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DISSERTATIONES MATHEMATICAE

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Trend estimation problems in time-series analysis

WARSZAWA 1973

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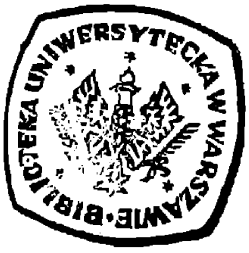

PRINTED IN POLAND

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1. Introduction

This paper presents some statistical problems and solutions in the field of strictly stationary discrete-parameter processes called *linear processes*. It is assumed that the only source of information we have is the knowledge of a realization of a linear process with a superimposed trend, i. e. a determinate real function defined on the set of arguments of this process.

The paper concerns essentially trend estimation problems. In most general and intuitive terms, I suggest considering estimators of trend such that the residuals which arise after the elimination of the estimated values from the observed ones behave in some sense like a linear process. Somewhat more precisely, if in the case of a trend in a considered class a certain test F rejects with probability equal asymptotically to one the hypothesis H_0, which states the absence of trend, then we take into account estimators producing residuals with the following property: the test F applied to residuals rejects H_0 asymptotically as rarely as it occurs when this rest is applied to the pure linear process.

The definition of those trend estimators, which we propose to call *F-estimators*, is given in Section 2. In Section 3.1 the J^*- and T^*- estimators, based on the tests J^* and T^* introduced by S. K. Zaremba in [38] and [36], are discussed, while in Section 3.2 these estimators are used to construct an interval estimator of a polynomial trend of an unknown degree.

In Section 4.1 a test is suggested which at first sight has nothing to do with the trend estimation problems. Its main role is to test the hypothesis that two linear processes have the same spectral density functions. However, as explained in Section 4.2, there exists a distinct link connecting this test with the trend estimation problems, according to the ideas given in Section 2, and with the model verification problems present in any statistical enterprise.

In Section 5 the suggested statistical algorithms are discussed on the ground of the existing results and the conjectural practical needs in the time series analysis.

2. F-estimators

To start with, we describe our model formally. The notations $\{a_t\}$ and $\{a_t\}_N$ will be used to describe sequences $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ and $\{a_1, a_2, \ldots, a_N\}$, respectively. We consider a stochastic process $\{z_t\} = \{x_t + y_t\}$, where $\{y_t\}$ is a sequence of real numbers and $\{x_t\}$ is a linear process, i.e.

$$x_t = \sum_{q=-\infty}^{\infty} h_{t-q}\varepsilon_q \quad (t = 0, \pm 1, \ldots),$$

the coefficients $\{h_k\}$ being real, equal to zero for $k < 0$, and satisfying $\sum_{k=0}^{\infty}|h_k| < \infty$ and $\sum_{k=1}^{\infty} kh_k^2 < \infty$, and $\{\varepsilon_t\}$ being a sequence of mutually independent identically distributed random variables with zero means, positive variances and finite moments up to the order $2L$ $(L \geqslant 1)$. Adopting the notation used in [36] and [24], we shall denote the above assumptions about $\{x_t\}$ by $\tilde{H}_{2L}$. In some cases it will be also required that $\sum_{j=1}^{\infty}|jR_j| < \infty$, where $R_j = E(x_t x_{t+j})$.

We assume that we can observe the realization $\{z_t\}_N$ for any sample size N, and we want to estimate the trend $\{y_t\}$ using the observed sequence $\{z_t\}_N$. More precisely, let $\mathscr{M}$ be a class of sequences $\{y_t\}$ which does not contain the sequence $\{0\}$ consisting of zeros only. This class represents all admissible trends. Let G be a subset of $\mathscr{M}$. It will be convenient to adopt here the following definition: a sequence of N-dimensional random variables $\{\hat{y}_t^{(N)}\}$, $N \geqslant N_0$, will be called a *trend estimator in* G, if for any $N \geqslant N_0$ and $t = 0, \pm 1, \ldots$, there exists a function $h_t^{(N)}\colon R^N \to R$ such that $h_t^{(N)}(\{z_t\}_N) = \hat{y}_t^{(N)}$ and that $\{h_t^{(N)}(a)\} \in G$ for any $N \geqslant N_0$ and any $a \in R^N$. It is clear that in some problems of trend estimation this definition may be too restrictive, the last condition being likely to be replaced by one postulating that the value of $\{\hat{y}_t^{(N)}\}$ is in some sense "close" to an element of G. However, in the problems considered in this work the above definition is sufficiently general. N_0 denotes the smallest admissible sample size.

Now we ask which trend estimator in class G is "the best" or at least "sufficiently good". Many such problems were investigated for some classes G with additional restrictions imposed on $\{x_t\}$, especially when $\{x_t\}$ is a sequence of independent identically distributed random variables, and different properties of a "good" trend estimator were formulated; among them, the best known are consistency, unbiasedness and effectiveness of the estimator when y_t is constant for

every t, that is when G is a set of sequences with equal real terms(1). In more general cases of G there are no generally adopted properties to be required; in some cases a distance $\delta_N(a_t, b_t)$ between any two sequences $\{a_t\}_N$ and $\{b_t\}_N$ is defined for any N and it is postulated that $\{\delta_N(\hat{y}_t^{(N)}, y_t)\}$ tends to zero in probability or in the mean with $N \to \infty$ or that $\lim\limits_{N\to\infty} E\,\delta_N(\hat{y}_t^{(N)}, y_t) = 0$, etc.

Below we define a new requirement which, as will be explained later, is relevant to the problems of constructing an estimator in some classes G.

Let $F = \{F_N\}$ be a sequence of functions $F_N\colon R^N \to R$ for $N \geqslant N_0$ such that for every α $(0 < \alpha < \frac{1}{2})$ there exists a set $\omega_\alpha \subset R$ satisfying the conditions:

$$\lim_{N\to\infty} P\big(F_N(\{x_t\}_N) \in \omega_\alpha\big) \leqslant \alpha \tag{2.1}$$

and

$$\lim_{N\to\infty} P\big(F_N(\{z_t\}_N) \in \omega_\alpha\big) = 1 \quad \text{if } \{y_t\} \in M \tag{2.2}$$

for any process $\{z_t\}$ described above.

In other words, we can speak of a test F (with a rejection set consisting of all $\{z_t\}_N$ such that $F_N(\{z_t\}_N) \in \omega_\alpha$) for testing the absence of a trend, and α is the chosen upper bound of the limiting probability of rejecting this hypothesis when it is true. The power of F satisfies (2.1) and (2.2), i. e. for small α and large N we can discriminate very well between the situations with no trend and with a trend from class $\mathscr{M}$.

A trend estimator in G, $\{\hat{y}_t^{(N)}\}$ $(N \geqslant N_0)$, will be called an *F-estimator* of trends in G if

$$\lim_{N\to\infty} P\big(F_N(\{z_t - \hat{y}_t^{(N)}\}_N) \in \omega_\alpha\big) \leqslant \alpha \quad \text{for } \{y_t\} \in G. \tag{2.3}$$

It means that — with reference to F — subtracting $\{\hat{y}_t^{(N)}\}$ is asymptotically equivalent to a no-trend situation, that is, the residua behave like a linear process, because after replacing $\{z_t - \hat{y}_t^{(N)}\}_N$ with $\{x_t\}_N$ formula (2.3) is identical with (2.1).

A trend estimator which does not satisfy (2.3) for given G and F and for a suitably chosen α usually cannot be recommended in practice. However, it is obvious that postulating (2.3) as the only requirement can lead to quite unsatisfactory solutions, which is illustrated by the following example. Let us choose for any α a sequence of real numbers $\{u_t\}$ such

(1) This set of assumptions corresponds, in our terminology, to a standard situation of estimating a mean value of a random variable on the basis of repeated independent observations of this random variable. It is the simplest case of "trend" estimation.

that, for every N, $F_N(\{u_t\}_N) \notin \omega_a$. Of course such a sequence always exists and can be easily constructed. Then putting $\hat{y}_t^{(N)} = z_t - u_t$ for $t = 1, \ldots, N$ and for every N we can immediately check that $\{\hat{y}_t^{(N)}\}$ $(N = 1, 2, \ldots)$ is an F-estimator of trend in $\mathscr{M}$ for any $a \geqslant 0$; obviously enough, it is useless in practice.

The realization of the event $F_N(\{z_t - \hat{y}_t^{(N)}\}_N) \in \omega_a$, $\{\hat{y}_t^{(N)}\}$ being an F-estimator of trend in G, should be considered as a cause of doubts about the assumed model. By (2.3), the probability of such an event when the doubts the model are groundless, i. e. when $\{y_t\} \in G$, is asymptotically not greater than a.

It is hoped that F-estimators can be helpful particularly in those cases when we have a sequential process of trend estimation and an appropriate stopping rule is needed. Then such a stopping rule can be introduced with the help of the asymptotic rejection set of a test F: the sequential process of estimation is to be stopped when for the first time the value of F_N for the residuals does not belong to ω_a or when a preassigned maximal number of steps is reached. Roughly speaking, we think that an estimation procedure with such a stopping rule is usually an F-estimator.

The case of a polynomial trend of an unknown degree provides an example of this point of view. Let S be the set of all polynomials of degrees not exceeding a fixed integer D and having a root-mean-square norm bigger than a fixed positive number Ξ for any N bigger than some fixed integer N^*. Then, for any $\{y_t\} \in S$, $y_t = \sum_{q=0}^{D} A_q \varphi_{q,N}(t)$, where $\{\varphi_{q,N}(t)\}$ is a family of orthogonal polynomials such that for any N $\sum_{t=1}^{N} [\varphi_{q,N}(t)]^2 = N$ (the exact formula for $\varphi_{q,N}(t)$ is given by (3.1.16)) so that $\sum_{q=0}^{D} A_q^2 > \Xi^2$. For any given D and Ξ, let S' be a subset of S such that $\{y_t\} \in S'$ if, for some m $(0 \leqslant m \leqslant D)$ and for any $N > N^*$, $|A_m| > \Xi$ and $A_{m+1} = \ldots = A_D = 0$ when $m < D$.

A trend estimator in S' would be easy to find: we would take $\{\hat{y}_{t,D}\}$, i. e. the least square estimator of polynomials of degree D, where

$$\begin{aligned} \hat{y}_{t,i} &= \sum_{q=0}^{i} \hat{A}_q \varphi_{q,N}(t) \quad \text{for any } i \geqslant 0, \\ \hat{A}_q &= N^{-1} \sum_{t=1}^{N} z_t \varphi_{q,N}(t). \end{aligned} \tag{2.4}$$

However, assuming an upper bound for the true degree m without testing this assumption could lead to entirely wrong conclusions if in fact the degree of the trend were higher than the preassigned bound, while estimating terms of a degree higher than that of the true trend

would introduce unnecessary stochastic and computational errors and increase the bias in the estimators, for instance, of the covariances of the residua (see [23]).

On the other hand, it is clear that we cannot properly estimate polynomial trends of degrees which are too high in relation to the sample size. Hence we have to assume in advance an upper bound of the degree of polynomials which we hope to estimate. It should be stressed that this is not an assumption about the true degree of the trend, but merely an expression of the limitations of the possibilities of stochastic inference. As opposed to the case when an upper bound is assumed for the true degree of the polynomial trend, this approach will not lead (except for events with small asymptotic probability) to erreneous conclusions about the trend when its degree exceeds D, but merely to the admission that we are not able to fit the appropriate polynomial.

The above considerations lead us to the following: given a test F satisfying (2.1) and (2.2) for some class $\mathscr{M} \supset S'$, we want to construct a trend estimator in S' such that (i) it would be an F-estimator in S', and (ii) it would be a polynomial of degree $\hat{m}$, where $\hat{m}$ is a function of $\{z_t\}_N$ satisfying the condition $\lim\limits_{N\to\infty} P(\hat{m} = m) \geqslant 1-a$ for $\{y_t\} \in S'$ and for any chosen D and a. Then, if $\{y_t\} \in S'$, i. e. when the true degree is smaller than the chosen D, the inequality $\hat{m} \leqslant D$ holds with probability which is asymptotically sufficiently big. On the other hand, (ii) is true for any given D and, therefore, if the true trend is a polynomial of a degree higher than the chosen D, then the asymptotic probability of fitting a polynomial with $\hat{m} \leqslant D$ is smaller than a.

Let us introduce the following F-rule of defining $\hat{m}$ for any chosen test F, a and D:

$$(2.5)\quad \hat{m} = \begin{cases} 0 & \text{if } F_N(\{z_t - \hat{y}_{t,0}\}_N) \text{ does not belong to } \omega_a; \\ j & \text{if } F_N(\{z_t - \hat{y}_{t,i}\}_N) \text{ belongs to } \omega_a \text{ for } i < j \leqslant D \\ & \qquad \text{and does not belong to } \omega_a \text{ for } i = j; \\ D+1 & \text{if } F_N(\{z_t - \hat{y}_{t,i}\}_N) \text{ belongs do } \omega_a \text{ for } 0 \leqslant i \leqslant D. \end{cases}$$

Then $\{\hat{y}_{t,\hat{m}}\}$, i. e. the least-square estimator of degree $\hat{m}$, is a trend estimator proposed here. Its asymptotic properties depend on those of the involved test F, as stated in the following theorem:

THEOREM 2.1. *Let $F = \{F_N\}$ be a test satisfying* (2.1) *and* (2.2) *for some $\mathscr{M} \supset S'$ and for a suitably chosen $a \in (0, \frac{1}{2})$, and such that for any $\{y_t\} \in S'$*

$$(2.6)\qquad \lim_{N\to\infty} P[F_N(\{z_t - \hat{y}_{t,i}\}_N) \in \omega_a] = 1 \quad \textit{for } 0 \leqslant i < m$$

and

$$(2.7)\qquad p\lim_{N\to\infty}[F_N(\{z_t-\hat{y}_{t,i}\}_N)-F_N(\{x_t\}_N)]=0 \quad \textit{for } m\leqslant i\leqslant D.$$

Moreover, assume that either the limiting distribution of $F_N(\{x_t\}_N)$ *exists and is continous or* $p\lim_{N\to\infty}F_N(\{x_t\}_N)=\infty$. *Then for any* $\{y_t\}\in S'$ *the trend estimator* $\{\hat{y}_{t,\hat{m}}\}$ *based on the F-rule of defining* $\hat{m}$ *is an F-estimator in* S', *and*

$$(2.8)\qquad \begin{aligned}&\lim_{N\to\infty}P(0\leqslant\hat{m}\leqslant D,\ \hat{m}\neq m)=0;\\ &\lim_{N\to\infty}P(\hat{m}=m)\geqslant 1-\alpha;\\ &\lim_{N\to\infty}P(\hat{m}=D+1)\leqslant\alpha.\end{aligned}$$

Proof. The notation A_i will be used, in this proof only, to denote the event $F_N(\{z_t-\hat{y}_{t,i}\}_N)\in\omega_a$; moreover, we shall write $F_N^{(i)}$ instead of $F_N(\{z_t-\hat{y}_{t,i}\}_N)$ and F_N instead of $F_N(\{x_t\}_N)$. Let β denote the left-hand side of (2.1); thus $\beta\leqslant\alpha$. We have $\lim_{N\to\infty}P(A_i)=1$ for $i=0,\ldots,m-1$ in view of (2.6), and $\lim_{N\to\infty}P(A_i)=\beta$ for $i=m,\ldots,D$ in view of (2.7) and of the assumptions about the asymptotic behaviour of F_N. Then, by (2.5)

$$\lim_{N\to\infty}P(\hat{m}<m)=\lim_{N\to\infty}P(\bar{A}_1\cup\ldots\cup\bar{A}_{m-1})\leqslant\sum_{i=0}^{m-1}\lim_{N\to\infty}P(\bar{A}_i)=0.$$

Hence $\lim_{N\to\infty}P\overline{(A_1\cap\ldots\cap A_{m-1})}=0$; but by (2.5)

$$\begin{aligned}P(\hat{m}=m)&=P(A_1\cap\ldots\cap A_{m-1}\cap\bar{A}_m)\\ &\geqslant 1-P\overline{(A_1\cap\ldots\cap A_{m-1})}-P(A_m).\end{aligned}$$

Consequently, $\lim_{N\to\infty}P(\hat{m}=m)\geqslant 1-\beta$ and $\lim_{N\to\infty}P(\hat{m}>m)\leqslant\beta$. If the asymptotic distribution exists and is continous, by (2.7) and the convergence theorem given for instance in [5], Section 20.6, the joint asymptotic distribution of $(F_N^{(i)},F_N^{(m)})$ for $m\leqslant i\leqslant D$ is the same as that of (F_N,F_N). Hence $\lim_{N\to\infty}P(A_i|A_m)=1$ for $i=m+1,\ldots,D$. Consequently, for any such i,

$$P(\hat{m}=i)=P(A_1\cap\ldots\cap A_{i-1}\cap\bar{A}_i)\leqslant P(\bar{A}_i|A_m)P(A_m)\to 0$$

when $N\to\infty$, and

$$\lim_{N\to\infty}P(\hat{m}=D+1)=1-\lim_{N\to\infty}P(\hat{m}\leqslant D)\leqslant\beta,$$

so that (2.8) is satisfied since $\beta\leqslant\alpha$.

If $p\lim_{N\to\infty}F_N=+\infty$, then ω_a must be bounded from above since β cannot be equal to 1 in view of (2.1); therefore β is equal to 0 in this

case so that $\lim_{N\to\infty} P(\hat{m} = m) = 1$ and (2.8) is satisfied. The same argument can be applied when $p \lim_{N\to\infty} F_N = -\infty$.

To prove the first part of the theorem, we notice that by (2.5) the events $A_{\hat{m}}$ and $\hat{m} \leqslant D$ are disjoint and hence for any $\{y_t\} \in S'$ and any $0 < a < \frac{1}{2}$, in view of (2.8),

$$\lim_{N\to\infty} P(A_{\hat{m}}) = \lim_{N\to\infty} P(A_{\hat{m}} \cap \hat{m} = D+1) \leqslant \lim_{N\to\infty} P(\hat{m} = D+1) \leqslant a;$$

then (2.3) is satisfied and $\{\hat{y}_{t,\hat{m}}\}$ is an F-estimator in S'.

Theorem 2.1 ensures that if $\{y_t\} \in S'$, then the asymptotic probability of an unjustified rejection of this assumption when $\{\hat{y}_{t,\hat{m}}\}$ is investigated is not greater than a, and $\hat{m}$ is equal to m with a high asymptotic probability; on the other hand, if $\{y_t\}$ is a polynomial of a degree higher than D, fitting a polynomial of a degree not greater than D is asymptotically very improbable.

Trend estimators $\{\hat{y}_{t,\hat{m}}\}$ have some interesting asymptotic properties under the condition that $\hat{m} \leqslant D$. The investigation of $\{\hat{y}_{t,\hat{m}}\}$ under this condition seems reasonable in view of the fact that $\{\hat{y}_{t,\hat{m}}\}$ is being used as an estimator of $\{y_t\}$ only when $\hat{m} \leqslant D$, since $\hat{m} = D+1$ is a signal for doubts about the assumed model. The following theorem describes $\{\hat{y}_{t,\hat{m}}\}$ as an asymtotically good estimator in the maximum absolute error sense.

THEOREM 2.2. *If for some given test F and any chosen a (2.8) is satisfied and if $\{y_t\} \in S'$, then under $\tilde{H}_2$, for any $\delta > 0$,*

$$\lim_{N\to\infty} P(\max_{1\leqslant t\leqslant N} |\hat{y}_{t,\hat{m}} - y_t| < \delta \mid \hat{m} \leqslant D) = 1.$$

Proof. It follows immediately from Lemma 4.11 in [24] (after easy alterations caused by the definition of $\varphi_{q,N}(t)$ adopted here according to (3.1.16) which differs from formula (4.1) in [24] defining the orthogonal polynomials used there) that for any $\delta > 0$ and any $\{y_t\} \in S'$

$$\lim_{N\to\infty} P(\max_{1\leqslant t\leqslant N} |\hat{y}_{t,m} - y_t| > \delta) = 0. \tag{2.9}$$

We have

$$\begin{aligned}
&P([\max_{1\leqslant t\leqslant N} |\hat{y}_{t,m} - y_t| + \max_{1\leqslant t\leqslant N} |\hat{y}_{t,\hat{m}} - \hat{y}_{t,m}|] > \delta \mid \hat{m} \leqslant D) \\
&\leqslant P\left(\max_{1\leqslant t\leqslant N} |\hat{y}_{t,m} - y_t| > \frac{\delta}{2} \mid \hat{m} \leqslant D\right) + P\left(\max_{1\leqslant t\leqslant N} |\hat{y}_{t,\hat{m}} - \hat{y}_{t,m}| > \frac{\delta}{2} \mid \hat{m} \leqslant D\right) \\
&\leqslant \frac{P\left(\max_{1\leqslant t\leqslant N} |\hat{y}_{t,m} - y_t| > \frac{\delta}{2}\right)}{P(\hat{m} \leqslant D)} + \\
&\quad + P\left(\max_{1\leqslant t\leqslant N} |\hat{y}_{t,\hat{m}} - \hat{y}_{t,m}| > \frac{\delta}{2} \cap \hat{m} = m \mid \hat{m} \leqslant D\right) + P(\hat{m} \neq m \mid \hat{m} \leqslant D).
\end{aligned} \tag{2.10}$$

The first term of the right-hand side of (2.10) tends to zero in view of (2.8) and (2.9), and the third term also tends to zero by (2.8). In the second term there is a conjuction of two events which are disjoint so that this term is equal to zero. But of course

$$(2.11)\qquad P\,(\max_{1\leqslant t\leqslant N}|\hat{y}_{t,m}-y_t|+\max_{1\leqslant t\leqslant N}|\hat{y}_{t,\hat{m}}-\hat{y}_{t,m}|<\delta\,|\,\hat{m}\leqslant D)$$
$$\leqslant P(\max_{1\leqslant t\leqslant N}|y_t-\hat{y}_{t,\hat{m}}|<\delta\,|\hat{m}\leqslant D)\leqslant 1.$$

Thus, since the left-hand sides of (2.11) and (2.10) sum up to 1 and since the latter has been proved to be zero in the limit, the theorem follows from (2.11).

The conditional distribution of $N^{\frac{1}{2}}(\hat{y}_{t,\hat{m}}-y_t)$ under the condition $\hat{m}\leqslant D$ will be derived in Section 3.2 and asymptotic confidence intervals of y_t for any t and any $\{y_t\}\in S'$ will be constructed on the basis of the studentisized version of this distribution. In the next section two tests F are considered which satisfy the conditions of Theorem 2.1 so that the corresponding estimators $\{\hat{y}_{t,\hat{m}}\}$ are F-estimators in S' and (2.8) holds.

3. The role of the tests J^* and T^* in polynomial trend estimation problems

3.1. J^*- and T^*- estimator in S'. Tests satisfying (2.1) and (2.2) for suitable sets $\mathscr{M}$ are nearly non-existing unless $\{x_t\}$ is supposed to be white noise or unless other restricting assumptions are made, as discussed in Section 5. The tests J^* and T^* introduced by S. K. Zaremba in [38] and [36] form very useful exception. In this section we shall prove that these tests provide J^*- and T^*- estimators in S' and that (2.8) is satisfied when $\hat{m}$ is based on these tests.

It will be convenient to introduce here the following rule concerning any quantity, say Q_N, which depends on N and is to be defined in what follows throughout this paper. Let $\mathscr{N}$ denote the subset of sample sizes for which there exists at least one pair of integers (p, q) such that $N = p\cdot q$ and $N^{2/5}\leqslant p < N^{\frac{1}{2}}$. Then we define Q_N for any $N\in\mathscr{N}$ while for any N which does not belong to $\mathscr{N}$ we assume that $Q_N = Q_{N'}$, where N' is the biggest integer less than N and belonging to $\mathscr{N}$. It means in practice that we reduce the sample to the greatest sample which is suitably factorizable. Since this reduction is possible only for $N\geqslant 6$, 6 being the smallest integer in $\mathscr{N}$, we always assume that $N\geqslant 6$.

According to this rule, we define μ and ν for any $N\in\mathscr{N}$ so that we choose the pair of (p, q) with the smallest p and put $\nu = p$ and $\mu = q$; then μ, ν and $\mu\nu^{-\frac{1}{2}}$ tend to $+\infty$ when $N\to\infty$.

The test function of the test J^* was denoted in [38] by $J^*_{\mu,\nu}$; we shall use here the notations J^*_N for $J^*_{\mu,\nu}(\{z_t\}_N)$ and J_N for $J^*_{\mu,\nu}(\{x_t\}_N)$; according to [38],

$$J^*_N = \mu^{\frac{1}{2}}\left[\frac{\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}(M^*_{\nu,r})^2}{\frac{\nu}{2(\mu-2)}\sum_{r=1}^{\mu-2}(M^*_{\nu,r+1}-M^*_{\nu,r-1})^2}-1\right], \tag{3.1.1}$$

where

$$M^*_{\nu,r} = \nu^{-1}\sum_{t=1}^{\nu} z_{t+r\nu}, \tag{3.1.2}$$

and obviously J_N is given by a formula identical to (3.1.1) with $M^*_{\nu,r}$ replaced by

$$M_{\nu,r} = \nu^{-1}\sum_{r=1}^{\nu} x_{t+r\nu}. \tag{3.1.3}$$

The set ω_a in the test J^* was defined in [38] as $\{a \in R\colon a > \zeta_{1-a}\}$, where, for any $0 \leqslant u \leqslant 1$, ζ_u satisfies

$$\Phi(\zeta_u) = u, \tag{3.1.4}$$

Φ being the standarized normal distribution function.

According to Proposition 8.1 in [38], under $\tilde{H}_4$ and when $\sum_{j=1}^{\infty}|jR_j| < +\infty$, J_N has an asymptotic distribution which is normal with a zero mean and a unit variance; hence $\lim_{N\to\infty} P(J_N > \zeta_{1-a}) = a$. Moreover, by Theorem 11.1 in [38], under the same assumptions, the power of the test J^* tends to 1 uniformly with respect to trends in S. Now we shall check the remaining assumptions of Theorem 2.1. This implies that we need to consider $J^*_{\mu,\nu}(\{z_t-\hat{y}_{t,i}\}_N)$ for $i = 0, \ldots, D$, where by (2.4)

$$z_t-\hat{y}_{t,i} = x_t+\sum_{q=0}^{D}A_q\varphi_{q,N}(t)-\sum_{q=0}^{i}\hat{A}_q\varphi_{q,N}(t).$$

Let

$$\begin{aligned} a_q &= \hat{A}_q - A_q = N^{-1}\sum_{t=1}^{N}(z_t-y_t)\varphi_{q,N}(t) = N^{-1}\sum_{t=1}^{N}x_t\varphi_{q,N}(t);\\ \gamma_{t,i} &= -\sum_{q=0}^{i}a_q\varphi_{q,N}(t). \end{aligned} \tag{3.1.5}$$

Thus for any $\{y_t\} \in S'$ and any $1 \leqslant t \leqslant N$

$$z_t-\hat{y}_{t,i} = \begin{cases} x_t+\gamma_{t,i}+\sum_{q=i+1}^{m}A_q\varphi_{q,N}(t) & \text{for } i = 0, \ldots, m-1;\\ x_t+\gamma_{t,i} & \text{for } i = m, \ldots, D. \end{cases} \tag{3.1.6}$$

In view of the last formula some properties of the test J^* which we shall prove now entail conditions (2.6) and (2.7) for this test. More precisely, we shall consider $J_{N,i} = J^*_{\mu,\nu}(\{x_t+\gamma_{t,i}+y_t\}_N)$ and we shall prove that for any $i = 0, \ldots, D$ $p \lim\limits_{N\to\infty} J_{N,i} = +\infty$, uniformly with respect to $\{y_t\}$ belonging to S, and, moreover, that $\{y_t\} = 0$ implies that $p \lim\limits_{N\to\infty} (J_{N,i} - J_N) = 0$. In other words, we shall prove that the stochastic component $\{\gamma_{t,i}\}$ does not affect the asymptotic behaviour of J^*_N and J_N, i. e. that theorems similar to Proposition 8.1 and Theorem 11.1 in [38] hold when this component is introduced. From this, we deduce easily (2.6) and (2.7).

Let

$$Y_{\nu,r} = \nu^{-1} \sum_{t=1}^{\nu} y_{t+r\nu},$$

$$\Gamma_{\nu,r,i} = \nu^{-1} \sum_{t=1}^{\nu} \gamma_{t+r\nu,i},$$

$$a_{N,i} = \frac{\nu}{\mu} \sum_{r=0}^{\mu-1} (M_{\nu,r} + \Gamma_{\nu,r,i} + Y_{\nu,r})^2,$$

$$b_{N,i} = \frac{\nu}{2(\mu-2)} \sum_{r=1}^{\mu-2} [(M_{\nu,r+1} - M_{\nu,r-1}) + (\Gamma_{\nu,r+1,i} - \Gamma_{\nu,r-1,i}) + (Y_{\nu,r+1} - Y_{\nu,r-1})]^2, \tag{3.1.7}$$

$$a_N = \frac{\nu}{\mu} \sum_{r=0}^{\mu-1} M^2_{\nu,r},$$

$$b_N = \frac{\nu}{2(\mu-2)} \sum_{r=1}^{\mu-2} (M_{\nu,r+1} - M_{\nu,r-1})^2.$$

Consequently, according to the definitions of $J_{N,i}$ and J_N,

$$J_{N,i} = \mu^{\frac{1}{2}} \left(\frac{a_{N,i}}{b_{N,i}} - 1 \right) \tag{3.1.8}$$

and

$$J_N = \mu^{\frac{1}{2}} \left(\frac{a_N}{b_N} - 1 \right). \tag{3.1.9}$$

LEMMA 3.1. *Under* $\tilde{H}_4$, *if* $\sum\limits_{j=1}^{\infty} |jR_j| < +\infty$ *and* $\{y_t\} = \{0\}$

$$p \lim_{N\to\infty} (J_{N,i} - J_N) = 0 \quad \textit{for } i = 0, \ldots, D.$$

Proof. We shall show first that if $\{y_t\} = \{0\}$, then under $\tilde{H}_4$ and for any $i = 0, \ldots, D$

$$p \lim_{N\to\infty} \mu^{\frac{1}{2}}(a_{N,i} - a_N) = 0 \tag{3.1.10}$$

and

$$p \lim_{N\to\infty} \mu^{\frac{1}{2}}(b_{N,i} - b_N) = 0. \tag{3.1.11}$$

In view of (3.1.7), $\{y_t\} = \{0\}$ implies that

$$\mu^{\frac{1}{2}}(a_{N,i} - a_N) = \nu\mu^{-\frac{1}{2}} \sum_{r=0}^{\mu-1} \Gamma^2_{\nu,r,i} + 2\nu\mu^{-\frac{1}{2}} \sum_{r=0}^{\mu-1} M_{\nu,r}\Gamma_{\nu,r,i}. \tag{3.1.12}$$

Dealing with the first term of the right-hand side of (3.1.12), we shall make use of the fact that $\{\gamma_{t,i}\}$ is a polynomial of degree i with stochastic coefficients. Hence some results concerning $\{y_t\}$ which were obtained in [38] can be easily obtained for $\{\gamma_{t,i}\}$. In particular, proceeding exactly like in the proof of Theorem 11.1 in [38], where an upper bound of $\nu\mu^{-1}\sum_{r=0}^{\mu-1} Y^2_{\nu,r}$ was found, we see that

$$\nu\mu^{-\frac{1}{2}} \sum_{r=0}^{\mu-1} \Gamma^2_{\nu,r,i} \leqslant \nu\mu^{\frac{1}{2}}\Big(\sum_{q=0}^{i} a_q^2\Big)(1 + |\varepsilon|),$$

where $\varepsilon \to 0$ uniformly when $N \to \infty$. Of course, $\sum_{q=0}^{i} a_q^2$ is a random variable. But it is known that asymptotic distribution of $N^{\frac{1}{2}}a_q$ is normal with a zero mean and a variance equal to $\sum_{j=-\infty}^{\infty} R_j$, and that $N\operatorname{cov}(a_p, a_q)$ tends to zero for $p \neq q$ when $N \to \infty$; this was proved in Lemmas 4.3 and 4.4 in [24], where, however, as it was explained before, the notations differ from those introduced here. Consequently,

$$\begin{cases} \text{for } i = 0, \ldots, D, \text{ the limiting distribution of} \\ \Big(\sum_{j=-\infty}^{\infty} R_j\Big)^{-1} N \sum_{q=0}^{i} a_q^2 \text{ is } \chi^2 \text{ with } (i+1) \text{ degrees of freedom,} \end{cases} \tag{3.1.13}$$

and hence for any $i = 0, \ldots, D$

$$\lim_{N\to\infty} E\Big(\nu\mu^{\frac{1}{2}} \sum_{q=0}^{i} a_q^2\Big)^2 = 0. \tag{3.1.14}$$

It follows that

$$\lim_{N\to\infty} E\Big(\nu\mu^{-\frac{1}{2}} \sum_{r=0}^{\mu-1} \Gamma^2_{\nu,r,i}\Big)^2 = 0. \tag{3.1.15}$$

Turning to the second term of the right-hand side of (3.1.12), we observe that the polynomial

$$(3.1.16)\qquad \varphi_{q,N}(t) = \left[\frac{2q+1}{(N^2-1^2)\cdot\ldots\cdot(N^2-q^2)}\right]^{\frac{1}{2}} \sum_{s=0}^{q} (-1)^{q-s} \frac{(q+s)!(N-s-1)!(t-1)^{(s)}}{(s!)^2(q-s)!(N-q-1)!},$$

where $u^{(s)} = u(u-1)\cdot\ldots\cdot(u-s+1)$, is bounded over $t = 1, \ldots, N$ for any $N > q$ (this follows for instance from Proposition 10.1 in [38], because the polynomial $\tilde{p}_q$ introduced there is bounded over $[0, 1]$). Let

$$(3.1.17)\qquad Q_D = \max_{0\leqslant q\leqslant D} \max_{1\leqslant t\leqslant N} |\varphi_{q,N}(t)|.$$

Thus for any $i = 0, \ldots, D$, by (3.1.3), (3.1.7) and (3.1.5),

$$(3.1.18)\qquad E\Big(\nu\mu^{-\frac{1}{2}} \sum_{r=0}^{\mu-1} M_{\nu,r}\Gamma_{\nu,r,i}\Big)^2$$

$$= N^{-3}\nu^{-1} \sum_{q_1,q_2=0}^{i} \sum_{p,r=0}^{\mu-1} \sum_{t_1,\ldots,t_4=1}^{\nu} \sum_{s_1,s_2=1}^{N} \{E(x_{s_1}x_{s_2}x_{t_1+p\nu}x_{t_2+r\nu}) \times$$

$$\times \varphi_{q_1,N}(s_1)\varphi_{q_2,N}(s_2)\varphi_{q_1,N}(t_3+p\nu)\varphi_{q_2,N}(t_4+r\nu)\}$$

$$\leqslant (i+1)^2 Q_D^4 \mu^{-1} N^{-2} \sum_{p,r=0}^{\mu-1} \sum_{t_1,t_2=1}^{\nu} \sum_{s_1,s_2=1}^{N} |E(x_{s_1}x_{s_2}x_{t_1+p\nu}x_{t_2+r\nu})|$$

$$= (i+1)^2 Q_D^4 \mu^{-1} N^{-2} \sum_{s_1,\ldots,s_4=1}^{N} |E(x_{s_1}x_{s_2}x_{s_3}x_{s_4})|.$$

But the right-hand side of (3.1.18) tends to zero when $N \to \infty$ in view of the following formula:

Under $\tilde{H}_{2s}$, for any $s = 1, 2, \ldots$ and any set of integers $i(1), \ldots, i(2s)$,

$$(3.1.19)\qquad \sum_{i(1)=1}^{N_1} \cdots \sum_{i(2s)=1}^{N_{2s}} |E(x_{i(1)}\cdot\ldots\cdot x_{i(2s)})| \leqslant I_{2s}\varkappa^{(2s)} \Big(\sum_{k=0}^{\infty} |h_k|\Big)^{2s} N_{u(1)}N_{u(3)}\cdot\ldots\cdot N_{u(2s-1)},$$

where I_r is the number of possible groupings of the set $\{i(1), \ldots, i(r)\}$, $\varkappa^{(r)} = \max(|\varkappa_2|, \ldots, |\varkappa_r|)$, $\varkappa_i$ is the i-th cumulant of ε_t, and $N_{u(1)}, \ldots, N_{u(2s)}$ is a permutation of $N_1, \ldots, N_{2s}$ such that $N_{u(1)} \leqslant \ldots \leqslant N_{u(2s)}$.

This formula is a slightly stronger form of (4.11) in [24]: the only difference is that in (4.11) the left-hand side of (3.1.19) above is replaced by

$$\Big|\sum_{i(1)=1}^{N_1} \cdots \sum_{i(2s)=1}^{N_{2s}} E(x_{i(1)}\cdot\ldots\cdot x_{i(2s)})\Big|.$$

It is easy to show that (3.1.19) is true. To start with, let

$$(3.1.20)\quad V_s[i(1), \ldots, i(s)] = \sum_{q=-\infty}^{\infty} h_{i(1)-q} \cdot \ldots \cdot h_{i(s)-q} \qquad (s = 2, 3, \ldots).$$

Thus

$$(3.1.21)\quad \sum_{j(1)=1}^{N_1} \ldots \sum_{j(l)=1}^{N_l} \left|V_l(j(1), \ldots, j(l))\right| \leqslant \min(N_1, \ldots, N_l) \Big(\sum_{k=-\infty}^{\infty} |h_k|\Big)^l.$$

Indeed, given any sequence $u_1, \ldots, u_s$ of indices of the h's in the summand of (3.1.20), the value of any argument of V determines the value of the internal summation variable q and hence the values of the remaining arguments; therefore $\min(N_1, \ldots, N_l)$ is an upper bound of the number of repetitions of any sequence $u_1, \ldots, u_s$ when $i(k)$ belongs to $[1, N_k]$ for $k = 1, \ldots, l$, and (3.1.21) holds. Now, [24] contained a weaker form of (3.1.21), in which the left-hand side of (3.1.21) was replaced by

$$\Big|\sum_{j(1)=1}^{N_1} \ldots \sum_{j(l)=1}^{N_l} V_l(j(1), \ldots, j(l))\Big|,$$

and it was incorrectly stated there in the proof of (4.11) that this weaker formula was used instead of (3.1.21). If, however, this mistake is corrected, it is easily seen that the proof given in [24] yields (3.1.19) (which validates (4.11) in [24], too).

Thus, by (3.1.19) and (3.1.18) for $i = 0, \ldots, D$

$$(3.1.22)\qquad \lim_{N\to\infty} E\Big(\nu\mu^{-\frac{1}{2}} \sum_{r=0}^{\mu-1} M_{\nu,r}\Gamma_{\nu,r,i}\Big)^2 = 0,$$

and (3.1.10) follows, by the Schwarz inequality, from (3.1.12), (3.1.15) and (3.1.22).

Let r_1 and r_2 take independently the value $r+1$ or $r-1$. Since

$$\Big|\sum_{r=1}^{\mu-2} \Gamma_{\nu,r_1,i}\Gamma_{\nu,r_2,i}\Big| \leqslant \sum_{r=0}^{\mu-1} \Gamma_{\nu,r,i}^2,$$

by (3.1.15) for $i = 0, \ldots, D$

$$(3.1.23)\qquad \lim_{N\to\infty} E\Big(\nu\mu^{-\frac{1}{2}} \sum_{r=1}^{\mu-2} \Gamma_{\nu,r_1,i}\Gamma_{\nu,r_2,i}\Big)^2 = 0.$$

Moreover, the proof of

$$(3.1.24)\qquad \lim_{N\to\infty} E\Big(\nu\mu^{-\frac{1}{2}} \sum_{r=1}^{\mu-2} M_{\nu,r_1}\Gamma_{\nu,r_2,i}\Big)^2 = 0 \qquad \text{for } i = 0, \ldots, D$$

is entirely similar to that of (3.1.22). Hence, since $\{y_t\} = \{0\}$ implies

$$\mu^{\frac{1}{2}}(b_{N,i}-b_N)$$
$$= \frac{\nu\mu^{\frac{1}{2}}}{2(\mu-2)}\sum_{r=1}^{\mu-2}[2(M_{\nu,r+1}-M_{\nu,r-1})(\Gamma_{\nu,r+1,i}-\Gamma_{\nu,r-1,i})+(\Gamma_{\nu,r+1,i}-\Gamma_{\nu,r-1,i})^2],$$

(3.1.11) follows, by the Schwarz inequality, from (3.1.23) and (3.1.24). But (3.1.11) entails

$$p\lim_{N\to\infty} b_{N,i} = \sum_{j=-\infty}^{\infty} R_j \quad \text{for } \{y_t\} = \{0\}, \tag{3.1.25}$$

since by Lemma 4.4 in [38]

$$p\lim_{N\to\infty} b_N = \sum_{j=-\infty}^{\infty} R_j \quad \text{for } \{y_t\} = \{0\}. \tag{3.1.26}$$

Finally, by Theorem 8.1 in [38],

$$p\lim_{N\to\infty} \mu^{-\frac{1}{2}} J_N = 0$$

and consequently, by (3.1.8) through (3.1.11), (3.1.26) and (3.1.25), $\{y_t\} = \{0\}$ implies that for $i = 0, \ldots, D$

$$p\lim_{N\to\infty}(J_{N,i}-J_N)$$
$$= p\lim_{N\to\infty}\left\{\frac{\mu^{\frac{1}{2}}(a_N-b_N)}{b_{N,i}}+\frac{\mu^{\frac{1}{2}}[(a_{N,i}-a_N)-(b_{N,i}-b_N)]}{b_{N,i}}-\frac{\mu^{\frac{1}{2}}(a_N-b_N)}{b_N}\right\}$$
$$= p\lim_{N\to\infty}\left[(\mu^{-\frac{1}{2}}J_N)\frac{\mu^{\frac{1}{2}}(b_N-b_{N,i})}{b_{N,i}}\right] = 0;$$

thus Lemma 3.1 is proved.

LEMMA 3.2. *Under* $\tilde{H}_4$ *and* $\sum_{j=1}^{\infty}|jR_j| < +\infty$, *for any* $i = 0, \ldots, D$,

$$\lim_{N\to\infty} E\left[\left(\sum_{q=0}^{D} A_q^2\right)^{-1}\nu^{-\frac{1}{2}}\left(a_{N,i}-\frac{\nu}{\mu}\sum_{r=0}^{\mu-1} Y_{\nu,r}^2\right)\right]^2 = 0, \tag{3.1.27}$$

uniformly with respect to $\{y_t\}$ *belonging to* S.

Proof. According to (3.1.7), the expression in square brackets in (3.1.27) is the sum of

$$\left(\sum_{q=0}^{D} A_q^2\right)^{-1}\frac{\nu^{\frac{1}{2}}}{\mu}\sum_{r=0}^{\mu-1} M_{\nu,r}^2, \tag{3.1.28}$$

$$\left(\sum_{q=0}^{D} A_q^2\right)^{-1}\frac{2\nu^{\frac{1}{2}}}{\mu}\sum_{r=0}^{\mu-1} M_{\nu,r}Y_{\nu,r}, \tag{3.1.29}$$

$$\left(\sum_{q=0}^{D} A_q^2\right)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu} \sum_{r=0}^{\mu-1} \Gamma_{\nu,r,i}^2, \tag{3.1.30}$$

$$\left(\sum_{q=0}^{D} A_q^2\right)^{-1} \frac{2\nu^{\frac{1}{2}}}{\mu} \sum_{r=0}^{\mu-1} M_{\nu,r} \Gamma_{\nu,r,i}, \tag{3.1.31}$$

and

$$\left(\sum_{q=0}^{D} A_q^2\right)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu} \sum_{r=0}^{\mu-1} \Gamma_{\nu,r,i} Y_{\nu,r}. \tag{3.1.32}$$

By the Schwarz inequality, it suffices, therefore, to show that (3.1.28) through (3.1.32) tend to zero in the mean uniformly in $\{y_t\} \in S$. Moreover, since $\sum_{q=0}^{D} A_q^2$ is uniformly bounded from below, the corresponding factor in (3.1.28), (3.1.30), and (3.1.31) can be neglected.

But by (4.1) in [38] $E\left(\nu^{\frac{1}{2}}\mu^{-1}\sum_{r=0}^{\mu-1} M_{\nu,r}^2\right) = O(\nu^{-\frac{1}{2}})$, while by Lemma 4.2 in [38] $\operatorname{var}\left(\nu\mu^{-1}\sum_{r=0}^{\mu-1} M_{\nu r}^2\right) = O(\mu^{-1})$; thus

$$\lim_{N\to\infty} E\left(\nu^{\frac{1}{2}}\mu^{-1}\sum_{r=0}^{\mu-1} M_{\nu,r}^2\right) = 0. \tag{3.1.33}$$

Furthermore, according to an argument serving to prove Theorem 9.1 in [38], we can write

$$E\left(\frac{2\nu^{\frac{1}{2}}}{\mu}\sum_{r=0}^{\mu-1} Y_{\nu,r} M_{\nu,r}\right)^2 \leqslant \frac{12}{\mu^{\frac{1}{2}}\nu} \sum_{q=-\infty}^{\infty} |R_q| \left(\frac{\nu}{\mu}\sum_{r=0}^{\mu-1} Y_{\nu,r}^2\right),$$

while an argument in the proof of Theorem 11.1 in the same paper shows that

$$\frac{\nu}{\mu}\sum_{r=0}^{\mu-1} Y_{\nu,r}^2 \leqslant \sum_{q=0}^{D} A_q^2(1+\varepsilon),$$

where $\varepsilon \to 0$ uniformly. Hence, taking into account the cancellation of $\sum_{q=0}^{D} A_q^2$, we find that (3.1.29) also tends to 0 in the mean uniformly.

By (3.1.15) and (3.1.22), (3.1.30) and (3.1.31) tend to zero uniformly in the mean. Turning to (3.1.32), we need the following inequalities obtained from (3.1.5), (3.1.7) and (3.1.17):

$$\max_{0\leqslant r\leqslant \mu-1} |Y_{\nu,r}| \leqslant \max_{1\leqslant t\leqslant N} |y_t| \leqslant Q_D \sqrt{(D+1)\sum_{q=0}^{D} A_q^2} \quad \text{for } \{y_t\} \in S \tag{3.1.34}$$

and

$$\max_{0\leqslant r\leqslant \mu-1} |\Gamma_{\nu,r,i}| \leqslant \max_{1\leqslant l\leqslant N} |\gamma_{l,i}| \leqslant Q_D \sqrt{(D+1)\sum_{q=0}^{D} a_q^2} \quad \text{for } i = 0, \ldots, D. \tag{3.1.35}$$

These inequalities imply

$$E\Big[\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu} \sum_{r=0}^{\mu-1} \Gamma_{\nu,r,i} Y_{\nu,r}\Big]^2 \leqslant \Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} Q_D^4 (D+1)^2 E\Big(\nu \sum_{q=0}^{D} a_q^2\Big) \quad \text{for } i = 0, \ldots, D. \tag{3.1.36}$$

By (3.1.13), $E\big(\nu \sum_{q=0}^{D} a_q^2\big)$ and, therefore, the right-hand side of (3.1.36), tend to zero uniformly in $\{y_l\} \in S$. This completes the proof of Lemma 3.2.

LEMMA 3.3. *Under $\tilde{H}_4$, for any $i = 0, \ldots, D$,*

$$\lim_{N\to\infty} E\Big[\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} \nu^{-\frac{1}{2}} b_{N,i}\Big]^2 = 0$$

uniformly in $\{y_l\} \in S$.

Proof. By (3.1.7), $\big(\sum_{q=0}^{D} A_q^2\big)^{-1} \nu^{-\frac{1}{2}} b_{N,i}$ is a sum of

$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} \frac{\nu^{\frac{1}{2}}}{2(\mu-2)} \sum_{r=1}^{\mu-2} [(M_{\nu,r+1} - M_{\nu,r-1}) + (Y_{\nu,r+1} - Y_{\nu,r-1})]^2, \tag{3.1.37}$$

$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu-2} \sum_{r=1}^{\mu-2} [(M_{\nu,r+1} - M_{\nu,r-1})(\Gamma_{\nu,r+1,i} - \Gamma_{\nu,r-1,i}) + \tfrac{1}{2}(\Gamma_{\nu,r+1,i} - \Gamma_{\nu,r-1,i})^2], \tag{3.1.38}$$

and

$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu-2} \sum_{r=1}^{\mu-2} [(\Gamma_{\nu,r+1,i} - \Gamma_{\nu,r-1,i})(Y_{\nu,r+1} - Y_{\nu,r-1})]. \tag{3.1.39}$$

By Lemma 4.4 in [38], $\frac{\nu^{\frac{1}{2}}}{\mu-2} \sum_{r=1}^{\mu-2} (M_{\nu,r+1} - M_{\nu,r-1})^2$ tends to 0 in the mean. Furthermore, according to an argument in the proof of Theorem 11.1 in the same paper,

$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} \frac{\nu^{\frac{1}{2}}}{2(\mu-2)} \sum_{r=1}^{\mu-2} (Y_{\nu,r+1} - Y_{\nu,r-1})^2 < 2\frac{\nu^{\frac{1}{2}}}{\mu^2} \big(3^{-1}(D+3)^{[3]} + \varepsilon\big),$$

where $\varepsilon \to 0$ uniformly when $N \to \infty$. Therefore, by the Schwarz inequality, (3.1.37) tends to 0 in the mean uniformly in $\{y_t\} \in S$. The same result is also immediately obtained for (3.1.38) in view of (3.1.23) and (3.1.24). Finally, expanding the Γ's and Y's in (3.1.39), we find that

$$E\left[\left(\sum_{q=0}^{D} A_q^2\right)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu} \sum_{r=1}^{\mu-2} \Gamma_{\nu,r_1,i}\, Y_{\nu,r_2}\right]^2$$

is not greater than the right-hand side of (3.1.36) which was shown to tend to zero uniformly. Hence this is also true for (3.1.39) and Lemma 3.3 follows by the Schwarz Inequality.

COROLLARY 3.4. *Under* $\tilde{H}_4$, *if* $\sum_{j=1}^{\infty} |jR_j| < +\infty$ *and* $i = 0, \ldots, D$, $\operatorname{plim}_{N\to\infty} J_{N,i} = +\infty$ *uniformly in* $\{y_t\} \in S$.

Proof. By an argument in the proof of Theorem 9.1 in [38], for any $\{y_t\} \in S$,

$$\left(\sum_{q=0}^{D} A_q^2\right)^{-1} \frac{\nu^{\frac{1}{2}}}{\mu} \sum_{r=0}^{\mu-1} Y_{\nu,r}^2 \geqslant \nu^{\frac{1}{2}}(1-|\varepsilon|),$$

where $\varepsilon \to 0$ uniformly when $N \to \infty$. Therefore, by Lemma 3.2,

$$\operatorname{plim}_{N\to\infty} \left(\sum_{q=0}^{D} A_q^2\right)^{-1} \nu^{-\frac{1}{2}} a_{N,i} = +\infty,$$

uniformly in $\{y_t\} \in S$, and hence by Lemma 3.3 $a_{N,i}/b_{N,i}$ tends in probability to $+\infty$, still uniformly in S. Thus Corollary 3.4 follows by (3.1.8).

COROLLARY 3.5. *Under* $\tilde{H}_4$, *if* $\sum_{j=1}^{\infty} |jR_j| < +\infty$ *and if* $\hat{m}$ *is defined by the* J^**-rule, then* $\{\hat{y}_{t,\hat{m}}\}$ *is a* J^**-estimator in* S' *and* (2.8) *holds.*

Proof. By the definition of S', if $\{\sum_{q=0}^{m} a_q t^q\}$ is any element of S', then, for any constants $\beta_0, \ldots, \beta_{m-1}$, $\{a_m t^m + \sum_{q=0}^{m-1} \beta_q t^q\} \in S'$, since the coefficient of $\varphi_{m,N}(t)$ is the same in both polynomials. On the other hand, if A_j $(j = 0, \ldots, m)$ denote the coefficients of $\varphi_{j,N}(t)$ in the polynomial $\{\sum_{q=0}^{m} a_q t^q\}$, then, for any given N and $i < m$, there exist constants $\beta_0, \ldots, \beta_{m-1}$ such that

$$\sum_{q=i+1}^{m} A_q \varphi_{q,N}(t) \equiv a_m t^m + \sum_{q=0}^{m-1} \beta_q t^q.$$

Consequently, by (3.1.6) and Corollary 3.4, for $i < m$ and for any $\varepsilon > 0$

and any positive number M there exists an N' such that for any $\{y_t\} \in S' \subset S$ and $N > N'$

$$P\{J_N^*(\{z_t - \hat{y}_{t,i}\}_N) > M\} = P\Big\{J_N^*\Big(\Big\{x_t + \gamma_{t,i} + \sum_{q=i+1}^{m} A_q \varphi_{q,N}(t)\Big\}_N\Big) > M\Big\}$$
$$> 1 - \varepsilon.$$

Hence by the definition of ω_a for the test J^* condition (2.6) is satisfied. Moreover, (3.1.6) and Lemma 3.1 imply (2.7). Hence Corollary 3.5 follows from Theorem 2.1, since the remaining assumptions of the latter theorem in the case of the test J^* are satisfied in view of Theorem 11.1 and Proposition 8.1 in [38].

The same result will now be obtained for the test T^*. Let

$$\begin{aligned}
C_{j,N}^* &= (N - |j|)^{-1} \sum_{t=1}^{N-|j|} z_t z_{t+|j|} \qquad (|j| = 0, 1, \ldots, N-1),\\
S_N^* &= \tfrac{1}{2} \sum_{j=1-N}^{N-1} \Big(1 - \frac{|j|}{N}\Big)^2 C_{j,N}^{*2},\\
C_{0,\nu,p}^* &= \nu^{-1} \sum_{t=1}^{\nu} z_{t+p\nu}^2 \qquad (p = 0, 1, \ldots, \mu-1),\\
U_N^* &= \nu(\mu-1)^{-1} \sum_{p=0}^{\mu-1} (C_{0,\nu,p}^* - C_{0,N}^*)^2.
\end{aligned} \tag{3.1.40}$$

The test function of the test T^* with arguments $\{z_t\}_N$ will be denoted by $T_N^*(k)$, k being a real number from the interval $[-2, \varkappa_4 \varkappa_2^{-2}]$; according to [36],

$$T_N^*(k) = \Big(\frac{\mu}{2}\Big)^{\frac{1}{2}} \Big[1 - \frac{2S_N^* + kC_{0,N}^{*2}}{U_N^*}\Big]. \tag{3.1.41}$$

Let $C_{j,N}$, S_N, $C_{0,\nu,p}$, U_N and $T_N(k)$ denote the expressions which are formed with $\{x_t\}_N$ in the same way as the corresponding starred expressions are formed with $\{z_t\}_N$, and let $C_{j,N}^{(i)}$, $S_{N,i}$, $C_{0,\nu,p}^{(i)}$, $U_{N,i}$ and $T_{N,i}(k)$ denote the corresponding expressions formed with $\{x_t + \gamma_{t,i} + y_t\}_N$.

The set ω_a is defined here as $\{a \in R\colon a < \zeta_a\}$, where $0 < a < \frac{1}{2}$ and ζ_a is given by (3.1.4). Of course, the left-hand side of (2.1) (i. e. the asymptotic probability of claiming the existence of trend when it is absent), which was denoted before by β, depends on k. By Proposition 7.1 in [36], $T_N(\varkappa_4 \varkappa_2^{-2})$ is normally distributed with a zero mean and a unit variance and hence in this case $\beta = a$. For $k < \varkappa_4 \varkappa_2^{-2}$ $\beta = 0$, since by (2.1) in [24] for any $k < \varkappa_4 \varkappa_2^{-2}$ $\operatorname*{plim}_{N\to\infty} T_N(k) = +\infty$. Thus condition (2.1) is satisfied

for $-2 \leqslant k \leqslant \varkappa_4 \varkappa_2^{-2}$. In practice, the value of $\varkappa_4 \varkappa_2^{-2}$ is usually unknown, so that we choose $k = -2$ since it is known that $\varkappa_4 \varkappa_2^{-2} \geqslant -2$. This choice does not affect condition (2.1); of course, it will reduce the power of the test for finite values of N. However, by Theorem 3.2 in [24], the power of the test $T^*(k)$ tends to 1 with $N \to \infty$ for any constant k and any trend which belongs to a class $\mathscr{M}$ composed of all trends $\{y_t\}$ such that

$$\lim_{N\to\infty} N^{-3}(\max_{1\leqslant t\leqslant N}|y_t|)^{-4} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2 = Q,$$

where Q is a positive constant. Thus the test $T^*(k)$ and this class of trends satisfies (2.2).

By Lemma 3.1 in [24], all real polynomials restricted to integral values of the argument belongs to $\mathscr{M}$. Moreover, the second part of Theorem 4.7 in [24] is a counterpart of condition (2.6) in case of the test T^* with S' replaced by all polynomials. However, the proof of Theorem 4.7 overlooked the fact that $\sum_{q=i+1}^{m} A_q \varphi_{q,N}(t)$ depends on N, so that a uniform stochastic convergence with respect to some class of polynomials was required, exactly like in the case of the test J^*.

There is no difficulty with the counterpart of Lemma 3.1 for the test T^*, since the question of uniformity of convergence does not arise when in fact there is no trend. Hence, by the first part of Theorem 4.7 in [24], we have

LEMMA 3.6. *Under $\tilde{H}_8$, if $\{y_t\} = \{0\}$, $\operatorname{plim}_{N\to\infty}[T_{N,i}(k) - T_N(k)] = 0$ for any constant k and for any i.*

Lemmas 3.8 and 3.9 jointly form the counterpart of Lemma 3.2, and Lemma 3.10 is the counterpart of Lemma 3.3. The following lemma ensures that, if

$$\Pi = [N^a]+1, \quad \tfrac{1}{2} < a < 1, \tag{3.1.42}$$

then the non-stochastic part of $\Pi^{-1}\big(\sum_{q=0}^{D} A_q^2\big)^{-2} S_{N,i}$ is uniformly bounded from below by a positive number for sufficiently large values of N.

LEMMA 3.7. *There exist an N' and a $\Lambda > 0$ such that for $N > N'$ and for any $\{y_t\} \in S$*

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_2^q\Big)^{-2} N^{-2} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2 > \Lambda.$$

Proof. By (3.1.42),

$$\sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2 \geqslant \Big(\sum_{t=1}^{N} y_t^2\Big)^2 + 2\sum_{j=1}^{\Pi} \Big(\sum_{t=1}^{N-j} y_t y_{t+j}\Big)^2.$$

The last sum can be represented as

$$\sum_{j=1}^{\Pi}\Big(\sum_{t=1}^{N-j} y_t(y_t+y_{t+j}-y_t)\Big)^2 = \sum_{j=1}^{\Pi}\Big(\sum_{t=1}^{N-j} y_t^2+\sum_{s=1}^{N-j} y_s(y_{s+j}-y_s)\Big)^2$$
$$\geqslant \sum_{j=1}^{\Pi}\Big(\sum_{t=1}^{N-j} y_t^2\Big)^2+2\sum_{j=1}^{\Pi}\sum_{t=1}^{N-j} y_t^2\sum_{s=1}^{N-j} y_s(y_{s+j}-y_s).$$

The first sum in the last expression can be further transformed, namely

$$\sum_{j=1}^{\Pi}\Big(\sum_{t=1}^{N-j} y_t^2\Big)^2 = \sum_{j=1}^{\Pi}\Big(\sum_{t=1}^{N} y_t^2-\sum_{s=N-j+1}^{N} y_s^2\Big)^2$$
$$\geqslant \sum_{j=1}^{\Pi}\Big(\sum_{t=1}^{N} y_t^2\Big)^2-2\sum_{j=1}^{\Pi}\sum_{t=1}^{N} y_t^2\sum_{s=N-j+1}^{N} y_s^2.$$

Consequently, since, for $\{y_t\}\in S$, $N^{-1}\sum_{t=1}^{N} y_t^2 = \sum_{q=0}^{D} A_q^2$,

$$\text{(3.1.43)}\qquad \Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2}N^{-2}\sum_{|j|<N}\Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2$$
$$\geqslant \Pi^{-1}+2-4\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1}N^{-1}\sum_{j=1}^{\Pi}\sum_{t=N-j+1}^{N} y_t^2+$$
$$+4\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2}N^{-2}\sum_{j=1}^{\Pi}\sum_{t=1}^{N-j} y_t^2\sum_{s=1}^{N-j} y_s(y_{s+j}-y_s).$$

Apart from $\Pi^{-1}+2$, the right-hand side of (3.1.43) is a sum of two terms; we shall show that these terms tend to zero uniformly in $\{y_t\}\in S$.

For the first term, by (3.1.34),

$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1}N^{-1}\Pi^{-1}\sum_{j=1}^{\Pi}\sum_{t=N-j+1}^{N} y_t^2 \leqslant N^{-1}\Pi^{-1}(D+1)Q_D^2\sum_{j=1}^{\Pi} j,$$

which tends to zero uniformly in $\{y_t\}\in S$, since by (3.1.42) $\Pi\cdot N^{-1}\to 0$ when $N\to\infty$.

Dealing with the second term, we use

$$\text{(3.1.44)}\qquad \hat{P}_q(u) = (2q+1)^{\frac{1}{2}}\sum_{r=0}^{q}(-1)^{q-r}\frac{(q+r)!}{(r!)^2(q-r)!}u^r \qquad (0\leqslant u\leqslant 1).$$

The derivatives of $\hat{P}_q(u)$ are bounded and it is known (e. q. [38], Proposition 10.2) that for any $0\leqslant q\leqslant D$ and any $1\leqslant t\leqslant N$ $\left|N\frac{d}{dt}\varphi_{q,N}(t)\right|\leqslant C_D+\varepsilon$, where $C_D = \max\limits_{q=0,\ldots,D}\ \max\limits_{0\leqslant u\leqslant 1}|\hat{P}_q'(u)|$ and ε tends to zero uniformly in t.

Hence for $1 \leqslant s \leqslant N-j$ there exist such $\xi_0, \ldots, \xi_D$ in $(s, s+j)$ that

$$(3.1.45) \qquad |y_{s+j}-y_s| \leqslant \sum_{q=0}^{D} |A_q|\,|\varphi_{q,N}(s+j)-\varphi_{q,N}(s)|$$
$$= \sum_{q=0}^{D} |A_q j \dot{\varphi}_{q,N}(\xi_q)| \leqslant N^{-1} \sum_{q=0}^{D} |A_q|\,|j|(C_D+\varepsilon).$$

Consequently, the second term in (3.1.43) tends uniformly to zero, since by (3.1.45) and (3.1.17)

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} N^{-2} \Big|\sum_{j=1}^{\Pi}\sum_{t=1}^{N-j} y_t^2 \sum_{s=1}^{N-j} y_s(y_{s+j}-y_s)\Big|$$
$$\leqslant \Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} N^{-1} \sum_{j=1}^{\Pi}\sum_{s=1}^{N-j}\sum_{p,q,=0}^{D} |A_p A_q|\,|\varphi_{p,N}(s)|\,N^{-1} j(C_D+\varepsilon)$$
$$\leqslant \Pi^{-1} N^{-2} \sum_{j=1}^{\Pi} (N-j)j(D+1)(C_D+\varepsilon)Q_D \leqslant \Pi N^{-1}(D+1)Q_D(C_D+\varepsilon).$$

Hence for any $\varepsilon > 0$ there exists an N' such that for $N > N'$ and any $\{y_t\} \in S$

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2 > 2-\varepsilon,$$

which completes the proof of Lemma 3.7.

LEMMA 3.8. *Under $\tilde{H}_8$, for any $i = 0, \ldots, D$,*

$$(3.1.46) \qquad \Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1}\Big[2S_{N,i} - N^{-2} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2\Big]$$

tends to zero in the mean, uniformly in $\{y_t\} \in S$.

Proof. By the definition of $S_{N,i}$, (3.1.46) is equal to a sum of

$$(3.1.47) \qquad \Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N} \Big\{\sum_{t=1}^{N-|j|} [x_t(x_{t+|j|}+y_{t+|j|}+\gamma_{t+|j|,i})+$$
$$+ y_t(x_{t+|j|}+\gamma_{t+|j|,i}) + \gamma_{t,i}(x_{t+|j|}+y_{t+|j|}+\gamma_{t+|j|,i})]\Big\}^2$$

and of

$$(3.1.48) \qquad 2\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N}\sum_{t=1}^{N-|j|} y_t y_{t+|j|} \sum_{s=1}^{N-|j|} [x_s(x_{s+|j|}+y_{s+|j|}+$$
$$+\gamma_{s+|j|,i}) + y_s(x_{s+|j|}+\gamma_{s+|j|,i}) + \gamma_{s,i}(x_{s+|j|}+y_{s+|j|}+\gamma_{s+|j|,i})].$$

To prove Lemma 3.9, it suffices to show that (3.1.47) and (3.1.48) tend to zero in the mean, uniformly in $\{y_t\} \in S$. But (3.1.47) is a sum of

eight terms corresponding to the squares of eight expressions in square brackets in (3.1.47), namely

$$(3.1.49)\qquad \Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} x_t x_{t+|j|}\Big)^2, \ldots,$$
$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} \gamma_{t,i}\gamma_{t+|j|,i}\Big)^2,$$

and of all the terms corresponding to particular mixed products of these expressions, and it is easy to see that, by the Schwarz inequality repeatedly used, the uniform convergence in the mean to 0 follows for (3.1.47) from that for each term in (3.1.49); of course, the coefficient $\big(\sum_{q=0}^{D} A_q^2\big)^{-2}$ can be omitted for those terms of (3.1.49) which do not depend on y's. Similarly, the uniform convergence in the mean to zero follows for (3.1.48) from that for each of the following terms which add up to one-half of (3.1.48):

$$(3.1.50)\qquad \Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N} \sum_{s,t=1}^{N-|j|} y_t y_{t+|j|} x_s x_{s+|j|}, \ldots,$$
$$\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N} \sum_{s,t=1}^{N-|j|} y_t y_{t+|j|} \gamma_{s,i}\gamma_{s+|j|,i}.$$

The first term of (3.1.49) is equal to

$$2\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} S_N$$

and it tends uniformly to zero in the mean since ([22])

$$\lim_{N\to\infty} ES_N^2 = \Big(\sum_{j=-\infty}^{\infty} R_j^2\Big)^2.$$

By an argument in the proof of Theorem 2.1 in [24], the expectation of the square of the second and the fourth term in (3.1.49) and of the first, the second and the fourth term in (3.1.50) (i. e. of an yterm constructed with x's and y's only) is bounded by

$$C\Big(\sum_{q=0}^{D} A_q^2\Big)^{-4} \Pi^{-2} N (\max_{1\leqslant t\leqslant N} |y_t|)^6,$$

where C is a constant which depends on the parameters of the process $\{x_t\}$ only. But $\Pi^{-2}N$ tends to zero in view of (3.1.42). Therefore, by (3.1.34), these terms tend to zero in the mean, uniformly in $\{y_t\} \in S$.

The third, the sixth and the eighth term in (3.1.49) (i. e. terms con-

structed with α's and γ's only), apart from the normalizing coefficient, were considered in the proof of Lemma 4.5 in [24] and shown to tend to zero in the mean. There remain the fifth and the seventh term of (3.1.49) and the third, the fifth, the sixth, the seventh and the eighth term of (3.1.50).

By (3.1.13), the fifth term of (3.1.49) converges to zero in the mean uniformly, since, by (3.1.34) and (3.1.35), for $i = 0, \ldots, D$

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} N^{-2} \sum_{|j|<N} \Big(\sum_{s,t=1}^{N-|j|} y_s y_t \gamma_{s+|j|,i}\gamma_{t+|j|,i}\Big)$$
$$\leqslant \Big(\sum_{q=0}^{D} A_q^2\Big)^{-1}(D+1)^2 Q_D^4 \sum_{|j|<N} (N-|j|)^2 N^{-2}\Pi^{-1}\sum_{q=0}^{D} a_q^2,$$

and a similar argument leads to the same result for the seventh term of (3.1.49).

The third term of (3.1.50) converges to 0 in the mean uniformly in view of formula (3.1.19) which implies

$$\lim_{N\to\infty} \Pi^{-2} N^{-2} \sum_{s_1,\ldots,s_4=1}^{N} |E(x_{s_1}\cdot\ldots\cdot x_{s_4})| = 0,$$

and in view of the following inequality obtained from (3.1.5) and (3.1.34):

$$E\Big\{\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} N^{-2} \sum_{|j|<N}\sum_{s,t=1}^{N-|j|} y_t y_{t+|j|} x_s \gamma_{s+|j|,i}\Big\}^2$$
$$= \Pi^{-2}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-4} N^{-4} \sum_{|j|,|l|<N}\sum_{s,t=1}^{N-|j|}\sum_{u,w=1}^{N-|l|} y_t y_{t+|j|} y_w y_{w+|l|} \times$$
$$\times E\Big(x_s x_u \sum_{p,q=0}^{i} a_p a_q\Big)\varphi_{p,N}(s+|j|)\varphi_{q,N}(w+|l|)$$
$$\leqslant \Pi^{-2}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} N^{-6} Q_D^6 (D+1)^2 \Big[\sum_{|j|<N}(N-|j|)\Big]^2 \times$$
$$\times \sum_{s_1,\ldots,s_4=1}^{N}\sum_{p,q=0}^{i}\Big|E(x_{s_1}\cdot\ldots\cdot x_{s_4})\varphi_{p,N}(s_3)\varphi_{q,N}(s_4)\Big|$$
$$\leqslant \Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} Q_D^8 (D+1)^4 \Big[N^{-4}\sum_{|j|<N}(N-|j|)^2\Big] \times$$
$$\times \Big[\Pi^{-2} N^{-2} \sum_{s_1,\ldots,s_4=1}^{N} |E(x_{s_1}\cdot\ldots\cdot x_{s_4})|\Big].$$

The reasoning for the sixth term of (3.1.50) is the same as that for the third one; moreover, the fifth term converges uniformly to zero in

the mean since by (3.1.42) and (3.1.13)

$$N^2\Pi^{-2}E\Big(\sum_{q=0}^{D} a_q^2\Big)$$

tends to zero and since by (3.1.34) and (3.1.35), for $i = 0, \ldots, D$,

$$E\Big\{\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} N^{-2} \sum_{|j|<N} \sum_{s,t=1}^{N-|j|} y_t y_{t+|j|} y_s \gamma_{s+|j|,i}\Big\}^2$$
$$\leqslant \Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} (D+1)^4 Q_D^8 \Big[N^{-6}\Big(\sum_{|j|<N} (N-|j|)\Big)^2\Big]\Big[N^2\Pi^{-2}E\Big(\sum_{q=0}^{D} a_q^2\Big)\Big].$$

Obviously, the same reasoning can be applied also to the seventh term. Finnally, the proof for the eighth term of (3.1.50) follows from (3.1.13) and from the following inequality based on (3.1.34) and (3.1.35):

$$\Big|\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Pi^{-1} N^{-2} \sum_{|j|<N} \sum_{s,t=1}^{N-|j|} y_t y_{t+|j|} \gamma_{s,i} \gamma_{s+|j|,i}\Big|$$
$$\leqslant \Big(\sum_{q=0}^{D} A_q^2\Big)^{-1} (D+1)^2 Q_D^4 \Big[N^{-3} \sum_{|j|<N} (N-|j|)^2\Big]\Big[\Pi^{-1} N \sum_{q=0}^{D} a_q^2\Big].$$

LEMMA 3.9. *Under $\tilde{H}_8$, for any $i = 0, \ldots, D$,*

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} C_{0,N}^{(i)2}$$

tends to zero in the mean, uniformly in $\{y_t\} \in S$.

Proof. It should be noticed that $S_{N,i}$ reduces to $C_{0,N}^{(i)2}$ if the summation with respect to j is omitted and j is replaced by 0. Hence the reasoning in the proof of Lemma 3.8 with obvious simplifications can be applied to show that

$$\lim_{N\to\infty} E\Big\{\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2}\Big[C_{0,N}^{(i)2} - N^{-2}\Big(\sum_{t=1}^{N} y_t^2\Big)^2\Big]\Big\}^2 = 0$$

uniformly in $\{y_t\} \in S$. But

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} N^{-2}\Big(\sum_{t=1}^{N} y_t^2\Big)^2 = \Pi^{-1} \tag{3.1.51}$$

and hence Lemma 3.9 is proved by the Schwarz inequality.

LEMMA 3.10. *Under $\tilde{H}_8$, for any $i = 0, \ldots, D$,*

$$\lim_{N\to\infty} E\Big[\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} U_{N,i}\Big]^2 = 0$$

uniformly in $\{y_t\} \in S$.

Proof. By definition,

$$(3.1.52)\qquad U_{N,i} = \frac{\nu}{\mu-1}\sum_{p=0}^{\mu-1}\Big\{\nu^{-1}\sum_{t=1}^{\nu}(x_{t+p\nu}+y_{t+p\nu}+\gamma_{t+p\nu,i})^2 - N^{-1}\sum_{t=1}^{N}(x_{t+p\nu}+y_{t+p\nu}+\gamma_{t+p\nu,i})^2\Big\}^2$$

$$= \frac{\nu}{\mu-1}\sum_{p=0}^{\mu-1}\Big\{\Big[\nu^{-1}\sum_{t=1}^{\nu}(x_{t+p\nu}+y_{t+p\nu})^2 - N^{-1}\sum_{t=1}^{N}(x_{t+p\nu}+y_{t+p\nu})^2\Big] + \Big[\nu^{-1}\sum_{t=1}^{\nu}(2x_{t+p\nu}\gamma_{t+p\nu,i}+\gamma_{t+p\nu,i}^2) - N^{-1}\sum_{t=1}^{N}(2x_{t+p\nu}\gamma_{t+p\nu,i}+\gamma_{t+p\nu,i}^2)\Big] + 2\Big[\nu^{-1}\sum_{t=1}^{\nu}y_{t+p\nu}\gamma_{t+p\nu,i} - N^{-1}\sum_{t=1}^{N}y_{t+p\nu}\gamma_{t+p\nu,i}\Big]\Big\}^2.$$

The term corresponding to the square of the expression in the first square bracket in the right-hand side of (3.1.52) is equal to U_N^*. By an argument in the proof of Theorem 2.1 in [24],

$$EU_N^* \leqslant C\nu\,(\max_{1\leqslant t\leqslant N}|y_t|)^4$$

and

$$\operatorname{var} U_N^* \leqslant C'\nu\mu^{-1}\,(\max_{1\leqslant t\leqslant N}|y_t|)^4,$$

where C and C' are constants which depend on the parameters of the process $\{x_t\}$ only. Since by (3.1.42) $\nu\Pi^{-1}$ tends to zero when $N\to\infty$, we have by (3.1.34)

$$(3.1.53)\qquad \lim_{N\to\infty} E\Big[\Pi^{-1}\Big(\sum_{q=0}^{D}A_q^2\Big)^{-2}U_N^*\Big]^2 = 0 \quad \text{uniformly in } \{y_t\}\in S.$$

Moreover, using one of the results in the proof of Lemma 4.1 in [24], we find

$$(3.1.54)\qquad \lim_{N\to\infty} E\Big\{\frac{\nu}{\mu-1}\sum_{p=0}^{\mu-1}\Big[\nu^{-1}\sum_{t=1}^{\nu}(2x_{t+p\nu}\gamma_{t+p\nu,i}+\gamma_{t+p\nu,i}^2) - N^{-1}\sum_{t=1}^{N}(2x_{t+p\nu}\gamma_{t+p\nu,i}+\gamma_{t+p\nu,i}^2)\Big]^2\Big\} = 0;$$

since this expression does not depend on y's, the question of the uniformity of convergence does not arise.

Finally, by (3.1.34) and (3.1.35), for $i = 0, \ldots, D$,

$$\Big|\Pi^{-1}\Big(\sum_{q=0}^{D}A_q^2\Big)^{-2}\frac{\nu}{\mu-1}\sum_{p=0}^{\mu-1}\nu^{-2}\sum_{t,s=1}^{\nu}y_{t+p\nu}y_{s+p\nu,i}\gamma_{t+p\nu,i}\gamma_{s+p\nu,i}\Big| \leqslant \Big(\sum_{q=0}^{D}A_q^2\Big)^{-1}Q_D^4(D+1)^2\Big[\Pi^{-1}\nu\sum_{q=0}^{D}a_q^2\Big]$$

and the same bound obviously applies to

$$\left|\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \frac{\nu}{\mu-1} \sum_{p=0}^{\mu-1} N^{-2} \sum_{t,s=1}^{N} y_{t+p\nu} y_{s+p\nu} \gamma_{t+p\nu,i} \gamma_{s+p\nu,i}\right|,$$

so that by (3.1.13) and by the Schwarz inequality

$$\text{(3.1.55)} \qquad \lim_{N\to\infty} E\Big\{\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \frac{\nu}{\mu-1} \sum_{p=0}^{\mu-1} \Big[\nu^{-1} \sum_{t=1}^{\nu} y_{t+p\nu}\gamma_{t+p\nu,i} - N^{-1} \sum_{t=1}^{N} y_{t+p\nu}\gamma_{t+p\nu,i}\Big]^2\Big\}^2 = 0,$$

uniformly in $\{y_t\} \in S$. Thus Lemma 3.10 follows from (3.1.52) through (3.1.55) by a repeated use of the Schwarz inequality.

COROLLARY 3.11. *Under $\tilde{H}_8$, for any $i = 0, \ldots, D$ and any constant k, $\operatorname{plim}_{N\to\infty} T_{N,i}(k) = -\infty$ uniformly in $\{y_t\} \in S$.*

Proof. By Lemmas 3.8, 3.9 and 3.10,

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} \Big[S_{N,i} + kC_{0,N}^{(i)2} - \tfrac{1}{2} N^{-2} \sum_{|j|<N} \Big(\sum_{t=1}^{N-|j|} y_t y_{t+|j|}\Big)^2\Big]$$

and

$$\Pi^{-1}\Big(\sum_{q=0}^{D} A_q^2\Big)^{-2} U_{N,i}$$

tend to 0 uniformly in the mean and hence in probability. Hence, by Lemma 3.7,

$$\operatorname{plim}_{N\to\infty} \frac{S_{N,i} + kC_{0,N}^{(i)2}}{U_{N,i}} = +\infty$$

uniformly in $\{y_t\} \in S$ and Corollary 3.11 follows by the definition of $T_{N,i}$.

COROLLARY 3.12. *Under $\tilde{H}_8$, for any constant k, $\operatorname{plim}_{N\to\infty} T_N^*(k) = -\infty$ uniformly in $\{y_t\} \in S$.*

Proof. The counterparts of Lemmas 3.8, and 3.9 and 3.10 with $S_{N,i}$, $C_{0,N}^{(i)}$ and $U_{N,i}$ replaced by S_N^*, $C_{0,N}^*$ and U_N^* are obviously true, the proofs being only simplified. Consequently, by Lemma 3.7, the corollary follows by an argument similar to that used in the preceding proof.

COROLLARY 3.13. *Under $\tilde{H}_8$, if $\hat{m}$ is defined by the T^*-rule for any $k \in [-2, \varkappa_1\varkappa_2^{-2}]$, then $\{\hat{y}_{t,\hat{m}}\}$ is a T^*-estimator in S' and (2.8) holds.*

Proof. Corollary 3.11 is the counterpart of Corollary 3.4 with the test J^* replaced by the test T^*. Therefore, proceeding exactly like in the proof of Corollary 3.5, we state that by (3.1.6), Corollary 3.11 and the definition of ω_a for the test T^*, condition (2.6) is satisfied by the latter test. Moreover, (3.1.6) and Lemma 3.6 imply (2.7). Hence Corollary 3.13 follows from Theorem 2.1, since the remaining assumptions of the latter theorem in the case of the test T^* are satisfied in view of Proposition 7.1 in [36], formula (2.2) in [24] and Lemma 3.1 in [24].

3.2 Interval estimation in S'. In the preceding section we proved that condition (2.8) was satisfied for any chosen D when $\hat{m}$ was defined by the J^*- or T^*-rule. Now we shall show that (2.8) leads to an interval estimation of trend based on $\{\hat{y}_{t,\hat{m}}\}$. We start with the following lemma which depends on the polynomials $\hat{P}_q(a)$ defined by (3.1.44).

LEMMA 3.14. *Under $\tilde{H}_2$, for any $\{y_t\} \in S$ and any real $a, b \in [0, 1]$*

$$\lim_{\substack{N\to\infty;\, t/N\to a;\\ s/N\to b}} N\operatorname{cov}(y_s-\hat{y}_{s,m}, y_t-\hat{y}_{t,m}) = \sum_{k=-\infty}^{\infty} R_k \sum_{q=0}^{m} \hat{P}_q(a)\hat{P}_q(b). \tag{3.2.1}$$

Proof. By (3.1.5), $Ea_p a_q = N^{-2} \sum_{i,j=1}^{N} R_{i-j}\varphi_{p,N}(i)\varphi_{q,N}(j)$. Therefore, by (3.1.5) and (3.1.6), for any $1 \leqslant s, t \leqslant N$ and any $\{y_t\} \in S$,

$$\begin{aligned} N\operatorname{cov}(y_s-\hat{y}_{s,m}, y_t-\hat{y}_{t,m}) &= N\operatorname{cov}\Big(\sum_{p=0}^{m} a_p\varphi_{p,N}(s), \sum_{q=0}^{m} a_q\varphi_{q,N}(t)\Big) \\ &= N\sum_{p,q=0}^{m} E(a_p a_q)\varphi_{p,N}(s)\varphi_{q,N}(t) \\ &= N^{-1}\sum_{p,q=0}^{m}\sum_{i,j=1}^{N} R_{i-j}\varphi_{p,N}(i)\varphi_{q,N}(j)\varphi_{p,N}(s)\varphi_{q,N}(t) \\ &= \sum_{|k|<N} R_k\Big[N^{-1}\sum_{p,q=0}^{m}\sum_{j=1}^{N-|k|}\varphi_{p,N}(j)\varphi_{q,N}(j+|k|)\varphi_{p,N}(s)\varphi_{q,N}(t)\Big] \\ &= \sum_{k=-\infty}^{\infty} R_k\psi_{k,N}, \quad \text{say.} \end{aligned} \tag{3.2.2}$$

Obviously, for $|k| \geqslant N$, $\psi_{k,N} = 0$ and for $|k| < N$, by (3.1.17), $|\psi_{k,N}| < Q_D^4(m+1)^2$. Moreover, for any given k

$$\lim_{\substack{N\to\infty;\, t/N\to a;\\ s/N\to b;}} \psi_{k,N} = \sum_{q=0}^{m} \hat{P}_q(a)\hat{P}_q(b). \tag{3.2.3}$$

Indeed, for $|k| < N$

$$N^{-1}\sum_{j=1}^{N-|k|}\varphi_{p,N}(j)\varphi_{q,N}(j+|k|) = N^{-1}\sum_{j=1}^{N-|k|}\varphi_{p,N}(j)\varphi_{q,N}(j)+$$
$$+N^{-1}\sum_{j=1}^{N-|k|}\varphi_{p,N}(j)[\varphi_{q,N}(j+|k|)-\varphi_{q,N}(j)] = N^{-1}\sum_{j=1}^{N}\varphi_{p,N}(j)\varphi_{q,N}(j)-$$
$$-N^{-1}\sum_{j=N-|k|+1}^{N}\varphi_{p,N}(j)\varphi_{q,N}(j)+N^{-1}\sum_{j=1}^{N-|k|}\varphi_{p,N}(j)\,|k|\dot{\varphi}_{q,N}(\xi_q),$$

where $j \leqslant \xi_q \leqslant j+|k|$. The first term in the last expression is equal to 1 when $p = q$ and to 0 when $p \neq q$. The absolute value of the second term for any given k is not greater than $Q_D^2\frac{|k|}{N}$ and thus tends to zero with $N \to \infty$. Moreover (see e. q. Proposition 10.2 in [38]), for any ξ_q, $\dot{\varphi}_{q,N}(\xi_q)$ tends to zero and hence the absolute value of the last term tends to zero for any given k. Finally, it is known (see e. q. Proposition 10.1 in [38]) that

$$\lim_{N\to\infty;\,t/N\to a}\varphi_{p,N}(t) = \hat{P}_p(a).$$

Therefore (3.2.3) holds and Lemma 3.14 follows from Lemma 18.1 in [22].

It should be noted (see e. q. [25], part VII, No. 23) that for any $0 \leqslant a \leqslant 1$ and for $q = 0, 1, \ldots$

$$|\hat{P}_q(a)| \leqslant \sqrt{2q+1}$$

and hence for $0 \leqslant a, b \leqslant 1$,

$$\sum_{q=0}^{m}|\hat{P}_q(a)\hat{P}_q(b)| \leqslant (m+1)^2. \tag{3.2.4}$$

LEMMA 3.15. *Under $\tilde{H}_4$, for any given $\{y_t\} \in S$,*

$$p\lim_{N\to\infty}\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}\left[\frac{1}{\nu}\sum_{t=1}^{\nu}(z_{t+r\nu}-\hat{y}_{t+r\nu,m})\right]^2 = \sum_{k=-\infty}^{\infty}R_k.$$

Proof. Consider first $\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}\left[\frac{1}{\nu}\sum_{t=1}^{\nu}x_{t+r\nu}\right]^2$. In view of (3.1.3), the last expression is equal to $\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}M_{\nu,r}^2$; by Lemma 4.2 in [38] its variance tends to zero, and by (4.1) in the same paper its expectation tends to $\sum_{q=-\infty}^{\infty}R_q$. Hence

$$p\lim_{N\to\infty}\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}M_{\nu,r}^2 = \sum_{q=-\infty}^{\infty}R_q. \tag{3.2.5}$$

By (3.1.6), $z_t - \hat{y}_{t,m} = x_t + \gamma_{t,m}$ and hence by (3.1.7)

$$\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}\left[\frac{1}{\nu}\sum_{t=1}^{\nu}(z_{t+r\nu}-\hat{y}_{t+r\nu,m})\right]^2 = \frac{\nu}{\mu}\sum_{r=0}^{\mu-1}(M_{\nu,r}+\Gamma_{\nu,r,m})^2.$$

But by (3.1.15) and (3.1.22)

$$\operatorname*{plim}_{N\to\infty}\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}[(M_{\nu,r}+\Gamma_{\nu,r,m})^2-M_{\nu,r}^2] = 0.$$

Thus, in view of (3.2.5), Lemma 3.15 is proved.

Let

$$\eta_{t,j} = \frac{N^{\frac{1}{2}}(y_t-\hat{y}_{t,j})}{(j+1)\sqrt{\frac{\nu}{\mu}\sum_{r=0}^{\mu-1}\left[\frac{1}{\nu}\sum_{t=1}^{\nu}(z_{t+r\nu}-\hat{y}_{t+r\nu,j})\right]^2}}, \tag{3.2.6}$$

or, after easy transformations,

$$\eta_{t,j} = \frac{N(y_t-\hat{y}_{t,j})}{(j+1)\sqrt{\sum_{r=0}^{\mu-1}[\sum_{t=1}^{\nu}(z_{t+r\nu}-\hat{y}_{t+r\nu,j})]^2}}. \tag{3.2.7}$$

THEOREM 3.16. *Under $\tilde{H}_4$, for any $r = 1, 2, \ldots$, and any real numbers $b_1, \ldots, b_r$ in $[0, 1]$, if $\lim_{N\to\infty} t_{i/N} = b_i$ $(i = 1, \ldots, r)$, the joint distribution of $\eta_{t_1,m}, \ldots, \eta_{t_r,m}$ tends, with $N \to \infty$, to the normal distribution specified by zero means and the covariance matrix*

$$\left\| (m+1)^{-2}\sum_{q=0}^{m}\hat{P}_q(b_i)\hat{P}_q(b_j)\right\|.$$

Proof. By (3.1.5) and (3.1.6), for $1 \leqslant t \leqslant N$ and for any given $\{y_t\} \in S$,

$$N^{\frac{1}{2}}(y_t-\hat{y}_{t,m}) = -N^{\frac{1}{2}}\sum_{q=0}^{m}a_q\varphi_{q,N}(t) = -\sum_{s=1}^{N}x_s\Big[N^{-1}\sum_{q=0}^{m}\varphi_{q,N}(s)\varphi_{q,N}(t)\Big], \tag{3.2.8}$$

i. e. this expression is a linear combination of x's. The absolute value of any coefficient of x_s $(s = 1, \ldots, N)$ is bounded by $N^{-1}(m+1)Q_D^2$ and, therefore, tends to 0 with $N \to \infty$; hence any linear combination of

$$N^{\frac{1}{2}}(y_{t_1}-\hat{y}_{t_1,m}), \ldots, N^{\frac{1}{2}}(y_{t_r}-\hat{y}_{t_r,m}) \tag{3.2.9}$$

is also a linear combination of $x_1, \ldots, x_n$ with coefficients of x_s tending to 0 for $s = 1, \ldots, N$, and by Lemma 3.14 its variance tends to a constant. Therefore, by Lemma 2.2 in [36], the limiting distribution of any linear

combination of the elements of (3.2.9) is normal with a zero mean since $Ea_q = 0$. Hence, by Theorem 7.1 in [21], the limiting joint distribution of (3.2.9) is normal with zero means and with the covariance matrix

$$\Big\| \sum_{k=-\infty}^{\infty} R_k \sum_{q=0}^{m} \hat{P}_q(b_i)\hat{P}_q(b_j) \Big\|.$$

Thus, by the convergence theorem ([5], Section 20.6), by (3.2.6) and by Lemma 3.15, Theorem 3.16 is proved.

If $\zeta_{1-\gamma/2}$ is defined by (3.1.4), then, by (3.2.4) and Theorem 3.16, for $1 \leqslant t \leqslant N$

$$\lim_{N\to\infty} P(|\eta_{t,m}| > \zeta_{1-\gamma/2}) \leqslant \gamma. \tag{3.2.10}$$

COROLLARY 3.17. *Under $\tilde{H}_4$, if $\{y_t\}$ is any given element of S and if* (2.8) *is satisfied for some chosen a, then, for $t = 1, \ldots, N$ and $0 < \gamma < \frac{1}{2}$,*

$$\lim_{N\to\infty} P(|\eta_{t,\hat{m}}| \geqslant \zeta_{1-\gamma/2} \hat{m} \leqslant D) \leqslant \frac{\gamma}{1-a}.$$

Proof. For any positive a, $P(|\eta_{t,\hat{m}}| \geqslant a|\ \hat{m} \leqslant D) = P(|\eta_{t,\hat{m}}| \geqslant a,\ \hat{m} = m|\ \hat{m} \leqslant D) + P(|\eta_{t,\hat{m}}| \geqslant a,\ \hat{m} \neq m|\ \hat{m} \leqslant D)$. The second term of the last expression is not greater than $P(\hat{m} \neq m|\ \hat{m} \leqslant D)$ which tends to 0 with $N \to \infty$ in view of (2.8). Since $|\eta_{t,\hat{m}}| \geqslant a$ and $\hat{m} = m$ imply $|\eta_{t,m}| > a$, the first term of this expression is not greater than

$$P(|\eta_{t,m}| \geqslant a|\hat{m} \leqslant D) \leqslant \frac{P(|\eta_{t,m}| \geqslant a)}{P(\hat{m} \leqslant D)}.$$

Thus Corollary 3.16 follows by (2.8) and (3.2.10).

Consequently, by (3.2.7), if the confidence limits are

$$y_{t,\hat{m}} \pm N^{-1}\zeta_{1-\gamma/2}(j+1)\sqrt{\sum_{r=0}^{\mu-1}\Big[\sum_{t=1}^{\nu}(z_{t+r\nu} - \hat{y}_{t+r\nu,\hat{m}})\Big]^2}$$

and the assumptions of Corollary 3.17 are satisfied, the asymptotic confidence level is not smaller than $1 - \dfrac{\gamma}{(1-a)}$.

4. Testing the equivalence of two linear processes

4.1. The proposed test and its properties. We assume that we have two observable linear processes, $\{x_t\}$ and $\{\tilde{x}_t\}$, $x_t = \sum_{q=-\infty}^{\infty} h_{t-q}\varepsilon_q$ and $\tilde{x}_t$

$= \sum_{q=-\infty}^{\infty} \tilde{h}_{t-q}\tilde{\varepsilon}_q$, $t = 0, \pm 1, \ldots$ Let us denote by $\tilde{H}_{2L_1,2L_2}$ the assertion that $\{x_t\}$ satisfies $\tilde{H}_{2L_1}$, $\{\tilde{x}_t\}$ satisfies $\tilde{H}_{2L_2}$ and that $\{\varepsilon_t\}$ and $\{\tilde{\varepsilon}_t\}$ (and, consequently, $\{x_t\}$ and $\{\tilde{x}_t\}$) are mutually independent. Let f and $\tilde{f}$, $\{R_k\}$ and $\{\tilde{R}_k\}$ denote the spectral density and autocorrelation function of $\{x_t\}$ and $\{\tilde{x}_t\}$, respectively.

The parameter $\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(t)-\tilde{f}(t)]^2 dt$ is proposed as a measure of the difference between the two processes. Thus $\theta = 0$ if and only if $f = \tilde{f}$ (or $\{R_k\} = \{\tilde{R}_k\}$ in view of $\theta = \sum_{k=-\infty}^{\infty} (R_k - \tilde{R}_k)^2$). In this section we present a test based on $\{x_t\}_N$ and $\{\tilde{x}_t\}_N$ of the null hypothesis H_0: $\theta = 0$ against H_{alt}: $\theta > 0$. The power of this test tends to 1 with $N \to \infty$ for every $\theta > 0$.

The definition of the test function is based on $C_{k,N}$ and S_N defined in Section 3.1, on their counterparts for the process $\{\tilde{x}_t\}$, namely

$$\tilde{C}_{k,N} = (N-k)^{-1} \sum_{k=1}^{N-|k|} \tilde{x}_t \tilde{x}_{t+|k|}$$

and

$$\tilde{S}_N = \tfrac{1}{2} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 \tilde{C}_{k,N}^2,$$

and on

$$M_N = \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 C_{k,N} \tilde{C}_{k,N}. \tag{4.1.1}$$

Let

$$A_N = N^{1/2}(S_N + \tilde{S}_N - 2M_N). \tag{4.1.2}$$

$N^{-1/2}A_n$ can be considered as an estimator of θ; this is justified by Theorem 4.2 below which states that the distribution of $A_N - N^{1/2}\theta$ is asymptotically normal with a zero mean and a constant positive variance. It follows that the test function defined as A_N divided by a consistent estimator of its standard deviation is asymptotically normally distributed with a zero mean and a unit variance for $\theta = 0$ and tends in probability to $+\infty$ for $\theta > 0$, which implies the definition of the rejection set of our test.

The following parameters:

$$\sigma_{m,n} = \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(t)]^m [\tilde{f}(t)]^n dt \quad (m, n = 0, 1, \ldots) \tag{4.1.3}$$

play an important part in what follows. They form a generalized version of

$$\sigma_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(t)]^n dt$$

introduced in [22], p. 146. In view of Lemma 18.2 in [22]

$$\sigma_{m,n} = \sum_{\substack{p_1,\ldots,p_{m+n}=-\infty \\ p_1+\ldots+p_{m+n}=0}}^{\infty} R_{p_1}\ldots R_{p_m}\tilde{R}_{p_{m+1}}\ldots\tilde{R}_{p_{m+n}}. \tag{4.14}$$

THEOREM 4.1. *Under $\tilde{H}_{8,8}$, the random variable*

$$\{2N^{\frac{1}{2}}(S_N-ES_N),\ 2N^{\frac{1}{2}}(\tilde{S}_N-E\tilde{S}_N),\ N^{\frac{1}{2}}(M_N-EM_N)\} \tag{4.1.5}$$

is asymptotically normally distributed with a zero mean and a covariance matrix

$16\varkappa_4\varkappa_2^{-2}\sigma_{2,0}^2+\frac{112}{3}\sigma_{4,0}$	0	$4\varkappa_4\varkappa_2^{-2}\sigma_{1,1}\sigma_{2,0}+8\sigma_{3,1}$
0	$16\tilde{\varkappa}_4\tilde{\varkappa}_2^{-2}\sigma_{0,2}^2+\frac{112}{3}\sigma_{0,4}$	$4\tilde{\varkappa}_4\tilde{\varkappa}_2^{-2}\sigma_{1,1}\sigma_{0,2}+8\sigma_{1,3}$
$4\varkappa_4\varkappa_2^{-2}\sigma_{1,1}\sigma_{2,0}+8\sigma_{3,1}$	$4\tilde{\varkappa}_4\tilde{\varkappa}_2^{-2}\sigma_{1,1}\sigma_{0,2}+8\sigma_{1,3}$	$(\varkappa_4\varkappa_2^{-2}+\tilde{\varkappa}_4\tilde{\varkappa}_2^{-2})\sigma_{1,1}^2+\frac{16}{3}\sigma_{2,2}$

Moreover, under $\tilde{H}_{4L_1,4L_2}$ $(L_1, L_2 \geqslant 2)$ all the univariate and mixed moments up to the order L_1 or L_2, respectively, tend to the respective moments of the limiting distribution.

Proof. The general scheme of the proof follows the pattern introduced in [21] and [22] and then frequently used by the authors. Assuming the existence of all moments of ε_t and $\tilde{\varepsilon}_t$, it is only needed to check that the limits of all univariate and mixed moments:

$$N^{n/2}E\{(2S_N-2ES_N)^{n_0}(2\tilde{S}_N-2E\tilde{S}_N)^{n_1}(M_N-EM_n)^{n_2}\}, \tag{4.1.6}$$
$$n \geqslant 2,\ n_0+n_1+n_2 = n,\ n_0, n_1, n_2 \geqslant 0,$$

form the respective moments of the normal distribution specified by Theorem 4.1. The further proof is based on a multivariate version of the Second Limit Theorem for a normal distribution and a suitable truncating device, which allows to replace the assumption about moments by $\tilde{H}_{8,8}$ ([21], [36]). The standard way of computing the limits of (4.1.6) proposed in [21] involves a suitable factorization of sums which leads to an assertion that these limits are constructed from the limits of the elements of the covariance matrix of (4.1.5) only.

We consider first this term of the expansion of (4.1.6) which contains none of the expectations ES_N, $E\tilde{S}_N$, EM_N. We obtain a multiple sum

with summation variables k_i, $i = 1, \ldots, n$ (corresponding to the summation variables k in S_N, $\tilde{S}_N$ and M_N) and α_i, β_i, $i = 1, \ldots, n$ (corresponding to the variables t appearing in the first and second factors of $C^2_{k_i,N}$, $\tilde{C}^2_{k_i,N}$ and $C_{k_i,N}\tilde{C}_{k_i,N}$, respectively). The α's and β's can be divided into two groups,

$$\alpha_1, \ldots, \alpha_{n_0}, \beta_1, \ldots, \beta_{n_0}, \alpha_{n_0+n_1+1}, \ldots, \alpha_n$$

on the one hand, and

$$\alpha_{n_0+1}, \ldots, \alpha_{n_0+n_1}, \beta_{n_0+1}, \ldots, \beta_{n_0+n_1}, \beta_{n_0+n_1+1}, \ldots, \beta_n$$

on the other, the variables of the first group referring to the process $\{x_t\}$ and those of the second referring to $\{\tilde{x}_t\}$.

Let $\gamma_i = \alpha_i$ or β_i. An expression $x_{\gamma_i} x_{\gamma_i+|k_i|}$ or $\tilde{x}_{\gamma_i}\tilde{x}_{\gamma_i+|k_i|}$ can be assigned to each of the α's and β's, the former in the case of a variable in the first group and the latter in the case of a variable of the second group. All these expressions connected with the first group can be arranged in any order into a sequence of $2n_0+n_1$ elements which are denoted by u_j $(j = 1, \ldots, 2n_0+n_1)$. Proceeding in the same way with the remaining elements, we obtain a sequence $\{\tilde{u}_j\}_{2n_1+n_2}$. Then the considered term of (4.1.6) may be written as

$$N^{-\frac{3}{2}} \sum_{\{k_1,\ldots,k_n;|k_i|\leqslant N\}} \sum E\Big(\prod_{j=1}^{2n_0+n_1} u_j\Big) \sum E\Big(\prod_{j=1}^{2n_1+n_2} \tilde{u}_j\Big), \tag{4.1.7}$$

where two last sums are extended to all the variables from the first and from the second group, respectively.

A further expansion of (4.1.7) is based on Proposition 2.1 in [36], reproduced below for the convenience of the reader:

If the functions V_p are defined by (3.1.20), then, under $\tilde{H}_{2M}$ $(M \geqslant 1)$, for every r $(1 \leqslant r \leqslant 2M)$ and for every set of integers $i(1), \ldots, i(r)$

$$\begin{aligned} E(x_{i(1)} \cdot \ldots \cdot x_{i(r)}) \\ = \sum \varkappa_{r_1} V_{r_1}\big(i(l_1), \ldots, i(l_{r_1})\big) \cdot \ldots \cdot \varkappa_{r_k} V_{r_k}\big(i(l_{r-r_k+1}), \ldots, i(l_r)\big), \end{aligned} \tag{4.1.8}$$

where the sum is extended over all different groupings of the set $I = \langle i(1), \ldots, i(r)\rangle$, i. e. over all sets of disjoint subsets of I whose union is I, two groupings being regarded as distinct if, for some m and n $(m, n = 1, \ldots, r)$, $i(m)$ and $i(n)$ belong to the same subset in one grouping, and to two different ones in another. Here $\langle l_1, \ldots, l_r\rangle$ denotes any permutations of the numbers $\langle 1, \ldots, r\rangle$ corresponding to a given grouping, i. e. such that the sets

$$\langle i(l_1), \ldots, i(l_{r_1})\rangle, \ldots, \langle i(l_{r-r_k+1}), \ldots, i(l_r)\rangle$$

form this particular grouping. In view of the symmetry of the V's with respect to their arguments, any two permutations corresponding to the

same grouping would yield the same term, so that, for any grouping, the choice of the permutation representing it does not matter, and, therefore, the sum is well defined. The variables $i(1), \ldots, i(r)$ must be regarded as distinct although in the end any values, not necessarily distinct ones, can be substituted for them. It should be noted that in the right-hand side of (4.1.8) we can confine ourselves to groupings in which each set is composed of more than one element, since $\varkappa_1 = 0$.

Of course, (4.1.8) can be applied to $E(\tilde{x}_{i(1)} \cdot \ldots \cdot \tilde{x}_{i(s)})$ if V's are replaced by $\tilde{V}$'s, where

$$(4.1.9) \quad \tilde{V}_p\big(i(1), \ldots, i(p)\big) = \sum_{q=-\infty}^{\infty} \tilde{h}_{i(1)-q} \cdot \ldots \cdot \tilde{h}_{i(p)-q} \quad (p = 2, 3, \ldots).$$

Now we assume the existence of all the moments of ε_t and $\tilde{\varepsilon}_t$ and according to (4.1.8) we transform (4.1.7) into a finite sum of terms; each of these terms is similar to (4.1.7), with the expectation of the product of u's replaced by a product of $\varkappa$'s and V's, and the expectation of the product of $\tilde{u}$'s replaced by a product of $\tilde{\varkappa}$'s and $\tilde{V}$'s. The set of arguments of the V's consists of all indices of x's, i. e. of all the α's and β's from the first group and of all the related $\alpha + |k|$'s and $\beta + |k|$'s, while the set of arguments of $\tilde{V}$'s consists of all the indices of $\tilde{x}$'s.

We divide the set of arguments of all the V's and $\tilde{V}$'s in each term into disjoint subsets in such a way that

(a) all the arguments of one V or $\tilde{V}$ appear in the same subset,

(b) for any $1 \leqslant i \leqslant n$ $\alpha_i, \alpha_i + |k_i|, \beta_i, \beta_i + |k_i|$ appear in the same subset, and

(c) any subset cannot be further divided into disjoint subsets satisfying the above conditions.

Consequently, each term can be rewritten so that the summation signs with respect to α's, β's and k's in one subset can be put just before the products of those V's and $\tilde{V}$'s whose arguments belong to this subset. A multiple sum of a product of V's and $\tilde{V}$'s related to one subset, multiplied by a product of the respective $\varkappa$'s and $\tilde{\varkappa}$'s and by $N^{-\frac{3}{2}\varrho}$ (where 4ϱ stands for the number of arguments in the subset), will be regarded as a factor of the whole sum. Thus any term of the expansion of (4.1.7) is a product of factors.

Apart from the grouping which has been just described, we shall consider another one which we obtain by a slight modification of the rule (b). Namely, four arguments associated with any given i should not necessarily appear in the same subset; instead, α_i and $\alpha_i + |k_i|$ must be in the same subset for any i and the same is required for β_i and $\beta_i + |k_i|$. The resulting grouping gives rise to subfactors, i. e. to products of $\varkappa$'s

or $\tilde{\varkappa}$'s and of V's or $\tilde{V}$'s with arguments associated with one subset. It is clear that apart from the normalizing coefficient the summand of each factor is a product of one or more subfactors.

We shall say that a subfactor or a factor is of degree q if the number of arguments of the V's and $\tilde{V}$'s appearing in it is equal to q. If follows that the degree of a subfactor is always even while that of a factor is a multiple of four. It should be noted that a quadratic subfactor (i. e. a subfactor of degree 2) is either $\varkappa_2 V_2(\gamma_i, \gamma_i+|k_i|)$ or $\tilde{\varkappa}_2 \tilde{V}_2(\gamma_i, \gamma_i+|k_i|)$, where $\gamma_i = \alpha_i$ or β_i, and hence by (4.1.8) it is R_{k_i} or $\tilde{R}_{k_i}$.

Returning to the consideration of all the terms of the expansion of (4.1.6), it will be seen that taking into account $-ES_N$, $-E\tilde{S}_N$ and $-EM_N$ results in the cancellation of all the terms of the expansion of (4.1.7) containing quartic factors.

It is clear that the expansion of (4.1.6) is a sum of a finite number of terms similar to those of (4.1.7), the only difference being that now in certain terms some of the sums with respect to triplets (k_i, α_i, β_i) disappear and are replaced by powers of $-2N^{\frac{1}{2}}ES_N$, $-2N^{\frac{1}{2}}E\tilde{S}_N$, and $-N^{\frac{1}{2}}EM_N$. Each of the expressions

$$
\begin{aligned}
2N^{\frac{1}{2}}ES_N &= N^{-\frac{3}{2}} \sum_{k=1-N}^{N-1} \sum_{\alpha,\beta=1}^{N-|k|} E(x_\alpha x_{\alpha+|k|} x_\beta x_{\beta+|k|}),\\
2N^{\frac{1}{2}}E\tilde{S}_N &= N^{-\frac{3}{2}} \sum_{k=1-N}^{N-1} \sum_{\alpha,\beta=1}^{N-|k|} E(\tilde{x}_\alpha \tilde{x}_{\alpha+|k|} \tilde{x}_\beta \tilde{x}_{\beta+|k|}),\\
N^{\frac{1}{2}}EM_N &= N^{-\frac{3}{2}} \sum_{k=1-N}^{N-1} \sum_{\alpha,\beta=1}^{N-|k|} E(x_\alpha x_{\alpha+|k|} \tilde{x}_\beta \tilde{x}_{\beta+|k|}),
\end{aligned}
\tag{4.1.10}
$$

can be related to one of the missing triplets of summation variables. We can, then, replace, in (4.1.10), the summation variables by the appropriate triplet (k_i, α_i, β_i) and substitute, in the expansion of (4.1.6), the right-hand sides of these equalities for their left-hand sides; of course, we choose $i \in [1, n_0]$ in the case of $(-2N^{\frac{1}{2}}ES_N)$, $i \in [n_0+1, n_0+n_1]$ in the case of $(-2N^{\frac{1}{2}}E\tilde{S}_N)$ and $i \in [n_0+n_1+1, n]$ in the case of $(-N^{\frac{1}{2}}EM_N)$. Then we proceed as before, applying (4.1.8) to every expectation of x's and $\tilde{x}$'s and factorizing the sums. Hence, finally, the expansion of (4.1.6) is, as before, a finite sum of terms which are products of factors, where the factors arising from the expansion of the left-hand terms in (4.1.10) are quartic and multiplied by (-1).

Clearly, given an arbitrary grouping of all the summation variables k_i with the corresponding α_i and β_i $(i = 1, \ldots, n)$, the factors which can be formed with any particular subset of the grouping do not depend on

the patterns of the factors formed with the other subsets thereof. The sum of all the factors formed with a given subset of the grouping will be described as a grand factor. The sum of all the terms corresponding to a given grouping of the summation variables can, therefore, be conveniently represented as a single product of grand factors. Clearly, such a product vanishes whenever at least one grand factor is equal to zero. But each quartic grand factor is composed of two factors which are easily seen to cancel out. Hence every product containing at least one quartic grand factor is equal to zero. Consequently in the sequel only products of factors free from quartic factors will be considered.

Let ω denote any factor of degree 4ϱ $(2 \leqslant \varrho \leqslant n)$ which can appear in any term of the expansion of (4.1.6). In view of (3.1.17) and (4.1.9), ω can be considered as a multiple sum with respect to α's, β's, k's and to the inner summation variables in the V's and $\tilde{V}$'s, so that the summand, apart from the cumulants and the normalizing coefficient $N^{-\frac{3}{2}\varrho}$, is a product of h's and $\tilde{h}$'s. The indices of the h's and $\tilde{h}$'s in ω depend on the summation variables. Let $p(\omega)$ denote an upper bound of the number of repetitions of any sequence of indices for all the admissible combinations of the summation variables. Since $\sum_{k=-\infty}^{\infty} h_k$ and $\sum_{k=-\infty}^{\infty} \tilde{h}_k$ are absolutely convergent, the order of magnitude of ω is at most $p(\omega)N^{-\frac{3}{2}\varrho}$. But for any given sequence of index values, the differences of arguments inside any of the corresponding V's and $\tilde{V}$'s are determined and hence the knowledge of one argument only in each V and $\tilde{V}$ is sufficient to determine the remaining arguments and the values of the internal summation variables of each V and $\tilde{V}$.

Now suppose that we are given the values of all the k's and of one α or β arbitrarily chosen in each subfactor in ω. Since any pair of V's or of $\tilde{V}$'s in the same subfactor is linked by at least one k_i, i. e. γ_i ($\gamma_i = \alpha_i$ or β_i) appears as an argument in one of them and $\gamma_i + |k_i|$ in another one, all the arguments of V's and $\tilde{V}$'s in ω, with all the remaining α's and β's among them, are determined under the above assumptions. Let $\mu(\omega)$ denote the total number of subfactors in ω. In view of the ranges of the α's, β's and k's, the number of different sequences of values of k's is $(2N-1)^\varrho$ and that of the values of one α or β in the $\mu(\omega)$ subfactors is $N^{\mu(\omega)}$, so that $p(\omega) \leqslant N^{\mu(\omega)}(2N-1)^\varrho$.

However, the knowledge of the values of all k's is not necessary in each ω. Let $\Lambda(\omega)$ denote the smallest number of k's in ω such that their absolute values together with the values of one α or β arbitrarily chosen in each subfactor and with the differences of arguments within any V or $\tilde{V}$ in ω allow to determine the values of all the remaining α's, β's and

$|k|$'s in ω. Then

$$p(\omega) \leqslant N^{\mu(\omega)}(2N-1)^{\Lambda(\omega)}2^{\varrho-\Lambda(\omega)},$$

where $2^{\varrho-\Lambda(\omega)}$ is due to the fact that two values of k (one positive and one negative) correspond to every determined non-zero value of k. Consequently, for any ω there exists a positive constant g_ϱ such that

$$|\omega| \leqslant g_\varrho N^{\mu(\omega)+\Lambda(\omega)-\frac{3}{2}\varrho}. \tag{4.1.11}$$

Let

$$A(\omega) = \mu(\omega)+\Lambda(\omega)-\tfrac{3}{2}\varrho. \tag{4.1.12}$$

By (4.1.11) ω is bounded if $A(\omega) \leqslant 0$ while factors with $A(\omega) < 0$ tend to zero. We shall show that $A(\omega) \leqslant 0$ for any ω and that all ω's such that $A(\omega) = 0$ are to be found among octic factors (i. e. of degree 8).

To start with, we note that each quadratic subfactor in ω (i. e. R or $\tilde{R}$) corresponds to one and only one non-quadratic subfactor in ω such that the same summation variable k appears in both subfactors. This follows from the fact that the same k cannot appear in two quadratic subfactors, since the two would form a quartic factor and these have been eliminated. A set consisting of a non-quadratic subfactor and of all the corresponding quadratic subfactors will be called an *extended subfactor*. Of course, the subfactors in ω can be grouped into extended subfactors in exactly one way.

It will be convenient to distinguish the summation variables k appearing in quadratic subfactors from the remaining ones. The former k's will be called *bound*, the latter *free*. We shall first consider factors with only one extended subfactor. The extended subfactors forming any factor of the considered type can be described as those in which all free k's appear twice, because otherwise the factor would be composed of more than one extended subfactor. If ω is of degree 4ϱ and the number of quadratic subfactors in ω is q, then the number of free k's in ω is equal to $\varrho-q$. It is easy to see that this implies $\Lambda(\omega) \leqslant \varrho-q$. Indeed, the absolute value of each bound k is determined by the difference of arguments of the V_2 or $\tilde{V}_2$ forming the corresponding quadratic subfactor and hence by definition $\Lambda(\omega)$ cannot exceed the total number of free k's in ω. Moreover, in the considered factors $\mu(\omega) = 1+q$ and hence by (4.1.12)

$$A(\omega) \leqslant 1+q+\varrho-q-\varrho = 1-\varrho/2.$$

Since $\varrho \geqslant 2$, $A(\omega) \leqslant 0$, with equality holding if and only if $\varrho = 2$ and $\Lambda(\omega) = 2-q$. Then three possibilities corresponding to $q = 0, 1$ or 2 are to be considered:

(1) ω is composed of one octic subfactor, $\Lambda(\omega) = 2$; such a factor will be referred to as an "8" factor;

(2) ω is composed of one quadratic subfactor and one sextic subfactor, $\Lambda(\omega) = 1$; such a factor will be referred to as an "$2-6$" factor:

(3) ω is composed of two quadratic subfactors and one quartic subfactor (and $\Lambda(\omega)$ is necessarily equal to zero); such a factor will be referred to as an "$2-2-4$" factor.

Thus we have shown that:

(4.1.13) $\left\{\begin{array}{l}\text{The only factors composed of one extended subfactor which} \\ \text{may not tend to zero are of the types "8", "2 — 6" and "2 — 2 —} \\ \text{— 4".}\end{array}\right.$

Now we turn to factors with at least two extended subfactors; each of the latter depends necessarily on at least one solitary free k (i. e. a k which appears in two different nonquadratic subfactors in ω). Let E be the set (with at least two elements) of all extended subfactors in ω and let e be any element of E consisting of a non-quadratic subfactor of degree $4r$ (so that r is a multiple of $\frac{1}{2}$) and of q quadratic subfactors. Then $1 \leqslant r \leqslant \varrho$ and $0 \leqslant q < 2r$ (the case $q = 2r$ is excluded since in this case all k's in e are bound). For any e we define λ as the sum of the number of free k's appearing twice in e and of one-half of the number of the solitary k's in e, i. e. $\lambda = r - \frac{1}{2}q$. Obviously, $\sum_{e \in E} \lambda$ is equal to the total number of free k's in ω and in view of the previous argument concerning $\Lambda(\omega)$, this entails

$$\Lambda(\omega) \leqslant \sum_{e \in E} \lambda. \tag{4.1.14}$$

Hence $\Lambda(\omega) \leqslant \mu(\omega) + \sum_{e \in E} \lambda - \frac{3}{2}\varrho$. Given e, let

$$a = 1 + q + \lambda - \tfrac{3}{2}r - \tfrac{3}{4}q = 1 - \tfrac{1}{4}q - \tfrac{1}{2}r. \tag{4.1.15}$$

Of course, the sum of $(1+q)$ over all e in E is equal to $\mu(\omega)$ and that of $(r + q/2)$ is equal to ϱ. Consequently,

$$\sum_{e \in E} a = \mu(\omega) + \sum_{e \in E} \lambda - \tfrac{3}{2}\varrho$$

and hence

$$\Lambda(\omega) \leqslant \sum_{e \in E} a. \tag{4.1.16}$$

By (4.1.15), a is negative if and only if $q = 0$ and $r > 2$ or $q = 1$ and $r > \frac{3}{2}$ or $q = 2$ and $r > 1$ or $q > 2$ and $r > \frac{1}{2}$. Let E_1 denote the set consisting of all those elements of E for which a is negative, and let $E_2 = E - E_1$. Since $q < 2r$, for any element of E_2 either $q = 0$ and $1 \leqslant r \leqslant 2$ or $q = 1$ and $1 \leqslant r \leqslant \frac{3}{2}$. By (4.1.16) $\Lambda(\omega) < 0$ if $E = E_1$. If $E \neq E_1$, we must replace (4.1.14) by a stronger inequality. More precisely, let μ_2 denote the number of elements in E_2; we shall prove that for any factor

ω with at least two extended subfactors

$$\Lambda(\omega) \leqslant \sum_{c \in E} \lambda - \mu_2/2. \tag{4.1.17}$$

To prove (4.1.17), we need to analyse more closely the relationships among k's appearing in any $c \in E_2$. It is clear that this reduces to considering only the non-quadratic subfactor, say $\mathfrak{s}$, of such an extended subfactor. Necessarily, $\mathfrak{s}$ depends on at least one solitary free k; moreover, it follows from the discussion concerning the r's and q's in any element of E_2 that $\mathfrak{s}$ is quartic or sextic with at most one bound k (i. e. $q = 0$ or 1 and $r = 1$ or $\frac{3}{2}$) or octic with no bound k (i. e. $q = 0$ and $r = 2$).

A set of k's appearing in the same subfactor in ω will be called redundant in this subfactor if

(1) it contains at least one free variable;

(2) the absolute value of any element of this set is determined by those of the remaining elements and by the differences of arguments in the V's or $\tilde{V}$'s forming this subfactor. If this set consists of only one (necessarily free) k, this k itself will be called *redundant* in this subfactor.

It will be shown that $\mathfrak{s}$ depends on at least one redundant set of k's. Since no subfactor can contain both V's and $\tilde{V}$'s, it is clear by symmetry that we can assume that $\mathfrak{s}$ is formed with V's.

Let k^* denote an arbitrarily chosen solitary free k appearing in $\mathfrak{s}$ and let γ^* denote this variable α or β which corresponds to k^* and appears in $\mathfrak{s}$. If γ^* and $\gamma^* + |k^*|$ are arguments of the same V in $\mathfrak{s}$, then k^* is redundant in $\mathfrak{s}$. Otherwise, γ^* and $\gamma^* + |k^*|$ are arguments of two different V's, so that k^* forms a link between them. Two cases have to be considered.

To start with, assume that there is another k, say k', which also forms a link between these V's. Let γ' denote an α or β such that γ' is an argument of one of the V's linked by k^* and $\gamma + |k'|$ is an argument of the other one. Then among the differences or arguments of these V's we find some

$$d_1 = \gamma^* + \delta^* |k^*| - \gamma' - \delta' |k'|$$

and

$$d_2 = \gamma' + (1 - \delta') |k'| - \gamma^* - (1 - \delta^*) |k^*|,$$

where δ^* and δ' are each equal to 0 or 1. Thus

$$d_1 + d_2 = (1 - 2\delta') |k'| - (1 - 2\delta^*) |k^*| \tag{4.1.18}$$

and hence the set $\{k^*, k'\}$ is redundant in $\mathfrak{s}$.

Now let k^* be the only link between two V's in $\mathfrak{s}$. Then it cannot be a link between two V_2's forming a subfactor. Hence $\mathfrak{s}$ must be of a degree higher than 4, i. e. it is necessarily sextic or octic. In a sextic $\mathfrak{s}$ k's appear in three arguments of V's; they will be denoted by k^*, k' and k''. Since at most one of them can be bound, k^* and k' can be assumed to be free;

since k^* is solitary, it is necessarily different from k' and k''. The variable k'' is either bound or free, and in the latter case it is distinct or identical with k' (i. e. k' can appear twice in $\mathfrak{s}$). In an octic $\mathfrak{s}$, k's appear in four arguments of V's and will be denoted by k^*, k', k'' and k''', where k^*, k' and k'' are assumed to be free and distinct, while k''' is also free but can be distinct from, or identical with, k''. Obviously, under the assumption that k^* is the only link between two V's in $\mathfrak{s}$, $\mathfrak{s}$ cannot consist of V_6 or of (V_4, V_2) or of V_8 or of (V_6, V_2) or of (V_4, V_4) (in the two last cases, either V would contain an odd number of k's other than k^* and at least one of these would form a second link). Hence the only possible combinations of indices of V's are (3,3), (2,2,2), (5,3), (4,2,2), (3,3,2) and (2,2,2,2).

Considering the case (3,3), we see that $|k'|$ and $|k''|$ are respectively equal to the differences of the appropriate arguments inside both V_3's; therefore k' is redundant in $\mathfrak{s}$ (and so is k'', which, however, is not necessarily different from k'). A similar argument leads to the conclusion that in the cases (5,3) and (4,2,2) at least one of k', k'' and k''' is redundant in $\mathfrak{s}$.

In the case (2,2,2) the differences of arguments inside the three V_2's, denoted by d_1, d_2 and d_3, are

$$d_1 = \gamma^* + \delta^* |k^*| - \gamma' - \delta' |k'|,$$

$$d_2 = \gamma' + (1-\delta')|k'| - \gamma'' - \delta''|k''|,$$

$$d_3 = \gamma'' + (1-\delta'')|k''| - \gamma^* - (1-\delta^*)|k^*|,$$

where δ^*, δ' and δ'' are each equal to 0 or 1. Hence

$$d_1 + d_2 + d_3 = (1-2\delta')|k'| + (1-2\delta'')|k''| - (1-2\delta^*)|k^*|. \tag{4.1.19}$$

Consequently, the set $\{k^*, k', k''\}$, which reduces to a pair if $k' = k''$, is redundant in $\mathfrak{s}$, since in any event the coefficient of $|k^*|$ in (4.1.19) is different from 0.

Similar considerations in the case (2,2,2,2) lead to the conclusion that the set $\{k^*, k', k'', k'''\}$ contains a set (composed of two, three, or four elements) which is redundant in $\mathfrak{s}$.

There remains to investigate the case (3,3,2). If k^* is the only link between the two V_3's, then necessarily V_2 has two links with one V_3 and no links with the other one. Hence one of the γ's corresponding in $\mathfrak{s}$ to k', k'' or k''' appears in the latter V_3 twice and this k is redundant in $\mathfrak{s}$.

If k^* is the only link between one of the V_3's and V_2, then there must also be exactly one link between V_2 and the other V_3, and two or no links between both V_3's. If the V_3's are not linked, then in each of them the difference of those arguments in which γ^* does not appear is equal to $|k'|$ or $|k''|$ or $|k'''|$, and one of them is redundant in $\mathfrak{s}$. If the V_3's are linked by two distinct k's, then these k's form a redundant set. Finally,

if the V_3's are doubly linked by $k'' = k'''$, then the arguments of V_3's and V_2 are respectively:

$$(\gamma^* + \delta^* |k^*|,\ \alpha'' + \delta'' |k''|,\ \beta'' + \delta''' |k'''|),$$
$$\big(\gamma' + \delta' |k'|,\ \alpha'' + (1-\delta'') |k''|,\ \beta'' + (1-\delta''') |k'''|\big),$$
$$\big(\gamma' + (1-\delta') |k'|,\ \gamma^* + (1-\delta^*) |k^*|\big),$$

where γ' denotes α or β corresponding in $\mathfrak{s}$ to k', (α'', β'') is a pair of summation variables corresponding to k'', and δ^*, δ', δ'' and d''' are each equal to 0 or 1. Denote the differences of the two first arguments in both V_3's and V_2 by d_1, d_2, and d_3, respectively. Then

$$d_1 - d_2 + d_3 = (1-2\delta')|k'| + (1-2\delta'')|k''| - (1-2\delta^*)|k^*|. \tag{4.1.20}$$

Hence the set $\{k^*, k', k''\}$ is redundant in $\mathfrak{s}$.

Since all the possible subfactors $\mathfrak{s}$ have been considered, each $\mathfrak{s}$ depends on at least one redundant set. These redundancies obviously reduce further the number of k's the absolute values of which determine those of the other k's in ω when the differences of arguments of the V's composing ω are known. Let K_0 denote the set of k's appearing in ω. Given any subset B of K_0, let $K(B)$ denote the set of all those elements of $K_0 - B$ the absolute values of which are determined by those of the elements of B and by the differences of arguments of the V's in ω. It will be seen that we can construct by a recurrent procedure a set $G \subset K_0$ such that each subfactor $\mathfrak{s}$ in ω depends on at least one element of $K(G)$ and that $K_0 = G \cup K(G)$.

We begin by putting $G_0 = \emptyset$. We assume now that a set G_j composed of free k's is determined for a particular value of j. Equations like (4.1.18), (4.1.19), and (4.1.20) allow us to find the corresponding set $K(G_j)$ which obviously contains all the bound and redundant k's belonging to K_0. If $G_j \cup K(G_j) = K_0$, we stop the construction of the sequence $\{G_j\}$. Otherwise, at least one (free) variable k in one of the subfactors does not belong to $G_j \cup K(G_j)$. We form, then, G_{j+1} by adjoining one of those variables to G_j. If this variable belongs to a redundant set in any non-quadratic subfactor in ω, this set contained another free k, for otherwise, by the definition of redundant sets, the chosen variable would already belong to $K(G_j)$. Hence if each subfactor $\mathfrak{s}$ contained at least one free element of $K_0 - G_j$, this holds also for G_{j+1}, and since this was obviously true for G_0, this property holds for all the sequence $\{G_j\}$. However, since K_0 is finite, a finite number of steps will lead to the formation of a set G_j satisfying $G_j \cup K(G_j) = K_0$, and this will be the required set G.

Since each subfactor $\mathfrak{s}$ in ω depends on one free element of $K(G)$, since the number of such subfactors in ω is equal to μ_2 and since the same

k can appear in at most two different subfactors, the number of free elements in $K(G)$ is not smaller than $\mu_2/2$, so that (4.1.17) follows.

But (4.1.17) can be rewritten in the form

$$\Lambda(\omega) \leqslant \sum_{e \in E_1} \lambda + \sum_{e \in E_2} (\lambda - \tfrac{1}{2}) \tag{4.1.21}$$

and hence in the considered factors

$$A(\omega) \leqslant \sum_{e \in E_1} a + \sum_{e \in E_2} (a - \tfrac{1}{2}). \tag{4.1.22}$$

By (4.1.15), $a - \frac{1}{2} = \frac{1}{2} - \frac{1}{4}q - \frac{1}{2}r$ and hence $a - \frac{1}{2} \leqslant 0$ for $r \geqslant 1$ and $q \geqslant 0$, i. e. for any $e \in E$. Hence, since $a < 0$ for $e \in E_1$, (4.1.22) implies that $A(\omega) \leqslant 0$ for any ω composed of at least two extended subfactors; moreover, if $A(\omega) = 0$, then, for each extended subfactor in ω, $q = 0$ and $r = 1$. Consequently, all ω's of this type are bounded and they tend to zero except possibly those which are composed exclusively of (at least two) quartic subfactors.

It is easy to see that in the latter factors

$$\Lambda(\omega) \leqslant 1. \tag{4.1.23}$$

Indeed, the quartic subfactors which form ω can be given a cyclic ordering in such a way that any two neighbouring subfactors are linked by a k appearing in both. This is best seen if we interpret the subfactors as points, and each k as a segment connecting the two subfactors in which it appears, and which are necessarily distinct. We have, then, a graph in which every vertex belongs exactly to two segments. Moreover, the graph is connected, since otherwise the subfactors would form more than one factor. Such a graph is necessarily a cycle, and either of the two cyclic orderings of its vertices has the required property. But any quartic subfactor with two different free k's is either formed by one V_4 or $\tilde{V}_4$, and then each k in this subfactor is redundant, or it is formed by two V_2's or $\tilde{V}_2$'s, and then the two corresponding k's form a redundant set. Therefore, owing to the cyclic ordering of the subfactors, (4.1.23) holds.

Moreover, in ω's composed of quartic subfactors, $\mu(\omega) = \varrho$ and hence, by (4.1.12) and (4.1.23),

$$A(\omega) \leqslant \varrho + 1 - \tfrac{3}{2}\varrho = 1 - \varrho/2. \tag{4.1.24}$$

Consequently, for factors of this kind $A(\omega) = 0$ if and only if $\varrho = 2$ and $\Lambda(\omega) = 1$. An octic factor composed of two quartic subfactors and such that $\Lambda(\omega) = 1$ will be referred to as a "4 – 4" factor.

Thus by (4.1.13) all factors tend to zero except possibly the octic factors "8", "2 – 6", "2 – 2 – 4" and "4 – 4", which are bounded. It

will be convenient to accept a general term "I-factor" for any of the latter factors. The limit of any octic grand factor is necessarily equal to the sum of the limits of the appropriate I-factors. Obviously, each octic grand factor is an element of the covariance matrix of (4.1.5) (for instance octic grand factors which depend on two k's belonging to $\{k_1, \ldots, k_{n_0}\}$ form the variance of $2N^{\frac{1}{2}}S_N$). We proceed now to the computation of the limits of these grand factors.

The limit of $\operatorname{var} N^{\frac{1}{2}}S_N$ is given in [22] and the same formula with obvious substitutions yields $\lim\limits_{N\to\infty} \operatorname{var} N^{\frac{1}{2}}\tilde{S}_N$. Furthermore, owing to the mutual independence of the random variables generating $\{x_t\}$ and $\{\tilde{x}_t\}$, $\operatorname{cov}(S_N, \tilde{S}_N) = 0$. There remain the limits of $\operatorname{var} N^{\frac{1}{2}}M_N$ and $\operatorname{cov}(2N^{\frac{1}{2}}S_N, N^{\frac{1}{2}}M_N)$; the expression for the latter will also yield, with the appropriate substitutions, $\lim\limits_{N\to\infty} \operatorname{cov}(2N^{\frac{1}{2}}\tilde{S}_N, N^{\frac{1}{2}}M_N)$.

Let W_N be the set of all the systems of integral values of the variables $(\alpha_1, \beta_1, k_1, \alpha_2, \beta_2, k_2)$ satisfying

$$|k_i| \leqslant N-1, \quad 1 \leqslant \alpha_i,\ \beta_i \leqslant N-|k_i|, \quad i = 1, 2. \tag{4.1.25}$$

We start with the I-factors which form $N \operatorname{var} M_N$ (i. e. we consider (4.1.6) with $n_0 = n_1 = 0$ and $n_2 = 2$). Since they are necessarily quartic in each of $\{h_k\}$ and $\{\tilde{h}_k\}$, they can only be of the types "4 – 4" and "4 – 2 – 2". In the former factors $\Lambda(\omega) = 1$ and therefore factors containing any subfactor formed with one V_4 or $\tilde{V}_4$ obviously cannot be of the type "4 – 4"; hence we have only to consider quartic subfactors arising out of the process $\{x_t\}$ which are of the form

$$R_{\alpha_1-\alpha_2}R_{\alpha_1-\alpha_2+|k_1|-|k_2|} \quad \text{or} \quad R_{\alpha_1-\alpha_2-|k_2|}R_{\alpha_1-\alpha_2+|k_1|},$$

and their counterparts arising out of $\{\tilde{x}_t\}$. However, for factors like

$$N^{-3}\sum_{W_N} R_{\alpha_1-\alpha_2}R_{\alpha_1-\alpha_2+|k_1|-|k_2|}\tilde{R}_{\beta_1-\beta_2-|k_2|}\tilde{R}_{\beta_1-\beta_2+|k_1|}$$

or

$$N^{-3}\sum_{W_N} R_{\alpha_1-\alpha_2-|k_2|}R_{\alpha_1-\alpha_2+|k_1|}\tilde{R}_{\beta_1-\beta_2}\tilde{R}_{\beta_1-\beta_2+|k_1|-|k_2|}$$

we find $\Lambda(\omega) = 0$, because here the indices of the R's and $\tilde{R}$'s determine both $|k_1| - |k_2|$ and $|k_1| + |k_2|$, and, therefore, $|k_1|$ and $|k_2|$. Consequently, in the considered case the only "4 – 4" factors are of the form:

$$N^{-3}\sum_{W_N} R_{\alpha_1-\alpha_2}R_{\alpha_1-\alpha_2+|k_1|-|k_2|}\tilde{R}_{\beta_1-\beta_2}\tilde{R}_{\beta_1-\beta_2+|k_1|-|k_2|} \tag{a}$$

and

$$N^{-3}\sum_{W_N} R_{\alpha_1-\alpha_2-|k_2|}R_{\alpha_1-\alpha_2+|k_1|}\tilde{R}_{\beta_1-\beta_2-|k_2|}\tilde{R}_{\beta_1-\beta_2+|k_1|}, \tag{b}$$

while the "2 — 2 — 4" factors are of the form

(c) $$N^{-3}\sum_{W_N}\varkappa_4 V_4(\alpha_1-\alpha_2,\ \alpha_1-\alpha_2+|k_1|,\ 0,\ |k_2|)\tilde{R}_{k_1}\tilde{R}_{k_2},$$

(d) $$N^{-3}\sum_{W_N}R_{\alpha_1-\alpha_2}R_{\alpha_1-\alpha_2+|k_1|-|k_2|}\tilde{R}_{k_1}\tilde{R}_{k_2},$$

(e) $$N^{-3}\sum_{W_N}R_{\alpha_1-\alpha_2-|k_2|}R_{\alpha_1-\alpha_2+|k_1|}\tilde{R}_{k_1}\tilde{R}_{k_2},$$

or the "mirror reflections" of these three in which the parts of the two processes are permuted.

Factors forming $N\operatorname{cov}(2S_N, M_N)$ are sextic in the h's and quadratic in the $\tilde{h}$'s, which excludes the "8" and "4 — 4" factors. Thus we have only "2 — 6" and "2 — 2 — 4" factors. The former are easily seen to be

(f) $$N^{-3}\sum_{W_N}R_{\alpha_1-\beta_1}R_{\alpha_1-\alpha_2+|k_1|-|k_2|}R_{\beta_1-\alpha_2+|k_1|}\tilde{R}_{k_2},$$

(g) $$N^{-3}\sum_{W_N}R_{\alpha_1-\beta_1}R_{\alpha_2-\beta_1+|k_2|}R_{\alpha_1-\alpha_2}\tilde{R}_{k_2},$$

(h) $$N^{-3}\sum_{W_N}\varkappa_4 V_4(\beta_1-\alpha_2+|k_1|,\ \alpha_1-\alpha_2+|k_1|,\ 0,\ |k_2|)R_{\alpha_1-\beta_1}\tilde{R}_{k_2},$$

(i) $$N^{-3}\sum_{W_N}\varkappa_4 V_4(\alpha_1-\alpha_2,\ \beta_1-\alpha_2,\ 0,\ |k_2|)R_{\beta_1-\alpha_1}\tilde{R}_{k_2},$$

and those which arise out of (f) and (g) by permuting α_1 and β_1. The latter are

(j) $$N^{-3}\sum_{W_N}\varkappa_4 V_4(\alpha_1-\alpha_2,\ \alpha_1-\alpha_2+|k_1|,\ 0,\ |k_2|)R_{k_1}\tilde{R}_{k_2},$$

(k) $$N^{-3}\sum_{W_N}R_{\alpha_1-\alpha_2}R_{\alpha_1-\alpha_2+|k_1|-|k_2|}R_{k_1}\tilde{R}_{k_2},$$

(l) $$N^{-3}\sum_{W_N}R_{\alpha_1-\alpha_2-|k_2|}R_{\alpha_1-\alpha_2+|k_1|}R_{k_1}\tilde{R}_{k_2},$$

and those which arise out of them by permuting α_1 and β_1.

The limits of (a) through (l) can be found according to the general idea introduced in [22]. The whole procedure is based on Lemma 18.1 in [22] which states that

(4.1.26)

$$\lim_{N\to\infty}\sum_{u_1,\ldots,u_s=-\infty}^{\infty}c^{(N)}_{u_1,\ldots,u_s}a_{u_1,\ldots,u_s}=\sum_{u_1,\ldots,u_s=-\infty}^{\infty}\Big(\lim_{N\to\infty}c^{(N)}_{u_1,\ldots,u_s}\Big)a_{u_1,\ldots,u_s}$$

provided that $\sum_{u_1,\ldots,u_s=-\infty}^{\infty} a_{u_1,\ldots,u_s}$ is absolutely convergent, that there exists a positive constant A such that $|c^{(N)}_{u_1,\ldots,u_s}| < A$ for any N and any $u_1, \ldots, u_s$, and that the limits of $c^{(N)}_{u_1,\ldots,u_s}$ exist.

All (a) through (l) factors can be presented in the form

$$\sum_{p,q,s=-\infty}^{\infty} c^{(N)}_{p,q,s}\, a_{p,q,s}, \tag{4.1.27}$$

where

$$(p, q, s) = \begin{cases} (\alpha_1-\alpha_2, \beta_1-\beta_2, |k_1|+|k_2|) & \text{in (a)},\\ (\alpha_1-\alpha_2-|k_2|, \beta_1-\beta_2-|k_2|, |k_1|-|k_2|) & \text{in (b)},\\ (\alpha_1-\alpha_2, k_1, k_2) & \text{in (c), (d), (e), (j), (k), (l)},\\ (\alpha_1-\beta_1, \beta_1-\alpha_2+|k_1|, k_2) & \text{in (f)},\\ (\beta_1-\alpha_1, \alpha_1-\alpha_2, k_2) & \text{in (g)},\\ (\beta_1-\alpha_2+|k_1|, \alpha_1-\beta_1, k_2) & \text{in (h)},\\ (\alpha_1-\alpha_2, \beta_1-\alpha_1, k_2) & \text{in (i)}, \end{cases} \tag{4.1.28}$$

$N^3 c^{(N)}_{p,q,s}$ for any given p, q, s is the number of elements of W_N satisfying (4.1.28), and

$$a_{p,q,s} = \begin{cases} R_p R_{p+s} \tilde{R}_q \tilde{R}_{q+s} & \text{in (a) and (b)},\\ \varkappa_4 V_4(p, p+q, 0, |s|) \tilde{R}_q \tilde{R}_s & \text{in (c)},\\ R_p R_{p+q+s} \tilde{R}_q \tilde{R}_s & \text{in (d)},\\ R_{p-s} R_{p+q} \tilde{R}_q \tilde{R}_s & \text{in (e)},\\ R_p R_{p+q+s} R_q \tilde{R}_s & \text{in (f), (g) and (k)},\\ \varkappa_4 V_4(p, p+q, 0, |s|) R_q \tilde{R}_s & \text{in (h), (i), (j)},\\ R_{p-s} R_{p+q} R_q \tilde{R}_s & \text{in (l)}. \end{cases} \tag{4.1.29}$$

The final form of $a_{p,q,s}$ given in (4.1.29) is due to the fact that for any u

$$R_u = R_{-u} = R_{|u|}. \tag{4.1.30}$$

In order to apply the lemma about the limits of multiple series we must find the limits of $c^{(N)}_{p,q,s}$ and verify that $c^{(N)}_{p,q,s}$ are collectively bounded in each of the cases (a) through (l).

Cases (a) and (b) with a slightly changed set W_N were considered in [22]. By nearly identical considerations we see that $|c^{(N)}_{p,q,s}| \leqslant 4$ in both cases and that

$$\lim_{N\to\infty} c^{(N)}_{p,q,s} = \begin{cases} 0 & \text{in (a)},\\ 4/3 & \text{in (b)}. \end{cases}$$

Taking (c) into account, we see that the number of elements of W_N satisfying (4.1.28) for any given p, q, s and $N > \max(|q|, |s|) + |p|$ is contained between $\big(N - \max(|q|, |s|) - |p|\big)(N - |q|)(N - |s|)$ and $(N - \max(|q|, |s|)\ (N - |q|)(N - |s|)$; multiplied by N^{-3}, both expressions are equal to 1 in the limit. Hence $\lim\limits_{N\to\infty} c_{p,q,s}^{(N)} = 1$ and $|c_{p,q,s}^{(N)}| \leqslant 1$ for any p, q, s.

For (f) and (g),

$$N^{-3}(N-|s|) \sum_{k_1=1-N}^{N-1} (N - |p| - |s| - |q| - |k_1|)$$
$$\leqslant c_{p,q,s}^{(N)} \leqslant N^{-3}(N-s) \sum_{k_1=1-N}^{N-1} (N - |k_1|)$$

so that $\lim\limits_{N\to\infty} c_{p,q,s}^{(N)} = 1$ and $|c_{p,q,s}^{(N)}| \leqslant 2$. The same conclusion can be drawn for (h) and (i), because these cases differ from (f) and (g) only in having p and q permuted.

Since in all the cases the limit of $c_{p,q,s}^{(N)}$ does not depend upon p, q, s and since $c_{p,q,s}^{(N)}$ are collectively bounded, it remains to compute $\sum\limits_{p,q,s=-\infty}^{\infty} a_{p,q,s}$. But by (4.1.4) and (4.1.30) $\sum\limits_{p,q,s=-\infty}^{\infty} R_p R_{p+s} \tilde{R}_q \tilde{R}_{q+s}$, $\sum\limits_{p,q,s=-\infty}^{\infty} R_p R_{p+q+s} \tilde{R}_q \tilde{R}_s$ and $\sum\limits_{p,q,s=-\infty}^{\infty} R_{p-s} R_{p+q} \tilde{R}_q \tilde{R}_s$ are equal to $\sigma_{2,2}$, while $\sum\limits_{p,q,s=-\infty}^{\infty} R_p R_{p+q+s} R_q \tilde{R}_s$ and $\sum\limits_{p,q,s=-\infty}^{\infty} R_{p-s} R_{p+q} R_q \tilde{R}_s$ are equal to $\sigma_{3,1}$. Furthermore,

$$\varkappa_4 \sum_{p,q,s=-\infty}^{\infty} V_4(p, p+q, 0, |s|) \tilde{R}_q \tilde{R}_s$$
$$= \varkappa_4 \sum_{q,s=-\infty}^{\infty} \tilde{R}_q \tilde{R}_s \sum_{j=-\infty}^{\infty} h_{-j} h_{|s|-j} \sum_{p=-\infty}^{\infty} h_{p-j} h_{p+q-j}$$
$$= \varkappa_4 \varkappa_2^{-2} \sum_{q=-\infty}^{\infty} R_q \tilde{R}_q \sum_{s=-\infty}^{\infty} R_s \tilde{R}_s = \varkappa_4 \varkappa_2^{-2} \sigma_{1,1}^2,$$

and proceeding in the same way we find that $\sum\limits_{p,q,s=-\infty}^{\infty} a_{p,q,s}$ in (h), (i) and (j) is equal to $\varkappa_4 \varkappa_2^{-2} \sigma_{2,0} \sigma_{1,1}$.

Consequently, $\lim\limits_{N\to\infty} N \operatorname{var} M_N$ is a sum of $\frac{4}{3}\sigma_{2,2}$ (from (b), the sum of (a) tending to 0), of $(\varkappa_4 \varkappa_2^{-2} + \tilde{\varkappa}_4 \tilde{\varkappa}_2^{-2}) \sigma_{1,1}^2$ (from (c) and its "mirror reflection") and of $4\sigma_{2,2}$ (from (d), (e) and their "mirror reflections"); $\lim\limits_{N\to\infty} N \operatorname{cov}(2S_N, M_N)$ is a sum of $8\sigma_{3,1}$ (from (f), (g), (k), (l) and their

counterparts having α_1 permuted with β_1) and of $4\varkappa_4\varkappa_2^{-2}\sigma_{1,1}\sigma_{2,0}$ (from (h), (i), (j) and the counterpart of (j) with β_1 instead of α_1).

Thus we have proved that the limit of the covariance of the i-th and j-th coordinate of (4.1.5), say $\mu_{i,j}$ $(i, j = 1, 2, 3)$, is as stated in this theorem. As to the higher moments, the limit of (4.1.6) for any odd n is obviously equal to zero since none of the terms of the expansion of (4.1.6) forms a product of octic factors. For an even n, the limit of (4.1.6) is a sum of products of limits of octic grand factors, the sum being extended over all the groupings of the set $K = \{k_1, \ldots, k_n\}$ into pairs. In other words, the limit of (4.1.6) is the sum of products of the appropriate elements of $\|\mu_{i,j}\|$. We shall now find its exact value.

Let J be the set of all sequences $j(1), \ldots, j(n)$ such that n_0 elements of the sequence are equal to 1, n_1 elements are equal to 2 and n_2 elements are equal to 3. Then to each permutation of $k_1, \ldots, k_n$ there corresponds exactly one element of J if we map K into $\{1, 2, 3\}$ so that 1 is assigned to each k belonging to $K_1 = \{k_1, \ldots, k_{n_0}\}$, 2 is assigned to each k belonging to $K_2 = \{k_{n_0+1}, \ldots, k_{n_0+n_1}\}$, and 3 is assigned to each k belonging to $K_3 = \{k_{n_0+n_1+1}, \ldots, k_n\}$. The same element of J corresponds to those and only those permutations of $k_1, \ldots, k_n$ in which only elements belonging to the same K_i $(i = 1, 2, 3)$ are permuted; thus there are $n_0!\, n_1!\, n_2!$ permutations corresponding to the same element of J. Consequently the sum of $\mu_{j(1),j(2)} \cdot \ldots \cdot \mu_{j(n-1),j(n)}$ over all the permutations of $k_1, \ldots, k_n$ is equal to $n_0!\, n_1!\, n_2! \sum_J \mu_{j(1),j(2)} \cdot \ldots \cdot \mu_{j(n-1),j(n)}$. But in view of $\mu_{i,j} = \mu_{j,i}$, the value of the summand of the last expression is the same for all the permutations $k_{u(1)}, \ldots, k_{u(n)}$ of $k_1, \ldots, k_n$ such that the same grouping of K is obtained by forming the pairs

$$(k_{u(1)}, k_{u(2)}), \ldots, (k_{u(n-1)}, k_{u(n)}),$$

and the number of those permutations is $(n/2)!\, 2^{n/2}$. Hence the limit of (4.1.6) for an even n is equal to

$$\frac{n_0!\, n_1!\, n_2!}{\left(\frac{n}{2}\right)!\, 2^{n/2}} \sum_J \mu_{j(1),j(2)} \cdot \ldots \cdot \mu_{j(n-1),j(n)}.$$

Consequently by the multivariate version of the Second Limit Theorem ([21], Corollary 7.2) the asymptotic distribution of (4.1.5) is normal with zero means and with a covariance matrix $\|\mu_{i,j}\|$. Using a truncation device similar to that introduced in [21], we can replace the assumption about the existence of all the moments of ε_t and $\tilde{\varepsilon}_t$ by $H_{8,8}$. It follows that under $\tilde{H}_{4L_1,4L}$ $(L_1, L_2 \geqslant 2)$ all the moments of (4.1.5) up to the order L_1

or L_2, respectively, tend to the respective moments of the limiting distribution.

THEOREM 4.2. *Under $\tilde{H}_{0,0}$, the distribution of $A_N - N^{\frac{1}{2}}\theta$ is asymptotically normal with a zero mean and a variance equal to*

$$\sigma_{\Delta}^2 = \tfrac{28}{3}(\sigma_{4,0}+\sigma_{0,4}) - 16(\sigma_{3,1}+\sigma_{1,3}) + \tfrac{64}{3}\sigma_{2,2} + \varkappa_4\varkappa_2^{-2}(4\sigma_{2,0}^2+4\sigma_{1,1}^2 - 8\sigma_{1,1}\sigma_{2,0}) + \tilde{\varkappa}_4\tilde{\varkappa}_2^{-2}(4\sigma_{0,2}^2+4\sigma_{1,1}^2-8\sigma_{1,1}\sigma_{0,2}).$$

Moreover, under $\tilde{H}_{4L,4L}$ the central moments of $A_N - N^{\frac{1}{2}}\theta$ up to the order L tend to the respective moments of the limiting distribution.

Proof. $A_N - EA_N$ is a linear combination of the coordinates of (4.1.5) and hence by Theorem 4.1 its limiting distribution is normal with the variance equal to σ_{Δ}^2 and the statement concerning the central moments is true. In view of the convergence theorem [5] it remains to prove that $EA_N - N^{\frac{1}{2}}\theta$ tends to zero with $N \to \infty$.

It is known ([22]) that $ES_N = \sum_{k=-\infty}^{\infty} R_k^2 + O(N^{-1})$. Consequently,

$$(4.1.31)\quad EA_N - N^{\frac{1}{2}}\theta = N^{\frac{1}{2}}\Big[ES_N + E\tilde{S}_N - 2EM_N - \sum_{k=-\infty}^{\infty}(R_k-\tilde{R}_k)^2\Big]$$

$$= 2N^{\frac{1}{2}}\Big(\sum_{k=-\infty}^{\infty} R_k\tilde{R}_k - EM_N\Big) + O(N^{-\frac{1}{2}}).$$

Now, changing slightly the proof of Lemma 2.3 in [36], we can prove that under $\tilde{H}_{4,4}$ $\sum_{k=1}^{\infty} k|R_k\tilde{R}_k| < \infty$. Indeed,

$$\sum_{r=0}^{\infty}|h_r|\sum_{s=0}^{\infty}|\tilde{h}_s|\sum_{k=1}^{\infty}k(h_k^2+\tilde{h}_k^2) = \sum_{r,s=0}^{\infty}|h_r\tilde{h}_s|\sum_{k=1}^{\infty}k(h_k^2+\tilde{h}_k^2)$$

$$\geqslant \tfrac{1}{2}\sum_{r,s=0}^{\infty}|h_r\tilde{h}_s|\Big(\sum_{k=1}^{\infty}(k+r)h_{k+r}^2 + \sum_{k=1}^{\infty}(k+s)\tilde{h}_{k+s}^2\Big)$$

$$\geqslant \tfrac{1}{2}\sum_{r,s=0}^{\infty}|h_r\tilde{h}_s|\sum_{k=1}^{\infty}k|h_{r+k}\tilde{h}_{s+k}| \geqslant \sum_{r,s=0}^{\infty}|h_r\tilde{h}_s|\sum_{k=1}^{\infty}k|h_{r+k}\tilde{h}_{s+k}|$$

$$= \sum_{k=1}^{\infty}k\sum_{r,s=0}^{\infty}|h_r\tilde{h}_s h_{r+k}\tilde{h}_{s+k}| = \sum_{k=1}^{\infty}k\sum_{r=0}^{\infty}|h_r h_{r+k}|\sum_{s=0}^{\infty}|\tilde{h}_s\tilde{h}_{s+k}| \geqslant \sum_{k=1}^{\infty}k|R_k\tilde{R}_k|,$$

and the three sums in the first line are convergent. Hence, owing to the independence of $C_{k,N}$ and $\tilde{C}_{k,N}$,

$$\begin{aligned}
&\Big|N^{\frac{1}{2}}\Big(\sum_{k=-\infty}^{\infty} R_k\tilde{R}_k - EM_N\Big)\Big| \\
&\quad = \Big|N^{\frac{1}{2}}\Big(\sum_{|k|\geqslant N} R_k\tilde{R}_k + 2\sum_{|k|<N}\frac{|k|}{N}R_k\tilde{R}_k - \sum_{|k|<N}\frac{k^2}{N^2}R_k\tilde{R}_k\Big)\Big| \\
&\quad \leqslant N^{-\frac{1}{2}}\Big\{\sum_{|k|\geqslant N} N|R_k\tilde{R}_k| + 2\sum_{|k|<N}|k|\,|R_k\tilde{R}_k| + \sum_{|k|<N}\frac{|k|}{N}|k|\,|R_k\tilde{R}_k|\Big\} \\
&\quad \leqslant N^{-\frac{1}{2}}\Big\{\sum_{|k|\geqslant N}|k|\,|R_k\tilde{R}_k| + 3\sum_{|k|<N}|k|\,|R_k\tilde{R}_k|\Big\} \\
&\quad \leqslant 6N^{-\frac{1}{2}}\sum_{k=1}^{\infty} k|R_k\tilde{R}_k| = O(N^{-\frac{1}{2}})
\end{aligned}$$

and in view of (4.1.31) $\lim\limits_{N\to\infty}(EA_N - N^{\frac{1}{2}}\theta) = 0$, which was to be proved.

Now A_N must be suitably normed and, therefore, a sequence of statistics is needed which converges in probability to σ_A.

Put

$$C_{k,\nu,r} = \frac{1}{\nu-|k|}\sum_{t=1}^{\nu-|k|} x_{t+r\nu}x_{t+r\nu+|k|}, \qquad |k|<\nu,\ r = 0,\ldots,\mu-1,$$

and let $\tilde{C}_{k,\nu,r}$ be defined as $C_{k,\nu,r}$ with x's replaced by $\tilde{x}$'s. Furthermore, we replace $C_{k,N}$, $\tilde{C}_{k,N}$ and N appearing in S_N, $\tilde{S}_N$ and M_N by $C_{k,\nu,r}$, $\tilde{C}_{k,\nu,r}$ and ν, respectively, and denote the three thus transformed expressions by $S_{\nu,r}$, $\tilde{S}_{\nu,r}$ and $M_{\nu,r}$. Then let

$$A_{\nu,r} = \nu^{\frac{1}{2}}(S_{\nu,r} + \tilde{S}_{\nu,r} - 2M_{\nu,r}), \qquad B_{\mu,\nu} = \frac{1}{\mu}\sum_{r=0}^{\mu-1}\Big(A_{\nu,r} - \frac{1}{\mu}\sum_{t=0}^{\mu-1}A_{\nu,t}\Big)^2.$$

LEMMA 4.3. *Under* $\tilde{H}_{8,8}$

$$\operatorname*{plim}_{N\to\infty} B_{\mu,\nu} = \sigma_A^2.$$

Proof. Obviously, the expectation and variance of $A_{\nu,r}$ for any r are respectively equal to EA_ν and $\operatorname{var} A_\nu$, where A_ν is given by (4.1.2) with N replaced by ν. Let

$$B'_{\mu,\nu} = \mu^{-1}\sum_{r=0}^{\mu-1}(A_{\nu,r} - EA_\nu)^2.$$

Hence $EB'_{\mu,\nu} = \mu^{-1}\sum_{r=0}^{\mu-1} \operatorname{var} A_{\nu,r} = \operatorname{var} A_\nu$ and therefore by Theorem 4.2

$$\lim_{N\to\infty} EB'_{\mu,\nu} = \sigma_A^2, \tag{4.1.32}$$

because $\nu \to \infty$ when $N \to \infty$.

It can be checked that after easy transformations

$$E(B'_{\mu,\nu} - B_{\mu,\nu})^2 = \mu^{-4} \sum_{r_1,r_2,r_3,r_4=0}^{\mu-1} E\Big[\prod_{i=1}^{4} (A_{\nu,r_i} - EA_\nu)\Big]. \tag{4.1.33}$$

The summand of the right-hand side of (4.1.33) is one of the fourth mixed central moments of $(A_{\nu,r_1}, \ldots, A_{\nu,r_4})$ and it differs only slightly from the fourth central moment of A_ν; but the latter is a linear combination of fourth moments (mixed and univariate) of $v^{\frac{1}{2}}(2S_\nu, 2\tilde{S}_\nu M_\nu)$ which can be expanded in the manner described in the proof of Theorem 4.1. Consequently, the summand in (4.1.33) can be presented as a linear combination with constant coefficients and a finite number of terms consisting either of two octic factors or of one factor of degree 16, the factors being formed according to the rules (a), (b) and (c) given in the proof of Theorem 4.1, with N replaced by ν and with α_i and β_i in the arguments of V's and $\tilde{V}$'s replaced by $\alpha_i + r_i\nu$ and $\beta_i + r_i\nu$ $(i = 1, \ldots, 4)$. Hence the right-hand side of (4.1.33) is a linear combination, with constant coefficients multiplied by μ^{-4}, of terms consisting of a fourfold sum with respect to $r_1, \ldots, r_4$ and of a product of one or two factors described above. Proceeding as before in the proof of Theorem 4.1 we can easily show that all these terms are bounded and that they are zero in the limit except possibly those which contain two factors identical to some I-factors subject to the previously described modifications concerning α's, β's and N. But obviously in each of the latter terms the sums with respect to r's can be factorized and μ^{-4} can be apportioned among both factors. Consequently, the terms under consideration are products of two octic factors, each of them being a modified I-factor with an associated double sum with respect to r's and with the coefficient μ^{-2}. Any such factor tends to zero; the proof of this statement will be now outlined for a factor corresponding to the I-factor described in case (c) in the proof of Theorem 4.1. Under this assumption we can present the considered factor in the form $\sum_{p,q,s=-\infty} d^{(\nu)}_{p,q,s} a_{p,q,s}$, where $a_{p,q,s}$ is given by the third line of (4.1.29) and $\mu^2\nu^2 d^{(\nu)}_{p,q,s}$ is the number of all the systems of integral values of the variables $(r_1, \alpha_1, \beta_1, k_1, r_2, \alpha_2, \beta_2, k_2)$ satisfying

$$p = \alpha_1 - \alpha_2 + (r_1 - r_2)\nu, \quad q = k_1, \; s = k_2 \tag{4.1.34}$$

and such that $0 \leqslant r_i \leqslant \mu - 1$, $1 \leqslant \alpha_i, \beta_i \leqslant \nu - |k_i|$, $|k_i| < \nu$, $i = 1, \ldots, 4$. We can easily see that $\mu^2\nu^3 d^{(\nu)}_{p,q,s} = O(\mu\nu^3)$ and $0 \leqslant d^{(\nu)}_{p,q,s} \leqslant 2$ for any

given p, q, s and ν. Hence by (4.1.26) the investigated factor tends to zero.

For the other factors corresponding to the remaining I-factors the proofs are almost identical because in each case the formulae expressing p, q and s in terms of α's, β's and r's contain a condition similar to the first one in (4.1.34) and therefore the corresponding $d^{(\nu)}_{p,q,s}$ is $O(\mu\nu^3)$. This is easily seen for any I-factor considered in the proof of Theorem 4.1 and can be also checked for any I-factor arising in $N\operatorname{var}S_N$ or $N\operatorname{var}\tilde{S}_N$ in view of the fact that the involved "8" and "4 — 4" factors were considered in the proofs of Propositions 18.4 and 19.1 in [22], and slightly changed "2 — 2 — 4" and "2 — 6" factors were considered in the same paper in the proofs of Propositions 20.1 and 21.1, respectively.

Consequently $\lim\limits_{N\to\infty} E(B'_{\mu,\nu}-B_{\mu,\nu})^2 = 0$ and hence

$$p\lim_{N\to\infty}(B'_{\mu\nu}-B_{\mu\nu}) = 0. \tag{4.1.35}$$

It remains to prove that $p\lim\limits_{N\to\infty} B'_{\mu,\nu} = \sigma_A^2$.

Consider the limit of

$$\begin{aligned}(4.1.36)\qquad & EB'^2_{\mu,\nu}-(EB'_{\mu,\nu})^2\\ &= \mu^{-2}\sum_{r,t=0}^{\mu-1}\{E[(A_{\nu,r}-EA_\nu)^2(A_{\nu,t}-EA_\nu)^2]-E(A_{\nu,r}-EA_\nu)^2E(A_{\nu,t}-EA_\nu)^2\}.\end{aligned}$$

For any r and t, the summand of the right-hand side of (4.1.36) consists of two expressions: $E[(A_{\nu,r}-EA_\nu)^2(A_{\nu,t}-EA_\nu)^2]$ and $-E(A_{\nu,r}-EA_\nu)^2E(A_{\nu,t}-EA_\nu)^2$.

The first expression is one of fourth central moments of $(A_{\nu,r}, A_{\nu,t})$. Proceeding as before in the case of the summand of (4.1.33) we can present this expression as a linear combination with constant coefficients and a finite number of terms consisting either of two octic factors or of one factor of degree 16; these factors are formed according to the rules (a), (b) and (c) given in the proof of Theorem 4.1, with the following differences: (1) α_i and β_i in the arguments of V's and $\tilde{V}$'s are replaced by $\alpha_i+r\nu$ and $\beta_{i+r\nu}$ for $i = 1, 2$ and by $\alpha_i+t\nu$ and $\beta_i+t\nu$ for $i = 3, 4$; (2) N is replaced by ν.

The second expression is a product of $-\operatorname{var}A_{\nu,r}$ and $\operatorname{var}A_{\nu,t}$. Obviously, $\operatorname{var}A_{\nu,r}$ is a linear combination with constant coefficients of octic factors which are identical with the octic factors in the expansion of (4.1.6) in Theorem 4.1 if only N is replaced by ν and α_i and β_i in the arguments of V's and $\tilde{V}$'s are replaced by $\alpha_i+r\nu$ and $\beta_i+r\nu$ $(i = 1, 2)$; $\operatorname{var}A_{\nu,t}$ can be expanded in a similar way, but it is convenient to use summation variables (k_i, α_i, β_i) with $i = 3, 4$.

It is easily seen that those terms in the first expression which do

not contain any factor with r and t appearing both in the arguments of V's and $\tilde{V}$'s forming this factor cancel out with the appropriate terms in the second expression appearing with the opposite sign. It follows that all the terms in the second expression are cancelled for any r and t. Consequently, after the cancellations, the right-hand side of (4.1.36) is a linear combination, with constant coefficients multiplied by μ^{-2}, of terms consisting of a double sum with respect to r and t and of a product of one or two factors such that r and t appear both in each factor. Proceeding as before in the proof of Theorem 4.1 we show that all these terms with their coefficients are bounded and they are zero in the limit except possibly terms containing two modified I-factors (i. e. I-factors subject to modifications concerning α's, β's and N which were previously described). A product of any two modified I-factors with the corresponding summation signs with respect to r and t and with the coefficient μ^{-2} can be presented in the form $\sum e_{\Pi}^{(\nu)} a_{\Pi}$, where $\Pi = (p_1, q_1, s_1, p_2, q_2, s_2)$ with p_i, q_i, s_i $(i = 1, 2)$ assuming any integral values in $(-\infty, \infty)$, the sum being extended to all the possible combinations of arguments of Π. The exact form of $e_{\Pi}^{(\nu)}$ and a_{Π} depends on the precise nature of the two involved modified I-factors. It is clear that all the possible pairs of modified I-factors have to be taken into account. For example, let us investigate the situation when both modified I-factors correspond to case (c) described in the proof of Theorem 4.1. Moreover, assume that $\alpha_1, \beta_1, k_1, \alpha_3, \beta_3, k_3$ are the summation variables of one factor and $\alpha_2, \beta_2, k_2, \alpha_4, \beta_4, k_4$ are those of the other one; in this way both r and t will appear in each factor since α_i and β_i correspond to r for $i = 1, 2$ and to t for $i = 3, 4$. Then, proceeding as in the proof of Theorem 4.1, we find

$$\begin{aligned} p_1 &= \alpha_1 - \alpha_3 + (r-t)\nu, \quad & q_1 &= k_1, \quad & s_1 &= k_3, \\ p_2 &= \alpha_2 - \alpha_4 + (r-t)\nu, \quad & q_2 &= k_2, \quad & s_2 &= k_4 \end{aligned} \tag{4.1.37}$$

(and this is the counterpart of the third line of (4.1.28)), and

$$a_{\Pi} = \varkappa_4^2 V_4(p_1, p_1+q_1, 0, |s_1|)\, V_4(p_2, p_2+q_2, 0, |s_2|) \cdot \tilde{R}_{q_1} \tilde{R}_{q_2} \tilde{R}_{s_1} \tilde{R}_{s_2}$$

(and this is the counterpart of the second line of (4.1.29)), while $\mu^2 \nu^6 e_{\Pi}^{(\nu)}$ is the number of different values of Π satisfying (4.1.37) and such that

$$0 \leqslant r, \quad t \leqslant \mu - 1, \quad 1 \leqslant \alpha_i, \beta_i \leqslant \nu - |k_i|, \quad |k_i| < \nu \quad (i = 1, \ldots, 4).$$

Now, in view of the first equation of (4.1.37), for $\nu > \max(|p_1|, |p_2|) + \max(|q_1|, |q_2|, |s_1|, |s_2|)$ and for any given value of Π, at most 2μ pairs of (r, t) must be considered so that $\mu^2 \nu^6 e_{\Pi}^{(\nu)} \leqslant 2\mu\nu^6$. Hence in this case $\lim_{N\to\infty} e_{\Pi}^{(\nu)} = 0$ and $0 \leqslant e_{\Pi}^{(\nu)} \leqslant 2$ for any Π and ν; therefore by (4.1.26) the involved term tends to zero when $N \to \infty$.

The same conclusion holds for any other pair of modified I-factors; the reasoning is quite similar to that described above. This follows from the fact that in each case the formula replacing (4.1.37) contains at least one expression similar to the first expression of (4.1.37), the only alterations being that p_1 can be replaced by p_2 or q_1 or q_2 and $\alpha_1-\alpha_3$ can be replaced by $\alpha_1-\beta_3$ or $\beta_1-\beta_3$ or $\alpha_4-\beta_4$ or $\beta_2-\beta_4$. Because of this at most 2μ pairs of (r, t) must be taken into account, and since the number of admissible sequences of values of $(\alpha_i, \beta_i, k_i;\ i = 1, 2)$ for any pair (r, t) is smaller than ν^6, $\lim\limits_{N\to\infty} e_{\Pi}^{(\nu)} = 0$ for any Π and $e_{\Pi}^{(\nu)}$ are collectively bounded. Therefore the limit of the left-hand side of (4.1.36) is zero and hence by (4.1.32)

$$\lim_{N\to\infty} EB'^2_{\mu,\nu} = \sigma_A^4. \tag{4.1.38}$$

It follows that by (4.1.32) and (4.1.38)

$$\lim_{N\to\infty} E(B'_{\mu,\nu}-\sigma_A^2)^2 = \lim_{N\to\infty} EB'^2_{\mu,\nu} - 2\lim_{N\to\infty} EB'_{\mu,\nu}\sigma_A^2 + \sigma_A^4 = 0$$

and therefore $p\lim\limits_{N\to\infty} B'_{\mu,\nu} = \sigma_A^2$ which completes the proof in view of (4.1.35).

Now we can define the test A which rejects H_0: $\theta = 0$ when $A_N(B_{\mu,\nu})^{-\frac{1}{2}} > \zeta_{1-\alpha}$, $\zeta_{1-\alpha}$ being determined by (3.1.4).

THEOREM 4.4. *Under $\tilde{H}_{8,8}$, the power of the test A is asymptotically equal to* 1 *when $\theta > 0$, and to α when $\theta = 0$.*

Proof. By Theorem 4.2, Lemma 4.3 and the convergence theorem ([5]) $(A_N - N^{\frac{1}{2}}\theta)(B_{\mu,\nu})^{-\frac{1}{2}}$ is asymptotically normally distributed with a zero mean and a unit variance. Hence for $\theta = 0$

$$\lim_{N\to\infty} P\big(A_N(B_{\mu,\nu})^{-\frac{1}{2}} > \zeta_{1-\alpha}\big) = \alpha$$

and the second part of the theorem is proved. Furthermore, by Theorem 4.2 $p\lim\limits_{N\to\infty} N^{-\frac{1}{2}}A_N = \theta$ so that in view of Lemma 4.3 it is easy to check that for $\theta > 0$ $\{A_N(B_{\mu,\nu})^{-\frac{1}{2}}\}$ tends in probability to $+\infty$ and consequently

$$\lim_{N\to\infty} P\big(A_N(B_{\mu,\nu})^{-\frac{1}{2}} > \zeta_{1-\alpha}\big) = 1.$$

4.2. Connections with trend estimation problems. Although the test A is directly applicable in practice in the cases considered in Section 4.1 only, it is hoped to be also useful in trend estimation and post trend elimination analysis.

Let $\{z_t\} = \{x_t + y_t\}$ and $\{\tilde{x}_t\}$ be any processes such that $\{x_t\}$ and $\{\tilde{x}_t\}$ satisfy $\tilde{H}_{8,8}$ and $\{y_t\}$ is a determinate trend. We assume that we can independently observe $\{z_t\}_N$ and $\{\tilde{x}_t\}_N$. For instance, $\{z_t\}$ is an output

of a technical device when an experiment of some kind is performed and $\{\tilde{x}_t\}$ is an output of the same device in a control situation; more generally, $\{\tilde{x}_t\}$ is the noise which is expected to exist in the investigated process $\{z_t\}$ and which can be separately examined.

One question which arises belongs to post trend elimination problems. We ask whether $\{x_t\}$ and $\{\tilde{x}_t\}$ are identical, i. e. whether $\{\tilde{x}_t\}$ is really the only stochastic component of $\{z_t\}$. It is natural to solve this problem eliminating trend from $\{z_t\}_N$ and testing the equivalence of $\{x_t\}$ and $\{\tilde{x}_t\}$ on the basis of the residuals $\{z_t - \hat{y}_t^{(N)}\}_N$ and $\{\tilde{x}_t\}_N$. It would be interesting to know whether, for a chosen class of trends and a chosen trend estimator $\{\hat{y}_t^{(N)}\}$, the power of the test A applied to $\{z_t - \hat{y}_t^{(N)}\}_N$ and $\{\tilde{x}_t\}_N$ preserves its asymptotic properties. If we confine ourselves to $\{y_t\} \in S'$ and to the trend estimator $\{\hat{y}_{t,\hat{m}}\}$ when $\hat{m}$ satisfies (2.8), we only have to prove that the replacement of $\{x_t\}_N$ by $\{x_t + \gamma_{t,m}\}_N$ does not affect the asymptotic power of the test A. It is believed that this is true although the proof has not been carried out.

Alternatively, suppose that $\{z_t\} = \{\tilde{x}_t\} + \{y_t\}$ and that we want to estimate $\{y_t\}$ on the basis of independent realizations of $\{z_t\}$ and of $\{\tilde{x}_t\}$. Let $\{F_N\}$ be a sequence of test functions which depend on $\{\tilde{x}_t\}_N$ and $\{z_t\}_N$. The idea of F-estimators introduced in Section 2 can be extended to cover this case; it seems that the test A might be useful in this context too.

5. Comments

Mathematical models in which the investigated process $\{z_t\}$ is a sum of a systematic "trend" component $\{z_t\}$ and a random "noise" component $\{x_t\}$ have been frequently considered in statistical literature from its early beginnings until the most recent times. These "error" models differ in assumptions concerning $\{x_t\}$, $\{y_t\}$ and the set T of values of parameter t. T is usually specified as $\{0, \pm 1, \ldots\}$ or $\langle 0, 1\rangle$ or as a multidimensional counterpart of any of these sets. The noise component $\{x_t\}$ is a stationary process of some kind, one — or multi-dimensional: "white noise" with a known variance in the simplest case, and a linear process with unknown spectrum in very general cases. The trend component $\{y_t\}$ was usually assumed to be linear with respect to some real parameters $a_0, \ldots, a_m$ (the number m being assumed to be known) and the properties of $\hat{a}_i$, i. e. of the estimators of a_i $(i = 0, \ldots, m)$ based on the sample $z_{t(1)}, \ldots, z_{t(N)}$, $t(j) \in T$ for $j = 1, \ldots, N$, were extensively investigated (see e. q. [1], [7], [13], [17], [31]), especially when $\hat{a}_i$ was a least-square or maximum likelihood estimate of a_i. Moreover, in these models the properties of $\hat{a}_i$ implied in an obvious way some properties of trend estimator $\{\hat{y}_t^{(N)}\}$ derived from $\{y_t\}$ when a_i were replaced by $\hat{a}_i$.

Other trends in error models attracted much less attention; the most frequently investigated case was that of trends of the form of

$$y_{t,m}(a_0, \ldots, a_m) = \sum_{i=0}^{m} a_i f_i(t), \qquad a_m \neq 0, \tag{5.1}$$

where the infinite sequence of functions $\{f_i\}$ was given and the number m was unknown. Polynomial trends of unknown degrees are a special case of the latter trends. The only serious attempts at estimating such trends seem to be those described in [1] and [15]. Anderson in [1] assumes that the upper bound of m is known and defines an estimator $\hat{m}$ of m by a multiple decision procedure which has some optimal properties for any finite N when $T = \{0, \pm 1, \ldots\}$ and $\{x_t\}$ is a sequence of independent normal variables with zero means and a common variance which is known; this procedure, slightly modified, still possesses asymptotically some optimal properties under somewhat less restrictive assumptions about $\{x_t\}$. Also Hoel's model [15] assumes normality and independence of x's, T being equal to $\langle 0, 1\rangle$. For more general assumptions about $\{x_t\}$ the problem is open; moreover, it seems that the multiple decision approach according to [19] is not quite satisfactory here until the properties of $\{y_{t,\hat{m}}(\hat{a}_0, \ldots, \hat{a}_m)\}$ are considered instead of those of $\hat{a}_0, \ldots, \hat{a}_m$ and $\hat{m}$. The last remark concerns not only models with linear trend given by (5.1): for any error models the decision theory approach will be desirable but at the stage of estimating $\{y_t\}$ and not only at the stage of estimating the parameters of trend. It is also well known that any attempt of this kind implies the introduction of many additional parameters characterizing our desires and the choice of these parameters is difficult. Finally, it should be noted that various concepts derived from the estimation theory of the parameters of the distributions of random variables: unbiasedness, effectiveness, confidence regions etc. are often useless as applied to trend parameters. For instance, unbiasedness of the number m in (5.1) is meaningless, since the repercussions of overestimating and underestimating m are quite different.

One gets the impression that, once out of the beaten track of models with linear trends when m is given, there is no general satisfactory approach towards trend estimation problems. On the other hand, there is an area of statistical inference which often interlaces with these problems: it is the analysis of residuals, which belongs to the oldest statistical traditions and is even more attractive nowadays because of the development of computer techniques.

The word "interlaces" was used here not by an accident, because it is difficult to qualify more precisely what kind of relationship exists. Residuals are traditionally applied in practice to verify error models,

but the theoretical background is lacking. Most often various tests for detecting trends ("no trend" tests) are used which verify the hypothesis H_0 that $\{z_t\}$ consists of $\{x_t\}$ only; if H_0 is rejected, then we are authorized to claim the presence of a trend. It is not correct, however, to verify the model by replacing groundlessly the observations of $\{z_t\}$ by those of the residual process $\{z_t - \hat{y}_t^{(N)}\}$ for a given trend estimator $\{\hat{y}_t^{(N)}\}$, and considering the model as verified if H_0 was not rejected. There exist also tests constructed to verify some detailed assumptions about noise, the test functions of which depend on the residuals $\{z_t - \hat{y}_t^{(N)}\}_N$ only. Among them the best known are the tests of Durbin and Watson [6] and of Anscombe [2]. However, these tests are also of the "significance" type so that they cannot protect against an erroneous positive verification of the model. Moreover, these tests apply only to very simple models.

F-estimators introduced in this paper present an attempt to express more closely the relationship between the model verification and trend estimation problems. They are obviously related to the tests discussed above ([2] and [6]). The need of appropriate concepts of that kind is also evident for instance in the history of trigonometric trend estimation. Many authors ([8], [12], [14], [34]) considered a "no trend" hypothesis for gradually more general assumptions about $\{x_t\}$ and constructed tests which were consistent against the hypothesis that a trigonometric trend is added. In one of these papers ([14], 1949) a sequential estimator of trigonometric trend based on residuals appears, presumably for the first time in the statistical literature, although this paper does not mention estimation problems: the estimator is only numerically calculated as an example which aims at demonstrating in an unprecised way the capabilities of the proposed test. And this is not a separate case that an author has difficulties in saying explicitly what he expects from the suggested test of the "no trend" type. It seems that the history of the trigonometric trend estimation (including [26] and [27]) might be to a large extent summarized in the form of a sequence of various F-estimators.

Of course, F-estimators do not solve the problem of model verification by means of residuals but it seems that they constitute some progress in this direction. In any case, they throw some light on the requirements which should be imposed on "no trend" tests.

It should be noted that the latter tests based on residuals are needed not only in error models. First of all, Cox and Snell in [4] introduced an interesting definition of a "residual" in models more general than error models, although they did not discuss further inference. The model considered by Box and Jenkins in [3] is another example. The process $\{z_t\}$ in [3] is supposed to be such that the m-th difference of its variables is a special case of a linear process. The authors of [3] suggest that the parameter m should be "identified", i. e. estimated in some informal

and subjective way. But it seems that the situation here is parallel to that in the case of estimating the number of terms in the polynomial trend and that a sequential estimator of m based on an appropriate "no trend" test is possible. The power of such a test applied to the subsequent differences of $\{z_t\}$ should have required properties parallel to those possessed by the powers of the tests T^* and J^* in the case of a polynomial trend.

Generally, interesting although controversial ideas given in [3] concerning "model diagnostic checking" and "model identification" illustrate the need of appropriate "no trend" tests in this kind of models. A fruitful application of such tests in the latter models seems possible. On the other hand, this is rather dubious in the case of a model described in [33]. In this model $\{z_t\}$ is an autoregressive process with time-dependent parameters, i. e. for any $t = 0, \pm 1, \ldots$ z_t satisfies

$$z_t - a_1(t)z_{t-1} - \ldots - a_p(t)z_{t-p} = \varepsilon_t, \tag{5.2}$$

where p is known and $\{\varepsilon_t\}$ is a white noise. The author of [33] considers the estimation of $a_i(t)$ for a prescribed value of t. As the starting point he takes a test for non-stationarity described in [28]. This test is a kind of two-way analysis of variance on logarithmically transformed estimators of the so-called evolutionary (i. e. time dependent) spectral density functions of $\{z_t\}$ for a number of arbitrarily chosen values of the parameter t and of the frequencies ω. However, the well-known and only partly solved difficulties with the estimation of the ordinary spectral density function (see the discussion in [37]) are increased in the case of time-dependent spectra. It seems that it is extremely difficult to treat (5.2) properly by statistical methods.

In view of their applications in error and related models, "no trend" tests should be constructed for possibly general null hypotheses about the noise component. The number of tests for independent normally distributed x's is enormous, and there still exist various tests referring to the case when normality is not assumed, but the situation changes when the assumption of independence is to be dropped. Hence the tests J^* and T^* proposed by S. K. Zaremba in [36] and [38] are so valuable suggestions.

Except for the simplest cases of white noise, the properties established for various tests are only asymptotic. That this is not satisfactory, is well illustrated by the results of Section 3 in this paper: the same results are obtained there for the test T^* and the test J^* with respect to polynomial trend estimation. On the other hand, numerical examples which were worked out on both tests and the general investigation of the powers of both tests under a special alternative hypothesis ([38]) suggest that the test J^* is more suitable for polynomial trends than the test T^*. The latter was designed to deal mainly with a trigonometric trend. We believe

that a superposition of these two tests will unable us to deal with more complicated trends as well. It is only doubtful whether a confidence interval for such a trend will be obtainable. The one described in Section 3.2 provides an important contribution to polynomial trend estimation. It should be noted that when the estimated value of m is smaller than the true one, the confidence interval is usually excessively wide, which diminishes the risk of wrong conclusions. On the other hand, the confidence interval does not protect us against an erroneous assumption that the x's are independent: such an assumption leads, as a rule, to a drastic underestimating of the confidence interval.

The estimation of $\sum R_k$ easily deducible from Section 3.2 is an example of post trend elimination problems. Many other problems of this kind might be investigated for F-estimators in S' when the assumptions of Theorem 2.1 are satisfied. For instance, the autocorrelation coefficients based on the residuals in question have the same joint asymptotic distribution as those based on $\{x_t\}_N$. A similar result can be established for a truncated periodogram (formula (2.6) in [20]) when the truncation point is chosen a priori. If, however, the truncation point is estimated from the observations as proposed in [20], any proof that the asymptotic distribution is not affected by the replacement of $\{x_t\}_N$ by $\{z_t - \hat{y}_{t,\hat{m}}\}_N$ is still not available.

As mentioned in Section 4.2, the test A is also not yet included into post trend elimination inference. This test seems to be valuable, mainly because it is based on efficient estimators of $\sum R_k^2$ and $\sum \tilde{R}_k^2$, i. e. on S_N and $\tilde{S}_N$; S_N, introduced by S. K. Zaremba in [22], seems to be a key statistic in time series analysis. The possibility of testing the equivalence of two noise components is obviously desirable, and tests similar to the test A under so general assumptions about $\{x_t\}$ are nearly non-existent. The only one we have found is that described in [11], p. 107, which, however, has a test function so complicated that it is difficult to calculate its value and it would be not easy to investigate its properties. Actually, the asymptotic distribution of the test function under the null hypothesis was found, but only when $\varkappa_4$ and $\tilde{\varkappa}_4$ are equal to zero; otherwise a knowledge a priori of the spectra densities of $\{x_t\}$ and $\{\tilde{x}_t\}$ is required.

The last remark concerns a matter which must be regarded as one of the most important in methodological approach to statistical analysis of time series: namely, that it is useless to consider methods assuming the existence of an a priori information which cannot be available or verified in practical applications. In trend estimation problems, we are often confronted with a situation in which the estimation of some unknown noise parameters is possible only after the elimination of the trend while the estimation of trend is impossible without the estimation of these

parameters; and in any case the influence of the replacement of the true values by estimated ones is very difficult to investigate.

We believe that the methods proposed in this paper are free of drawbacks of this kind, the considered model with a linear process as a noise component being thought to be sufficiently general to have an useful application in practice.

I am greatly indebted to Professor S. K. Zaremba, Université de Montréal, for his guidance in the field of the time series analysis and for his most valuable criticisms and continuing interest in my research. His results, in particular those concerning S_N, the tests T^* and J^* and the method of proving asymptotic normality when linear processes are concerned, have been the starting point of my present work. He also helped me to clarify and simplify some proofs, specially that of Theorem 4.1.

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