leqslant j \leqslant k.$$

Contradiction. Thus X is consistent and the space $\mathscr{S}$ is l-compact by Theorem II.2. Suppose, conversely, that $\mathscr{S}$ is l-compact and $\mathscr{S}^*$ $\langle S^*, \mathrm{Cn}^*\rangle$ an arbitrary dual space for $\mathscr{S}$. Let $\mathscr{F} \subseteq \mathscr{C}(\mathrm{Cn}^*)$, $\bigcap\mathscr{F} = \mathrm{Cn}^*(\emptyset)$ and $\bigcap\mathscr{F}_f \neq \mathrm{Cn}^*(\emptyset)$ for every finite $\mathscr{F}_f \subseteq \mathscr{F}$. Define $\mathscr{G}$ as the family of all Cn^*-closed sets Z such that $Z = \bigcap\mathscr{F}_f$ for some finite $\mathscr{F}_f \subseteq \mathscr{F}$. Clearly, $\mathscr{G}$ is a downward directed family, and by hypothesis, $\check{\varrho}(Z)$ is a Cn-consistent closed set for every $Z \in \mathscr{G}$. Moreover, $\mathrm{Inf}\,\mathscr{G} = \bigcap\mathscr{G} = \bigcap\mathscr{F} = \mathrm{Cn}^*(\emptyset)$. Hence $\check{\varrho}(\mathrm{Inf}\,\mathscr{G}) = \mathrm{Sup}\{\check{\varrho}(Z) \mid Z \in \mathscr{G}\} = S$. But, $\{\check{\varrho}(Z) \mid Z \in \mathscr{G}\}$ is an upward directed family of Cn-consistent sets and, by Theorem II.2, the union of it is Cn-consistent, that is, $\mathrm{Sup}\{\check{\varrho}(Z) \mid Z \in \mathscr{G}\} \neq S$. Contradiction.

Definition 4. A Cn-closed set $X \subseteq S$ is called *clopen* in the closure space $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ if there exists a Cn-closed set Y such that $\mathrm{Sup}\{X, Y\} = S$ and $\mathrm{Inf}\{X, Y\} = \mathrm{Cn}(\emptyset)$. The closure space $\mathscr{S}$ is called *0-dimensional* if it has a basis of clopen sets.

Theorem 6. *If $\mathscr{S} = \langle S, \mathrm{Cn}\rangle$ has the* NEG-*property then every dual space for $\mathscr{S}$ is 0-dimensional.*

Proof. Let $\mathscr{S}^* = \langle S^*, \mathrm{Cn}^*\rangle$ be a dual space for $\mathscr{S}$ and ϱ be the dual isomorphism between $\mathscr{C}(\mathrm{Cn})$ and $\mathscr{C}(\mathrm{Cn}^*)$. Then, the family $\{\varrho(\mathrm{Cn}(a)) \mid a \in S\}$ is a basis of S^*. All the sets $\varrho(\mathrm{Cn}(a))$ appear to be clopen if $\mathscr{S}$ has the NEG-property. To see it, suppose that for all $X \subseteq S$, $a \in \mathrm{Cn}(X)$ iff $\mathrm{Cn}(X \cup \{b\}) = S$. Then, by Theorem II.6,

$$\mathrm{Inf}(\mathrm{Cn}(a), \mathrm{Cn}(b)) = \mathrm{Cn}(\emptyset) \quad \text{and} \quad \mathrm{Sup}(\mathrm{Cn}(a), \mathrm{Cn}(b)) = S.$$

Hence,

$$\mathrm{Sup}(\varrho(\mathrm{Cn}(a)), \varrho(\mathrm{Cn}(b))) = \varrho(\mathrm{Inf}(\mathrm{Cn}(a), \mathrm{Cn}(b))) = \varrho(\mathrm{Cn}(\emptyset)) = S^*$$

and

$$\mathrm{Inf}(\varrho(\mathrm{Cn}(a)), \varrho(\mathrm{Cn}(b))) = \varrho(\mathrm{Sup}(\mathrm{Cn}(a), \mathrm{Cn}(b))) = \varrho(S) = \mathrm{Cn}^*(\emptyset).$$

We now turn to the relationship between closure spaces and their natural duals. For $i = 1, 2$, let $\mathscr{S}_i = \langle S_i, \mathrm{Cn}_i\rangle$ be a closure space, $\mathscr{B}_i$ = the family of all (consistent) Cn_i-closed sets and $\mathscr{S}_i^* = \langle \mathscr{B}_i, \mathrm{Cn}_i^*\rangle$ be the corresponding (proper) natural dual for $\mathscr{S}_i$.

Theorem 7. *If $f \in \mathrm{Hom}(\mathscr{S}_1, \mathscr{S}_2)$ then the mapping $F_f\colon \mathscr{B}_2 \to \mathscr{B}_1$ defined as $F_f(V) = \check{f}(V) = \{a \in S_1 \mid f(a) \in V\}$ for $V \in \mathscr{B}_2$, is in* $\mathrm{Hom}(S_2^*, S_1^*)$.

Proof. Suppose, $\mathcal{F} \subseteq \mathcal{B}_1$ is Cn_1^*-closed. Then, there exists $X \in \mathcal{B}_1$ such that $\mathcal{F} = \{Z \in \mathcal{B}_1 \mid X \subseteq Z\}$. Then, $\breve{F}_f(\mathcal{F}) = \{W \in \mathcal{B}_2 \mid F_f(W) \in \mathcal{F}\}$ $= \{W \in \mathcal{B}_2 \mid X \subseteq F_f(W)\}$. Let $U = \bigcap \breve{F}_f(\mathcal{F})$ and let $\mathcal{G} = \{V \in \mathcal{B}_2 \mid U \subseteq V\}$. By their definitions, $\breve{F}_f(\mathcal{F}) \subseteq \mathcal{G}$. If $V \in \mathcal{G}$ then $X \subseteq \mathcal{F}_f(U) \subseteq \mathcal{F}_f(V)$ which implies that $V \in \breve{F}_f(\mathcal{F})$. Therefore, $\mathcal{G} = \breve{F}_f(\mathcal{F})$. But, $\mathcal{G}$ is a Cn_2^*-closed set, hence, $\breve{F}_f(\mathcal{F})$ also is.

COROLLARY 7.1. *If the closure spaces* $\mathcal{S}_1, \mathcal{S}_2$ *are homeomorphic then their (proper) natural dual spaces* $\mathcal{S}_1^*$ *and* $\mathcal{S}_2^*$ *also are homeomorphic.*

COROLLARY 7.2. *If* Σ *is the category of closure spaces then the map* $\Phi(\mathcal{S}) = \mathcal{S}^* =$ *the proper natural dual of* $\mathcal{S}$ *and* $\Phi(f) = F_f$ *for all* $\mathcal{S} \in \Sigma$ *and all* $f \in \mathrm{Hom}(\mathcal{S}_1, \mathcal{S}_2)$ *is a contravariant functor.*

V. Abstract logics

DEFINITION 1. An *abstract logic* of similarity type τ is a pair $\langle \mathcal{A}, \mathrm{Cn} \rangle$ or $\langle \mathcal{A}, \mathcal{C} \rangle$ consisting of a finitary abstract algebra $\mathcal{A}$ of similarity type τ, together with a closure operator Cn on A or closure system $\mathcal{C}$ on A where $A = |\mathcal{A}| =$ the carrier of $\mathcal{A}$. Although we require $\mathcal{A}$ to be a finitary algebra, most of our results can be extended to partial and infinitary algebras.

Two abstract logics are said to be *similar* if they are of the same similarity type. If $\mathcal{L}$ is an abstract logic, $\langle \mathcal{A}, \mathrm{Cn} \rangle$, $\langle \mathcal{A}, \mathcal{C} \rangle$ then $|\mathcal{L}|$ is the closure space underlying $\mathcal{L}$, i.e., $\langle |\mathcal{A}|, \mathrm{Cn} \rangle$, $\langle |\mathcal{A}|, \mathcal{C} \rangle$.

DEFINITION 2. Let $\mathcal{L}_i = \langle \mathcal{A}_i, \mathcal{C}_i \rangle$ be similar abstract logics for $i = 1, 2$. A mapping $f\colon A_1 \to A_2$ is called a *logical morphism* if (i) f is a homomorphism of $\mathcal{A}_1$ to $\mathcal{A}_2$, $f \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ and (ii) f is continuous, i.e., $f \in \mathrm{Hom}(|\mathcal{L}_1|, |\mathcal{L}_2|)$. $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$ will denote the set of all l-morphisms between similar logics $\mathcal{L}_1$ and $\mathcal{L}_2$. If $f\colon A_1 \to A_2$ is bijective, $f \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$ and $\breve{f} \in \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)$ then f is called an *l-isomorphism* and the logics $\mathcal{L}_1, \mathcal{L}_2$ are said to be *l-isomorphic*.

If $\mathcal{A}$ is a Boolean algebra and $\mathcal{C}$ is the family of all filters of $\mathcal{A}$ then $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ is an abstract logic, called *Boolean logic*. If $\mathcal{L}_i = \langle \mathcal{A}_i, \mathcal{C}_i \rangle$ are Boolean logics for $i = 1, 2$ then $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, i.e., every homomorphism of $\mathcal{A}_1$ to $\mathcal{A}_2$ is a logical morphism of $\mathcal{L}_1$ to $\mathcal{L}_2$.

Most of the concepts and theorems concerned with closure spaces can easily be modified and applied to abstract logics.

THEOREM 1. *The class of all abstract logics of a given similarity type constitutes a category whose morphisms are logical morphisms.*

THEOREM 2. *Given two similar abstract logics* $\mathcal{L}_1$ *and* $\mathcal{L}_2$, *let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be the corresponding (proper) natural dual spaces. If* $f \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$

then the f-counter-image map $F_f \in \mathrm{Hom}(\mathscr{S}_2, \mathscr{S}_1)$. *If* $\mathscr{L}_1, \mathscr{S}_2$ *are l-isomorphic then* $\mathscr{S}_1, \mathscr{S}_2$ *are homeomorphic.*

THEOREM 3. *If* Λ *is the category of abstract logics of a given similarity type and* Σ *is the category of closure spaces then the map* $\Phi\colon \Lambda \to \Sigma$ *defined by* $\Phi(\mathscr{L}) = \mathscr{S}^* =$ *the (proper) natural dual of* $|\mathscr{L}|$ *and* $\Phi(f) = F_f$ *for all* $\mathscr{L} \in \Lambda$ *and all* $f \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$ *is a contravariant functor.*

It is clear that the map $\Phi_0\colon \Lambda \to \Sigma$ such that $\Phi_0(\mathscr{L}) = |\mathscr{L}|$ and $\Phi_0(f) = f$ for all $\mathscr{L} \in \Lambda$ and all $f \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$ is a "forgetful" functor. Furthermore, many notions and theorems concerned with abstract logics may easily be reformulated for closure spaces ("forget" the algebraic structure). This applies, in particular, to the projective and inductive generating operations, considered in the next chapters.

VI. Projective generation of abstract logics

DEFINITION 1. Given a non-empty index set I, let $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ be an abstract logic of type τ for each $i \in I$. If $\mathscr{A}$ is an abstract algebra of type τ and $f_i \in \mathrm{Hom}(\mathscr{A}, \mathscr{A}_i)$ for $i \in I$ then $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is *projectively generated* from $\{\mathscr{L}_i\}$ by $\{f_i\}$ if and only if $\mathscr{C}$ is the coarsest (smallest) closure system on $A = |\mathscr{A}|$ such that $f_i \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}_i)$ for all $i \in I$. In order to see that $\mathscr{C}$ exists, define it to be the infimum of the appropriate family and, then, show that this infimum is the minimum.

First, we consider the case of projective generation by a single homomorphism. We write simply $\mathscr{C}_i$ and $\overline{\mathscr{C}}_i$ for $\mathscr{C}(\mathrm{Cn}_i)$ and $\overline{\mathscr{C}}(\mathrm{Cn}_i)$, respectively.

THEOREM 1. *Let* $\mathscr{L}_1 = \langle \mathscr{A}_1, \mathscr{C}_1 \rangle$, $\mathscr{L}_2 = \langle \mathscr{A}_2, \mathscr{C}_2 \rangle$ *and* $h \in \mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2)$. *Then* $\mathscr{L}_1$ *is projectively generated from* $\mathscr{L}_2$ *by* h *if and only if*

$$\mathscr{C}_1 = \{\check{h}(V) \mid V \in \mathscr{C}_2\}. \tag{1}$$

Furthermore, (1) *is equivalent to each of the following conditions: for all* $X \subseteq A_1$,

$$\mathrm{Cn}_1(X) = \check{h}\big(\mathrm{Cn}_2(h(X))\big), \tag{2}$$

$$\mathrm{Cn}_1(X) = \bigcap \{\check{h}(V) \mid h(X) \subseteq V \in \mathscr{C}_2\}, \tag{3}$$

$$\mathrm{Cn}_1(X) = \{a \in A_1 \mid a \in \mathrm{Cn}_V(X),\ \textit{all}\ V \in \mathscr{C}_2\} \tag{4}$$

$$\textit{where}\quad \mathrm{Cn}_V(X) = \{a \in A_1 \mid h(X) \subseteq V \Rightarrow h(a) \in V\},$$

$$\mathrm{Cn}_1(X) = \bigcap \{\mathrm{Cn}_V(X) \mid V \in \mathscr{C}_2\}. \tag{5}$$

Obviously, one may replace $\mathscr{C}_i$ *by* $\overline{\mathscr{C}}_i$ *in* (1), (3), (4), *and* (5).

Proof. If (1) holds then clearly $h \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. If $\mathscr{L} = \langle \mathscr{A}_1, C \rangle$

and $\breve{h}(V)\in\mathscr{C}$ for every $V\in\mathscr{C}_2$ then $\mathscr{C}_1\subseteq\mathscr{C}$. Thus, $\mathscr{C}_1$ is the coarsest closure system on $\mathscr{A}_1$ such that $h\in\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. If on the other hand, $\mathscr{L}_1$ is projectively generated from $\mathscr{L}_2$ by h then $\breve{h}(V)\in\mathscr{C}_1$ for every $V\in\mathscr{C}_2$. Moreover, if $\mathscr{C}$ is any closure system on $\mathscr{A}_1$ such that all sets $h(V)$, for $V\in\mathscr{C}_2$, are in $\mathscr{C}$ then $\mathscr{C}_1\subseteq\mathscr{C}$. Hence, (1) holds. Equivalencies (3) $\Leftrightarrow$ (4) and (4) $\Leftrightarrow$ (5) are obvious. (1) $\Rightarrow$ (3):

$$\begin{aligned}\mathrm{Cn}_1(X) &= \bigcap\{Y\in\mathscr{C}_1 \mid X\subseteq Y\} = \bigcap\{\breve{h}(V) \mid X\subseteq\breve{h}(V),\ V\in\mathscr{C}_2\}\\ &= \bigcap\{\breve{h}(V) \mid h(X)\subseteq V\in\mathscr{C}_2\}.\end{aligned}$$

(3) $\Rightarrow$ (2):

$$\begin{aligned}\mathrm{Cn}_1(X) &= \bigcap\{\breve{h}(V) \mid h(X)\subseteq V\in\mathscr{C}_2\} = \breve{h}\big(\bigcap\{V\in\mathscr{C}_2 \mid h(X)\subseteq V\}\big)\\ &= \breve{h}\big(\mathrm{Cn}_2(h(X))\big).\end{aligned}$$

(2) $\Rightarrow$ (1). If $X\in\mathscr{C}_1$ then $X=\mathrm{Cn}_1(X)=\breve{h}\big(\mathrm{Cn}_2(h(X))\big)$. Thus $X=\breve{h}(V)$ where $V\in\mathscr{C}_2$. If, conversely, $X=\breve{h}(V)$ for some $V\in\mathscr{C}_2$ then $X=h\big(\mathrm{Cn}_2(V)\big)$ $=h\big(\mathrm{Cn}_2(h(X))\big)=\mathrm{Cn}_1(X)$. Hence, $X\in\mathscr{C}_1$.

THEOREM 2. *Suppose that* $\mathscr{L}=\langle\mathscr{A},\mathscr{C}\rangle$, $\mathscr{L}_i=\langle\mathscr{A}_i,\mathscr{C}_i\rangle$, $h_i\in\mathrm{Hom}(\mathscr{A},\mathscr{A}_i)$ *and* $\mathscr{C}^{(i)}=\{\breve{h}_i(V) \mid V\in\mathscr{C}_i\}$. *If* $\mathscr{L}$ *is projectively generated from* $\{\mathscr{L}_i\}$ *by* $\{h_i\}$ *then* $\mathscr{C}=\operatorname*{Sup}_i\mathscr{C}^{(i)}$, *and by duality*, $\mathrm{Cn}=\operatorname*{Inf}_i\mathrm{Cn}^{(i)}$.

Proof. Since $h_i\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_i)$, we have that $\mathscr{C}^{(i)}\subseteq\mathscr{C}$ for $i\in I$. Hence, $\operatorname*{Sup}_i\mathscr{C}^{(i)}\subseteq\mathscr{C}$. But each h_i is a logical morphism with respect to $\operatorname*{Sup}_i\mathscr{C}^{(i)}$. This implies that $\mathscr{C}\subseteq\operatorname*{Sup}_i\mathscr{C}^{(i)}$.

Theorems 3 and 4 together comprise the basic theorems on projective generation.

THEOREM 3. *Suppose that* $\mathscr{L}_0=\langle\mathscr{A}_0,\mathscr{C}_0\rangle$ *is projectively generated from* $\{\mathscr{L}_i\}$ *by* $\{h_i\}$ *where* $\mathscr{L}_i=\langle\mathscr{A}_i,\mathscr{C}_i\rangle$ *and* $h_i\in\mathrm{Hom}(\mathscr{A}_0,\mathscr{A}_i)$ *for* $i\in I$. *Then, for any logic* $\mathscr{L}=\langle\mathscr{A},\mathscr{C}\rangle$ *and any* $g\in\mathrm{Hom}(\mathscr{A},\mathscr{A}_0)$, $g\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_0)$ *if and only if* $(h_i\circ g)\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_i)$ *for all* $i\in I$.

Proof. *If* $g\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_0)$ then each $h_i\circ g$ is a logical morphism. Let $\mathscr{C}^{(i)}=\{\breve{h}_i(V) \mid V\in\mathscr{C}_i\}$, then by Theorem 2, $\mathscr{C}_0=\operatorname*{Sup}_i\mathscr{C}^{(i)}$. By Theorem 1.7, $\mathscr{C}_0=\big[\bigcup_i\mathscr{C}^{(i)}\big]$. Hence, if $X\in\mathscr{C}_0$ then $X=\bigcap_i Y_i$ where $Y_i=\breve{h}_i(W_i)$ and $W_i\in\mathscr{C}_i$. This implies that $\breve{g}(X)=\bigcap_i\breve{g}(Y_i)=\bigcap_i\breve{g}\big(\breve{h}_i(W_i)\big)$. But if $h_i\circ g\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_i)$ for all $i\in I$ then $\breve{g}\big(\breve{h}_i(W_i)\big)\in\mathscr{C}$ for all $i\in I$. Therefore, $\breve{g}(X)\in\mathscr{C}$ for all $X\in\mathscr{C}_0$.

THEOREM 4. *Let* $\mathscr{L}_0=\langle\mathscr{A}_0,\mathscr{C}_0\rangle$, $\mathscr{L}_i=\langle\mathscr{A}_i,\mathscr{C}_i\rangle$ *and* $h_i\in\mathrm{Hom}(\mathscr{A}_0,\mathscr{A}_i)$ *for* $i\in I$. *Suppose that for any* $\mathscr{L}=\langle\mathscr{A},\mathscr{C}\rangle$ *and any* $g\in\mathrm{Hom}(\mathscr{A},\mathscr{A}_0)$, $g\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_0)$ *if and only if* $(h_i\circ g)\in\mathrm{Hom}(\mathscr{L},\mathscr{L}_i)$ *for all* $i\in I$. *Then*, $\{h_i\}$ *projectively generates* $\mathscr{L}_0$ *from* $\{\mathscr{L}_i\}$.

Proof. Put $\mathscr{A} = \mathscr{A}_0$ and $g = E_A$ = the identity map on A. Since E_A is a logical morphism, $\mathscr{C}_0 \subseteq \mathscr{C}$ iff all $h_i \circ E_A = h_i$ are logical morphisms.

THEOREM 5. *Let* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0 \rangle$, $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ *and* $h_i \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_i)$ *for* $i \in I$. *Suppose that, for each* $i \in I$, $\mathscr{L}_i$ *is projectively generated from* $\{\mathscr{L}_{ij}\}$ *by* $\{f_{ij}\}$ *where* $\mathscr{L}_{ij} = \langle \mathscr{A}_{ij}, \mathscr{C}_{ij} \rangle$ *and* $f_{ij} \in \mathrm{Hom}(\mathscr{A}_i, \mathscr{A}_{ij})$ *for* $j \in J_i$. *Then,* $\mathscr{L}_0$ *is projectively generated by* $\{h_i\}$ *from* $\{\mathscr{L}_i\}$ *iff the family* $\{f_{ij} \circ h_i \mid j \in J_i, i \in I\}$ *projectively generates* $\mathscr{L}_0$ *from* $\{\mathscr{L}_{ij} \mid j \in J_i, i \in I\}$.

Proof. Suppose $\{h_i\}$ projectively generates $\mathscr{L}_0$, $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is similar to $\mathscr{L}_0$ and $g \in \mathrm{Hom}(\mathscr{A}, \mathscr{A}_0)$. Put $k_{ij} = f_{ij} \circ h_i$. Since each $\mathscr{L}_i$ is projectively generated by $\{f_{ij}\}$, we have by Theorem 3, that $(h_i \circ g) \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}_i)$ for all $i \in I$. If $\mathscr{L}_0$ is projectively generated by $\{h_i\}$, again by Theorem 3, we have that $g \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}_0)$ and, consequently, $k_{ij} \circ g \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}_{ij})$. Therefore, by Theorem 4, $\{k_{ij}\}$ projectively generates $\mathscr{L}_0$. Now suppose that $\{k_{ij}\}$ projectively generates $\mathscr{L}_0$, $\mathscr{L}$ is similar to $\mathscr{L}_0$ and $g \in \mathrm{Hom}(\mathscr{A}, \mathscr{A}_0)$. Then, using Theorem 3, we obtain that $g \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}_0)$ iff for all $i \in I$, $(h_i \circ g) \in (\mathscr{L}, \mathscr{L}_i)$. Therefore, by Theorem 4, $\{h_i\}$ projectively generates $\mathscr{L}_0$.

The most important examples of projective generation are product logics.

DEFINITION 2. Let $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ and $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ for $i \in I$. Then, $\mathscr{L}$ is said to be the *product logic* of $\{\mathscr{L}_i\}$, $\mathscr{A} = \underset{i}{X}\mathscr{A}_i$, iff $\mathscr{A} = \underset{i}{X}\mathscr{A}_i$ = product algebra and $\mathscr{L}$ is projectively generated from $\{\mathscr{L}_i\}$ by the family of projections, $\{\pi_i\}$.

By definition the product logic is a particular instance of projective generation. The next proposition shows that the construction of a projectively generated logic can be reduced to the construction of a product logic and a logic projectively generated by a simple mapping.

THEOREM 6. *Let* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0 \rangle$, $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ *and* $h_i \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_i)$ *for* $i \in I$. *Suppose that* $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle = \underset{i}{X}\mathscr{L}_i$ *and* $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$ *where* $\pi_i(h(a)) = h_i(a)$ *for all* $a \in |A_0|$ *and* $i \in I$. *Then,* $\{h_i\}$ *projectively generates* $\mathscr{L}_0$ *from* $\{\mathscr{L}_i\}$ *if and only if* $\mathscr{L}_0$ *is projectively generated from* $\mathscr{L}$ *by* h.

Proof. The theorem is a corollary to Theorem 5 since $\mathscr{L}$ is projectively generated from $\{\mathscr{L}_i\}$ by $\{\pi_i\}$.

Another example of projective generation are sublogics.

DEFINITION 3. Let $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0 \rangle$, $\mathscr{A}$ a subalgebra of $\mathscr{A}_0$ and, $\mathscr{C} = \{A \cap V \mid V \in \mathscr{C}_0\}$. Then, $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is said to be a *sublogic* of $\mathscr{L}_0$.

THEOREM 7. $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ *is a sublogic of* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0 \rangle$ *iff* $\mathscr{A}$ *is a subalgebra of* $\mathscr{A}_0$ *and* $\mathscr{L}$ *is projectively generated from* $\mathscr{L}_0$ *by the identity map* E_A *on* A.

Remark. There is a generalized notion of a sublogic which might be useful in another investigation: $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn}\rangle$ is a sublogic of $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn}_0\rangle$ if A is subalgebra of $\mathscr{A}_0$ and $\mathrm{Cn} \leqslant \mathrm{Cn}_0$.

VII. Inductive generation of abstract logics

DEFINITION 1. Given a non-empty index set I, let $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i\rangle$ be an abstract logic of type τ for each $i \in I$. If $\mathscr{A}$ is an abstract algebra of type τ and $f_i \in \mathrm{Hom}(\mathscr{A}_i, \mathscr{A})$ for $i \in I$ then $\mathscr{L} = \langle \mathscr{A}, \mathscr{C}\rangle$ is *inductively generated* from $\{\mathscr{L}_i\}$ by $\{f_i\}$ if and only if $\mathscr{C}$ is the finest (greatest) closure system on $A = |\mathscr{A}|$ such that $f_i \in \mathrm{Hom}(\mathscr{L}_i, \mathscr{L})$ for all $i \in I$. In order to see that $\mathscr{C}$ exists, define it to be the supremum of the appropriate family and, then, show that this supremum is the maximum.

THEOREM 1. *Let* $\mathscr{L}_1 = \langle \mathscr{A}_1, \mathscr{C}_1\rangle$, $\mathscr{L}_2 = \langle \mathscr{A}_2, \mathscr{C}_2\rangle$ *and* $h \in \mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2)$. *Then* $\mathscr{L}_2$ *is inductively generated from* $\mathscr{L}_1$ *by* h *if and only if*

$$(*)\qquad \mathscr{C}_2 = \{V \subseteq A_2 \mid \check{h}(V) \in \mathscr{C}_1\}.$$

Obviously, one may replace $\mathscr{C}_i$ *by* $\overline{\mathscr{C}}_i$.

Proof. If $(*)$ holds then clearly $h \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. If $\mathscr{L} = \langle \mathscr{A}_1, \mathscr{C}\rangle$ and $\check{h}(V) \in \mathscr{C}_1$ for every $V \in \mathscr{C}$ then $\mathscr{C} \subseteq \mathscr{C}_2$. Thus, $\mathscr{C}_2$ is the finest closure system on A_2 such that $h \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. If, on the other hand, $\mathscr{L}_2$ is inductively generated from $\mathscr{L}_1$ by h then $\check{h}(V) \in \mathscr{C}_1$ for every $V \in \mathscr{C}_2$. Moreover, if $\mathscr{C}$ is any closure system on A_2 such that all sets $\check{h}(V)$, for $V \in \mathscr{C}$, are in $\mathscr{C}_1$ then $\mathscr{C} \subseteq \mathscr{C}_2$. Hence, $(*)$ holds.

THEOREM 2. *Suppose that* $\mathscr{L} = \langle \mathscr{A}, \mathscr{C}\rangle$, $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i\rangle$, $h_i \in \mathrm{Hom}(\mathscr{A}_i, \mathscr{A})$ *and* $\mathscr{C}^{(i)} = \{V \subseteq A \mid \check{h}_i(V) \in \mathscr{C}_i\}$ *where* $i \in I$. *If* $\mathscr{L}$ *is inductively generated from* $\{\mathscr{L}_i\}$ *by* $\{h_i\}$ *then* $\mathscr{C} = \operatorname*{Inf}_i \mathscr{C}^{(i)}$ *and, by duality,* $\mathrm{Cn} = \operatorname*{Sup}_i \mathrm{Cn}^{(i)}$. *Note that*

$$\operatorname*{Inf}_i \mathscr{C}^{(i)} = \{V \subseteq A \mid \check{h}_i(V) \in \mathscr{C}_i \text{ for all } i \in I\}.$$

Proof. Since $h_i \in \mathrm{Hom}(\mathscr{L}_i, \mathscr{L})$, we have that $\mathscr{C} \subseteq \mathscr{C}^{(i)}$ for $i \in I$. Hence, $\mathscr{C} \subseteq \operatorname*{Inf}_i \mathscr{C}^{(i)}$. But each h_i is a logical morphism with respect to $\operatorname*{Inf}_i \mathscr{C}^{(i)}$. This implies that $\operatorname*{Inf}_i \mathscr{C}^{(i)} \subseteq \mathscr{C}$.

Theorems 3 and 4 are the basic theorems on inductive generation.

THEOREM 3. *Suppose that* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0\rangle$ *is inductively generated from* $\{\mathscr{L}_i\}$ *by* $\{h_i\}$ *where* $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i\rangle$ *and* $h_i \in \mathrm{Hom}(\mathscr{A}_i, \mathscr{C}_0)$ *for* $i \in I$. *Then, for any logic* $\mathscr{L} = \langle \mathscr{A}, \mathscr{C}\rangle$ *and any* $g \in \mathrm{Hom}(\mathscr{A}_i, \mathscr{A})$, $g \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L})$ *if and only if* $(g \circ h_i) \in \mathrm{Hom}(\mathscr{L}_i, \mathscr{L})$ *for all* $i \in I$.

Proof. If $g \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L})$ then each $g \circ h_i$ is a logical morphism. On the other hand, suppose that $(g \circ h_i) \in \mathrm{Hom}(\mathscr{L}_i, \mathscr{L})$ for all $i \in I$. If $W \in \mathscr{C}$

and $V = \breve{g}(W)$ then, by Theorem 2, $V \in \mathscr{C}_0$ iff $\breve{h}_i(V) \in \mathscr{C}_i$ for all $i \in I$. But, $\breve{h}_i(V) = (\breve{h}_i \circ \breve{g})(W) \in \mathscr{C}_i$ for all $i \in I$. Hence $g \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L})$.

THEOREM 4. *Let* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0 \rangle$, $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ *and* $h_i \in \mathrm{Hom}(\mathscr{A}_i, \mathscr{A}_0)$ *for* $i \in I$. *Suppose that for any* $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ *and any* $g \in \mathrm{Hom}(\mathscr{A}, \mathscr{C}_0)$, $g \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L})$ *if and only if* $(g \circ h_i) \in \mathrm{Hom}(\mathscr{L}_i, \mathscr{L})$ *for all* $i \in I$. *Then,* $\{h_i\}$ *inductively generates* $\mathscr{L}_0$ *from* $\{\mathscr{L}_i\}$.

Proof. Put $\mathscr{A} = \mathscr{A}_0$ and $g = E_{A_0}$ = the identity map on A_0. Since E_{A_0} is a logical morphism, $\mathscr{C} \subseteq \mathscr{C}_0$ iff all $E_{A_0} \circ h_i = h_i$ are logical morphisms.

The most important examples of inductive generation are quotient logics.

DEFINITION 2. Suppose that $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$, $\sim$ is a congruence of $\mathscr{A}$, $\mathscr{A}/\sim$ = the quotient of $\mathscr{A}$ under $\sim$ and k is the corresponding canonical map, $k(a) = a/\sim$ for $a \in A$. Then, let $\mathscr{L}/\sim = \langle \mathscr{A}/\sim, \mathscr{C}/\sim \rangle$ where $\mathscr{C}/\sim$ is the finest closure system on $A/\sim$ such that k is a logical morphism. The logic $\mathscr{L}/\sim$ is inductively generated from $\mathscr{L}$ by k and called the *quotient* of $\mathscr{L}$ under $\sim$. Note that k is logical epimorphism anp $\mathscr{C}/\sim = \{V \subseteq A/\sim \mid \breve{k}(V) \in \mathscr{C}\}$.

The next theorem is the analogue of the Homomorphism Theorem of Universal Algebra.

THEOREM 5. *If* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathscr{C}_0 \rangle$ *is inductively generated from* $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ *by an epimorphism* $h \in \mathrm{Hom}(\mathscr{A}, \mathscr{A}_0)$ *and* $\sim$ *is the kernel of* h *then there exists a logical isomorphism* $g \in \mathrm{Hom}(\mathscr{L}/\sim, \mathscr{L}_0)$ *such that* $h = g \circ k$ *where* k *is the canonical map.*

Proof. Let $g(a/\sim) = h(a)$. Then, g is an algebraic isomorphism of $\mathscr{A}/\sim$ and $\mathscr{A}_0$. Both h and k are inductively generating mappings and, $h = g \circ k$ and $k = \breve{g} \circ h$. Hence, by Theorem 3, $g \in \mathrm{Hom}(\mathscr{L}/\sim, \mathscr{L}_0)$ and $\breve{g} \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}/\sim)$.

VIII. Logical congruences and bi-logical morphisms

DEFINITION 1. A congruence $\sim$ of $\mathscr{A}$ is said to be a *logical congruence* of the logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$, $\sim \in \Theta_{\mathscr{L}}$, if for all $a, b \in A$, $\mathrm{Cn}(a) = \mathrm{Cn}(b)$ whenever $a \sim b$ or, equivalently, $a \sim b$ and $a \in \mathrm{Cn}(X)$ imply $b \in \mathrm{Cn}(X)$ for alle $X \subseteq A$.

THEOREM 1. *Let* $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ *be similar logics for* $i = 1, 2$. *If* $h \in \mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2)$ *then the kernel of* h, $\underset{h}{\sim}$, *is a* l-*congruence iff* $\breve{h}(h(X)) = X$ *for all* $X \in \mathscr{C}_1$.

Proof. If $\underset{h}{\sim}$ is a l-congruence, $X \in \mathscr{C}_1$ and $b \in h(\breve{h}(X))$ then $h(b) \in h(X)$ and, hence, there exists $a \in X$ such that $a \underset{h}{\sim} b$. Therefore, $b \in X$. Thus,

$\check{h}(h(X)) \subseteq X$, i.e., $\check{h}(h(X)) = X$. Suppose, $\check{h}(h(X)) = X$ whenever $X \in \mathscr{C}_1$. If $a \in X$ and $a \underset{h}{\sim} b$ then $h(b) = h(a) \in h(X)$. Hence $b \in \check{h}(h(X)) = X$.

COROLLARY 1.1. *If* $h \in \mathrm{Epi}(\mathscr{A}_1, \mathscr{A}_2)$ *and* $\underset{h}{\sim}$ *is a* l*-congruence of* $\mathscr{L}_1 = \langle \mathscr{A}_1, \mathscr{C}_1 \rangle$ *then* $\mathscr{L}_2 = \langle \mathscr{A}_2, \mathscr{C}_2 \rangle$ *is inductively generated by* h *from* $\mathscr{L}_1$ *iff* $\mathscr{C}_2 = \{h(X) \mid X \in \mathscr{C}_1\}$.

Proof. In view of Theorem 1 we have that $\{V \mid \check{h}(V) \in \mathscr{C}_1\} = \{h(X) \mid X \in \mathscr{C}_1\}$.

DEFINITION 2. Let $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ be similar logics for $i = 1, 2$ and let $h \in \mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2)$. Then, h is said to be an *l^*-morphism*, $h \in \mathrm{Hom}^*(\mathscr{L}_1, \mathscr{L}_2)$, if h projectively generates $\mathscr{L}_1$ from $\mathscr{L}_2$, that is, $\mathscr{C}_1 = \{\check{h}(V) \mid V \in \mathscr{C}_2\}$. In particular, if h is an l^*-epimorphism then h is called a *bi-logical morphism*, $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$. Note that every l^*-morphism is an l-morphism and l-isomorphisms are l^*-morphisms.

THEOREM 2. *If* $h \in \mathrm{Hom}^*(\mathscr{L}_1, \mathscr{L}_2)$ *then its kernel* $\underset{h}{\sim}$ *is a* l*-congruence.*

Proof. Suppose that $h(a) = h(b)$, $X \in \mathscr{C}_1$ and $a \in X$. Then, $X = \check{h}(V)$ for some $V \in \mathscr{C}_2$ and $h(a) \in h(X) \subseteq V$. Hence, $h(b) \in V$, that is $b \in \check{h}(V) = X$.

THEOREM 3. *If* $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$ *then* $\mathscr{L}_2$ *is inductively generated by* h *from* $\mathscr{L}_1$.

Proof. Suppose, $h \in \mathrm{Epi}(\mathscr{A}_1, \mathscr{A}_2)$ and $\mathscr{C}_1 = \{\check{h}(V) \mid V \in \mathscr{C}_2\}$. If $\check{h}(W) \in \mathscr{C}_1$ then $\check{h}(W) = \check{h}(V)$ for some $V \in \mathscr{C}_2$. But h is onto map. Hence, $W = h(\check{h}(W)) = (h(\check{h}V)) = V$ and $W \in \mathscr{C}_2$. Therefore, $\mathscr{C}_2 = \{W \mid h(W) \in \mathscr{C}_1\}$ since h is a l-morphism.

THEOREM 4. *If* $h \in \mathrm{Epi}(\mathscr{A}_1, \mathscr{A}_2)$, $\mathscr{C}_2 = \{h(X) \mid X \in \mathscr{C}_1\}$ *and* $\check{h}(h(X)) = X$ *for all* $X \in \mathscr{C}_1$ *then* $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$.

Proof. The hypotheses clearly imply that $h(\bigcap_i X_i) = \bigcap_i h(X_i)$ if $X_i \in \mathscr{C}_1$ for $i \in I$. That is, $\mathscr{C}_2$ is a closure system on A_2 whenever $\mathscr{C}_1$ is a closure system on A_1. Consequently, by Corollary 1.1, $\mathscr{L}_2 = \langle \mathscr{A}_2, \mathscr{C}_2 \rangle$ is inductively generated by h from $\mathscr{L}_1 = \langle \mathscr{A}_1, \mathscr{C}_1 \rangle$, $\mathscr{C}_2 = \{V \mid \check{h}(V) \in \mathscr{C}_1\}$, and $\mathscr{L}_1$ is projectively generated by h from $\mathscr{L}_2$, $\mathscr{C}_1 = \{\check{h}(V) \mid V \in \mathscr{C}_2\}$.

COROLLARY 4.1. *If* $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ *are similar logics for* $i = 1, 2$ *and* $h\colon A_1 \to A_2$ *then* $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$ *iff* $h \in \mathrm{Epi}(\mathscr{A}_1, \mathscr{A}_2)$, *the kernel of* h *is an* l*-congruence of* $\mathscr{L}_1$ *and* $\mathscr{C}_2 = \{h(X) \mid X \in \mathscr{C}_1\}$.

Proof. By Theorem 2, Corollary 1.1, Theorem 3 and Theorem 4.

THEOREM 5. *Let* $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ *be similar logics. Suppose that* $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$ *and* $g \in \mathrm{Hom}^*(\mathscr{L}_1, \mathscr{L}_3)$. *Then, a necessary and sufficient condition for the existence of morphism* $f \in \mathrm{Hom}^*(\mathscr{L}_2, \mathscr{L}_3)$ *such that* $g = f \circ h$ *is that* $\underset{h}{\sim} \subseteq \underset{g}{\sim}$.

Proof. Assume that a morphism $f \in \mathrm{Hom}^*(\mathscr{L}_2, \mathscr{L}_3)$ exists such that

$g = f \circ h$. If $a \underset{h}{\sim} b$ then $g(a) = f(h(a)) = f(h(b)) = g(b)$, i.e., $a \underset{g}{\sim} b$. Thus, $\underset{h}{\sim} \subseteq \underset{g}{\sim}$. Suppose that $\underset{h}{\sim} \subseteq \underset{g}{\sim}$ and define $f(x) = g(a)$ where $x \in A_2$ and $a \in \breve{h}(x)$. This well defines $f \in \mathrm{Hom}(\mathscr{A}_2, \mathscr{A}_3)$ such that $g = f \circ h$. We have to show that f projectively generates $\mathscr{L}_2$ from $\mathscr{L}_3$. If $W \in \mathscr{C}_3$ then $\breve{g}(W) \in \mathscr{C}_1$ since g is an l-morphism. But $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$. Hence, by Corollary 4.1, $\breve{f}(W) = h(\breve{h}(\breve{f}(W))) = h(\breve{g}(W)) \in \mathscr{C}_2$. On the other hand, if $V \in \mathscr{C}_2$ then $\breve{h}(V) \in \mathscr{C}_1$ since h is an l-morphism. But, $\mathscr{C}_1 = \{\breve{g}(W) \mid W \in \mathscr{C}_3\}$ since g is an l^*-morphism. Therefore, $\breve{h}(V) = \breve{g}(W)$ for some $W \in \mathscr{C}_3$. However, $V = h(\breve{h}(V)) = h(\breve{g}(W)) = h(\breve{h}(\breve{f}(W))) = \breve{f}(W)$. Hence $V = \breve{f}(W)$ where $W \in \mathscr{C}_3$. Thus, $\mathscr{C}_2 = \{\breve{f}(W) \mid W \in \mathscr{C}_3\}$.

COROLLARY 5.1. *If $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ are similar logics, $h \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_2)$ and $g \in \mathrm{Epi}^*(\mathscr{L}_1, \mathscr{L}_3)$ then there exists an l-isomorphism between $\mathscr{L}_2$ and $\mathscr{L}_3$ such that $g = f \circ h$ iff $\underset{h}{\sim} = \underset{g}{\sim}$.*

THEOREM 6. *Let $\mathscr{L}/\sim = \langle \mathscr{A}/\sim, \mathscr{C}/\sim \rangle$ be the quotient of $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ in the sense of Definition* VII. 2 *where $\sim$ is an l-congruence of $\mathscr{L}$. Then, $\mathscr{C}/\sim = \{a/\sim \mid a \in A\}$ and the canonical morphism defined by $\sim$ is a bilogical morphism. In this case we say that $\mathscr{L}/\sim$ is a logical quotient of $\mathscr{L}$.*

Proof. By Corollary 1.1 and Theorem 4.

THEOREM 7. *If h is a bi-logical morphism of $\mathscr{L}_1$ to $\mathscr{L}_2$ and k is the canonical morphism defined by $\underset{h}{\sim}$, the kernel of h, then there exists an l-isomorphism $f \in \mathrm{Hom}(\mathscr{L}_1/\underset{h}{\sim}, \mathscr{L}_2)$ such that $h = f \circ k$.*

Proof. By Theorem 6, k is a bi-logical morphism. Obviously, the kernels of h and k are the same. Hence, by Corollary 5.1 the required l-isomorpism exists.

Theorem 7 is the Homomorphism Theorem for bi-logical morphisms and logical quotients. Two bi-logical morphisms are called *equivalent* whenever they have the same kernels. Then, Theorem 7 shows that there is a one-one correspondence between the equivalence classes of bi-logical morphisms of $\mathscr{L}$ and $\Theta_{\mathscr{L}}$. We will now discuss some properties of $\Theta_{\mathscr{L}}$.

IX. The structure of $\Theta_{\mathscr{L}}$

THEOREM 1. *For any logic $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ the logical congruences of $\mathscr{L}$ exist, $\Theta_{\mathscr{L}} \neq \emptyset$.*

Proof. $I_{\mathscr{A}}$, the identity congruence of $\mathscr{A}$, is in $\Theta_{\mathscr{L}}$.

THEOREM 2. *If $\underset{i}{\sim} \in \Theta_{\mathscr{L}}$ for $i \in \mathscr{I} \neq \emptyset$ and $\sim = \bigcap_i \underset{i}{\sim}$ then $\sim \in \Theta_{\mathscr{L}}$.*

Proof. The relation $\sim$ is a congruence of $\mathscr{A}$. If $a \in X \in \mathscr{C}$ and $a \underset{i}{\sim} b$ for each $i \in \mathscr{J}$ then $b \in X$.

THEOREM 3. (Robert Quackenbush). *$\Theta_{\mathscr{L}}$ is inductive.*

Proof. Let $\Theta = \{\underset{i}{\sim}\} \subseteq \Theta_{\mathscr{L}}$ be a non-empty chain. Then, $\sim = \bigcup_i \underset{i}{\sim}$ is a congruence of $\mathscr{A}$. Suppose that $a \in X \in \mathscr{C}$ and $a \sim b$, that is, $a \underset{i}{\sim} b$ for some i. If follows that $b \in X$.

THEOREM 4. *To every $\sim \in \Theta_{\mathscr{L}}$ there exists maximal $\approx \in \Theta$ such that $\sim \leqslant \approx$.*

Proof. By Theorem II. 3 and Theorem 3. Note that the universal congruence of $\mathscr{A}$ is not a logical congruence of $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$, in general.

DEFINITION 1. A logic $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is said to be *simple* if $\Theta_{\mathscr{L}} = \{I_{\mathscr{A}}\}$.

THEOREM 5. *If $\sim \in \Theta_{\mathscr{L}}$ then $\mathscr{L}/\sim$ is simple iff $\sim$ is maximal in $\Theta_{\mathscr{L}}$.*

Proof. Suppose, $\sim$ is maximal in $\Theta_{\mathscr{L}}$ and there exists $\approx$ in $\mathscr{L}/\sim$ such that $\approx \neq I_{\mathscr{L}/\sim}$. Then, by Theorem VIII.5, there exists $\overset{*}{\sim} \in \Theta_{\mathscr{L}}$ such that $\sim \subset \overset{*}{\sim}$, a contradiction. Suppose, $\sim \in \Theta_{\mathscr{L}}$, $\approx$ is a maximal extension in Θ of $\sim$ and $\sim \neq \approx$. Then, by Theorem VIII.5 again, there exists a l^*-morphism $f \in \mathrm{Hom}^*(\mathscr{L}/\sim, \mathscr{L}/\approx)$ such that $a/\approx\, = f(a/\sim)$ for every $a \in A$. Clearly, f is not an l-isomorphism. Hence, $I_{\mathscr{L}/\sim}$ is not the kernel of f and $\mathscr{L}/\sim$ is not simple.

The next theorem is a generalization of an observation made by J. Porte [1], pp. 49 and 64. If $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ is any logic and $P = \{p_i\}$ a non-empty set of binary polynomials of $\mathscr{A}$, $i \in I \neq \emptyset$. The *analytical relation* of $\mathscr{L}$, determined by P, is the binary relation $\overset{P}{\sim}$ on A defined as follows: $a \overset{P}{\sim} b$ iff $p_i(a, b) \in \mathrm{Cn}(\emptyset) = \bigcap \mathscr{C}$ for all $i \in I$.

LEMMA. *If $\overset{P}{\sim}$ is reflexive then $\sim \subseteq \overset{P}{\sim}$ for each $\sim \in \Theta_{\mathscr{L}}$.*

Proof. Let $\sim \in \Theta_{\mathscr{L}}$ and $a \sim b$. Then, $p_i(a, a) \sim p_i(a, b)$ for all $i \in I$. By reflexivity of $\overset{P}{\sim}$, $p_i(a, a) \in \mathrm{Cn}(\emptyset)$ for all $i \in I$. Hence, $p_i(a, b) \in \mathrm{Cn}(\emptyset)$ for all $i \in I$, that is, $a \overset{P}{\sim} b$.

THEOREM 6. *If the analytic relation $\overset{P}{\sim}$ of $\mathscr{L}$ is in $\Theta_{\mathscr{L}}$ then $\overset{P}{\sim}$ is the greatest member of $\Theta_{\mathscr{L}}$ and, consequently, $\Theta_{\mathscr{L}}$ is a complete lattice.*

DEFINITION 2. Let $\mathscr{A}$ be an algebra, $A = |\mathscr{A}|$, $X \subseteq A$ and $R \subseteq A \times A$. The set X is called *invariant* if the condition

$$a \in X \quad \text{implies} \quad e(a) \in X$$

holds for all $a \in A$ and all $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. The relation R is called *invariant* if the condition

$$a\, R\, b \text{ implies } e(a)\, R\, e(b)$$

holds for all $a, b \in A$ and all $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$.

THEOREM 7. (B. H. Neumann [1]). *If $\mathscr{A}$ is an algebra, freely generated by $X \subseteq A$ and $\sim$ is a congruence of $\mathscr{A}$ then the quotient algebra $\mathscr{A}/\sim$ is freely generated by $X/\sim\, = \{a/\sim \mid a \in X\}$ iff $\sim$ is invariant.*

COROLLARY 7.1. *If $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is a logic such that $\mathscr{A}$ is freely generated by $X \subseteq A$, the least closed set $\bigcap \mathscr{C}$ is invariant and some analytical relation $\sim$ of $\mathscr{L}$ is in $\Theta_{\mathscr{L}}$ then the logical quotient $\mathscr{L}/\sim\, = \langle \mathscr{A}/\sim, \mathscr{C}/\sim \rangle$ is a simple logic and the quotient algebra $\mathscr{A}/\sim$ is freely generated by $X/\sim$.*

THEOREM 8 (J. Porte [1]). *Given an abstract logic $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$, $\Theta_{\mathscr{L}}$ is a principal ideal of $\Theta_{\mathscr{A}}$ = the set of all congruences of the algebra.*

Proof. It is well known that $\Theta_{\mathscr{A}}$ is a complete sublattice of the lattice of all equivalence relations on $|\mathscr{A}|$. We prove that $\operatorname{Sup}\Theta_{\mathscr{L}}$ is in $\Theta_{\mathscr{L}}$. Denote $\operatorname{Sup}\Theta_{\mathscr{L}}$ as $\approx$. Then, $\approx\, \in \Theta_{\mathscr{A}}$ and, $a \approx b$ iff there exist elements $a = x_0, x_1, \ldots, x_n = b$ and congruences $\underset{1}{\sim}, \ldots, \underset{n}{\sim}$ in $\Theta_{\mathscr{L}}$ such that $x_{i-1} \underset{i}{\sim} x_i$ for $i = 1, \ldots, n$. Since, $\mathrm{Cn}(x_{i-1}) = \mathrm{Cn}(x_i)$ for $i = 1, \ldots, n$ it follows that $\mathrm{Cn}(a) = \mathrm{Cn}(b)$, i.e., $\approx$ is in $\Theta_{\mathscr{L}}$. To complete the proof we observe that if $\underset{1}{\sim} \in \Theta_{\mathscr{A}}$, $\underset{2}{\sim} \in \Theta_{\mathscr{L}}$ and $\underset{1}{\sim} \leqslant \underset{2}{\sim}$ then $\underset{1}{\sim} \in \Theta_{\mathscr{L}}$.

X. Logical matrices

DEFINITION 1. Let $\mathscr{A}$ be an abstract algebra of similarity type τ and $\mathscr{F}$ a non-empty family of subsets of $A = |\mathscr{A}|$ then the pair $\mathscr{M} = \langle \mathscr{A}, \mathscr{F} \rangle$ is called a *generalized logical matrix* of type τ (Wójcicki [1]). We call $\mathscr{M}$ *proper* if $\emptyset \neq T \neq A$ for some T in $\mathscr{F}$. If, in particular, $\mathscr{F}$ contains single set, $\mathscr{F} = \{T\}$, then $\mathscr{M}$ is simply called *logical matrix* of type τ (Łukasiewicz and Tarski, [1]; see Tarski [1], Chap. IV, pp. 38-59).

Abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$ and quasi-logics $\langle \mathscr{A}, \overline{\mathscr{C}} \rangle$ are the most important examples of generalized matrices.

DEFINITION 2. If $\mathscr{M} = \langle \mathscr{A}, \mathscr{F} \rangle$ and $\mathscr{C} = [\mathscr{F}]$ then $[\mathscr{M}] = \langle \mathscr{A}, \mathscr{C} \rangle$ is the abstract logic *associated* with the generalized matrix $\mathscr{M}$. Obviously, $[\mathscr{M}]$ is a proper logic iff $\mathscr{M}$ is a proper generalized matrix. Two generalized matrices $\mathscr{M}_1, \mathscr{M}_2$ are said to be *l-equal*, if $[\mathscr{M}_1] = [\mathscr{M}_2]$.

One will later see that in logical application, *l*-equal generalized matrices have the same properties. Hence, in these applications, we can always assume that the generalized matrices under discussion are abstract logics (or quasi-logics). On the other hand, generalized matrices may, in a sense, be reduced to logical matrices. If $\mathscr{A}$ is an abstract algebra then each class of logical matrices, $\mathscr{K} = \{\mathscr{M}_i\}$ where $\mathscr{M}_i = \langle \mathscr{A}, T_i \rangle$, is called a *bundle* of logical matrices on $\mathscr{A}$. Clearly, there is a one-one correspondence between generalized matrices $\langle \mathscr{A}, \mathscr{F} \rangle$ and bundles of matrices on $\mathscr{A}$. Thus, instead of generalized matrices, one can always consider bundles of logical matrices.

If $\mathscr{M} = \langle \mathscr{A}, T\rangle$ is a logical matrix then the associated abstract logic is *elementary* one, that is, $\langle \mathscr{A}, \mathscr{C}\rangle$ where $\mathscr{C} = \{T, A\}$ and $T \subseteq A$. Conversely, to every elementary logic $\langle \mathscr{A}, \{T, A\}\rangle$ there is the unique corresponding logical matrix $\langle \mathscr{A}, T\rangle$. Clearly, this correspondence is one-one. In view of this fact, many notions and operations concerned with abstract logics may be applied to logical matrices.

DEFINITION 3. Logical congruences, logical quotients, l^*-morphisms and bi-logical morphisms, if restricted to the class of elementary logics of type τ, are called correspondingly *m-congruences, m-quotients, m-morphisms* $(\mathrm{Hom}^*(\mathscr{M}_1, \mathscr{M}_2))$ and *m-epimorphisms* $(\mathrm{Epi}^*(\mathscr{M}_1, \mathscr{M}_2))$ of logical matrices of type τ. The notion of m-morphism has been introduced by Łoś [1].

The following theorems are immediate corollaries to Chapter VIII.

THEOREM 1. *The kernel of an m-morphism is a m-congruence.*

THEOREM 2. *The class of logical matrices of a given type form a category with morphisms being m-morphisms.*

THEOREM 3. *If* $h \in \mathrm{Epi}^*(\mathscr{M}_1, \mathscr{M}_2)$ *and* k *is the canonical morphism defined by* $\underset{h}{\sim}$, *the kernel of* h, *then there exists an m-isomorphism* $f \in \mathrm{Hom}^*(\mathscr{M}_1, \mathscr{M}_2)$ *such that* $h = f \circ k$.

Remark. Let $\{\mathscr{M}_i\}$ be a family of similar logical matrices and $\{\mathscr{L}_i\}$ be the corresponding family of elementary logics. The product matrix of all $\mathscr{M}_i$ is a logical matrix again, obtained in by the usual product construction applied to logical matrices considered as so called *relational systems*. On the other hand, the product logic of all $\mathscr{L}_i$ is not an elementary one, in general.

XI. Generating logics by matrices

DEFINITION 1. Given a logical matrix $\mathscr{M} = \langle \mathscr{A}, T\rangle$ and an algebra $\mathscr{A}_0$ similar to $\mathscr{A}$, let $\mathscr{D}(\mathscr{M})$ be the subset of $A_0 = |\mathscr{A}_0|$, defined as follows:

$$D(\mathscr{M}) = \bigcap \{\check{h}(T) \mid h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})\}.$$

Subsequently, define two mappings of the power set $\mathscr{P}(A_0)$ into itself, $\mathrm{Cn}_{\mathscr{M}}$ and $C_{\mathscr{M}}$:

(1) $a \in \mathrm{Cn}_{\mathscr{M}}(X)$ iff $X \subseteq D(\mathscr{M})$ implies $a \in D(\mathscr{M})$,

(2) $a \in C_{\mathscr{M}}(X)$ iff for all $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$, $X \subseteq \check{h}(T)$ implies $a \in \check{h}(T)$.

THEOREM 1. $\mathrm{Cn}_{\mathscr{M}}$ *and* $C_{\mathscr{M}}$ *are closure operators on* A_0, *called consequence operations generated by* $\mathscr{M}$. *Clearly,* $C_{\mathscr{M}} \leqslant \mathrm{Cn}_{\mathscr{M}}$.

Proof. Łoś and Suszko [1], Suszko [3].

Generalized matrices and even classes of matrices also generate consequence operations.

DEFINITION 2. Let $\mathcal{K} = \{\mathcal{M}_i\}$ be a class of similar matrices of type τ and $\mathcal{A}_0$ an algebra of type τ. Then, we have two closure operators on A_0:

$$\mathcal{K}-C = \operatorname*{Inf}_i C_{\mathcal{M}_i}, \qquad \mathcal{K}-\mathrm{Cn} = \operatorname*{Inf}_i \mathrm{Cn}_{\mathcal{M}_i}.$$

If $\mathcal{K}$ is the bundle of matrices, corresponding to the generalized matrix $\mathcal{M}$ then we set:

$$\mathrm{Cn}_{\mathcal{M}} = \mathcal{K}-\mathrm{Cn} \quad \text{and} \quad C_{\mathcal{M}} = \mathcal{K}-C.$$

Obviously, we get Definition 1, if $\mathcal{M}$ is a logical matrix.

THEOREM 2. *If* $\mathcal{L} = [\mathcal{M}]$ *is the abstract logic associated with the generalized matrix* $\mathcal{M} = \langle \mathcal{A}, \mathcal{F}\rangle$ *then* $C_{\mathcal{M}} = C_{\mathcal{L}}$ *and* $\mathrm{Cn}_{\mathcal{M}} = \mathrm{Cn}_{\mathcal{L}}$, *for every algebra* $\mathcal{A}_0$ *similar to* $\mathcal{A}$.

Proof. It is obvious that $C_{\mathcal{L}} \leqslant C_{\mathcal{M}}$ and $\mathrm{Cn}_{\mathcal{L}} \leqslant \mathrm{Cn}_{\mathcal{M}}$. Suppose, $a \notin C(X)$. Then, there exists $W \in \mathcal{C}$ and $h \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A})$ such that $h(X) \subseteq W$ and $h(a) \notin W$. But, $W = \bigcap g$ for some $W \subseteq \mathcal{F}$. Hence, there exists $V \in \mathcal{F}$ such that $h(X) \subseteq V$ and $h(a) \notin V$, i.e., $a \notin C_{\mathcal{M}}(X)$. Suppose now that $a \notin \mathrm{Cn}_{\mathcal{L}}$. Then, $h(X) \subseteq W$ and $h(a) \notin W$ for some $W \in \mathcal{C}$ and $h \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A})$. Again, there exists $V \in \mathcal{F}$ such that $h(X) \subseteq V$ and $h(a) \notin V$, i.e., $a \notin \mathrm{Cn}_{\mathcal{M}}$.

COROLLARY 2.1. *If* $\mathcal{M}_1, \mathcal{M}_2$ *are l-equal generalized matrices of some type then for every algebra* $\mathcal{A}_0$ *of the same type, we have:*

$$\mathrm{Cn}_{\mathcal{M}_1} = \mathrm{Cn}_{\mathcal{M}_2} \quad \textit{and} \quad C_{\mathcal{M}_1} = C_{\mathcal{M}_2}.$$

LEMMA 1. (Birkhoff [1]). *Suppose,* $\mathcal{A}_0$ *is a free algebra in a class* $\mathcal{K}_0$ *of algebras, the algebras* $\mathcal{A}_1, \mathcal{A}_2$ *are in* $\mathcal{K}_0$ *and* $f \in \mathrm{Epi}(\mathcal{A}_1, \mathcal{A}_2)$. *Then, to every* $g \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_2)$ *there exists* $h \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$ *such that* $g = f \circ h$.

THEOREM 3. *Let* $\mathcal{A}_0$ *be free in* $\mathcal{K}_0$ *and* $\mathcal{A}_1, \mathcal{A}_2$ *be in* $\mathcal{K}_0$. *If* $\mathcal{M}_i = \langle \mathcal{A}_i, T_i\rangle$ *for* $i = 1, 2$ *and there is an m-epimorphism* f *of* $\mathcal{M}_1$ *to* $\mathcal{M}_2$ *then* $C_{\mathcal{M}_1} = C_{\mathcal{M}_2}$ *and* $\mathrm{Cn}_{\mathcal{M}_1} = \mathrm{Cn}_{\mathcal{M}_2}$.

Proof. If $a \notin C_{\mathcal{M}_1}(X)$ then $h(X) \subseteq T_1$ and $h(a) \notin T_1$ for some $h \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$. Let $g = f \circ h$. Then, $g \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_2)$, $g(X) = f(h(X)) \subseteq T_2$ and $g(a) = f(h(a)) \notin T_2$. Hence, $a \notin C_{\mathcal{M}_2}$. Therefore, $C_{\mathcal{M}_2} \leqslant C_{\mathcal{M}_1}$. If, on the other hand, $a \notin C_{\mathcal{M}_2}$ then there exists $g \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_2)$ such that $g(X) \subseteq T_2$ and $g(a) \notin T_2$. Then, by the Lemma, there exists a $h \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$ such that $g = f \circ h$. Since, $g(X) = f(h(X)) \subseteq T_2$ hence $h(X) \subseteq T_1$. Similarly, $h(a) \notin T_1$ because $g(a) = f(h(a)) \notin T_2$. Thus, $a \notin C_1$. Hence, $C_{\mathcal{M}_1} \leqslant C_{\mathcal{M}_2}$.

For the second part of the theorem, suppose that $a \notin \mathrm{Cn}_{\mathcal{M}_1}(X)$, that is, $h(X) \subseteq T_1$ for all $h \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$ but $h_0(a) \notin T_1$ for some $h_0 \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$. Then, $g_0 = f \circ h_0 \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$ and $g_0(a) = f(h_0(a)) \notin T_2$.

If $g \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_2)$ then, by the lemma, $g = f \circ h$ for some $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_1)$ and, by hypothesis, $g(X) = f(h(X)) \subseteq T_2$. Thus, $a \notin \mathrm{Cn}_{\mathscr{M}_2}(X)$. Hence $\mathrm{Cn}_{\mathscr{M}_2} \leqslant \mathrm{Cn}_{\mathscr{M}_1}$. Suppose, conversely, that $a \notin \mathrm{Cn}_{\mathscr{M}_2}(X)$. We have $g(X) \subseteq T_2$ for all $g \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_2)$ and $g_0(a) \notin T_1$ for some $g_0 \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_2)$. Take any $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_1)$. Then, $h(X) \subseteq T_1$ since $(f \circ h) \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_2)$ and $f(h(X)) \subseteq T_2$. On the other hand, there exists, by the Lemma, an $h_0 \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_1)$ such that $g_0 = f \circ h_0$. It follows that $h_0(a) \notin T_2$. So, $a \notin \mathrm{Cn}_{\mathscr{M}_1}(X)$. Therefore, $\mathrm{Cn}_{\mathscr{M}_1} \leqslant \mathrm{Cn}_{\mathscr{M}_2}$.

XII. Structurality and invariance

DEFINITION 1. An abstract logic $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ is said to be *structural* if $e(\mathrm{Cn}(X)) \subseteq \mathrm{Cn}(e(X))$ for all $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$ and all $X \subseteq A$.

THEOREM 1. *$\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ is structural iff $\mathrm{Hom}(\mathscr{A}, \mathscr{A}) = \mathrm{Hom}(\mathscr{L}, \mathscr{L})$, that is, every endomorphism is continuous.*

Proof. Theorem III.1.

COROLLARY 1.1. *If all $\mathscr{L}_i = \langle \mathscr{A}, \mathrm{Cn}_i \rangle$ are structural then $\mathscr{L} = \langle \mathscr{A}, \operatorname{Inf}_i \mathrm{Cn}_i \rangle$ also is.*

THEOREM 2. *Let $\mathscr{L}_i = \langle \mathscr{A}_i, \mathscr{C}_i \rangle$ for $i = 1, 2$. If $\mathscr{L}_1$ is projectively generated from $\mathscr{L}_2$ by $\mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2)$ then $\mathscr{L}_2$ is structural.*

Proof. $\mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2) = \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. Hence, $(h \circ e) \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$ whenever $h \in \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$ and $e \in \mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_1)$. Therefore, by Theorem VI.3, $\mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_1) = \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_1)$.

THEOREM 3. *If $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$ is any (generalized) logical matrix of type τ and $\mathscr{A}_0$ is an algebra of type τ then the logic $\mathscr{L}_0 = \langle \mathscr{A}_0, C_{\mathscr{M}} \rangle$ is structural.*

Proof. Let $\mathscr{L} = [\mathscr{M}]$. Then, by Theorem XI.2 and Theorem VI.1, $\mathscr{L}_0$ is projectively generated from $\mathscr{L}$ by $\mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$. Hence, by Theorem 2, $\mathscr{L}_0$ is structural.

THEOREM 4. *Let $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn} \rangle$ and $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$. Then, $\mathrm{Cn} = C_{\mathscr{M}}$ iff $\mathscr{L}_0$ is projectively generated from $[\mathscr{M}]$ by $\mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$.*

Proof. Theorems XI.2 and VI.1.

THEOREM 5. *Let $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn} \rangle$ and $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$. If $\mathrm{Cn} = C_{\mathscr{M}}$ then $\langle B, R \rangle$, where $B = \{\langle h, T_i \rangle \mid h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}), \text{ all } i\}$ and $R(h, T_i) = \check{h}(T_i)$ is a Galois representation for $\mathscr{L}_0$.*

Proof. $\overset{*}{R}(X) = \{\langle h, T_i \rangle \mid X \subseteq \check{h}(T_i)\}$. Suppose, $a \notin \overset{+}{R}(\overset{*}{R}(X))$. Then, there exists $\langle h, T_i \rangle$ such that $a \notin R(h, T_i) = \check{h}(T_i)$ and $\langle h, T_i \rangle \in \overset{*}{R}(X)$, i.e., $X \subseteq R(h, T_i) = \check{h}(T_i)$. Therefore, $a \notin C_{\mathscr{M}}(X)$. Suppose, $a \notin C_{\mathscr{M}}(X)$.

Then there exists an $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$ such that $X \subseteq \breve{h}(T_i)$ and $a \notin \breve{h}(T_i)$ for some i. Hence, $a \notin \overset{+}{R}\big(\overset{*}{R}(X)\big)$. Thus $C_{\mathscr{M}} = \mathrm{Cn}^R_{A_0}$.

Corollary 5.1. *Let* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn} \rangle$ *and* $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$. *Define* B *and* R *as in Theorem* 5. *If* $\mathrm{Cn} = C_{\mathscr{M}}$ *then* $\mathscr{S} = \langle B, \mathrm{Cn}^R_B \rangle$ *is a dual space for* $\mathscr{L}_0$.

We now turn to invariant logics.

Definition 2. Let $\mathscr{A}$ be an abstract algebra and $\Phi \subseteq \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Then, $\mathrm{Sb}_\Phi(X) = \bigcup \{e(X) \mid e \in \Phi\}$. If $\Phi = \mathrm{Hom}(\mathscr{A}, \mathscr{A})$ then Sb_Φ will be denoted simply as Sb.

Corollary. *If* Φ *is closed under composition and the identity map on* A *is in* Φ *then* Sb_Φ *is a consequence operator on* A. *Moreover,* $\mathrm{Sb}_\Phi(\emptyset) = \emptyset$ *and* $\mathrm{Sb}_\Phi(X \cup Y) = \mathrm{Sb}_\Phi(X) \cup \mathrm{Sb}_\Phi(Y)$.

Definition 3. An abstract logic $\langle \mathscr{A}, \mathrm{Cn} \rangle$ is said to be *invariant* (with respect to endomorphisms) if $\mathrm{Sb} \leqslant \mathrm{Cn}$.

Corollary. *If all* $\mathscr{L}_i = \langle \mathscr{A}, \mathrm{Cn}_i \rangle$ *are invariant then* $\mathscr{L} = \langle \mathscr{A}, \operatorname{Inf}_i \mathrm{Cn}_i \rangle$ *is also.*

Theorem 6. *If* $\langle \mathscr{A}, \mathrm{Cn} \rangle$ *is a structural logic then* $\mathrm{Sb} \circ \mathrm{Cn} \leqslant \mathrm{Cn} \circ \mathrm{Sb}$.

Proof. If $a \in \mathrm{Sb}\big(\mathrm{Cn}(X)\big)$ then $a \in e\big(\mathrm{Cn}(X)\big)$ for some $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Hence, by structurality, $a \in \mathrm{Cn}\big(e(X)\big) \subseteq \mathrm{Cn}\big(\mathrm{Sb}(X)\big)$.

Theorem 7. *If the logic* $\langle \mathscr{A}, \mathrm{Cn} \rangle$ *is structural then* $\mathrm{Cn} \circ \mathrm{Sb} = \mathrm{Sup}(\mathrm{Cn}, \mathrm{Sb})$ *and, hence,* $\langle \mathscr{A}, \mathrm{Cn} \circ \mathrm{Sb} \rangle$ *is an invariant logic.*

Proof. Theorem 6 and Theorem I.8, I.9.

Theorem 8. *If* $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$ *is any (generalized) logical matrix of type* τ *and* $\mathscr{A}_0$ *is an algebra of type* τ *then the logic* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn}_{\mathscr{M}} \rangle$ *is invariant.*

Proof. Suppose, $a \in X \subseteq \breve{h}(T_i)$ for all $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$ and $e \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A}_0)$. Then, $g\big(e(a)\big) \in T_i$, i.e., $e(a) \in \breve{g}(T_i)$ for all $g \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$. Thus, $\mathrm{Sb}(X) \subseteq \mathrm{Cn}_{\mathscr{M}}(X)$.

Theorem 9. *If* $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ *is structural and* $\mathscr{M} = \langle \mathscr{A}, \mathscr{C}(\mathrm{Cn}) \rangle$ *then* $\mathrm{Cn}_{\mathscr{M}} = \mathrm{Cn} \circ \mathrm{Sb}$.

Proof. Suppose, $a \notin \mathrm{Cn}_{\mathscr{M}}(X)$. Then, there exists $Y \in \mathscr{C}(\mathrm{Cn})$ and $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$ such that $e(a) \notin Y$ and $f(X) \subseteq Y$ for all $f \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Hence, by structurality of $\mathscr{L}$, $a \notin \mathrm{Cn}(X)$. Thus, $\mathrm{Cn} \leqslant \mathrm{Cn}_{\mathscr{M}}$. Suppose that $Y \in \mathscr{C}(\mathrm{Cn})$, $a \in \mathrm{Sb}(X)$ and $f(X) \subseteq Y$ for all $f \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Then, $a = e(b)$ for some $b \in X$ and $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Then, $a = e(b)$ for some $b \in X$ and $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Therefore, $f(a) = f\big(e(b)\big) \in Y$ for all $f \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Hence, $\mathrm{Sb} \leqslant \mathrm{Cn}_{\mathscr{M}}$. Combining these two results we have $\mathrm{Sup}(\mathrm{Cn}, \mathrm{Sb}) \leqslant \mathrm{Cn}_{\mathscr{M}}$. That is, $\mathrm{Cn} \circ \mathrm{Sb} \leqslant \mathrm{Cn}_{\mathscr{M}}$, by Theorem 7. Suppose now, $a \notin \mathrm{Cn}\big(\mathrm{Sb}(X)\big)$. Clearly, $\mathrm{Cn}\big(\mathrm{Sb}(X)\big) \in \mathscr{C}(\mathrm{Cn})$ and $\mathrm{Sb}(X) \subseteq \mathrm{Cn}\big(\mathrm{Sb}(X)\big)$. But, by hypothesis, it is not true that $\mathrm{Sb}(a) \subseteq \mathrm{Cn}(\mathrm{Sb}(X))$. Thus, $a \notin \mathrm{Cn}_{\mathscr{M}}$. Hence, $\mathrm{Cn} \leqslant \mathrm{Cn} \circ \mathrm{Sb}$.

THEOREM 10. *Let* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn} \rangle$ *and* $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$. *If* $\mathrm{Cn} = \mathrm{Cn}_{\mathscr{M}}$ *then* $\langle \mathscr{B}, R \rangle$ *is a Galois representation for* $\mathscr{L}_0$, *where* $\mathscr{B} = \{T_i\}$ *and* $R(T_i) = \bigcap \{\breve{h}(T_i) \mid h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})\}$.

Proof. For all $X \subseteq A_0$, $T_i \in \overset{*}{R}(X)$ iff $X \subseteq \breve{h}(T_i)$ for all $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$. On the other hand, if $a \in A_0$ and $\mathscr{F} \subseteq \mathscr{B}$ then $a \in \overset{+}{R}(\mathscr{F})$ iff $a \in \breve{h}(T)$ for all $T \in \mathscr{F}$ and all $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$. Suppose, $a \notin \overset{+}{R}(\overset{*}{R}(X))$. Then, there exists $T_i \in \overset{*}{R}(X)$ and $g \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$ such that $a \notin \breve{g}(T_i)$. But since $T_i \in \overset{*}{R}(X)$, hence $X \subseteq \breve{h}(T_i)$ for all $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$. Therefore, $a \notin \mathrm{Cn}_{\mathscr{M}}(X)$. Suppose, $a \notin \mathrm{Cn}_{\mathscr{M}}(X)$. Then, there exists T_i and $g \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$ such that $a \notin \breve{g}(T_i)$ but $X \subseteq \breve{h}(T_i)$ for all $h \in \mathrm{Hom}(\mathscr{A}_0, \mathscr{A})$. Hence, $T_i \in \overset{*}{R}(X)$ and $a \notin \overset{+}{R}(\overset{*}{R}(X))$. Thus, $\mathrm{Cn}_{\mathscr{M}} = \mathrm{Cn}^{R}_{A_0} = \overset{+}{R} \circ \overset{*}{R}$.

COROLLARY 10.1. *Let* $\mathscr{L}_0 = \langle \mathscr{A}_0, \mathrm{Cn} \rangle$ *and* $\mathscr{M} = \langle \mathscr{A}, \{T_i\} \rangle$. *Define* $\mathscr{B}$ *and* R *as in Theorem* 10. *If* $\mathrm{Cn} = \mathrm{Cn}_{\mathscr{M}}$ *then* $\mathscr{S} = \langle \mathscr{B}, \mathrm{Cn}^{R}_{\mathscr{B}} \rangle$ *is a dual spaoe for* $\mathscr{L}_0$.

XIII. Adequacy and completeness

The distinction between structural and invariant logics and corresponding operations

$$\mathscr{M} \to C_{\mathscr{M}} \quad \text{and} \quad \mathscr{M} \to \mathrm{Cn}_{\mathscr{M}}$$

has important application in mathematical logic. First, structurality and invariance will appear incompatible, in a sense. Subsequently, we will show that the so called completeness problem of mathematical logic, for structural or invariant consequence operations has a positive solution as an immediate application of two adequacy theorems for abstract logics.

DEFINITION 1. An abstract logic $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is said to be *free logic* of type τ if $\mathscr{A}$ is absolutely free algebra of the similarity type τ.

THEOREM 1. *Let* $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ *be a free logic with at least two free generators of* $\mathscr{A}$. *If* $\mathscr{L}$ *is structural and invariant then* Cn *is not proper.*

Proof. By structurality, for every $e \in \mathrm{Hom}(\mathscr{A}, \mathscr{A})$, $e(a) \in \mathrm{Cn}(e(b))$ whenever $a \in \mathrm{Cn}(b)$. On the other hand, by invariance, $x \in \mathrm{Cn}(y)$ for any free generators x, y. Therefore, if $x \neq y$ then $a \in \mathrm{Cn}(b)$ for all $a, b \in A$.

DEFINITION 2. Let $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ be a free logic. A $\mathscr{L}$*-interpretation* is a matrix $\mathscr{M} = \langle \mathscr{B}, T \rangle$ where $\mathscr{B}$ is similar to $\mathscr{A}$. $\mathrm{Hom}(\mathscr{A}, \mathscr{B})$ is the set of *$\mathcal{M}$-valuations* of $\mathcal{A}$. A valuation h *satisfies* a set $X \subseteq A$ if $h(X) \subseteq T$. The set of all $a \in A$ such that $h(a) \in T$ for every $\mathcal{M}$-valuation h is denoted by $\mathrm{TR}(\mathcal{M})$, and read the set of formulas of $\mathcal{L}$, true in $\mathcal{M}$.

DEFINITION 3. Let $\mathcal{L} = \langle \mathcal{A}, \mathrm{Cn} \rangle$ be a free logic and let $\mathcal{K}$ be a family of $\mathcal{L}$-interpretations. The logic $\mathcal{L}$ is said to be *$\mathcal{K}^{(s)}$-complete* if for all $a \in A$ and all $X \subseteq A$, $a \in \mathrm{Cn}(X)$ iff for every $\mathcal{M} \in \mathcal{K}$, any $\mathcal{M}$-valuation h satisfies a whenever h satisfies X. The logic $\mathcal{L}$ is said to be *$\mathcal{K}^{(i)}$-complete* if for all $a \in A$ and all $X \subseteq A$, $a \in \mathrm{Cn}(X)$ iff $X \subseteq \mathrm{TR}(\mathcal{M})$ implies $a \in \mathrm{TR}(\mathcal{M})$.

COROLLARY 1. *If $\mathcal{L} = \langle \mathcal{A}, \mathrm{Cn} \rangle$ is a free logic and $\mathcal{K}$ is a family of $\mathcal{L}$-interpretations then $\mathcal{L}$ is $\mathcal{K}^{(s)}$-complete iff* $\mathrm{Cn} = \mathcal{K} - C$ *and $\mathcal{L}$ is $\mathcal{K}^{(i)}$-complete iff* $\mathrm{Cn} = \mathcal{K} - \mathrm{Cn}$.

COROLLARY 2. *If $\mathcal{L}$ is $\mathcal{K}^{(s)}$-complete then $\mathcal{L}$ is structural. If $\mathcal{L}$ is $\mathcal{K}^{(i)}$-complete then $\mathcal{L}$ is invariant.*

COROLLARY 3. *If $\mathcal{L}$ is a proper free logic with at least two free generators then it is impossible to find familiar of $\mathcal{L}$-interpretations, $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{L}$ is $\mathcal{K}_1^{(s)}$-complete and $\mathcal{K}_2^{(i)}$-complete.*

The completeness problem for a free logic $\mathcal{L}$, structural or invariant, consists in the question whether there exists a family $\mathcal{K}$ of $\mathcal{L}$-interpretations such that $\mathcal{L}$ is $\mathcal{K}^{(s)}$-complete or $\mathcal{K}^{(i)}$-complete, respectively. Two adequacy theorems for abstract logics (Theorems 3 and 4) imply that the completeness problem for structural and invariant free logics, always has a positive solution. Moreover, the required class of interpretations may always be a bundle of logical matrices, i.e. a single generalized matrix. We turn now to abstract logics.

DEFINITION 4. Let $\mathcal{L} = \langle \mathcal{A}, \mathrm{Cn} \rangle$ be an abstract logic of type τ and $\mathcal{M}$ be a generalized matrix of type τ. We say that $\mathcal{M}$ is *s-adequate* or *i-adequate* for $\mathcal{L}$ if, correspondingly, $\mathrm{Cn} = C_{\mathcal{M}}$ or $\mathrm{Cn} = \mathrm{Cn}_{\mathcal{M}}$.

DEFINITION 5. Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$. Then, $\mathcal{M}_{\mathcal{L}} =$ the *canonical* (generalized) $\mathcal{L}$-matrix and $\mathcal{K}_{\mathcal{L}} = \{\langle \mathcal{A}, T \rangle \mid T \in \overline{\mathcal{C}}\}$ is the *canonical bundle* of $\mathcal{L}$-matrices.

THEOREM 1. *A logic $\mathcal{L} = \langle \mathcal{A}, \mathrm{Cn} \rangle$ is structural iff $\mathcal{L}$ is projectively generated from $\mathcal{L}$ by* $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$.

Proof. Suppose, $\mathcal{L}$ is structural and $g \in \mathrm{Hom}(\mathcal{A}_0, \mathcal{A})$ where $\mathcal{L}_0 = \langle \mathcal{A}_0, \mathrm{Cn}_0 \rangle$ is some other similar logic. Since, $\mathrm{Hom}(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}(\mathcal{L}, \mathcal{L})$, it follows that $g \in \mathrm{Hom}(\mathcal{L}_0, \mathcal{L})$ iff $(h \circ g) \in \mathrm{Hom}(\mathcal{L}_0, \mathcal{L})$ for all $h \in \mathrm{Hom}(\mathcal{A}, \mathcal{A})$. Hence, by Theorem VI.4, $\mathcal{L}$ is projectively generated from $\mathcal{L}$ by $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$. The other way around, by Theorem XII.2.

THEOREM 2. *A generalized matrix $\mathcal{M} = \langle \mathcal{B}, \mathcal{F} \rangle$ is s-adequate for the logic $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ iff $\mathcal{L}$ is projectively generated by* $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ *from the logic $[\mathcal{M}]$ associated with $\mathcal{M}$.*

Proof. Definition XI.2, Theorem XI.2 and Theorem VI.1.

THEOREM 3. Adequacy theorem. *If* $\mathscr{L}$ *is an abstract logic then the canonical (generalized)* $\mathscr{L}$*-matrix* $\mathscr{M}_{\mathscr{L}}$ *is s-adequate for* $\mathscr{L}$ *iff* $\mathscr{L}$ *is structural.*

Proof. Theorem 1 and Theorem 2.

COROLLARY (Wójcicki [1]). *If* $\mathscr{L}$ *is a free logic then* $\mathscr{L}$ *is* $\mathscr{K}^{(s)}$*-complete iff* $\mathscr{L}$ *is structural.*

We have an analogous theorem for i-adequacy. First, an obvious lemma.

LEMMA. *A logic* $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ *is invariant iff every* Cn*-closed set is invariant* (Definition IX.2).

THEOREM 4. Adequacy theorem (in collaboration with S. L. Bloom). *If* $\mathscr{L}$ *is an abstract logic then the canonical (generalized)* $\mathscr{L}$*-matrix* $\mathscr{M}_{\mathscr{L}}$ *is i-adequate iff* $\mathscr{L}$ *is invariant.*

Proof. If $\mathscr{M}_{\mathscr{L}}$ is i-adequate for $\mathscr{L}$ then, by Theorem XII. 8, the logic $\mathscr{L}$ is invariant. Suppose, $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ is invariant. If $a \in \mathrm{Cn}(X)$ and Y is any Cn-closed set such that $\mathrm{Sb}(X) \subseteq Y$ then, by the Lemma, $\mathrm{Sb}(a) \subseteq \mathrm{Cn}(X) \subseteq \mathrm{Cn}(\mathrm{Sb}(X)) \subseteq \mathrm{Cn}(Y) \subseteq Y$. Thus, $\mathrm{Cn} \leqslant \mathrm{Cn}_{\mathscr{L}} \leqslant \mathrm{Cn}_{\mathscr{M}_{\mathscr{L}}}$. If $a \notin \mathrm{Cn}(X)$ then $\mathrm{Cn}(X)$ is a consistent Cn-closed set. By the Lemma again, $\mathrm{Cn}(X)$ is invariant, that is, $\mathrm{Sb}(\mathrm{Cn}(X)) \subseteq \mathrm{Cn}(X)$. Hence, $\mathrm{Sb}(X) \subseteq \mathrm{Cn}(X)$. However, the inclusion $\mathrm{Sb}(a) \subseteq \mathrm{Cn}(X)$ does not hold. Thus $\mathrm{Cn}_{\mathscr{M}_{\mathscr{L}}} \leqslant \mathrm{Cn}$.

COROLLARY. *If* $\mathscr{L}$ *is a free logic then* $\mathscr{L}$ *is* $\mathscr{K}^{(i)}$*-complete iff* $\mathscr{L}$ *is invariant.*

In the last chapter, we consider some known examples of free structural logics. Free invariant logics will not be investigated. They occur in mathematical logic, as a rule, under the form $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \circ \mathrm{Sb}_{\Phi} \rangle$ where $\langle \mathscr{A}, \mathrm{Cn} \rangle$ is structural and $\Phi \subseteq \mathrm{Hom}(\mathscr{A}, \mathscr{A})$. Hence, the following theorem may be offered for possible applications in mathematical logic.

THEOREM 5. *Let* $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ *be a structural abstract logic and* $\Phi \subseteq \mathrm{Hom}(\mathscr{A}, \mathscr{A})$ *where* Φ *is closed under composition and the identity map on* A *is in* Φ. *Then,* $\mathrm{Cn} \circ \mathrm{Sb}_{\Phi} = \mathrm{Sup}(\mathrm{Cn}, \mathrm{Sb}_{\Phi})$ *and*

$$\mathrm{Cn} \circ \mathrm{Sb}_{\Phi} = \mathrm{Inf}\{\mathrm{Cn}_{\mathscr{M}}^{(\Phi)} \mid \mathscr{M} = \langle \mathscr{L}, T \rangle, T \in \overline{\mathscr{C}}\}$$

where $a \in \mathrm{Cn}_{\mathscr{M}}^{(\Phi)}(X)$ *iff* $h(a) \subseteq T$ *for all* $h \in \Phi$ *whenever* $h(X) \subseteq T$ *for all* $h \in \Phi$.

Proof. Similar as for Theorems XII.6, XII.7, XII.9.

XIV. Some applications to mathematical logic

Most free structural logics studied in mathematical logic have the peculiar property that some analytical relation is a logical congruence.

The following definition and theorem are fundamental in studying such logics.

DEFINITION 1. Let $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ be a structural abstract logic such that $\Theta_{\mathscr{L}}$ is a complete lattic and $\sim$ is the unit of $\Theta_{\mathscr{L}}$. Then the generalized matrix $\mathscr{M}^*_{\mathscr{L}} = \langle \mathscr{A}/\sim, \overline{\mathscr{C}}/\sim \rangle = \langle \mathscr{A}/\sim, \mathscr{C}/\sim - \{A/\sim\} \rangle$ is called the *canonical quotient $\mathscr{L}$-matrix*. Similarly, the bundle of logical matrices which corresponds to $\mathscr{M}^*_{\mathscr{L}}$ is called the *canonical quotient bundle* of $\mathscr{L}$-matrices and denoted by $\mathscr{K}^*_{\mathscr{L}}$.

THEOREM 1. *If $\mathscr{L} = \langle \mathscr{A}, \mathscr{C} \rangle$ is a free structural logic with the set X of free generators of $\mathscr{A}$ and some analytical relation $\sim$ of $\mathscr{L}$ is a logical congruence of $\mathscr{L}$ then:*

(1) *the logical quotient $\mathscr{L}/\sim$ is a simple logic,*

(2) *the quotient algebra $\mathscr{A}/\sim$ is freely generated by $X/\sim$,*

(3) *the canonical quotient $\mathscr{L}$-matrix $\mathscr{M}^*_{\mathscr{L}}$ is adequate for $\mathscr{L}$ and*

(4) *the logic $\mathscr{L}$ is $\mathscr{K}^*_{\mathscr{L}}$-complete.*

Proof. Structurality implies that the set $\bigcap \mathscr{C} = \mathrm{Cn}(\emptyset)$ is invariant. Hence, by Corollary 7.1, $\mathscr{L}/\sim$ is a simple logic and $\mathscr{A}/\sim$ is freely generated by $X/\sim$. By Theorem XIII.3, $\mathscr{M}_{\mathscr{L}}$ is adequate for $\mathscr{L}$. But, $[\mathscr{M}_{\mathscr{L}}] = \mathscr{L}$, hence, by Theorem XI.2, $\mathscr{L}$ is adequate for $\mathscr{L}$. Therefore, by Theorem VIII, Definition XI.2 and Theorem XI.3, $\mathscr{L}/\sim$ is adequate for $\mathscr{L}$. Since, $\mathscr{L}/\sim = [\mathscr{M}^*_{\mathscr{L}}]$, hence, $\mathscr{M}^*_{\mathscr{L}}$ is adequate for $\mathscr{L}$, by Theorem XI.2 again. It follows, by Definition XIII.4, that $\mathscr{L}$ is $\mathscr{K}^*_{\mathscr{L}}$-complete.

We shall now consider several examples of structural logics which appear in Rasiowa and Sikorski [1].

Let $\mathscr{R}$ be the class of all abstract algebra $\mathscr{B} = \langle B, f_1, f_2, f_3 \rangle$ with three binary operations f_1, f_2, f_3. Let $\mathscr{F}_p = \langle F_p, \wedge, \vee, \Rightarrow \rangle$ be free in $\mathscr{R}$, with the set V of free generators. The elements of V are called *sentential variables* and $\mathscr{F}_p$ is called the *algebra of positive formulas*.

Let A_p be the set of formulas of the form:

(τ_1) $a \Rightarrow (b \Rightarrow a)$,
(τ_2) $\big(a \Rightarrow (b \Rightarrow c)\big) \Rightarrow \big((a \Rightarrow b) \Rightarrow (a \Rightarrow c)\big)$,
(τ_3) $(a \wedge b) \Rightarrow a$,
(τ_4) $(a \wedge b) \Rightarrow b$,
(τ_5) $(c \Rightarrow a) \Rightarrow \Big((c \Rightarrow b) \Rightarrow \big(c \Rightarrow (a \wedge b)\big)\Big)$,
(τ_6) $a \Rightarrow (a \vee b)$,
(τ_7) $b \Rightarrow (a \vee b)$,
(τ_8) $(a \Rightarrow c) \Rightarrow \Big((b \Rightarrow c) \Rightarrow \big((a \vee b) \Rightarrow c\big)\Big)$

where a, b, c are arbitrary formulas of $\mathscr{F}_p$.

A consequence operation will be defined on F_p syntactically. There is one rule of inference, modus ponens, that is $a, a \Rightarrow b/b$. For all $X \subseteq F_p$, $\mathrm{Cn}_p(X)$ is the set of all formulas of $\mathscr{F}_p$ which have a finite derivation from

$A_p \cup X$ by means of modus ponens. As is well known, Cn_p is a consequence operator on F_p. The free abstract logic $\mathscr{F}_p = \langle \mathscr{F}_p, \mathrm{Cn}_p \rangle$ is called the *positive sentential calculus.*

THEOREM 2 (Łoś and Suszko). *The positive sentential calculus is a structural logic.*

COROLLARY. *$\mathscr{L}_p$ is $\mathscr{K}_{\mathscr{L}_p}$-complete.*

Proof. Corollary to the Theorem XIII.3.

THEOREM 3 (Rasiowa and Sikorski [1]). *Let $\mathscr{M} = \langle \mathscr{F}_p, U \rangle$ be a canonical $\mathscr{L}_p$-matrix, i.e., $F_p \neq \mathrm{Cn}_p(U) = U \subseteq F_p$. If $\mathrm{Cn}_p(T) = T \subseteq U$ then the relation $\underset{T}{\sim}$ on F_p, defined as follows $a \underset{T}{\sim} b$ iff $(a \Rightarrow b) \wedge (b \Rightarrow a) \in T$ is an m-congruence of $\mathscr{M}$.*

COROLLARY 3.1. *The analytical relation $\underset{T_0}{\sim}$ where $T_0 = \mathrm{Cn}_p(\emptyset)$ is a logical congruence of $\mathscr{L}_p$.*

COROLLARY 3.2. *The logic $\mathscr{L}_p / \underset{T_0}{\sim}$ is simple, the algebra $\mathscr{A} / \underset{T_0}{\sim}$ is freely generated by $X / \underset{T_0}{\sim}$, the generalized canonical quotient $\mathscr{L}_p$-matrix $\mathscr{M}_p^*$ is adequate for $\mathscr{L}_p$ and the logic $\mathscr{L}_p$ is $\mathscr{K}^*_{\mathscr{L}_p}$-complete.*

THEOREM 4 (Rasiowa and Sikorski [1]). *For $\mathscr{M} = \langle \mathscr{F}_p, U \rangle$ as defined in Theorem 3, let $\mathscr{M} / \underset{U}{\sim} = \langle \mathscr{F}_p / \underset{U}{\sim}, U / \underset{U}{\sim} \rangle$. Then $\mathscr{F}_p / \underset{U}{\sim}$ is a relatively pseudo-complemented lattice (with unit) and $U / \underset{U}{\sim} = \{1\}$ where 1 is the unit of $\mathscr{F}_p / \underset{U}{\sim}$.*

COROLLARY. *Let $\mathscr{K}_p$ be the class of all matrices $\mathscr{M} / \underset{U}{\sim}$ as defined in Theorem 4. Then, $\mathscr{L}_p$ is $\mathscr{K}_p$-complete.*

Proof. Observe that $\mathscr{M}$ and $\mathscr{M} / \underset{U}{\sim}$ are l-equivalent.

THEOREM 5. (Rasiowa and Sikorski [1]). *If $\mathscr{A}$ is a relatively pseudo-complemented lattice and $\mathscr{M} = \langle \mathscr{A}, \{1\} \rangle$ where 1 is the unit of $\mathscr{A}$ then $\mathrm{Cn}_p \leqslant C_{\mathscr{M}}$.*

COROLLARY 5.1. *Let $\mathscr{K}$ be the class of all relatively pseudo-complemented lattices with distinguished unit element. Then $\mathscr{L}_p$ is $\mathscr{K}$-complete.*

COROLLARY 5.2 (Rasiowa and Sikorski [1]). $\mathrm{Cn}_p(\emptyset) =$ *the set of all those formulas of $\mathscr{F}_p$ which are true in every relatively pseudo-complemented lattice with distinguished unit elements.*

Similarly, it can be shown that the classical (truth functional), modal and intuitionistic sentential calculi which appear in Rasiowa and Sikorski [1] are structural, hence $\mathscr{K}$-complete with respect to their bundle of canonical matrices. Again, an l-congruence and a family of m-congruences are defined for each of these calculi. Consequently, we infer analogous completeness theorems where instead of the class of relatively pseudo-complemented lattices we have the class of Boolean, topological Boolean and pseudo-Boolean algebras, respectively, where in each case $\{1\}$ is the distinguished set.

We have seen that the bundle of canonical matrices, $\mathscr{K}_{\mathscr{L}}$, provides a solution to the completeness problem for structural logics $\mathscr{L}$. Often there exists a proper sub bundle $\mathscr{H}$ of $\mathscr{K}_{\mathscr{L}}$ such that $\mathscr{L}$ is $\mathscr{H}$-complete. A sufficient condition for the existence of such an $\mathscr{H}$ is given by the next theorem.

THEOREM 6. *Let* $\mathscr{L} = \langle \mathscr{A}, \mathrm{Cn} \rangle$ *be structural and* $\mathscr{H}$ *a basis for* $\mathscr{C} = \mathscr{C}(\mathrm{Cn})$. *Then,* $\mathscr{L}$ *is* $\mathscr{K}_{\mathscr{H}}$*-complete where* $\mathscr{K}_{\mathscr{H}}$ *is the bundle of matrices* $\{\langle \mathscr{A}, V \rangle \mid V \in \mathscr{H}\}$.

Proof. The generalized canonical $\mathscr{L}$-matrix, $\mathscr{M}_{\mathscr{L}} = \langle \mathscr{A}, \overline{\mathscr{C}} \rangle$ and the generalized matrix $\mathscr{M}_{\mathscr{H}} = \langle \mathscr{A}, \mathscr{H} \rangle$ are l-equival. Hence, $\mathscr{M}_{\mathscr{L}}$ and $\mathscr{M}_{\mathscr{H}}$ generate the same structural consequence operation $C_{\mathscr{M}}$. But, $\mathscr{M}_{\mathscr{L}}$ is adequate for $\mathscr{L}$. Therefore, $\mathscr{M}_{\mathscr{H}}$ also is adequate for $\mathscr{L}$. Thus $\mathscr{L}$ is $\mathscr{K}_{\mathscr{H}}$-complete.

It is well known that the classical (truth-functional) sentential calculus is regular, i.e., the family of all Cn-complete sets constitute a basis. The quotient matrices defined by Cn-complete sets are the two-element Boolean algebras. Since they are all m-isomorphic, we conclude that the single matrix defined by the two-element Boolean algebra with distinguished set $\{1\}$ is adequate for the classical (truth functional) sentential calculus, a well known result.

Because of the m-congruences which are defined for the calculi we have just discussed, the role of matrices is often hidden in most discussions concerning the completeness of the various calculi. Granted $\mathscr{M} = \langle \mathscr{A}, \{1\} \rangle$, where $\mathscr{A}$ is a lattice with unit 1, is a matrix, but because of its special nature most questions can be answered within a lattice-theoretic framework.

However, there is a sentential calculus (Suszko [4], Bloom and Suszko [1]), called the *sentential calculus with identity connective*, where the standard lattice-theoretic techniques of algebraic logic do not provide a "semantics" or complete class of interpretations. But, SCI is a structural logic and therefore we can apply the adequacy Theorem XIII.3 and the corollary to it to SCI.

We only present a fragment of SCI, called the *positive* SCI. Let $\mathscr{R}$ be the class of abstract algebras $\mathscr{B} = \langle B, f_1, f_2, f_3, f_4 \rangle$ with four binary operations f_1, f_2, f_3, f_4. Let $\mathscr{F}_p^* = \langle F_v^*, \wedge, \vee, \Rightarrow, \equiv \rangle$ be free in $\mathscr{R}$, with the set V of free generators, called *sentential variables*. $\mathscr{F}_p^*$ is called the *algebra of positive* SCI formulas.

Let A_p^* be the set of formulas of $\mathscr{F}_p^*$ of the form (A_1) through (A_8) given above and those of the form:

(A$_9$) $a \equiv a$,

(A$_{10}$) $(a \equiv b) \Rightarrow \big((a \Rightarrow b) b \wedge (b \Rightarrow a)\big)$,

(A$_{11}$) $\big((a \equiv b) \wedge (c \equiv d)\big) \Rightarrow \big((a \wedge c) \equiv (b \wedge d)\big)$,

(A$_{12}$) $\big((a \equiv b) \wedge (c \equiv d)\big) \Rightarrow \big((a \vee c) \equiv (b \vee d)\big)$,

(A$_{13}$) $\big((a \equiv b) \wedge (c \equiv d)\big) \Rightarrow \big((a \Rightarrow c) \equiv (b \Rightarrow d)\big)$,

(A$_{14}$) $\big((a \equiv b) \wedge (c \equiv d)\big) \Rightarrow \big((a \equiv c) \equiv (b \equiv d)\big)$

where a, b, c, d are arbitrary formulas of $\mathscr{F}_p^*$.

A consequence operation will be defined on F_p^* syntactically. Again there is only one rule of inference, modus ponens, and for all $X \subseteq F_p^*$, $\mathrm{Cn}_p^*(X)$ is the set of all formulas of $\mathscr{F}_p^*$ which have a finite derivation from $A_p^* \cup X$ by means of modus ponens. The free abstract logic $\mathscr{L}_p^* = \langle F_p^*, \mathrm{Cn}_p^* \rangle$ is called the *positive sentential calculus with identity*. Again, $\mathscr{L}_p^*$ is a structural logic. Hence, $\mathscr{L}_p^*$ is $\mathscr{K}_{\mathscr{L}_p^*}$-complete.

All the techniques applied previously to $\mathscr{L}_p$ may also be applied to $\mathscr{L}_p^*$. There is, however, an important difference between $\mathscr{L}_p$ and $\mathscr{L}_p^*$ which reduces completely to their l-congruences and m-congruences. The following propositions concerned with $\mathscr{L}_p^*$ may be obtained by adjusting certain results of Bloom and Suszko [1] to $\mathscr{L}_p^*$.

THEOREM 7. *Let* $\mathscr{M} = \langle \mathscr{F}_p^*, U \rangle$ *be a canonical* $\mathscr{L}_p^*$*-matrix, i.e.,* $F_p^* \neq \mathrm{Cn}_p^*(U) = U \subseteq F_p$. *If* $\mathrm{Cn}_p(T) = T \subseteq U$ *then the relation* $\underset{T}{\approx}$ *on* F_p *defined as follows,* $a \underset{T}{\approx} b$ *iff* $(a \equiv b) \in T$, *is an* m*-congruence of* $\mathscr{M}$.

We infer, as previously (Corollary 3.1'), that the analytical relation $\underset{T_0}{\sim}$ where $T_0 = \mathrm{Cn}_p^*(\emptyset)$, is a logical congruence of $\mathscr{L}_p^*$. However, the relation $\underset{T_0}{\sim}$ appears to be the identity relation, i.e., the logic $\mathscr{L}_p^*$ is simple. Therefore, an analogue to Corollary 3.2, obtained by Theorem 1 gives nothing new. One may, however, apply the procedure used in Theorem 4 and the corollary to it and get the class $\mathscr{K}_p^*$ of all quotient matrices $\mathscr{M}/\underset{U}{\approx} = \langle \mathscr{F}_p^*/\underset{U}{\approx}, U/\underset{U}{\approx} \rangle$ such that $\mathscr{L}_p^*$ is $\mathscr{K}_p^*$-complete.

In case of the full SCI the matrices are also indispensable. The full SCI is a simple logic again. However, it is regular and one may combine the procedure used in Theorem 4 and Corollary to it with the Theorem 6. Thus, we get a class $\mathscr{K}$ of matrices, known as Bloom's normal models for SCI, such that SCI is $\mathscr{K}$-complete.

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