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Preface

Both papers in this volume are concerned with abstract logics. The first one is an introduction to the general theory of abstract logics and the second deals with a particular kind of logics, called classical logics.

An abstract logic is a pair $\langle \mathscr{A}, \mathscr{C} \rangle$ where $\mathscr{A}$ is an algebra and $\mathscr{C}$ is a closure system on the carrier of $\mathscr{A}$. Obviously, $\mathscr{C}$ may be replaced by the closure operator Cn corresponding to $\mathscr{C}$. Abstract logics are obtained as generalization of various logical notions. If $\mathscr{C} = \{A, B\}$ where $B \subseteq A$ = the carrier of $\mathscr{A}$ then $\mathscr{C}$ is called *elementary closure system* and the abstract logic $\langle \mathscr{A}, \mathscr{C} \rangle$ forms a logical matrix, a concept introduced by Łukasiewicz and Tarski in 1930. At about the same time, Tarski introduced the consequence operation (or entailment relation) as the fundamental logical notion. It is a special case of a closure operator. Also, as observed first by Lindenbaum, formalized languages are algebraic systems, i.e., sets supplied with (free) operations determined by formation rules. Hence, a formalized language conceived as an algebra of sentential formulas together with a consequence operation generated by logical axioms and rules of inference or, perhaps defined otherwise, is an abstract logic.

Since the 1930's several abstract generalizations, either algebraic or closure theoretic, arose from the logical field. I only mention cylindric algebras of Tarski and polyadic algebras of Halmos and, on the other hand, *Théorie metamathematique des ideaux* of A. Robinson (1955).

Note that Boole's mathematical thought already reached the level of abstract logics. Any Boolean algebra with the family of all its filters is a paradigm of abstract logics. Similarly, a Boolean algebra together with a single ultrafilter may be taken as paradigm of a logical matrix (elementary abstract logic).

The class of abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$, with algebras $\mathscr{A}$ of a fixed similarity type, constitute a category when supplied with suitably defined morphisms. Here, the analogy with general topology is the guiding idea so that the morphisms of abstract logics are defined as "continuous" homomorphisms. It is natural to continue the said analogy and, to introduce projective and inductive generation of abstract logics as well as the notion of a dual space. Thus, we construct a general framework of the theory of abstract logics.

Certainly, like the category of all topological spaces, categories of abstract logics are too large to be interesting or important when considered as a whole. One may prefer to study certain classes of abstract logics possessing certain properties, distinguished from this or that point of view. For example, topological separation properties, which originated in geometry, are of no use from the purely logical point of view adopted here. We are concerned with other properties: some of these are set-theoretical properties of closure systems (finiteness, logical compactness, regularity), the others involve the closure system and the underlying algebra, simultaneously (structurality, invariance, negativity, disjunctivity).

Properties of abstract logics which are distinguished from the logical point of view are those of "good" logics. These are either

(I) logics we actually use (!) or, at least, are able to use or

(II) logics which appear as semantical interpretations of the former ones.

The primary logics (I) seem to be necessarily free in the general algebraic sense. On the other hand, the most natural secondary logics (II) are logical quotients of primary logics modulo logical congruences. They are, in several particular cases, known as Lindenbaum-Tarski algebras. Therefore the theory of abstract logics cannot overlook free logics and must consider logical congruences in a general setting.

The theory of abstract logics constitutes a framework for semantical investigations. Here, we only consider the problem of the so called completeness theorem, which is the generally accepted minimal requirement on what semantics provide. Given a logic $\langle \mathscr{A}, \mathrm{Cn} \rangle$, in the semantics we look for a class $\mathscr{K}$ of interpretations (logical matrices) with the following property (completeness of the logic with respect to $\mathscr{K}$): $p \notin \mathrm{Cn}(X)$ iff there is an interpretation in $\mathscr{K}$ which makes p invalid (false, unsatisfied) while making all elements of X valid (true, satisfied). Our idea, partly due to R. Wójcicki is to make the completeness problem trivial. Indeed, if one views the logic $\langle \mathscr{A}, \mathscr{C} \rangle$ as a bundle of logical matrices, one may generalize the old Lindenbaum construction to show that abstract logics, if structural or invariant *are* complete with respect to themselves and, hence, with respect to their logical quotients.

However undesirable it might seem, the trivial completeness theorem underlies the investigations of all the diverse (sentential) logics by H. Rasiowa and R. Sikorski. On the other hand, their exemplary studies may suggest that the framework of abstract logics is the result of an over-eagerness to generalize beyond reasonable understanding. Indeed, all the logical quotients obtained by Rasiowa and Sikorski are "nice" algebras, e.g. lattices with unit, together with a closure system of filters (or, in particular, I-filters). One may argue that this closure system is inherent in the corresponding algebra and what seemingly matters here is the *unit* which again is inseparable from the algebra (lattice with unit). If the free logics considered by Rasiowa and Sikorski are typical for primary logics then as secondary logics one may only expect algebras involving an "internal" closure system. Therefore, it is a useless luxury to climb to the level of abstract logics $\langle \mathscr{A}, \mathscr{C} \rangle$ where the closure system $\mathscr{C}$ cannot be easily manufactured from the algebra $\mathscr{A}$.

There is indeed some point in this reasoning. However, what matters here is the question whether there exists a primary logic such that in logical quotients the algebra and the closure system are independent, so to speak.

The answer is an emphatic YES and constitutes the ultimate justification for the theory of abstract logics. This answer comes from the non-Fregean logic (NFL), and in particular, its basic level called the sentential calculus with identity (SCI). The SCI does not fit into the lattice-theoretic framework and requires a general theory of logical matrices and abstract logics. You will find some details at the end of the first paper.

The theory of abstract logics is kept here on the level which corresponds to sentential calculi. However, it easily generalizes to the level which corresponds to open (quantifier-free) logics with both sentential and nominal variables. Again, the open non-Fregean logic calls necessarily for a general theory of algebraic structures together with closure systems, possibly elementary. It pleases me very much to acknowledge that J. Łoś was the first (1949) to introduce such structures.

The roots of the second paper may again be found in non-Fregean logic. NFL is extremely weak (as a closure operator) and extremely rich (as closure system). Hence, the non-Fregean logic (or SCI, in particular) forces one to consider an uncountable lattice of logics, that is, the complete lattice of all its extensions (stronger closure operator and smaller closure system, the language being fixed). We face a genuine embarrassment of riches. The first attempt to deal with it consists in dividing all the extensions of NFL (or SCI) into (1) elementary ones, obtained by adding new axioms and (2) the non-elementary ones. All the elementary extensions have very good properties and have been labelled classical logics. The Fregean logic, known from textbooks of mathematical logic, *is* an elementary extension of NFL and in a sense a maximal one. Furthermore, logical quotients of sufficiently strong elementary extensions of SCI appear as nice algebras, e.g., Boolean algebras. On the other hand, all non-elementary extensions possess rather bad properties and are, at least, very strange. The question arises whether we can formulate in general terms the deep difference between elementary and non-elementary extensions of NFL (or SCI, at least). An attempt to do that is the content of the paper on classical logics. A very simple definition of classical abstract logics is assumed and a pleasant result is obtained: an abstract logic is classical iff it is equivalent in a sense to a Boolean logic, i.e., a Boolean algebra together with all (!) filters.

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