A REPRESENTATION
OF RELATIVELY COMPLEMENTED DISTRIBUTIVE LATTICES

BY

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It is known [1] that every distributive lattice can be imbedded in a Boolean lattice. It is shown in this paper that, for every relatively complemented distributive lattice $L$, there exists a Boolean lattice $B$ such that $L$ is equal to the intersection of a prime ideal and an ultrafilter of $B$ (Theorem 1.4 (b)). We also show that the free relatively complemented extension of any distributive lattice exists. Using this result, we give a characterization of free relatively complemented distributive lattices (Theorem 2.5).

0. Background. The following categories will be considered in this paper:

the category $\mathcal{D}$ whose objects are distributive lattices and whose morphisms are lattice homomorphisms;

the category $\mathcal{R}$ whose objects are relatively complemented distributive lattices and whose morphisms are lattice homomorphisms;

the category $\mathcal{B}$ whose objects are Boolean lattices and whose morphisms are lattice homomorphisms.

Let $L \in \mathcal{D}$ and let $a, b \in L$. Then $a + b$ will denote the join of $a$ and $b$, $ab$ the meet of $a$ and $b$, $a^*$ the set of all proper prime ideals of $L$ which do not contain $a$, and $L^* = \{a^*: a \in L\}$. The lattice $L$ itself will be considered as a prime ideal and an ultrafilter of $L$.

It is well known (see [1]) that the ring of sets $L^*$ is lattice-isomorphic to $L$ and that if $\hat{B}(L)$ is the field of sets generated by $L^*$, then $\langle \hat{B}(L), i(L) \rangle$ is the free Boolean lattice extension of $L$, where $i(L): L \to \hat{B}(L)$ is the natural imbedding of $L$ in $\hat{B}(L)$. That is, if $B_1 \in \mathcal{B}$ and $f \in \text{Hom}_\mathcal{D}[L, B_1]$, then there exists a unique $f^* : \hat{B}(L) \to B_1$ such that $f^* \circ i(L) = f$. Then $\hat{B} : \mathcal{D} \to \mathcal{B}$ can be extended to a reflector functor. $L^*$ will be identified with
$L$, and $x \in \hat{B}(L)$ if and only if

$$x = a + \sum_{i=1}^{n} a_i \bar{b}_i + \bar{b},$$

where $a, a_i, b_i, b \in L$ and $\bar{b}_i$ is the complement of $b_i$ in $\hat{B}(L)$.

1. Characterization of a relatively complemented distributive lattice.

**Theorem 1.1.** If $L \in \mathcal{R}$, then $L$ is convex in $\hat{B}(L)$.

**Proof.** Suppose $a \leq b \leq c$, where $a, c \in L$ and $b \in \hat{B}(L)$. Since $b \in \hat{B}(L)$, $b$ can be written in the form

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i + \bar{v},$$

where $u, u_i, v_i, v \in L$ and $u, \bar{v}$ or $u_i \bar{v}_i$ cannot occur. Now $a \leq b$ implies

$$b = a + b = a + u + \sum_{i=1}^{n} u_i \bar{v}_i + \bar{v},$$

and $b \leq c$ implies

$$b = a + cu + \sum_{i=1}^{n} cu_i \bar{v}_i + c\bar{v}.$$  

Thus we can assume

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i,$$

where $u$ always occurs and $u_i \bar{v}_i$ can or cannot occur. If no $u_i \bar{v}_i$ occurs, then $b = u \in L$. Assume that

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i$$

and that $u_i \bar{v}_i$ occurs. Then $uv_i \leq v_i \leq u_i + v_i$ for every $i$ and there exists $v_i' \in L$ such that $v_i'$ is the complement of $v_i$ in $[uv_i, u_i + v_i]$. Hence

$$v_i' = (u_i + v_i) \bar{v}_i + uv_i = u \bar{v}_i + uv_i$$

and

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i = u + \sum_{i=1}^{n} (u_i \bar{v}_i + uv_i) = u + \sum_{i=1}^{n} v_i' \in L.$$  

**Lemma 1.2.** Let $B \in \mathcal{B}$ and let $L \in \mathcal{R}$ be a sublattice of $B$ such that $0 \in L$ and $\hat{B}(L) = B$. Then $L$ is a prime ideal of $B$.

**Proof.** If $1 \in L$, then $L \in \mathcal{B}$ implies $\hat{B}(L) = L$, so $L = B$. Assume $1 \notin L$. By Theorem 1.1, $L$ is convex in $\hat{B}(L)$, and $0 \in L, 1 \notin L$ imply $L$ is a proper ideal of $B$. 


We claim that \( L \) is a prime ideal of \( B \). Indeed, given \( a \in B - L \), we show that \( \bar{a} \in L \). The assumption \( B = \hat{B}(L) \) yields

\[
a = u + \sum_{i=1}^{n} u_i \bar{v}_i + \bar{v},
\]

where \( u, u_i, v_i, v \in L \) and \( u, v \) or \( u_i \bar{v}_i \) need not occur. If \( u_i \bar{v}_i \) occurs, we get \( 0 \leq u_i \bar{v}_i \leq u_i \); \( 0, u_i \in L \) implies \( u_i \bar{v}_i \in L \) by convexity of \( L \). Thus \( a = u + \bar{v} \), where \( u, v \in L \) and \( \bar{a} = \bar{u}v \). Since \( v \in L \), \( \bar{a} = \bar{u}v \in L \) by convexity of \( L \).

A similar argument yields

**Lemma 1.3.** Let \( B \in \mathcal{B} \) and let \( L \in \mathcal{R} \) be a sublattice of \( B \) such that \( 1 \in L \) and \( \hat{B}(L) = B \). Then \( L \) is an ultrafilter of \( B \).

**Theorem 1.4.** Let \( B \in \mathcal{B} \).

(a) If \( P \) and \( F \) are a non-principal prime ideal and ultrafilter, respectively, such that \( F \neq B - P \), then \( P, F \) and \( P \cap F \) are relatively complemented convex sublattices of \( B \) and \( \hat{B}(P) = \hat{B}(F) = \hat{B}(P \cap F) = B \).

(b) If \( L \in \mathcal{R} \) is a sublattice of \( B \) such that \( \hat{B}(L) = B \), then \( L = P \cap F \), where \( P \) is either a non-principal prime ideal of \( B \) or \( P = B \) and \( F \) is either a non-principal ultrafilter of \( B \) or \( F = B \).

**Proof.** (a) \( P \) is obviously a relatively complemented lattice and, clearly, \( P \) is a sublattice of \( B \) such that \( 0 \in P \). Suppose \( a \in B - P \). Then \( \bar{a} \in P \) implies \( \hat{B}(P) = B \). Thus \( P \) is convex in \( B \) by Theorem 1.1.

Similarly, \( F \) is convex in \( B \).

Now consider \( P \cap F \). Clearly, \( P \cap F \) is a relatively complemented sublattice of \( B \). We show that \( \hat{B}(P \cap F) = B \). Let \( a \in B - P \cap F \). Then there are the following three possibilities:

(i) \( a \notin P \) and \( a \notin F \). Then \( \bar{a} \in P \cap F \) and \( a = \bar{a} \hat{B}(P \cap F) \).

(ii) \( a \notin P \) and \( a \in F \). Then, for any \( b \in P \cap F \), we have \( ab \in P \cap F \) and \( \bar{a} + b \in P \cap F \). Thus \( \bar{a} = \bar{a} + b \hat{B}(P \cap F) \) and \( ab \in P \cap F \) implies

\[
a = ab + ab \hat{B}(P \cap F).
\]

(iii) \( a \in P \) and \( a \notin F \) — dual to (ii).

Thus \( \hat{B}(P \cap F) = B \) and, by Theorem 1.1, \( P \cap F \) is convex in \( B \).

(b) If \( 0 \in L \), then \( L \) is a prime ideal by Lemma 1.2. If \( 1 \in L \), then \( L \) is an ultrafilter of \( B \) by Lemma 1.3.

Suppose \( 0 \notin L \) and \( 1 \notin L \). Then \( \hat{B}(L) = B \), so that \( L \) is convex in \( B \) by Theorem 1.1. Let

\[
P = \{ x \in B : x \leq a \text{ for some } a \in L \}.
\]
Then $P$ is a proper ideal of $B$, and $P \supseteq L$ implies $\hat{B}(P) = B$. Hence $P$ is a prime ideal of $B$ by Lemma 1.2. Dually,

$$F = \{x \in B : x \supseteq a \text{ for some } a \in L\}$$

is an ultrafilter of $B$. Now $L \subseteq F$ and $L \subseteq P$ imply $L \subseteq P \cap F$. Let $x \in P \cap F$. Then there exist $a \in L$ such that $a \leq x$ and $b \in L$ such that $x \leq b$. Since $L$ is convex in $B$, $x \in L$.

**Corollary 1.5.** Every lattice $L \in \mathfrak{L}$ can be imbedded in a Boolean lattice $B$ so that $L = P \cap F$, where $P$ is a prime ideal of $B$ and $F$ is an ultrafilter of $B$.

**Proof.** The proof follows if we take $B = \hat{B}(L)$ and apply Theorem 4 (b).

### 2. The free relatively complemented extension of a distributive lattice.

**Lemma 2.1.** If $L \in \mathfrak{D}$ is a convex sublattice of $B \in \mathfrak{B}$, then $L \in \mathfrak{R}$.

**Proof.** Suppose $a, b, c \in L$ and $a \leq b \leq c$. Then there exists $b' \in B$ such that $b + b' = c$ and $bb' = a$. Since $a \leq b' \leq c$, $b' \in L$ by convexity of $L$. Thus $L \in \mathfrak{R}$.

**Theorem 2.2.** Let $L \in \mathfrak{D}$ and consider $L$ as a sublattice of $\hat{B}(L)$. Then

$$R(L) = \{x \in \hat{B}(L) : x \in L \text{ or } x = \sum_{i=1}^{n} a_i \bar{b}_i, \text{ where } a, a_i, b_i \in L\}$$

is the smallest relatively complemented sublattice of $\hat{B}(L)$ that contains $L$ as a sublattice.

**Proof.** First we show that $R(L)$ is a relatively complemented lattice. Clearly, $R(L)$ is a lattice. Suppose $x \leq y \leq z$, where $x, z \in R(L)$ and $y \in \hat{B}(L)$. Then

$$x = a + \sum_{i=1}^{n} b_i \bar{c}_i \quad \text{and} \quad z = e + \sum_{j=1}^{n} f_j \bar{g}_j,$$

where $a, b_i, c_i, e, f_j, g_j \in L$ and $b_i \bar{c}_i, f_j \bar{g}_j$ cannot occur. Since $y \in \hat{B}(L)$,

$$y = u + \sum_{k=1}^{n} u_k \bar{v}_k + \bar{v} \quad \text{for } u, u_k, v_k, v \in L,$$

where $u, u_k \bar{v}_k$ or $\bar{v}$ cannot occur. If $a \in L$ and $a \leq x$, then $a \leq y$, so $a + y = y$. Similarly, since $yz = y$, we have

$$y = u'' + \sum_{k=1}^{n} u_k' \bar{v}_k'',$$

where $u'', u_k', v_k' \in L$, and $u''$ must occur and $u_k' \bar{v}_k'$ can or cannot occur.
Therefore, \( y \in R(L) \). Thus \( R(L) \) is convex in \( \hat{B}(L) \) and \( R(L) \in \mathcal{R} \) by Lemma 2.1.

Next we show that \( R(L) \) is the smallest relatively complemented distributive lattice containing \( L \) as a sublattice.

Let \( L' \in \mathcal{R} \) be a sublattice of \( \hat{B}(L) \) containing \( L \). We show that if

\[
x = a + \sum_{i=1}^{n} b_i \bar{c}_i \quad \text{for } a, a_i, b_i \in L,
\]

then \( x \in L' \). It suffices to show that, for any \( a, b, c \in L \), we have \( a+b\bar{c} \in L' \).

Now \( a \leq a+c \leq a+b+c \) and \( a, a+c, a+b+\bar{c} \in L' \). Since \( L' \in \mathcal{R} \), there exists \( c' \in L' \) such that \( (a+c)c' = a \) and \( a+c+c' = a+b+c \).

But \( c' \in \hat{B}(L) \) implies that

\[
c' = a + (a+b+c)(a+c) = a + (a+b+c)\bar{a}\bar{c} = a + b\bar{a}\bar{c} = a + b\bar{c} \in L'.
\]

Note. \( R(L) \) contains 0 if and only if \( L \) contains 0; similarly for 1.

Definition. Let \( B \in \mathcal{B} \) and let \( L \in \mathcal{D} \) be a sublattice of \( B \). The convex hull of \( L \) in \( B \), denoted by \( L_B^* \), is the smallest convex sublattice of \( B \) containing \( L \).

**Theorem 2.3.** If \( L \in \mathcal{D} \), then \( L_B^* = R(L) \).

**Proof.** Clearly, \( \hat{B}(R(L)) = \hat{B}(L) \) since \( L \subseteq R(L) \subseteq \hat{B}(R(L)) \). Thus, by Theorem 1.1, \( R(L) \) is convex in \( \hat{B}(L) \), and \( L_B^* \) is a sublattice of \( R(L) \).

Also, \( L_B^* \in \mathcal{R} \), and \( L \) is a sublattice of \( L_B^* \). Hence \( R(L) \) is a sublattice of \( L_B^* \) by Theorem 2.2.

**Definition.** Let \( L \in \mathcal{D} \). Then \( \langle R(L), \lambda(L) \rangle \) is the free relatively complemented extension of \( L \) if \( R(L) \in \mathcal{R} \), \( \lambda(L) : L \rightarrow R(L) \) is an imbedding map and whenever \( M \in \mathcal{R} \) and \( f \in \text{Hom}_{\mathcal{D}}[L, M] \), then there exists a unique \( f^* \in \text{Hom}_{\mathcal{R}}[R(L), M] \) such that \( f^* \circ \lambda(L) = f \).

**Theorem 2.4.** Let \( L \in \mathcal{D} \) and let \( \lambda(L) : L \rightarrow R(L) \) be the imbedding map. Then \( \langle R(L), \lambda(L) \rangle \) is the free relatively complemented extension of \( L \).

**Proof.** Let \( M \in \mathcal{R} \) and let \( f \in \text{Hom}_{\mathcal{D}}[L, M] \). Then

\[
\hat{B}(f) \in \text{Hom}_{\mathcal{R}}[\hat{B}(L), \hat{B}(M)]
\]

is the unique extension of \( f \) to \( \hat{B}(L) \). Let \( f^* = \hat{B}(f)|_{R(L)} \). We claim that \( \text{Im} f^* \subseteq M \). Indeed, let \( x \in R(L) \). If \( x \in L \), then \( f^*(x) = f(x) \in M \). Now suppose

\[
x = b + \sum_{i=1}^{n} c_i \bar{d}_i, \quad \text{where } b, c_i, d_i \in L.
\]

Then

\[
f^*(x) = f^*(b + \sum_{i=1}^{n} c_i \bar{d}_i) = f^*(b) + \sum_{i=1}^{n} f^*(c_i)f^*(\bar{d}_i)
\]

\[
= f(b) + \sum_{i=1}^{n} f(c_i)f(\bar{d}_i) \in R(f(L)).
\]
Also, \( f(L) \subseteq M \) implies \( R(f(L)) \subseteq M \) by Theorem 2.2. Thus \( \text{Im} f^* \subseteq M \).

Clearly, \( f^* \in \text{Hom}_D[R(L), M] \) and \( f^* \circ \lambda(L) = f \). The uniqueness of \( f^* \) follows from the uniqueness of \( \hat{B}(f) \).

Remarks. (1) By Theorem 2.4, \( R : \mathcal{D} \to \mathcal{R} \) can be extended to a reflector functor.

(2) It is well known (see [1]) that the free distributive lattice on \( n \) generators, \( n \) being finite, has the length \( 2^n - 2 \). If \( L \in \mathcal{D} \), \( |L| < \aleph_0 \) and \( \mathcal{C} \) is a maximal chain of \( L \), then \( \hat{B}(L) = \hat{B}(\mathcal{C}) \). If the length of \( L \) is \( k \), then \( |\hat{B}(L)| = 2^k \).

(3) Let \( \mathcal{A} \) and \( \mathcal{C} \) be categories that have free objects. Let \( F : \mathcal{A} \to \mathcal{C} \) be a reflector functor. Then it is known (see [2]) that if \( S \) is a set and \( A_S \) is free on \( S \) in \( \mathcal{A} \), then \( F(A_S) \) is free on \( S \) in \( \mathcal{C} \).

From Remarks (2) and (3) we get the following characterization of the free objects in \( \mathcal{R} \).

**Theorem 2.5.** Let \( S \) be a non-empty set. If \( F_S \) is the free object on \( S \) in \( \mathcal{D} \), then \( R(F_S) \) is the free object on \( S \) in \( \mathcal{R} \). If \( |S| = n \), then \( R(F_S) \in \mathcal{B} \) and \( |R(F_S)| = 2^{2^n - 2} \).

**References**


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