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Confluent mappings of fans

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1. Introduction

The paper is devoted to investigations of confluent mappings of fans.

Since 1964, when defined in [8], confluent mappings play an important role in continua theory: not only as a tool, being a common generalization of monotone and of open mappings, but also by their own interest, both because of results shown as well as of hard problems related to them. As examples we recall here a characterization of hereditarily indecomposable continua (as such continua Y for which any continuous mapping from a continuum onto Y is confluent, see [34], Theorem 4, p. 243 and [70], 5.7, p. 111) and a question whether arc-likeness of continua is preserved under confluent mappings ([66], Problem 4, p. 94; cf. [69], Problem II, p. 53).

According to the title of the present paper, the notion of a confluent mapping is fundamental among those discussed here. Therefore the reader is referred not only to [8] for basic properties of these mappings, but also to [17], [24], [29], [46], [48], [52], [65], [67]–[71], [75], [78], [84], [90], [95]–[98], [108]–[110], [117] and [118], where various interesting facts on confluent mappings can be found. In particular, we recall that Table II of [84], p. 28 shows relations between many kinds of mappings considered here.

It is natural that investigations of mappings of dendroids have been started from a class of rather simple ones, namely from fans (see e.g. [9], [45] and [49]). And although some problems that are still open for all dendroids have been (completely or partially) solved for fans, the class of fans has been recognized as large enough (and interesting enough) to be investigated for its own sake. In particular, some preliminary facts concerning confluent mappings of fans have been shown in § 9 of [9].

It is understandable that studying mapping properties of fans, the authors were forced to exploit various structural properties of these curves and of dendroids. However, they tried to treat these properties as tools only, not as a separate goal of the paper. The reader who is interested especially in structural properties of dendroids (in particular of fans) as for example a manner of convergence of arcs to the limit continua, the structure of the set of ramification or of end points and the like, is referred to e.g. [2], [5], [6], [7], [9], [25]–[28], [30], [32], [33], [35]–[42], [45], [46], [49], [51], [53], [58], [63], [64], [72]–[75], [80], [81], [83]–[85], [87]–[89], [91], [100]–[107] and to many other papers not quoted here.

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The paper consists of 15 chapters. After Preliminaries some general properties concerning the behaviour of fans under confluent mappings are proved in the third chapter. In particular it is recalled that the image, when nondegenerate, can be either a fan or an arc. According to this, the next two chapters are devoted to confluent mappings of fans with range space a fan or an arc, respectively. It is shown that some particular properties of these mappings differ much in both cases. A list of such (necessary for the confluence of the mapping) properties is presented, and some of them, properly chosen, are shown to be sufficient also. In this way several characterizations of confluent mappings of fans are obtained. These two chapters, the fourth and the fifth, seem to form an important part of the paper.

The sixth chapter contains a rather particular result, namely a characterization of the top of the fan, stated in terms of mappings.

Most of known examples of confluent mappings, if not monotone, are light (especially open ones). Some exceptions (i.e. nonlight open mappings of fans) are presented in the seventh chapter, where the authors study relationships between openness and lightness of a mapping defined on a fan.

In the next two chapters results related to limits of inverse sequences of fans with confluent bonding mappings are recalled and collected, and some facts on confluent mappings between locally connected fans are presented.

The rest of the paper, except for the last chapter, is devoted to the study of some particular properties of fans, especially to investigations of the invariance of these properties when the fan undergoes a confluent mapping.

So, planability is studied in the tenth chapter. In particular, it is shown that planability of fans is, while nonplanability is not, invariant under monotone mappings. The same is discussed for other subclasses of the class of confluent mappings, e.g. for open, light open, light confluent mappings etc., but in fact only some minor results and examples are presented; a number of questions remain open, so the reader can see that the problem of finding a reasonable characterization of confluent mappings that preserve planability of fans needs further investigations.

We have a better situation for smoothness (the eleventh chapter). After recalling some necessary and sufficient conditions for a fan to be smooth, images of the Cantor fan F_C are characterized under arbitrary mappings, or light ones, or monotone relative to the top of F_C (or both). It is shown that for the family of all smooth fans the Cantor fan is both a universal element and a common model with respect to the above-mentioned classes of mappings. These mappings are not confluent in general. By an earlier result, smoothness of fans is invariant under confluent mappings.

Here it is shown that this property is co-invariant under mappings that are light and confluent. The chapter ends with some results related to limits of inverse sequences of smooth fans with confluent bonding mappings.

However, it should be stressed here that a number of results strictly connected with smoothness of fans have been moved to the next (i.e. twelfth) chapter, which deals with the property of Kelley. This is so because each fan having the property of Kelley is smooth. A special attention is paid to those smooth fans for which the set of their end points, or its union with the top, are closed. This property is shown to be related to both the property of Kelley and the existence of some particular confluent mappings from the fan onto an arc. Conditions are found under which a fan can be retracted onto an arc under a light open or under a confluent mapping, and onto an arbitrary n-od contained in it in such a way that the retraction in question is both confluent and light. A result obtained in this area has been applied to extend the list of conditions which characterize the Lelek fan (i.e. a smooth fan with one-dimensional set of end points). Characterizations of all confluent, of all open and of all monotone images of the Cantor fan, obtained recently in [24], are recalled here. Finally, results are obtained which describe the behaviour of fans with the property of Kelley under continuous mappings.

In the thirteenth chapter contractibility of fans is considered. Since contractible fans are characterized as ones satisfying certain four conditions ([105]; see Theorem 13.1 here), the behaviour of each of these conditions is studied with respect to particular classes of mappings as monotone, open, light open, light confluent, etc. Also hereditary contractibility of fans as well as their noncontractibility are discussed in that chapter. In particular, we study how some other conditions that imply noncontractibility of fans, viz. containing R^i-continua, are transformed when a fan is confluently mapped onto another one (or onto an arc).

The next property is selectibility (the fourteenth chapter). Some relations between this property and contractibility are recalled and, as in the previous chapter, the behaviour of properties that either imply or are implied by selectibility of fans is studied if the fan is mapped under a particular confluent mapping.

The last chapter concerns various questions on the structure of fans, asked in [9]. All these questions have been answered in the literature. The answers are collected here for the reader's convenience.

The authors would like to express their gratitude to all participants of the two Seminars in Geometric Topology in Wrocław, who contributed to these investigations with their attention, discussions and advice.

2. Preliminaries

All considered spaces are assumed to be metric. An open ball centered at a point a and having radius r is denoted by $B(a,r)$; and we put $B(A,r) = \bigcup\{B(a,r) : a \in A\}$ for a given subset A of the space.

The symbols $\operatorname{Li} A_n$, $\operatorname{Ls} A_n$ and $\operatorname{Lim} A_n$ stand for the lower limit, the upper limit and the limit of a sequence of sets A_n respectively, in the sense of Kuratowski's monograph ([61], § 29, I–VI, pp. 335–340). Given an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, we denote by $\underleftarrow{\lim}\{X^i, f^i\}$ its (inverse) limit. And for a fixed index $i \in \mathbb{N}$ the natural projection from the limit onto X^i is denoted by π^i.

An *arc* is understood as a homeomorphic image of a closed unit interval of the real line. If any two points of a space X can be joined by an arc lying in X, then X is said to be *arcwise connected*. If, moreover, there is always exactly one such arc (i.e., if X is arcwise connected without containing any simple closed curve), then X is said to be *uniquely arcwise connected*.

Given two points u and w in a space, the symbol uw stands for an arc joining these points. Since the considered spaces are mostly uniquely arcwise connected, this notation should cause no confusion. If u and w lie in a Euclidean space, we denote by $\overline{uw}$ the straight line segment with end points u and w.

A space X is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that every arc in X contains k points that cut it into subarcs of diameters less than ε.

The union of three arcs emanating from a point v is called a *simple triod* provided the singleton $\{v\}$ is the intersection of any two of the arcs. The point v is then called the *top* of the simple triod, and the arcs are called its *arms*.

Let a space X be a union of arcs in the sense that each point of X lies in some arc contained in X. A point x in X is called an *end point* of X if x is an end point of each arc containing x and contained in X. The set of all end points of a space X is denoted by $E(X)$. A point x is called a *ramification point* of X if there exists a simple triod contained in X and having x as its top. Given a space X, we denote by $L(X)$ the set of all points of X at which X is locally connected.

A *continuum* is a compact connected space. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A continuum which is both arcwise connected and hereditarily unicoherent is called a *dendroid*. A locally connected dendroid is called a *dendrite*, and a dendrite with finitely many end points is called a *tree*. By a *fan* we mean a dendroid which has exactly one ramification point; we call this point the *top* of the fan (other authors sometimes use "vertex"), and we denote it by v. If a fan X with the top v is given, then we put $S(X) = \{v\} \cup E(X)$. A fan is said be ***countable*** or ***finite*** provided the set of its end points is countable or finite, respectively. If a fan has exactly n end points, it is called an ***n-od***, and denoted by F^n. The symbol F^ω stands for a countable locally connected fan. In other words, F^ω denotes the union of countably many straight line segments in the plane, any two of which intersect at their common point v only and such that for each $\varepsilon > 0$ at most finitely many segments have lengths greater than ε (see Fig. 1).

Fig. 1

Denote by H the closure of the harmonic sequence of reals, i.e., $H = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. The cone over H is called the ***harmonic fan,*** and is denoted by F_H. Similarly, the cone over the Cantor ternary set C is called the ***Cantor fan***, and is denoted by F_C. Thus we have

$$F_H = (H \times [0,1])/(H \times \{0\}),$$

and

$$F_C = (C \times [0,1])/(C \times \{0\}).$$

Usually we understand the above fans as subspaces of the Euclidean plane $\mathbb{R}^2$. If $\mathbb{R}^2$ is equipped with the Cartesian coordinate system, then putting $v = (0,0)$, $e_0 = (0,1)$ and $e_n = (1/n, 1)$ for $n \in \mathbb{N}$ we can write

$$F_H = \bigcup\{\overline{ve_n} : n \in \{0,1,2,\ldots\}\};$$

similarly, putting $v = (1/2, 0)$ and considering $C \times \{1\}$ as the set of end points of F_C, we can write

$$F_C = \bigcup\{\overline{ve} : e \in C \times \{1\}\}.$$

The union F_{HP} of straight line segments in the plane connecting $v = (0,0)$ with points of the set $\{(0,2)\} \cup \{(1/n, 1) : n \in \mathbb{N}\}$ is called the ***harmonic prolonged fan***. We call the ***Lelek fan*** a subfan F_L of the Cantor fan F_C such that the set $E(F_L)$ of its end points is dense in F_L, and $S(F_L) = \{v\} \cup E(F_L)$ is a connected set (where v denotes the top of the two fans F_C and $F_L \subset F_C$). Thus $E(F_L)$ is one-dimensional. For a geometrical definition and other properties of F_L see [64], § 9, pp. 314–318. It is known that any two Lelek fans are homeomorphic, see [33].

The next fan we are going to define is situated in the Euclidean 3-space (equipped with the Cartesian coordinate system). Let $v = (0,0,0)$, $a_1 = (1,0,0)$, $a_2 = (0,1,0)$ and $a_3 = (0,-1,0)$. For $k \in \{1,2,3\}$ put $L_{0k} = \overline{va_k}$, and consider three sequences of arcs $\{L_{kn}\}_{n=1}^{\infty}$ emanating from v and disjoint outside v, such that each sequence smoothly approximates the union of some two arms of the triod $T = L_{01} \cup L_{02} \cup L_{03}$ as follows: $\operatorname{Lim} L_{1n} = L_{01} \cup L_{02}$, $\operatorname{Lim} L_{2n} = L_{01} \cup L_{03}$ and $\operatorname{Lim} L_{3n} = L_{02} \cup L_{03}$. Furthermore we assume that $T \cap L_{kn} = \{v\}$ for each k and n. Thus the union

$$F_B = T \cup \bigcup \{L_{kn} : n \in \mathbb{N} \text{ and } k \in \{1,2,3\}\}$$

is a countable fan, and it is called the *Borsuk fan*. The reader is referred to [5], pp. 233 and 234 for a detailed geometrical description of L_{kn} and for a proof that F_B is nonplanable (i.e. it cannot be topologically embedded into the plane).

A fan X with top v is said to be *smooth* provided that for each $x \in X$ and for each sequence x_n in X the condition $x = \lim x_n$ implies $vx = \operatorname{Lim} vx_n$ ([9], § 3, p. 7). A continuum X is said to have the *property of Kelley* provided that for each x in X, for each sequence x_n converging to x and for each continuum $L \subset X$ with $x \in L$ there exists a sequence of continua $L_n \subset X$ such that $x_n \in L_n$ for each $n \in \mathbb{N}$ and $L = \operatorname{Lim} L_n$ ([54], (3.2), p. 26).

A *mapping* is a continuous transformation. A mapping $f : X \to Y$ is said to be:

— a *local homeomorphism* if each point in X has an open neighborhood U such that $f(U)$ is open in Y and $f|U : U \to f(U)$ is a homeomorphism;

— *interior at* $x \in X$ if for each open set U around x the image $f(x)$ is an interior point of $f(U)$ ([116], p. 149);

— *open* if f maps each open set in X onto an open set in Y (note that f is open if and only if it is interior at each point of X);

— *monotone* if for each subcontinuum Q of Y, $f^{-1}(Q)$ is a continuum in X (see [62], p. 123); or, which is equivalent provided X is a continuum, if the inverse image of each point of Y is connected (see [116], p. 127);

— *monotone relative to* $x \in X$ if for each subcontinuum Q of Y such that $f(x) \in Q$, $f^{-1}(Q)$ is connected ([75], p. 720);

— *quasi-monotone* if for each subcontinuum Q of Y with nonempty interior, $f^{-1}(Q)$ has a finite number of components and each of them is mapped under f onto Q ([112], p. 136);

— *confluent* if for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped under f onto Q ([8], p. 213; cf. [65], p. 223);

— *weakly confluent* if for each subcontinuum Q of Y there is a component of $f^{-1}(Q)$ which is mapped under f onto Q ([67], p. 98; cf. [70], p. 106);

- *semi-confluent* if for each subcontinuum Q of Y and for any two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$ ([78], p. 252);
- *locally confluent* if each point of Y has a closed neighborhood V such that $f|f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V ([48], p. 239);
- *light* if for each point of Y its inverse image has one-point components ([116], (4.4), p. 130; cf. [47], pp. 450 and 454 for a terminology discussion). Note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional ([116], p. 130; [47], p. 450);
- an *OM-mapping* if there exist a space Z and two mappings: $f_1 : X \to Z$ and $f_2 : Z \to Y$ such that f_1 is monotone, f_2 is open, and $f = f_2 f_1$ ([70], p. 104).

Recall that two mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be *topologically equivalent* if there exist homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $h_Y f_1 = f_2 h_X$ (see [116], p. 127, footnote).

Given a property $\mathcal{P}$ of spaces and a class $\mathcal{M}$ of mappings, we say that $\mathcal{P}$ is *invariant* (*co-invariant*) under mappings belonging to $\mathcal{M}$ provided for each surjection $f : X \to Y$ with $f \in \mathcal{M}$ if X has $\mathcal{P}$ then Y has $\mathcal{P}$ (if Y has $\mathcal{P}$ then X has $\mathcal{P}$, respectively).

Given a family $\mathcal{S}$ of spaces and a class $\mathcal{M}$ of mappings, a member X of $\mathcal{S}$ is said to be:

- a *common model* for $\mathcal{S}$ with respect to $\mathcal{M}$ provided that for each space Y in $\mathcal{S}$ there exists a mapping f in $\mathcal{M}$ such that $Y = f(X)$;
- a *universal element* for $\mathcal{S}$ provided that for each space Y in $\mathcal{S}$ there exists a homeomorphism h such that $h : Y \to h(Y) \subset X$.

3. General properties

The following well-known fact will frequently be used in the sequel without any special quotation. For its proof see [8], V and VI, p. 214.

FACT 3.0. *Let f be a surjective mapping between continua. If f is either monotone or open, then it is confluent.*

We start our investigation of confluent mappings with the following proposition.

PROPOSITION 3.1. *A surjective mapping defined on a dendroid is confluent if and only if it is locally confluent.*

Proof. Indeed, each confluent mapping is locally confluent by definition. If a surjection defined on a dendroid is locally confluent, then the range space is also a dendroid ([84], Theorem (7.32), p. 67), thus it is hereditarily arcwise connected ([7], T26, p. 197). Since all locally confluent mappings onto hereditarily arcwise connected spaces are confluent (see Theorem 5.3 of [70], p. 110), the conclusion follows.

Remarks 3.2. The hypothesis of Proposition 3.1 saying that the domain space is a dendroid can be neither removed nor weakened to the requirement that it is a λ-*dendroid* (i.e. a hereditarily decomposable and hereditarily unicoherent continuum, see [10], p. 15 and Theorem 1, p. 16), even under the additional assumption of its arc-likeness: see [70], Example (4.2), p. 106. See also [84], Example (7.17), p. 63; [70], Remarks, p. 107; and [109], Theorem 5, p. 237.

The following proposition is a corollary to two more general results concerning λ-dendroids (Theorem 9 and Corollary 10 of [76], pp. 857 and 858).

PROPOSITION 3.3. *A surjective mapping between dendroids is a homeomorphism if and only if it is a local homeomorphism.*

Therefore, by virtue of Propositions 3.1 and 3.3 we conclude that it makes no sense to consider local homeomorphisms and locally confluent mappings as separate classes of mappings between dendroids, in particular between fans.

PROPOSITION 3.4. *Let a surjective confluent mapping* $f : X \to Y$ *be defined on a fan* X *with top* v. *Then*

(1) *the range space* Y *is a fan, an arc, or a point;*
(2) *for each* $e \in E(X) \setminus f^{-1}(E(Y))$, *the mapping* $f|ve$ *is constant;*
(3) $f(E(X)) \subset \{f(v)\} \cup E(Y)$.

Proof. Note that (1) has been proved as Theorem 12 of [9], p. 32, and (3) is an immediate consequence of (2). Thus we have to prove (2) only. So suppose on the contrary that there is an end point e of X such that $f(e) \in Y \setminus E(Y)$ and $f(ve)$ is nondegenerate. Then there is an arc Q in Y such that $f(e) \in Q \setminus E(Q)$ and $f(ve) \setminus Q \neq \emptyset$. Denote by K the component of $f^{-1}(Q)$ containing e. Thus K is a proper subarc of ve ending at e. Since $f|K$ is confluent (see [8], I, p. 213), it maps K onto either a singleton $\{f(e)\}$ or an arc $f(K)$ (see [9], Corollary 20, p. 32) having $f(e)$ as one of its end points, whence $f(K) \neq \emptyset$, a contradiction.

Remark 3.5. Conclusion (1) of Proposition 3.4 holds under a weaker assumption about the mapping f, with "confluent" replaced by "semi-confluent" (see [78], Theorem 5.6, p. 263). But neither (2) nor (3) is true for semi-confluent mappings f, as Example 5.7 of [78], p. 263 shows.

Remark 3.6. Let $\mathcal{M}$ be a class of mappings between continua. Then a mapping $f : X \to Y$ is said to be ***hereditarily*** $\mathcal{M}$ if $f|K$ is in $\mathcal{M}$ for each subcontinuum K in X ([82], p. 124; cf. [84], Chapter 4, Part B, p. 16). Recall that conclusion (1) of Proposition 3.4 holds if f is assumed to be hereditarily weakly confluent (see [82], Corollary (5.23), p. 149), but again — similarly to semi-confluent mappings (see the previous remark) — neither (2) nor (3) of 3.4 remains true for this class of mappings, which can be seen by the same Example 5.7 of [78], p. 263. Heredity of weak confluence is essential in the above result, because weakly confluent mappings do not preserve acyclicity of a continuum, even if the domain space is locally connected, as can be seen from the well-known mapping $x \mapsto \exp(2\pi i x)$ from the real line onto the unit circle, restricted to the closed interval $[-1, 1]$.

The next example shows that the property of being a fan is not invariant under weakly confluent mappings even if the range space is a tree.

EXAMPLE 3.7. *There exists a weakly confluent mapping from the Cantor fan onto a tree with two ramification points.*

Proof. Indeed, for $i \in \{1, 2\}$ let $g_i : X_i \to T_i$ be a weakly confluent mapping of the Cantor fan X_i with top v_i onto a simple triod T_i with an end point e_i such that $g_i(v_i) = e_i$ (see [46], Example III.1, p. 413). Identifying v_1 with v_2 and e_1 with e_2, i.e. putting $X = X_1 \cup X_2$, where $X_1 \cap X_2 = \{v_1\} = \{v_2\}$, and $Y = T_1 \cup T_2$, with $T_1 \cap T_2 = \{e_1\} = \{e_2\}$, we see that X is homeomorphic to the Cantor fan and Y is homeomorphic to the capital H. Now let $g : X \to Y$ be the mapping defined by $g|X_i = g_i$ for $i \in \{1, 2\}$. Then obviously g is weakly confluent.

Remark 3.8. Example (7.35) of [84], p. 68 shows a quasi-monotone mapping from a fan which is the one-point union of two harmonic fans onto a dendroid having two ramification points. Furthermore, the mapping is also monotone relative to the top of the domain fan (and, obviously, is not confluent according to (1) of Proposition 3.4). Thus the property of being a fan is not invariant under mappings that are both quasi-monotone and monotone relative to the top of the domain.

In the light of Remarks 3.5, 3.6 and 3.8 the following question seems to be interesting.

QUESTION 3.9. For what mappings f defined on a fan X the image $f(X)$ is a fan (an arc, or a point)?

Remark 3.10. For wider classes of continua preserved under confluent mappings see [8], Corollary 1, p. 219 and XIV, p. 217 (dendroids and λ-dendroids), [90], Theorem 2.1, p. 468 and Corollary 2.2, p. 472 (tree-like curves), and [69], (2.7), p. 53 (acyclic curves and acyclic continua). Recall that invariance of arc-likeness of curves under confluent mappings is still an open problem ([66], Problem 4, p. 94; cf. [69], Problem II, p. 53).

Recall that $L(X)$ stands for the set of all points of a given space X at which X is locally connected. The following proposition is certainly known.

PROPOSITION 3.11. *If a mapping* $f : X \to Y$ *is open, then* $f(L(X)) \subset L(f(X))$.

Indeed, if a family is a local base composed of connected sets at $x \in X$, then their images form a local base at $f(x)$.

It is known that all locally confluent mappings from compact spaces onto locally connected ones are OM-mappings ([70], Corollary 5.2, p. 109). Since local connectedness of continua is invariant under (continuous) mappings, the next two propositions are consequences of the above quoted result.

PROPOSITION 3.12. *Let a fan* X *be given and let a surjective mapping* $f : X \to Y$ *be confluent. If either* X *or* Y *is locally confluent, then* f *is an* OM*-mapping.*

PROPOSITION 3.13. *Let a fan* X *be given and let a surjective mapping* $f : X \to Y$ *be light confluent. If either* X *or* Y *is locally confluent, then* f *is open.*

4. Mappings onto fans

According to (1) of Proposition 3.4 the image of a fan under a confluent mapping is either a fan, or an arc, or a point. In the present chapter we consider confluent mappings of fans satisfying the additional assumption that the range space is a fan. Necessary conditions are found for such mappings in Theorem 4.1, and it is shown that satisfying some of them suffices for the mapping in question to be confluent (Theorem 4.6 and Remarks 4.7 and 4.8), so that characterizations of such mappings are obtained (Theorem 4.10). Similar conditions characterize light confluent mappings between fans (Theorem 4.17).

We start with the following result.

THEOREM 4.1. *Let a surjective confluent mapping* $f : X \to Y$ *be defined on a fan* X *with top* v. *If* Y *is a fan, then*

(4) $f(v)$ *is the top of* Y;
(5) f *is monotone relative to* v;
(6) $f^{-1}(f(v))$ *is a continuum;*
(7) *for each* x *in* X, *the mapping* $f|vx$ *is monotone;*

(8) *for each end point e of X, the mapping $f|ve$ is monotone;*

(9) *for each end point e of X either $f(ve)$ is a singleton $\{f(v)\}$ or there exists an end point e' of Y such that $f|ve : ve \to f(v)e'$ is a monotone surjection;*

(10) $E(Y) \subset f(E(X))$;

(11) $f(S(X)) = S(Y)$.

Proof. Conclusions (4) and (5) are known (see [9], Theorem 12 and Lemma 4, p.. 32). Further, (6) is an immediate consequence of (4) and (5). To see (7) we conclude from (4), (5) and Theorem 2.5 of [75], p. 721 that $f(vx) = f(v)f(x)$ for each point x of X. Now (7) follows from Corollary 2.10 of [75], p. 722; and (8) is a particular case of (7). By (8) we see that $f(ve)$, as a monotone image of the arc ve, is either a singleton $\{f(v)\}$ or an arc having $f(v)$ as one of its end points (compare [62], § 48, I, Theorem 3, p. 192). In the latter case it is contained in the arc $f(v)e'$ for some $e' \in E(Y)$. Suppose that $f(ve)$ is a proper nondegenerate subarc of the arc $f(v)e'$ and denote by z its end point distinct from $f(v)$. Therefore $z = f(e) \in f(v)e' \setminus \{f(v), e'\}$. Let K be the component of $f^{-1}(ze')$ containing e. So v is not in K, whence $K \subset ve \setminus \{v\}$; thus $f(K) \subset f(ve \setminus \{v\}) \subset f(v)z$. Since f is confluent, we have $f(K) = ze'$, a contradiction. So, (9) is proved. To show (10) take an end point e_1 of Y and a point x of X with $f(x) = e_1$. Thus $x \neq v$, and there is an end point e of X with $x \in ve$. By (7), $f|vx : vx \to f(v)e_1$ is a monotone surjection, and by (9) there is an end point $e' \in E(Y)$ such that $f|ve : ve \to f(v)e'$ also is a monotone surjection. Since $x \in ve$, we conclude $e_1 = e'$ and $f(xe) = \{e_1\}$. In particular $f(e) = e_1$ and the desired inclusion follows. Finally (11) is an immediate consequence of (3) in Proposition 3.4, of (4) and (10). The proof is complete.

The assumption of Theorem 4.1 saying that the domain space X is a fan cannot by neglected because neither (10) nor (11) holds for arbitrary dendroids, even if they are finite trees (if X is a dendroid, then the symbol $S(X)$ appearing in (11) can be understood as the set composed of all end points and all ramification points of X). To see this, put in the Cartesian coordinates in the plane $\mathbb{R}^2$:

$$X = \{(x,0) : |x| \leq 1\} \cup \{(x,y) : |x| = 1 \text{ and } |y| \leq 1\},$$
$$Y = \{(x,0) : 0 \leq x \leq 1\} \cup \{(1,y) : |y| \leq 1\} \subset X,$$

and let $f : X \to Y$ be defined by $f((x,y)) = (|x|, y)$. Then X is a tree having four end points and two ramification points, Y is a simple triod, f is a light open retraction, and we have

$$(0,0) \in E(Y) \setminus f(S(X)) \subset S(Y) \setminus f(S(X)),$$

so neither (10) nor (11) holds true.

Further, keeping the same X, putting

$$Y' = X \setminus \{(-1, y) : -1 \leq y < 0\},$$

and defining $f' : X \to Y'$ by $f'((x,y)) = (x, |y|)$ if $(x,y) \in X \setminus Y'$ and $f'((x,y)) = (x,y)$ if $(x,y) \in Y'$ we see that Y' is a simple triod, and f' is an open retraction again, and $(-1,0) \in f'(S(X)) \setminus S(Y)$. So neither of the two possible inclusions between members of (11) holds true in general.

The other assumption of Theorem 4.1, saying that the range space Y is a fan, cannot be omitted either, even in the case when the mapping f under consideration is open. This is seen from the following example.

EXAMPLE 4.2. *There exists an open retraction of a simple triod onto an arc such that no one of statements* (5)–(10) *hold.*

Proof. In the rectangular coordinates in the plane put $v = (0,0)$, $a = (1,1)$, $b = (0,1)$, $e_1 = (1,0)$, $e_2 = (1,1/2)$ and $e_3 = (1,2)$, and $X = \overline{ve_1} \cup \overline{ve_2} \cup \overline{va} \cup \overline{ab} \cup \overline{be_3}$. Then the orthogonal projection of X onto $\overline{ve_1}$ is the needed mapping.

Remark 4.3. Conclusion (4) of Theorem 4.1 holds true if f is assumed to be either semi-confluent (this generalizes that part of 4.1, see [78], Theorem 5.6, p. 263) or hereditarily weakly confluent ([82], Corollary (5.23), p. 149). But even if f is both semi-confluent and hereditarily weakly confluent, conclusions (5), (7), (8), (9), (10), and (11) need not hold, as Example 5.7 of [78], p. 263 shows. An easy modification of that example shows that (6) need not hold either.

As a consequence of Proposition 3.4 and Theorem 4.1 we get the following corollaries.

COROLLARY 4.4. *If fans X and Y are such that $E(X)$ is closed and* $\operatorname{int}\operatorname{cl} E(Y) \neq \emptyset$, *then there is no confluent mapping of X onto Y.*

Proof. Suppose $f : X \to Y$ is a confluent surjection. Since $E(X)$ is closed, so is $S(X)$, and therefore $S(Y)$, by (11) of Theorem 4.1. Thus we get $\operatorname{int} E(Y) \neq \emptyset$, which is an absurd.

Note that the condition $\operatorname{int}\operatorname{cl} E(Y) \neq \emptyset$ holds in particular when $E(Y)$ is dense. So we can apply Corollary 4.4 to the Cantor fan F_C as X and the Lelek fan F_L as Y, to obtain the following particular result.

COROLLARY 4.5. *There is no confluent mapping from the Cantor fan onto the Lelek fan.*

The above result is also a consequence of Theorem 12.3 below, in which a characterization is presented of confluent images of the Cantor fan. Some other characterizations, which concern images of the Cantor fan under light confluent, light open, open, and monotone mappings, are stated in the twelfth chapter, as Theorems 12.4 and 12.5. The reader can find their proofs in [24].

The next three results are converse, in a sense, to the implication of Theorem 4.1. It is shown that the conjunction of some of the conditions (4) through (11) implies the confluence of the mapping.

THEOREM 4.6. *Let a surjective mapping* $f : X \to Y$ *be defined on a fan* X *with top* v. *If*

(1) Y *is a fan, an arc, or a point,*
(3) $f(E(X)) \subset \{f(v)\} \cup E(Y)$,
(4) *in case* Y *is a fan,* $f(v)$ *is its top, and*
(8) *for each end point* e *of* X, *the mapping* $f|ve$ *is monotone,*

then f *is confluent.*

Proof. To prove that f is confluent take a subcontinuum Q of Y and consider two cases.

First, let $f(v)$ be in Q. Then v is in $f^{-1}(Q)$ and we have

$$\begin{aligned} f^{-1}(Q) = f^{-1}(Q) \cap X = f^{-1}(Q) \cap \bigcup\{ve : e \in E(X)\} \\ = \bigcup\{ve \cap f^{-1}(Q) : e \in E(X)\} \\ = \bigcup\{(f|ve)^{-1}(Q) : e \in E(X)\}. \end{aligned}$$

Now note by (8) that, for each $e \in E(X)$ the set $(f|ve)^{-1}(Q)$ is a continuum, and since $f(v) \in Q$, this continuum contains v. Thus $f^{-1}(Q)$ is connected, i.e., it consists of one component only, which obviously is mapped onto Q.

Second, let $f(v)$ be outside Q. Denote by K an arbitrary component of $f^{-1}(Q)$. Thus v is not in K, and hence there is $e \in E(X)$ such that $K \subset ve \setminus \{v\}$. Note that $f(e)$ differs from $f(v)$ because otherwise $f|ve$ is a constant mapping of the arc ve onto the singleton $\{f(v)\}$ by (8), and consequently $f(K) = \{f(v)\} \subset Q$, a contradiction. Thus $f(e)$ is an end point of Y by (3), and we conclude that $f|ve : ve \to f(v)f(e)$ is a monotone surjection. Since $\emptyset \neq f(K) \subset Q \cap f(ve) = Q \cap f(v)f(e)$ and since Q is an arc as a subset of the set $Y \setminus \{f(v)\}$ whose components by (4) are arcs without one end point, we conclude $Q \subset f(v)f(e)$. Further, we see by (8) that $(f|ve)^{-1}(Q) = ve \cap f^{-1}(Q)$ is a connected set which contains K, thus $K = (f|ve)^{-1}(Q)$. Consequently, we have

$$f(K) = (f|ve)(K) = (f|ve)((f|ve)^{-1}(Q)) = Q,$$

and the proof is complete.

Remark 4.7. In Theorem 4.6 condition (8) can be replaced by (5) (since (5) implies (8) by Corollary 2.10 of [75], p. 722).

Remark 4.8. In Theorem 4.6 conditions (3) and (8) can be replaced by (9) (since (9) implies (3) and (8)).

Remark 4.9. Example (7.35) of [84], p. 68 shows that condition (1) cannot be omitted in Theorem 4.6 and Remarks 4.7 and 4.8. Also condition (4) is essential. Namely take a simple triod X with top v and end points e_1, e_2, e_3, and define a surjection f from X onto itself as follows: $f(v) = e_3$; for every $i \in \{1,2,3\}$ put $f(e_i) = e_i$; let $f|ve_1$ and $f|ve_2$ be homeomorphisms and $f|ve_3$ the constant mapping. Then f is not confluent, (4) is not satisfied but all other assumptions are. The reader can easily find appropriate examples showing that no other hypotheses of Theorem 4.6 can be left out.

As a consequence of Proposition 3.4, Theorems 4.1 and 4.6 and Remarks 4.7 and 4.8 we have the following result that gives some characterizations of confluent mappings between fans.

THEOREM 4.10. *Let a surjective mapping $f : X \to Y$ between fans X and Y with tops v and v' respectively be given such that $f(v) = v'$. Then the following conditions are equivalent:*

(a) *f is confluent,*
(b) *f is monotone relative to v and inclusion* (3) *holds,*
(c) *inclusion* (3) *holds and f satisfies* (8),
(d) *f satisfies* (9).

Remark 4.11. The equivalence of (a) and (c) in Theorem 4.10 has been shown in [86].

The rest of this chapter is devoted to those confluent mappings between fans that are light. The following result is an immediate consequence of Theorem 4.1 and of the definition of a light mapping.

PROPOSITION 4.12. *Let a surjective light confluent mapping $f : X \to Y$ be defined on a fan X with top v. If Y is a fan, then conditions* (5)–(10) *can be strengthened to the following:*

(12) $f^{-1}(f(v)) = \{v\}$;
(13) *for each end point e of X there exists an end point e' of Y such that $f|ve : ve \to f(v)e'$ is a homeomorphism;*
(14) $f(E(X)) = E(Y)$.

COROLLARY 4.13. *Let a surjective light confluent mapping $f : X \to Y$ be defined between fans X and Y with tops v and v'. Then*

$$f^{-1}(v') = \{v\}, \quad f^{-1}(E(Y)) = E(X) \quad \text{and} \quad f^{-1}(S(Y)) = S(X).$$

Proof. The first equality follows from (4) of Theorem 4.1 and (12) of Proposition 4.12. Conditions (13) and (14) of that proposition give the second equality. The third one is a consequence of the previous two.

For further discussion concerning the conditions considered in the conclusion of Corollary 4.13 see Example 7.10 and Proposition 11.18.

Now we shall prove some converses to the previous proposition. They correspond to Theorem 4.6 and Remarks 4.7 and 4.8 giving sufficient conditions for a mapping between fans to be both light and confluent.

PROPOSITION 4.14. *Let a surjective mapping $f : X \to Y$ be defined on a fan X with top v. If conditions* (1), (4) *and* (13) *hold, then f is light and confluent.*

Proof. The confluence of f follows from Corollary 4.8. Suppose f is not light. Then there is y in Y such that a component K of $f^{-1}(y)$ is nondegenerate. Thus $ve \cap K$ must be nondegenerate for some $e \in E(X)$. So, $f|ve$ is not a homeomorphism, a contradiction with (13).

COROLLARY 4.15. *Let a surjective mapping $f : X \to Y$ be defined on a fan X with top v. If conditions* (1), (4) *and* (14) *are satisfied, and if*

(15) *for each end point e of X, the mapping $f|ve : ve \to f(ve)$ is a homeomorphism,*

then f is light and confluent.

Indeed, note that (14) and (15) imply (13), and then apply Proposition 4.14.

REMARK 4.16. The reader can verify that all assumptions of Proposition 4.14 and of Corollary 4.15 are indispensable.

The following result, giving characterizations of light confluent mappings between fans, is a consequence of Theorem 4.10, Propositions 4.12 and 4.14 and Corollary 4.15.

THEOREM 4.17. *Let a surjective mapping $f : X \to Y$ between fans X and Y with tops v and v' respectively be given such that $f(v) = v'$. Then the following conditions are equivalent:*

(a) *f is light and confluent,*
(b) *f satisfies* (13),
(c) *f satisfies* (14) *and* (15).

Some particular kinds of confluent mappings between fans, as e.g. open, light open or monotone open ones will be examined in Chapter 7.

5. Mappings onto an arc

The following two propositions are consequences of Propositions 3.12 and 3.13.

PROPOSITION 5.1. *The class of confluent mappings from a fan onto an arc coincides with the class of OM-mappings.*

PROPOSITION 5.2. *The class of light confluent mappings from a fan onto an arc coincides with the class of light open mappings.*

As the reader has already seen from Example 4.2, a confluent (an open even) mapping of a (locally connected) fan onto an arc can have no property listed in the conclusion of Theorem 4.1. We shall now show some other properties of such mappings. To this end we need a sequence of lemmas.

LEMMA 5.3. *Let a surjective mapping $f: X \to Y$ be defined on a fan X with top v, and let a subcontinuum Q of Y be given. Then:*

(a) *for each end point e of X, an arbitrary component K of $(f|ve)^{-1}(Q)$ that does not contain v coincides with a component of $f^{-1}(Q)$ (thus, if f is confluent, then $f(K) = Q$);*

(b) *for each component K of $f^{-1}(Q)$ that does not contain v there exists an end point e of X such that K coincides with a component of $(f|ve)^{-1}(Q)$.*

Proof. (a) At most one component of $(f|ve)^{-1}(Q)$ contains v. Any other component K of $(f|ve)^{-1}(Q)$, if it does exist, is contained in the half-closed arc $ve \setminus \{v\}$, which in turn is a component of $X \setminus \{v\}$. Thus K coincides with the corresponding component of $f^{-1}(Q)$.

(b) The argument is quite similar to the previous one.

LEMMA 5.4. *If a surjective mapping $f: X \to Y$ from a fan X with top v onto an arc $Y = ab$ is confluent, then for each end point e of X and for any two points x' and x'' of two distinct components of $ve \cap f^{-1}(a)$ there is $x \in x'x''$ such that $f(x) = b$.*

Proof. Order the arc ab from a to b and note that $f|x'x''$ is not constant, so there is $x \in x'x''$ at which it takes its maximum y. If $y < b$, let Q be a subcontinuum of ab such that $y \in \operatorname{int} Q \subset Q \subset ab \setminus \{a, b\}$. Denote by K the component of $(f|ve)^{-1}(Q)$ that contains x. We claim that v is not in K. Indeed, if $v \in K$ then $xv \subset K$, but since either x' or x'' is in xv, we have $a \in f(xv) \subset f(K) \subset Q$, a contradiction. Thus Lemma 5.3 (a) can be applied, whence $f(K) = Q$, which contradicts the definition of x. Therefore $f(x) = y = b$.

The next lemma is an immediate consequence of the previous one.

LEMMA 5.5. *Let a surjective mapping $f : X \to Y$ from a fan X with top v onto an arc $Y = ab$ be confluent. Then for each end point e of X the number of components of $ve \cap f^{-1}(a)$ is finite, and between any two consecutive components of $ve \cap f^{-1}(a)$ lies a component of $ve \cap f^{-1}(b)$.*

LEMMA 5.6. *Let a surjective mapping $f : X \to Y$ from a fan X with top v onto an arc $Y = ab$ be confluent, and let an end point e of X be fixed. If A and B are two consecutive components of $ve \cap f^{-1}(a)$ and $ve \cap f^{-1}(b)$ respectively, then for any $x_1 \in A$ and $x_2 \in B$, the mapping $f|x_1x_2 : x_1x_2 \to Y$ is a monotone surjection.*

Proof. Since $f(x_1) = a$ and $f(x_2) = b$, $f|x_1x_2$ is a surjection. To see it is monotone suppose on the contrary that there is $c \in ab$ such that $(f|x_1x_2)^{-1}(c) = x_1x_2 \cap f^{-1}(c)$ is not connected. Let x' and x'' lie in distinct components of the intersection. Thus $f|x_1x_2$ is not constant, so it takes its values either in ac or in cb (or both). Without loss of generality we can assume $cb \cap f(x'x'') \neq \emptyset$, and therefore there is $x \in x'x''$ at which $f|x_1x_2$ takes its maximum y (we consider ab ordered from a to b). Since A and B are consecutive components of $ve \cap f^{-1}(\{a, b\})$, we have $y < b$. Then taking a subcontinuum Q of ab such that $y \in \operatorname{int} Q \subset Q \subset cb \setminus \{c, b\}$ and arguing as in the final part of the proof of Lemma 5.4 we get a contradiction.

Now we are able to show the first main result of this chapter.

THEOREM 5.7. *Let a surjective confluent mapping $f : X \to Y$ be defined on a fan X with top v. If Y is an arc, then*

(16) *for each end point e of X either $f|ve$ is constant, or there is a finite sequence of distinct points $v = x_0, x_1, \ldots, x_n = e$ in ve ordered from v to e such that for each $k \in \{0, 1, \ldots, n-1\}$, the mapping $f|x_kx_{k+1} : x_kx_{k+1} \to Y$ is monotone; if $k \neq 0$, then $f|x_kx_{k+1}$ is a surjection onto the whole arc Y, and moreover, if $f(v) \in E(Y)$, then also $f|x_0x_1$ is a surjection.*

Proof. Take an arbitrary end point e of X and order the arc ve from v to e. Assume $f(ve)$ is nondegenerate. Since $f(e) \in E(Y)$ by (2) of Proposition 3.4, $ve \cap f^{-1}(E(Y))$ is a nonempty closed set. Note that, by Lemma 5.5, this intersection has a finite number of components. So let $K_1, \ldots, K_n$ be consecutive components of $ve \cap f^{-1}(E(Y))$ numbered in such a way that K_1 is the first one to which v does not belong, and $e \in K_n$. Put $x_0 = v$ and let $x_k = \inf K_k$ for $k \in \{1, \ldots, n\}$. Now the conclusion follows easily from Lemma 5.6.

Our second main result is a converse to the first one.

THEOREM 5.8. *If a surjective mapping $f : X \to Y$ from a fan X with top v onto an arc Y satisfies condition (16), then f is confluent.*

Proof. Let Q be a nondegenerate subcontinuum of Y and let K be an arbitrary component of $f^{-1}(Q)$. Consider three cases.

1) There is an end point e of X such that $K \subset ve$, and no x_k is in K, where $k \in \{0, 1, \ldots, n\}$. Then, in particular, $x_0 = v$ is not in K, and we have $K \subset x_i x_{i+1} \setminus \{x_i, x_{i+1}\}$ for some $i \in \{0, 1, \ldots, n\}$. Thus by Lemma 5.3(b), K coincides with the corresponding component of $(f|ve)^{-1}(Q)$. Since $f|x_i x_{i+1}$ is monotone by (16), we have $f(K) = Q$.

2) There is an end point e of X such that $K \subset ve$, and x_i is in K for some $i \in \{1, \ldots, n\}$, while $x_0 = v$ is not. Thus Lemma 5.3(b) can be applied. If more than one x_k is in K, then since $f|x_k x_{k+1}$ is a surjection onto Y by (16), we have $f(K) = Y \supset Q$. So let exactly one x_i be in K. Then K is the union of two subarcs:

$$K = (K \cap x_{i-1}x_i) \cup (K \cap x_i x_{i+1}),$$

and each of these subarcs is mapped onto Q under f, since the partial mappings $f|x_{i-1}x_i$ and $f|x_i x_{i+1}$ are both monotone by (16).

3) For no end point e of X we have $K \subset ve$. Then $v \in K$ and K is a union of arcs of the form $K \cap ve$ for some $e \in E(X)$. So $f(v) \in Q$ and, if $Y = ab$, we have

$$Q = (Q \cap af(v)) \cup (Q \cap f(v)b).$$

Since Q is nondegenerate, at least one member of the above union is nondegenerate, too. Without loss of generality we may assume $Q \cap af(v)$ is nondegenerate. Thus there is an end point e_a of X such that

$$(Q \cap af(v) \setminus \{f(v)\}) \cap f(K \cap ve_a) \neq \emptyset,$$

whence we conclude that

$$(Q \cap af(v) \setminus \{f(v)\}) \cap f(K \cap vx_1) \neq \emptyset,$$

where x_1 is defined in (16) for $e = e_a$. Now since $f|vx_1 : vx_1 \to af(v)$ is monotone by (16), it is also a surjection onto the arc $af(v)$ (with $f(x_1) = a$). Therefore $f(K \cap vx_1) = Q \cap af(v)$, whence $f(K \cap be_a) = Q \cap af(v)$. Analogously we can find $e_b \in E(X)$ such that $f(K \cap ve_b) = Q \cap f(v)b$. So

$$f(K \cap (ve_a \cap ve_b)) = (Q \cap af(v)) \cup (Q \cap f(v)b) = Q,$$

which implies $f(K) = Q$. The proof is complete.

As a consequence of Theorems 5.7 and 5.8 and of Proposition 5.1 we obtain the following result.

COROLLARY 5.9. *Let a surjective mapping $f : X \to Y$ from a fan X with top v onto an arc Y be given. Then f is confluent (equivalently: is an OM-mapping) if and only if it satisfies condition* (16).

It is easy to verify, simply by the definition, that if a confluent mapping f from a fan X with top v onto an arc Y is additionally assumed to be light, then all partial mappings $f|x_k x_{k+1}$ claimed in (16) to be monotone turn out to be also one-to-one, i.e., they are homeomorphisms. And conversely, if all these partial mappings are homeomorphisms, then (using the same argument as in the proof of Proposition 4.14) one can see that f is light (and confluent by Theorem 5.8). Thus the following result follows from Corollary 5.9 by the above argument and by Proposition 5.2.

THEOREM 5.10. *Let a surjective mapping $f : X \to Y$ from a fan X with top v onto an arc Y be given. Then f is light confluent (equivalently: light open) if and only if the following condition holds:*

(17) *for each end point e of X there is a finite sequence of distinct points $v = x_0, x_1, \dots, x_n = e$ in the arc ve ordered from v to e such that for each $k \in \{0, 1, \dots, n-1\}$, the mapping $f|x_k x_{k+1} : x_k x_{k+1} \to Y$ is one-to-one; if $k \neq 0$, then $f|x_k x_{k+1}$ is a surjection onto the whole arc Y, i.e. it is a homeomorphism; and moreover, if $f(v) \in E(Y)$, then also $f|x_0 x_1$ is a surjection, so a homeomorphism onto Y.*

Conditions are considered in the above theorems under which a given mapping from a fan onto an arc is confluent (or light confluent). One can ask if these conditions can be satisfied for an arbitrary fan or, in other words, whether any fan admits a confluent mapping onto an arc. An answer to this question is negative. Namely, Proposition 2 of [33] says that each confluent image of the Lelek fan F_L is homeomorphic to F_L (see below, Theorem 12.14). In particular, there is no confluent mapping from the Lelek fan onto an arc. Thus the following problem seems to be natural.

PROBLEM 5.11. Characterize fans that admit a confluent (i.e. an OM-) mapping onto an arc.

Below we give some partial results related to this problem.

PROPOSITION 5.12. *If a fan X contains an open set U of the form $Z \times]0,1[$, where Z is a zero-dimensional compact set, then X admits a confluent mapping onto an arc.*

Proof. Assume that X contains a set $U = Z \times]0,1[\subset Z \times [0,1] \subset X$. Define a surjection $f : X \to [0,1]$ putting $f(x) = 1 - |2t-1|$ for $x = (z,t) \in U$ and letting $f(X \setminus U) = \{0\}$. Since Z is compact and U is open, we have

$$\operatorname{bd} U = \operatorname{cl} U \setminus U = (Z \times \{0\}) \cup (Z \times \{1\}).$$

Thus the points $x = (z,t) \in U$ tend to a point of $\operatorname{bd} U$ if and only if their second coordinates, t, tend either to 0 or to 1, and then the values

$f(x)$ tend to 0 by the definition of $f|U$. So we see f is continuous. It is straightforward to verify that f is confluent.

The next example indicates that Proposition 5.12 cannot be sharpened to get existence of an open mapping from the considered fan onto an arc.

EXAMPLE 5.13. *The one-point union of the Cantor fan and the Lelek fan (having as top the only common point) does not admit any open mapping onto an arc.*

Proof. Let F_C and F_L be homeomorphic copies of the Cantor and of the Lelek fans respectively, with their common top v as the only intersection point. Put $X = F_C \cup F_L$ and note that X contains an open set U of the form discussed in Proposition 5.12. Suppose on the contrary that there is an open mapping f from X onto an arc Y. Thus $F_L \setminus \{v\}$ is an open subset of X, so $f(F_L \setminus \{v\})$ is an open subset V of Y. Since $E(F_L)$ is a dense subset of $F_L \setminus \{v\}$, its image $f(E(F_L))$ is dense in V. Hence there are end points of X that are mapped neither onto end points of Y, nor onto the image of the top of X, a contradiction with (3) of Proposition 3.4.

The implication of Proposition 5.12 cannot be reversed, as the following example shows.

EXAMPLE 5.14. *There exists a fan X containing no open set U of the form $Z \times]0,1[$, where Z is a zero-dimensional compact set, and such that X admits an OM-mapping onto* $[0,1]$.

Proof. In the Cantor fan F_C consider a countable dense set $D \subset F_C \setminus S(F_C)$ such that $\operatorname{card}(D \cap ve) \leq 1$ for each $e \in E(F_C)$. Order D in a sequence $d_1, d_2, \dots, d_n, \dots$ and for each d_n denote by e_n an end point of F_C such that $d_n \in ve_n$. Next choose $a_n \in vd_n \setminus \{v, d_n\}$ and $b_n \in d_ne_n \setminus \{d_n, e_n\}$ satisfying the condition $\lim \operatorname{diam} a_nb_n = 0$. For each $n \in \mathbb{N}$ define a surjective mapping $\phi_n : ve_n \to ve_n$ putting $\phi_n(v) = v$, $\phi_n(a_n) = b_n$, $\phi_n(b_n) = a_n$, $\phi_n(e_n) = e_n$ and assuming that $\phi_n|va_n$, $\phi_n|a_nb_n$ and $\phi_n|b_ne_n$ are linear.

Let a relation $\sim$ on F_C be defined by $x \sim y$ if and only if $\phi_n(x) = \phi_n(y)$ for some $n \in \mathbb{N}$. Then $\sim$ is an equivalence relation. Note that all but countably many equivalence classes of $\sim$ are singletons, and that the non-degenerate ones consist of at most three points belonging to a straight line segment $a_nb_n \subset ve_n$ for some n. Since the lengths of these segments tend to zero, it follows that the decomposition of F_C into the equivalence classes of $\sim$ is upper semicontinuous. Hence the quotient space $X = F_C/\sim$ is a metric continuum ([116], Chapter 7, (2.2), (3.11) and (4.1), pp. 123–127). Let $q : F_C \to X$ be the quotient mapping. Since for each $e \in E(F_C)$ the set $q(ve)$ is an arc in X from $q(v)$ to $q(e)$ and since for any two distinct end points e and e' of F_C the arcs $q(ve)$ and $q(ve')$ are disjoint outside their common end point $q(v)$, the continuum X is both arcwise connected and hereditarily unicoherent, so it is a dendroid. Moreover, we conclude from the above argument that the dendroid X has only one ramification point, namely $q(v)$. Thus X is a fan.

Now let U be an open subset of X. Then $q^{-1}(U)$ is an open subset of F_C. Since D is dense in F_C and since the distances from a_n to b_n tend to zero, there is an index n_0 such that $a_{n_0}b_{n_0} \subset q^{-1}(U)$. Hence $q(a_{n_0}b_{n_0})$ is a subarc of $q(ve_{n_0})$ contained in U, and so it follows from the definitions of ϕ_{n_0} and of q that $q(v) < q(b_{n_0}) < q(a_{n_0}) < q(e_{n_0})$, where $<$ is the natural ordering of the arc $q(ve_{n_0}) = q(v)q(e_{n_0})$ from $q(v)$ to $q(e_{n_0})$. In the set $E(F_C)\setminus\{e_n : n \in \mathbb{N}\}$ choose a sequence $\{e^{(m)}\}$ tending to e_{n_0} and note that $q|ve^{(m)} : ve^{(m)} \to q(v)q(e^{(m)})$ is a homeomorphism for each m. Therefore there are $a^{(m)}$ and $b^{(m)}$ in $ve^{(m)}$ with $a^{(m)} \in vb^{(m)}$ satisfying $a^{(m)} \to a_{n_0}$ and $b^{(m)} \to b_{n_0}$ as $m \to \infty$. By continuity of q we see that for sufficiently large m the arc $q(v)q(e^{(m)})$ starts from $q(v)$, goes to $q(a^{(m)})$ which is close to $q(a_{n_0})$ (passing nearby $q(b_{n_0})$), next goes back from $q(a^{(m)})$ to $q(b^{(m)})$ which is close to $q(b_{n_0}) \in q(v)q(a_{n_0})$, and finally goes from $q(b^{(m)})$ to $q(e^{(m)})$ (passing again nearby $q(a_{n_0})$). So the arc $q(v)q(e^{(m)})$ forms a zig-zag. The sequence of these zig-zags contained in U tends to the arc $q(b_{n_0})q(a_{n_0})$. Hence U is not homeomorphic to the family of arcs $\{\{z\} \times]0,1[: z \in Z\}$, where Z is a zero-dimensional compact set.

To see that X admits an OM-mapping onto [0,1], define a monotone mapping g on X, shrinking, for each $n \in \mathbb{N}$ separately, the arc $q(b_n)q(a_n)$ to a point. Since the diameters of these arcs tend to zero as n tends to infinity, continuity of g follows.

Note that $gq : F_C \to gq(F_C) = g(X)$ is a nontrivial monotone mapping of F_C, whence it follows by (1) of Proposition 3.4 that $g(X)$ is either a fan or an arc. However, it cannot be an arc by Theorem 9 of [24] (quoted here as Theorem 5.18, see below). Thus $g(X)$ is a fan. Observe further that no end point of F_C is mapped onto the top of $g(X)$ under gq, whence we conclude by Theorem 10 of [24] (quoted here as Theorem 12.5) that $g(X)$ is homeomorphic to the Cantor fan F_C. Assuming $g(X) = F_C$ for shortness, consider the natural projection $p : g(X) \to [0,1]$ which maps the top of $g(X)$ to 0 and $E(g(X))$ to 1 and which is linear on each arc from the top to an end point of $g(X)$. Then p is open, and so $pg : X \to [0,1]$ is an OM-mapping as required. The proof is complete.

However, the existence of an open subset U having the discussed form in a fan X can be derived from the assumption that the considered confluent mapping of X onto an arc is light (thus it is light open according to Proposition 5.2).

PROPOSITION 5.15. *If a fan X admits a light confluent (equivalently: a light open) mapping onto an arc, then X contains an open set U having the form of the product $Z \times]0,1[$, where Z is a zero-dimensional compact set.*

Proof. Let a fan X admit a light confluent mapping onto an arc. Without loss of generality we may consider the arc to be the closed unit interval $[0,1]$, and moreover, it can be assumed that the top v of X is mapped to an end of the interval. Indeed, if $f : X \to [0,1]$ is confluent with $f(v) \in]0,1[$, we define an open surjective mapping $g : [0,1] \to [0,1]$ putting $g(0) = 1 = g(1)$ and $g(f(v)) = 0$, and taking $g|[0, f(v)]$ and $g|[f(v), 1]$ to be linear. Then the composition $gf : X \to X$ is confluent ([8], VI and III, p. 214) and we have $gf(v) = 0$.

So, let a mapping $f : X \to [0,1]$ be light confluent, with $f(v) = 0$. Consider the set $Z = f^{-1}(1/2)$ which is obviously compact and zero-dimensional.

According to condition (17) of Theorem 5.10, each component of $f^{-1}(1/2)$ is contained in an arc $x_k x_{k+1} \subset ve$ for some $e \in E(X)$ and for some $k \in \mathbb{N}$, such that $f|x_k x_{k+1} : x_k x_{k+1} \to [0,1]$ is a homeomorphism. Thus either $f(x_k) = 0$ and $f(x_{k+1}) = 1$ or $f(x_k) = 1$ and $f(x_{k+1}) = 0$. In any case $x_k x_{k+1} \cap f^{-1}(]0,1[)$ is an arc without its end points, and so it has the form $\{z\} \times]0,1[$, where z represents that component of $f^{-1}(1/2)$ which is contained in $x_k x_{k+1}$. Therefore $U = f^{-1}(]0,1[)$ is an open subset of X which has the form of the product $Z \times]0,1[$. Indeed, the needed homeomorphism $h : U \to Z \times]0,1[$ is defined by $h(x) = (z, f(x))$, where $x \in U$ and the point $z \in Z$ is uniquely determined by the condition that both x and z belong to the same component of U. The proof is complete.

Remarks 5.16. 1) It can be shown that the assumption of lightness of the mapping in Proposition 5.15 (for both versions) is essential. An example showing this for light open mappings will be constructed in Chapter 7 (Example 7.8).

2) The reader is referred to Theorem 12.14, where the same condition of existence of a confluent mapping from a fan onto an arc is considered under the additional assumption of smoothness of the fan, and where a further structural characterization is shown. For some related results see 12.7, 12.8, 12.9 and 12.16.

After the above discussion related to Problem 5.11 it is very natural to ask for a similar (i.e. intrinsic, structural) characterization of those fans which can be mapped onto an arc under a mapping belonging to any particular subclass of the class of all confluent mappings (monotone, open or light open, for example). For monotone mappings the problem has already been solved (see Theorem 5.18 below). But for other classes of (confluent) mappings we have no full answer.

PROBLEMS 5.17. Characterize fans that admit (a) an open, (b) a light open mapping onto an arc.

Only a partial answer to 5.17 is known, under the additional assumption of smoothness of the considered fans (see the eleventh and twelfth chapters, where besides smoothness of fans, a restriction is put for mappings to be retractions: 11.16, 12.9–12.12, 12.18 and 12.19).

As was said above, we know a solution of the considered problem for monotone mappings. To formulate it we need a definition.

An arc ab contained in a continuum X is said to be *free* provided $ab \setminus \{a, b\}$ is an open subset of X. The following characterization of fans that admit a monotone mapping onto an arc is shown as Theorem 9 of [24].

THEOREM 5.18. *A fan X can be mapped onto an arc under a monotone mapping if and only if X contains a free arc.*

It will be shown in Theorem 7.11 that no fan can be mapped onto an arc under a mapping which is both monotone and open.

6. A characterization of the top

Given a point x of a dendroid X, let us denote by $\mathrm{Ord}_x X$ the (cardinal) number of arc components of the set $X \setminus \{x\}$ (for another form of this definition see [6], p. 230). Therefore if X is a fan with top v, then obviously $\mathrm{Ord}_v X = \operatorname{card} E(X)$. Hence by (1) of Proposition 3.4 and by (9) of Theorem 4.1 we have the following result.

PROPOSITION 6.1. *If a confluent mapping f is defined on a fan X with top v, then*

$$\mathrm{Ord}_{f(x)} f(X) \le \mathrm{Ord}_v X . \tag{18}$$

Now we shall prove a converse implication.

PROPOSITION 6.2. *Let a fan X be given and let a point v of X be such that inequality* (18) *holds true for each monotone mapping f defined on X and having just one nondegenerate point-inverse. Then v is the top of X.*

Proof. We shall show that if a point x of X is not the top, then there exists a monotone mapping f on X with exactly one nondegenerate point-inverse and such that

$$\mathrm{Ord}_x X < \mathrm{Ord}_{f(x)} f(X) . \tag{19}$$

Indeed, denote by v the top of X. Thus $\mathrm{Ord}_v X \geq 3$. Since $x \neq v$, we have $\mathrm{Ord}_x X < 3$. Identify the arc vx to the point v and let $f : X \to Y = X/vx$ be the identification mapping. Thus f is monotone and for each singleton of Y distinct from the image of vx under f its inverse image is again a singleton. If $\mathrm{Ord}_x X = 2$, then $\mathrm{Ord}_{f(x)} Y = \mathrm{Ord}_v X$, so (19) is true. If $\mathrm{Ord}_x X = 1$ (i.e., if $x \in E(X)$), then $\mathrm{Ord}_{f(x)} Y \geq 2$, so again (19) holds. Thus the proof is complete.

Since each mapping with exactly one nondegenerate point-inverse (that is connected) is obviously monotone, and each monotone mapping is confluent, Propositions 6.1 and 6.2 imply the following theorem.

THEOREM 6.3. *A point v of a fan X is the top of X if and only if inequality* (18) *holds for each mapping f defined on X which is either* 1° *monotone with just one nondegenerate point-inverse, or* 2° *monotone, or* 3° *confluent.*

Remark 6.4. A version of Theorem 6.3, namely for confluent mappings, has earlier been obtained in [86].

Remark 6.5. Consider an arbitrary open mapping f defined on a simple triod X. For an end point v of X we have $\mathrm{Ord}_v X = 1$ and $\mathrm{Ord}_{f(v)} f(X) = 1$ by Corollary (7.31) of [116], p. 147. Therefore (18) holds true for each open mapping f, while v is not the top. Thus the class of open mappings can neither be substituted for the one considered in Proposition 6.2 nor joined to those listed in Theorem 6.3.

7. Open mappings and their lightness

In Chapters 4 and 5 some conditions were presented under which a mapping defined on a fan is both light and confluent (see Proposition 4.12, Corollaries 4.13 and 4.15 and Theorem 5.10). Light confluent mappings defined on some (rather simple) continua coincide with open ones. Such a coincidence holds e.g. for a simple closed curve and for an arc: the implication one way is shown in [116], Chapter 10, (1.2) and (1.3), p. 184, the other one follows from a result saying that all locally confluent mappings from compacta onto locally connected spaces are OM-mappings (see [70], Corollary 5.2, p. 109). Hence a simple closed curve and an arc are examples of continua on which each open mapping is light. It is well known that all linear graphs have this property (see [116], Chapter 10, Theorem (1.1), p. 182). On the other hand as early as 1935 B. Knaster constructed a nontrivial monotone open mapping of an irreducible continuum onto an arc ([55], Section 5, p. 574), and E. Dyer showed in 1953 that each such mapping must possess indecomposable point-inverses ([43], Theorem, p. 591).

A very interesting example of a nontrivial monotone open retraction of a (smooth) dendroid onto an arc has been constructed by L. G. Oversteegen (see [108], Example 3.2, p. 118; some other examples of nontrivial monotone open mappings are also given there).

Thus it is natural to ask how the situation looks like for fans: must each open mapping on a fan be light or are there open mappings of fans having nondegenerate components of point-inverses? Analyzing examples of mappings known from the literature one can conjecture that openness implies lightness for any fan as the domain space. It will be shown later, in Theorem 9.6 below, that the implication holds indeed, provided the mapping is defined on a locally connected fan. However, it is not true in general. We now give an example (constructed by M. Morayne) showing that local connectedness is an essential assumption in the discussed implication. The same conclusion also follows from Example 7.7.

EXAMPLE 7.1. (Morayne) *There exists a nonlight open mapping from the harmonic fan onto an arc.*

Proof. Recall that the harmonic fan F_H in the Cartesian coordinates in the plane has the top $v = (0,0)$ and end points $e_0 = (0,1)$ and $e_n = (1/n, 1)$ for $n \in \mathbb{N}$, so that $F_H = \bigcup\{\overline{ve_n} : n \in \{0,1,2,\ldots\}\}$. To describe the required mapping $f : F_H \to \overline{ve_0}$ we need some auxiliary notations.

Arrange all rationals lying in the open interval $]1/3, 2/3[$ in a sequence $\{\rho_n : n \in \mathbb{N}\}$. Put $a = (0, 1/3)$, $b = (0, 2/3)$ and for each $n \in N$ take points a_n, r_n, b_n in the segment $\overline{ve_n}$ such that their second coordinates are $1/3$, ρ_n and $2/3$ respectively. Thus, if $<$ denotes the natural order on $\overline{ve_n}$ from v to e_n, then $v < a_n < r_n < b_n < e_n$. Since $[1/3, 2/3] = \operatorname{cl}\{\rho_n : n \in \mathbb{N}\}$, we have

$$\overline{ab} = \operatorname{cl}\{r_n : n \in \mathbb{N}\} \setminus \{r_n : n \in \mathbb{N}\}\,. \tag{20}$$

Finally, take in $\overline{ab}$ two sequences of points, c_n^- and c_n^+, converging to the midpoint c of ab from below and from above respectively, e.g. such that their second coordinates are $1/2 - 1/(6n)$ and $1/2 + 1/(6n)$, correspondingly.

We shall define f on F_H consecutively. Put $f(v) = v$, $f(e_0) = e_0$, $f(\overline{ab}) = \{c\}$, and let the partial mappings

$$f|\overline{va} : \overline{va} \to \overline{vc} \qquad \text{and} \qquad f|\overline{be_0} : \overline{be_0} \to \overline{ce_0}$$

be linear. Thus $f|\overline{ve_0} : \overline{ve_0} \to \overline{ve_0}$ is a nontrivial monotone mapping. Further, for each $n \in \mathbb{N}$, put $f(e_n) = e_0$, $f(a_n) = c_n^-$, $f(r_n) = c$, $f(b_n) = c_n^+$, and extend the mapping linearly to the whole segment $\overline{ve_n}$, i.e., the partial mappings

$$f|\overline{va_n} : \overline{va_n} \to \overline{vc_n^-}\,, \qquad f|\overline{a_nr_n} : \overline{a_nr_n} \to \overline{c_n^-c}\,,$$

$$f|\overline{r_nb_n} : \overline{r_nb_n} \to \overline{cc_n^+}\,, \qquad f|\overline{b_ne_n} : \overline{b_ne_n} \to \overline{c_n^+e_0}$$

are linear. Therefore, for each $n \in \mathbb{N}$, $f|\overline{ve_n} : \overline{ve_n} \to \overline{ve_0}$ is a homeomorphism.

The mapping $f : F_H \to \overline{ve_0}$ is now completely described. It is evident that f is continuous and not light. We shall show that it is open. To this end it is enough to verify that f is interior at each x in F_H. Since for each segment $\overline{ve_n}$ in F_H with $n \neq 0$ the partial mapping $f|ve_n$ is a homeomorphism, the interiority of f at each $x \in \overline{ve_n}$ is obvious. So, let $x \in \overline{ve_0} \setminus \{v\}$ and let V be a neighborhood of x in F_H. If x is not in $\overline{ab}$, then there exists an arc uw such that $x \in uw \setminus \{u, w\} \subset uw \subset (\overline{ve_0} \cap V) \setminus \overline{ab}$. Since $f|\overline{va}$ and $f|\overline{be_0}$ are linear, $f(x)$ is an interior point of $f(uw)$, hence of $f(V)$. If $x \in \overline{ab}$, there exists by (20) a point $r_n \in V$. Take an arc uw such that $r_n \in uw \setminus \{u, w\} \subset uw \subset \overline{ve_n} \cap V$. Since $f(x) = f(r_n) = c$ and since $f|\overline{ve_n}$ is a homeomorphism, $f(x)$ is again an interior point of $f(V)$. So f is open, and thus the proof is finished.

A simple modification of the example above leads to a mapping from the harmonic fan onto a finite fan having the same properties. We shall show this constructing a mapping from F_H onto a simple triod contained in F_H.

EXAMPLE 7.2. *There exists a nonlight open mapping from the harmonic fan onto a simple triod.*

Proof. Indeed, it is enough to define $g : F_H \to \overline{ve_0} \cup \overline{ve_1} \cup \overline{ve_2} \subset F_H$ such that $g|\overline{ve_1}$ and $g|\overline{ve_2}$ are identities, and to put $g(x) = f(x)$ for $x \in F_H \setminus (\overline{ve_1} \cup \overline{ve_2})$, where f denotes the mapping defined in Example 7.1.

For a modification of Examples 7.1 and 7.2 see Example 11.17 below. Observe that the range space in all these examples is locally connected. In fact, it can be shown that each open mapping from the harmonic fan onto a not locally connected fan is light. Before proving the result we need a lemma. Recall that $L(X)$ is the set of all points of local connectedness of a space X.

LEMMA 7.3. *Let fans X and Y be such that $X \subset F_H$ and Y is not locally connected, and let a surjection $f : X \to Y$ be open. If $x_0 \in X \setminus L(X)$ and $x_1 \in L(X) \setminus \{v\}$, where v is the top of X, then $f(x_0) \neq f(x_1)$.*

Proof. Let $d_0, d_1 \in E(X)$ be such that $x_0 \in vd_0$ and $x_1 \in vd_1$, and denote by v' the top of Y. Suppose that $f(x_0) = f(x_1)$. Then $f(vd_0) = f(vd_1) \subset L(Y)$. Indeed, $f|vd_0$ and $f|vd_1$ are monotone onto $v'f(d_0)$ and $v'f(d_1)$ respectively, and $f(d_0), f(d_1) \in E(Y)$ by (9) of Theorem 4.1. Moreover, $f|vd_1$ is a homeomorphism by openness of f, so $f(x_1) = f(x_0) \neq v'$ and therefore the arcs $v'f(d_0)$ and $v'f(d_1)$ have two distinct points in common, so they must coincide. Since $vd_1 \subset L(X)$, we have

$f(vd_0) = f(vd_1) \subset L(Y)$ by Proposition 3.11. Therefore we have shown that the image of any point in X is in $L(Y)$, so Y is locally connected, a contradiction.

PROPOSITION 7.4. *Each open mapping from a subfan of the harmonic fan onto a not locally connected fan is light.*

Proof. Take two fans X and Y such that $X \subset F_H$ and Y is not locally connected. Suppose $f : X \to Y$ is a nonlight open surjection. Apply notation used in 7.1. It is clear that for each $n \neq 0$, $f|(X \cap ve_n)$ is a homeomorphism. Since f is nonlight and $f|(X \cap ve_0)$ is monotone by (9) of Theorem 4.1, there is an arc $a'b' \subset X \cap ve_0$ such that $a'b' = (X \cap ve_0) \cap f^{-1}(y)$ for some $y \in Y$. Take an open set U in X such that $a'b' \cap U \neq \emptyset$ and $a', b' \in X \setminus U$. Since $a'b' \neq X \cap ve_0$ (otherwise Y is locally connected), we can assume that a' is not an end point of X and differs from the top v. Let $a_n \in (X \cap ve_0) \setminus a'b'$ be a sequence tending to a'. By Lemma 7.3 we see that $f^{-1}(f(a_n)) \subset X \cap ve_0$, and therefore $f(a_n)$ is outside $f(U)$. So we have constructed a sequence $f(a_n)$ tending to $f(a') = y \in f(U)$, thus $f(U)$ is not open, a contradiction.

Remarks 7.5. 1) The requirement regarding the domain space to be contained in F_H cannot be omitted in Proposition 7.4. In fact, let F_H^1 and F_H^2 be two copies of the harmonic fan such that $F_H^1 \cap F_H^2 = \{v\}$. Let e_0^1 denote the only accumulation point of $E(F_H^1)$. Put $X = F_H^1 \cup F_H^2$ and $Y = ve_0^1 \cup F_H^2$, and define a surjection $f : X \to Y$ as follows. The partial mapping $f|F_H^1 : F_H^1 \to ve_0^1$ is the nonlight open mapping of Example 7.1, while $f|F_H^2 : F_H^2 \to F_H^2$ is the identity. Then X cannot be embedded into the harmonic fan, Y is homeomorphic to F_H and thus it is not locally connected, and f is nonlight and open.

2) Examples 7.1 and 7.2 show that the two assumptions on the range space in Proposition 7.4 (of being a fan and of local connectedness) are also essential.

Now consider again the harmonic fan $F_H = \bigcup\{\overline{ve_n} : n \in \{0,1,2,\ldots\}\}$. Let $\{\rho_n : n \in \mathbb{N}\}$ be all rationals in the open unit interval $]0,1[$. For each $n \in \mathbb{N}$ take $x_n \in \overline{ve_n}$ such that $d(v, x_n) = \rho_n \cdot d(v, e_n)$, where d denotes the Euclidean metric in the plane, put $x_0 = e_0$ and define

$$F_{HS} = \bigcup\{\overline{vx_n} : n \in \{0,1,2,\ldots\}\} \subset F_H .$$

We call F_{HS} the *harmonic shredded fan*. Note that $\overline{vx_0} \setminus \{x_0\} = \mathrm{cl}\, E(F_{HS}) \setminus E(F_{HS})$. The reader can verify that the definition does not depend on the sequence $\{\rho_n : n \in \mathbb{N}\}$, i.e. that all harmonic shredded fans are homeomorphic.

PROPOSITION 7.6. *Each open mapping defined on the harmonic shredded fan is light.*

Proof. Suppose there is an open nonlight surjective mapping $f : F_{HS} \to Y$. By Proposition 7.4 the range space Y is either a locally connected fan or an arc. If the image $f(\overline{vx_0})$ of the limit segment $\overline{vx_0}$ of F_{HS} is nondegenerate, then it is an arc, because if Y is a fan, then $f|\overline{vx_0} : \overline{vx_0} \to f(v)f(x_0)$ is monotone and maps $\overline{vx_0}$ onto the arc $f(v)f(x_0)$ (see (9) of Theorem 4.1). Take $y \in f(v)f(x_0) \setminus \{f(v), f(x_0)\}$. Thus there is $x \in \overline{vx_0}$ with $f(x) = y$. Since $\overline{vx_0} \subset \operatorname{cl} E(F_{HS})$, there is a sequence x_{n_m} of end points of F_{HS} with limit x. Thus the sequence $f(x_{n_m})$ tends to y. Since Y is locally connected at y, the points $f(x_{n_m})$ lie in the arc $f(\overline{vx_0})$ far from its end points, which contradicts (3) of Proposition 3.4.

Therefore $f(\overline{vx_0})$ must be degenerate. Thus there exists an end point e' of Y which is outside $f(\overline{vx_0})$. Consider two cases. If Y is a (locally connected) fan, there is an open neighborhood U of x_0 having the form

$$U = \{x \in F_{HS} : d(v, x) > 1 - \varepsilon \text{ and if } x \in \overline{vx_n} \text{ then } n \geq n_0\}$$

for some $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, and such that $U \cap f^{-1}(e') = \emptyset$. Then $f(v) \in f(U)$, and since $f(U)$ is open, there is a point y in $f(U) \cap (f(v)e' \setminus \{f(v)\})$. Let $x \in U \cap f^{-1}(y) \cap \overline{vx_n}$ for some $n \geq n_0$ with $x_n \in E(F_{HS})$. Then $x_n \in U$ by the definition of U. Thus by (9) of Theorem 4.1 we have $f(x_n) = e'$, which contradicts the definition of U. If Y is an arc, let V be an open neighborhood of $f(v)$ that contains no end point of Y (possibly except $f(v)$, in case $f(v) \in E(Y)$). Then the open set $f^{-1}(V)$ contains the whole segment $\overline{vx_0}$. Since $\overline{vx_0} \subset \operatorname{cl} E(F_{HS})$, each end point x_n of F_{HS} which is in $f^{-1}(V)$ is mapped to $f(v)$ by (3) of Proposition 3.4, and we see that for such x_n the segments $\overline{vx_n}$ are mapped onto the singleton $\{f(v)\}$ by (2) of Proposition 3.4, so f is not open. The proof is complete.

Besides Examples 7.1 and 7.2 we give another example of an open but not light mapping of a fan onto an arc. The example concerns the Cantor fan and is related to the Lelek fan located in the previous one in a special way.

EXAMPLE 7.7. *There is a surjective mapping $f : F_C \to [0,1]$ such that:* 1) *f is open;* 2) *$f^{-1}(0)$ is the Lelek fan;* 3) *for all $y \in\,]0,1]$, $f^{-1}(y)$ is homeomorphic to the Cantor set.*

Proof. Take F_C to be the cone in the Euclidean plane with Cartesian coordinates with vertex $v = (1/2, 0)$ over $C \times \{1\}$. Further, let F_L denote a copy of the Lelek fan diminished twice with respect to the original size of D in [64], Fig. 4, p. 316, and embedded into F_C. Thus both fans have the common top v, and $F_L \subset F_C$ is located in the lower half of F_C, i.e., the image of F_L under the orthogonal projection onto the y-axis is the interval $[0, 1/2]$. Since the set of end points of F_L is dense in F_L, we see that F_L is a boundary subset of F_C.

Consider the monotone decomposition of F_C whose only nondegenerate element is $F_L \subset F_C$, and let $q : F_C \to F_C/F_L$ be the quotient mapping. So $q(F_C)$ is the image of the Cantor fan under a monotone mapping with a closed set of end points (since $q|E(F_C)$ is a homeomorphism onto $E(q(F_C))$). Therefore by Theorem 10 of [24] (quoted here as Theorem 12.5 below; compare also [102], Lemma 3, p. 101) $q(F_C)$ is homeomorphic to F_C, and we will simply write $q : F_C \to F_C$. Now let p be the orthogonal projection of F_C to the y-axis. Note that p is light and open. Finally, put $f = pq : F_C \to [0,1]$. Therefore $f^{-1}(0) = F_L$, and for each $y \in [0,1]$ the set $f^{-1}(y)$ is closed (thus compact) zero-dimensional without any isolated points, so it is homeomorphic to the Cantor set (see e.g. [47], Exercise 6.2.A(c), p. 455). To see that f is open, take an arbitrary open set $U \subset F_C$ and note that $f(U) = \bigcup\{f(U \cap ve) \subset [0,1] : e \in E(F_C)\}$. If $U \cap F_L = \emptyset$, then $f(U)$ is a union of open subsets of $[0,1]$. Otherwise $f(U)$ is a union of open subsets of $[0,1]$ and of $\{0\}$. Since $E(F_L)$ is dense in F_L, and $U \cap F_L \neq \emptyset$, we can find a point $u \in E(F_L) \cap U$. Denoting by $e \in E(F_C)$ the point such that $u \in ve$, we see $(U \cap ve) \setminus F_L \neq \emptyset$, and therefore $f(U \cap ve)$ is an open neighborhood of 0. So $f(U)$ is open, which concludes the proof.

Our next example also shows a nonlight open mapping from a fan X onto $[0,1]$. Unlike the previous one, the set of points of the range having non-zero-dimensional preimages is dense in $[0,1]$. The example is related to the existence of an open subset U of X having a special form, discussed in Propositions 5.12 and 5.15, and it shows that lightness is essential in Proposition 5.15, as already said in 1) of Remarks 5.16. The method of its construction consists in, roughly speaking, the condensation of singularities that occur in Examples 7.1 and 5.14.

EXAMPLE 7.8. *There exist a fan X not containing any open set U of the form $Z \times \,]0,1[$, where Z is a zero-dimensional compact set, and a surjective mapping $f : X \to [0,1]$ such that* 1) *f is open;* 2) *f is nonlight;* 3) *the set $\{y \in [0,1] : f^{-1}(y)$ is not zero-dimensional$\}$ is a countable dense subset of $]0,1[$;* 4) *$f(v) = 0$ and $f(e) = 1$ for each $e \in E(X)$;* 5) *$f|ve : ve \to [0,1]$ is a monotone surjection for each $e \in E(X)$.*

Proof. Let $F_C = (C \times [0,1])/(C \times \{0\})$ be the standard Cantor fan, whose top is denoted by v. Hence each $x \in F_C \setminus \{v\}$ can be written in the form (c,t), where $c \in C$ and $t \in \,]0,1]$. Let $p_1 : F_C \setminus \{v\} \to C$ and $p_2 : F_C \setminus \{v\} \to \,]0,1]$ be the natural projections. We also put $p_2(v) = 0$.

Let a subset E of $E(F_C)$ be defined by $e = (c,1) \in E$ provided that either $c = 1$ or c is the left end of a contiguous interval of C (i.e., there is a $\gamma > 0$ such that no point of C is in $]c, c+\gamma[$). Hence E is a countable dense subset of $E(F_C)$. Order its points in a sequence $e_1, e_2, \dots, e_n, \dots$, and choose, for each $n \in \mathbb{N}$, a point $d_n \in ve_n \setminus \{v, e_n\}$ in such a way that $D = \{d_n : n \in \mathbb{N}\}$ is dense in F_C. Let $c_n = p_1(d_n) \in C$.

Now put $C_0 = C$ and take, for each $n \in \mathbb{N}$, closed and open subsets C_n of C with $\operatorname{diam} C_n < 1/2^n$ such that $C_n = \{c_n\} \cup \bigcup\{C_n^m : m \in \mathbb{N}\}$, where the C_n^m are closed and open, pairwise disjoint (for a fixed n) subsets having $\{c_n\}$ as their limit (if m tends to infinity and n is fixed), and such that for each $m \in \mathbb{N}$ and $i < j$ the intersection $C_i^m \cap C_j$ is either empty or C_j. We shall use the sequence $\{C_n\}$ to construct, by induction, open sets V_n of F_C and closed nowhere dense sets $A_n \subset \operatorname{cl} V_n$ having some special properties that will be employed to define both the needed fan X and the required open mapping from X onto $[0,1]$.

To start the inductive procedure take $d_1 \in D$ and $0 < \varepsilon_1 < 1/2$ such that $\alpha_1 = p_2(d_1) - \varepsilon_1 > 0$ and $\beta_1 = p_2(d_1) + \varepsilon_1 < 1$, and $\alpha_1, \beta_1 \in\ ]0,1[\ \setminus p_2(D)$. Put

$$V_1 = C_1 \times\ ]\alpha_1, \beta_1[\quad \text{and} \quad I_1 = \{c_1\} \times [\alpha_1, \beta_1] .$$

Choose a sequence of numbers δ_1^m, and next find a sequence of positive numbers ε_1^m tending to zero if m tends to infinity in such a way that $\{\delta_1^m - \varepsilon_1^m, \delta_1^m, \delta_1^m + \varepsilon_1^m\} \subset\ ]\alpha_1, \beta_1[\ \setminus p_2(D)$ and $\operatorname{cl}\{\delta_1^m : m \in \mathbb{N}\} = [\alpha_1, \beta_1]$. Define

$$A_1 = I_1 \cup (C_1 \times \{\alpha_1, \beta_1\}) \cup \bigcup\{C_1^m \times \{\delta_1^m - \varepsilon_1^m, \delta_1^m, \delta_1^m + \varepsilon_1^m\} : m \in \mathbb{N}\} .$$

Thus A_1 is a closed nowhere dense subset of F_C contained in $\operatorname{cl} V_1$, and we see that $A_1 \cap D = \{d_1\}$. The first step of the induction is finished. We additionally put $V_0 = F_C$ and $A_0 = S(F_C) = \{v\} \cup E(F_C)$.

Now fix $n \in \mathbb{N}$ and assume that for each $i \in \{1, \dots, n\}$ we have defined:

1° positive numbers $\varepsilon_i < 1/2^i$, $\alpha_i = p_2(d_i) - \varepsilon_i$ and $\beta_i = p_2(d_i) + \varepsilon_i < 1$ (where $d_i \in D$), with $\alpha_i, \beta_i \in\ ]0,1[\ \setminus p_2(D)$,

2° the open subsets V_i of F_C having the form

$$V_i = C_i \times\ ]\alpha_i, \beta_i[,$$

3° the straight line segments $I_i = C_i \times [\alpha_i, \beta_i]$,

4° the sequences of positive numbers δ_i^m and $\varepsilon_i^m \to 0$ if $m \to \infty$ satisfying $\{\delta_i^m - \varepsilon_i^m, \delta_i^m, \delta_i^m + \varepsilon_i^m\} \subset\ ]\alpha_i, \beta_i[\setminus p_2(D)$ with $\operatorname{cl}\{\delta_i^m : m \in \mathbb{N}\} = [\alpha_i, \beta_i]$, and

5° the closed nowhere dense subsets A_i of F_C defined by

$$A_i = I_i \cup (C_i \times \{\alpha_i, \beta_i\}) \cup \bigcup\{C_i^m \times \{\delta_i^m - \varepsilon_i^m, \delta_i^m, \delta_i^m + \varepsilon_i^m\} : m \in \mathbb{N}\} ,$$

and such that $(A_0 \cup A_1 \cup \ldots \cup A_{i-1}) \cap \operatorname{cl} V_i = \emptyset$.

Note that $A_i \subset \operatorname{cl} V_i$, and that $A_i \cap D = \{d_i\}$, whence we conclude $(A_0 \cup A_1 \cup \ldots \cup A_n) \cap D = \{d_1, \ldots, d_n\}$ and so $d_{n+1} \in D \setminus (A_0 \cup A_1 \cup \ldots \cup A_n)$. Since $A_0 \cup A_1 \cup \ldots \cup A_n$ is a closed and nowhere dense subset of F_C, and d_{n+1} is outside this union, there is an open neighborhood V_{n+1} of d_{n+1} satisfying all the above conditions with $n+1$ in place of n. Thus A_{n+1} can be defined in the same way. So the inductive procedure is finished.

Now we are ready to define the fan X. We follow the idea of Example 5.14. For each $n \in \mathbb{N}$ put $a_n = (c_n, \alpha_n)$, $b_n = (c_n, \beta_n)$ and note that $d_n \in a_n b_n = I_n \subset v e_n \setminus \{v, e_n\}$, whence $\lim \operatorname{diam} a_n b_n = 0$. Consider a surjection $\phi_n : v e_n \to v e_n$ such that

$$\phi_n(v) = v\,, \quad \phi_n(a_n) = b_n\,, \quad \phi_n(b_n) = a_n\,, \quad \phi_n(e_n) = e_n\,,$$

and $\phi_n | v a_n$, $\phi_n | a_n b_n$ and $\phi_n | b_n e_n$ are linear. The relation $\sim$ on F_C defined by the condition $x \sim y$ if and only if $\phi_n(x) = \phi_n(y)$ for some $n \in \mathbb{N}$ is an equivalence relation. Thus the quotient space $X = F_C/\!\sim$ is a fan constructed in the same way as that of Example 5.14. Therefore X does not contain any open set U of the form $Z \times]0,1[$, where Z is zero-dimensional and compact. However, the needed open mapping $f : X \to [0,1]$ has to be defined much more carefully than the OM-mapping of 5.14.

Consider, as in Example 5.14, the quotient mapping $q : F_C \to X$ and recall the monotone mapping g of X which shrinks, for each $n \in \mathbb{N}$ separately, the arc $q(I_n)$ to a point. Arguing as in the corresponding part of the proof of 5.14 we conclude that $g(X)$ is homeomorphic to F_C. Assuming $g(X) = F_C$ for shortness, we see that $gq : F_C \to F_C$ is a monotone mapping. For further purposes we now describe a mapping $h : F_C \to F_C$ that is topologically equivalent to gq. The mapping h will be defined as the limit mapping of a uniformly convergent sequence of mappings $h_n : F_C \to F_C$ with $n \in \{0, 1, 2, \ldots\}$ which in turn will be defined by induction on n in such a way that for each n and for each $e \in E(F_C)$, the partial mapping $h_n | ve : ve \to ve$ is monotone and piecewise linear. We stress that each segment ve is mapped onto itself under each h_n.

Let $h_0 : F_C \to F_C$ be the identity mapping. Now fix $n \in \{0, 1, 2, \ldots\}$ and assume that for each $i \in \{0, 1, \ldots, n\}$ we have defined mappings $h_i : F_C \to F_C$ in such a way that:

1° for each $e \in E(F_C)$, the mapping $h_i | ve : ve \to ve$ is monotone and piecewise linear, with $h_i(v) = v$ and $h_i(e) = e$,

2° $h_n | (A_0 \cup A_1 \cup \ldots \cup A_i) = h_i | (A_0 \cup A_1 \cup \ldots \cup A_i)$, and

3° $h_i(V_i) \subset V_i$.

Take the open set V_{n+1} and note that $p_1(V_{n+1}) = C_{n+1}$. Recall that $(A_0 \cup A_1 \cup \ldots \cup A_n) \cap \operatorname{cl} V_{n+1} = \emptyset$ and that $A_0 \cup A_1 \cup \ldots \cup A_n$ is a closed nowhere dense subset of F_C. Define

$$s = \max p_2\big((A_0 \cup A_1 \cup \ldots \cup A_n) \cap (C_{n+1} \times [0, \alpha_{n+1}])\big)$$

and

$$u = \min p_2\big((A_0 \cup A_1 \cup \ldots \cup A_n) \cap (C_{n+1} \times [\beta_{n+1}, 1])\big).$$

It follows from the definition of the system of sets C_i^m with $i \in \{0, 1, \ldots, n\}$ and $m \in \mathbb{N}$, and from the definition of the sets A_i that $C_{n+1} \times \{s, u\} \subset A_0 \cup A_1 \cup \ldots \cup A_n$ and $(A_0 \cup A_1 \cup \ldots \cup A_n) \cap (C_{n+1} \times]s, u[) = \emptyset$. Put

$$h_{n+1}|(F_C \setminus C_{n+1} \times]s, u[) = h_n|(F_C \setminus C_{n+1} \times]s, u[),$$

whence it follows in particular that the values $h_{n+1}((c, s)) = h_n((c, s))$ and $h_{n+1}((c, u)) = h_n((c, u))$ with $c \in C_{n+1}$ are already defined. We have to define $h_{n+1}|(C_{n+1} \times]s, u[)$. To this end fix $c \in C_{n+1}$. If $c = c_{n+1}$, let $h_{n+1}(I_{n+1}) = \{d_{n+1}\}$ and define

$$h_{n+1}|(c_{n+1}, s)a_{n+1} : (c_{n+1}, s)a_{n+1} \to h_{n+1}((c_{n+1}, s))d_{n+1}$$

and

$$h_{n+1}|b_{n+1}(c_{n+1}, u) : b_{n+1}(c_{n+1}, u) \to d_{n+1}h_{n+1}((c_{n+1}, u))$$

to be linear. If $c \in C_{n+1} \setminus \{c_{n+1}\}$, then there is $m \in \mathbb{N}$ with $c \in C_{n+1}^m$. Consider five points: (c, α_{n+1}), $(c, \delta_{n+1}^m - \varepsilon_{n+1}^m)$, (c, δ_{n+1}^m), $(c, \delta_{n+1}^m + \varepsilon_{n+1}^m)$, and (c, β_{n+1}), forming the intersection $A_{n+1} \cap ve$, where $e = (c, 1)$, and recall that $0 \le s < \alpha_{n+1} < \delta_{n+1}^m - \varepsilon_{n+1}^m < \delta_{n+1}^m < \delta_{n+1}^m + \varepsilon_{n+1}^m < \beta_{n+1} < u \le 1$. Since $c_{n+1} = \max C_{n+1}$, we have $0 < c_{n+1} - c \le 1$. Put

$$\begin{aligned}
h_{n+1}((c, \alpha_{n+1})) &= (c, p_2(d_{n+1}) - (c_{n+1} - c)\varepsilon_{n+1}), \\
h_{n+1}((c, \delta_{n+1}^m - \varepsilon_{n+1}^m)) &= (c, p_2(d_{n+1}) - (c_{n+1} - c)\varepsilon_{n+1}/2), \\
h_{n+1}((c, \delta_{n+1}^m)) &= (c, p_2(d_{n+1})), \\
h_{n+1}((c, \delta_{n+1}^m)) &= (c, p_2(d_{n+1}) + (c_{n+1} - c)\varepsilon_{n+1}/2), \\
h_{n+1}((c, \beta_{n+1})) &= (c, p_2(d_{n+1}) - (c_{n+1} - c)\varepsilon_{n+1}).
\end{aligned}$$

And finally we extend h_{n+1} linearly over the whole segment $\{c\} \times [s, u] \subset ve$. In other words, since the values of h_{n+1} are already determined at seven points of the considered segment, namely at the points (c, t) where $t \in \{s, \alpha_{n+1}, \delta_{n+1}^m - \varepsilon_{n+1}^m, \delta_{n+1}^m, \delta_{n+1}^m + \varepsilon_{n+1}^m, \beta_{n+1}, u\}$, we define the six partial mappings to be linear on the consecutive straight line segments contained in $\{c\} \times [s, u]$ and with end points (c, t), where t is in the above seven-element set. Thus the definition of h_{n+1} is complete, and it is straightforward to verify that $h_{n+1} : F_C \to F_C$ is continuous and satisfies all the required conditions, i.e. 1°, 2° and 3° above with $n + 1$ in place of n. The inductive procedure is therefore finished.

As a consequence of the above definition of h_n one can see that for each $n \in \{0, 1, 2, \ldots\}$ we have $h_n(V_n) \subset V_n$, whence it follows that for each

$e \in E(F_C)$, the mapping h_n moves no point of $A_n \cap ve$ at a distance greater than $2\varepsilon_n$ along ve. Since $h_n|ve$ is linear outside A_n, we conclude that for any $x \in F_C$ the distance from $h_n(x)$ to $h_{n+1}(x)$ is less than $4\varepsilon_n < 1/2^{n-2}$. Thus the sequence of mappings h_n converges uniformly to a limit mapping $h : F_C \to F_C$. The reader can verify that the decompositions of F_C into the preimages of points are the same for h and for gq. Hence h is topologically equivalent to gq.

Note that condition 2° implies $h|A_n = h_n|A_n$ for each $n \in \{0, 1, 2, \ldots\}$.

Now we prove that the composite mapping $p_2h : F_C \to [0,1]$ is open. It is enough to verify that p_2h is interior at each $b \in F_C$. Choose $e \in E(F_C)$ such that $b \in ve$, and let W be an open neighborhood of b in F_C. Consider two cases. First, let $b \in F_C \setminus \bigcup\{I_n : n \in \mathbb{N}\}$. Since at most one I_n can be contained in ve, the set $W \cap ve$ contains an arc, say xy, such that $b \in xy \setminus \{x, y\} \subset xy \subset F_C \setminus \bigcup\{I_n : n \in \mathbb{N}\}$. Then $p_2h(b)$ is an interior point of $p_2h(xy)$, thus of $p_2h(W)$. Second, let $b \in I_n$ for some $n \in \mathbb{N}$. Since $C_n \to \{c_n\}$ if m tends to infinity, and since $\{\delta_n^m : m \in \mathbb{N}\}$ is dense in $p_2(I_n)$, there is $m \in \mathbb{N}$ such that

$$C_n^m \times [\delta_n^m - \varepsilon_n^m, \delta_n^m + \varepsilon_n^m] \subset W\,.$$

By the definition of h_n we have $h_n(b) = d_n$ and

$$p_2(h_n(C_n^m \times \{\delta_n^m\})) = p_2(d_n)\,.$$

Pick up $c' \in C_n^m$ and put $b' = (c', \delta_n^m) \in C_n^m \times \{\delta_n^m\} \subset A_n$. Thus $h_n(b) = h_n(b')$. Since $h|A_n = h_n|A_n$, we conclude $h(b) = h(b')$. Put $e' = (c', 1)$. The set $W \cap ve'$ contains an arc $B = \{c'\} \times [\delta_n^m - \varepsilon_n^m, \delta_n^m + \varepsilon_n^m]$. Then by the definition of h_n we see that $h(B) = h_n(B)$ is an arc containing $h(b') = h_n(b') = h(b)$ in its interior. So $p_2h(b)$ is an interior point of $p_2h(W)$, and therefore p_2h is interior at b. Thus p_2h is open.

Since h is equivalent to gq, we infer that $p_2gq : F_C \to [0,1]$ is open. Now recall that if $f = f_2f_1 : X \to Y$ is open (where $f_1 : X \to f_1(X)$ and $f_2 : f_1(X) \to Y$ are surjections and X is compact), then f_2 is open, too (see [116], (3.1), p. 140). Hence we infer from openness of p_2gq that $p_2g : X \to [0,1]$ is open. Putting $f = p_2g$ we see that also conditions 2) and 5) of the conclusion hold true. So the proof is complete.

The following problem is natural in the light of results presented above.

PROBLEM 7.9. Characterize the class of fans X having the property that each open mapping defined on X is light.

The above and other examples of open mappings between fans suggest the conjecture that some conditions obtained in Theorem 4.1 can be sharpened somewhat if f is an open mapping (similarly to the case of light confluent mappings, see Corollary 4.13). However, this is not so, because of the next example, which is due to J. R. Prajs.

EXAMPLE 7.10. (Prajs) *There exists a fan X with top v, a simple triod $T \subset X$ and an open retraction $f : X \to T$ such that no one of the equalities of Corollary* 4.13 *holds, i.e.,*

$$f^{-1}(v) \neq v\,, \quad f^{-1}(E(T)) \neq E(X) \quad \textit{and} \quad f^{-1}(S(T)) \neq S(X)\,.$$

Proof. In $\mathbb{R}^3$ with Cartesian coordinates, take the Lelek fan F_L with top v and a simple triod T contained in the plane $z = 0$ such that F_L and T have their tops in common only. Denote by e^0, e^1, e^2 the end points of T. Consider a dense countable subset G of $E(F_L)$ and arrange its points in a sequence $e_1, e_2, \ldots, e_n, \ldots$ with $e_m \neq e_n$ provided $m \neq n$ in such a way that the diameters of three point-sets $\{e_{3n}, e_{3n+1}, e_{3n+2}\}$ tend to zero as $n \to \infty$. For each point e_n of G let A_n denote the straight line segment of length $1/n$ situated in the upper half space $\{(x, y, z) : z \geq 0\}$, parallel to the z-axis and ending at e_n. Further, let B_n denote an arc in the plane $z = 1/n$ which projects onto the arc $e^i e_n \subset F_L \cup T$ with $i \equiv n (\mathrm{mod}\, 3)$, i.e.,

$$B_n = \{(x, y, 1/n) : (x, y, 0) \in e^i e_n \text{ and } i \equiv n (\mathrm{mod}\, 3)\}$$

for each $n \in \mathbb{N}$. Thus $A_n \cap F_L = \{e_n\}$ and $A_n \cap B_n = \{(x, y, 1/n)\}$, where $(x, y, 0) = e_n$. Then the union

$$X = F_L \cup T \cup \bigcup \{A_n \cup B_n : n \in \mathbb{N}\}$$

is a fan with top v.

Consider now the natural quotient mapping $q : X \to X/F_L$ that shrinks the Lelek fan F_L to the top v, and observe that $q|T$ is a homeomorphism and that for a fixed $i \in \{0, 1, 2\}$ the set $q(F_L \cup T \cup \bigcup\{A_{3n+i} \cup B_{3n+i} : n \in \mathbb{N}\})$ is homeomorphic to the harmonic fan having the arm $q(ve^i)$ of $q(T)$ as the limit arc. Thus $q(X) = X/F_L$ is a countable fan homeomorphic to the union of three harmonic fans each having one arm of $q(T)$ as the limit arc. Define a mapping $p : q(X) \to q(T)$ as the natural projection of each of the above mentioned harmonic fans onto its limit arc. Thus $pq|T$ is again a homeomorphism. We show that the composition $pq : X \to pq(X) = pq(T)$ is open. To this end it is enough to verify that pq is interior at each $b \in X$. This fact is obvious if $b \in (A_n \cup B_n) \setminus F_L$ for some $n \in \mathbb{N}$, because $pq|[(A_n \cup B_n) \setminus F_L]$ is a homeomorphism according to the definition. Similarly in case $b \in T \setminus \{v\}$. If $b \in F_L$, then for each open neighborhood U of b in X there is a three-point set $\{e_{3n}, e_{3n+1}, e_{3n+2}\}$ contained in U. Thus there are three nondegenerate arcs $b_n^i e_{3n+i} \subset e^i e_{3n+i} \cap U$ for $i \in \{0, 1, 2\}$. Then $pq(b_n^0 e_{3n} \cup b_n^1 e_{3n+1} \cup b_n^2 e_{3n+2}) \subset pq(U)$ is a simple triod contained in the triod $pq(X) = pq(T)$, and therefore it forms a neighborhood of the top $pq(b) = pq(v)$ of the range. Thus our proof of the openness of pq is finished.

Finally let us note that $(pq)^{-1}(v) = F_L$, whence all three inequalities of the conclusion follow with $pq = f$. The reader can observe that pq is topologically equivalent to the retraction f of X onto T. So the proof is complete.

We close this chapter with the following result.

THEOREM 7.11. 1) *Each monotone open mapping between fans is a homeomorphism.* 2) *There is no monotone open mapping from a fan onto an arc.*

Proof. Let a surjective mapping $f : X \to Y$ from a fan X onto a continuum Y be both monotone and open. Recall that the family $\{f^{-1}(y) : y \in Y\}$ is a continuous monotone decomposition of X (see [116], Theorem (4.1), p. 127 and Theorem (4.31), p. 130). If f is not a homeomorphism, there is $y \in Y$ such that $f^{-1}(y)$ is a nondegenerate proper subcontinuum of X. Take $x \in f^{-1}(y)$ and $x' \in X \setminus f^{-1}(y)$. Thus there exists $c \in xx'$ such that $x'c \cap f^{-1}(y) = \{c\}$. Choose a sequence of distinct points $c_n \in x'c$ which tends to c such that the continua $K_n = f^{-1}(f(c_n))$ are pairwise disjoint. Then $c_n \in K_n$ for each $n \in \mathbb{N}$, and we have $f^{-1}(y) = \operatorname{Lim} K_n$ by openness of f. Since, for sufficiently great n, each continuum which contains c_n and is close to $f^{-1}(y)$ must contain the arc $c_n c$ (because $f^{-1}(y)$ is nondegenerate and X is a fan), these continua cannot be disjoint, a contradiction.

Remark 7.12. Note that the above result cannot be generalized to mappings between dendroids, because there exists a nontrivial monotone open retraction of a (smooth) dendroid onto an arc (see [108], Example 3.2, p. 118). For some related results see [59] (especially Theorems 2.7 and 2.8, p. 117, and a remark added in proof, p. 128); compare [44].

Other relations between openness and lightness of mappings defined on (locally connected) fans are considered in Theorem 9.6 below.

8. Inverse limits

Bellamy ([2], Lemma 1, p. 192 and Nadler ([93], Theorem 4, p. 229; see also [92]) have proved the following result about inverse limits of dendroids.

THEOREM 8.1. (Bellamy, Nadler) *Let X denote the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where X^i is a dendroid for each $i \in \mathbb{N}$.*

1. *If X is arcwise connected, then X is a dendroid.*
2. *If X is locally connected, then X is a dendrite.*
3. *If X^i is a dendrite and $f^i : X^{i+1} \to X^i$ is monotone for each $i \in \mathbb{N}$, then X is a dendrite.*

4. *If $f^i : X^{i+1} \to X^i$ is monotone for each $i \in \mathbb{N}$, then X is a dendroid.*

As a consequence of Statement 1 of the above theorem we have the following observation.

COROLLARY 8.2. *Let X be the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where X^i is a fan for each $i \in \mathbb{N}$. If X is locally connected, then X is a fan.*

Proof. By Theorem 8.1 the limit X is a dendroid. Suppose X contains two ramification points. Then there are two distinct simple triods T_1 and T_2. It is well known that, given any $\varepsilon > 0$, for sufficiently great $i \in \mathbb{N}$ the projection $\pi^i : X \to X^i$ is an ε-mapping. Hence there is $n \in \mathbb{N}$ such that $\pi^n(T_1)$ and $\pi^n(T_2)$ are disjoint simple triods in X^n. Thus X^n fails to be a fan.

Statement 4 in Theorem 8.1 has been generalized as follows (see [21], Corollary 3, p. 145).

THEOREM 8.3. *Let X denote the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where for each $i \in \mathbb{N}$ the following conditions are satisfied. 1° X^i is a dendroid, and 2° there exists a thread $p = \{p^i\}$ such that the mapping f^i is monotone relative to p^{i+1}. Then X is a dendroid.*

A similar result holds true for fans (see [21], Theorem 3, p. 146).

THEOREM 8.4. *Let X denote the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where for each $i \in \mathbb{N}$ the following conditions are satisfied. 1° X^i is a fan with top v^i; 2° the points v^i form a thread $v = \{v^i\}$; and 3° f^i is monotone relative to v^{i+1}. Then X is a fan with top v.*

If we consider the inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ of fans X^i with confluent bonding mappings f^i, then conditions 2° and 3° of the above theorem hold by (4) and (5) of Theorem 4.1, respectively. Hence we have the following corollary (a similar result is shown in [20], Theorem 2, p. 11 for monotone bonding mappings only).

COROLLARY 8.5. *The limit of an inverse sequence of fans with confluent bonding mappings is a fan having as its top a thread consisting of the tops of the consecutive terms of the sequence.*

Other results and questions concerning inverse sequences of fans with some special bonding mappings are related to such properties of fans as smoothness, property of Kelley or contractibility. Therefore they are discussed in the corresponding chapters of the paper, devoted to those particular subjects (see Theorem 11.20, Corollary 11.21, Theorems 12.6 and 12.7, and Questions 13.29).

Finally recall the following nice and important result, due to J.B. Fugate (see [49], Theorem 2, p. 125) that goes, in a sense, in the opposite direction than the results considered in this chapter.

THEOREM 8.6. (Fugate) *Each fan is the limit of an inverse sequence of finite fans.*

Some special inverse sequences of finite fans are considered in Theorem 11.20 below. In the light of these results the following general question seems to be both natural and interesting.

QUESTION 8.7. What conditions regarding bonding mappings have to be satisfied for each fan to be representable as the limit of an inverse sequence of finite fans with bonding mappings satisfying these conditions? In other words, characterize the class $\mathcal{B}$ of bonding mappings such that, given any fan X, there is an inverse sequence of finite fans with bonding mappings in $\mathcal{B}$ having X as its limit.

Remarks 8.8. (a) The fixed point property has been shown in [87] for the inverse limits of sequences of some fans, namely of the cones over zero-dimensional compacta with isolated points, provided edges of the cones are images of edges under bonding mappings.

(b) Similarly, in [88] the same property has been shown for the inverse limits of n-ods provided the bonding mappings preserve the tops of the n-ods as well as maximal sub-j-ods for all $j \leq n$.

(c) For further results in this direction see [89].

9. Local connectedness

Recall that F^n and F^ω stand for an n-od and for the countable locally connected fan, respectively. It is evident that each locally connected fan is homeomorphic either to F^n for some integer $n \geq 3$ or to F^ω. Thus the following fact can readily be observed.

FACT 9.1. *The family of all locally connected fans can be ordered in a sequence*

$$F^3, \quad F^4, \quad \dots, \quad F^\omega$$

such that for each integer $n \geq 3$ the following two conditions are satisfied.

1) *The fan F^n can be considered to be a subset of F^{n+1} (of F^ω) under a natural embedding such that $E(F^n)$ is a subset of $E(F^{n+1})$ (of $E(F^\omega)$, respectively).*

2) *If F^n is naturally embedded into F^{n+1} (into F^ω), then there exists a uniquely determined monotone retraction from F^{n+1} (from F^ω, respectively) onto F^n.*

As easy consequences of the above fact we get two corollaries.

COROLLARY 9.2. *For the family of all locally connected fans the fan* F^ω *is both a universal element and a common model with respect to monotone mappings.*

COROLLARY 9.3. *For each integer* $n \geq 3$ *assume* F^n *is naturally embedded into* F^{n+1}, *and let* $f^n : F^{n+1} \to F^n$ *be the monotone retraction. Then* $F^\omega = \varprojlim\{F^n, f^n\}$.

The rest of this chapter is devoted to open mappings. To begin with, observe that for locally connected fans the concepts of an end point and of a point of Menger–Urysohn order one coincide (see e.g. [62], § 51, I, p. 274). Since the Menger–Urysohn order of a point cannot be increased under open mappings of continua ([116], Corollary (7.31), p. 147), an end point of a locally connected fan is mapped to an end point of the image under an open mapping. Further, since the top of F^ω lies in the closure of $E(F^\omega)$, we deduce from Proposition 3.4 the following result (compare [14], p. 493).

PROPOSITION 9.4. *Each open image of the countable locally connected fan* F^ω *is homeomorphic to* F^ω.

In connection with Proposition 9.4 the next problem seems to be both natural and interesting (compare [14], Problem, p. 493 and [15], Problem 11, p. 29).

PROBLEM 9.5. What fans X have the property that each open image of X is homeomorphic to X?

Note that the Lelek fan F_L enjoys this property ([33], Corollary, Parts (a) and (d)), while the Borsuk fan F_B does not (see Examples 10.6 below).

Coming back to locally connected fans, we have the following result.

THEOREM 9.6. *A confluent mapping defined on a locally connected fan is open if and only if it is light.*

Proof. To prove that openness implies lightness suppose on the contrary that there are an open mapping $f : X \to Y$ of a locally connected fan X onto a nondegenerate range space Y, and a point y in Y such that there is a nondegenerate component K of $f^{-1}(y)$. Since X is either F^n for some $n \geq 3$ or F^ω, we see that each nondegenerate subcontinuum of X has nonempty interior. In particular $\emptyset \neq \operatorname{int} K \subset K \subset f^{-1}(y)$ and thus $f(\operatorname{int} K) = \{y\}$, so the singleton $\{y\}$ has to be open, a contradiction with the connectedness of Y. Since the opposite implication is a consequence of Proposition 3.13, the proof is complete.

Remarks 9.7. 1) Observe that another argument for the "if" part of Theorem 9.6 can run as follows. Since X is locally connected, f is quasi-monotone ([8], IX, p. 215). This property is known to be equivalent to openness for light mappings of locally connected continua ([116], Theorem 8.2, p. 152).

2) Recall that there are fans, not necessarily locally connected, on which each open mapping is light. Such is, e.g., the harmonic shredded fan (see Proposition 7.6).

3) Recently K. Omiljanowski has shown that each open mapping defined on a local dendrite is light.

10. Planability

A continuum X is said to be *planable* provided there is a homeomorphism of X into the Euclidean plane. The harmonic fan and the Cantor fan are planable by their definitions, while the Borsuk fan F_B is not (see [5]). As early as 1959 B. Knaster raised the problem (which is still open) of finding an intrinsic characterization of planable dendroids (see [56]). Some conditions related to planability of dendroids can be found e.g. in [28], [30], [53], [80] and [81]. An important step in the study of planable dendroids was made by L. G. Oversteegen. Answering a problem posed in [30] he has proved ([106], Theorem 5.2, p. 502) the following result.

THEOREM 10.1. (Oversteegen) *If a fan is locally connected at its top, then it is planable.*

Since planability of dendroids is preserved under monotone mappings ([11], Corollary 2, p. 172), the next proposition follows from (1) of Proposition 3.4.

PROPOSITION 10.2. *A monotone image of a planable fan is a planable fan, an arc, or a point.*

It is known that planability is not an invariant property under local homeomorphisms (and thus under open mappings) for the class of linear graphs ([116], Example, p. 189) and it is not invariant under confluent mappings for the class of dendroids (see [11], the example — due to T. Maćkowiak — following Corollary 2 on p. 172). However, neither of these examples is related to mappings of fans. Hence a natural question arises how far monotonicity of the mapping is essential in Proposition 10.2. We will show that light confluent mappings do not preserve planability of fans, even if the cardinality of each point-inverse is at most two.

EXAMPLE 10.3. *There is a light confluent mapping from a plane fan onto the Borsuk nonplanable fan, such that each point-inverse consists of one or two points.*

Proof. Take a polar coordinate system (ρ, ϕ) in the plane with origin v, and put for each $n \in \mathbb{N}$:

$$a_n = (1, -(3n)^{-1}), \quad b_n = (1 + (3n)^{-1}, 0), \quad c_n = (1, (3n)^{-1}),$$
$$d_n = ((3n)^{-1}, \pi/6), \quad e_n = (1, \pi/3 - (3n)^{-1}),$$

and $a_0 = (1, 0)$, $e_0 = (1, \pi/3)$. Define

$$F = \overline{va_0} \cup \overline{ve_0} \cup \bigcup \left\{ \overline{va_n} \cup \overline{a_n b_n} \cup \overline{b_n c_n} \cup \overline{c_n d_n} \cup \overline{d_n e_n} : n \in \mathbb{N} \right\},$$

and note that F is a countable fan with top v and end points a_0, e_0, e_1, e_2, ... (see Fig. 2).

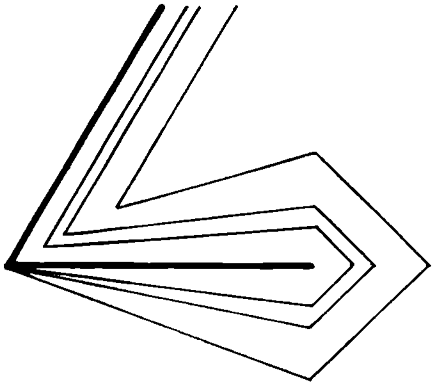

Fig. 2

Now for $k \in \{0, 1, 2\}$ let r_k denote the rotation of the plane through $2k\pi/3$ around v. Putting $F_k = r_k(F)$ and $X = F_0 \cup F_1 \cup F_2$ we see that X is a countable plane fan with top v. This fan is pictured here as Fig. 3.

Define now an equivalence relation on X that identifies each pair of opposite (i.e. antipodal) points of the limit straight line segments only, i.e., for $\rho \in \,]0, 1]$ and $\phi \in \{0, \pi/3, 2\pi/3\}$ the two points (ρ, ϕ) and $(\rho, \phi + \pi)$ form one equivalence class, while all other equivalence classes are singletons. Let $f : X \to Y$ be the quotient mapping and let $v' = f(v)$. Then the resulting quotient space Y is homeomorphic to the Borsuk fan F_B and has v' as its top. Since the preimage of each point of Y consists of two points at most, f is light. To see f is confluent it is enough to observe that the assumptions of Theorem 4.6 are satisfied. The proof is complete.

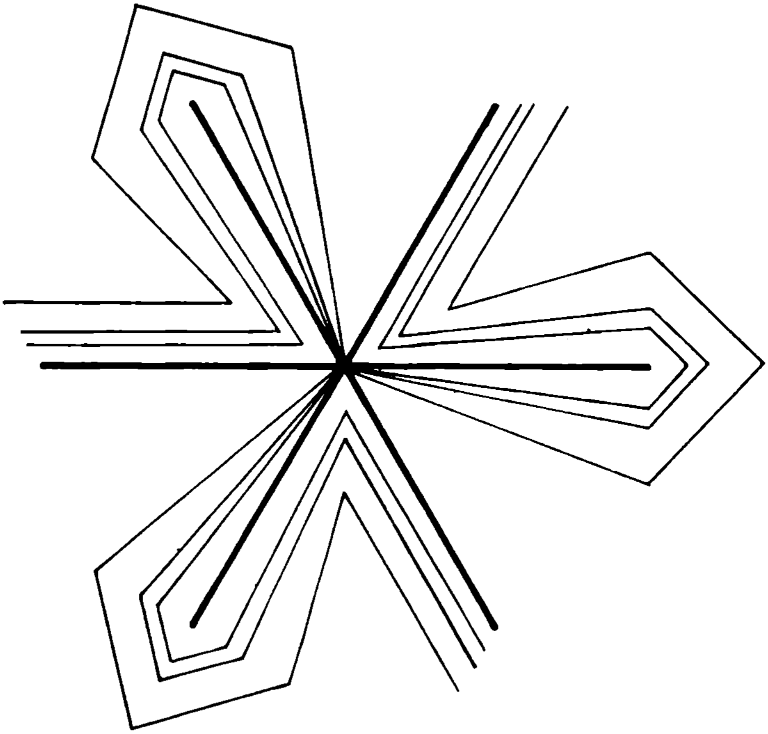

Fig. 3

However, in connection with Proposition 10.2 and Example 10.3 we have the following questions.

QUESTIONS 10.4. Is planability of fans invariant under mappings which are 1° light open, 2° open?

The same set of problems as for planable fans can be considered for nonplanable ones.

PROPOSITION 10.5. *Nonplanability of fans is not invariant under monotone mappings.*

Proof. Indeed, it is enough to shrink the arm L_{01} of the limit simple triod T of the Borsuk fan F_B to its top v (see Preliminaries), i.e., to consider the monotone decomposition of F_B whose only nondegenerate element is L_{01}. The resulting decomposition space is a planable fan.

For open mappings we have the following two examples.

EXAMPLES 10.6. *There exist light open mappings of the Borsuk nonplanable fan onto* (a) *a plane fan such that each point-inverse consists of at* *most three points, and* (b) *onto an arc, such that each point-inverse is at most countable.*

Proof. First we define a light open mapping $f : F \to Y$ from the Borsuk fan F_B onto a countable fan Y, such that for each point y of Y the set $f^{-1}(y)$ has at most three points, and next we describe a light open retraction $g : Y \to L_0$ onto an arc L_0, such that the point-inverses are at most countable. Then $gf : F_B \to L_0$ will be the needed light open mapping of F_B onto an arc.

So, take the Borsuk fan F_B described as in Preliminaries, and define an equivalence relation on F_B such that the equivalence class of the top v of F_B is the singleton $\{v\}$, while all other equivalence classes consist of exactly three points, and are defined as follows. For each point of the limit simple triod T (different from v) its equivalence class is composed of three points belonging to distinct arms of the triod and having the same distance from v. Similarly, for each point of $F_B \setminus T$ its equivalence class is composed of three points, say u_1, u_2, u_3, such that $u_k \in L_{kn}$ for $k \in \{1,2,3\}$ and some fixed $n \in \mathbb{N}$ (the same for all three points u_k) and that the lengths of the arcs $vu_k \subset L_{kn}$ are equal.

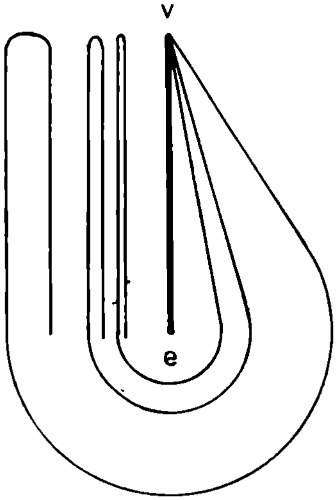

Fig. 4

Therefore the quotient mapping $f : F_B \to Y$ glues together the three arms L_{01}, L_{02} and L_{03} of the limit triod T of F_B into a limit segment L_0 and for each fixed $n \in \mathbb{N}$ it glues together the three arms L_{1n}, L_{2n} and L_{3n} of the n-th triod $L_{1n} \cup L_{2n} \cup L_{3n}$ of F_B into one arc, say L_n. Thus Y can be considered as a fan which is the union of the straight line segment L_0 and of a sequence of arcs L_n for $n \in \mathbb{N}$ which approximates L_0 in the following way. Each L_n emanates from the top v' of Y which is an end point of L_0.

For sufficiently great n the arc L_n goes smoothly near to L_0 in the direction from v' to the other end point e of L_0; next at a place not far from e it turns back, approximates L_0 in the direction from e to v', and finally at a point which is close enough to v' it turns back again, approximating L_0 for the third time, in the direction from v' to e.

It is obvious that Y can be embedded into the plane (see Fig. 4). Further, since the equivalence classes are finite, the quotient mapping f is light. Finally, the decomposition of F_B into equivalence classes is continuous by definition, and hence f is open ([116], Theorem 4.31, p. 130).

Now we define $g : Y \to L_0$ as a natural projection which is the identity on L_0 and which folds thrice each arc $L_n \subset Y$, where $n \in \mathbb{N}$, mapping it openly in an obvious way onto L_0. So, g is a retraction. This concludes the proof.

COROLLARY 10.7. *Nonplanability of fans is not invariant under light open (thus under open) mappings, even if all point-inverses are at most countable.*

Remark 10.8. As the reader already knows from Theorem 7.11 and Remark 7.12, monotone open mappings defined on fans are homeomorphisms (if the range space is nondegenerate), and the result cannot be extended to arbitrary dendroids. However, it is known that if a monotone open surjection f is defined on a dendroid X which is additionally assumed to be planar, then $f(X)$ must be a dendrite. In fact, Theorem 3 of [44], p. 358, says that $f(X)$ is (hereditarily) locally connected, and Corollary 2 of [8], p. 219, asserts us that $f(X)$ is a dendroid. Thus it is a dendrite, being a locally connected dendroid.

Remark 10.9. It has been shown in Corollary 9.2 that the class of all locally connected fans (which obviously are planar) has both a universal element and a common model with respect to monotone mappings (and therefore with respect to any class of mappings that contains monotone ones). A similar result will be shown in Corollary 11.10 for the class of smooth fans (which are planar, too): the Cantor fan is a universal element as well as a common model with respect to certain two classes of mappings. However, since for each hereditarily decomposable continuum X there exists a plane fan Y such that no subcontinuum of X can be mapped onto Y ([57], Theorem 4.1, p. 740), the following classes have no universal element: fans, dendroids, λ-dendroids and hereditarily decomposable continua as well as their respective subclasses of planable elements ([57], Corollary 4.2, p. 740). Another consequence of the above theorem is the nonexistence of a common model for any of the above classes (and corresponding subclasses of their planable members) with respect to all (continuous) mappings (Corollary 4.3 of [57], p. 740).

11. Smoothness

Recall that a fan X with top v is said to be *smooth* provided that if a sequence of points x_n of X tends to a limit point x, then the sequence of arcs vx_n is convergent, and it has the arc vx as its limit. This concept, originally defined in [9], § 3, p. 7, for fans only, has been extended first to dendroids in [25], p. 298. A dendroid X is said to be *smooth at a point* v provided that the above condition is satisfied, and *smooth* if it is smooth at some point. For a nonmetric analog of smooth dendroids, i.e. generalized trees, see [113], p. 801. The concept of smoothness has next been generalized to Hausdorff continua which are hereditarily unicoherent at a point ([50], p. 52; see also [73] and [77]), and finally to arbitrary metric continua in [79], p. 81. For a characterization of smooth dendroids in terms of the existence of some special homotopies and selections see [91], Theorems 1.14 and 1.16, pp. 370–371, [26], Corollary, p. 93, and [114], Theorem 2, p. 1043. A nonnegative real-valued function of the first class of Baire is used in [58], Theorem 3.1 and Corollary 3.2, pp. 233–234 to characterize smooth dendroids.

Coming back to fans, recall that two characterizations of their smoothness, the first of which being rather close to the definition, are given in [9], Theorems 1 and 2, pp. 7 and 9. Another one is a consequence of Corollary 15 of [9], p. 28, and Corollary 4 of [45], p. 90. It runs as follows.

PROPOSITION 11.1. *A fan X with top v is smooth if and only if there is a mapping $f : X \to [0,1]$ such that $f(v) = 0$ and for each end point e of X, the mapping $f|ve : ve \to f(ve) \subset [0,1]$ is a homeomorphism.*

Note that the above mapping f is light and monotone with respect to v, but it need not be confluent.

It is known that if a smooth dendroid X is a fan, then its top v can be taken as a point at which X is smooth, and that X is locally connected at v (see [25], Corollary 9, p. 301). Thus by Oversteegen's result (Theorem 10.1 above) each smooth fan is planable. However, a much stronger result holds true. Namely, using the same argument as above for Proposition 11.1, and additionally Theorem 9 of [9], p. 27 and Corollary 6 of [25], p. 299, one gets the next equivalence.

PROPOSITION 11.2. *A fan is smooth if and only if it can be homeomorphically embedded into the Cantor fan.*

The above characterization of smooth fans has many important applications and, as the reader will see from the text of this chapter, it is a very good tool in investigation of various (mapping as well as structural) properties of smooth fans. At the moment let us observe that the uniform arcwise connectedness of each smooth fan is a consequence of Proposition 11.2 (this property also holds for all smooth dendroids, see [25], Corollary 16, p. 318). Since for uniquely arcwise connected spaces uniform arcwise connectedness coincides with uniform pathwise connectedness ([60], p. 316), the discussed property leads to further results. Namely let us bring to the reader's attention a nice and important result by W. Kuperberg ([60], Theorem 3.5, p. 322) that characterizes the images of the Cantor fan under arbitrary (continuous) mappings as uniformly pathwise connected continua. As a consequence a corollary is obtained (viz. Corollary 3.6 of [60], p. 322) saying that the Cantor fan is a common model for the families of all uniformly arcwise connected fans and of all uniformly arcwise connected dendroids with respect to the class of all continuous mappings. Along these lines of ideas see Corollaries 11.10 and 11.11 below. Recall that uniform arcwise connectedness of dendroids (hence, equivalently, the property of being the image of the Cantor fan under a mapping) has been characterized in Theorem 3.6 and Corollary 3.7 of [58], pp. 236–237 by means of a nonnegative real-valued function of the first class of Baire.

Recently one more condition has been joined to the two given by W. Kuperberg, so that the following theorem holds (see [24], Theorem 12).

THEOREM 11.3. *The following conditions are equivalent for a nondegenerate continuum Y:*

1° *Y is the image of the Cantor fan under a light mapping;*

2° *Y is the image of the Cantor fan under a mapping;*

3° *Y is uniformly arcwise connected.*

COROLLARY 11.4. *Each smooth fan is the image of the Cantor fan under a light mapping.*

Remarks 11.5. 1) The converse implication to that of Corollary 11.4 is not true. First, observe that light mappings do not preserve the property of being a dendroid: identify the end points of the unit closed interval to get a circle. Second, note that even if the range space is a fan, smoothness is not preserved under light mappings. To see this, let us map the Cantor fan F_C onto the harmonic fan F_H in such a way that, if F_C is represented as the union of the segment $\overline{ve_0}$ (where $e_0 = (0,1)$), and of "portions" of F_C that are more and more narrow and tend to $\overline{ve_0}$, then these portions are projected onto consecutive segments joining the top of F_H to consecutive end points of F_H, and $\overline{ve_0}$ is projected onto the limit segment of F_H. Obviously, such a mapping is light. Next take a (light) mapping of F_H onto a nonsmooth fan as described in Example 5.7 of [78], p. 263. Then the composition of these two mappings transforms the Cantor fan onto a nonsmooth fan.

2) Corollary 11.4 makes more exact an earlier result saying that each smooth fan is the image of the Cantor fan under a (continuous) mapping

(see [9], Theorem 10, p. 28 and [45], Corollary 4, p. 90; compare Corollary 11 of [25], p. 311). Now we give another specification of this result.

The following proposition is a consequence of a similar result for dendroids, due to T. Maćkowiak, viz. Corollary 2.11 of [75], p. 722, saying that a continuum Y is a smooth dendroid if and only if there is a mapping f of the Cantor fan F_C with top v onto Y which is monotone relative to v, and then Y is smooth at $f(v)$.

PROPOSITION 11.6. *A fan is smooth if and only if it is the image of the Cantor fan F_C under a mapping that is monotone relative to the top of F_C.*

Remark 11.7. The assumption that the continuum under consideration is a fan is indispensable in the "if" part of Proposition 11.6. In other words, the existence of a surjective mapping $f : F_C \to Y$ that is monotone relative to the top of F_C guarantees that Y is a smooth dendroid (according to Corollary 2.11 of [75], p. 722), but it does not guarantee that it is a fan. Namely an example of a mapping f from F_C onto a smooth dendroid with two ramification points such that f is monotone relative to the top of F_C can be obtained as an obvious modification of Example (7.35) of [84], p. 68: we replace the harmonic fans A and B used there by the left and the right halves of the Cantor fan F_C, and define the needed mapping f exactly as in that example. Note that f is monotone relative to the top of F_C and quasi-monotone but not confluent.

The following result is related to Proposition 11.6 and Corollary 11.4.

THEOREM 11.8. *A continuum Y is the image of the Cantor fan F_C under a mapping which is light and monotone relative to the top of F_C if and only if Y is a dendroid that is smooth at a point $v' \in Y \setminus \operatorname{cl} E(Y)$.*

Proof. Let v be the top of F_C and let a surjection $f : F_C \to Y$ be light and monotone relative to v. Put $v' = f(v)$. Then Y is a dendroid that is smooth at v' ([75], Corollary 2.8, p. 721). Suppose on the contrary that $v' \in \operatorname{cl} E(Y)$. Then there exists a sequence of end points e'_n of Y having v' as its limit. For each $n \in \mathbb{N}$ take an end point e_n of F_C with $f(e_n) = e'$ (in fact, if $x \in f^{-1}(e'_n)$ and $x \neq v$, then $x \in \overline{ve}$ for some $e \in E(F_C)$; since $f|\overline{ve}$ is monotone ([75], Corollary 2.10, p. 722) we have $f(xe) = e'_n$ and we put $e_n = e$). Without loss of generality one can assume the sequence $\{e_n\}$ is convergent; let $e_0 \in E(F_C)$ denote its limit. Since for each n we have $f(ve_n) = v'e'_n$, and since $v'e'_n \to \{v'\}$ by smoothness of Y at v', we conclude $f(ve) = \operatorname{Lim} f(ve_n) = \{v'\}$, and thus f is not light.

To prove the other implication let us recall that if a dendroid is smooth at a point, then it admits an equivalent metric which is radially convex with respect to this point ([25], Theorem 10, p. 310). So let metrics d and d' on F_C and Y be radially convex with respect to v and v' respectively, and assume that $d(v,e) = 1$ for each $e \in E(F_C)$. Consider an arbitrary mapping g from the Cantor set $E(F_C)$ onto $\operatorname{cl} E(Y)$ and extend it to a mapping $f : F_C \to Y$ such that $f(v) = v'$, and for each $e \in E(F_C)$ the partial mapping $f|ve : ve \to v'g(e) \subset Y$ is determined by the formula

$$d'(v', f(x)) = d(v,x) \cdot d'(v', g(e))$$

for each $x \in ve \subset F_C$. Now the continuity of f is a consequence of the smoothness of Y at v' and of the above formula (note that the condition $d'(v', \operatorname{cl} E(Y)) > 0$ is exploited here). Further, the formula implies that $f|ve$ is monotone for each $e \in E(F_C)$, and therefore f is monotone relative to v ([75], Corollary 2.10, p. 722). Finally, $f^{-1}(y)$ is closed for each $y \in Y$, and by the above formula the set $(f|ve)^{-1}(y) = ve \cap f^{-1}(y)$ consists of one point at most, for each $e \in E(F_C)$. Thus $f^{-1}(y)$ is zero-dimensional, i.e., f is light. The proof is then complete.

COROLLARY 11.9. *Each smooth fan with top outside the closure of the set of all its end points is the image of the Cantor fan F_C under a light mapping that is monotone relative to the top of F_C.*

Propositions 11.2 and 11.6 and Corollaries 11.4 and 11.9 lead to the following two corollaries which are analogs of the corresponding property of F^{ω} for the class of locally connected fans (see Proposition 9.2).

COROLLARY 11.10. *For the family of all smooth fans the Cantor fan is both a universal element and a common model with respect to the following two classes of mappings: monotone with respect to the top and light.*

COROLLARY 11.11. *For the family of all smooth fans with tops not in the closures of the sets of their end points the Cantor fan is both a universal element and a common model with respect to the following three classes of mappings:* 1) *monotone with respect to the top,* 2) *light,* 3) *both light and monotone with respect to the top.*

Remark 11.12. Note that the class of confluent mappings does not satisfy the conclusion of the part of Corollary 11.10 related to the common model. Indeed, the Lelek fan F_L is smooth by Proposition 11.2, but it is not the image of F_C under a confluent mapping, according to Corollary 4.5. Thus a natural question arises to give an internal characterization of those (smooth) fans which are the images of the Cantor fan under confluent (and under some related, viz. open, light open, light confluent and monotone) mappings. The question has been answered in [24] as Theorems 6, 8 and 10 there. Since these results are closely related to the property of Kelley, we discuss them in the next chapter (see Theorems 12.3, 12.4 and 12.5).

The following result is known (see [9], Theorem 13, p. 33). For its further generalizations see [75], Corollary 3.4, p. 724; [78], Example 5.7, p. 263; and [79], § 6, pp. 89–94.

THEOREM 11.13. *A nondegenerate image of a smooth fan under a confluent mapping is either a smooth fan or an arc.*

Perhaps it is worth noticing that the above theorem is a direct consequence of Proposition 11.6 and of Lemma 4 of [9], p. 32 saying that a confluent mapping between fans is monotone relative to the top of the domain (this lemma was also used in the original proof, presented in [9]). Namely, if a confluent surjection $f : X \to Y$ is defined on a smooth fan X, then, by (1) of Proposition 3.4, only the smoothness of Y has to be proved in the case when Y is a fan. According to Proposition 11.6 there is a surjective mapping $f_0 : F_C \to X$ such that f_0 is monotone relative to the top of F_C. By Lemma 4 of [9], p. 32 we see that the composition $ff_0 : F_C \to Y$ is monotone relative to the top of F_C. Thus Y is smooth again by 11.6.

If the mapping is additionally assumed to be light, then also the smoothness of the domain space can be derived from that of the range. Namely we have the following proposition.

PROPOSITION 11.14. *Let a surjective light confluent mapping be defined between fans X and Y. Then the smoothness of Y implies the smoothness of X.*

Proof. Let $f : X \to Y$ be the considered mapping and let v be the top of X. By (4) of Theorem 4.1 the top of Y is $f(v)$; and by (13) of Proposition 4.12 for each $e \in E(X)$ there is $e' \in E(Y)$ such that $f|ve : ve \to f(v)e'$ is a homeomorphism. Since Y is smooth, by virtue of Proposition 11.1 there is a mapping $g : Y \to [0,1]$ such that $g(f(v)) = 0$ and that for each $e' \in E(Y)$, the mapping $g|f(v)e' : f(v)e' \to g(f(v)e') \subset [0,1]$ is a homeomorphism. Consider $h = gf : X \to [0,1]$. Thus $h(v) = 0$, and it is clear that for each $e \in E(X)$, the mapping $h|ve : ve \to h(ve) \subset [0,1]$ is a homeomorphism. Applying Proposition 11.1 once more we get the conclusion.

Theorem 11.13 and Proposition 11.14 imply the following theorem.

THEOREM 11.15. *If a surjective confluent mapping between fans is light, then the smoothness of one of them implies the smoothness of the other.*

Easy examples show that a confluent (even monotone) image of a nonsmooth fan can be a smooth fan. Indeed, take the fan F described in Example 10.3 (Fig. 2) and shrink its limit continuum to a point; the obtained fan is homeomorphic to F^ω (Fig. 1). Thus the lightness assumption cannot be omitted in Proposition 11.14 and, consequently, in Theorem 11.15. Also the assumption that Y is a fan is essential in that proposition, and it cannot be replaced by the condition that Y is an arc, or either an arc or a fan. Namely we have the following example.

EXAMPLE 11.16. *There exists a countable plane nonsmooth fan X with $S(X)$ closed and $E(X)$ nonclosed such that for each $e \in E(X)$ and for each arc $A \subset ve$ there is a light open retraction of X onto A.*

Proof. Put, in the Cartesian rectangular coordinates in the plane, for any $n \in \mathbb{N}$:

$$\begin{array}{lll} v = (0,0)\,, & e_0 = (0,1)\,, & a_n = (1/n,1)\,, \\ b_n = (0,1+1/n)\,, & c_n = (-1/n,1)\,, & e_n = (-1/n,0) \end{array}$$

and define

$$X = \overline{ve_0} \cup \bigcup \{\overline{va_n} \cup \overline{a_nb_n} \cup \overline{b_nc_n} \cup \overline{c_ne_n} : n \in \mathbb{N}\}\ . \tag{21}$$

Evidently X is a nonsmooth fan for which $S(X)$ is closed while $E(X)$ is not (see Fig. 5).

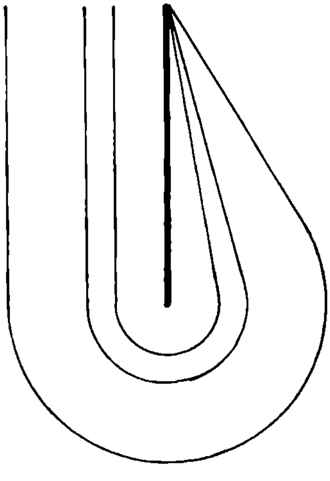

Fig. 5

Define $m \in \{0,1,2,\ldots\}$ by the condition $A \subset ve_m$, and consider a mapping $g : X \to e_0v \cup ve_m$ such that $g|(e_0v \cup ve_m)$ is the identity, and for all arcs $vb_n = \overline{va_n} \cup \overline{a_nb_n}$ and $b_ne_n = \overline{b_nc_n} \cup \overline{c_ne_n}$, where $0 \neq n \neq m$, the partial mappings $g|vb_n : vb_n \to ve_0$ and $g|b_ne_n : b_ne_n \to ve_0$ are linear, with $g(v) = v$, $g(b_n) = e_0$, and $g(e_n) = v$. Then readily g is a well-defined, continuous light retraction, and its range space is an arc, namely $e_0e_m = e_0v \cup ve_m$ if $m \neq 0$, and e_0v if $m = 0$. Note that if $m = 0$ the mapping g is open; otherwise it is interior at any point of $X \setminus E(X)$. If $m \neq 0$, we consider an auxiliary mapping $h : e_0e_m \to ve_m$ such that $h|ve_0 : ve_0 \to ve_m$

is a homeomorphism onto ve_m with $h(v) = v$ and $h(e_0) = e_m$, and $h|ve_m$ is the identity. Let $g' : X \to ve_m$ be defined by $g' = hg$ if $m \neq 0$, and $g' = g$ if $m = 0$. Thus g' is a light open retraction of X onto ve_m.

Further note that the arc A is a subset of the range of g'. Thus, in the case when the end points of A coincide with the end points of the range, the proof is finished. Otherwise denote by u and w the end points of A, with $u \in vw$. In the case when $u \neq v$ and $w \neq e_m$, let us fold ve_m onto $uw = A$ by means of a light open retraction $f : ve_m \to A$ such that $f|uw$ is the identity, and $f|uv$ and $f|we_m$ are both linear, with $f(v) = w$ and $f(e_m) = u$. In the case when either $u = v$ or $w = e_m$ the definition is even simpler. Thus f is a light open retraction from the range of g' onto A. Therefore $fg' : X \to A$ is a light open retraction, too, and thus the proof is complete.

Let us note that the assumption $A \subset ve$ for some $e \in E(X)$ is indispensable in 11.16. Namely it can be observed that if an arc $A \subset X$ has nondegenerate intersections with ve_{n_1} and ve_{n_2} for some $n_1, n_2 \in \mathbb{N}$, then there is no open retraction from X onto A.

Note that Example 12.11 below is related to Example 11.16.

As the reader can see from Example 7.10, openness cannot be substituted for lightness and confluence in Proposition 11.14. Below we present another, simpler example showing this, the main idea coming from Morayne's Example 7.1.

EXAMPLE 11.17. *There exists a nonlight open mapping of a nonsmooth fan onto a simple triod.*

Proof. Put, in the plane,

$$v = (0,0)\,, \quad a = (0,1/3)\,, \quad b = (0,2/3)\,, \quad e_0 = (0,1)$$

and, for any $n \in \mathbb{N}$:

$$\begin{aligned} a_n &= (1/(2n),1/3)\,, & b_n &= (1/(2n),2/3)\,, \\ d_n &= (1/(2n+1),1/3)\,, & e_n &= (1/(2n+1),1)\,. \end{aligned}$$

Define (see Fig. 6)

$$X = \overline{ve_0} \cup \bigcup \left\{ \overline{va_n} \cup \overline{a_nb_n} \cup \overline{b_nd_n} \cup \overline{d_ne_n} : n \in \mathbb{N} \right\}\,.$$

Thus X is a nonsmooth fan, and we have

$$\lim a_n = \lim d_n = a\,, \quad \lim b_n = b\,, \quad \lim e_n = e_0\,. \tag{22}$$

Consider a simple triod $T \subset X$ with vertex v and end points e_0, e_1, e_2. For each $n \in \mathbb{N}$ choose $r_n \in \overline{b_nd_n}$ such that $\overline{ab} \subset \mathrm{cl}\,\{r_n : n \in \mathbb{N}\}$. Finally put, for $n \in \mathbb{N}$,

$$c = (0,1/2)\,, \quad c_n^- = (0,1/2 - 1/(6n))\,, \quad c_n^+ = (0,1/2 + 1/(6n))$$

and define $f : X \to T$ as follows. The partial mapping $f|(ve_1 \cup ve_2)$ is the identity; $f(\overline{ab}) = \{c\}$; $f|\overline{va} : \overline{va} \to \overline{vc}$ and $f|\overline{be_0} : \overline{be_0} \to \overline{ce_0}$ are linear, and for each $n \in \{3, 4, 5, \ldots\}$ the four partial mappings

$$f|(\overline{va_n} \cup \overline{a_n b_n}) : \overline{va_n} \cup \overline{a_n b_n} \to \overline{vc_n^-}\,, \qquad f|\overline{b_n r_n} : \overline{b_n r_n} \to \overline{c_n^- c}\,,$$

$$f|\overline{r_n d_n} : \overline{r_n d_n} \to \overline{cc_n^+}\,, \qquad f|\overline{d_n e_n} : \overline{d_n e_n} \to \overline{c_n^+ e_0}$$

also are linear surjections. Since $f^{-1}(c)$ contains $\overline{ab}$, f is nonlight. Its openness can be verified as in Example 7.1. So the proof is finished.

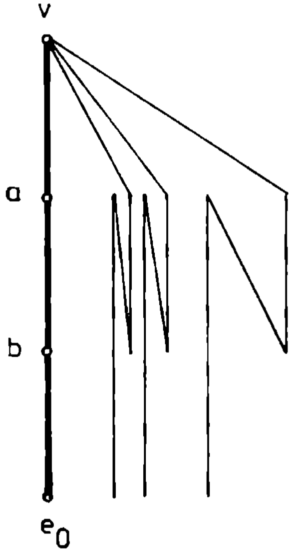

Fig. 6

The following proposition is related to conditions discussed for light confluent mappings in Corollary 4.13 and for open mappings in Example 7.10. The result is proved as Proposition 3 of [24].

PROPOSITION 11.18. *Let a surjective open mapping $f : X \to Y$ be defined on a smooth fan X. If Y is a fan and if v and v' denote the tops of X and Y respectively, then*

$$f^{-1}(v') = \{v\} \quad \text{and} \quad f(E(X)) = E(Y)\,.$$

Example 7.10 shows that the smoothness of X cannot be omitted in the above proposition.

Remark 11.19. A concept related to smoothness is pointwise smoothness. A dendroid X is said to be *pointwise smooth* if for each x in X there exists $v(x)$ in X such that for each sequence x_n converging to x, the sequence of arcs $v(x)x_n$ converges to the arc $v(x)x$ (see [37]; cf. [39]). However, for fans this concept coincides with smoothness ([37], Theorem 1, p. 170), and therefore it need not be discussed separately. After all, let us mention that pointwise smoothness of dendroids is an invariant of open, but not of monotone, mappings ([40], Theorem 5, p. 119 and Example 7, p. 121).

We close this chapter with some results related to inverse limits. Namely we show a characterization of smooth fans in terms of inverse sequences of finite (or of smooth) fans with bonding mappings monotone relative to the corresponding tops.

THEOREM 11.20. *The following three conditions are equivalent for a space X.*

1° *X is the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where for each $i \in \mathbb{N}$ the following are satisfied:*

1) *X^i is a finite fan with top v^i;*

2) *the points v^i form a thread $v = \{v^i\}$;*

3) *f^i is monotone relative to v^i.*

2° *X is the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where for each $i \in \mathbb{N}$ the following are satisfied:*

1′) *X^i is a smooth fan with top v^i;*

2) *and* 3) *as above in* 1°.

3° *X is a smooth fan with top v.*

Proof. Since 1° ⇒ 2° is trivial and 2° ⇒ 3° is just Corollary 5 of [21], p. 146, the only implication that needs a proof is 3° ⇒ 1°.

For each $i \in \{0\} \cup \mathbb{N}$ let F_i be the 2^i-od understood as the cone with vertex v^i over the discrete space $A_i = \{0,1\}^i$, i.e., $F_i = (A_i \times [0,1])/(A_i \times \{0\})$. The projections $C \to A_i$ and $A_{i+1} \to A_i$ (where C denotes the Cantor ternary set) induce open mappings $p^i : F_C \to F_i$ and $u^i : F_{i+1} \to F_i$. Observe that $\{F_i, u^i\}_{i=1}^{\infty}$ is an inverse sequence having F_C as its limit ([32], p. 165). Now let $h : X \to h(X) \subset F_C$ be an embedding (Proposition 11.2). Putting $X^i = p^i(h(X))$ and $f^i = u^i|X^{i+1}$ we see that X^i is a finite fan, and f^i is monotone with respect to v^{i+1}. Further, it can be observed that $\{X^i, f^i\}_{i=1}^{\infty}$ is an inverse sequence having $h(X)$ as its limit. So the proof is finished.

For a result similar to the implication 2° ⇒ 3° above (i.e., to Corollary 5 of [21], p. 146) that is related to smoothness of arbitrary continua see [31], Theorem 1, p. 187 and [21], Theorem 1, p. 144 (for smooth dendroids see the main theorem of [32], p. 164).

Arguing exactly in the same way as in the proof of Corollary 8.4 we get the following corollary to Theorem 11.20.

COROLLARY 11.21. *The limit of an inverse sequence of smooth fans with confluent bonding mappings is a smooth fan.*

12. The property of Kelley

Let a continuum X with a metric d be given. We denote by $C(X)$ the family of all subcontinua of X equipped with the Hausdorff distance dist (see [62], § 42, II, p. 47), and we use the symbol $C(p, X)$ to denote the subfamily of $C(X)$ consisting of all continua in X to which a fixed point p belongs.

A continuum X is said to have the *property of Kelley* ([54], (3.2), p. 26; cf. [97], (16.10), p. 538) provided that given any $\varepsilon > 0$ there exists $\delta > 0$ such that if a, $b \in X$ with $d(a, b) < \delta$ and $A \in C(a, X)$, then there exists $B \in C(b, X)$ with $\operatorname{dist}(A, B) < \varepsilon$. In other words X has the property of Kelley if for each x in X, for each sequence x_n converging to x and for each continuum $L \in C(x, X)$ there exists a sequence of continua $L_n \in C(x_n, X)$ that has L as its limit. Note that the harmonic prolonged fan F_{HP} (see the definition in Preliminaries) is an example of a smooth fan which does not have the property of Kelley. On the other hand, it is known that each dendroid having the property of Kelley is smooth ([41], Corollary 5, p. 730). Further, as we know from some previous investigation (see [22] for example), in the realm of fans the property of Kelley is related not only to smoothness, but also to the structure of the set of end points, in particular to the condition demanding that the set of end points (or of end points together with the top) is closed. Therefore we begin this chapter with these problems.

J. Nikiel has proved in [102], Lemma 3, p. 101, that if two smooth fans X_1 and X_2 have closed sets of their end points, then each homeomorphism between these sets can be extended to a homeomorphism between the whole fans. The authors thank J. Nikiel for pointing out that a stronger result can be shown with $S(X_1)$ and $S(X_2)$ in place of the sets of end points. This is done below.

PROPOSITION 12.1. *Let fans X_1 and X_2 with tops v_1 and v_2 be smooth, and let $S(X_1)$ be closed. Then X_1 is homeomorphic to X_2 if and only if $S(X_1)$ is homeomorphic to $S(X_2)$ and the homeomorphism maps v_1 on v_2.*

Proof. Let a homeomorphism $f : S(X_1) \to S(X_2)$ be given with $f(v_1) = v_2$. Denote by $p : X_1 \setminus \{v_1\} \to E(X_1)$ the projection of $X_1 \setminus \{v_1\}$ onto $E(X_1)$, that is, if $x \in X_1 \setminus \{v_1\}$, then $p(x)$ is the only point of $E(X_1)$ such that $x \in vp(x)$. Then p is continuous. Further, for $i \in \{1, 2\}$, let d_i denote a convex metric on X_i with respect to v_i (it exists by Theorem 10 of [25], p. 310). Then the needed homeomorphism $g : X_1 \to X_2$ which is an extension of f can be defined by the same formula as in the proof of Lemma 3 of [102], p. 101, namely by

$$d_2(g(x), v_2) = d_1(x, v_1) \cdot \frac{d_2(f(p(x)), v_2)}{d_1(p(x), v_1)},$$

and the further argument concerning the properties of g remains the same.

Remarks 12.2. We shall verify that no assumption of Proposition 12.1 can be deleted.

1) Smoothness is necessary because the harmonic fan F_H and the fan X of Example 11.17 (pictured on Fig. 6), while not homeomorphic, have $S(F_H)$ and $S(X)$ homeomorphic, and the homeomorphism takes one top to the other.

2) The closedness of $S(X_1)$ is essential as can be seen from the following example. Consider the harmonic prolonged fan F_{HP} and two harmonic fans F'_H and F''_H, all three lying in the plane, with common top $v = (0,0)$ and defined as follows. F_{HP} is the union of straight line segments joining v to points of the set $\{(0,2)\} \cup \{(1/n, 1) : n \in \mathbb{N}\}$; to obtain F'_H we join v by segments to points of $\{(0,2)\} \cup \{(-1/n, 2) : n \in \mathbb{N}\}$; and F''_H is defined as the union of segments from v to points of the set $\{(-1,2)\} \cup \{(-1-1/n, 2) : n \in \mathbb{N}\}$. Putting

$$X_1 = F'_H \cup F_{HP} \quad \text{and} \quad X_2 = F''_H \cup F_{HP}$$

we see that X_1 and X_2 are smooth nonhomeomorphic fans, the sets $S(X_1)$ and $S(X_2)$ are not closed and homeomorphic, and a homeomorphism between these sets can be chosen such that it maps v to v.

3) Finally observe that the assumption of $S(X_1)$ and $S(X_2)$ being homeomorphic is not enough to attain the existence of a homeomorphism between the fans, and it must be supplied with the condition of taking one top to the other. Indeed, the harmonic fan F_H and the locally connected fan F^ω are both smooth, $S(F_H)$ and $S(F^\omega)$ are closed and homeomorphic, but the top is an isolated point of $S(F_H)$, but not of $S(F^\omega)$, so no homeomorphism between these sets can map one top onto the other.

As was said in Remark 11.12, the property of Kelley for smooth fans is also related to the property of being the image of the Cantor fan under confluent (and under some related) mappings. The results, obtained in [24], are presented below.

Confluent images of the Cantor fan have been characterized in Theorem 6 of [24] as follows.

THEOREM 12.3. *The following conditions are equivalent for a nondegenerate continuum Y:*

1° *Y is the image of the Cantor fan F_C under a confluent mapping;*

2° *Y is embeddable into F_C, and for each (for some) embedding h of Y into F_C the image $h(Y)$ is either a confluent retract, or a light retract, or a retract of F_C;*

3° *Y is either a fan having the property of Kelley or an arc;*

4° *Y is either a smooth fan with $S(Y)$ closed or an arc.*

For further equivalent conditions in the realm of (smooth) fans see Theorems 12.6 and 12.16.

The next characterizations concern images of the Cantor fan under open (light open, light confluent) mappings. For the proof see Theorem 8 of [24].

THEOREM 12.4. *The following conditions are equivalent for a nondegenerate continuum Y:*

1° *Y is the image of the Cantor fan F_C under either an open, or a light open, or a light confluent mapping;*

2° *there exists an embedding h of Y into F_C such that $h(Y)$ is either an open retract, or a light open retract, or a light confluent retract of F_C;*

3° *Y is either a smooth fan with $E(Y)$ closed, or an arc.*

Some other conditions that are equivalent to those of Theorem 12.4 will be discussed later: see Theorem 12.7, Remark 12.8, Theorem 12.13, Proposition 12.18 and Theorem 12.19.

For each $n \in \mathbb{N}$ let F_C^n denote a copy of the Cantor fan F_C with $\operatorname{diam}(F_C^n) < 1/n$, and let F_C^ω be the one-point union of all F_C^n with their tops identified. In other words, we replace each arc ve_n of the locally connected countable fan F^ω (where $e_n \in E(F^\omega)$ and $e_n \to v$) by F_C^n. The continuum F_C^ω thus obtained can also be described as being homeomorphic to F_C/ve for some fixed $e \in E(F_C)$.

The next result concerns images of F_C under monotone mappings. For its proof see [24], Theorem 10.

THEOREM 12.5. *The following conditions are equivalent for a nondegenerate continuum Y:*

1° *Y is the image of the Cantor fan F_C under a monotone mapping;*

2° *there exists an embedding h of Y into the Cantor fan F_C such that the image $h(Y)$ is a monotone retract of F_C;*

3° *Y is homeomorphic either to F_C or to F_C^ω.*

The following two theorems, which we quote after [22] and [23], are closely related to Theorems 12.3 and 12.4. They contain characterizations, in the realm of all fans, of those X that have either $S(X)$ or $E(X)$ closed. As we already know, conditions of this kind are related to the property of Kelley. The reader is referred to Theorem 3 of [22], p. 75 and to Theorem 3 of [23], p. 171, for the proof of Theorem 12.6, and to Theorem 1 together with Corollary 4 of [22] for the proof of Theorem 12.7.

THEOREM 12.6. *Let a fan X with top v be given. Then the following conditions are equivalent:*

1° *X is smooth and $S(X)$ is closed;*

2° *X is smooth, and for no countable subset of end points e_n of X the union $\bigcup\{ve_n : n \in \mathbb{N}\}$ is homeomorphic to the harmonic prolonged fan F_{HP};*

3° *X is embeddable into F_C^ω in such a way that end points of X are mapped to end points of F_C^ω;*

4° *X is the limit of an inverse sequence of either*

(a) *finite fans with confluent bonding mappings, or*

(b) *locally connected fans with open bonding mappings;*

5° *X has the property of Kelley.*

THEOREM 12.7. *Let a fan X with top v be given. Then the following conditions are equivalent:*

1° *X is smooth and $E(X)$ is closed;*

2° *there exists an embedding $h : X \to h(X) \subset F_C$ such that $h(E(X)) \subset E(F_C)$;*

3° *there exists a surjective mapping $f : X \to [0,1]$ such that $f(v) = 0$ and for each $e \in E(X)$, the mapping $f|ve : ve \to [0,1]$ is a (surjective) homeomorphism;*

4° *X is the limit of an inverse sequence of finite fans with open bonding mappings;*

5° *X has the property of Kelley and v is not in the closure of $E(X)$.*

Remark 12.8. As the reader has certainly observed, for fans conditions 4° of Theorem 12.3 and 1° of Theorem 12.6 as well as 3° of Theorem 12.4 and 1° of Theorem 12.7 coincide. Consequently, if the continuum under consideration is a fan, all conditions of Theorems 12.3 and 12.6 are equivalent, as are those of Theorems 12.4 and 12.7.

Our next result is also related to Theorems 12.4 and 12.7. It involves the condition (which has already been considered in Example 11.16) of each arc in the fan considered being a retract of the fan under a special retraction.

PROPOSITION 12.9. *If a fan X with top v is smooth and $E(X)$ is closed, then for each arc $A \subset X$ there exists a light open retraction of X onto A (i.e., each arc in X is a light open retract of X).*

Proof. By the implication 1° $\Rightarrow$ 2° of Theorem 12.7 there exists a homeomorphism $h : X \to h(X) \subset F_C$ with $h(E(X)) \subset E(F_C)$. Choose two points e_1 and e_2 of X such that $A \subset e_1e_2$ and decompose the set $E(F_C)$ into two disjoint open and closed subsets E_1 and E_2 with $h(e_1) \in E_1$ and $h(e_2) \in E_2$. Consider a projection r of F_C onto the union of two straight line segments $L = h(v)h(e_1) \cup h(v)h(e_2) \subset F_C$ such that a straight line segment $h(v)e \subset F_C$ is mapped onto either $h(v)h(e_1)$ or $h(v)h(e_2)$ depending on whether e is in E_1 or in E_2, and such that for each point u of F_C the second Cartesian coordinates of u and $r(u)$ are equal. Thus r is obviously a light open retraction of F_C onto L. Then $g = h^{-1}\,(r|h(X))\,h$ maps X onto e_1e_2. Next fold e_1e_2 onto $A \subset e_1e_2$ under a light open retraction $f : e_1e_2 \to A$

which sends e_1 and e_2 to end points of A. Then the composition fg is the needed mapping.

Remarks 12.10. 1) Smoothness is essential in Proposition 12.9. Namely the fan X described in Example 11.17 (Fig. 6) is not smooth, it has $E(X)$ closed, and no retraction of X onto the limit arc ve_0 is even confluent. To see this last statement observe that for each retraction $f : X \to ve_0$ and for each sufficiently large $n \in \mathbb{N}$ the images $f(b_n)$ and $f(d_n)$ lie close to b and a respectively, and therefore $f|ve_n$ does not satisfy condition (16) of Theorem 5.7.

2) Also the assumption that $E(X)$ has to be closed is necessary in Proposition 12.9, as the example of the countable locally connected fan F^{ω} shows by Proposition 9.4.

3) The condition formulated in the conclusion of Proposition 12.9 cannot be joined to the five equivalent conditions of Theorem 12.7, i.e., the converse to 12.9 is not true, as can be seen by Example 12.11 below. However, some similar conditions expressed in terms of existence of special retractions of a smooth fan X onto an arc $A \subset X$ are shown to be equivalent to ones saying that either $E(X)$ or $S(X)$ is closed: see Theorems 12.16 and 12.19. In particular, for smooth fans X the condition formulated in the conclusion of 12.9 is equivalent to $E(X) = \operatorname{cl} E(X)$ by Proposition 12.18.

EXAMPLE 12.11. *There exists a countable plane nonsmooth fan F with $E(F)$ closed having the property that each arc contained in F is a light open retract of F.*

Proof. The fan F is defined in Example 10.3 (see Fig. 2). We will use notation from that example. Put

$$B = ve_0 \cup va_0 \cup \bigcup\{ve_n : A \cap (ve_n \setminus \{v\}) \neq \emptyset\},$$

and observe that B is either an arc (if $A \subset ve_0 \cup va_0$), or a triod, or a 4-od. Define a retraction $r_1 : F \to B$ by the following conditions. The partial mapping $r_1|B$ is the identity. For each $n \in \mathbb{N}$ such that $e_n \in F \setminus B$ we put $r_1(b_n) = a_0$, $r_1(d_n) = v$ and $r_1(e_n) = e_0$, and we define $r_1|vb_n$, $r_1|b_nd_n$ and $r_1|d_ne_n$ to be suitable homeomorphisms. Note that r_1 is a light retraction which is interior at each point different from the d_n's. Let A' be the union of those arcs from v to some end points of B whose intersections with A contain more than one point. Thus A' is an arc containing A and contained in B. We will describe a light open retraction r_2 from B onto A' such that the composition $r_2r_1 : F \to A'$ is also light open. We consider two cases. If A' contains only one end point e of B, then we fold B onto $A' = ve$ in such a way that r_2 restricted to an arc from v to an end point of B is a homeomorphism. If A' contains two end points of B, then we fold B onto A'

in a similar manner with the additional condition $r_2(va_0) \cap r_2(ve_0) = \{v\}$. The reader can verify that $r_2 r_1$ is an open (and light) retraction. Since A is a light open retract of A', we are done.

In connection with Example 11.16, Proposition 12.9, Remarks 12.10 and Example 12.11 the following problems seem to be interesting.

PROBLEMS 12.12. Characterize fans X (with top v) having the property that for each end point e of X (for each arc $A \subset X$) there exists a light open retraction of X onto ve (onto A, respectively).

A partial solution of the above problems (namely under the extra assumption of smoothness of the considered fans) is presented below, in Theorem 12.19.

Now we give two consequences of Proposition 12.9. The first concerns conditions under which an n-od Y contained in a fan X is a light confluent (equivalently: light open, see Proposition 3.13) retract of X. Recall that retractions of fans onto finite fans are investigated in [49], where it is shown among other things that each fan can be ε-retracted onto a finite subfan (for each $\varepsilon > 0$). The other consequence of 12.9 is a new characterization of the Lelek fan F_L. It extends Proposition 2 of [33].

THEOREM 12.13. *For $n \geq 3$ let an n-od Y be contained in a fan X. Then there exists a light confluent (equivalently: light open) retraction of X onto Y if and only if X is smooth, $E(X)$ is closed, and $E(Y) \subset E(X)$.*

Proof. To prove the "if" part put $E(Y) = \{e_1, e_2, \dots, e_n\}$ and note that we can represent X as the union of n fans $X_1, X_2, \dots, X_n$ such that for any two distinct indices i and j from $\{1, 2, \dots, n\}$ we have $X_i \cap X_j = \{v\}$ (where v is the top of X), and $ve_i \subset X_i$. Since these fans X_i are smooth, being contained in a smooth fan X (see [9], Corollary 2, p. 7), Proposition 12.9 can be applied for every $i \in \{1, 2, \dots, n\}$ separately, and so we conclude that there are light open retractions $f_i : X_i \to ve_i$. It is easy to observe that the common extension of all the f_i is a light confluent retraction of X onto Y.

To show the "only if" part, assume that a light confluent retraction f of X onto Y does exist. Then the smoothness of X is a consequence of that of Y by Proposition 11.14. Further, it follows from Corollary 4.13 that $E(X) = f^{-1}(E(Y))$, thus $E(X)$ is closed by continuity of f. Finally the inclusion $E(Y) \subset E(X)$ is a consequence of (c) of Theorem 4.17.

THEOREM 12.14. *The following conditions are equivalent for a smooth fan X:*

1° *X is homeomorphic to the Lelek fan F_L;*

2° *$E(X)$ is dense;*

3° *$S(X)$ is connected;*

4° *each image of X under a confluent mapping is homeomorphic to X;*

5° *each image of X under a monotone mapping is homeomorphic to X;*

6° *there is no monotone mapping from X onto a smooth fan Y with $E(Y)$ closed and there is no monotone mapping from X onto an arc;*

7° *there is no confluent (equivalently OM-) mapping from X onto an arc.*

Proof. Conditions 1° through 5° are shown to be equivalent in Proposition 2 of [33]. The two conditions mentioned in 7° are equivalent by Proposition 5.1. Obviously 4° implies 7°. Assume 7° and suppose 6° does not hold. Let $f : X \to Y$ be as in the negation of 6°. Then Y cannot be an arc by 7°, so it is a fan, and by Proposition 12.9 there exists a light open retraction g from Y onto an arc $ve \subset Y$. Hence $gf : X \to ve$ is an OM-mapping of X onto an arc, contrary to 7°. So 7° implies 6°. To complete the proof we have to show that 6° implies 2°. The idea of the proof comes from a part of Proposition 3 of [33]. So, let a smooth fan X be embedded into the Cantor fan $F_C = (C \times [0,1])/(C \times \{0\})$, where C is the Cantor set (see 11.2). Suppose that 2° does not hold. Thus there are an open and closed set $B \subset C$ and $t_0, t_1 \in [0,1]$ such that $B \times [t_0, t_1] \subset F_C$ contains some points of X and is disjoint from $E(X)$. Then shrink $X \cap ((C \setminus B) \times [0,1] \cup B \times [0,t_0])$ to the top of X, and every arc of the form $X \cap (\{c\} \times [t_1, 1])$, for $c \in B$, to a point. Denote the resulting space by Y. Thus Y is either a smooth fan with $E(Y)$ closed or an arc, and the quotient mapping from X onto Y is monotone by definition. This contradicts 6°. The proof is complete.

Remarks 12.15. 1) Let us recall that condition 7° of Theorem 12.14 is related to Propositions 5.12, 5.15, Examples 5.13, 5.14 and 7.8 and Theorem 5.18, where fans admitting confluent mappings onto an arc are considered.

2) Observe that the condition saying that the image of X under an open mapping is homeomorphic to X cannot be joined to conditions 4° and 5° of Theorem 12.14 as the example of the countable locally connected fan F^{ω} shows (see Proposition 9.4).

To prove further results on smooth fans with closed sets of end points we need the equivalences between the condition saying that the set $S(X)$ is closed for a given smooth fan X and the existence of confluent (equivalently OM-, see Proposition 5.1) retractions of X onto arcs $A \subset X$. These equivalences, which are related to the property of Kelley by conditions 3° and 4° of Theorem 12.3, run as follows.

THEOREM 12.16. *Let a fan X with top v be smooth. The following conditions are equivalent:*

1° *$S(X)$ is closed;*

2° for each arc $A \subset X$ there exists a confluent (equivalently an OM-) retraction of X onto A;

3° for each end point e of X there exists a confluent (equivalently an OM-) retraction of X onto ve.

Proof. Since the implication 2° $\Rightarrow$ 3° is obvious, we need only show that 1° $\Rightarrow$ 2°, and 3° $\Rightarrow$ 1°. By Proposition 11.2 we can consider X to be contained in the Cantor fan, and therefore we may assume that the arcs ve, where $e \in E(X)$, are straight line segments, and that the metric d on X coincides with the usual Euclidean metric on the plane.

1° $\Rightarrow$ 2°. Let A be an arc. If $E(X)$ is closed, an open retraction from X onto $A \subset X$ exists by Proposition 12.9. So let $E(X)$ be nonclosed. Since $S(X) = \{v\} \cup E(X)$ is closed, the top v of X is the only point of $\mathrm{cl}\, E(X) \setminus E(X)$. Further, since components of the compact set $S(X)$ are continua, no one of them is nondegenerate, and thus $S(X)$ is zero-dimensional. Take a positive $\varepsilon < \operatorname{diam} A$. Since $\dim S(X) = 0$, there is an open ε-neighborhood V of v such that $V \cap S(X)$ is an open and closed subset of $S(X)$. Then define a mapping g on X as follows. For an end point e of X such that $ve \cap A$ is nondegenerate, and for each $e \in E(X) \setminus V$, let $g|ve : ve \to ve$ be the identity. Otherwise $g|ve : ve \to \{v\}$ is a constant mapping. Thus g is a monotone retraction of X onto $g(X)$, and we see that $A \subset g(X)$. If $g(X)$ is an arc, we can fold $g(X)$ onto A under a light open mapping f, i.e. in such a way that the closures of (at most two) components of $g(X) \setminus A$ are linearly mapped onto A. If $g(X)$ is not an arc, it is a fan with $E(g(X))$ closed, and therefore Proposition 12.9 can be applied. Thus there is a light open retraction $f : g(X) \to A$ of $g(X)$ onto A. In both these cases the composition fg, with g monotone and f open, is the needed retraction of X onto A.

3° $\Rightarrow$ 1°. Suppose on the contrary that $S(X)$ is not closed. Thus there exist $x \in \mathrm{cl}\, S(X) \setminus S(X)$ and a sequence of end points e_n of X tending to x. Take an end point e of X such that $x \in ve$. Let $r : X \to ve$ be a confluent retraction of X onto ve. Since $v \neq x \neq e$ by assumption, there is an arc ab in ve such that $a \in ex \setminus \{e, x\}$ and $b \in xv \setminus \{x, v\}$. Further, take a positive $\varepsilon < \min\{d(a,x), d(x,b), d(b,v)\}$. Since r is uniformly continuous, there exists a positive δ such that, for any u and u' in X the inequality $d(u,u') < \delta$ implies $d(r(u), r(u')) < \varepsilon$. The fan X being smooth, the condition $e_n \to x$ implies $\operatorname{Lim} ve_n = vx$ and thus we can choose n so large that $\operatorname{dist}(ve_n, vx) < \delta$. This implies in particular that $d(e_n, x) < \delta$, whence $d(r(e_n), r(x)) < \varepsilon$, i.e., $d(r(e_n), x) < \varepsilon$, since $r(x) = x$. Thus $r(e_n) \in ab$ by the choice of ε. Denote by K_n the component of $r^{-1}(ab)$ containing e_n, and observe that, since $r(v) = v$, the top v is not in K_n. Thus $K_n \subset ve_n \setminus \{v\}$. Since r is confluent, we have $r(K_n) = ab$. Hence there is $u \in K_n$ with $r(u) = a$. Since $u \in K_n \subset ve_n \subset B(vx, \delta)$, there is $u' \in vx$ such that $d(u,u') < \delta$.

Thus $d(r(u), r(u')) < \varepsilon$, i.e., $d(a, u') < \varepsilon$. Since $d(a, x) \leq d(a, u')$, we have $d(a, x) < \varepsilon$ contrary to the choice of ε. The proof is complete.

Remark 12.17. Smoothness is necessary in 12.16 by the same Example 11.17 (Fig. 6) as previously (see Remark 1 of 12.10 for an argument).

Now we are able to prove a partial converse to Proposition 12.9. Next we shall apply it to obtain in Theorem 12.19 equivalences between certain four conditions expressed in terms of the existence of special retractions of a smooth fan X onto arcs $A \subset X$ on one hand, and the equality $E(X) = \operatorname{cl} E(X)$ on the other.

PROPOSITION 12.18. *Let a fan X with top v be smooth. If for each end point e of X there exists an open retraction of X onto the arc ve, then $E(X)$ is closed.*

Proof. Since X is smooth, we may assume by Proposition 11.2 that X is contained in the Cantor fan. Suppose on the contrary that $E(X)$ is not closed. Since $S(X)$ is closed by $3° \Rightarrow 1°$ of Theorem 12.16, by the same arguments as at the beginning of the proof of the implication $1° \Rightarrow 2°$ of Theorem 12.16 we see that v is the only point of $\operatorname{cl} E(X) \setminus E(X)$ and that $S(X)$, and thus $E(X)$, is zero-dimensional. Hence there exists a sequence of open sets $V_n \subset X$ having the singleton $\{v\}$ as its limit, and such that the intersections $V_n \cap E(X)$ are both open and closed in $E(X)$. For each $n \in \mathbb{N}$ let K_n denote the cone with vertex v over $V_n \cap E(X)$. Thus each K_n is a subcontinuum of X. Note that since $V_n \cap E(X)$ is an open subset of $E(X)$, the set $K_n \setminus \{v\}$ is open.

Take an arbitrary end point e of X and let $r : X \to ve$ be an open retraction. Denote by c the center of the straight line segment ve. By continuity of r there is a $\delta > 0$ such that if $K_n \subset B(v, \delta)$, then $r(K_n) \subset vc$. On the other hand, since $r(K_n \setminus \{v\})$ is open by openness of r, the set $r(K_n)$ is not the singleton $\{v\}$. Thus there exists an end point e_1 of X such that $r(ve_1)$ is a nondegenerate proper subarc vu of ve. Take a small arc Q around u in ve, and let K be the component of $r^{-1}(Q)$ containing e_1. Then $K \subset ve_1$, and so $r(K) \subset r(ve_1) = vu$, whence $r(K) \neq Q$, a contradiction to the confluence of the open mapping r. This finishes the proof.

THEOREM 12.19. *Let a fan X with top v be smooth. The following conditions are equivalent:*

1° *$E(X)$ is closed;*

2° *for each arc $A \subset X$ there exists a light open retraction of X onto A;*

3° *for each arc $A \subset X$ there exists an open retraction of X onto A;*

4° *for each end point e of X there exists a light open retraction of X onto ve.*

5° *for each end point e of X there exists an open retraction of X onto ve.*

Proof. The implications 2° $\Rightarrow$ 3°, 2° $\Rightarrow$ 4°, 3° $\Rightarrow$ 5° and 4° $\Rightarrow$ 5° are obvious, and 1° $\Rightarrow$ 2° and 5° $\Rightarrow$ 1° form Propositions 12.9 and 12.18 respectively. This finishes the proof.

The next theorem describes the behaviour of fans having the property of Kelley under confluent mappings. The theorem is an immediate consequence of (1) of Proposition 3.4 and of Theorem (4.3) of [115], p. 296 (cf. [97], (16.29), p. 554). However, it can also be derived from the equivalence between conditions 1° and 5° of Theorem 12.6 and from (4) and (11) of Theorem 4.1.

THEOREM 12.20. *The nondegenerate image of a fan with the property of Kelley under a confluent mapping is either a fan with the property of Kelley or an arc.*

The above result resembles a similar one for smoothness of fans (Theorem 11.13). Thus a natural question arises whether the property of Kelley can be substituted in place of smoothness in Proposition 11.14. The next result gives an affirmative answer to this.

PROPOSITION 12.21. *Let a surjective light confluent mapping be defined between fans X and Y. If Y has the property of Kelley, then so has X.*

Proof. By Theorem 12.6 (viz. the equivalence of 1° and 5°) the fan Y is smooth and has $S(Y)$ closed. Now Proposition 11.14 implies the smoothness of X, and from Corollary 4.13 we infer that $S(X)$ is closed, too. Applying Theorem 12.6 once more we get the conclusion.

Theorem 12.20 and Proposition 12.21 imply the next result.

THEOREM 12.22. *If a surjective confluent mapping between fans is light, then one of them has the property of Kelley if and only if the other one has.*

Remarks 12.23. 1) Lightness is indispensable in Proposition 12.21 and Theorem 12.22. Namely it is easy to find a monotone retraction from the harmonic prolonged fan F_{HP} (which does not have the Kelley property) onto the harmonic fan F_H contained in F_{HP} (which obviously has the property). Thus light confluent mappings cannot be replaced by monotone ones in 12.21 and 12.22.

2) Similarly, "open" cannot be substituted for "light confluent" in the two preceding results, by Example 11.17. In fact, since the property of Kelley implies smoothness of fans by Theorem 12.6, the fan X of Example 11.17 does not have the property of Kelley, while its open image, the simple triod, does.

The above remarks lead to the following questions.

QUESTIONS 12.24. Suppose a continuum (a dendroid, a fan) X does not have the property of Kelley. Under what mappings f the image $f(X)$ does not have this property either?

13. Contractibility

Recall that a topological space X is called *contractible* if there is a homotopy $H : X \times [0,1] \to X$ from the identity to a constant mapping. X is said to be *monotone (confluent, weakly confluent) contractible* provided there exists a homotopy H as above with $H|(X \times \{t\})$ monotone (confluent, weakly confluent respectively) for each $t \in [0,1]$. It is known that each one-dimensional contractible continuum is a uniformly arcwise connected dendroid (see e.g. [12], Propositions 1 and 4, p. 73).

Although a number of papers by various authors are known dealing with contractibility of dendroids (see e.g. [3], Section 8 of [9], [16], [26], [27], [35], [38], [51], [85], [91], [104], [105] and [107]), only some necessary conditions for this property have been found till now, and no internal characterization of contractible dendroids appeared in the literature. But if restricted to fans only, the problem of finding such a characterization has already been solved by L. G. Oversteegen in [105], Theorem 3.4, p. 393 (his results are quoted below as Theorem 13.1; compare also [51] and [107]). To formulate them we need the following four notions, the first of which is due to R. B. Bennett ([4]; see also [51], p. 78, where the term P-point is used; and [105], p. 392).

A point p of a dendroid X is called a *Q-point* if there is a sequence x_n in X converging to p with $\operatorname{Ls} px_n \neq \{p\}$ and such that if $x_n p_n$ denotes the arc irreducible between x_n and $\operatorname{Ls} px_n$, then the sequence p_n converges to p. For example, the tops of the fans of Figures 2, 3, 4 and 5 are Q-points.

The next two concepts are due to B. G. Graham ([51], p. 78; see also [105], p. 393). Let a point r of a dendroid X, and two sequences r_n^1 and r_n^2 of points of X, both converging to r, be given. We say that the former sequence *dominates* the latter provided that whenever there is s in X and a sequence s_n^1 of points converging to s such that the arcs $r_n^1 s_n^1$ converge to rs, then there exists a sequence s_n^2 in X converging to s such that the arcs $r_n^2 s_n^2$ converge to rs. A dendroid is said to be *pairwise smooth* provided that whenever a pair of sequences converge to a common point, then one of them dominates the other. Examples of fans that are not pairwise smooth are presented in Figures 3 and 4 of [51], p. 91 (the former is redrawn here, see Figs. 10 and 11).

We say that a dendroid X *contains a zig-zag* provided there exist in X: an arc A with end points p and q and a sequence of arcs A_n with end points

p_n and q_n, and points $p'_n, q'_n \in A_n \setminus \{p_n, q_n\}$ such that

$$A = \operatorname{Lim} A_n\,, \quad p = \lim p_n = \lim p'_n\,, \quad q = \lim q_n = \lim q'_n$$

and $p_n q'_n \subset p_n p'_n$ for each $n \in \mathbb{N}$. Figures 4 through 7 of [51], pp. 91-93 show examples of fans containing zig-zags. In particular, the fan in Fig. 7, p. 93 of [51] is homeomorphic to that of Example 11.16 above (see our Fig. 5).

The fourth concept related to the internal structure of a fan and used in the characterization of contractible fans is that of a dendroid of type N. This concept is due to L. G. Oversteegen ([105], p. 392; see also [104], p. 837). A dendroid X is said to be *of type* N provided there exist in X: an arc A with end points p and q; two sequences of arcs A_n and B_n with end points p_n, p'_n and q_n, q'_n respectively; and points $p''_n \in B_n \setminus \{q_n, q'_n\}$ and $q''_n \in A_n \setminus \{p_n, p'_n\}$ such that the following conditions are satisfied: $A = \operatorname{Lim} A_n = \operatorname{Lim} B_n$, $p = \lim p_n = \lim p'_n = \lim p''_n$ and $q = \lim q_n = \lim q'_n = \lim q''_n$.

Note that each dendroid containing a zig-zag is of type N ([105], p. 393).

The mentioned characterizations, proved in [105], Theorem 3.4, p. 393, are as follows.

THEOREM 13.1. (Oversteegen) *The following conditions are equivalent for a fan* X*:*

(i) X *is contractible;*
(ii) X *is weakly confluent contractible;*
(iii) X *is confluent contractible;*
(iv) X *is monotone contractible;*
(v) X *contains no* Q*-point,* X *is not of type* N *and* X *is pairwise smooth;*
(vi) X *contains no* Q*-point and no zig-zag, and* X *is pairwise smooth.*

A problem is related to the above result.

PROBLEM 13.2. Give an internal characterization of those contractible dendroids for which conditions (i) through (iv) of Theorem 13.1 are equivalent.

The main goal of this chapter is to study some relations between contractibility of fans and contractibility of their images under confluent and related mappings. However, very little is known about mappings of fans in reltion to their contractibility. The next example concerns this topic.

EXAMPLE 13.3. *There exist a contractible fan* X *and a monotone mapping* f *defined on* X *such that only one point-inverse is nondegenerate and that the resulting fan* $f(X)$ *contains both a* Q*-point and a zig-zag (thus it is of type* N*).*

Proof. In polar coordinates in the plane with pole $v = (0,0)$, consider for each $n \in \mathbb{N}$

$$a_0 = (1,0), \quad a_n = (1, 1/n), \quad b_0 = (1/2, 0) \quad \text{and} \quad b_n = (1/2, 1/n).$$

Putting

$$X = \overline{va_0} \cup \bigcup\{(\overline{va_{2n}} \cup \overline{a_{2n}b_{2n+1}}) : n \in \mathbb{N}\} \tag{23}$$

we obtain the harmonic hooked fan (compare [9], p. 31 and Fig. 7 here). It is known to be contractible ([9], p. 31). Consider the monotone decomposition of X whose only nondegenerate element is the segment $\overline{vb_0}$, and let $f : X \to f(X) = Y$ be the quotient mapping. Thus $Y = X/\overline{vb_0}$ is homeomorphic to the fan of Example 11.16 (Fig. 5) which is known to be noncontractible ([26], p. 95; see also Example 1.2 of [104], p. 838 and a remark following it).

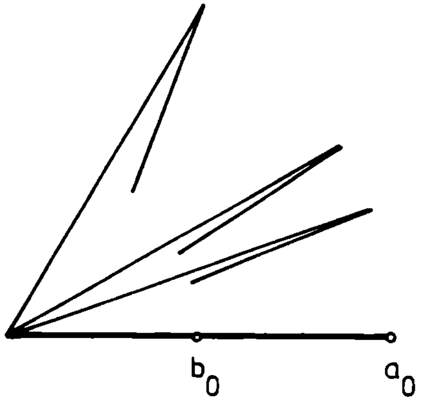

Fig. 7

The example above justifies the following corollary.

COROLLARY 13.4. *The image of a contractible fan under a monotone mapping need not be contractible, even if all point-inverses but one are degenerate.*

Example 13.3 shows that the properties of not containing a Q-point or a zig-zag are not invariant under monotone mappings. But the mapping f defined in that example is neither open nor (obviously) light. So we have

QUESTIONS 13.5. Is the property of not containing a) a Q-point, b) a zig-zag, invariant under 1° light open, 2° open, 3° light confluent mappings?

The next example shows that also the third property used in the characterizations (v) and (vi) of Theorem 13.1, namely pairwise smoothness, is not invariant under monotone mappings. This gives another proof of Corollary 13.4.

EXAMPLE 13.6. *There exist a contractible fan X' and a monotone mapping f defined on X' such that only one point-inverse is nondegenerate and that the resulting fan $f(X')$ is not pairwise smooth.*

Proof. To describe X' we use the same notation as in Example 13.3. Thus, again in polar coordinates in the plane, put $c_0 = (1/4, 0)$ and $c_n = (1/4, -1/n)$ for each $n \in \mathbb{N}$. Keeping in mind the fan X defined by (23) observe that the union

$$X' = X \cup \bigcup \{\overline{vc_n} : n \in \mathbb{N}\}$$

(see Fig. 8) is a contractible fan (its contractibility can be shown exactly as for the harmonic hooked fan X, see [9], p. 31).

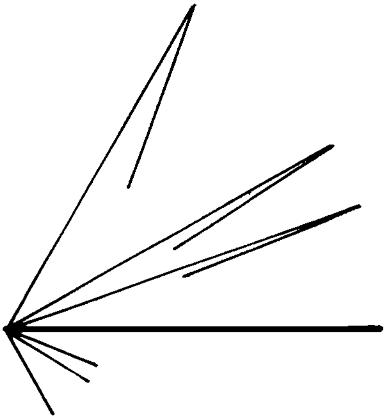

Fig. 8

Now f is defined on X' as the quotient mapping of the monotone decomposition of X' whose only nondegenerate element is the subsegment $\overline{c_0b_0}$ of the limit segment $\overline{va_0}$ of X'. Thus the resulting quotient space $f(X') = X'/\overline{c_0b_0}$ is homeomorphic to a not pairwise smooth fan pictured here in Fig. 10 (the same as Fig. 3 of [51], p. 91).

For light confluent mappings we have the following example.

EXAMPLE 13.7. *There exist a contractible fan X'' and a light confluent mapping f on X'' with all point-inverses consisting of two points at most, and such that the resulting fan $f(X'')$ is not pairwise smooth.*

Proof. We use the same notation in polar coordinates as in Examples 13.3 and 13.6. Again let X be the harmonic hooked fan of Example 13.3 defined by (23). Put $d_0 = (1, -\pi/2)$ and $d_n = (1/2, -\pi/2 - 1/n)$ for each $n \in \mathbb{N}$. Then the union $U = \bigcup\{\overline{vd_n} : n \in \{0\} \cup \mathbb{N}\}$ is homeomorphic to the harmonic prolonged fan F_{HP}. Putting

$$X'' = X \cup U$$

we see that X and U have v as the only common point, and therefore X'' is a contractible fan (Fig. 9). Now we define f on X'' as the identity mapping on X and as the rotation through $\pi/2$ about v (i.e., $(\rho, \phi) \mapsto (\rho, \phi + \pi/2)$)

on U. Thus $f(d_n) = (1/2, -1/n)$ and $f(d_0) = (1,0) = a_0 \in X$, so

$$f(X'') = X \cup \bigcup \{\overline{vf(d_n)} : n \in \mathbb{N}\}$$

is homeomorphic to the not pairwise smooth fan $f(X')$ of Example 13.6, pictured here in Fig. 10 ([51], Fig. 3, p. 91). By the definition of f we see that for each y in $va_0 \setminus \{v\} \subset f(X)$ its point-inverse consists of two points, while all other y in $f(X)$ have singletons as point-inverses.

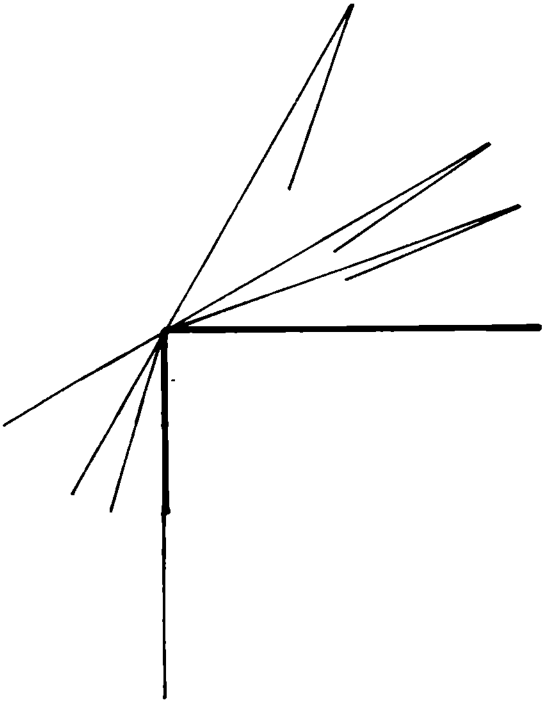

Fig. 9

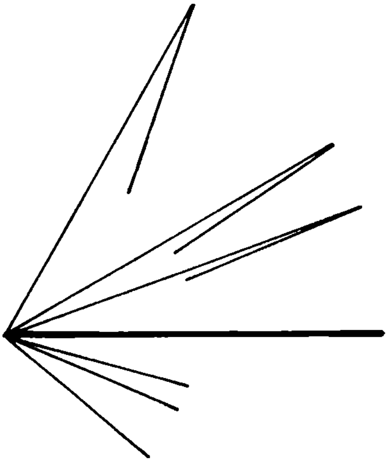

Fig. 10

COROLLARY 13.8. *The image of a contractible fan under a light confluent mapping need not be contractible, even if all point-inverses consist of one or two points.*

Similarly to Questions 13.5, since the mapping f of the previous example is again not open, we have the following.

QUESTIONS 13.9. Is the property of being a pairwise smooth fan invariant under light open or open mappings?

The second of the following two questions is a particular case of the first.

QUESTIONS 13.10. What kinds of confluent mappings preserve contractibility of fans?

QUESTIONS 13.11. Is contractibility of fans invariant under light open or under open mappings?

A concept we recall now is used in the next remark to show the noncontractibility of a dendroid. Consider an arc ab contained in a dendroid X and a point x of X. If x is in $\operatorname{cl} B(ab, r)$ for some $r > 0$, and $xa \cap \operatorname{cl} B(ab, r)$ is disconnected, then we denote by $x(r)$ the first point of the arc xa ordered from x to a which lies in $\operatorname{bd} B(ab, r)$, i.e., $x(r) \in xa \cap \operatorname{bd} B(ab, r)$ and $xx(r) \cap \operatorname{bd} B(ab, r) = \{x(r)\}$. An arc ab contained in a dendroid X is called an *R-arc* if 1° there are two sequences $\{u_n\}$ and $\{v_n\}$ of end points of X such that $\lim u_n = a$ and $\lim v_n = b$; 2° there is $r > 0$ such that for almost all $n \in \mathbb{N}$ the intersections $u_n b \cap \operatorname{cl} B(ab, r)$ and $v_n a \cap \operatorname{cl} B(ab, r)$ are disconnected (thus the points $u_n(r)$ and $v_n(r)$ are well defined) and the sets $u_n u_n(r) \setminus \{u_n(r)\}$ and $v_n v_n(r) \setminus \{v_n(r)\}$ contain no ramification points of X; and 3° the sequences of arcs $u_n u_n(r)$ and $v_n v_n(r)$ are convergent and ab is the intersection of their limits. The case of a degenerate R-arc (i.e. when $a = b$) is also acceptable, and the term *R-point* is then used (see [27], Definition 4, p. 230). It is known that if a dendroid contains an R-arc, then it is noncontractible ([27], Corollary 6, p. 232). The concept of an R-arc has been generalized in three different ways to so called R^i-continua, which will be discussed later (see below, after Question 13.22).

Remark 13.12. Note that the harmonic hooked fan defined above by formula (23) in Example 13.3 is contractible but not hereditarily contractible. Namely it contains a subfan

$$X''' = \overline{va_0} \cup \bigcup\{(\overline{va_{2n}} \cup \overline{a_{2n}b_{2n+1}}) : n \in \{1,3,5,\ldots\}\} \\ \cup \bigcup\{(\overline{vb_{2n}}) : n \in \{2,4,6,\ldots\}\} \tag{24}$$

(see Fig. 11; cf. [16], Fig. 2, p. 112) which is known to be noncontractible because $b_0 = (1/2, 0)$ is an R-point of X''', which suffices for noncontractibility by Corollary 6 of [27], p. 232 (see [16], Proposition 4, p. 111).

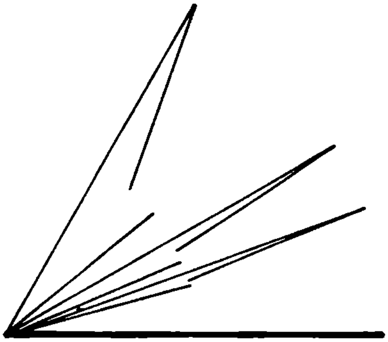

Fig. 11

Note that X''' is homeomorphic to the range spaces considered in Examples 13.6 and 13.7, so it is not pairwise smooth, which gives another reason for its noncontractibility. Moreover, since hereditary contractibility of fans is equivalent to their smoothness ([27], Corollary 17, p. 237) which is preserved under confluent mappings (Theorem 11.13 here) and which is known to be co-invariant under light confluent mappings (Proposition 11.14) one can reformulate 11.13 and 11.14 as follows.

THEOREM 13.13. *Hereditary contractibility is invariant under confluent mappings and co-invariant under light confluent mappings.*

Remarks 13.14. 1) It is easy to construct examples which show that the above theorem cannot be generalized to arbitrary dendroids (not being fans). Note that no internal characterization of hereditarily contractible dendroids is known till now, and that there is a conjecture that this property is equivalent to pointwise smoothness ([37], Proposition 2 and Question 1, p. 170; cf. [39]; see also Remark 11.19). The reader can find a wide discussion and several new and important results concerning hereditary contractibility of dendroids in the forthcoming paper [42].

2) B. G. Graham in [51], Appendix, pp. 89–90, describes a contractible dendroid X (containing exactly two ramification points, see Fig. 1 of [51], p. 89) such that for each homotopy $H : X \times [0,1] \to X$ with $H_0 =$ identity, $H_t(X)$ is a noncontractible subdendroid of X for some $t \in [0,1]$ (see Fig. 2 of [51], p. 91). It would be interesting to know if there is a fan having the above property.

EXAMPLE 13.15. *There exist a fan X containing a Q-point and a zig-zag (thus being of type N), and a monotone mapping $f : X \to F^\omega$ from X onto the locally connected countable fan F^ω, such that the only one point-inverse under f is nondegenerate.*

Proof. Let X denote the fan of Example 11.16 defined by (21). Consider the monotone decomposition of X whose only nondegenerate element is the limit segment $\overline{ve_0}$. If $f : X \to f(X)$ is the quotient mapping of this decomposition, then the resulting quotient space $f(X) = X/\overline{ve_0}$ is homeomorphic to F^ω, which obviously is contractible.

COROLLARY 13.16. *The image of a noncontractible fan under a monotone mapping need not be noncontractible, even if all but one point-inverses are degenerate.*

EXAMPLE 13.17. *There exist a not pairwise smooth fan X containing an R-point and a monotone mapping f from X onto the harmonic fan such that only one point-inverse under f is nondegenerate.*

Proof. In fact, such a fan X is pictured in Fig. 10, and it is homeomorphic to the fan X''' of Remark 13.12 defined by (24). Shrinking the subsegment $\overline{b_0a_0}$ of the limit segment $\overline{va_0}$ of X''' to a point, we get a monotone mapping f of X having just one nondegenerate point-inverse, and we see that $f(X) = X'''/\overline{b_0a_0}$ is homeomorphic to the harmonic fan.

Note that Corollary 13.16 follows also from Example 13.17.

We are now going to discuss open mappings of fans and their connections with noncontractibility. To this end let us recall Examples 11.16 and 11.17. The former shows a fan X (defined by (21)) that contains both a Q-point and a zig-zag, and a light open retraction of X onto an arc (note that the Borsuk fan F_B and its mapping onto an arc of Example 10.6 have the same properties). The latter shows a fan X containing a zig-zag, and a nonlight open mapping of X onto a simple triod. The examples mentioned above justify the following corollary.

COROLLARY 13.18. *Neither the existence of a Q-point nor the existence of a zig-zag is invariant under light open mappings of fans.*

QUESTIONS 13.19. Does there exist a light open mapping from a fan containing a) a Q-point, b) a zig-zag, onto a fan that contains neither a Q-point nor a zig-zag?

QUESTIONS 13.20. Does there exist an open (or even a light open) mapping from a not pairwise smooth fan onto a fan that is pairwise smooth (contractible)?

As a consequence of the remarks that precede Corollary 13.18 we conclude the following.

COROLLARY 13.21. *The image of a noncontractible fan under a light open mapping need not be noncontractible.*

One can ask a question similar to 13.10.

QUESTION 13.22. What kinds of confluent mappings preserve noncontractibility of fans?

Up to now, we discussed the behaviour of four concepts (viz. Q-points, pairwise smoothness, zig-zags and being of type N) under confluent mappings. It was so because these concepts were used to characterize contractible fans (see (v) and (vi) of Theorem 13.1). But there are also some other concepts known from the literature which are related to contractibility of dendroids. Such is, for example, the concept of an R-arc (discussed above; see [27], p. 230), later generalized to an R-continuum ([35], Definition 1, p. 300; see also [36]), and finally renamed as an R^1-continuum, to distinguish it from similarly constructed notions of R^2- and R^3-continua ([38], Definitions 1.1, 1.2 and 1.3, p. 75). We recall here these three concepts.

A nonempty proper subcontinuum K of a dendroid X is called an *R^1-continuum* (*R^2-continuum*, *R^3-continuum*) if there exist an open set U containing K and two sequences C_n^1, C_n^2 (two sequences C_n^1, C_n^2, a sequence C_n, respectively) of components of U such that $\mathrm{Ls}\, C_n^1 \cap \mathrm{Ls}\, C_n^2 = K$ ($\mathrm{Lim}\, C_n^1 \cap \mathrm{Lim}\, C_n^2 = K$, $\mathrm{Li}\, C_n = K$ respectively). The main result connected with these concepts says that if a dendroid X contains an R^i-continuum, where $i \in \{1,2,3\}$, then X is not contractible ([38], Theorem 9, p. 78; [35], Corollary 4, p. 302). This criterion of noncontractibility of dendroids entails that of [3], Corollary 1, p. 48 (compare [13], Theorem 2, p. 272) expressed in terms of a set function T (see [36], Theorem 7, p. 305 and Corollary 8, p. 306).

Unfortunately, in the papers where these three notions of R^i-continua are introduced, they are neither considered especially for fans nor is there any discussion about mappings related to them. However, a fan X is defined in Example 5 of [35], p. 302 (pictured there on p. 301) that contains a one-point R^1-continuum (which is also an R^2- and an R^3-continuum, see [38], Corollary 11, p. 78). It is easy to see that shrinking the limit continuum of X to a point we get a monotone mapping f from X with $f(X)$ homeomorphic to the fan F^ω. So, we have the following result.

COROLLARY 13.23. *Containing an R^i-continuum for $i \in \{1,2,3\}$ is not invariant under monotone mappings of fans, even if all but one point-inverses are degenerate.*

Observe that the above corollary gives another proof of Corollary 13.16.

QUESTIONS 13.24. Let a fan (a dendroid) X be given. What confluent mappings f defined on X have the property that if X contains an R^i-continuum, where $i \in \{1,2,3\}$, then $f(X)$ also contains an R^i-continuum?

Further, note that the fan X'' of Example 13.7 does not contain any R^i-continuum, $i \in \{1,2,3\}$, while its image under the light confluent mapping f defined there does contain an R-point $(1/2, 0)$. Hence we have

COROLLARY 13.25. *Containing an R^i-continuum for $i \in \{1,2,3\}$ is not co-invariant under light confluent mappings of fans, even if all point-inverses consist of one or two points.*

QUESTIONS 13.26. Let a fan (a dendroid) X be given. What confluent mappings f defined on X have the property that if $f(X)$ contains an R^i-continuum, where $i \in \{1,2,3\}$, then X also contains an R^i-continuum?

Another notion that is related to contractibility of fans is due to T. Maćkowiak [83]. A set B is called a ***bend set of a subcontinuum*** A contained in a dendroid X provided there exist two sequences A_n and A'_n of subcontinua of X such that (i) $A_n \cap A'_n \neq \emptyset$ for each $n \in \mathbb{N}$; (ii) $\operatorname{Lim} A_n = A = \operatorname{Lim} A'_n$; and (iii) $B = \operatorname{Lim}(A_n \cap A'_n) \neq \emptyset$. We say that a dendroid X has the ***bend intersection property*** provided for each subcontinuum A of X the intersection of all bend sets of A is nonempty.

Recently T. J. Lee has shown in [63] the following result.

THEOREM 13.27. (Lee) *Every contractible fan has the bend intersection property.*

The next questions are related to this result.

QUESTIONS 13.28. Does every contractible (or confluent contractible, or weakly confluent contractible, or monotone contractible) dendroid have the bend intersection property?

Note that the equivalence between contractibility, weakly confluent contractibility, confluent contractibility and monotone contractibility is proved for fans only (Theorem 13.1).

Finally, let us recall some questions related to inverse limits (see [21], Problem 2, p. 148).

QUESTIONS 13.29. Let X be the limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$ such that each X^i is a contractible dendroid (fan), and each f^i is monotone. Is then the dendroid (fan) X contractible?

14. Selectibility

Let a continuum X be given. By a *continuous selection* for the hyperspace $C(X)$ of all nonempty subcontinua of X equipped with the Hausdorff distance we mean a mapping $\sigma : C(X) \to X$ such that $\sigma(A) \in A$ for each $A \in C(X)$. A continuum is said to be *selectible* provided it admits a continuous selection for $C(X)$. Important results on structural properties of selectible continua are obtained in [99]. In particular, the following fact is proved ([99], Corollary, p. 371).

PROPOSITION 14.1. (Nadler and Ward) *A locally connected continuum is selectible if and only if it is a dendrite.*

As a consequence of Lemma 3 of [99], p. 370 and of Proposition 2 of [16], p. 110 we have the next proposition.

PROPOSITION 14.2. *Each selectible continuum is a dendroid which is a continuous image of the Cantor fan, and therefore is uniformly arcwise connected.*

The inverse implication is not true in general. Namely L. E. Ward, Jr. in [114] has characterized smoothness of dendroids in terms of a special selection, called rigid. A selection $\sigma : C(X) \to X$ is said to be *rigid* provided that for any two subcontinua A and B of X with $\sigma(A) \in A \subset B$ the equality $\sigma(A) = \sigma(B)$ holds. The following is proved in [114] (Theorem 2, p. 1043).

PROPOSITION 14.3. (Ward) *A continuum X is a smooth dendroid if and only if there exists a rigid selection for $C(X)$.*

However, examples are known in the literature of nonsmooth dendroids X which admit a (nonrigid) selection for $C(X)$ (see [99], Theorem 3, pp. 372–374; [16], Propositions 3 and 4, pp. 110–112). And furthermore, there are (uniformly arcwise connected) dendroids X which admit no continuous selection for $C(X)$ at all ([99], Theorem 2, p. 372). So the following two open problems seem to be of some importance for further study of selectible continua.

PROBLEMS 14.4. Give internal characterizations of selectible dendroids and of selectible fans.

Only some necessary *or* sufficient conditions of selectibility of dendroids are known till now in the literature. For necessary ones, besides Proposition 14.2, see [16], Theorem, p. 114. Along the same lines of ideas let us recall a result due to T. Maćkowiak [83] that is related to the bend intersection property (defined at the end part of the previous chapter). The following statement is a corollary ([83], p. 548) to a more general result (viz. Theorem on p. 547 of [83]).

PROPOSITION 14.5. (Maćkowiak) *If a dendroid X is selectible, then it has the bend intersection property.*

The converse implication is not true: an example is given in [83], p. 548. However, it is not a fan (it has two ramification points). So we have a question.

QUESTION 14.6. Does there exist a nonselectible fan with the bend intersection property?

Let us now come back to contractible spaces discussed in the previous chapter, and recall that every contractible curve is a uniformly arcwise connected dendroid ([12], Propositions 1 and 4, p. 72). This result corresponds to Proposition 14.2 for selectible continua. Similarly, it is easy to observe that a locally connected contractible curve is a dendrite — a result that corresponds to Proposition 14.1. So it is tempting — especially in view of many known examples — to conjecture that in the realm of dendroids the existence of a continuous selection for the hyperspace of subcontinua is related in some way to the property of being contractible. The reader is referred to [16] for some results on this topic. S. B. Nadler, Jr. asked (see [94]; compare also [97], Question (5.11), p. 259) whether contractibility of dendroids implies their selectibility. Answering this question in the negative, T. Maćkowiak has constructed a suitable example (see [85], Example, p. 321), which is, however, neither planable nor a fan (cf. [18]). So, we have the next two questions.

QUESTION 14.7. Is it true that if a planable dendroid is contractible, then it is selectible?

QUESTION 14.8. Does contractibility of fans imply their selectibility?

Remark 14.9. Recall that the converse implications do not hold: there exists a plane fan (viz. the fan pictured in Fig. 10) which is noncontractible (being not pairwise smooth, see [51], p. 78 and use Theorem 13.1), but selectible ([16], Proposition 4, p. 111).

As the reader certainly remembers from Theorem 13.1, if a fan contains a Q-point, then it is not contractible. One can ask whether a similar implication holds if contractibility is replaced by selectibility. However, this is not so, in view of the following example (see [19], Example 3.10).

EXAMPLE 14.10. *There exists a countable plane fan contains a Q-point, is not of type N, is not pairwise smooth, and is selectible.*

The following concept extends that of a dendroid of type N (see the previous chapter). A dendroid X is said to be *of type N generalized* provided all conditions of the definition of a dendroid of type N are satisfied with only one change: the limit arc $A = pq$ is replaced by an arbitrary nondegenerate continuum which contains p and q. The next result is quoted after [83], p. 548.

PROPOSITION 14.11. (Maćkowiak) *If a dendroid is of type N generalized (or is of type N, or contains a zig-zag), then it is nonselectible.*

Finally let us recall that the third property used in Theorem 13.1 to characterize contractible fans, namely pairwise smoothness, which is implied by contractibility of fans, is not implied by their selectibility, as has been indicated above in Remark 14.9.

Going now to mappings and their relations to selectibility, we formulate the main problem which is rather a research program.

PROBLEMS 14.12. Let $\mathcal{M}$ be a class of mappings and let $\mathcal{D}$ be a class of dendroids. For what classes $\mathcal{M}$ and $\mathcal{D}$ is it true that if a selectible (nonselectible) dendroid is in $\mathcal{D}$ and f is in $\mathcal{M}$, then $f(X)$ is selectible (nonselectible, respectively)?

In this direction we have the following results. There exist a selectible fan X and a monotone mapping f defined on X such that only one point-inverse is nondegenerate and $f(X)$ is nonselectible (see Example 13.3 above; for a proof see [83], Example 2, p. 549). Similarly, there exist a nonselectible fan Y and a monotone mapping g defined on Y such that only one point-inverse is nondegenerate and $g(Y)$ is locally connected, thus selectible. Namely such are the fan $Y = f(X)$ of the previous example and the mapping f shrinking the limit segment of Y to a point, see Example 13.15. Consequently we have the following result.

COROLLARY 14.13. *The image of a selectible (nonselectible) fan under a monotone mapping need not be selectible (nonselectible), even if all but one point-inverses are degenerate.*

As regards open mappings, there exist a selectible dendroid and an open mapping defined on it such that the image is a nonselectible dendroid ([83], Example 3, p. 549). The dendroid in question is not a fan, and a question is asked in [83], p. 550, whether an open image of a selectible fan is selectible. More generally, answers to the following questions are not known to the authors.

QUESTION 14.14. Is selectibility invariant under mappings of fans that are 1° light and open, 2° open, 3° light and confluent?

On the other hand, nonselectibility is not invariant under open mappings from a fan onto an arc, even if the mapping is a light retraction — see Examples 11.16 and 10.6, nor is it invariant under nonlight open mappings from a fan onto a simple triod — see Example 11.17. So we have the next corollary.

COROLLARY 14.15. *The image of a nonselectible fan under an open mapping need not be nonselectible.*

QUESTION 14.16. Do there exist a nonselectible fan and a light open mapping defined on it such that the image is a selectible fan?

The above question is a particular case of a more general one.

QUESTION 14.17. What kind of confluent mappings preserve selectibility (nonselectibility) of fans?

15. Final remarks

As the reader has observed, many concepts discussed in the present paper have been investigated for the first time (or even defined) in the first-named author's paper [9] (written more than twenty years ago), and many problems considered here have their sources in, or have been derived from, some investigations started in [9]. This is just the case with the topic of the present paper, i.e., with confluent mappings of fans: their study began in § 9 of [9].

However, the authors do not consider the present paper to be a continuation, or prolongation, of [9]. That paper concerns fans in general, while this one concerns their confluent mappings only. Nevertheless, the areas of both papers are very close to each other, so the reader can expect that the authors will give an information concerning open problems posed in [9] and solved in the meantime. This is just the aim of this chapter.

Recall that the following seven classes of fans have been considered in [9]: smooth (S), uniformly arcwise connected (UAC), geometrical (G), folding (F), embeddable into the Cantor fan (E), contractible by a retracting homotopy (CR) and confluent (C). They are all listed on p. 35 of [9], where in Table 1 implications are shown by continuous arrows, and open problems by dotted ones. All these problems have been solved, as follows.

Each smooth fan is folding ([45], Corollary 4, p. 90), so by Corollary 15 of [9], p. 28, the classes (S) and (F) coincide (this is just the content of Proposition 11.1 above). It has been shown in [26], Corollary, p. 93, that smoothness of dendroids is equivalent to their contractibility by a retracting homotopy. Thus (S) and (CR) coincide.

The above results imply that five among the seven considered classes of fans coincide: (S), (E), (G), (F) and (CR). Each of them implies (UAC). The converse implication is known to be false ([9], p. 14).

Concerning confluent fans (i.e. fans admitting a confluent mapping onto an arc, see [9], p. 35), it is known that neither (S) nor (UAC) implies (C), as has been mentioned on p. 35 of [9], using the fans F_{Sk} (see (6.1) of [9], p. 21) as an example. To see that (C) does not imply (UAC) (and, consequently, (S)), one can take a fan F_{Pk} of [7], (49), p. 201, shrink to a point a half of the limit segment containing the top, and observe that the fan obtained is still not uniformly arcwise connected, and it can easily be projected in a natural way onto its limit segment; the natural projection is even open. Consequently, (C) neither implies nor is implied by any other property of Table 1 of [9], p. 35. However, it is known that a fan is in (S)\(C) if and only if it is homeomorphic to the Lelek fan (see Theorem 12.14). So the final form of the discussed table is the following.

$$\begin{tikzcd}
(S) \arrow[r, shift right] \arrow[r, leftarrow, shift left] \arrow[d] & (F) \arrow[r, shift right] \arrow[r, leftarrow, shift left] & (E) \arrow[r, shift right] \arrow[r, leftarrow, shift left] & (CR) \\
(UAC) & (C) & &
\end{tikzcd}$$

Since (S) and (F) are equivalent, the dotted arrows in Table 2 of [9], p. 36, should be replaced by continuous ones.

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