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On the foundations of k-group theory

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CONTENTS

1. Introduction

Hausdorff (T_2) k-spaces have been investigated for the past 20 or so years by a number of authors, e.g. Brown [2], Noble [11], and Steenrod [15]. More recently, weakly Hausdorff (t_2) k-spaces were introduced by McCord [10] by demanding that the diagonal be closed in the k-product. Even though T_2 k-spaces behave better under certain operations than their more general topological counterparts, it appears that for k-spaces t_2 separation is the appropriate concept serving the same end purpose in most instances as T_2 separation does for topological spaces.

A k-group is a group with a k-topology such that inversion is continuous and multiplication is continuous on the k-product (instead of the topological product as in a topological group). While receiving less attention than k-spaces, k-groups, specifically T_2 k-groups, have been studied by a few authors. Dubuc and Portia [4] looked at k-algebras, and Ordman [13] is investigating k_ω-groups (topological groups which are k_ω-spaces, hence, k-groups). Noble [12] has introduced another concept of a k-group which differs significantly from that given above; indeed for him they need not even be k-spaces, only topological groups for which every k-continuous group morphism is continuous.

Any topological group can be made into a k-group simply by introducing the standard k-refinement of its topology. We note, however, that not every k-group is the k-refinement of a topological group. In fact, the k-refinement of a topological group must either have no separation or be functionally Hausdorff (continuous functions into the unit interval separate points). Thus, any t_2, non-Hausdorff k-group is not the k-refinement of a topological group topology; furthermore, such groups exist in plentitude. Indeed, a k-group is T_0 if and only if it is t_2, but a t_2 k-group may well not be T_2-quite a different situation from that of topological groups. Therefore, what was said regarding separation in k-spaces is particularly valid in k-groups. That is, t_2 separation instead of T_2 separation is the better notion. For example, if H is a closed normal subgroup of a k-group G, then G/H is t_2 if G is t_2; however, G/H may well fail to be T_2 when G is T_2.

We first cover the general properties of non-separated (possibly not T_0) k-spaces, this material being essential for a clear and self-contained development of k-group theory. However, since some of this material is only the straight-forward generalizing of the analogous material in the literature on T_2 k-spaces, we will attempt to keep the number of proofs here to a minimum. Next, we present some theorems for k-groups which are analogous to theorems for topological groups, for example necessary and sufficient conditions for subgroups to be closed, which, as for topological groups, yield that locally compact subgroups of t_2 k-groups are closed.

If any one result has to be identified as a core result, it would be that any t_2 k-space X generates a free t_2 k-group FX into which it is topologically embe ded as a closed set of free generators. We investigate when FX is T_2, noting that X being non-Hausdorff certainly requires that FX be non-Hausdorff; a sufficient condition for FX to be T_2 is that X is functionally T_2. One of several examples given in this section shows, however, that the Hausdorff separation of X is not sufficient to guarantee that of FX. Here we also investigate when FX is, in fact, a topological group, and note the quite surprising fact that for this to be true, every t_2 quotient of X must be a normal space. In particular, if X is t_2, FX can only be a topological group when, in actuality, X is a normal k-space. The converse of this, however, does not hold, since and example is provided when X is a normal k-space, but FX is not a topological group.

Using the t_2 free k-group, we show that the free product of any collection of t_2 k-groups can be given a t_2 k-group topology which makes it into the categorical coproduct of the collection. As with the construction of the free t_2 k-groups, the major difficulty here is showing that the topology is indeed t_2. Following this, we again employ the free t_2 k-group along with a k-group semi-direct product to prove that epics in either the category of t_2 k-groups or the category of k_ω-groups are morphisms with dense range. We might note that this question is still unanswered for the categorie of T_2 topological groups. See [17] for the solution in T_2 k-groups or T_2 abelian k-groups.

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1. k-spaces

A quasi-compact space is a topological space with the property that every open cover has a finite subcover; a compact space is a T_2 quasi-compact space. A space is called *locally compact* if its topology has a base of compact sets. Note that locally compact spaces are not required to be T_2. By *TOP* we shall denote the category of topological spaces and continuous functions. A continuous $\varphi\colon T \to X$ is called a *test map* if T is compact. For a topological space X, let kX be the space with the same underlying set as X and the topology containing all k-open sets, where U is k-open (closed) if for all tests $\varphi\colon T \to X$, $\varphi^{-1}(U)$ is open (closed). If kX is topologically isomorphic to X via the identity, then X is called a *k-space.*

Note, if X is T_2, then X is a k-space if and only if U is open in X whenever $U \cap C$ is open in C for every compact C in X.

Like most of the material in this section, the following is well known in the T_2 case. We present the proof only as an indication of how one changes from compact subsets in the T_2 case to test maps in the non-separated case.

Often a space Y with topology t will be denoted by the pair (Y, t).

1.1. PROPOSITION. *First countable spaces and locally compact spaces are k-spaces.*

Proof. Assume that X is first countable and that A is a k-closed subset of X. To show that $A = \bar{A}$, let $x \in \bar{A}$ with $\{a_n \mid n \in Z^+\}$ a sequence in A converging to x. For $T = \{a_n \mid n \in Z^+\} \cup \{x\}$ and t the topology on T such that each $\{a_n\} \in t$ and $\{a_n \mid n > n_0\} \cup \{x\} \in t$ for each fixed n_0, T is the one point compactification of the positive integers. Thus, the inclusion $\varphi\colon T \to X$ is a test map; therefore, $\varphi^{-1}(A)$ is closed in T; consequently, $x \in \varphi^{-1}(A)$ showing that $x \in A$.

For X locally compact, the proof is like that of XI, 9.1 in [5].

Let K denote the category of k-spaces and continuous functions. The following is trivial.

1.2. PROPOSITION. *The correspondence $X \to kX$ extends to a functor $k\colon TOP \to K$ which is right adjoint to the inclusion functor, the counit of the adjunction being the identity $kX \to X$.*

Note, for $X \in K$ and $Y \in TOP$ that $f\colon X \to Y$ is continuous if and only if $f \circ \varphi$ is continuous for all tests φ into X.

A continuous surjection f is called a *quotient* if U is open whenever $f^{-1}(U)$ is open.

1.3. PROPOSITION. *If $X \in K$ and $f\colon X \to Y$ is a quotient, then $Y \in K$.*

The following is an example of a k-space in which the compact subsets do not suffice in determining the topology, as they do for T_2 k-spaces.

1.4. EXAMPLE. Let X be the quotient space of the real interval $[0, 1]$, where all points of the open subinterval $]1/3, 2/3[$ have been collapsed to one point x_0. Clearly, any T_2 (hence any compact) subspace of X that contains x_0 can not also contain the image of $1/3$ or $2/3$. It follows that $\{x_0\} \cap C$ is closed in C for every compact C in X; however, $\{x_0\}$ is not closed in X since its inverse image in $[0, 1]$ is not closed.

1.5. PROPOSITION. *X is T_1 if and only if kX is T_1.*

Proof. Certainly kX is T_1 if X is T_1. Assume now that X is not T_1. Then there exists $a \in X$ with $\{a\}$ not closed; thus, there exists $b \in X$ such that each open U in X containing b must also contain a. Let $T = \{a_n | n \in Z^+\} \cup \{b\}$ be the compact space in 1.1. and define $\varphi: T \to X$ by $\varphi(a_n) = a$ for all n and $\varphi(b) = b$. Since it is continuous, φ is a test; and since $\varphi^{-1}(a) = T \setminus \{b\}$ which is not closed in T, $\{a\}$ is not closed in kX. Therefore, kX is not T_1.

In the definition of the k-topology on a space X, the class of test maps is certainly not a set — a circumstance not very satisfying. In what follows we show, however, that a set of tests is always sufficient.

1.6. PROPOSITION. *For a k-space there always exists a set of test maps sufficient to determine the topology.*

Proof. Let Γ be the class of all pairs (φ, T), where $\varphi: T \to X$ is a test. Since X is a set, there is at most a set of distinct images of these maps; also, on each of these images A there is at most a set of possible topologies. In particular, there is only a set of topologies σ on A with σ the quotient topology induced by some test $\varphi: T \to \varphi(T) = A \subset X$. Let S be the set of all such pairs (A, σ).

As a corollary to the axiom of choice, one shows for each $(A, \sigma) \in S$ that a pair $(\varphi, T) \in \Gamma$ can be chosen with $\varphi(T) = A$ and σ the quotient topology induced by φ. Let Γ' be this set. To see that Γ' is sufficient in determining the topology on kX, let $U \subset X$ with $\varphi^{-1}(U)$ open for all $(\varphi, T) \in \Gamma'$. We need to show that U is k-open, that is, for any test $\psi: C \to X$, $\psi^{-1}(U)$ must be open. But by the definition of S there is $(A, \sigma) \in S$ with $\psi(C) = A$ and σ the quotient topology determined by ψ. Also there exists $(\varphi, T) \in \Gamma'$ such that $\varphi(T) = A$ and σ is the quotient topology induced by φ. Then $\varphi^{-1}(U)$ open in T implies that $\psi^{-1}(U)$ is open in C.

As a consequence of this, we have the following analog of XI, 9.4 in [5] in the absence of separation.

1.7. PROPOSITION. *A topological space is a k-space if and only if it is the quotient of a T_2 locally compact space.*

Proof. By 1.1 and 1.3 any quotient of a locally compact space is a k-space. Conversely, let X be a k-space and let $\{(\varphi_i, T_i) | i \in I\}$ be a set of test maps determining its topology. Clearly the disjoint union $L_X = \coprod \{T_i | i \in I\}$ given the sum topology is locally compact T_2. One checks that $\varrho: L_X \to X$ given by $\varrho(y) = \varphi_i(y)$ when $y \in T_i$ is a quotient.

Combining this with VI, 2.1 of [5] and a trivial observation yield:

1.8. PROPOSITION. *If $X \in K$ and A is the intersection of an open subset and a closed subset of X, then A with the subspace topology is also in K.*

Of course, not every subset of a k-space is a k-space. For Ω the first uncountable ordinal, let $X = [0, \Omega]$, and let A be the subspace of X

with all limit ordinals except Ω removed. Then A with the subspace topology is not discrete; however, since only finite subsets of A are compact, kA is discrete.

Since compact spaces are k-spaces, one might think that quasi-compact spaces are also k-spaces. We see, though, that this is not the case. Let A be as above; then, A with the subspace topology is T_2, but is not in K. Let Y be its one point compactification, a quasi-compact space; Y, however, is not a k-space, since, if it were, A being an open subset of Y would then be in K.

For $A \subset X$ and X not a k-space, we have two ways of giving A a k-topology. The following proposition, the straightforward proof of which is omitted, tells us that they yield equivalent topologies on A. For t the topology on X, $t|A$ denotes the subspace topology on A.

1.9. Proposition. *If A is a subset of X and X has topology t,* $(A, k[t|A]) \approx (A, k[(kt)|A])$.

The next two propositions are simple categorical bookkeeping, easily verified with the possible exception of the equivalence of quotients and coequalizers. Only small limits and colimits are considered.

1.10. Proposition. *The category TOP is complete and cocomplete. Equalizers are precisely the topological embeddings, products are the usual topological products, coequalizers are precisely the quotient maps, and coproducts are the disjoint unions.*

1.11. Proposition. *The category K is complete and cocomplete. All limits, in particular products and equalizers, are computed by reflecting the limits in TOP into K by k. All colimits, in particular coproducts and coequalizers, are the same in K as in TOP.*

The following notational convention will be followed for the remainder of this paper. To distinguish between the categorical product in *TOP* and that of K, for X and Y in *TOP*, let $X \times_c Y$ denote their topological product; for X and Y in K, let $X \times Y$ be the k-product. Thus, $k[X \times_c Y] = kX \times kY$. For X and Y in either of these categories, the coproduct of X and Y is denoted by $X \amalg Y$, it being the topological sum in both instances. If the indexing set is large, $\prod_{\mathfrak{A}}\{X_i | i \in I\}$ and $\coprod_{\mathfrak{A}}\{X_i | i \in I\}$ will denote the product and coproduct, respectively, of the collection of objects X_i in a category $\mathfrak{A}$.

We should remark, for those unfamiliar with k-spaces, that the topological product and the k-product of two k-spaces need not be the same. See, for example, VI, 8, Example 5 of [5]; also, in our 2.1 $kG \times kG \neq kG \times_c kG$ topologically. The proof of the following is routine.

1.12. Proposition. *For X and Y in K, to determine the topology on X Y it is sufficient to consider only tests of the form $\varphi \times \psi$, where φ and ψ are tests into X and Y, respectively.*

The next two results will be needed in the next chapter. The proofs, while differing from those for the topological product, are not difficult and are omitted.

1.13. PROPOSITION. *For X and Y in K with $A \subset X$ and $B \subset Y$, and $\overline{A \times B}$ denoting the closure taken in $X \times Y$, $\bar{A} \times \bar{B} = \overline{A \times B}$*

1.14. PROPOSITION. *For X and Y in K, the product projections p_X: $X \times Y \to X$ and p_Y: $X \times Y \to Y$ are open maps.*

An injective continuous function f: $X \to Y$ in K is called a *k-embedding* if every function g: $A \to X$ with A in K and $f \circ g$ continuous is continuous.

1.15. PROPOSITION. *An injective f: $X \to (Y, t)$ in K is a k-embedding if and only if f: $X \to (f(X), k[t|\ f(X)])$ is an isomorphism. Furthermore, k-embeddings are precisely the equalizers in K.*

Given X and Y in TOP, define the test-open topology on the set of continuous functions $TOP(X, Y)$ to be the topology generated by subbasic open sets $W(\varphi, U) = \{f|\ f(\varphi(T)) \subset U\}$, where φ: $T \to X$ is a test and U is open in Y. We denote this space by $TOP_{t\text{-}o}(X, Y)$. For X and Y in K, define $\underline{K}(X, Y) = k[K_{t\text{-}o}(X, Y)]$. One notes for a T_2 space X that the test-open topology is the usual compact-open topology, which we denote by $TOP_{c\text{-}o}(X, Y)$. For X and Y in TOP, the evaluation map ev: $TOP(X, Y) \times X \to Y$ is defined by $ev(f, x) = f(x)$.

Note that here $TOP(X, Y)$ is just a set, and $\times$ is used to denote the set product; therefore, $\times$ can either be the product in K or in Sets. But $\times_c$ is always the topological product.

1.16. PROPOSITION. *For X and Y in TOP and φ: $T \to X$ a test, the composition $ev \circ (1 \times \varphi)$: $TOP_{t\text{-}o}(X, Y) \times_c T \to Y$ is continuous.*

Proof. For U open in Y, suppose that $ev \circ (1 \times \varphi)(f, t)$ is in U; then, $f(\varphi(t)) \in U$ and $t \in (f \circ \varphi)^{-1}(U)$. Since T is compact, there is an open V in T with $t \in V \subset \bar{V} \subset (f \circ \varphi)^{-1}(U)$. Then $\psi = \varphi | \bar{V}$: $\bar{V} \to X$ is a test, and $W(\psi, U) \times_c V$ is an open set containing (f, t) which is mapped into U by ev.

From this and 1.12 we have:

1.17. PROPOSITION. *For X and Y in K, ev: $\underline{K}(X, Y) \times X \to Y$ is continuous.*

Dropping the Hausdorff requirement for locally compact spaces in [5], one has the following trivial generalization of XII, 3.1 there.

1.18. PROPOSITION. *For Y locally compact, the functor $- \times_c Y$: $TOP \to TOP$ is left adjoint to $TOP_{c\text{-}o}(Y, \_)$. That is, for any X and Z in TOP, φ: $TOP(X \times_c Y, Z) \to TOP(X, TOP_{c\text{-}o}(Y, Z))$ given by $\varphi(f)(x)(y) = f(x, y)$ is a bijection, natural in X and Z.*

1.19. PROPOSITION. *For X and Y in K with Y locally compact, $X \times_c Y = X \times Y$.*

Proof. By 1.7 there is a quotient $\varrho\colon L_X \to X$ with L_X locally compact T_2. Since quotients and coequalizers are the same in *TOP* and since $-\times_c Y$ is a left adjoint, $\varrho\times 1\colon L_X\times_c Y \to X\times_c Y$ is a quotient. Thus, since $L_X\times_c Y$ is locally compact, $X\times_c Y$ is a k-space.

1.20. Proposition. *For Y in K, for a set $\{(\varphi_i, T_i) \mid i\in I\}$ of tests determining its topology, and for the canonical quotient $\varrho\colon L_Y = \amalg T_i \to Y$, the map $TOP(\varrho, Z)\colon TOP_{t\text{-}o}(Y, Z)\to TOP_{c\text{-}o}(L_Y, Z)$ is an embedding for any Z in TOP.*

Proof. Let $\bar\varrho$ denote the map. Then $\bar\varrho$ is clearly a continuous injection. And for $W(\psi, U)$ a subbasic open set in the t-o topology with $\psi\colon C\to Y$ a test, there is a test φ_i with $\varphi_i(T_i) = \psi(C)$. Noting that $\varrho(a) = \varphi_i(a)$ for $a\in T_i$, one has that $\bar\varrho\big(W(\psi, U)\big) = \bar\varrho\big(TOP(Y, Z)\big)\cap W(T_i, U)$, showing that $\bar\varrho$ is open onto its image.

Omitting the idea of V-functors, we modify the definition in [3] of an enriched adjunction; in its place we have the following more specialized one. For $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ subcategories of *TOP* such that all the hom-sets $\mathfrak{A}(A, A')$ and $\mathfrak{B}(B, B')$ are provided with topologies in $\mathfrak{C}$, an adjunction $F\colon \mathfrak{A}\rightleftarrows\mathfrak{B}\colon G$ is called a *$\mathfrak{C}$-adjunction* if each natural bijection $\mathfrak{B}(FA, B) \approx \mathfrak{A}(A, GB)$ is, in fact, an isomorphism in $\mathfrak{C}$.

The next proposition is the version of [15], 5.6, covering the non-separated case. Various proofs could be given; we give one using 1.7 and 1.18.

1.21. Proposition. *For any Y in K, the functor $\_\times Y\colon K\to K$ is K-left adjoint to $\underline{K}(Y, \_)$. That is, for any X and Z in K, $\varphi\colon \underline{K}(X\times Y, Z)\to \underline{K}\big(X, \underline{K}(Y, Z)\big)$ given by $\varphi(f)(x)(y) = f(x, y)$ is a topological isomorphism, natural in X and Z.*

Proof. We first show that φ is well defined, then that it is a bijection, and finally that it is an isomorphism. Let $f\colon X\times Y\to Z$ be continuous. For $\varrho\colon L_Y\to Y$ as in 1.7, $f\circ(1\times\varrho)\colon X\times L_Y\to Z$ is continuous. Letting φ' denote the φ of 1.18, by 1.18 $\varphi'\big(f\circ(1\times\varrho)\big)\colon X\to TOP_{c\text{-}o}(L_Y, Z)$ is continuous. And, for each $x\in X$, $\varphi(f)(x)\circ\varrho = \varphi'\big(f\circ(1\times\varrho)(x)\big)$; therefore, since ϱ is a quotient, $\varphi(f)(x)$ is continuous. To see that $\varphi(f)$ is continuous, we note that $TOP(\varrho, Z)\circ\varphi(f) = \varphi'\big(f\circ(1\times\varrho)\big)\colon X\to TOP_{c\text{-}o}(L_Y, Z)$, and recall from 1.20 that $TOP(\varrho, Z)$ is an embedding. Thus, $\varphi(f)$ is continuous into $K_{t\text{-}o}(Y, Z)$, hence, into $\underline{K}(Y, Z)$.

Now define $\psi\colon K\big(X, \underline{K}(Y, Z)\big)\to K(X\times Y, Z)$ by $\psi(g)(x, y) = g(x)(y)$. Since $\psi(g) = ev\circ(g\times 1)$, each $\psi(g)$ is continuous. Trivially, φ and ψ are inverses. The naturality is straightforward.

To show that the adjunction is a K-adjunction, apply the work done so far several times to get for any $A\in K$ that $K\big(A, \underline{K}(X\times Y, Z)\big)$ is bijectively isomorphic to $K\big(A, \underline{K}\big(X, \underline{K}(Y, Z)\big)\big)$. Having done this, invoke the appropriate form of the Yonneda lemma.

This, as noted by many authors in the T_2 case, is the main reason that k-spaces are more convenient than topological spaces. Still more reasons follow.

1.22. PROPOSITION. *If $f\colon X\to A$ and $g\colon Y\to B$ are quotients in K, then $f\times g\colon X\times Y\to A\times B$ is also a quotient.*

Proof. Since the composition of quotients is a quotient, we need only that $f\times 1$ is a quotient. However, quotients are the same as coequalizers in K by 1.10 and 1.11. Then $\_\times Y$ being a left adjoint preserves coequalizers; therefore, $f\times 1$ is a quotient.

1.23. PROPOSITION. *For Y in K, the functor $\underline{K}(Y, \_)\colon K\to K$ preserves k-embeddings.*

Proof. By 1.15 equalizers in K are the same as k-embeddings. Then, since right adjoints preserve equalizers, invoking 1.21 we are finished.

Using 1.21 and the obvious isomorphism between $X\times Y$ and $Y\times X$, we have:

1.24. PROPOSITION. *For Z in K, the functor $\underline{K}(\_, Z)\colon K\to K^{op}$ is K-left adjoint to the functor $\underline{K}(\_, Z)\colon K^{op}\to K$. That is, for X and Y in K, $\varphi\colon \underline{K}(X, \underline{K}(Y,Z))\to\underline{K}(Y, \underline{K}(X,Z))$ given by $\varphi(f)(y)(x) = f(x)(y)$ is a topological isomorphism, natural in X and Y.*

1.25. PROPOSITION. *If $f\colon X\to Y$ is a quotient in K, then $\underline{K}(f,Z)\colon \underline{K}(Y,Z)\to\underline{K}(X,Z)$ is a k-embedding for any Z in K.*

Proof. Since f is a coequalizer by 1.10 and 1.11, it is preserved by the left adjoint $\underline{K}(\_, Z)$. However, coequalizers in K^{op} are equalizers in K which by 1.15 are k-embeddings.

We will later see that this k-embedding is a closed embedding, hence, a topological embedding when Z is weakly Hausdorff. A topological space X is called *weakly Hausdorff* (t_2) if for each test $\varphi\colon T\to X$, $\varphi(T)$ is closed in X.

1.26. PROPOSITION. *Topological products of t_2 spaces are t_2, as are subspaces of t_2 spaces; also, t_2 separation is between T_1 separation and T_2 separation. In addition, if X is t_2, then kX is t_2.*

The next propositions are, respectively, 2.1 and 2.3 of [10].

1.27. PROPOSITION. *If X is t_2, then for every test $\varphi\colon T\to X$, $\varphi(T)$ is compact in the subspace topology. In particular, the topology of kX is determined by compact subsets.*

1.28. PROPOSITION. *For X in K, X is t_2 if and only if the diagonal $\Delta(X)$ is closed in $X\times X$.*

This last proposition indicates the similarity between weakly Hausdorff separation in K and Hausdorff separation in TOP. Let WHK denote the full subcategory of K consisting of t_2 k-spaces. The next result follows immediately from 1.28.

1.29. PROPOSITION. *Equalizers in WHK are precisely the closed embeddings, that is, closed subsets.*

1.30. PROPOSITION. *For X in WHK and $R \subset X \times X$ defining an equivalence relation on X, the quotient X/R is in WHK if and only if R is a closed subset.*

Proof. Let $f\colon X \to X/R$ be the canonical quotient. Since quotients of k-spaces are also in K, we need to show that X/R is t_2 if and only if R is closed. By 1.22 $f \times f\colon X \times X \to X/R \times X/R$ is a quotient; therefore, X/R is t_2 if and only if $\Delta(X/R)$ is closed in $X/R \times X/R$ if and only if $[f \times f]^{-1}(\Delta(X/R))$ is closed in $X \times X$. However, $[f \times f]^{-1}(\Delta(X/R)) = R$.

1.31. COROLLARY. *If X is in WHK and A is a closed subset, the quotient X/A realized by collapsing A to one point is also in WHK.*

This along with the following example tells us that WHK is a nicer category as regards quotients than are T_2 topological spaces or T_2 k-spaces.

1.32. EXAMPLE. Let X be a T_2 k-space which is not regular, e.g. example 78 of [14]. Then there is a closed subset A such that the quotient X/A is not T_2 — but certainly t_2.

This example will later yield a t_2 k-group which is not T_2, a situation not occurring for topological groups, where T_0 separation implies T_2 separation. We note, however, that for first countable spaces weak Hausdorff separation is equivalent to Hausdorff separation.

1.33. PROPOSITION. *Let X be first countable; then X is t_2 if and only if it is T_2.*

Proof. For X first countable, $X \times X = X \times_c X$; therefore, $\Delta(X)$ is closed in $X \times X$ if and only if it is closed in $X \times_c X$.

The following two propositions are straightforward; the third proposition follows from the previous two.

1.34. PROPOSITION. *Let X and Y be in K with each Y_x denoting a copy of Y; then, $f\colon \underline{K}(X, Y) \to \prod_K \{Y_x \mid x \in X\}$ given by $f(g) = \prod \{g(x) \mid x \in X\}$ is a continuous injection.*

1.35. PROPOSITION. *If $f\colon A \to B$ is a continuous injection and B is t_2, then A is also t_2.*

1.36. PROPOSITION. *For X and Y in K with Y t_2, $\underline{K}(X, Y)$ is t_2.*

We remark that for any X its weak Hausdorffization (for lack of a better word) can be formed, namely, X/R for R the smallest equivalence relation such that X/R is t_2. With this one has that WHK is cocomplete, with colimits being simply the weak Hausdorffization of the colimits in K. In particular, coequalizers and quotients are still the same. Also, replacing K with WHK, 1.21 and 1.24 still hold. From this it follows that if $f\colon X \to Y$ is a quotient in WHK, then $WHK\,(f, Z)\colon \underline{WHK}(Y, Z) \to \underline{WHK}(X, Z)$ is an equalizer in WHK, that is, a closed embedding. In fact, noting for any Z in WHK that $\underline{K}(\_\,, Z)$: $K \to WHK^{op}$ is a left adjoint, we have strengthened 1.25 to get:

1.37. Proposition. *For f: $X \to Y$ a quotient in K, $K(f, Z)$: $\underline{K}(Y, Z) \to \underline{K}(X, Z)$ is a closed embedding for any Z in WHK.*

2. k-groups

A k-group is a group G with a k-topology such that inversion is continuous and such that the multiplication is continuous on the k-product. Note that multiplication is not required to be continuous on the topological product (if it were we would have a topological group). Certainly if $G \times_c G = G \times G$, as for locally compact groups or first countable groups, then G is both a topological group and a k-group. We also note that if G is a topological group, then kG is a k-group; though, as we see below, kG may not be a topological group. In addition, following 2.27 it will be observed that there are k-groups which are not the k-refinement of any topological group.

2.1. Example. Let G be the topological product of an uncountable number of copies of the reals. Since G is a topological group, kG is a k-group. And since kG is T_2, to show that kG is not a topological group it sufficies to show that kG is not regular. This is acomplished by piecing together several of the results in [11] as follows: By 5.5 of [11] G is not a k-space; therefore, the identity f: $G \to kG$ is not continuous. Assume now that kG is regular. Using the terminology of [11], since f is k-continuous and the reals are of type $\mathfrak{C}$, 2.4 of [11] says that f is Σ-continuous, hence, Σ^0-continuous. Then, since k-continuity implies 2-continuity, 1.1 of [11] requires that f be continuous, a contradiction.

We look now at a few properties of k-groups in general but soon switch our attention only to those k-groups that are t_2.

2.2. Proposition. *In a k-group, translation by a fixed element and inversion are topological isomorphisms; also, multiplication is an open map.*

2.3. Proposition. *If H is a subgroup of the k-group G, then the closure of H is also a k-group.*

Proof. Since closed subspaces of k-spaces are k-spaces, $\bar{H}$ is in K. Let m be the multiplication on G and $\overline{H \times H}$ the closure of $H \times H$ in $G \times G$. By 1.13 $\overline{H \times H} = \bar{H} \times \bar{H}$. Then $m(\bar{H} \times \bar{H}) = m(\overline{H \times H}) \subset \overline{m(H \times H)} = \bar{H}$; thus, $\bar{H}$ is closed under multiplication. That it is closed under inversion is trivial.

2.4. Proposition. *For H a subgroup of a k-group G, the following are equivalent:*

1° H *is closed.*

2° *There exists an identity neighborhood* U *in* G *with* $\bar{U} \cap H$ *closed in* G.

3° *There exists and identity neighborhood* U *in* G *with* $U \cap H$ *closed in* U.

Proof. Clearly 1° implies both 2° and 3°. To see that 3° implies 1°, suppose that U is an identity neighborhood with $U \cap H$ closed in U. Let $x \in \bar{H}$ with $\{x_i\}$ a net in H converging to x. Since $\bar{H}$ is a group, $x^{-1} \in \bar{H}$; thus, there exists $y \in Ux^{-1} \cap H$. Letting m be the multiplication on G, this says that $(y, x) \in m^{-1}(U)$. Then, by the continuity of $a \mapsto (y, a)$: $G \to G \times G$, there exists i_0 such that $i > i_0$ implies that $(y, x_i) \in m^{-1}(U)$, that is, $yx_i \in U$. Since $\{(y, x_i)\}$ converges to (y, x) in $G \times G$, and since m is continuous, $\{yx_i\}$ converges to yx in U. Now, $yx_i \in H$ along with $yx_i \in U$ for $i > i_0$ imply that that $yx_i \in H \cap U$ for $i > i_0$. However, $H \cap U$ is closed in U; thus, $yx \in U \cap H \subset H$. Then $x = y^{-1}yx$ is in $HH = H$ showing that $H = \bar{H}$.

The proof that 2° implies 1° is essentially the same.

One recalls that there is a similar result for topological groups; however, the usual proof given, for example 5.9 of [7], does not carry over to k-groups. The reason for this is that in a topological group, given any identity neighborhood U, there exists another identity neighborhood V with $VV \subset U$. This does not hold, in general, for k-groups, since then the k-group would actually be a topological group.

In [1], for X in *TOP*, a subset A is called *locally closed* if for each $a \in A$ there is a neighborhood U of a in X with $A \cap U$ closed in U. Thus, 2.4 says that locally closed subgroups of k-groups are closed. A slight generalization of Proposition 12 of I. 66 in [1] yields that locally compact subsets of t_2 spaces are locally closed. Combining all of this we have:

2.5. Corollary. *Open subgroups and locally compact (hence also discrete) subgroups of* t_2 *k-groups are closed.*

For notational purposes, let *KG* and *WHKG* denote, respectively, the category of k-groups and k-group morphisms and the category of t_2 k-groups and k-group morphisms.

A routine verification yields the following proposition regarding separation in a k-group.

2.6. Proposition. *For* G *in KG, the following are equivalent:*

1° G *is* T_1,

2° $\{1\}$ *is closed,*

3° G *is* t_2,

4° G *is* T_0.

In 2.14 we shall see that t_2 separation does not imply T_2 separation in k-groups, as it does in topological groups.

2.7. Proposition. *For G in KG and H a normal subgroup, G/H given the quotient topology is also in KG, and the quotient map is open. In addition, if G is t_2, then H is closed if and only if G/H is t_2.*

Proof. For m the multiplication on G, m' the induced multiplication on G/H, and π the canonical quotient, $m' \circ (\pi \times \pi) = \pi \circ m$, and by 1.22 $\pi \times \pi$ is a quotient; therefore, m' is continuous. Similarly, inversion is shown to be continuous. That π is open follows exactly as with topological groups.

Suppose now that G is t_2. If G/H is t_2, then $\{1\}$ is closed in G/H; therefore, $H = \pi^{-1}(1)$ is closed. Conversely, suppose that H is closed. To see that G/H is t_2, it is sufficient that the equivalence relation R determined by H is closed in $G \times G$. However, for m and σ the multiplication and inversion, respectively, on G, $R = [m \circ (1 \times \sigma)]^{-1}(H)$ which is closed.

2.8 Proposition. *The correspondence $G \mapsto G/\overline{\{1\}}$ extends to a functor from KG to $WHKG$ which is K-left adjoint to the inclusion functor.*

Proof. Let $\pi_G \colon G \to G/\overline{\{1\}}$ be the canonical quotient. For any t_2 k-group H, one checks that $KG(\pi_G, H) \colon \underline{WHKG}(G/\overline{\{1\}}, H) \to \underline{KG}(G, H)$ is a natural bijection. That it is, in fact, a topological isomorphism follows from 2.37 – which does not depend on the present proposition.

This functor is called the *weak Hausdorffization* (again for lack of a better name) *functor* for k-groups. The following is again categorical bookkeeping; its proof is not difficult and is omitted.

2.9. Proposition. *The categories KG and $WHKG$ have coequalizers, the coequalizers in $WHKG$ being the weak Hausdorffization of those in KG. In both categories, the coequalizers are precisely the quotient morphisms. Also both categories are complete, with the grounding functors to K and WHK preserving limits.*

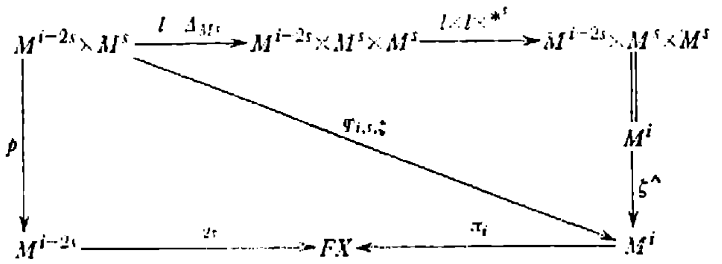
In 2.38 the coproducts in these two categories will also be identified.

We have now arrived at one of our principal goals: Showing for any t_2 k-space X that the free group generated by X can be given a t_2 k-group topology with X embedded as a closed subset. We begin by establishing some new notation.

2.10 Notation. For X in K, let X^* be a k-space such that there is an isomorphism $* : X \to X^*$ with $x \mapsto x^*$. Let $M = X \sqcup X^*$ and extend $*$ to $* : M \to M$ by $x^{**} = x$. Set $M^0 = \{0\}$; and for i any positive integer, let M^i be the k-product of i copies of M. Then, for FX the free group generated by X, one has the canonical surjection $\pi : \coprod \{M^i \mid 0 \leqslant i < \infty\} \to FX$, where $\pi(m_1 m_2 \ldots m_i)$ is the reduced word associated with $m_1 m_2 \ldots m_i$. For $\eta_X : X \to FX$ the canonical injection, where $\eta_X(x)$ is the singleton word x, one has $\eta_X(x) = \pi(x)$; also, $\pi(x^*) = \pi(x)^{-1}$, the inverse in FX of $\pi(x)$; and $\pi(M^0) = 1$, the identity of FX. Finally, let $\pi_i = \pi \mid M^i \colon M^i \to FX$.

For i a positive integer, let $S(i)$ be the set of all permutations on i symbols. And given $\zeta \in S(i)$, define $\zeta^{\wedge}$: $M^i \to M^i$ by the correspondence $m_1 m_2 \ldots m_i \mapsto m_{\zeta^{-1}(1)} m_{\zeta^{-1}(2)} \ldots m_{\zeta^{-1}(i)}$, where the use of ζ^{-1} is of no significance, other than one traditionally lets $S(i)$ act as a group. Certainly, each $\zeta^{\wedge}$ is a k-space isomorphism.

For i a positive integer and s an integer with $0 \leqslant s \leqslant i/2$, $\zeta \in S(i)$, and p the product projection onto the first $i-2s$ coordinates, consider the following diagram:



where $*^s(m_1 m_2 \ldots m_s) = m_1^* m_2^* \ldots m_s^*$ and Δ_A: $A \to A \times A$ is the diagonal map. Define $\Gamma_{i,s}$ to be the set of all ζ in $S(i)$ such that this diagram commutes; also, let $\varphi_{i,s,\zeta}$ be the indicated composition when ζ is in $\Gamma_{i,s}$.

2.11. Remark. In 2.10 one notes that $\bigcup \{\varphi_{i,s,\zeta}(M^{i-2s} \times M^s) \mid \zeta \in \Gamma_{i,s}\}$ is the set of all words in i letters which can be reduced to words of length less than or equal to $i-2s$. If X is t_2, then M^s is t_2; therefore, by 1.28 Δ_{M^s} is a closed embedding. Consequently, each $\varphi_{i,s,\zeta}$ is a closed embedding when X is t_2.

We now show that FX provided with the quotient topology from π is a k-group when X is in K and a t_2 k-group when X is in WHK.

2.12. Theorem. *There exist functors F: $K \to KG$ and F: $WHK \to WHKG$ which are left adjoint to their respective grounding functors. Furthermore, in both cases the underlying group of FX is the free group generated by X, and when X is t_2, the canonical injection η_X is a closed embedding.*

Proof. First, we assume only that X is in K; the second part of the proof will deal with X in WHK. Since X is in K, each M^i and, therefore, $\coprod\{M^i \mid 0 \leqslant i < \infty\}$ are in K. So, if FX is given the quotient topology from π, 1.3 says that FX is also in K. We proceed now to show that this topology is a k-group topology on FX.

Note that g: $\coprod_{i=0}^{\infty} M^i \times \coprod_{i=0}^{\infty} M^i \to \coprod_{i=0}^{\infty} M^i$ defined as follows is continuous: for i and j not zero, $g\big((m_1 \ldots m_i)(n_1 \ldots n_j)\big) = m_1 \ldots m_i n_1 \ldots n_j \in M^{i+j}$; for $0 \in M^0$, $g\big((0)(m_1 \ldots m_i)\big) = g\big((m_1 \ldots m_i)(0)\big) = m_1 \ldots m_i$; and $g\big((0)(0)\big) = 0 \in M^0$. Letting g' denote the multiplication on FX, we then have $g' \circ (\pi \times \pi) = \pi \circ g$ which is continuous. By 1.22, $\pi \times \pi$ is a quotient; therefore, g': $FX \times FX \to FX$ is continuous.

Using a similar argument, one has that inversion is continuous. Then, having done this, one has that FX is a k-group.

Since the canonical injection η_X is $\pi|X$, η_X is clearly continuous. We need now to show for any k-group G and continuous $f\colon X\to G$ the existence of a unique k-group morphism $f'\colon FX\to G$ with $f'\circ\eta_X=f$. This will then complete the proof that the functor $F\colon K\to KG$ is left adjoint to the grounding functor. Since FX is the free group generated by X, there exists a unique group morphism f' satisfying this; therefore, we need only that f' is continuous. For a the inversion on G and $t\colon G^i\to G$ the mapping which multiplies the i coordinate entries together, define $p\colon \coprod_{i=0}^{\infty} M^i\to G$ by $p\,|\,M^i$ is the composition $t\circ[f\amalg(a\circ f)]^i\circ[1\amalg^*]^i\colon M^i=[X\amalg X^*]^i\to[X\amalg X]^i\to G^i\to G$ when i is not zero; for $i=0$, let $p(M^0)=1$, the identity of G. Since each $p\,|\,M^i$ is continuous, p is continuous. Then, since π is a quotient and $f'\circ\pi=p$, f' is continuous.

We now require X to be t_2 and show that FX is t_2 with η_X being a closed embedding. By 2.6, for FX to be t_2 it is only necessary that the identity of FX is closed, that is, $\pi^{-1}(1)\cap M^i$ is closed in M^i for each i. This is certainly true if i is odd, since then the intersection is empty. So, we may assume that $i=2s$ for some positive integer s. Then, in 2.10 we have $M^{i-2s}=M^0=\{0\}$, and $\bigcup\{\varphi_{i,s,\zeta}(M^{i-2s}\times M^s)\,|\,\zeta\in\Gamma_{i,s}\}$ is the set of all words in i letters which can be reduced to words of length zero, that is, $\pi^{-1}(1)\cap M^i$. Since each $\varphi_{i,s,\zeta}$ is a closed embedding, and since the union is taken over a finite set, $\pi^{-1}(1)\cap M^i$ is closed in M^i.

To show that η_X is a closed embedding, let A be closed in X. Then A is closed in M^1; and we need to show that $\pi^{-1}\big(\eta_X(A)\big)\cap M^i$ is closed in M^i for each i. Since the intersection is empty for even i, we need only consider $i=1+2s$, s a non-negative integer. To finish, one notes that $\pi^{-1}\big(\eta_X(A)\big)\cap M^i=\bigcup\{\varphi_{i,s,\zeta}(A\times M^s)\,|\,\zeta\in\Gamma_{i,s}\}$ which is a finite union of closed subsets of M^i.

In the construction of FX, note that the topology on FX is the colimit of the expanding sequence of subspaces $\pi(M^0)\subset\pi(M)\subset\pi(M^2)\subset\ldots$, where each $\pi(M^i)$ has the quotient topology from $\pi\,|\,\pi^{-1}\big(\pi(M^i)\big)$; from 2.16 it follows that the topology on $\pi(M^i)$ is also the quotient derived from $\pi\,|\,M^i$ and that each $\pi(M^i)$ is closed in FX.

2.13. Corollary. *For X in K, X is t_2 if and only if FX is t_2.*

2.14. Example. For X the non-Hausdorff, weakly Hausdorff k-space in 1.32, FX is a non-Hausdorff, weakly Hausdorff k-group. Since FX is T_0, it also follows that FX is not a topological group.

In 2.21 we shall present a T_2 k-space X for which FX is still only t_2.

In passing it should be noted that if the free abelian group $F_{Ab}X$

is used in place of FX, the method of proof in 2.12 yields the analogous result for abelian groups. We record this as:

2.15. COROLLARY. *For any t_2 k-space X, the free abelian group $F_{Ab}X$ generated by X, provided with the quotient topology from $\coprod_{i=0}^{\infty}(X \amalg X^*)^i$, is a t_2 abelian k-group with X topologically embedded as a closed subspace.*

The following will enable us to determine when certain subsets of FX are closed; unfortunately the proof is somewhat tedious.

2.16. PROPOSITION. *For X in WHK, let $A \subset M^r$ be closed in M^r with $A = \pi_r^{-1}(\pi_r(A))$; then, $\pi_i^{-1}(\pi_r(A))$ is closed in M^i for all i, that is, $\pi(A)$ is closed in FX.*

Proof. If $i-r$ is odd, $\pi_i^{-1}(\pi_r(A)) = \emptyset$; so we may assume that $i-r$ is even. By hypothesis the claim is true for $i = r$; thus, we have two cases to consider: $i > r$ and $i < r$. First, we assume that $i > r$ with s the positive integer such that $i-2s = r$. We now show that $\pi_i^{-1}(\pi_r(A)) = \bigcup\{\varphi_{i,s,\zeta}(A \times M^s) \mid \zeta \in \Gamma_{i,s}\}$ which, by 2.11, is a closed subset of M^i.

By the commuting of the diagram in 2.10, the right-hand side of the desired equality is contained in the left-hand side. For the opposite containment, assume that $y \in \pi_i^{-1}(\pi_r(A))$. Since $A \subset M^r$, $\pi_i^{-1}(\pi_r(A)) \subset \pi_i^{-1}(\pi_r(M^r))$. Next, we note that $\pi_i^{-1}(\pi_r(M^r)) \subset \bigcup\{\varphi_{i,s,\zeta}(M^r \times M^s) \mid \zeta \in \Gamma_{i,s}\}$. To realize this, let $z \in \pi_i^{-1}(\pi_r(M^r))$, that is, $\pi_i(z) = \pi_r(c)$ for some $c \in M^r$. Now, there exist $d \in M^s$ and $\zeta \in \Gamma_{i,s}$ with $\varphi_{i,s,\zeta}(c, d) = z$ by 2.11; thus, $z \in \bigcup\{\varphi_{i,s,\zeta}(M^r \times M^s) \mid \zeta \in \Gamma_{i,s}\}$. Combining these two containments, we have the existence of $\zeta \in \Gamma_{i,s}$ and $(u, b) \in M^r \times M^s$ with $\varphi_{i,s,\zeta}(u, b) = y$. We need that u is in A. Again by the commuting of the diagram in 2.10, $\pi_i(y) = \pi_i \circ \varphi_{i,s,\zeta}(u, b) = \pi_r(u)$. Thus, $u \in \pi_r^{-1}(\pi_i(y)) \subset \pi_r^{-1}\Big(\pi_i\big(\pi_i^{-1}(\pi_r(A))\big)\Big) \subset \pi_r^{-1}(\pi_r(A)) = A$.

Now assume that $i < r$ with s the integer such that $r = i+2s$. As before we wish to show that $\pi_i^{-1}(\pi_r(A))$ is closed in M^i. First, we shall show for each $\zeta \in \Gamma_{i,s}$ that $\varphi_{r,s,\zeta}^{-1}(A) = B_\zeta \times M^s$ for some $B_\zeta \subset M^i$. Set $B_\zeta = p(\varphi_{r,s,\zeta}^{-1}(A))$; then, certainly $\varphi_{r,s,\zeta}^{-1}(A) \subset B_\zeta \times M^s$. Conversely, let $(c, d) \in B_\zeta \times M^s$; then, there exists $d_0 \in M^s$ with $(c, d_0) \in \varphi_{r,s,\zeta}^{-1}(A)$. Thus, $\pi_r \circ \varphi_{r,s,\zeta}(c, d) = \pi_i \circ p(c, d) = \pi_i(c) = \pi_i \circ p(c, d_0) = \pi_r \circ \varphi_{r,s,\zeta}(c, d_0)$ which is in $\pi_r(A)$. So, $\varphi_{r,s,\zeta}(c, d) \in \pi_r^{-1}(\pi_r(A)) = A$, that is, $(c, d) \in \varphi_{r,s,\zeta}^{-1}(A)$. Thus, we have shown that $B_\zeta \times M^s = \varphi_{r,s,\zeta}^{-1}(A)$ which is closed in $M^i \times M^s$ since A is closed.

Define $B = \bigcup\{B_\zeta \mid \zeta \in \Gamma_{r,s}\}$; then, $B \times M^s$, being the finite union of closed sets $B_\zeta \times M^s$, is closed in M^{i+s}. Then, since $p\colon M^i \times M^s \to M^i$ is an open map by 1.14, $p(M^{i+s} \setminus (B \times M^s)) = p((M^i \setminus B) \times M^s) = M^i \setminus B$ is open in M^i, that is, B is closed in M^i. To finish, we show that $B = \pi_i^{-1}(\pi_r(A))$. Suppose that $b \in B_\zeta \subset B$ for some $\zeta \in \Gamma_{r,s}$. Then, there exists $c \in M^s$ with $(b, c) \in \varphi_{r,s,\zeta}^{-1}(A)$. And by the commuting of the diagram in 2.10, $\pi_i(b)$

$= \pi_r \circ \varphi_{r,s,\zeta}(b, c) \in \pi_r(A)$, that is, $b \in \pi_i^{-1}(\pi_r(A))$. Conversely, suppose that $y \in \pi_i^{-1}(\pi_r(A))$; then, there exists $a \in A$ with $\pi_i(y) = \pi_r(a)$. Applying the remark in 2.11 (with i and r interchanged), there exist $c \in M^s$ and $\zeta \in \Gamma_{r,s}$ such that $\varphi_{r,s,\zeta}(y, c) = a$, which implies that $(y, c) \in \varphi_{r,s,\zeta}^{-1}(A)$, hence, that $y \in p(\varphi_{r,s,\zeta}^{-1}(A)) = B_\zeta \subset B$.

An immediate corollary of this is:

2.17. Corollary. *Let X be a t_2 k-space with A a closed subset of $\coprod_{i=0}^{\infty} M^i$ such that $A \cap M^i \neq \emptyset$ only if $i \in F$ for some finite F. If for each $i \in F$, $A \cap M^i = \pi_i^{-1}(\pi_i(A \cap M^i))$, then $\pi(A)$ is closed in FX.*

2.18. Corollary. *For X in WHK and any n, X^n is embedded as a closed subspace in FX.*

Proof. For the canonical injection $f\colon X^n \to M^n \to \coprod_{i=0}^{\infty} M^i$ and the canonical quotient $\pi\colon \coprod_{i=0}^{\infty} M^i \to FX$, clearly $\pi \circ f$ is a continuous injection. To see that it is, in fact, a closed embedding, let A be closed in X^n. Then $f(A)$ is closed in M^n; furthermore, $f(A) = \pi_n^{-1}(\pi_n(f(A)))$; and, invoking 2.16, $\pi \circ f(A)$ is closed in FX.

So, it is noteworthy that FX can have no property inherited by closed subsets which all finite k-products of X do not also share. For example, FX is not normal if $X \times X$ is not normal.

2.19. Corollary. *For any n, the set of all words of reduced length n or less is closed in FX.*

Proof. The set of words of reduced length less than or equal to n is $\bigcup\{\pi(M^i) \mid 0 \leqslant i \leqslant n\}$. By 2.16 each $\pi(M^i)$ is closed; therefore, the set in question is closed.

If X is a k-space such that for any $(x, y) \notin \Delta(X)$ there exist disjoint open sets U and V in $X \times X$ with $(x, y) \in U$ and $\Delta(X) \subset V$, X is called *diagonally separable.* Clearly, every regular T_2 k-space is diagonally separable, since $\Delta(X)$ is closed in $X \times_c X$ which is regular. Also, note that X is T_2 if it is diagonally separable. To see this, suppose that $a \neq b$ and define a continuous $g\colon X \to X \times X$ by $g(x) = (x, b)$. Then $g(a) \notin \Delta(X)$; therefore, there are disjoint open U and V in $X \times X$ with $(a, b) \in U$ and $\Delta(X) \subset V$ yielding that $a \in g^{-1}(U)$ and $b \in g^{-1}(V)$.

2.20. Proposition. *A k-group is diagonally separable if and only if it is T_2.*

Proof. By our previous observation, we need only that T_2 implies diagonal separability. For G a T_2 k-group, define $h\colon G \times G \to G$ by $h(x, y) = xy^{-1}$. Then $h^{-}(1) = \Delta(G)$, and the diagonal separability of G follows from the continuity of h and that G is T_2.

Trivially we note that if $f\colon X \to Y$ is a continuous injection in K

and Y is diagonally separable, then X is also. Thus, if FX is T_2, then X must be diagonally separable. So, the next example yields a T_2 k-space X for which FX is not T_2. This is example 74 of [14] with an additional observation.

2.21. Example. For R the reals, let $X = R \quad R \cup \{(0, 0)'\}$ with the topology defined as follows: Points of $X \setminus \{(0, 0), (0, 0)'\}$ have the same neighborhood bases as they would in $R \times R$; basic neighborhoods of $(0, 0)$ are $N_\varepsilon = \{(a, b) \mid b > 0 \text{ and } 0 \leqslant a^2 + b^2 < \varepsilon\}$, and basic neighborhoods of $(0, 0)'$ are $N'_\varepsilon = \{(a, b) \mid b < 0 \text{ and } 0 < a^2 + b^2 < \varepsilon\} \cup \{(0, 0)'\}$, where ε is any positive real number. As noted in [14], this topology is first countable, therefore, a k-space topology. We now show that $\big((0, 0), (0, 0)'\big)$ can not be separated from the diagonal. Suppose that $\big((0, 0), (0, 0)'\big) \in N_\varepsilon \times N'_{\varepsilon'}$ (note that $X \times X = X \times_c X$ since X is first countable) and suppose that $\Delta(X) \subset U$ for some open U. Let ε denote the smaller of ε and ε'. Now $\big((\varepsilon/2, 0), (\varepsilon/2, 0)\big) \in \Delta(X) \subset U$; so there exists an open ball $B = \{(a, b) \mid (a - \varepsilon/2)^2 + b^2 < \delta\}$ with $\delta < \varepsilon/4$, such that $\big((\varepsilon/2, 0), (\varepsilon/2, 0)\big) \in B \times B \subset U$. However, $B \cap N_\varepsilon \neq \emptyset$ and $B \cap N'_\varepsilon \neq \emptyset$; therefore, $N_\varepsilon \times N'_{\varepsilon'} \cap U \neq \emptyset$. Thus, X is not diagonally separable, and FX is not T_2.

We temporarily leave the question of the separation of FX to consider a type of t_2 k-space for which FX is also a topological group. Clearly, such a X would have to be completely regular T_2. A t_2-space X is called a *k_ω-space* if there is a sequence of compact subsets $X_1 \subset X_2 \subset \ldots$, with $X = \bigcup X_i$ such that U is open in X if and only if $U \cap X_i$ is open in X_i for each i. After one has verified that such a X is T_2, this definition then agrees with that given in [9]; the following facts are either obvious or are given in [9] without proof but with appropriate references.

2.22. Proposition. *For a k_ω-space X:*

1° *X is T_2.*

2° *X is a k-space.*

3° *Any compact subset is contained in some X_n.*

4° *Any t_2 quotient of X is a k_ω-space.*

5° *If Y is also a k_ω-space, $X \times_c Y$ is a k_ω-space, hence, $X \times_c Y = X \times Y$.*

Topological groups which are also k_ω-spaces will be called *k_ω-groups*. By 5° in the previous proposition, k_ω-groups are also k-groups.

2.23. Proposition. *For a k_ω-space X, FX is a topological group (k_ω-group).*

Proof. Noting that $\coprod_{i=0}^{\infty} M$ is a k_ω-space, FX, since it is the t_2 quotient of this space, is also a k_ω-space. But then $FX \times FX = FX \times_c FX$.

We are now ready to give a much weaker condition than k_ω necessitating that FX be T_2. A space X is functionally Hausdorff if for each $a \neq b$ in X, there exists a continuous $f\colon X \to [0, 1]$ with $f(a) = 0$ and

$f(b) = 1$. Clearly, if X is completely regular T_2, then it is functionally T_2, as is kX. Therefore, for $\prod R_i$ the product of an uncountable number of copies of the reals, $k[\prod R_i]$ is functionally T_2; however, as noted in 2.1, this space is not regular. Thus, functionally T_2 spaces need not be regular (this will be used in 2.33 to provide an example of a T_2 k-group G with a closed normal subgroup H such that G/H is not T_2).

Example 90 of [14] is a regular T_2, hence diagonally separable, space which is not functionally T_2. After one verifies that this space is indeed a k-space, we know that not all diagonally separated k-spaces are functionally T_2. One easily checks, though, that functionally T_2 k-spaces are diagonally separable.

Functionally Hausdorff spaces are characterized as follows:

2.24. PROPOSITION. *A space is functionally T_2 if and only if it can be continuously injected into a completely regular T_2 space if and only if it can be continuously injected into a compact space.*

Proof. This is 4.5 of [8] with each $Y_f = [0, 1]$.

2.25. LEMMA. *If $g\colon X \to Y$ is an injection in K, $Fg\colon FX \to FY$ is an injection in KG.*

Proof. For e_i either 1 or -1, note that $Fg(x_1^{e_1} \dots x_n^{e_n}) = g(x_1)^{e_1} \dots \dots g(x_n)^{e_n}$; then, from this the injectivity is immediate.

2.26. PROPOSITION. *For any X in K, X is functionally T_2 if and only if FX is functionally T_2.*

Proof. Since the converse implication is clear, assume that X is functionally T_2. Then, there is a continuous injection $g\colon X \to C$ with C compact; therefore, $Fg\colon FX \to FC$ is a continuous injection. However, FC is a topological group, since C is a k_ω-space; thus, since FC is t_2 and a topological group, it is completely regular T_2. Invoking 2.24, FX is functionally T_2.

An interesting question is whether or not FX is functionally T_2 when it is T_2; or, more generally, is any k-group G T_2 if and only if it is functionally T_2? This, of course, would hold if all T_2 k-groups were simply the k-refinement of some T_2 topological group topology on G. At present we do not even have an example of a T_2 k-group which is not functionally T_2. We do know, though, that only certain types of k-groups may be the k-refinement of a topological group:

2.27. PROPOSITION. *For H a topological group, kH is either not T_0 or kH is functionally T_2.*

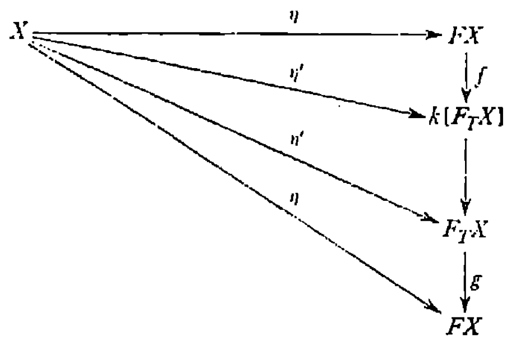
Proof. Certainly, H is either not T_0 or H is completely regular T_2. If H is completely regular T_2, then kH is functionally T_2. And if H is not T_0, it is not T_1 either; therefore, by 1.5 kH is not T_1 and, hence, not T_0.

Thus, any k-group G which is T_0 but not T_2 is not the k-refinement of a topological group topology on G.

Checking the definition in [16] of the free topological group $F_T X$ generated by a space X (where X need not be completely regular), we note, in particular, from our previous observation that for X a t_2, non-T_2 k-space, FX is not isomorphic to $kF_T X$. We also have:

2.28. Proposition. *For X in K, if FX is a topological group, then $FX \approx F_T X$.*

Proof. For η and η' the canonical injections of X into FX and $F_T X$, respectively, we have the commuting diagram:



where f arises from the universal property of FX, and g arises from the universal property of $F_T X$. By the uniqueness in the adjoint set-up, $g \circ f = 1_{FX}$, yielding that f is a quotient. Since f is easily verified to be a continuous injection, it is a k-group isomorphism.

A similar conclusion when $F_T X$ is a k-group does not appear to hold. At least its verification is not as straightforward.

2.29. Proposition. *If $g\colon X \to Y$ is a quotient in K, $Fg\colon FX \to FY$ is a quotient in KG.*

Proof. This is immediate from 1.10, 1.11, and 2.9, along with the fact that F being a left adjoint preserves coequalizers.

2.30. Proposition. *For X in K, if FX is a topological group, then every t_2 quotient of X is completely regular T_2, equivalently, every t_2 quotient is normal; in particular, if X is t_2, it is normal.*

Proof. Suppose that $g\colon X \to Y$ is a quotient in K with Y t_2; then, $Fg\colon FX \to FY$ is a quotient in KG. But quotients of topological groups under group morphisms are topological groups; therefore, FY is also a topological group. Since FY is t_2, it is then completely regular T_2. Thus, all t_2 quotients of X must be completely regular T_2.

To see that Y must, in fact, be normal, suppose that it is not. Then, there would be a closed subset A in Y with the quotient Y/A non-regular which, if Y is a t_2 quotient of X, yields a non-completely regular t_2 quotient of X in contradiction with the first part of the proof.

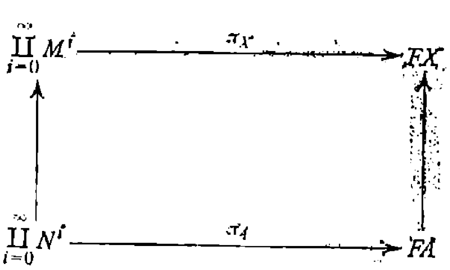
Thus, we have a necessary condition – much more stringent than anything like complete regularity or local compactness – for FX to be a topological group. Still lacking, though, is a sufficient condition. Also, indirectly we have that k_ω-spaces are normal. One would now like to have an example of a normal k-space A for which FA is not a topological group. For this it is sufficient to produce a normal k-space with a non-normal t_2 quotient. The following is such a space.

2.31. Example. First, consider example 82 of [14]. Set $Z = \{(x, y) \in R\times R \mid y \geqslant 0\}$ with points in $\{(x, y) \mid y > 0\}$ given the same neighborhood bases as in $R\times R$; neighborhoods of each $(x, 0)$ being $\{(x, 0)\}\cup D$ for D any open disc in $\{(x, y) \mid y > 0\}$ which is tangent to $(x, 0)$. As noted in [14], Z is a first countable (hence a k-space), non-normal T_2 space. We now realize Z as the quotient of a normal k-space as follows: For each $(x, 0)\in Z$, let $N(x, 0)$ be any fixed basic neighborhood of $(x, 0)$ as given above. Note that each $N(x, 0)$ in the subspace topology from Z is second countable and regular; therefore, by Theorem 17, chapter 4 of [8], each $N(x, 0)$ is a T_2 metric space, therefore, a normal k-space. Set $A = \coprod\{N(x, 0) \mid (x, 0)\in Z\}\amalg\{(x, y) \mid y > 0\}$ and define $f\colon A\to Z$ by $f(a) = a$. One, then, easily checks that A is a normal k-space and that f is an open map, therefore, a quotient.

We now extend 2.25 to preservation of closed embeddings when the spaces considered are in WHK.

2.32. Proposition. *The functor F: $WHK\to WHKG$ preserves closed embeddings, that is, equalizers.*

Proof. Let A be closed in X. Then, for $N = A\amalg A^*$, $M = X\amalg X^*$, and π_X and π_A the canonical quotients, we have the commuting diagram:



where the vertical arrows are the canonical injections. For any closed B in FA, identifying B with its image in FX, we need that B is closed in FX, that is, that $\pi_X^{-1}(B)\cap M^i$ is closed in M^i for each i. We have that $\pi_A^{-1}(B)\cap N^i$ is closed in N^i for each i, hence, in each M^i. To finish, using the notation of 2.10, note that $\pi_X^{-1}(B)\cap M^i = \bigcup\{\varphi_{i,s,\zeta}([\pi_A^{-1}(B)\cap M^{i-2s}]\times M^s) \mid 0\leqslant s\leqslant i/2,\ s \text{ an integer, and } \zeta\in\Gamma_{i,s}\}$, which, being the finite union of closed subsets of M^i, is closed in M^i.

Another way of saying this is: For X in WHK and A closed in X, given the subspace topology from FX, the subgroup of FX generated by A is isomorphic to FA. This is not the case for non-closed subsets. However, we defer until 2.49 the presentation of such an example, this being more easily acomplished there.

We have seen that quotients of t_2 k-groups by closed, normal subgroups are again t_2 k-groups. As the next example will show, though, quotients of T_2 k-groups by closed, normal subgroups are not always T_2. This, of course, is in marked contrast with topological groups; and again underscores the appropriateness of t_2 separation instead of T_2 separation for k-groups.

2.33. Example. Let X be the functionally Hausdorff, non-regular k-space in 2.1. Since X is non-regular, it contains a closed subset A with the quotient X/A non-Hausdorff. By 2.29, $FX \to F(X/A)$ is a quotient. And by 2.26, FX is T_2; however, $F(X/A)$ is non-Hausdorff since X/A is non-Hausdorff. Therefore, letting H be the kernel of this quotient, FX/H is the weakly Hausdorff, non-Hausdorff quotient of a Hausdorff k-group by a closed, normal subgroup.

The next two results are preliminary in establishing that the left adjoint F is, in fact, an enriched left adjoint. Letting $|\ |: KG \to K$ denote the grounding functor, we first have:

2.34. Lemma. *For G and H in KG, the inclusion $\underline{KG}(G, H) \to \underline{K}(|G|, |H|)$ is a k-embedding, which if H is t_2 is a closed embedding.*

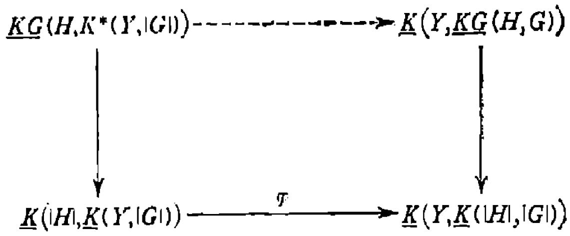
Proof. Since the topology on $\underline{KG}(G, H)$ is simply the k-refinement of the subspace topology from $K_{t\text{-}o}(|G|, |H|)$, the inclusion is a k-embedding as an application of 1.9 with $A = KG(G, H)$. Assuming that H is t_2, for x and y in G and 1 the identity of H, note that $B_{x,y} = \{f \mid f(x)f(y)f(xy)^{-1} = 1\}$ is closed in $\underline{K}(|G|, |H|)$. Thus, $KG(G, H) = \bigcap \{B_{x,y} \mid x$ and y are in $G\}$ is closed in $\underline{K}(|G|, |H|)$.

2.35. Proposition. *For G in KG, the functor $\underline{KG}(\_, G)$: $KG \to K^{op}$ is a K-left adjoint.*

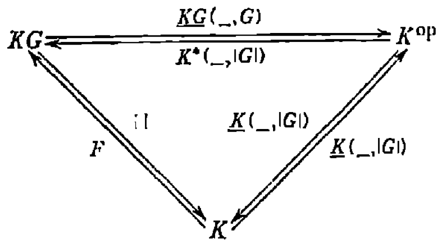
Proof. For X in K, note that $\underline{K}(X, |G|)$ has a natural k-group structure arising from that of G, where, for example, $\alpha\beta(x) = \alpha(x)\beta(x)$. Letting $K^*(X, |G|)$ denote this group, we now show that $K^*(\_, |G|)$ is K-right adjoint to $KG(\_, G)$. For Y in K and H in KG, consider the following diagram, where the left-hand vertical arrow is the k-embedding of 2.34, the right-hand vertical arrow is $\underline{K}(Y, \_)$ applied to the k-embedding of 2.34, which by 1.23 is also a k-embedding, and φ is the natural isomorphism of 1.24.

One checks that the induced map, denoted by the dotted arrow, is a topological isomorphism, natural in Y and H.

2.36. Proposition. *The free functors of 2.12 are, respectively, K and WHK left adjoints.*



Proof. We present the proof for the K case, the other being exactly the same. For G in KG consider the following diagram of adjoint pairs of functors, the exterior functor being left adjoint to the interior functor.



Clearly, the interior commutes; therefore, by the uniqueness of left adjoints, the exterior commutes. This yields that $\underline{KG}(FX, G)$ is naturally isomorphic to $K(X, |G|)$ for each X in K. To finish, one checks that this isomorphism is the canonical one.

2.37. Proposition. *If $f\colon A \to B$ is a quotient in KG, then for any G in KG, $KG(f, G)\colon \underline{KG}(B, G) \to KG(A, G)$ is a k-embedding, which if G is t_2 is a closed embedding.*

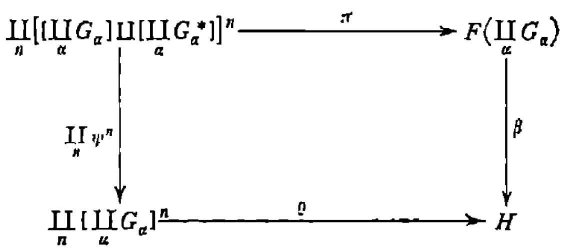
Proof. By 2.35 $\underline{KG}(\_, G)$ is a left adjoint, and from 2.9 f is a coequalizer in KG; therefore, $\underline{KG}(f, G)$ is a coequalizer in K^{op}, that is, an equalizer in K which is a k-embedding. For G in $WHKG$, one proceeds as in 1.37 to show that the embedding is, in fact, closed.

We now consider the algebraic free product of a collection of k-groups, give it a k-group topology, show that this topology is t_2 when each member of the collection is, and finally note that we have constructed the coproduct in KG (or $WHKG$) of this collection. For a collection $\{G_a \mid a \in A\}$ of k-groups, one recalls that the free product (see [6]) of this collection is the set of all symbols $H = \{g_1 g_2 \dots g_n \mid g_i \in G_i,\ g_i \neq e_i$, the identity of G_i, and g_i and g_{i+1} not in the same $G_a\} \cup \{1\}$, 1 the identity. Letting $\square$ denote multiplication on each G_a, multiplication on the free product is given by: $(g_1 g_2 \dots g_n)(h_1 h_2 \dots h_m) = g_1 g_2 \dots g_n h_1 h_2 \dots h_m$ if g_n and h_1 are not from the same G_a; it is $g_1 g_2 \dots g_n \square h_1 h_2 \dots h_m$ if g_n and h_1 are in the same G_a. Also, the inverse of $g_1 g_2 \dots g_n$ is $g_n^{-1} \dots g_2^{-1} g_1^{-1}$. For the canonical surjective group morphism $\beta\colon F(\coprod\{G_a \mid a \in A\}) \to H$, if H is given the quotient topology induced by β, H is clearly a k-group. In what follows, our main concern is the verification that this topology is t_2 when each G_α is t_2.

2.38. THEOREM. *For any collection of k-groups, their free product can be given a k-group topology such that it is the coproduct in KG of the collection. In addition, if each member of the collection is t_2, then the free product is t_2.*

Proof. Using the notation established above, since H is the quotient of a k-group, it is also a k-group. Employing a straightforward verification, one also has that H posseses all the requisite properties of the coproduct in KG of this collection. We now verify the non-trivial part of the proposition, namely, that H is t_2 whenever each G_α is t_2.

Consider the commuting diagram below, where ψ: $[\coprod_\alpha G_\alpha] \amalg [\coprod_\alpha G_\alpha^*] \to \coprod_\alpha G_\alpha$ is given by $\psi(g) = g$ for each $g \in G_\alpha$ and each $\alpha \in A$; $\psi(g^*) = g^{-1}$ for each $g^* \in G_\alpha^*$. Trivially, ψ is continuous, in fact, an open map. Also, π is the canonical quotient of 2.10, and ϱ is the induced surjection.



Since π and β are quotients, ϱ is a quotient.

To see that H is t_2, by 2.6 we need that $\{1\}$ is closed, that is, that $\varrho^{-1}(1) \cap [\coprod_\alpha G_\alpha]^n$ is closed in $[\coprod_\alpha G_\alpha]^n$ for each n. First, we note that $\coprod_n [\coprod_\alpha G_\alpha]^n = \coprod \{G_{f(1)} \times G_{f(2)} \times \quad \times G_{f(n)} \mid n$ is a positive integer and f: $\{1, 2, 3, \dots, n\} \to A\}$ under the obvious isomorphism, where A is the indexing set of the collection. We, thus, need that $\varrho^{-1}(1) \cap [G_{f(1)} \times \dots \times G_{f(n)}]$ is closed in $G_{f(1)} \times \quad \times G_{f(n)}$ for each n and f. The proof is by induction on n.

For $n = 1$ letting e_α denote the identity of G_α, $\varrho^{-1}(1) \cap G_{f(1)} = \{e_{f(1)}\}$ which, since $G_{f(1)}$ is t_2, is closed. Assume now that the statement is true for all $n \leqslant k$; we need to verify the statement for $n = k+1$. So, consider $G_{f(1)} \times \quad \times G_{f(k+1)}$ and divide the remainder of the proof into four cases.

Case 1. $f(1) \neq f(t)$ for $1 < t \leqslant k+1$. Here,

$$\varrho^{-1}(1) \cap [G_{f(1)} \times \quad \times G_{f(k+1)}] = \{e_{f(1)}\} \times \left[\varrho^{-1}(1) \cap [G_{f(2)} \times \quad \times G_{f(k+1)}]\right],$$

which is closed by the induction hypothesis.

Case 2. $f(1) \neq f(t)$ for $2 \leqslant t < s$, but $f(1) = f(s)$ with s fixed and $2 < s \leqslant k+1$. Here

$$\begin{aligned} &\varrho^{-1}(1) \cap [G_{f(1)} \times \quad \times G_{f(k+1)}] \\ &\quad = \{e_{f(1)}\} \times [\varrho^{-1}(1) \cap (G_{f(2)} \times \quad \times G_{f(k+1)})] \cup \tau\big([\varrho^{-1}(1) \cap (G_{f(2)} \times \\ &\qquad \ldots \times G_{f(s-1)})] \times [\varrho^{-1}(1) \cap (G_{f(1)} \times G_{f(s)} \times \quad \times G_{f(k+1)})]\big), \end{aligned}$$

where $\tau(g_2 \ldots g_{s-1} g_1 g_s \ldots g_{k+1}) = g_1 g_2 \ldots g_{k+1}$ in the usual order. By the induction hypothesis and the fact that τ is an isomorphism, this set is closed.

Case 3. $f(1) = f(t)$ for $2 \leqslant t \leqslant s$, but $f(1) \neq f(s+1)$ for some fixed $s \leqslant n$. Here,

$$\begin{aligned} \varrho^{-1}(1) \cap [G_{f(1)} \times \quad \times G_{f(k+1)}] &= [\varrho^{-1}(1) \cap (G_{f(1)} \times \quad \times G_{f(s)})] \times \\ &\times [\varrho^{-1}(1) \cap (G_{f(s+1)} \times \quad \times G_{f(k+1)})] \cup \tau\big([\varrho^{-1}(1) \cap [(G_{f(1)} \times \quad \times G_{f(s)}) \times \\ &\times (G_{f(s+j)} \times \quad \times G_{f(k+1)})]] \times [\varrho^{-1}(1) \cap (G_{f(s+1)} \times \ldots \times G_{f(s+j-1)})]\big), \end{aligned}$$

where j is the smallest positive integer such that $G_{f(1)} = G_{f(s+j)}$; if no such j exists, then the right-hand set of the union is the null set. Also, τ is given by $\tau(g_1 g_2 \ldots g_s g_{s+j} \ldots g_{k+1} g_{s+1} \ldots g_{s+j-1}) = g_1 g_2 \ldots g_{k+1}$ in the usual order. As before, the set in question is closed.

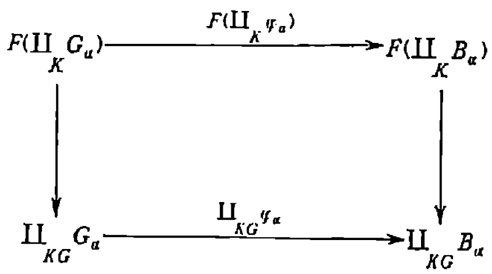
Case 4. $f(1) = f(t)$ for $2 \leqslant t \leqslant k+1$. Letting $\square$ denote multiplication on $G_{f(1)}$, define $\tau\colon G_{f(1)} \times \quad \times G_{f(k+1)} \to G_{f(1)}$ by $\tau(g_1 g_2 \ldots g_{k+1}) = g_1 \square g_2 \square \ldots \square g_{k+1}$. Then, $\varrho^{-1}(1) \cap [G_{f(1)} \times \quad \times G_{f(k+1)}] = \tau^{-1}(e_{f(1)})$ which is closed since τ is continuous.

Since these four cases are exhaustive, the statement is proved for $n = k+1$, hence, for all n.

Letting $\coprod_{KG} G_a$ denote the coproduct in KG of the collection, we have:

2.39. Proposition. *If each $\varphi_a\colon G_a \to B_a$ is a quotient in KG, then $\coprod_{KG} \varphi_a\colon \coprod_{KG} G_a \to \coprod_{KG} B_a$ is also a quotient in KG.*

Proof. Since $\coprod_K \varphi_a\colon \coprod_K G_a \to \coprod_K B_a$ is a quotient in K, $F(\coprod_K \varphi_a)\colon F(\coprod_K G_a) \to F(\coprod_K B_a)$ is a quotient in KG by 2.29. Then, for the vertical arrows being the canonical quotients of 2.38, the following commutes:



Therefore, $\coprod_{KG} \varphi_a$ is a quotient.

Immediate from 2.9 and 2.38 is the following bit of categorical bookkeeping.

2.40 PROPOSITION. *The categories KG and $WHKG$ are cocomplete. In addition, the grounding functor from KG to Groups preserves colimits, and the grounding functor to Groups from $WHKG$ preserves coproducts.*

In 2.38, if each G_α is a k_ω-group and the collection is countable, $F(\coprod G_\alpha)$ is a k_ω-group by 2.23; thus, $\coprod_{KG} G_\alpha$ is also a k_ω-group, since it is a t_2 quotient of $F(\coprod G_\alpha)$. It follows that the k-group coproduct is also the coproduct in topological groups, a result obtained by Ordman in [13].

As a final topic, we show that epics in either the category of t_2 k-groups or the category of k_ω-groups are morphisms having dense range. This, of course, is trivial for the abelian counterparts of these categories. However, in [17] we see that, to our surprise, epics need not have dense range in the category of T_2 k-groups or in the category of T_2 abelian k-groups.

We should point out that everything which follows except 2.41 can be carried out in the category of T_2 topological groups. Thus, if 2.41 could be adapted for T_2 topological groups, one would know that epics in that category also have dense range — someting which, to the best of our knowledge, is still unanswered.

2.41. LEMMA. *Any continuous $t\colon X \times Y \to Y$ in K extends to a continuous $\tau\colon X \times FY \to FY$, where*

$$\tau(x, y_1^{e_1} \dots y_n^{e_n}) = [t(x, y_1)]^{e_1} \dots [t(x, y_n)]^{e_n}.$$

Proof. Let $t^*\colon X \times Y^* \to Y^*$ be the continuous function induced by t and the isomorphism between Y and Y^* in 2.10, that is, $t^*(x, y^*) = [t(x, y)]^*$. Then for π the canonical quotient of 2.12, $\Delta_n\colon X \to X^n$ the diagonal map, $\alpha_n\colon X^n \times [Y \amalg Y^*]^n \to [X \times (Y \amalg Y^*)]^n$ the isomorphism given by $x_1 x_2 \dots x_n a_1 a_2 \dots a_n \mapsto x_1 a_1 x_2 a_2 \dots x_n a_n$, and $\beta\colon X \times (Y \amalg Y^*) \to (X \times Y) \amalg (X \times Y^*)$ the obvious isomorphism, consider the following commuting diagram:

$$\begin{array}{ccc}
X \times \coprod_n (Y \amalg Y^*)^n \cong \coprod_n X \times (Y \amalg Y^*)^n & \xrightarrow{\coprod_n (\Delta_n \times 1)} & \coprod_n X^n \times (Y \amalg Y^*)^n \\
\downarrow{\scriptstyle 1 \times \pi} & & \downarrow{\scriptstyle \coprod_n \alpha_n} \\
X \times FY & & \coprod_n [X \times (Y \amalg Y^*)]^n \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \coprod_n \beta^n} \\
FY \xleftarrow{\ \pi\ } \coprod [Y \amalg Y^*]^n & \xleftarrow{\coprod_n (t \amalg t^*)^n} & \coprod_n [(X \times Y) \amalg (X \times Y^*)]^n
\end{array}$$

Since the right-hand side is continuous, $\tau \circ (1 \times \pi)$ is continuous. But $1 \times \pi$ is a quotient; therefore, τ is continuous.

For $t\colon X \times Y \to Y$ and $x \in X$, let $t_x\colon Y \to Y$ denote the function $t_x(y) = t(x, y)$. Trivially, each t_x is continuous when t is continuous. And, since $\tau_x = Ft_x$, we have:

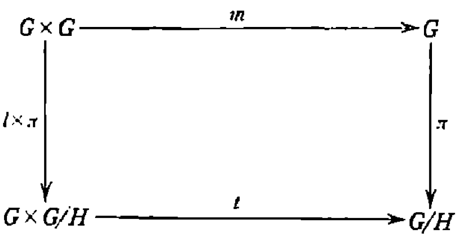
2.42. LEMMA. *If t_x is a k-space isomorphism, τ_x is a k-group isomorphism.*

2.43. LEMMA. *For H a closed subgroup of a t_2 k-group [k_ω-group] G and π: $G \to G/H$ defined via $\pi(a) = \pi(b)$ if $aH = bH$, G/H is a t_2 k-space [k_ω-space].*

Proof. Letting $R \subset G \times G$ denote the equivalence relation determined by π and noting that $R = \{(a, b) | b^{-1} a \in H\}$, we have that R is closed in $G \times G$, which if G is t_2 requires that G/H be t_2, and if G is k_ω requires that G/H be k_ω.

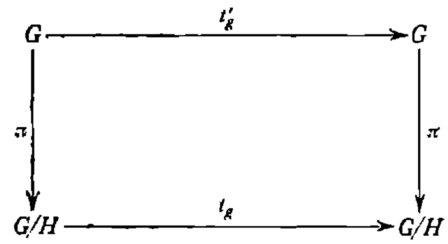
2.44. LEMMA. *For H a subgroup of a k-group G, t: $G \times G/H \to G/H$ defined by $t(g, xH) = gxH$ is continuous. Furthermore, each t_g: $G/H \to G/H$ is a k-space isomorphism.*

Proof. After one has checked that t is well-defined, to see that t is continuous, let m denote the multiplication on G and consider the following diagram:



Since this commutes, $t \circ (1 \times \pi)$ is continuous, which, since $1 \times \pi$ is a quotient, implies that t is continuous.

For g in G, denote by t'_g: $G \to G$ translation by g in G, which according to 2.2 is a k-space isomorphism. Considering the following diagram, it is then immediate that t_g is continuous, surjective, and a quotient.



To finish one checks that t_g is one-to-one.

Immediate from this, 2.41, and 2.42 is:

2.45. LEMMA. *For τ: $G \times F(G/H) \to F(G/H)$ the extension of t in 2.44, τ is continuous. In addition, each τ_g: $F(G/H) \to F(G/H)$ is a k-group isomorphism.*

As a final preliminary to forming a k-group semi-direct product with G and $F(G/H)$, we note that τ is mixed associative, that is, for any

u and v in G, $\tau_{uv} = \tau_u \circ \tau_v$. The next proposition's proof is straightforward and will be omitted (see [6] for the algebraic part).

2.46. PROPOSITION. *For k-groups G and A and a mixed associative τ: $G \times A \to A$ in K such that each τ_g is a k-group isomorphism on A, the semi-direct product $A \times G$ provided with the usual k-product topology is also a k-group.*

2.47. THEOREM. *If H is a proper closed subgroup of a t_2 k-group [k_ω-group] G, there exists a t_2 k-group [k_ω-group] E and k-group morphisms α, β: $G \to E$ with $\alpha(g) = \beta(g)$ if and only if $g \in H$.*

Proof. Let E be the semi-direct product $F(G/H) \times G$. By 2.43, 2.12, and 2.46, E is a t_2 k-group when G is t_2; by 2.43, 2.23, 2.46, and 2.22(5), E is a k_ω-group when G is k_ω. Recall now that multiplication in E is given by $(a, g)(a', g') = (a\tau_g(a'), gg')$. Also, for the singleton word H in $F(G/H)$, note that $\tau_g(H) = H$ if and only if $g \in H$.

We now define two subgroups of E as follows: $G_1 = \{1\} \times G$, and $G_2 = \{(H^* \tau_g(H), g) \mid g \in G\}$, where H^* denotes the inverse of H in $F(G/H)$. One checks that both canonical functions α: $G \to G_1$ and β: $G \to G_2$ are one-to-one k-group morphisms. In addition, $\alpha(g) = \beta(g)$ if and only if $H^* \tau_g(H) = 1$ if and only if $H^* gH = 1$ if and only if $g \in H$.

As a direct consequence of this, we have the result regarding epics.

2.48. COROLLARY. *If f: $B \to G$ is an epic in either the category of t_2 k-groups or the category of k_ω-groups, then $f(B)$ is dense in G.*

Proof. If the closure of $f(B)$ is not G, let H be the closure of $f(B)$; the result is then immediate from the previous theorem.

Here we must credit Karl Hofmann and Eric Nummela for an unpublished proof that epics have dense range in the category of t_2 k-groups. They too employed a free k-group and a semi-direct product; indeed their proof motivated ours. However, the present approach has the advantage of providing the additional result for k_ω-groups.

In 2.32 it was shown that the free functor F: $WHK \to WHKG$ preserved equalizers, that is, for A closed in X, $FA \to FX$ was a closed embedding. We promised the reader a counter example to this when A was not closed, and with the aid of 2.48 such an example is available with a minimum of effort.

2.49. EXAMPLE. Let $X = Z^+ \cup \{*\}$ be the one point compactification of the positive integers, topologized as in 1.1; let $A = Z^+$. Then the inclusion i: $A \to X$ has dense range, and, as is easily verified, this says that i is an epic in WHK. Since left adjoints preserve epics, Fi: $FA \to FX$ is an epic in $WHKG$, and by 2.48 Fi has dense range in FX. Since A is a discrete space, FA is a discrete group; thus, if Fi were a topological embedding, $Fi(FA)$ would be a proper discrete subgroup of FX. But discrete subgroups of t_2 k-groups are closed by 2.5; this contradicts $Fi(FA)$ being dense. Consequently, Fi is not an embedding.

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