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"BEST POSSIBLE" MAXIMUM PRINCIPLES

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Maximum principles for solutions of second order elliptic equations have been used in the mathematical literature for close to a century and a half. They have been used to study questions of existence, uniqueness, regularity, stability and error bounds. The book of Protter and Weinberger [16] gives a good survey of the vast literature on the subject up to 1967. This paper will deal with work done since that time.

The idea a "best possible" maximum principle is as follows. We are given a nonlinear elliptic partial differential equation of the form

$$F(x, u, Du, D^2 u) = 0 \quad \text{in} \quad \Omega \subset R^N \tag{0.1}$$

and are interested in solution of this equation which satisfies

$$B(x, u, Du) = 0 \quad \text{on} \quad \partial\Omega. \tag{0.2}$$

We say that a function $\phi(u, q^2)$ of u and $q^2 \equiv |\text{grad}\, u|$ satisfies a "best possible" maximum principle if the following two properties hold:

1) $\phi(u, q^2)$ satisfies a maximum principle for any Ω.

2) Exist some domain $\tilde{\Omega}$ (or a limiting domain) such that $\phi \equiv$ constant in $\tilde{\Omega}$. The condition 2 gives rise to the term "best possible".

Let me first indicate why such "best possible" inequalities might be useful. Many interesting problems in mathematics, physics, and engineering are modeled by nonlinear second order elliptic boundary value problems. For many such problems a bound for the solution is not nearly as important as a bound for the maximum of the absolute value of the gradient which in different contexts represents maximum stress, maximum velocity, maximum heat flux, etc. Our maximum principles will not only lead to bounds for the

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maximum value of $|\text{grad } u|$ but will also frequently yield some information about where this maximum value occurs [12]. Furthermore, the bounds will be sharp in the sense that the equality sign will hold for some $\tilde{\Omega}$.. Since the inequalities are sharp in this sense, then they may be integrated over Ω or subdomains of Ω and isoperimetric inequalites (in the sense of Polya and Szegö) will result. They also lead to isoperimetric comparison theorems which relate the solutions of nonlinear problems to those of associated linear or one dimensional problems (see e.g. Payne [9], [10]).

"Best possible" maximum principles have been used in solid and fluid mechanics [11], [12], [13], reactor theory [15], [14], plasma physics [8], [18], diffusion-reaction problems (see e.g. Sperb [17]), in geometry [13] and in other areas.

1. Some recent results

A "best possible" maximum principle for the equation

$$\begin{aligned} &\Delta\psi + 1 = 0 \quad \text{in} \quad \Omega \subset R^2 \\ &\psi = 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.1}$$

has been known for some time. Miranda [7] has shown that the quantity

$$\phi \equiv |\text{grad } \psi|^2 + \psi \tag{1.2}$$

takes its maximum value on $\partial\Omega$ and that ϕ is a constant throughout Ω, if Ω is the interior a circle. The analogue of this statement in higher dimensions is also well known.

Results of this type depend, of course, on the two Hopf maximum principles [4], [5]. We do not require the most general forms of these principles but state them for solutions of the inequality

$$Lu \equiv \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} \geqslant 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

where a_{ij} and b_i are assumed to be continuous in $\bar{\Omega}$, and the boundary $\partial\Omega$ is assumed to be a $C^{2+\alpha}$ surface for some $\alpha > 0$. Furthermore, it is assumed that the matrix with components $a_{ij}(x)$ is positive definite. Under these conditions the two Hopf principles may be stated as

H.1. If u assumes its maximum value M in Ω then $u \equiv M$ throughout $\bar{\Omega}$;

H.2 If u assumes its maximum value M at a point P_1 on $\partial\Omega$ then either $u \equiv M$ in Ω or $\frac{\partial u}{\partial n}(P_1) > 0$.

Here $\frac{\partial}{\partial n}$ denotes the normal derivate on $\partial\Omega$ directed outward from Ω.

We describe first some recent results of Payne and Philippin [14], which apply to classical solutions of the nonlinear problem

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[g(u, q^2)\frac{\partial u}{\partial x_i}\right]+h(u, q^2) = 0 \qquad \text{in} \quad \Omega \subset R^N,$$

(1.4)

$$u = 0 \qquad \text{on} \quad \partial\Omega.$$

We assume throughout that g is a C^1 function of its arguments and that h is a continuous function. These conditions can be relaxed, but for simplicity we make these somewhat restrictive hypotheses.

We further assume that

$$G(u, q^2) \equiv g(u, q^2)+2q^2\frac{\partial g(u, q^2)}{\partial q^2} > 0 \tag{1.5}$$

on solutions. Condition (1.5) guarantees that the equation is strongly elliptic. Our aim is to find a function $\Phi(u, q^2)$ which will satisfy a "best possible" maximum principle, and as a guide in making a suitable choice we consider the analogous one dimensional problem which can be reduced to a first order partial differential equation. This lead to the following candidate for Φ, i.e., a function which satisfies the first order partial differential equation

$$2H(u, q^2)\frac{\partial \Phi}{\partial q^2} = G(u, q^2)\frac{\partial \Phi}{\partial u} \tag{1.6}$$

where

$$H(u, q^2) = h(u, q^2)+q^2\frac{\partial g}{\partial u}, \tag{1.7}$$

together with the requirement

$$\frac{\partial \Phi}{\partial q^2} > 0 \tag{1.8}$$

on solutions. This latter condition imposes some further restriction on the function h, but in many interesting examples it can easily be satisfied. With this condition we might expect to be able to obtain a solution Φ in the form

$$\Phi = q^2-F(u). \tag{1.9}$$

We now indicate a proof of the following two theorems:

THEOREM I. *Let Φ be a solution of* (1.6), (1.8) *where u is a classical solution of* (1.4), *then Φ takes its maximum value either on $\partial\Omega$ or at a point in Ω at which* $|\text{grad } u| = 0$.

THEOREM II. *If the average curvature K is nonnegative at every point of $\partial\Omega$ then Φ cannot assume its maximum value on $\partial\Omega$.*

The proof of Theorem I involves showing that Φ satisfies an inequality of type (1.3). However, for the Φ satisfying (1.6) the functions $b_i(x)$ (which actually depend on the solution u) have singularities at those points in Ω at which $|\text{grad } u| = 0$. The conclusion is then obvious. The proof that Φ satisfies such an inequality is quite involved and we do not reproduce it here.

In establishing Theorem II it is shown that if $K \leqslant 0$ on $\partial\Omega$ then $\frac{\partial \Phi}{\partial n} \leqslant 0$ on $\partial\Omega$. An application of H.2 then establishes the theorem.

In the special case

$$g = g(q^2), \qquad h = f(u)\varrho(q^2), \tag{1.10}$$

the equation (1.6) separates and $\Phi(u, q^2)$ can be written explicitly as

$$\Phi(u, q^2) = \int_0^{q^2} \frac{G(s)}{\varrho(s)} ds + 2\int_0^u f(\eta)\, d\eta. \tag{1.11}$$

A number of interesting examples for $N = 2$ are of form (1.10).

(a) Surface of constant mean curvature: $g = (1+q^2)^{-1/2}$, $f = \text{const.}$, $\varrho = \text{const.}$

(b) Torsional creep: g arbitrary, $f = \text{const.}$, $\varrho = \text{const.}$

(c) Meniscus problem (capillary tube): $g = (1+q^2)^{-1/2}$, $f = -u$, $\varrho = \text{const.}$

(d) Extensible film: $g = (1+q^2)^{-1/2}$, $f = \text{const.}$, $\varrho = (1+q^2)^{-1/2}$.

Furthermore, various special problems in two dimensional nonlinear elasticity are of form (1.6) with g and h given by (1.10).

Theorems I and II yield a "best possible" maximum principle for domains with boundaries of nonnegative average curvature in that Φ is identically equal to constant in the one dimensional problem, the limiting case of a spheroid with fixed minor axis as the radius of the major cross section tends to infinity.

Remark. It is possible to treat other classes of boundary conditions (see e.g. [13], [14]).

The combination Φ which satisfies (1.6) is not the only combination which satisfies a maximum principle. It was shown in [14] that if instead of (1.6), Φ_1 satisfies the equation

$$2H(u, q^2)\frac{\partial \Phi_1}{\partial q^2} = NG(u, q^2)\frac{\partial \Phi_1}{\partial u}, \tag{1.12}$$

and furthermore

$$2gH\frac{\partial H}{\partial q^2} - NG\left(g\frac{\partial H}{\partial u} - h\frac{\partial g}{\partial u}\right) \geqslant 0 \tag{1.13}$$

on solutions, then Φ_1 must take its maximum value on $\partial\Omega$. (Here no sign condition on the curvature of the boundary is needed.) This maximum principle is sometimes "best possible" in that the equality sign holds when Ω is the interior of the N-sphere. However, the additional condition (1.13) reduces the applicability of this principle somewhat. On the other hand it may be applied for any region with sufficiently smooth boundary, and in fact it is not necessary that $\partial\Omega$ be a $C^{2+\alpha}$ surface.

Suppose g and h are such that (1.4) has a positive solution. Theorems I and II may then imply that

$$\Phi(u, q^2) \leqslant \Phi(u_M, 0). \tag{1.14}$$

In view of (1.8) it is then in theory possible to solve a bound for q^2 of the following type:

$$q^2 \leqslant \mathscr{F}(u, u_M). \tag{1.15}$$

Integrating this inequality along a ray from the interior point P_0 at which $u = u_M$ to the nearest boundary point P_1 we would then obtain

$$\int_0^{u_M} \frac{d\eta}{\sqrt{\mathscr{F}(\eta, u_M)}} \leqslant \delta(P_0, P_1) \leqslant d. \tag{1.16}$$

Here $\delta(P_0, P_1)$ is the distance between P_0 and P_1, while d is the radius of the largest inscribed N-sphere. In many cases this inequality can be solved for u_M to obtain

$$u_M \leqslant \sigma(d) \tag{1.17}$$

and the result inserted back into (1.15) to give

$$\max_{\partial\Omega} q^2 \leqslant \mathscr{F}(0, \sigma(d)). \tag{1.18}$$

From the inequality for Φ_1, it is also possible to obtain a bound for $\max_{\partial\Omega} q^2$, and in some cases to determine subsets of $\partial\Omega$ on which $\max_{\partial\Omega} q^2$ cannot occur (see e.g. [12], [13]).

EXAMPLE. Consider the special case $g \equiv g(q^2)$, $h \equiv k_1$, where k_1 is a positive constant. Then provided (1.5) is satisfied it follows that if $\partial\Omega$ has positive average curvature K then

(a) q takes its maximum value q_0 at a point P on $\partial\Omega$;
(b) $q_0 g(q_0^2) \leqslant k_1 (N K_{\min})^{-1}$;
(c) $K(P) \leqslant k_1$ meas $\partial\Omega/(N$ meas $\Omega)$;
(d) various special results for special value of g (see [13]).

In (b), $K_{\min}$ designates the minimum value of K on $\partial\Omega$.

In continuum mechanics contexts the quantity on the left of (b) is the maximum stress on $\partial\Omega$. But by (1.5) it follows that $qg(q^2)$ is an increasing function of q and thus that (b) gives an upper bound for the stress throughout $\bar{\Omega}$. The interesting feature of (c) is that it is independent of the particular form of g. It states that the maximum value of q cannot occur at a point on the boundary at which the average courvature of the boundary satisfies inequality (c).

A number of interesting applications of these maximum principles in plasma physics have been made by Stakgold [18], Mossino [8], and in some as yet unpublished results of Bandle and Sperb. As a mathematical model of a certain plasma confinement problem Grad, Hu and Stevens [3] have proposed the following equation:

$$\begin{aligned} &\Delta u + \frac{dp}{du} = 0 \quad \text{in} \quad \Omega \subset R^2, \\ &u = \gamma \quad \text{on} \quad \partial\Omega, \\ &-\oint_{\partial\Omega} \frac{\partial u}{\partial n} ds = I. \end{aligned} \tag{1.19}$$

The constant γ is not prescribed a priori but the number I is specified. In plasma physics one is interested in equations whose solutions have the property that if one plots the surface $z = u(x, y)$ over Ω it will have the shape of a hill with γ either positive or negative. This is to be realized by a function p of the following form

$$p = p\left(u, a(u), \frac{du}{da}\right), \tag{1.20}$$

where $a(\tilde{u})$ is the measure of the set of points for which $u > \tilde{u}$. This leads of course to a very nonlocal type of equation, and very little is known about solutions except for very special functions p (see e.g. Temam [20]). Whenever one is assured of a solution which is C^1 in Ω and C^2 in some neighborhood of $\partial\Omega$ then it can easily be shown that the quantity $\Phi(u) \equiv |\text{grad } u|^2 + 2p\left(u, a(u), \frac{du}{da}\right)$ takes its maximum value at a point in $\bar{\Omega}$ at which $|\text{grad } u| = 0$, provided Ω is convex. If in fact

$$p = \tilde{p}(u, a(u)), \tag{1.21}$$

and the solution has the shape of a hill then under mild assumptions one can conclude that

$$|\text{grad } u|^2 + 2\tilde{p}(u, a(u)) \leqslant 2\tilde{p}(u_M, 0). \tag{1.22}$$

A "best possible" maximum principle has been employed in the literature to study a few special cases, e.g.,

(a) $$p = \begin{cases} \frac{1}{2}\lambda u^2, & u \geqslant 0, \\ 0, & u < 0, \end{cases}$$ (Stakgold [18]),

(b) $$p = \mu \int_0^u g(a(\eta))\,d\eta,$$ (Mossino [8]),

(c) $$p = \begin{cases} \dfrac{\lambda u^\alpha}{\alpha}, & u \geqslant 0, \\ 0, & u < 0, \end{cases}$$ (Bandle and Sperb).

2. Comparison theorems

In this section we indicate how "best possible" maximum principles may be used to obtain sharp inequalities relating solutions of nonlinear elliptic problems with those of simple linear ones and/or analogous one dimensional problems (see Payne [9], [10]).

As an example suppose u is a positive solution of

$$\begin{aligned} &\Delta u + \lambda f(u) = 0 \quad \text{in} \quad \Omega \subset R^N, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where the function f satisfies

$$f(u) \geqslant 0, \qquad f'(u) \geqslant 0 \qquad \text{for} \quad u > 0. \tag{2.2}$$

We wish to "compare" the solution of (2.1) with that of the linear problem

$$\begin{aligned} &\Delta\psi + 1 = 0 \quad \text{in} \quad \Omega \subset R^N, \\ &\psi = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.3}$$

or to the positive solution w of

$$\begin{aligned} &\Delta w + \lambda_1 w = 0 \quad \text{in} \quad \Omega, \\ &w = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.4}$$

To this end we define the two functions

$$\phi_1 = \left[\int_0^{u_M} \frac{d\eta}{\sqrt{F(u_M) - F(\eta)}}\right]^2 - \left[\int_u^{u_M} \frac{d\eta}{\sqrt{F(u_M) - F(\eta)}}\right]^2 \tag{2.5}$$

and

$$\phi_2 = \cos\left[\sqrt{\frac{\lambda_1}{2\lambda}} \int_u^{u_M} \frac{d\eta}{\sqrt{F(u_M) - F(\eta)}}\right], \tag{2.6}$$

where

$$F(u) = \int_0^u f(\eta)\,d\eta. \tag{2.7}$$

It is then possible by use of the "best possible" maximum principle to establish the following two theorems:

THEOREM III. *Let u be a classical positive solution of* (1.1), (1.2) *in a region Ω whose boundary has nonnegative average curvature then*

$$\phi_1 \leqslant 4\lambda\psi \quad \text{in} \quad \Omega \tag{2.8}$$

and

$$|\text{grad } \phi_1| \leqslant 4\lambda |\text{grad } \psi| \quad \text{on} \quad \partial\Omega. \tag{2.9}$$

THEOREM IV. *Let u be a classical positive solution of* (1.1), (1.2) *in a region Ω whose boundary has nonnegative average curvature, then*

$$\int_0^{u_M} \frac{d\eta}{\sqrt{F(u_M) - F(\eta)}} \geqslant \frac{\pi^2 \lambda}{2\lambda_1}. \tag{2.10}$$

To prove (2.8) we use the "best possible" maximum principle for $|\text{grad } u|^2 + 2F(u)$ (where u is the solution of (1.1)) to show that

$$\Delta\phi_1 + 4\lambda \geqslant 0 \quad \text{in} \quad \Omega. \tag{2.11}$$

The result then follows from Hopf's first principle. Inequality (2.9) is then a standard results (see e.g. [16]). To establish (2.10) we employ the same maximum principle to show that

$$\Delta\phi_2 + \lambda_1 \phi_2 \leqslant 0 \quad \text{in} \quad \Omega \tag{2.12}$$

from which the result follows easily. A number of somewhat more involved alternative theorems are also established in [10] by use of "best possible" maximum principles.

We mention now a result in another direction. It was indicated earlier that the boundary value problem

$$\begin{aligned} &\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[g(q^2)\frac{\partial u}{\partial x_i}\right] + 2k_1 = 0 \quad \text{in} \quad \Omega \subset R^2, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.13}$$

is sometimes used to model the problem of torsional creep. The next theorem compares the solution of (2.13) for a convex region Ω to the solution of the analogous one dimensional problem

$$[g(p^2)V']' + 2k_1 = 0, \qquad 0 \leqslant x \leqslant \sqrt{2\psi_M}, \tag{2.14}$$
$$V'(0) = 0, \qquad V(\sqrt{2\psi_M}) = 0,$$

where ψ is the solution of (2.3) in Ω and $\psi_M = \max_\Omega \psi$. In (2.14), $p^2 = [V']^2$.

THEOREM V. *Let u be classical solution of* (2.13) *in a convex region* Ω. *If* $g(s)$ *is a* C^2 *function of its argument for* $s > 0$ *and satisfies the conditions*:
(i) $g'(s) < 0$, $s > 0$;
(ii) $g(s) + 2sg'(s) > 0$, $s > 0$;
(iii) $g'(s)[\{sg^2(s)\}']^{-1}$ *nonincreasing in s for* $s > 0$, *then*

$$u \leqslant V(\sqrt{2(\psi_M - \psi)}) \quad in \quad \Omega \tag{2.15}$$

and

$$|\text{grad } u| \leqslant \frac{|V'(\sqrt{2\psi_M})|}{\sqrt{2\psi_M}} |\text{grad } \psi| \qquad on \quad \partial\Omega. \tag{2.16}$$

In some contexts condition (i) is said to describe a softening material.

By a straight forward though complicated computation using a result of Makar-Limanov [6] which states that if Ω is convex all of the level curves of ψ (the solution of (2.3)) are convex, together with the "best possible" maximum principle for the solution of (2.3) (i.e., $|\text{grad } \psi|^2 \leqslant 2[\psi_M - \psi]$), it can be shown that $\phi_3 \equiv V(\sqrt{2(\psi_M - \psi)})$ satisfies the inequality

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[g(\varrho^2)\frac{\partial \phi_3}{\partial x_i}\right] + 2k_1 \leqslant 0 \quad \text{in} \quad \Omega, \tag{2.17}$$
$$\phi_3 = 0 \quad \text{on} \quad \partial\Omega.$$

Where $\varrho^2 = |\text{grad } \phi_3|^2 = \dfrac{|\text{grad } \psi|^2}{4(\psi_M - \psi)} p^2$. Inequality (2.15) then follows directly, and (2.16) is an immediate consequence of the fact that if $u \leqslant \phi$ in Ω and $u = \phi$ on $\partial\Omega$ then $|\text{grad } u| \leqslant |\text{grad } \phi|$ on $\partial\Omega$.

The inequalities in Theorems III, IV, and V are sharp in the sense that the equality sign holds in the limit for a thin strip as the ratio of thickness to length tends to zero. Thus as mentioned earlier the inequalities in the three theorems may be integrated over appropriate subdomains of Ω and the resulting inequalities will be isoperimetric.

As an example, it is known that for the solution of (2.13) the functional

$$S = \int_\Omega u \, dx \tag{2.18}$$

is proportional to the rigidity of the beam. By integration of (2.15) over Ω it is possible to establish the following isoperimetric inequality for S

$$S \leqslant [2\psi_M]^{-1/2} A \int_0^{\sqrt{2\psi_M}} V(\sigma)\, d\sigma, \tag{2.19}$$

where A is the area of Ω.

Theorem V may also be used to derive the following simple bound for the maximum shear strain intensity $Q_0 = \max_{\partial\Omega} q[g(q^2)]$.

$$Q_0 \leqslant k_1 \sqrt{2\psi_M}. \tag{2.20}$$

This inequality displays the remarkable fact that the isoperimetric result is completely independent of the form of g.

In the inequalities derived in this section many results involved ψ_M or $\max_{\partial\Omega} |\text{grad}\, \psi|$. Bounds for these quantities in terms of the geometry of Ω can be found in numerous ways, e.g., monotony principles, maximum principles, variational principles, etc. A simple bound derived in [11] is

$$\max_{\partial\Omega} |\text{grad}\, \psi|^2 \leqslant 4\psi_M \leqslant d^2, \tag{2.21}$$

where d is the radius of the largest inscribed circle in Ω. Sharper estimates have been given by Fu and Wheeler [2].

The results on torsional creep will appear in a forthcoming paper. A number of additional comparison results would follow if it were possible to determine classes of functions f for which positive solutions of

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega \subset R^2,$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$

satisfy the Makar-Limanov property (i.e., the convexity of Ω implies the convexity of all of the level curves of u). This is known to be the case for the positive solution of (2.4) (see Brascamp and Lieb [1]) but whether it holds for other functions f is still an open question.

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