SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 20
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1988

LIPSCHITZ STRATIFICATION OF REAL ANALYTIC SETS

ADAM PARUSIŃSKI

Institute of Mathematics, Gdańsk University,
Gdańsk, Poland

In this paper we shall prove the existence of a Lipschitz stratification for germs of real analytic sets. The concept of Lipschitz stratification was introduced in [1] by Mostowski, where he proved the existence in a complex case. Lipschitz stratifications are interesting, because they ensure a Lipschitz equisingularity along every stratum.

Our proof will be based on the result of Mostowski and a crucial lemma which estimates the distance to a complex analytic set by the distance to a real one.

Often we shall not distinguish between a set and its germ at the origin. The letter C will denote different constants.

1. Lipschitz stratifications

In this section we repeat without proofs the results of Mostowski [1]. By a stratification $\mathcal{S}$ of $(\mathbf{C}^n, 0)$ $((\mathbf{R}^n, 0)$ respectively) we shall mean a family of germs of analytic sets $X^j \subset \mathbf{C}^n$ (where $j = \dim X^j$) such that $X^j \subset X^{j+1}$ and $\mathring{X}^j = X^j \setminus X^{j-1}$ is smooth. Let $\varrho_j(q)$ denote $\operatorname{dist}(q, X^j)$.

Definition 1. A *chain* for a point $q \in \mathring{X}^j$ is a decreasing sequence of indices j_s and points $q_{j_s} \in \mathring{X}^{j_s}$ such that $j_1 = j$, $q_{j_1} = q$ and j_s is the greatest number for which

$$(1) \qquad \varrho_k(q) \geqslant 2\varrho_{j_s}(q) \quad \text{for all } k < j_s,$$

$$(2) \qquad |q - q_{j_s}| = \varrho_{j_s}(q).$$

It is easy to check for a chain the following inequalities:

$$(3) \qquad \tfrac{1}{2}\varrho_k(q) \leqslant \varrho_k(q_{j_s}) \leqslant \tfrac{3}{2}\varrho_k(q) \quad \text{for } k < j_s \text{ and all } s.$$

For $q \in \mathring{X}^j$ let P_q denote the orthogonal projection of C^n onto $T_q \mathring{X}^j$, $P_q^{\perp} = I - P_q$.

DEFINITION 2. A stratification $\mathscr{S}$ will be called a *Lipschitz stratification* if there exists a constant $C > 0$ such that for every $q \in \mathring{X}^j$, every chain $q_{j_1} = q, q_{j_2}, \ldots$ for q and every $q' \in \mathring{X}^j$ such that $|q - q'| \leqslant \frac{1}{2} \varrho_{j-1}(q)$ the following estimates hold for every k:

(L1k) $$|P_{q_{j_1}}^{\perp} P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant C |q_{j_1} - q_{j_2}| / \varrho_{j_k - 1}(q_{j_1}),$$

(L2k) $$|(P_q - P_{q'}) P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant C |q - q'| / \varrho_{j_k - 1}(q).$$

DEFINITION 3. We call a stratification $\mathscr{S}$ *compatible with an analytic set* X is a union of components of $\mathring{X}^j$s.

THEOREM 1 (Mostowski [1]). *For every finite family of germs of analytic subsets of C^n there is a Lipschitz stratification compatible with each of them.*

Theorem 1 has two interesting corollaries:

COROLLARY 1. *Let $\mathscr{S}$ be a Lipschitz stratification and v a Lipschitz vector field on X^j. Then there exists a Lipschitz vector field w on C^n such that $w | X^j = v$ and for every k and $q \in \mathring{X}^k$, $w(q) \in T_q \mathring{X}^k$.*

COROLLARY 2. *Let $X \subset C^n \times C^m$ be a germ of an analytic set at the origin and p: $C^n \times C^m \to C^m$ be the standard projection. There is a proper analytic subset T of C^m such that X is Lipschitz equisingular outside T* (i.e., *for every $t \in C^m \setminus T$ there exists an open neighbourhood U of t such that*

$$\big(p^{-1}(U),\, p^{-1}(U) \cap X,\, \{0\} \times U\big) \overset{\mathrm{Lip}}{\approx} \big(p^{-1}(t) \times U,\, (p^{-1}(t) \cap X) \times U,\, \{0\} \times U\big)$$

and this Lipschitz homeomorphism is compatible with the standard projections on U).

If we examine the proof of Theorem 1 more carefully we can obtain the following result:

Remark 1. Theorem 1 still holds if we require (L1k) for q_{j_1} arbitrary and $\{q_{j_2}, \ldots, q_{j_k}\}$ a chain and (L2k) for q' arbitrary and $\{q_{j_1}, \ldots, q_{j_k}\}$ a chain. Moreover, we can change the conditions in the definition of a chain as follows:

(i) j_2 is the greatest number for which
$$\varrho_k(q) \geqslant 2\varrho_{j_2}(q) \quad \text{for all } k < j_2;$$

(ii) $|q - q_{j_2}| = \varrho_{j_2}(q)$;

(iii) $\{q_{j_2}, \ldots, q_{j_k}\}$ is a chain (inductive definition), so each part of a chain is also a chain. For such chains we have also estimates similar to (3).

Remark 2. For a chain $\{q_{j_2}, \ldots, q_{j_k}\}$ and q_{j_1} arbitrary ($j_1 > j_2$) in (L1k) we obtain equivalent inequalities for the following right sides (constants may be different):

(A) $C|q_{j_1}-q_{j_2}|/\varrho_{j_k-1}(q_{j_1})$;

(B) $C|q_{j_1}-q_{j_2}|/\varrho_{j_k-1}(q_{j_2})$;

(C) $C|q_{j_1}-q_{j_2}|/\operatorname{dist}(\{q_{j_1}, q_{j_2}\}, X^{j_k-1})$;

(D) $C|q_{j_1}-q_{j_2}|/\varrho_{j_k-1}(q_{j_k})$.

A similar result holds for (L2k).

Proof. We prove the equivalence for (A) and (C). It is obvious that if the inequality holds for (A) then it holds for (C). We assume that the inequality holds for (C). We consider two cases:

Case 1. $4|q_{j_1}-q_{j_2}| \geqslant \varrho_{j_k-1}(q_{j_1})$, then the inequality holds for (A) trivially, because the left side of (L1k) is bounded by 1.

Case 2. $4|q_{j_1}-q_{j_2}| \leqslant \varrho_{j_k-1}(q_{j_1})$, then

$$(\operatorname{dist}(\{q_{j_1}, q_{j_2}\}, X^{j_k-1}))^{-1} \leqslant 2(\varrho_{j_k-1}(q_{j_1}))^{-1/2},$$

thus the inequality holds for (A).

The rest of the proof is left to the reader. □

We call a complex analytic subset of $\boldsymbol{C}^n$ "real" if it is a complexification of a real analytic subset of $\boldsymbol{R}^n$.

Remark 3. If a family of sets from the statement of Theorem 1 is "real" then a Lipschitz stratification can be chosen "real".

Remark 4. In [1] Mostowski approximates $\operatorname{dist}(\cdot, X^j)$ by some semi-analytic function. This is important for the proof but not for the result.

2. Estimates of distance

We want to prove the existence of a Lipschitz stratification for real analytic sets by complexification. First we must see how the sides of (L1k), (L2k) change after complexification.

If $Y \subset \boldsymbol{R}^n$ is an analytic subset of $\boldsymbol{R}^n$ then we denote by $\tilde{Y}$ its complexification. In general, the sign $\tilde{\ }$ will denote passing from the real to the complex case. For example, if $q \in Y \setminus \operatorname{Sing} Y$ then $\tilde{P}_q$ will denote the orthogonal projection of $\boldsymbol{C}^n$ onto $T_q\tilde{Y}$ and P_q the orthogonal projection of $\boldsymbol{R}^n$ onto $T_q Y$.

It is easy to see that if $\mathscr{S} = \{X^j\}_{0 \leqslant j \leqslant n}$ is a stratification of $(\boldsymbol{R}^n, 0)$, $\{q_{j_s}\}$ some sequence of points ($q_{j_s} \in \mathring{X}^{j_s}$) and $\tilde{\mathscr{S}} = \{\tilde{X}^j\}_{0 \leqslant j \leqslant n}$ a stratification of

$(C^n, 0)$ formed by the complexifications of X^j then

$$(4)\qquad |\tilde{P}^{\perp}_{q_{j_1}} \tilde{P}_{q_{j_2}} \ldots \tilde{P}_{q_{j_k}}| = |P^{\perp}_{q_{j_1}} P_{q_{j_2}} \ldots P_{q_{j_k}}|$$

and a similar equality holds for the left side of (L2k).

So if we want to construct a Lipschitz stratification in real case we must estimate $\operatorname{dist}(q, \tilde{X})$ ($q \in \boldsymbol{R}^n$, $X \subset \boldsymbol{C}^n$) by a distance to some real analytic set.

THEOREM 2. *Let $\tilde{X}$ be a germ of a k-dimensional analytic subset of $\boldsymbol{C}^n$ and $X = \tilde{X} \cap \boldsymbol{R}^n$. Let $\tilde{Y}$ be a germ of a complex analytic subset of $\tilde{X}$ and $\dim_{\boldsymbol{C}} \tilde{Y} \leqslant k-1$. Then there exists a germ Y of an analytic subset of X and a constant C such that*

(i) $\dim_{\boldsymbol{R}} Y \leqslant k-1$;

(ii) $\operatorname{dist}(q, Y) \leqslant C \operatorname{dist}(q, \tilde{Y})$ *for* $q \in X$.

THEOREM 3. *Let $\tilde{Y}$ be a germ of a complex analytic subset of $\boldsymbol{C}^n$. Then there is Y a germ of an analytic subset of $\boldsymbol{R}^n$ such that*

(i) $\dim_{\boldsymbol{R}} Y \leqslant \dim_{\boldsymbol{C}} \tilde{Y}$;

(ii) $\operatorname{dist}(q, Y) \leqslant C \operatorname{dist}(q, \tilde{Y})$ *for* $q \in \boldsymbol{R}^n$.

The distance from a real point to a complex analytic set is illustrated in Figure 1. The situation of Figure 2 cannot happen.

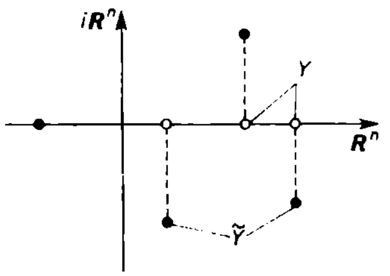

Fig. 1

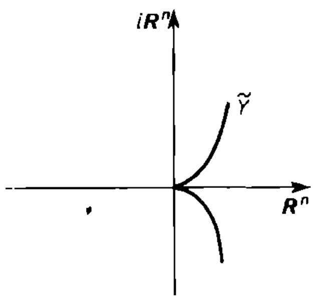

Fig. 2

The proof of Theorem 2 is based on the concept of regular projections introduced by Mostowski ([1]).

3. Regular projections

DEFINITION 4. Let $X \subset \boldsymbol{C}^n$ be a hypersurface with reduced equation $F = 0$. A *projection* $\pi = \pi_\xi$: $\boldsymbol{C}^n \to \boldsymbol{C}^{n-1}$ parallel to a vector $(\xi, 1)$ ($\xi \in \boldsymbol{C}^{n-1}$) will be *called* (C, ε)-regular with respect to X at $q \in \boldsymbol{C}^n$ if

(i) $\pi | X$ is finite;

(ii) there exist analytic functions $\lambda_j(\eta)$ defined for $|\eta - \xi| < \varepsilon$ such that $F(q + \lambda_j(\eta)(\eta, 1)) = 0$, every solution of $F(q + \lambda(\eta, 1)) = 0$ is of the form $\lambda = \lambda_j(\eta)$ and $\lambda_j(\eta) \neq \lambda_k(\eta)$ for all η and $j \neq k$, $|D\lambda_j| \leqslant C|\lambda_j|$.

Similarlly π will be called *regular* with respect to X along a curve q_t if for some C, ε it is (C, ε) regular with respect to X at every point of q_t (t small, $t \neq 0$).

Choose the x_n-axis so that F doesn't vanish on it. Let the germ at 0 $(x, \xi, \lambda) \rightsquigarrow F(x+\lambda(\xi, 1))$ be equivalent to a distinguished polynomial W with respect to λ. Let W be defined for $x \in K(a) = \{x \in C^n; |x| < a\}$, $\xi \in B(a) = \{\eta \in C^{n-1}; |\eta| < a\}$. The following proposition was proved in Mostowski [1].

PROPOSITION 1. *There is a finite subset of $B(a)$, $\xi_1, \ldots, \xi_N$, such that for every complex analytic curve q_t in a small neighbourhood of* 0 *there is a j such that π_{ξ_j} is regular along q_t. Furthermore, we can choose $\xi_1, \ldots, \xi_N$ real.*

Let $X \subset C^n$, $B(a)$ be as above, $\xi \in B(a)$. Let $\operatorname{dist}_\xi(x, X)$ denote $\operatorname{dist}\left(x, X \cap \pi_\xi^{-1}(\pi_\xi(x))\right)$ (the distance to X in $(\xi, 1)$ direction).

PROPOSITION 2. *Let X, $\xi_1, \ldots, \xi_N$ be as in Proposition* 1. *Then there exists a constant $C > 0$ such that*

$$\min_j \{\operatorname{dist}_{\xi_j}(x, X)\} \leqslant C \operatorname{dist}(x, X) \quad \textit{for } x \textit{ close to } 0.$$

Proof. Suppose the proposition were false. Then by the curve selection lemma (see Łojasiewicz [3]) we could find a real analytic curve q_t such that:

$$\min_j \{\operatorname{dist}_{\xi_j}(q(t), X)\}/\operatorname{dist}(q(t), X) \to \infty, \quad t \to 0, \tag{5}$$

and $q(t) \in X$ for $t \neq 0$. (We must approximate the distance function by some semi-analytic function φ satisfying

$$\tfrac{1}{2}\operatorname{dist}(x, X) \leqslant \varphi(x) \leqslant 2\operatorname{dist}(x, X).$$

The existence of a such function is clear from Łojasiewicz [3]).

We can extend this curve to a complex analytic curve which we also denote by q_t. By Proposition 1 we can choose ξ_j such that π_{ξ_j} is regular along q_t. Let us assume for simplicity that $\pi = \pi_{\xi_j}\colon C^n \to C^{n-1}$ is the standard projection. Let C_q be a cone $\{q+\lambda(\xi, 1);\ \lambda \in C,\ |\xi| < \varepsilon/2\}$ and $q = q_t$ for some $t = 0$. Then it is easy to see from the implicit function theorem that $C_q \cap X$ is the disjoint sum of the graphs of analytic functions $\varphi_j\colon K\left(\pi(q), |\lambda_j(0)|\,\delta\right) \to C$ (where the constant $\delta > 0$ depends only on C, ε, n) (see for details Mostowski [1]). Hence $\pi^{-1}\left(K\left(\pi(q), \min_i |\lambda_i(0)|\,\delta\right)\right) \setminus C_q$ cannot contain any point of X, so $\operatorname{dist}_{\xi_j}(q_t, X) \leqslant C \operatorname{dist}(q_t, X)$ for some $C > 0$ and every $t \neq 0$, but this contradicts (5). □

4. Proof of Theorem 2

We first prove the theorem for $X = \mathbf{R}^n$ ($\tilde{X} = \mathbf{C}^n$). We can assume that $\tilde{Y}$ is a hypersurface in $\mathbf{C}^n$. Let the vectors $\xi_1, \ldots, \xi_N \in \mathbf{R}^{n-1}$ be chosen for $\tilde{Y}$ as in the statement of Proposition 2. Let $\pi = \pi_{\xi_j}$ and assume for simplicity that π is the standard projection.

For the product $\mathbf{C}^n = \mathbf{C}^{n-1} \times \mathbf{C}$ let $\tilde{Y}_j = (\mathbf{R}^{n-1} \times \mathbf{C}) \cap \tilde{Y}$ and let $p_j\colon \mathbf{R}^{n-1} \times \mathbf{C} \to \mathbf{R}^{n-1} \times \mathbf{R}$ be the standard projection $(y, z) \leadsto (y, \mathrm{Re}(z))$. $p_j(\tilde{Y}_j)$ is a semi-analytic set (see Łojasiewicz [3]) and let Y_j be the smallest analytic set containing $p_j(\tilde{Y}_j)$ (then $\dim Y_j \leqslant k-1$).

Let us define $Y = \bigcup_j Y_j$. We see that for $x \in \mathbf{R}^n$

$$\operatorname{dist}(x, X) = \min_j \operatorname{dist}(x, Y_j) \overset{(i)}{\leqslant} \min_j \operatorname{dist}_{\xi_j}(x, \tilde{Y}_j)$$

$$= \min_j \operatorname{dist}_{\xi_j}(x, \tilde{Y}) \overset{(ii)}{\leqslant} C \operatorname{dist}(x, \tilde{Y}),$$

where (i) follows by construction of Y_j and (ii) by Proposition 2. This proves the theorem for $X = \mathbf{R}^n$.

Now we assume that $\dim X = k < n$ (if $\dim X < k$ the theorem is obvious). Let $\tilde{X} = \tilde{X}_n \subset \tilde{Z}^{n-1}$, where $\tilde{Z}^{n-1}$ is a hypersurface in $\mathbf{C}^n$, and let $\xi_1^{n-1}, \ldots, \xi_{N_{n-1}}^{n-1}$ be a family of real vectors as in the statement of Proposition 1. We denote by $\tilde{X}_{n-1}$ the set $\bigcup_j \pi_{\xi_j}(\tilde{X}_n)$ and let $\tilde{Z}^{n-2}$ be a hypersurface in $\mathbf{C}^{n-1}$ containing $\tilde{X}_{n-1}$. We can choose for $\tilde{Z}^{n-2}$ vectors $\xi_1^{n-2}, \ldots, \xi_{N_{n-2}}^{n-2}$ and so on. In this way we construct a sequence of sets $\tilde{X}_n, \ldots, \tilde{X}_{k+1}$, $\tilde{X}_k = \mathbf{C}^k$ each of (complex) dimension k and hypersurfaces $\tilde{Z}^{n-1}, \ldots, \tilde{Z}^k$ such that $\tilde{X}_j \subset \tilde{Z}^{j-1} \subset \mathbf{C}^j$. In the same way we define $\tilde{Y}_{n-1}, \ldots, \tilde{Y}_k$ ($\tilde{Y}_s = \bigcup_j \pi_{\xi_j}(\tilde{Y}_{s+1})$). Using the special case considered above, we can find an analytic subset Y_k' of $\mathbf{R}^k$ ($\dim Y_k' \leqslant k-1$) such that

$$\operatorname{dist}(x, Y_k') \leqslant C \operatorname{dist}(x, \tilde{Y}_k) \tag{6}$$

for $x \in \mathbf{R}^k$ and some $C > 0$.

Now we pass to the real domain.

We inductively define $X_n = X$, $X_{n-1} = \bigcup_j \pi_{\xi_j}(X_n)$, and so on. X_j is a k-dimensional semi-analytic subset of $\mathbf{R}^j$. We also define

$$T_{n-1} = \bigcup_j \{\text{critical values of } \pi_{\xi_j}|X_n\},$$

$$T_{n-2} = \bigcup_j \{(\text{critical values of } \pi_{\xi_j}|X_{n-1}) \cup \pi_{\xi_j}(T_{n-1})\},$$

and so on. Then $T = T_k \cup \operatorname{Sing}(X_k)$ is a semi-analytic subset of X_k and

$\dim T \leqslant k-1$. Let Y_k be the smallest analytic subset which contains $Y_k' \cup T$ ($\dim Y_k \leqslant k-1$), so for $x \in X_k$

$$\operatorname{dist}(x, X_k \cap Y_k) \leqslant \operatorname{dist}(x, Y_k') \leqslant C \operatorname{dist}(x, \tilde{Y}_k). \tag{7}$$

We define $Y_{k+1} = \bigcup_j \pi_{\xi_j}^{-1}(Y_k)$, and so on. We shall prove that for $x \in X$

$$\operatorname{dist}(x, Y_n \cap X) \leqslant C \operatorname{dist}(x, \tilde{Y}_n). \tag{8}$$

Suppose that (8) is false. Then by the curve selection lemma there is a real analytic curve q_t such that

$$\operatorname{dist}(q_t, \tilde{Y})/\operatorname{dist}(q_t, Y_n \cap X) \to 0, \qquad t \to 0, \tag{9}$$

and $q_t \notin Y_n \cap X$ for $t \neq 0$. We can extend q_t to a complex analytic curve and use Proposition 1. We choose a sequence of regular projections $\pi_{\xi^{n-1}}$ for q_t, $\pi_{\xi^{n-2}}$ for $\pi_{\xi^{n-1}}(q_t)$, and so on. We assume, for simplicity of notation, that these projections are standard, $\pi^{j-1} = \pi_{\xi^{j-1}}\colon C^j \to C^{j-1}$.

We fix $q = q_t$ for some small $t \neq 0$. Let us denote $\pi^j \circ \ldots \circ \pi^{n-1}(q)$ by q_j. Let y_k be the nearest point of $Y_k \cap X_k$ to q_k and l the interval joining these points. $l \setminus \{y_k\} \subset X_k \setminus T_k$, so we can lift l to l_{k+1} a curve in X_{k+1} joining q_{k+1} and some point y_{k+1} of $Y_{k+1} \cap X_{k+1}$. From the regularity of π^k at q_{k+1} we have

$$|y_{k+1} - q_{k+1}| \leqslant C |y_k - q_k|$$

(since y_{k+1} is outside a cone $\{q_{k+1} + \lambda(\eta, 1);\ \lambda \in C,\ |\eta| \leqslant \varepsilon/2\}$). Since $l_{k+1} \setminus \{y_{k+1}\} \subset X_{k+1} \setminus T_{k+1}$ we can continue a lifting and finally obtain

$$\operatorname{dist}(q, Y_n \cap X) \leqslant C \operatorname{dist}(q_k, Y_k \cap X_k) \tag{10}$$

so by (7)

$$\operatorname{dist}(q, Y_n \cap X) \leqslant C \operatorname{dist}(q_k, Y_k \cap X_k) \overset{(7)}{\leqslant} C' \operatorname{dist}(q_k, \tilde{Y}_k) \leqslant C'' \operatorname{dist}(q, \tilde{Y})$$

for $q = q_t$ ($t \neq 0$ arbitrary) and C'' doesn't depend on t, in contradiction to (9). □

5. Proof of Theorem 3

The proof is by induction on $n-j$ ($j = \dim_C \tilde{Y}$). For $n-j = 1$ the theorem follows from Theorem 2. Let $n-j > 1$. We can find an analytic set $\tilde{Z}$ such that $\tilde{Y} \subset \tilde{Z}$ and $\dim \tilde{Z} = \dim \tilde{Y} + 1$. Then by the inductive assumption there is an analytic subset T of $\boldsymbol{R}^n$ satisfying

$$\dim T \leqslant j+1,$$

$$\operatorname{dist}(q, T) \leqslant C \operatorname{dist}(q, \tilde{Z}) \qquad \text{for } q \in \boldsymbol{R}^n.$$

Let $\tilde{T}$ be the complexification of T. Using Theorem 2 for $\tilde{X} = \tilde{Z} \cup \tilde{T}$ and $\tilde{Y}$ we find $Y \subset \boldsymbol{R}^n$ such that

$$\dim Y \leqslant j,$$
$$\operatorname{dist}(x, Y) \leqslant \tilde{C} \operatorname{dist}(x, \tilde{Y}) \quad \text{for } x \in T \subset \tilde{X} \cap \boldsymbol{R}^n.$$

We shall prove that Y satisfies the statement of the theorem.

Let $q \in \boldsymbol{R}^n$, $\operatorname{dist}(q, T) = |q - x|$ for some $x \in T$. We consider two cases.

Case 1. $\operatorname{dist}(q, Y) \leqslant 2(1+\tilde{C})|x-q|$ then

$$\operatorname{dist}(q, Y) \leqslant 2(1+\tilde{C}) \operatorname{dist}(q, T) \leqslant \hat{C} \operatorname{dist}(q, \tilde{Z}) \leqslant \hat{C} \operatorname{dist}(q, \tilde{Y}).$$

Case 2. $\operatorname{dist}(q, Y) \geqslant 2(1+\tilde{C})|x-q|$ then

$$\operatorname{dist}(q, Y) \leqslant \operatorname{dist}(x, Y) + |x-q| \leqslant \tilde{C} \operatorname{dist}(x, \tilde{Y}) + |x-q|$$
$$\leqslant \tilde{C} \operatorname{dist}(q, \tilde{Y}) + (1+\tilde{C})|x-q| \leqslant \tilde{C} \operatorname{dist}(q, \tilde{Y}) + \tfrac{1}{2} \operatorname{dist}(q, Y),$$

hence $\operatorname{dist}(q, Y) \leqslant 2\tilde{C} \operatorname{dist}(q, \tilde{Y})$. □

6. Construction of Lipschitz stratifications

THEOREM 4. *For every finite family of germs at the origin of analytic subsets of $\boldsymbol{R}^n$ there is a Lipschitz stratification compatible with each of them.*

Proof. We shall give the proof only for one germ, the proof in the general case is similar.

Let X be an analytic subset of some open neighbourhood of 0 in $\boldsymbol{R}^n$. Let $\dim X = k$. We define a Lipschitz stratification of X inductively. Suppose that a stratification $X^n = \boldsymbol{R}^n, X^{n-1}, \ldots, X^{k+1}, X^k, X^{k-1}, \ldots$ is chosen in such a way that

(11) (L1k), (L2k) hold for every chain $\{q_{j_s} \in \mathring{X}^{j_s}\}$ $(j_s > j)$.

Let $\tilde{X}^n, \ldots, \tilde{X}^j$ be the complexifications of $X^n, \ldots, X^j$. Then by Theorem 1 we can construct a complex Lipschitz stratification $\hat{\mathscr{S}}$ compatible with $\tilde{X}^n, \ldots, \tilde{X}^j$. By Theorem 3 we can find an analytic subset X'^{j-1} of $\boldsymbol{R}^n$ such that for $x \in \boldsymbol{R}^n$ we have

$$\operatorname{dist}(x, X'^{j-1}) \leqslant C \operatorname{dist}(x, \hat{X}^{j-1}) \tag{12}$$

and $\dim X'^{j-1} \leqslant j-1$, $\hat{X}^{j-1} \in \hat{\mathscr{S}}$.

Now we must change $X^n, \ldots, X^j$ a little to obtain $X^j \supset X'^{j-1}$. We can do it without affecting the property (11) and the compability of the stratification with X.

In fact, we can define $Y^s = X^s \cup X'^{j-1}$ for $s \geqslant j-1$. If $\{q_{j_s} \in \mathring{Y}^{j_s}\}$ $(j_s > j)$ is a chain then it is a chain in the previous stratification and the estimates (L1k), (L2k) still hold. Changing strata of dimension less then $j-1$ we preserve compability. This new stratification will be denoted by $\{X^s\}_{0\le s\le n}$.

Now, we shall prove that this stratification satisfies (11) with $j-1$ instead of j. We prove only (L1k) (the proof of (L2k) is similar).

Let $\{q_{j_s}\in \mathring{X}^{j_s}\}$ $(j_s \geqslant j,\ s=1, 2, \dots, t,\ j_t = j)$ be a chain. After an arbitrary small perturbation we can assume that for each s, $q_{j_s}\in \hat{X}^{j_s}\setminus \hat{X}^{j_s-1}$. Let $\{\tilde{q}_{l_r}\}$ be a complex chain with respect to $\mathscr{S}' = \hat{X}^n, \hat{X}^{n-1}, \dots, \hat{X}^0$ for $\tilde{q}_{l_1} = q_{j_2}$ (in the strong sense of Remark 1).

Let $l_v \geqslant j > l_{v+1}$. We shall use the following notation:

$$\hat{P}_i = \begin{cases} \tilde{P}_{\tilde{q}_{l_r}} & \text{if } i = l_r \text{ for some } r \text{ or } i = j_1, \\ \mathrm{Id} & \text{otherwise;} \end{cases}$$

$$\tilde{P}_i = \begin{cases} \tilde{P}_{q_{j_s}} & \text{if } i = j_s \text{ for some } s, \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

Then we have

$$(13)\qquad |P^{\perp}_{q_{j_1}} P_{q_{j_2}} \dots P_{q_{j_t}}| \overset{(4)}{=} |\tilde{P}^{\perp}_{q_{j_1}} \tilde{P}_{q_{j_2}} \dots \tilde{P}_{q_{j_t}}| = |\tilde{P}^{\perp}_{j_1} \tilde{P}_{j_1-1} \dots \tilde{P}_j|$$

$$\leqslant |\hat{P}^{\perp}_{j_1} \hat{P}_{j_1-1} \dots \hat{P}_j| + \sum_{j\leqslant w\leqslant j_2-1} \underbrace{|P_{j_1} P_{j_1-1} \dots P_{w+1}|\,|(\hat{P}_w - \tilde{P}_w)\hat{P}_{w-1} \dots \hat{P}_j|}_{A_w}.$$

Now we consider three cases:

Case 1. $w = j_s = l_r$ (for some s, r); then by Remark 1, 2 and the properties of chains (the letter C denotes different constants):

$$(14)\qquad |(\hat{P}_w - \tilde{P}_w)\hat{P}_{w-1} \dots \hat{P}_j| = |(\tilde{P}^{\perp}_{q_{j_s}} - \tilde{P}_{\tilde{q}_{l_r}})\tilde{P}_{\tilde{q}_{l_{r+1}}} \dots \tilde{P}_{\tilde{q}_{l_v}}|$$

$$\leqslant \frac{C|q_{j_s} - \tilde{q}_{l_r}|}{\operatorname{dist}(\tilde{q}_{l_r}, \hat{X}^{l_v-1})} \leqslant \frac{C|q_{j_s} - \tilde{q}_{l_r}|}{\operatorname{dist}(\tilde{q}_{l_1}, \hat{X}^{j-1})}$$

so

$$(15)\qquad A_w \leqslant \frac{C|q_{j_1} - q_{j_2}|\,|q_{j_s} - q_{l_r}|}{\operatorname{dist}(q_{j_2}, X^{j_{s-1}-1})\operatorname{dist}(\tilde{q}_{l_1}, \hat{X}^{j-1})}$$

$$\overset{(12)}{\leqslant} \frac{C|q_{j_1} - q_{j_2}|\,|q_{j_s} - \tilde{q}_{l_r}|}{\operatorname{dist}(q_{j_2}, X^{j_s})\operatorname{dist}(q_{j_2}, X^{j-1})} \overset{(\#)}{\leqslant} \frac{C|q_{j_1} - q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j-1})},$$

$$(\#)\qquad |q_{j_s} - \tilde{q}_{l_r}| \leqslant C\operatorname{dist}(q_{j_2}, \hat{X}^{l_r}) + C\operatorname{dist}(q_{j_2}, X^{j_s}) \leqslant C\operatorname{dist}(q_{j_2}, X^{j_s}).$$

Case 2. $l_{r-1} > w = j_s > l_r \geqslant l_v$ (for some s, r)

$$(16)\qquad |(\hat{P}_w - \tilde{P}_w)\hat{P}_{w-1} \dots \hat{P}_j| = |\tilde{P}^{\perp}_{q_{j_s}} \tilde{P}_{\tilde{q}_{l_r}} \dots \tilde{P}_{\tilde{q}_{l_v}}|$$

$$\leqslant \frac{C|q_{j_s} - \tilde{q}_{l_r}|}{\operatorname{dist}(\tilde{q}_{l_r}, \hat{X}^{l_v-1})} \leqslant \frac{C|q_{j_s} - \tilde{q}_{l_r}|}{\operatorname{dist}(\tilde{q}_{l_1}, \hat{X}^{j-1})}$$

so

$$A_w \leqslant \frac{C|q_{j_1}-q_{j_2}|\,|q_{j_s}-q_{l_r}|}{\operatorname{dist}(q_{j_2}, X^{j_{s-1}-1})\operatorname{dist}(\tilde{q}_{l_1}, \hat{X}^{j-1})} \leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j-1})} \tag{17}$$

because

$$\begin{aligned}|q_{j_s}-q_{l_r}| &\leqslant C\operatorname{dist}(q_{j_2}, X^{j_s})+C\operatorname{dist}(q_{j_2}, \hat{X}^{l_r})\\ &\leqslant C\operatorname{dist}(q_{j_2}, X^{j_s})+C\operatorname{dist}(q_{j_2}, \hat{X}^{j_s}) \leqslant C\operatorname{dist}(q_{j_2}, X^{j_{s-1}-1}).\end{aligned}$$

If $w=j_s<l_v$ then

$$\begin{aligned}A_w &\leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j_{s-1}-1})} \leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j_s})}\\ &\leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, \hat{X}^{j_s})} \leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, \hat{X}^{j-1})}\\ &\leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j-1})}.\end{aligned} \tag{18}$$

Case 3. $j_s<w=l_r<j_{s-1}$ (for some s, r). In this case the proof of the estimate

$$A_w \leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j-1})} \tag{19}$$

is similar to the proof of (15), (17), (18) and is left to the reader.

Since

$$\begin{aligned}|\hat{P}_{j_1}^{\perp}\hat{P}_{j_1-1}\ldots\hat{P}_j| = |\tilde{P}_{q_{j_1}}^{\perp}\tilde{P}_{\tilde{q}_{l_1}}\ldots\tilde{P}_{\tilde{q}_{l_v}}| &\leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{l_v-1})}\\ &\leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, \hat{X}^{j-1})} \leqslant \frac{C|q_{j_1}-q_{j_2}|}{\operatorname{dist}(q_{j_2}, X^{j-1})}\end{aligned}$$

we have (L1k) from (13), (19).

In this way we can construct inductively a stratification which satisfies (11) for $j=-1$, in other words a Lipschitz stratification.

7. Some remarks

The same proof works for germs of closed semi-analytic sets, but it fails in the sub-analytic case. So the problem of the existence of a Lipschitz stratification for sub-analytic sets is still open. The second problem is to construct a global Lipschitz stratification, for compact sets for example. It seems to be difficult because the Lipschitz stratification constructed above is not "canonical". For example, if $X = \{(x, y, z) \in \mathbf{R}^3;\ y^2+z^2 = x^3\}$ then $X^2 = X$, $X^1 = X^0 = \{0\}$ is not a Lipschitz stratification. To obtain a Lipschitz stratification we must choose for X^1 any curve satisfying $C_0(X^1) \supset x$-axis

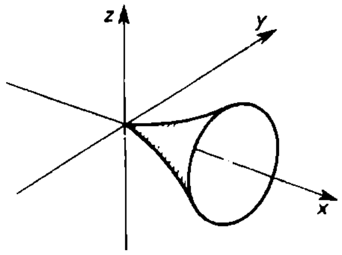

Fig. 3

($C_0(X^1)$ denotes the tangent cone to X^1 at 0). (The best general reference to this problem is Mostowski [2]). The methods of Mostowski [1] and the above enable us to construct a global Lipschitz stratification in the algebraic affine case.

I would like to thank Tadeusz Mostowski for calling my interest to the problem and many valuable suggestions.

References

[1] T. Mostowski, *Lipschitz equisingularity*, Dissertationes Math. (Rozprawy Mat.) 243 (1985).
[2] —, *Tangent cones and Lipschitz stratifications*, this volume, 303–322.
[3] S. Łojasiewicz, *Ensembles semi-analytiques*, Institute des Hautes Études Scientifiques, Bûres-sur-Yvette 1965.

Presented to the semester
Singularities
15 February–15 June, 1985