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WALTER SCHACHERMAYER

**On some classical measure-theoretic theorems for
non-sigma-complete Boolean algebras**

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CONTENTS

Introduction

If $\mathscr{F}$ is a σ-algebra, the following theorems hold for $\mathscr{F}$:

VITALI–HAHN–SAKS (VHS): *A sequence $\{\mu_n\}_{n=1}^{\infty}$ of finitely additive, bounded, real valued measures on $\mathscr{F}$, such that $\{\mu_n(A)\}_{n=1}^{\infty}$ converges for every $A \in \mathscr{F}$, is uniformly exhaustive.*

NIKODYM (N): *A family M of finitely additive, bounded, real valued measures on $\mathscr{F}$, such that $\{\mu(A): \mu \in M\}$ is bounded for every $A \in \mathscr{F}$, is uniformly bounded.*

ORLICZ–PETTIS (OP): *Every measure $\mu: \mathscr{F} \to X$ with values in a Banach space X which is σ-additive with respect to the weak topology of X is σ-additive with respect to the norm topology of X.*

GROTHENDIECK (G): *Let $B(\mathscr{F})$ be the Banach space of bounded $\mathscr{F}$-measurable functions equipped with the sup-norm. A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$ that converges weak* converges weakly, i.e. $B(\mathscr{F})$ is a Grothendieck space.*

ROSENTHAL (R): *Every continuous linear map from $B(\mathscr{F})$ to a Banach space X which is not weakly compact fixes a copy of l^{∞}, i.e. $B(\mathscr{F})$ is a Rosenthal-space.*

For proofs we refer to [D–U 77] and for an account of the historical development of these important and beautiful theorems from Vitali's paper of 1907 we refer to [F 76].

As regards (VHS) and (N), these theorems were originally stated in terms of σ-additive measures. T. Andô [A 61] was the first to realize that these theorems have their proper setting in the framework of finitely additive measures and since then many improvements and generalisations have been obtained. So the natural question has arisen as to whether these theorems could be extended to non-σ-complete Boolean algebras.

As regards (G), it was observed by Grothendieck that on the dual of $L^{\infty}(\mu)$ weak and weak* sequential convergence coincide. Lindenstrauss [L 64] proved that for a $C(K)$-space this property is equivalent to a certain extension property of separably valued operators (see Theorem 5.1 below). He raised the question of characterizing this property, which we call (G), in terms of the topological properties of K. Since then considerable progress has been made in the study of Grothendieck spaces but they still remain mysterious objects. A good reference is [D 73], in which a list of open problems is stated.

We take the validity of the theorems in question as a definition

(analogously to calling a locally convex space barrelled if the Banach–Steinhaus theorem holds on it): We say that a Boolean algebra $\mathscr{F}$ satisfies (VHS), (N), (OP), (G) or (R) if the corresponding theorem holds on $\mathscr{F}$.

First we note that an arbitrary Boolean algebra need not satisfy these properties. Let $\Phi(N)$ be the algebra of finite and cofinite subsets of N and let δ_n denote the Dirac measure located at $\{n\}$. The sequence $\{\delta_n\}_{n=1}^{\infty}$ provides a counterexample to (VHS), while the set $\{n(\delta_n - \delta_{n+1}): n = 1, 2, 3, \ldots\}$ provides a counterexample to (N). Concerning (OP) consider the measure μ from $\Phi(N)$ into c_0 (the Banach space of null-sequences), whose coordinates are the scalar measures $\delta_n - \delta_{n+1}$. For (G) note that $B(\Phi(N))$ may be identified with the Banach space c of convergent sequences and that $\{\delta_n\}_{n=1}^{\infty}$ (i.e., the evaluation in the nth coordinate is a sequence in c^* which converges weak* but not weakly. Finally note that the identity on c is not weakly compact but c does not contain a copy of l^∞, whence $\mathscr{F}$ does not have (R).

On the other hand, there exist classes of non-σ-complete algebras which have the properties stated above ([S 68], [D 78], [F 76]).

It has been realized that there are strong interrelations between these properties of a Boolean algebra. (The reader has probably noticed that the counterexemples in the case of $\Phi(N)$ are essentially the same.) So it has been asked ([S 68], [D–F–H 76], [F 76]) whether the four properties (VHS), (G), (N) and (OP) are equivalent for a Boolean algebra.

The only known equivalence result is due to Diestel, Faires and Huff [D–F–H 76]: A Boolean algebra has (VHS) iff it has (G) and (N).

It is proved that (G) implies (OP) (Theorem 6.3 below). Whether (G) implies (VHS) (or equivalently (N)) remains open([1]). None of the other possible implications between (VHS), (G), (N) and (OP) holds, as is shown by a series of examples (see § 3 below). As regards (R) we can only show the (trivial) observation that (R) implies (G). (During the final preparation of this paper I was informed that R. Haydon [H 79] and – independently – M. Talagrand [T 79] had constructed examples, showing that (G) is not equivalent to (R).)

As regards the organisation of the paper: After a preliminary § 1, in § 2 we give definitions and present the above mentioned Diestel–Faires–Huff theorem. We also show that $\mathscr{F}$ has (N) iff the normed space of $\mathscr{F}$-measurable simple functions is barreled, which allows some sharpening of the theorems of Dieudonné–Grothendieck and Seever. Finally we show that in the definition of (OP) one may reduce to the case of bounded σ-additive measures.

In § 3 we give examples. For instance the algebra $\mathscr{J}$ of Jordan-measur-

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[1] Note added in proof. M. Talagrand has constructed an example showing that assuming the continuum hypothesis (G) does not imply (VHS). (Oral communication).

able subsets of $[0,1]$ has (N) and (OP) but not (R), (VHS) and (G). However, there is a quotient algebra of $\mathscr{J}$ which does not have (OP). Also the Stone space of $\mathscr{J}$ has the property that every infinite closed subset contains a copy of βN.

In §4 we investigate two special classes of Boolean algebras:

(1) If $\mathscr{F}$ is a Boolean subalgebra of a σ-complete algebra Σ which is not too far away from Σ (the condition is that the Banach space $B(\mathscr{F})$ of bounded $\mathscr{F}$-measurable functions is a countable intersection of closed hyperplanes in $B(\Sigma)$), then $\mathscr{F}$ has all our properties.

(2) On the other hand, if $\mathscr{F}$ is a countable union of a strictly increasing sequence $\{\mathscr{F}_n\}_{n=1}^\infty$ of Boolean algebras (these objects arise for example in the theory of martingales), then $\mathscr{F}$ does not satisfy (VHS), (G), (N) or (R).

In §5 we investigate property (G) in detail. It is shown that (G) is equivalent to a condition very similar to (OP) (stated, however, in terms of finitely additive measures). We also characterize (G) in terms of convex weak *-compact subsets of $B(\mathscr{F})^*$.

Finally, in §6 we prove that (G) implies (OP).

My warmest thanks go to Joseph Diestel. Without him this article would never have been written. He suggested to me the open questions that are partially solved in this paper, he gave me the unpublished preprint [D–F–H 76] and I had the opportunity of some stimulating conversations with him. I also thank Barbara Faires for her kind collaboration. Finally, I am greatly indebted to the two referees for many valuable comments, simplifications of proofs and modifications of the poor style in which the first version of this paper was written. Most of the results were obtained during the academic year 1977/78, while the author was a Solomon Lefschetz – instructor at the Centro de Investigación del Instituto Politécnico Nacional in Mexico.

§ 1. Preliminaries

1.1. In this paper $\mathscr{F}$ is a Bodean algebra with the operations of sup, inf and complementation denoted by $\bigvee, \bigwedge$ and C. The smallest and the largest elements of $\mathscr{F}$ will be denoted by $\emptyset$ and Ω. The Stone representation theorem (see [S 71a], Theorem 16.2.3, for example) states that $\mathscr{F}$ has a unique representation as the field of clopen sets of a totally disconnected compact Hausdorff space, the "Stone representation space" of $\mathscr{F}$, which will also be denoted by Ω. Note, however, that $\mathscr{F}$ always has many representations as a field of sets (the Stone representation is one such representation, unique only in the sense stated above).

If $\{A_i\}_{i\in I}$ is a family of elements of $\mathscr{F}$, we write $\bigvee_{i\in I} A_i$ for the smallest element in $\mathscr{F}$ that majorises all A_i, if such an element exists. (A necessary

and sufficient condition for its existence is that, if $\mathscr{F}$ is Stone represented, the closure of the union of the A_i is open). $\mathscr{F}$ is σ-complete (resp. complete) if all countable (resp. all) suprema exist in $\mathscr{F}$.

The letters X, Y, Z will denote Banach spaces, which — only for simplicity — are assumed real, and X^*, Y^*, Z^* the topological duals. A function $\mu: \mathscr{F} \to X$ is called a *measure*, if it is additive, i.e. whenever A_1, A_2 in $\mathscr{F}$ are disjoint (this means that $A_1 \wedge A_2 = \emptyset$), and then $\mu(A_1 \vee A_2) = \mu(A_1) + \mu(A_2)$.

A measure μ is called *bounded* if $\mathrm{Rg}\,(\mu) = \{\mu(A): A \in \mathscr{F}\}$ is bounded, and is called σ-*additive* (resp. *weakly σ-additive*) if, for every sequence of mutually disjoint elements $\{A_n\}_{n=1}^\infty$, such that $\bigvee\limits_{n=1}^\infty A_n$ exists in $\mathscr{F}$, we have $\sum\limits_{n=1}^\infty \mu(A_n) = \mu(\bigvee\limits_{n=1}^\infty A_n)$, the series converging in the norm topology (resp. in the weak topology).

Denote by $B(\mathscr{F})$ the Banach algebra of real valued bounded $\mathscr{F}$-measurable functions equipped with the sup norm. $B(\mathscr{F})$ may be obtained in an abstract way as the completion of the normed algebra $B_s(\mathscr{F})$ of simple functions, i.e. expressions of the form $\sum\limits_{i=1}^n \lambda_i \chi_{A_i}$ with the obvious norm and algebra operations. A less formal approach is to note that $B(\mathscr{F})$ may be naturally identified with the Banach algebra $C(\Omega)$ of continuous functions on the Stone space Ω.

The dual $B(\mathscr{F})^*$ may be represented as the space of all real valued finitely additive measures μ on $\mathscr{F}$ with finite variation norms $\|\mu\| = \sup\{\sum\limits_{i=1}^n |\mu(A_i)|: \{A_i\}_{i=1}^\infty$ partition of $\mathscr{F}\}$. Note that a real valued measure has a bounded variation norm iff it is bounded, as the following estimate shows ([D–S 58], III. 1.5)

$$\sup\{|\mu(A)|: A \in \mathscr{F}\} \leqslant \|\mu\| \leqslant 2 \sup\{|\mu(A)|: A \in \mathscr{F}\}.$$

Another representation of $B(\mathscr{F})^*$ is to interpret it as the space of Radon measures on the Stone space Ω.

Our notation is that of [D–U 77] with the following exception. A measure $\mu: \mathscr{F} \to X$ is called *exhaustive* (in [D–U 77] "strongly additive") if, for every sequence $\{A_n\}_{n=1}^\infty$ of mutually disjoint members of $\mathscr{F}$, $\|\mu(A_n)\|$ tends to zero. A family M of X-valued measures is called *uniformly exhaustive* if $\|\mu(A_n)\|$ tends to zero uniformly in $\mu \in M$. The following result gives the connection between uniform exhaustivity and weak compactness in $B(\mathscr{F})^*$. Although this result is well known, I am unable to give a reference for the exact result that we need and shall therefore sketch a proof.

1.2. PROPOSITION. *A subset M of $B(\mathscr{F})^*$ is relatively weakly compact iff M is uniformly exhaustive and bounded on the members of $\mathscr{F}$.*

Proof. We consider $B(\mathscr{F})^*$ as the space of Radon-measures on the Stone space Ω.

If M is relatively weakly compact then M is bounded and, by Dunford's characterization of weak compactness in spaces of σ-additive measures ([D–U 77], th. IV. 2.5.) there is a positive Randon-measure η on Ω such that M is uniformly absolutely continuous with respect to η. So for every disjoint sequence $\{A_i\}_{i=1}^{\infty}$ of clopen sets in Ω, $\eta(A_i) \to 0$, whence $\mu(A_i) \to 0$ uniformly in $\mu \in M$.

Conversely, suppose M to be uniformly exhaustive and bounded on members of $\mathscr{F}$. Then M is a bounded subset of $B(\mathscr{F})^*$.

Indeed, define the measure $m: \mathscr{F} \to l^{\infty}(M, X)$ by $m(A) = \{\mu(A)\}_{\mu \in M}$. Then m is exhaustive and hence bounded, i.e. M is a bounded subset of $B(\mathscr{F})^*$. I want to thank the referee for pointing out this argument to me.

As M is bounded, we may apply Corollary I.5.4 of [D–U 77] to find a control-measure $\eta \in B(\mathscr{F})^*$ for M, i.e. a positive Randon-measure η on Ω such that

$$\lim_{\eta(A) \to 0} \sup \{|\mu(A)|: \mu \in M\} = 0,$$

where A ranges through $\mathscr{F}$. With the help of Lemma I.5.1 of [D–U 77] we may again apply Dunford's characterization of weak compactness to infer that M is relatively weakly compact in $B(\mathscr{F})^*$. ∎

§ 2. Definitions and a theorem of Diestel, Faires and Huff

2.1. DEFINITION. F *has* (VHS) if one of the following equivalent conditions is satisfied:

(VHS₁)   A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$ which converges pointwise on $\mathscr{F}$ (i.e., for all $A \in \mathscr{F}$, $\{\mu_n(A)\}_{n=1}^{\infty}$ converges) is uniformly exhaustive.

(VHS₂)   A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$ which converges with respect to the $\sigma(B(\mathscr{F})^*, B_s(\mathscr{F}))$-topology converges weakly, i.e. with respect to $\sigma(B(\mathscr{F})^*, B(\mathscr{F})^{**})$.

Proof. The equivalence is a direct consequence of Proposition 1.2 and the observation that a relatively weakly compact sequence that converges with respect to $\sigma(B(\mathscr{F})^*, B_s(\mathscr{F}))$ converges weakly. ∎

2.2. DEFINITION. A Banach space X is called a *Grothendieck space* if a sequence in X^* which converges weak* to zero converges weakly.

2.3. DEFINITION. $\mathscr{F}$ *has* (G) if one of the following equivalent conditions is satisfied:

(G₁)   A bounded sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$ which converges pointwise on $\mathscr{F}$ is uniformly exhaustive.

(G_2) $B(\mathscr{F})$ is a Grothiendieck space.

Proof. $B_s(\mathscr{F})$ is a dense subspace of $B(\mathscr{F})$. Whence a bounded $\sigma(B(\mathscr{F})^*, B_s(\mathscr{F}))$-convergent sequence is $\sigma(B(\mathscr{F})^*, B(\mathscr{F}))$-convergent. Conversely, it follows from the uniform boundedness principle that a $\sigma(B(\mathscr{F})^*, B(\mathscr{F}))$-convergent sequence is bounded. Hence the equivalence of (G_1) and (G_2) is again a consequence of Proposition 1.2. ∎

2.4. DEFINITION. $\mathscr{F}$ has (N) if one of the following equivalent conditions is satisfied:

(N_1) A subset M of $B(\mathscr{F})^*$ which is pointwise bounded on $\mathscr{F}$ (i.e., for all $A \in \mathscr{F}$ we have $\sup\{|\mu(A)|: \mu \in M\} < \infty$) is uniformly bounded.

(N_2) The normed space $B_s(\mathscr{F})$ (with the supremum norm) is barrelled.

Proof. Since pointwise boundedness of M on $\mathscr{F}$ is the same as boundedness with respect to the duality $\langle B_s(\mathscr{F}), B(\mathscr{F})^* \rangle$ the equivalence is a particular case of IV.5.2 of [S 71]. ∎

2.5. THEOREM ([D–F–H 76]). (VHS) $\Leftrightarrow$ (G)+(N), *i.e.* $\mathscr{F}$ *has* (VHS) *iff it has* (G) *and* (N).

Proof. $(VHS_1) \Rightarrow (G_1)$ and $(G_1)+(N_1) \Rightarrow (VHS_1)$ are evident. So it remains to prove (VHS) $\Rightarrow$ (N). Suppose non (N) and (VHS). Then we can find a sequence $\{\mu_n\}_{n=1}^\infty$ in $B(\mathscr{F})^*$ which is pointwise bounded on $\mathscr{F}$ but such that $\|\mu_n\| \to \infty$. Defining

$$\lambda_n = \|\mu_n\|^{-1/2} \mu_n$$

we exhibit a sequence tending pointwise to zero on $\mathscr{F}$ but such that $\|\lambda_n\| \to \infty$, a contradiction of (VHS). ∎

Despite its simplicity, the characterization of (N) by condition (N_2) (which is also essentially contained in [D–F–H 76]) gives more insight into the nature of property (N). For example the following proposition allows us to sharpen the results of Seever and Dieudonné–Grothendieck by using the "open mapping theorem technique" connected with the notion of barrelledness.

2.6. PROPOSITION. *Let* Y *be a dense subspace of a Banach space* Z. *The following conditions are equivalent:*

(i) Y *is barrelled.*

(ii) *Given any continuous linear map* T *from a Banach space* X *to* Z *such that* $T(X)$ *contains* Y, T *is onto* Z.

(iii) *Every linear map* T *from* Y *to a Banach space* X *with closed graph is continuous.*

(iv) *Every linear map* T *from* Y *to a Banach space* X *such that* $x^* \circ T \in Y^*$ *for each* x^* *in some total subset* Γ *of* X^* *is continuous.*

Proof. (i) $\Leftrightarrow$ (iii) follows from [S 71], IV.8.5. and IV.8.6.

(iii) $\Leftrightarrow$ (iv) is immediate because $T: Y \to X$ has a closed graph iff there

is a total subset Γ of X^* such that $x^* \circ T \in Y^* (\forall x^* \in \Gamma)$ (see [K 77], 34.5 (3), p. 34).

[(i), (iii)] $\Rightarrow$ (ii): We may assume that T is 1-1. Let $W = T(X)$; then $Y \subseteq W \subseteq Z$. As Y is barrelled and dense in W, W is barrelled. Since T is continuous, $T^{-1}: W \to X$ has a closed graph and so is continuous. Thus T is an isomorphism of X onto W, which implies that W is closed in Z; finally $W = Z$.

(ii) $\Rightarrow$ (i): If $B_s(\mathscr{F})$ is not barrelled, there exists a lower semicontinuous seminorm, say p, on $B_s(\mathscr{F})$ which is not continuous. Therefore the norm $q = p + \| \cdot \|_\infty$ is strictly stronger than $\| \cdot \|_\infty$ on $B_s(\mathscr{F})$. Let X be the completion of $(B_s(\mathscr{F}), q)$. As q is stronger than $\| \cdot \|_\infty$, X embeds continuously into $B(\mathscr{F})$. Moreover, $X \neq B(\mathscr{F})$ since otherwise q would be equivalent to $\| \cdot \|_\infty$. The identity $X \to B_s(\mathscr{F})$ contradicts (ii). ∎

Remark. Proposition 2.6, applied to $Y = B_s(\mathscr{F})$ and $Z = B(\mathscr{F})$, shows in particular that the theorems of Dieudonné–Grothendieck and Seever ([D–U 77], I.3.3 and I.3.4) hold for algebras with property (N) and that they only hold for those algebras. In the above proposition the word "Banach space" may be replaced by "Ptak space", whence in particular by locally convex "Fréchet space".

An interesting aspect of property (N) is that it furnishes natural examples of non-complete normed barrelled spaces with closed subspaces which are not barrelled. For example let Σ be a σ-algebra and $\mathscr{F}$ a subalgebra which does not have (N), e.g. $\Sigma = \{$all subsets of $N\}$ and $\mathscr{F} = \Phi(N)$. Then $B_s(\mathscr{F})$ is a closed non-barrelled subspaces of the barrelled space $B_s(\Sigma)$, which is a non-complete normed space.

2.7. Definition. $\mathscr{F}$ *has* (OP) if one of the following equivalent conditions is satisfied:

(OP$_1$)   A measure μ from $\mathscr{F}$ to a Banach space X which is weakly σ-additive is σ-additive (i.e., for the norm topology).

(OP$_2$)   A bounded measure μ from $\mathscr{F}$ to a Banach space X which is weakly σ-additive is σ-additive.

Clearly (OP$_1$) $\Rightarrow$ (OP$_2$). But the implication (OP$_2$) $\Rightarrow$ (OP$_1$) is not obvious since σ-additive (even real valued) measures on algebras are not bounded in general. This is due to the fact that there are algebras on which no non-trivial countable suprema exist (see 3.14 below). On such an algebra any measure (equivalently any linear map from $B_s(\mathscr{F})$ to X) is σ-additive (as the requirements are empty) but there are unbounded ones among them.

We now prove (OP$_2$) $\Rightarrow$ (OP$_1$). Suppose that $\mu: \mathscr{F} \to X$ is weakly σ-additive and let $\{A_n\}_{n=1}^\infty$ be a disjoint sequence in $\mathscr{F}$ such that $\bigvee_{n=1}^\infty A_n$

exists in $\mathscr{F}$. Then

$(*)$ there is an N such that μ restricted to $\mathscr{F} \wedge \bigvee_{n=N}^{\infty} A_n$ is bounded.

Assuming $(*)$ for the moment, we may apply (OP_2) to μ restricted to $\mathscr{F} \wedge \bigvee_{n=N}^{\infty} A_n$ and infer that $\sum_{n=N}^{\infty} \mu(A_n)$ is strongly convergent in X. This gives $(\mathrm{OP}_2) \Rightarrow (\mathrm{OP}_1)$.

To prove $(*)$, assume that it is not true. We shall construct inductively a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of integers and elements $B_k \in \mathscr{F}$ such that

$$B_k \subseteq \bigvee_{l=n_k+1}^{n_{k+1}} A_l$$

and

$$\|\mu(B_k)\| \to \infty.$$

Let

$$C_k = \bigvee_{l=n_k+1}^{n_{k+1}} A_l \backslash B_k.$$

Since $B_1 \vee C_1 \vee B_2 \vee C_2 \vee \ldots = \bigvee_{n=1}^{\infty} A_n \in \mathscr{F}$, the corresponding series $\mu(B_1) + \mu(C_1) + \mu(B_2) + \ldots$ converges weakly to $\mu\left(\bigvee_{n=1}^{\infty} A_n\right)$ by the weak σ-additivity of μ. Hence $\mu(B_k)$ tends to zero weakly and cannot be unbounded; a contradiction.

I n d u c t i o n : Let $n_1 = 0$ and suppose that $n_1, \ldots, n_k$ and $B_1, \ldots, B_{k-1}$ are defined. By assumption there exists $D_k \subseteq \bigvee_{l=n_k+1}^{\infty} A_l$ such that $\|\mu(D_k)\| > k$. Find x_k^* in X^* such that $\|x_k^*\| = 1$ and $|x_k^* \circ \mu(D_k)| > k$. By the σ-additivity of $x_k^* \circ \mu$ we may find n_{k+1} such that $\left|x_k^* \circ \mu(D_k \wedge \bigvee_{l=n_k+1}^{n_{k+1}} A_l)\right| > k$, whence $\left\|\mu(D_k \wedge \bigvee_{l=n_k+1}^{n_{k+1}} A_l)\right\| > k$. Putting $B_k = D_k \wedge \bigvee_{l=n_k+1}^{n_{k+1}} A_l$, we complete the induction step, thus finishing the proof of $(\mathrm{OP}_2) \Rightarrow (\mathrm{OP}_1)$. ∎

2.8. DEFINITION. We call a Banach space X a *Rosenthal space* if every non weakly compact operator $T: X \to Y$ fixes a copy of l^{∞}, i.e. there is a continuous linear map $j: l^{\infty} \to X$ such that $T \circ j$ is an isomorphism into Y.

2.9. DEFINITION. We say that $\mathscr{F}$ *has* (R) if $B(\mathscr{F})$ is a Rosenthal space.

2.10. PROPOSITION. (R) $\Rightarrow$ (G), *i.e. an algebra $\mathscr{F}$ having* (R) *has* (G).

Proof. It is immediate from the definition that the Banach space $B(\mathscr{F})$ is a Grothendieck space iff every map $T: B(\mathscr{F}) \to c_0$ is weakly compact (see Theorem 5.1 below). But, as c_0 does not contain an isomorphic copy of l^∞, it is plain that (R) implies (G). ■

To finish the section we state the following result on the stability of the properties in question. For the definition of subalgebras, quotients and ideals of a Boolean algebra we refer to [H 63] or [S 62].

2.11. PROPOSITION. (a) *None of the properties* (VHS), (G), (N), (OP) *or* (R) *is inherited by sub-algebras.*

(b) (VHS), (G), (N) *and* (R) *are inherited by quotient-algebras, while* (OP) *is not.*

Proof. (a) This follows from the example in the introduction. $\varPhi(N)$ is a sub-algebra of the complete Boolean algebra $\mathscr{P}(N)$ of all subsets of N.

(b) Let $\mathscr{I}$ be an ideal in $\mathscr{F}$ and $\mathscr{G} = \mathscr{F}/\mathscr{I}$ the quotient algebra. Note that $B(\mathscr{G})$ is in a natural way a quotient of $B(\mathscr{F})$, and $B_s(\mathscr{G})$ a quotient of $B_s(\mathscr{F})$. As the property of being a Grothendieck, a Rosenthal or a barrelled space is inherited by (separated) quotient spaces, we see that (R), (G) and (N) are inherited by quotient algebras and, by Theorem 2.5, we get the result for (VHS) too. The fact that (OP) is not inherited by quotients will be shown in the forthcoming example (Proposition 3.9 below). ■

§ 3. Examples

3.1. Let $\mathscr{J}$ denote the family of Jordan-measurable sets in $[0, 1]$, i.e. the $A \subseteq [0, 1]$ such that $\mathrm{bd}\,(A) = \bar{A}\backslash A^0$ is of Lebesgue measure 0. $\mathscr{J}$ forms a field of subsets of $[0, 1]$ and $B(\mathscr{J})$ is the Banach space of bounded Riemann-integrable functions on $[0, 1]$.

We shall show that $\mathscr{J}$ does not have (G) (and therefore it does not have (VHS) and (R) either) but does have (N) and (OP). Finally we construct a quotient algebra of $\mathscr{J}$ which does not have (OP).

3.2. PROPOSITION. $\mathscr{J}$ *does not have* (G).

Proof. Define a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{J})^*$ by

$$\mu_n = 2^{-n+1} \sum_{k=1}^{2^{n-1}} \delta_{\{(2k-1)/2^n\}}$$

where δ denotes the usual Dirac-measure. We shall show that, for $A \in \mathscr{J}$, $\mu_k(A)$ converges to $m(A)$, the Lebesgue measure of A. Indeed, $\int \chi_A \, d\mu_n$ is a sequence of Riemann integral sums. As χ_A is Riemann-integrable, $\mu_n(A)$ tends to $m(A)$. ■

3.3. PROPOSITION. $\mathscr{J}$ *has* (N).

Proof. Suppose there is a sequence $\{\mu_n\}_{n=1}^\infty \in B(\mathscr{J})^*$ such that $\{\mu_n(A)\}_{n=1}^\infty$ is bounded $\forall\, A \in \mathscr{J}$ and $\|\mu_n\| \to \infty$. By the compactness of $[0,1]$ there is a $t_0 \in [0,1]$ such that, for every $k \in N$,

$$\{|\mu_n|(]t_0 - 1/k,\, t_0 + 1/k[\, \cap\, [0,1])\}_{n=1}^\infty$$

is unbounded.

Now we adapt the proof of Nikodym's theorem as presented in [D–U 77]: We may find a partition (E_1, F_1) of $[0,1]$ into disjoint members of $\mathscr{J}$ and an integer n_1 such that

$$|\mu_{n_1}(E_1)|,\ |\mu_{n_1}(F_1)| > 2.$$

At least either

$$\inf_{k \in N}\ \sup_{n \in N}\ \sup_{E \in \mathscr{J}} |\mu_n(E \cap E_1 \cap\,]t_0 - 1/k,\, t_0 + 1/k[)|$$

or

$$\inf_{k \in N}\ \sup_{n \in N}\ \sup_{E \in \mathscr{J}} |\mu_n(E \cap F_1 \cap\,]t_0 - 1/k,\, t_0 + 1/k[)|$$

is infinite.

If the first of the two is infinite, set $S_1 = E_1$ and $T_1 = F_1$; otherwise set $S_1 = F_1$ and $T_1 = E_1$. In any case there is an $n_2 > n_1$ and (E_2, F_2), a disjoint partition of $S_1 \cap\,]t_0 - \tfrac{1}{2}, t_0 + \tfrac{1}{2}[$, such that

$$|\mu_{n_2}(E_2)|,\ |\mu_{n_2}(F_2)| > 3 + |\mu_{n_2}(T_1)|.$$

Now at least either

$$\inf_{k \in N}\ \sup_{n \in N}\ \sup_{E \in \mathscr{J}} |\mu_n(E \cap E_2 \cap\,]t_0 - 1/k,\, t_0 + 1/k[)|$$

or

$$\inf_{k \in N}\ \sup_{n \in N}\ \sup_{E \in \mathscr{J}} |\mu_n(E \cap F_2 \cap\,]t_0 - 1/k,\, t_0 + 1/k[)|$$

is infinite.

If the first of the two is infinite, set $S_2 = E_2$ and $T_2 = F_2$; otherwise, set $S_2 = F_2$ and $T_2 = E_2$.

Continue in this fashion, obtaining a sequence $\{T_n\}_{n=1}^\infty$ of pairwise disjoint members of $\mathscr{J}$, such that $T_n \subseteq\,]t_0 - 1/n, t_0 + 1/n[$ and a strictly increasing sequence of positive integers $\{n_k\}_{k=1}^\infty$ such that, for each $k \geqslant 1$,

$$|\mu_{n_k}(T_k)| > \sum_{j=1}^{k-1} |\mu_{n_k}(T_j)| + k + 1.$$

Now make the crucial observation: for any subsequence $\{T_{k_l}\}_{l=1}^\infty$, $\bigcup_{l=1}^\infty T_{k_l} \in \mathscr{J}$. Indeed,

$$\overline{\bigcup_{l=1}^\infty T_{k_l}} \subseteq \bigcup_{l=1}^\infty \overline{T_{k_l}} \cup \{t_0\} \quad \text{and} \quad \Big(\bigcup_{l=1}^\infty T_{k_l}\Big)^0 \supseteq \bigcup_{l=1}^\infty T_{k_l}^0.$$

So

$$\overline{\bigcup_{l=1}^{\infty} T_{k_l}} \setminus (\bigcup_{l=1}^{\infty} T_{k_l})^0 \subseteq (\overline{\bigcup_{l=1}^{\infty} T_{k_l} \cup \{t_0\}}) \setminus \bigcup_{l=1}^{\infty} T_{k_l}^0 \subseteq \bigcup_{l=1}^{\infty} (\overline{T_{k_l}} \setminus T_{k_l}^0) \cup \{t_0\}.$$

This implies $\bigcup_{l=1}^{\infty} T_{k_l} \in \mathcal{J}$. So we may form any union $\bigcup_{l=1}^{\infty} T_{k_l}$ without leaving $\mathcal{J}$, i.e. the σ-algebra Σ generated by $\{T_k\}_{k=1}^{\infty}$ is contained in $\mathcal{J}$. Apply the Nikodym boundedness theorem ([D–U 77], th. I.3.1) to the sequence μ_{n_k}, restricted to Σ, to arrive at a contradiction in view of $|\mu_{n_k}(T_k)| > k$. ∎

Remark. We have proved Proposition 3.3 for the Jordan-measurable sets with respect to the Lebesgue-measure on $[0, 1]$. But the only essential things that were needed were the compactness argument and the fact that the point $\{t_0\}$ has the Lebesgue-measure zero. Having this in mind, one can easily adapt the above proof to more general circumstances: Let S be a completely regular topological space and m a σ-finite positive Radon-measure on S without atoms (i.e. for $t \in S, m(\{t\}) = 0$). The algebra $\mathcal{J}_m(S)$ of subsets A of S with $m(\bar{A} \setminus A^0) = 0$ has (N).

But note that the situation changes drastically if m has atoms. For example, let $m = \delta_{\{0\}}$ be the Dirac-measure at zero on $\mathbf{R}$. Then $\mathcal{J}_m(S)$ are the subsets A of $\mathbf{R}$ such that $\{0\} \notin \bar{A} \setminus A^0$. In other words $A \in \mathcal{J}_m(S)$ iff $\{0\}$ is either an interior point of A or of CA. So the sequence $\mu_n = n \left(\delta \left\{ \frac{1}{n} \right\} - \delta \left\{ \frac{1}{n-1} \right\} \right)$ is unbounded in norm while $\{\mu_n(A)\}_{n=1}^{\infty}$ is finally zero for each $A \in \mathcal{J}_m(X)$.

Let us now state two corollaries of Proposition 3.3, which we formulate for the Lebesgue-measure on $[0, 1]$ but which of course may be generalized as above. It is interesting to note that the field of finite unions of intervals is not sufficient for the forthcoming corollary to hold.

3.4. COROLLARY. *A subset B of $L^1[0, 1]$ is bounded iff, for every $A \in \mathcal{J}$, the set $\left\{ \int_A f \, dm: f \in B \right\}$ is bounded.* ∎

3.5. COROLLARY. *The space of simple Riemann-integrable functions on $[0, 1]$, equipped with the supremum norm, is barrelled.* ∎

3.6. DEFINITION ([D 78]). An algebra $\mathcal{F}$ is called *up-down semi-complete* if, for every disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\bigvee_{n=1}^{\infty} A_n$ exists in $\mathcal{F}$, $\bigvee_{k=1}^{\infty} A_{n_k}$ also exists in $\mathcal{F}$ for every subsequence $\{n_k\}_{k=1}^{\infty}$.

For example, if Ω is an $\mathcal{F}$-space or, equivalently if $\mathcal{F}$ satisfies Seever's interpolation property, then $\mathcal{F}$ is up-down semi-complete [S 68]. Dashiell [D 78] has shown that an "up-down semi-complete" algebra which satisfies

a mild additional lattice property satisfies (R), (VHS), (G) and (N), and has applied these results to lattices of Baire functions.

It is clear that we also have

3.7. PROPOSITION. *An up-down semi-complete algebra $\mathscr{F}$ satisfies* (OP).

3.8. PROPOSITION. $\mathscr{J}$ *is up-down semi-complete.*

P r o o f. Let $\{A_n\}_{n=1}^{\infty}$ be a disjoint sequence in $\mathscr{J}$ such that $\bigvee_{n=1}^{\infty} A_n \in \mathscr{J}$.

First note that, since the one-point-sets $\{t\}$ belong to $\mathscr{J}$, $\bigvee_{n=1}^{\infty} A_n$, the minimal set in $\mathscr{J}$ majorizing all A_n, coincides with the set-theoretic union $\bigcup_{n=1}^{\infty} A_n$. By adding $[0,1] \setminus \bigcup_{n=1}^{\infty} A_n$ to the sequence $\{A_n\}_{n=1}^{\infty}$ we may suppose that $\bigcup_{n=1}^{\infty} A_n = [0,1]$. Hence

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(A_n^0) = 1.$$

For a subsequence $\{n_k\}_{k=1}^{\infty}$ the boundary of the set $\bigcup_{k=1}^{\infty} A_{n_k}$ is given by

$\mathrm{Bd} = \overline{\bigcup_{k=1}^{\infty} A_{n_k}} \setminus (\bigcup_{k=1}^{\infty} A_{n_k})^0$. For every $n \in N$, $\mathrm{Bd} \cap A_n^0 = \emptyset$. Indeed, if n occurs in $\{n_k\}_{k=1}^{\infty}$, then $A_n^0 \subseteq (\bigcup_{k=1}^{\infty} A_{n_k})^0$; if it does not, then $\overline{\bigcup_{k=1}^{\infty} A_{n_k}} \cap A_n^0 = \emptyset$. This shows that $m(\mathrm{Bd}) = 0$, i.e., $\bigcup_{k=1}^{\infty} A_{n_k}$ belongs to $\mathscr{J}$ and it is evidently the Boolean supremum of the sequence $\{A_{n_k}\}_{k=1}^{\infty}$. ∎

R e m a r k. This answers the question, raised in [D 78], whether up-down semi-completeness implies (R) in the negative. $\mathscr{J}$ also provides a negative answer to the question, raised in [S 68] and [D–F–H 76], whether (N) and (VHS) are equivalent. Finally, I want to note that Proposition 3.8 of course also generalizes to the hypothesis in the remark following Proposition 3.3.

3.9. PROPOSITION. *There is a quotient-algebra of $\mathscr{J}$ that does not satisfy* (OP) *and therefore cannot be up-down semi-complete. Hence neither* (OP) *nor up-down semi-completeness is inherited by quotients.*

P r o o f. Let $\mathscr{I}$ be the ideal in $\mathscr{J}$ of sets that contain no dyadic point other than 0 or 1. Consider $\{\mu_n\}_{n=1}^{\infty}$ as defined in the proof of Proposition 3.2 and note that the μ_n all vanish on $\mathscr{I}$.

So, if $\tilde{\mathscr{J}}$ denotes the quotient-algebra $\mathscr{J}/\mathscr{I}$, the μ_n are well defined

on $\tilde{\mathscr{J}}$. Define

$$\mu: \tilde{\mathscr{J}} \mapsto c_0$$
$$\tilde{A} \mapsto \{\mu_{2n}(\tilde{A}) - \mu_{2n-1}(\tilde{A})\}_{n=1}^{\infty}.$$

It follows from the proof of 3.2 that μ takes its values in c_0. Evidently μ is weakly σ-additive as on the bounded sets of c_0 the weak topology concides with the coordinate-wise topology.

Define $A_1 = \{\frac{1}{2}\}, \ldots, A_n = \left\{\dfrac{1}{2^n}, \dfrac{3}{2^n}, \ldots, \dfrac{2^n-1}{2^n}\right\}, \ldots$, which are elements

of $\mathscr{J}$ and denote by $\tilde{A}_n$ their images in $\tilde{\mathscr{J}}$. Then $\bigvee\limits_{n=1}^{\infty} \tilde{A}_n = [0, 1]$ but

$\sum\limits_{n=1}^{\infty} \mu(\tilde{A}_n)$ of course does not converge strongly in c_0. ■

3.10. COROLLARY. (N) $\not\Rightarrow$ (OP).

Proof. The algebra $\tilde{\mathscr{J}}$, constructed in 3.9, has (N) (by 3.3 and 2.11) but not (OP). ■

Summing up: (N) implies neither (VHS), nor (G), nor (OP). Also (OP) implies neither (VHS) nor (G). That (OP) does not imply (N) will be shown in 3.16. Unfortunately, we leave open the question whether (G) implies (VHS).[2]

We note one more property of the Stone space of $\mathscr{J}$.

3.11. PROPOSITION. *Let Ω be the Stone space of $\mathscr{J}$ and D an infinite subset of Ω. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ in D such that the closure of $\{x_n\}_{n=1}^{\infty}$ in Ω is homeomorphic to βN, the Stone–Čech-compactification of N. Hence every infinite closed subset of Ω contains a copy of βN.*
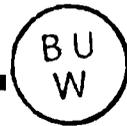
Proof. It will be convenient to make the following convention. If A is in $\mathscr{J}$, we write A if we consider it as a member of the field of subsets of $[0, 1]$, and write $\tilde{A}$ if we consider it as a clopen subset of Ω.

By the compactness of $[0, 1]$ there is for $x \in \Omega$ a unique "localization point", i.e. a point $t \in [0, 1]$ such that, for every neighbourhood V of t, $V \in \mathscr{J}$, x is contained in $\tilde{V}$. Hence we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ in D so that the localization points t_n of x_n converge to some $t_0 \in [0, 1]$. It is easy to construct a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and a disjoint sequence $\{A_k\}_{k=1}^{\infty}$ in $\mathscr{J}$ such that $x_{n_k} \in \tilde{A}_k$ and each A_k is contained in $]t_0 - 1/k$, $t_0 + 1/k[$. Relabel x_{n_k} by x_k.

If N_1, N_2 is a partition of N, let $B_1 = \bigcup \{A_k : k \in N_1\}$ and $B_2 = \bigcup \{A_k : k \in N_2\}$. Note that B_1 and B_2 are disjoint members of $\mathscr{J}$. Hence $\tilde{B}_1$ and $\tilde{B}_2$ are disjoint clopen sets in Ω such that $\{x_k\}_{k \in N_1} \subseteq \tilde{B}_1$ and $\{x_k\}_{k \in N_2} \subseteq \tilde{B}_2$. In particular,

$$\overline{\{x_k\}_{k \in N_1}} \cap \overline{\{x_k\}_{k \in N_2}} = \emptyset,$$

which proves that $\overline{\{x_k\}_{k=1}^{\infty}}$ is homeomorphic to βN. ■

[2] Note added in proof. See however the note added in the Introduction.

3.12. DEFINITION. An algebra $\mathcal{F}$ has property (n-σ) if no non-trivial countable suprema exist in $\mathcal{F}$, i.e., for no sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint, non-zero elements in $\mathcal{F}$, the supremum $\bigvee_{n=1}^{\infty} A_n$ exists in $\mathcal{F}$.

The following proposition is trivial.

3.13. PROPOSITION. (n-σ) $\Rightarrow$ (OP), *i.e. if $\mathcal{F}$ has* (n-σ), *$\mathcal{F}$ has* (OP). ∎

It is well known that the quotient algebra of $\mathcal{P}(N)$ modulo the ideal of finite subsets of N or, equivalently, the algebra of clopen subsets of $\beta N \setminus N$ has property (n-σ) (cf. [S 71a], prop. 16.5.6). However, this algebra also verifies all our other properties as $\beta N \setminus N$ is an F-space ([S 68]). So we have to consider more general cases.

3.14. PROPOSITION. *Let $\mathcal{X}$ be a set and $\mathcal{F}$ a field of subsets of $\mathcal{X}$. Suppose $\mathcal{F}$ contains all countable sets and let $\mathcal{I}$ be the ideal of finite sets. Then $\mathcal{F}/\mathcal{I}$ has property* (n-σ).

Proof. (cf. [S 71a], 16.5.6). Let $\tilde{A}_1 \leqslant \tilde{A}_2 \leqslant \tilde{A}_3 \leqslant \dots$ be a strictly increasing sequence in $\tilde{\mathcal{F}} = \mathcal{F}/\mathcal{I}$ and suppose $\tilde{A} = \bigcup_{n=1}^{\infty} \tilde{A}_n$ exists in $\tilde{\mathcal{F}}$. Choose $A_1, A_2, \dots$ and A to be representatives of $\tilde{A}_1, \tilde{A}_2, \dots$, and $\tilde{A}$ in $\mathcal{F}$. Of course we may choose the $\{A_n\}_{n=1}^{\infty}$ to be increasing in $\mathcal{F}$ too. Then, for every $n \in N$, $A_n \setminus A$ is finite, while $A_{n+1} \setminus A_n$ is infinite (the $\tilde{A}_n$ are strictly increasing). Therefore there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $x_n \in A_{n+1} \setminus A_n$. Let $B = A \setminus \{x_n\}_{n=1}^{\infty}$, which belongs to $\mathcal{F}$, and let $\tilde{B}$ be its image in $\tilde{\mathcal{F}}$. Then $\tilde{A}_n \leqslant \tilde{B}$ for every $n \in N$, while $\tilde{B}$ is strictly smaller than $\tilde{A}$, a contradiction. ∎

3.15. Let $\mathcal{F}$ be the field of subsets of $[0, 1]$ generated by the dyadic intervals $\left[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right]$ and the countable sets. Let $\mathcal{I}$ be the ideal of finite sets and $\tilde{\mathcal{F}} = \mathcal{F}/\mathcal{I}$ the quotient algebra. By 3.14 $\tilde{\mathcal{F}}$ has property (n-σ) and therefore, by 3.13, (OP).

3.16. PROPOSITION. *$\tilde{\mathcal{F}}$ does not have* (N) *or* (G). *Hence* (OP) $\not\Rightarrow$ (N).

Proof. For $n \geqslant 1$ define the measures

$$\mu_n = n^2 \cdot (\chi_{[0, 1/n]} \cdot m - \chi_{[1/n, 2/n]} \cdot m)$$

on $\mathcal{F}$ (m denoting the Lebesgue measure). Clearly $\{\mu_n(A)\}_{n=1}^{\infty}$ is eventually zero for every $A \in \mathcal{F}$, while $\|\mu_n\| = 2n$ tends to infinity. As the μ_n all vanish on $\mathcal{I}$, they factor through $\tilde{\mathcal{F}}$, which readily shows that $\tilde{\mathcal{F}}$ does not have (N). Considering $\nu_n = n^{-1}\mu_n$, we may conclude that $\tilde{\mathcal{F}}$ does not have (G) either. ∎

3.17. In the above example the property (OP) was implied by the fact that because of (n-σ) *every* measure is trivially σ-additive (with respect to

whichever topology). Another way of finding algebras having (OP) is to construct examples of algebras $\mathscr{F}$ on which *no* real valued measure (except $\mu = 0$) is σ-additive. Indeed, a Banach valued measure on $\mathscr{F}$ which is weakly σ-additive is then necessarily identically zero.

It is an old result (cf. [H 63], Lemma 15.4) that the complete Boolean algebra of regular open subsets of $[0, 1]$ has the property that every σ-additive real valued measure on it vanishes identically. Incidentally, this algebra, being complete, satisfies all our properties. But there are other algebras with this property.

3.18. PROPOSITION. *Let $\mathscr{F}$ be the algebra of clopen subsets of $\{0, 1\}^N$. Every σ-additive real valued measure on $\mathscr{F}$ vanishes identically. Hence $\mathscr{F}$ has* (OP).

Proof. Let $\mu \in B(\mathscr{F})^*$, $\mu \neq 0$, and suppose for simplicity that $\mu \geqslant 0$. As $B(\mathscr{F})$ may be identified with $C(\Delta)$, the space of continuous functions on $\Delta = \{0, 1\}^N$, μ is the restriction of a positive Radon-measure on Δ, denoted by $\bar{\mu}$, to the clopen sets.

If $\bar{\mu}$ has atoms, there exists an $x_0 \in \Delta$ such that $\bar{\mu}(\{x_0\}) > 0$. Represent $\Delta \setminus \{x_0\}$ as a countable disjoint union of clopen sets $\{A_n\}_{n=1}^\infty$. Note that $\bigvee_{n=1}^\infty A_n = \Delta$ (the supremum being taken in $\mathscr{F}$). But

$$\sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \bar{\mu}(A_n) = \mu(\Delta \setminus \{x_0\}) < \bar{\mu}(\Delta) = \mu(\Delta);$$

therefore μ is not σ-additive.

If $\bar{\mu}$ has no atom, let $\{x_n\}_{n=1}^\infty$ be a dense sequence in Δ and choose inductively clopen neighbourhoods A_n of x_n such that $\mu(A_n) < 2^{-n} \cdot \mu(\Delta)$. Putting $B_n = A_n \setminus (A_1 \vee \ldots \vee A_{n-1})$, we again obtain a disjoint sequence $\{B_n\}_{n=1}^\infty$ in $\mathscr{F}$, such that $\bigvee_{n=1}^\infty B_n = \Delta$, while $\sum_{n=1}^\infty \mu(B_n) < \mu(\Delta)$.

For the case of signed μ, the above argument is easily adapted. ∎

On the other hand, it is plain that $\mathscr{F}$ does not have (N) and (G). (Compare the proof of 3.16, noting that $\mathscr{F}$ may also be represented as the field of subsets of $[0, 1[$ generated by the dyadic intervals $[k/2^n, (k+1)/2^n[$.)

Hence again we get an example of an algebra having (OP) but none of our other properties.

§ 4. Some special classes of Boolean algebras

In the first part of this section we exhibit a relatively broad class of Boolean algebras satisfying all our properties, while in the second part we exibit another relatively broad class, which satisfies none of them except possibly (OP).

4.1. Many examples have been constructed of non-σ-complete algebras $\mathscr{F}$ on which (R) and (VHS) hold ([A 62], [I–S 63], [S 68]). One takes for example βN and glues together 2 points of $\beta N \backslash N$. The quotient space K thus obtained does not have nice disconnectedness-properties (it is not even an F-space), while the space $C(K)$ may be identified with a closed hyperplane of l^∞ and is therefore isomorphic to l^∞. The situation carries over to the respective Boolean algebras of clopen sets. We do not deal with the details here, as we shall consider a more general case below.

4.2. Definition. An algebra $\mathscr{F}$ has *property* (E) if, for every sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint elements of $\mathscr{F}$, there is a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ such that for every subsequence $\{A_{n_{k_l}}\}_{l=1}^{\infty}$ the supremum $\bigvee_{l=1}^{\infty} A_{n_{k_l}}$ exists in $\mathscr{F}$.

4.3. Proposition. (E) $\Rightarrow$ (R) *and* (E) $\Rightarrow$ (VHS), *i.e. an algebra having* (E) *has* (R) *and* (VHS).

Proof. Just copy the proofs of (R) and (VHS) in the case of σ-complete Boolean algebras (cf. [D–U 77], Theorem I.4.2 and Theorem I.4.8), except passing once more to a subsequence in the respective proofs. ∎

Remark. During the final preparation of this paper I was informed that R. Haydon [H 79] had introduced property (E) under the name "subsequential completeness property" and also established the above proposition.

4.4. Proposition. *Let Σ be a σ-complete Boolean algebra and $\mathscr{F}$ a subalgebra such that $B(\mathscr{F})$ may be represented as a countable intersection of hyperplanes of $B(\Sigma)$. Then $\mathscr{F}$ has* (E).

Remark. The proposition includes the above-mentioned examples ([A 62], [I–S 63], [S 68]) as special cases.

The following simple proof has been suggested by the referee.

Proof. By hypothesis there is a sequence $\{\mu_i\}_{i=1}^{\infty}$ in $B(\Sigma)^*$ such that

$$\mathscr{F} = \{A \in \Sigma : \mu_i(A) = 0, \ i \in N\}.$$

If $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ is disjoint, then applying a theorem of Drewnowski ([D–U 77], p. 38) we may find a subsequence $\{A_{n_k}\}$ such that each μ_i is σ-additive on the σ-algebra generated by $\{A_{n_k}\}$. It follows that $\mu_i\left(\bigvee_{l=1}^{\infty} A_{n_{k_l}}\right) = 0$ for each $i \in N$, i.e. $\bigvee_{l=1}^{\infty} A_{n_{k_l}} \in \mathscr{F}$ for all subsequences $\{n_{k_l}\}$ of $\{n_k\}$. ∎

4.5. We now consider a class of Boolean algebras which behaves badly with respect to the properties we are interested in, namely, algebras which are unions of a strictly increasing sequence of subalgebras. These objects arise naturally in the context of martingales. It is a standard exercise, which can be found in most probability text-books, to show that these

algebras may fail to be σ-complete. It was even shown [B–H 76] that these algebras are never σ-complete. Our approach furnishes an easy proof of this curious fact as well as stronger results. In fact, they never satisfy (N) or (G) (and therefore neither (VHS) nor (R)). On the other hand, they may verify (OP).

Let us first prove the latter assertion. The algebra $\mathscr{F}$ considered in 3.18 may be represented as $\bigcup_{n=1}^{\infty} \mathscr{F}_n$, where $\mathscr{F}_n$ is the algebra of subsets of $\varDelta$ depending only on the first n coordinates. We have seen in 3.18 that $\mathscr{F}$ has (OP).

4.6. PROPOSITION. *Let $\mathscr{F}$ be a Boolean algebra and suppose that $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where $\mathscr{F}_n$ is a strictly increasing sequence of subalgebras of $\mathscr{F}$. Then $\mathscr{F}$ does not have (N) or (G).*

For the proof we shall need some lemmas.

4.7. LEMMA. *Let $(X, \| \ \|)$ be a normed space and suppose that $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of closed subspaces of X. Then $(X, \| \ \|)$ is not barrelled.*

Proof. Let B_n be the ball in X_n around zero with radius $1/n$ and let T be the closed convex hull of $\{B_n\}_{n=1}^{\infty}$. Then T is a barrel but not a neighbourhood of zero in X. ■

4.8. LEMMA. *Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where $\mathscr{F}_n$ is a strictly increasing sequence of subalgebras. There is a sequence $n_0 < n_1 < \ldots$ and pairwise disjoint elements $E_k \in \mathscr{F}_{n_k} \setminus \mathscr{F}_{n_{k-1}}$ $(k = 1, 2, \ldots)$.*

Remark. The author is indebted to the referee for the formulation of Lemma 4.8 (which replaced a clumsy statement in the first version) and the following proof. The above formulation has also been suggested by F. K. Dashiell.

Proof. Let $A \in \mathscr{F}_2 \setminus \mathscr{F}_1$. Then one of the sets

$$\{n: \mathscr{F}_n \wedge A \neq \mathscr{F}_{n+1} \wedge A\}$$

and

$$\{n: \mathscr{F}_n \wedge CA \neq \mathscr{F}_{n+1} \wedge CA\}$$

is infinite; we may assume that it is the second set. Define $n_0 = 1$, $n_1 = 2$, $E_1 = A$ and choose $n_2 > n_1$ so that $\mathscr{F}_{n_1} \wedge CE_1 \neq \mathscr{F}_{n_2} \wedge CE_1$. Choose any B in $(\mathscr{F}_{n_2} \wedge CE_1) \setminus (\mathscr{F}_{n_1} \wedge CE_1)$. One of the sets

$$\{n: \mathscr{F}_n \wedge CE_1 \wedge B \neq \mathscr{F}_{n+1} \wedge CE_1 \wedge B\}$$

and

$$\{n: \mathscr{F}_n \wedge CE_1 \wedge CB \neq \mathscr{F}_{n+1} \wedge CE_1 \wedge CB\}$$

is infinite; we may assume that it is the second set. Define $E_2 = B$. Continuing the inductive construction in an obvious way, we establish Lemma 4.8. ■

Proof of Proposition 4.6. As regards (N), remark that $B_s(\mathscr{F})$ $= \bigcup_{n=1}^{\infty} B_s(\mathscr{F}_n)$ cannot be barrelled by 4.7, whence $\mathscr{F}$ does not have property (N_2) of Definition 2.4.

Now we show that $\mathscr{F}$ does not have (G). Choose a sequence $\{n_k\}_{k=1}^{\infty}$ and disjoint elements $E_k \in \mathscr{F}_{n_k} \setminus \mathscr{F}_{n_{k-1}}$ as in Lemma 4.8. By a well-known corollary to the Hahn–Banach-theorem ([D–S 58], II.3.12) and the fact that dist $(\chi_{E_k}, B(\mathscr{F}_{n_{k-1}})) = \frac{1}{2}$ we may find $\mu_k \in B(\mathscr{F})^*$ such that μ_k vanishes on $\mathscr{F}_{n_{k-1}}$, $\|\mu_k\| = 2$, and $\mu_k(E_k) = 1$. Then $\mu_k(A)$ is finally zero for every $A \in \mathscr{F}$ but $\{\mu_k\}_{k=1}^{\infty}$ is not uniformly exhaustive. Hence $\mathscr{F}$ does not have (G). ■

4.9. Remark. If $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ is a countable union of a strictly increasing sequence of subalgebras, one might ask (and F. K. Dashiell actually did so), whether there is necessarily a non trivial convergent sequence in the Stone space of $\mathscr{F}$. However in general this is not the case, as is shown by the following example.

4.10. Example. Let $\mathscr{F}$ be the algebra of subsets of N consisting of the sets $A \subseteq N$ such that for all, but finitely many k, the pair $\{2k-1, 2k\}$ is either in A or in the complement. Denote $\mathscr{F}_n = \{A \subseteq N$: for $k > n$ the pair $\{2k-1, 2k\}$ is either in A or in $CA\}$. Clearly $\mathscr{F}_n$ is strictly increasing and $\bigcup_{n=1}^{\infty} \mathscr{F}_n = \mathscr{F}$.

Let us look at the Stone-space Ω of $\mathscr{F}$. We may identify naturally N with a discrete subset of Ω, still denoted N. We claim that the remainder $\Omega \setminus N$ is homeomorphic to $\beta N \setminus N$. Indeed, denote u the surjection of N onto N that sends $2k-1$ and $2k$ onto k. Then u extends to a homeomorphism of $\Omega \setminus N$ onto $\beta N \setminus N$. From this fact it follows easily that there is no non trivial convergent sequence in Ω. ■

§ 5. The Grothendieck-property

We first recall known results.

5.1. Theorem. *For a Boolean algebra $\mathscr{F}$ the following statements are equivalent*:

(i) $\mathscr{F}$ *has* (G).

(ii) *Every continuous operator* $T: B(\mathscr{F}) \to c_0$ *is weakly compact.*

(ii)' *Every continuous operator* $T: B(\mathscr{F}) \to X$ *into a weakly compactly generated (abbreviated* WCG) *Banach space* X *is weakly compact.*

(iii) *If* $T_n: B(\mathscr{F}) \to X$ *is a sequence of weakly compact operators into a Banach space* X, *converging in the strong operator topology, then the limit* $T = \lim T_n$ *is weakly compact.*

(iii)' *As* (iii), *with "strong operator topology" replaced by "weak operator topology".*

(iv) *If* $T: B(\mathscr{F}) \to X$ *is a continuous operator into a separable Banach space* X *and* $B(\mathscr{F})$ *is a subspace of a Banach space* Z, *then there is a norm-preserving extension of* T *to* Z.

(v) *If* $T: B(\mathscr{F}) \to X$ *is a continuous operator to a Banach space* X *which is not weakly compact, then* T *fixes a copy of* $C[0, 1]$, *i.e. there is a subspace of* $B(\mathscr{F})$ *isomorphic to* $C[0, 1]$ *on which* T *reduces to an isomorphism.*

The equivalence of (i) to (iii)' may be found in [D 73] and the equivalence of (i) and (iv) in [L 64]. The remarkable result that (i) implies (v) is due to Diestel and Seifert [D–S 78], while it is plain that (v) implies (ii). Let us point out that (ii), (ii)', (iii) and (iii)' actually characterize general Banach spaces as Grothendieck spaces, while (iv) and (v) apply to $C(K)$-spaces.

We now show that property (G) is equivalent to a property very similar to (OP) (a result of the type "weak implies strong") but stable in terms of finitely additive measures.

5.2. THEOREM. $\mathscr{F}$ *has* (G) *iff*

(vi) *For every bounded measure* $\mu: \mathscr{F} \to X$ *such that there is some bounded positive measure* η *on* $\mathscr{F}$ *with* $x^* \circ \mu \ll \eta$ *for every* $x^* \in X^*$ *it follows that* $\mu \ll \eta$.

Here $x^* \circ \mu \ll \eta$ (resp. $\mu \ll \eta$) means that for $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu(A) < \delta$ implies $|x^* \circ \mu(A)| < \varepsilon$ (resp., $\|\mu(A)\| < \varepsilon$).

Proof. (v) $\Rightarrow$ (vi). Suppose μ, η, X given as above. We represent $\mathscr{F}$ as the algebra of clopen sets of the Stone space Ω, whence η and $x^* \circ \mu$ extend to Radon measures on Ω. Let $T: C(\Omega) \to X$ be the integration operator associated with μ.

(Note that T is bounded since, for every $x^* \in X^*$, $T^* x^* = x^* \circ \mu \ll \eta$; hence in particular $T^* x^* \in C(\Omega)^*$.)

CLAIM. T *is weakly compact.*

Indeed, if it were not so by (v), we could find a subspace Z of $C(\Omega)$ isomorphic to $C[0, 1]$ and such that $T|_Z$ is an isomorphism onto its image. Denote $T_0 = T|_Z$ and $Y = T_0(Z)$. By the Radon–Nikodym theorem the adjoint T^* transforms X^* into $L^1(\eta)$ which is a WCG space. On the other hand, $T_0^*(Y^*) = Z^*$ is isomorphic to $M[0, 1]$, where $M[0, 1]$ is the space

of Radon measures on $[0, 1]$, which is not WCG. Consider the continuous quotient map $q: M(\Omega) \to Z^*$ and $L^1(\eta) \subset M(\Omega)$. We have $q(L^1(\eta)) = Z^*$. Indeed, T^* factors through $L^1(\eta)$ and $q \circ T^*$ is onto Z^*. Hence Z^* must be WCG, a contradiction proving the claim.

Therefore $T: C(\Omega) \to X$ is weakly compact and T^* (ball (X^*)) is a weakly compact set in $L^1(\eta)$. Hence

$$\lim_{\eta(A) \to 0} |x^* \circ \mu(A)| = 0,$$

uniformly in $\|x^*\| \leqslant 1$; in other words

$$\lim_{\eta(A) \to 0} \|\mu(A)\| = 0.$$

(vi) $\Rightarrow$ (ii). A continuous operator $T: B(\mathscr{F}) \to c_0$ is given coordinate-wise by a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$ tending weak* to zero. Putting

$$\eta = \sum_{n=1}^{\infty} 2^{-n} |\mu_n|,$$

we find that η satisfies the assumptions of (vi). Hence

$$\lim_{\eta(A) \to 0} \|\mu(A)\| = 0,$$

which means that $\{\mu_n\}_{n=1}^{\infty}$ is uniformly exhaustive and thus weakly tending to zero (Proposition 1.2). This is equivalent to the fact that T is a weakly compact operator (cf. [D 73]). ■

5.3. PROPOSITION. $\mathscr{F}$ has (G) *iff*

(vii) *There is no subspace of $B(\mathscr{F})$ isometric to c_0 and complemented in $B(\mathscr{F})$.*

Proof. (ii) $\Rightarrow$ (vii) is trivial.

(vii) $\Rightarrow$ (i): If $\mathscr{F}$ does not have (G), we can find a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$ converging weak* to zero and a disjoint sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathscr{F}$ such that $|\mu_n(E_n)| > \varepsilon$ for some $\varepsilon > 0$. Multiplying a bounded sequence of scalars, we may assume $\mu_n(E_n) = 1$. Denote by $\mathscr{E}$ the ring generated by $\{E_n\}_{n=1}^{\infty}$ in $\mathscr{F}$. Then $B(\mathscr{E})$ is a subspace of $B(\mathscr{F})$ isometric to c_0. Define an operator $T: B(\mathscr{F}) \to c_0 = X$ on the indicator-functions by $T(\chi_A) = \{\mu_n(A)\}_{n=1}^{\infty}$ and extend it by linearity and continuity to $B(\mathscr{F})$. By [D 76] there is an infinite set $M \subseteq N$ such that $T_0 = T|_{B(\mathscr{E}_M)}$ is an isomorphism onto its image, where $\mathscr{E}_M$ is the ring generated by $\{E_n, n \in M\}$. By a theorem of Sobczyk (see e.g. [J 74], 29.22) there is a continuous projection P_1 from X onto $T_0(B(\mathscr{E}_M))$. Hence $T_0^{-1} P_1 T$ is a continuous projection from $B(\mathscr{F})$ onto $B(\mathscr{E}_M)$, an isometric copy of c_0. ■

Remark. Proposition 5.3 may be rephrased as follows: If $\mathscr{F}$ does not have (G), there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathscr{F})^*$, tending weak* to zero,

and a disjoint sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathscr{F}$ so that

$$\mu_n(E_m) = \delta_{n,m},$$

$\delta_{n,m}$ denoting the Kronecker-symbol. Indeed, the projection $T_0^{-1} P_1 T$ constructed above, viewed as an operator into c_0, defines such a sequence $\{\mu_n\}_{n=1}^{\infty}$.

Note, however, that it follows from 3.11 and 4.10 that $\{\mu_n\}_{n=1}^{\infty}$ cannot, in general, be replaced by a sequence $\{\delta_{x_n}\}_{n=1}^{\infty}$ of Dirac measures on Ω.

5.4. We say that a bounded set $\{f_i\}_{i \in I}$ in a Banach space X is *equivalent* to the l^1-basis iff the continuous operator $T: l_I^1 \to X$, taking the unit-vectors onto the corresponding f_i's, is an isomorphism; equivalently, if there exists a constant $\alpha > 0$ such that for every choice of scalars $\mu_1, \ldots, \mu_n$ and $f_{i_1}, \ldots, f_{i_n}$, we have $\left\| \sum_{k=1}^{n} \mu_k f_{i_k} \right\| \geq \alpha \sum_{k=1}^{n} |\mu_k|$. We say that $\{f_i\}_{i \in I}$ is equivalent to the l^1-basis and *complemented* if, in addition, the space spanned by $\{f_i\}_{i \in I}$ is complemented in X.

Note also that it follows from the special nature of the l^1-norm that, if $T: X \to Y$ is a continuous operator and $\{f_i\}_{i \in I}$ in Y is equivalent to the l^1-basis and there exists a *bounded* family $\{e_i\}_{i \in I}$ in X, such that $T(e_i) = f_i$, then $\{e_i\}_{i \in I}$ is also equivalent to the l^1-basis and T reduces to an isomorphism on the span of $\{e_i\}_{i \in I}$. Moreover, if $\{f_i\}_{i \in I}$ is assumed to be complemented, then $\{e_i\}_{i \in I}$ is complemented.

5.5. PROPOSITION. $\mathscr{F}$ *has* (G) *iff*

(viii) *Every weak* compact convex subset of $B(\mathscr{F})^*$ which is not weakly compact contains a family $\{f_t\}_{t \in [0,1]}$ equivalent to the l^1-basis and complemented in $B(\mathscr{F})^*$.*

Proof. (viii) $\Rightarrow$ (vii): If (vii) does not hold, there is a complemented subspace of $B(\mathscr{F})$ isomorphic to c_0. Hence l^1 is isomorphic to a weak*-complemented and therefore weak*-closed subspace of $B(\mathscr{F})^*$. The intersection of this space with the unit ball is weak*-compact and convex but contains no family $\{f_t\}_{t \in [0,1]}$ equivalent to the l^1-basis.

(v) $\Rightarrow$ (viii): Let K in $B(\mathscr{F})^*$ be weak*-compact convex and let $\bar{K}$ be its circled hull, which is weak*-compact again. Consider the Banach space X spanned by $\bar{K}$ in $B(\mathscr{F})^*$, $\bar{K}$ being the unit ball in X. As $\bar{K}$ is weak*-compact, X is a dual Banach space, i.e., there exists a Banach space Y such that X is isometric and weak*-isomorphic to Y^* (see e.g. [C 78]).

Consider the canonical injection from Y^* into $B(\mathscr{F})^*$, $j: Y^* \to B(\mathscr{F})^*$. The injection j is continuous with respect to the weak*-topologies; hence there exists a continuous operator $j_*: B(\mathscr{F}) \to Y$ such that $(j_*)^* = j$. If K is not weakly compact, j^* is not a weakly compact operator, and so by (v)

j_* fixes a copy of $C[0, 1]$, i.e. we have the commuting diagramm

$$\begin{array}{ccc} B(\mathcal{F}) & \xrightarrow{\ j_*\ } & Y \\ \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} \\ C[0,1] \xrightarrow{\ k\ } & V & \xrightarrow{\ l\ } W \end{array}$$

where i_1, i_2 are isometric injections and k and l are isomorphisms. Transposing, we get

$$\begin{array}{ccc} B(\mathcal{F})^* & \xleftarrow{\ j\ } & X \\ \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_2^*} \\ M[0,1] \xleftarrow{\ k^*\ } & V^* & \xleftarrow{\ l^*\ } W^* \end{array}$$

Note that $k^* i_1^* j$ is an open mapping onto, as $k^* l^* i_2^*$ is. Whence $k^* i_1^* j$ (ball (X)), which is just $k^* i_1^*(\bar{K})$, contains for some $\alpha > 0$ an α-ball around zero in $M[0, 1]$. The Dirac measures $\{\delta_t\}_{t\in[0,1]}$ form a complemented l^1-basis in $M[0, 1]$.

There exists a family $\{f_t\}_{t\in[0,1]}$ in $\bar{K}$ such that $k^* i_1^*(f_t) = \alpha \cdot \delta_{\{t\}}$. It is evident that we may find $\{\bar{f_t}\}_{t\in[0,1]}$ in K such that $k^* i_1^*(\bar{f_t}) = \alpha_t \cdot \delta_{\{t\}}$ where $\{\alpha_t\}_{t\in[0,1]}$ is a bounded family of scalars such that $|\alpha_t| \geqslant \alpha$. As the $\{\alpha_t \cdot \delta_{\{t\}}\}_{t\in[0,1]}$ are also equivalent to an l^1-basis and complemented in $M[0, 1]$, we conclude by the remark precedings the proposition that $\{\bar{f_t}\}_{t\in[0,1]}$ is equivalent to an l^1-basis and complemented in $B(\mathcal{F})^*$. ∎

5.6. COROLLARY. $\mathcal{F}$ has (G) iff one of the following two equivalent conditions holds.

(ix) *Every weak*-closed subspace Z of $B(\mathcal{F})^*$ which does not contain a complemented subspace isomorphic to $l^1[0, 1]$ is reflexive.*

(x) *Either every bounded measure $\mu: \mathcal{F} \to X$ is exhaustive or there is a bounded family $\{x_t^*\}_{t\in[0,1]}$ in X^* such that $\{x_t^* \circ \mu\}_{t\in[0,1]}$ is equivalent to the l^1-basis and complemented on $B(\mathcal{F})^*$.*

Proof. We only show (viii) $\Rightarrow$ (ix): The intersection of Z with the unit ball of $B(\mathcal{F})^*$ is weak*-compact convex. So if it is not weakly compact (i.e., Z is not reflexive), by (viii) it contains a complemented subspace isomorphic to $l^1[0, 1]$. ∎

To finish this section we remark that Proposition 5.5 (as well as Corrollary 5.6) may be strengthened if one assumes (R) instead of (G).

5.7. PROPOSITION. *If $\mathscr{F}$ has* (R), *then every weak*-compact convex subset of $B(\mathscr{F})^*$ which is not weakly compact contains a family $\{f_i\}_{i \in I}$ equivalent to the l^1-basis and complemented in $B(\mathscr{F})^*$, where I is a set of cardinality $2^{[0,1]}$.*

Proof. Just copy the proof of Proposition 5.5 replacing $C[0,1]$ by l^∞. The dual $(l^\infty)^*$ may be identified with the Radon measures on the Stone–Čech-compactification of N; the Dirac measures on βN are equivalent to the l^1-basis and complemented in $(l^\infty)^*$. Hence we can carry over the rest of the proof, replacing everywhere $[0,1]$ by βN, a set of cardinality $2^{[0,1]}$. ∎

§ 6. The Orlicz–Pettis property

In this section we shall show that (G) implies (OP). We first show the result for the special case in which the unit-ball of X^* is weak*-sequentially compact (e.g. if X is WCG [D 73]). In this case we may state a stronger result and the proof is very easy.

6.1. PROPOSITION. *Let $\mathscr{F}$ satisfy* (G), *and let X be a Banach space such that ball (X^*) is weak*-sequentially compact and let $\Gamma \subseteq X^*$ be a norming subset of X^*.*

If a measure $\mu: \mathscr{F} \to X$ is such that $x^* \circ \mu$ is σ-additive $\forall\, x^* \in \Gamma$, then μ is strongly σ-additive.

Proof. Let $\{A_n\}_{n=1}^\infty$ be a disjoint sequence such that $\bigvee_{n=1}^\infty A_n \in \mathscr{F}$. To show that $\sum_{n=1}^\infty \mu(A_n) \to \mu\left(\bigvee_{n=1}^\infty A_n\right)$ in norm it suffices to show that the series is norm-Cauchy.

Supposing it is not so, we may find blocks $B_k = \bigvee_{n=n_k+1}^{n_k+1} A_n$ and $\alpha > 0$ such that $\|\mu(B_k)\| > \alpha$. As Γ is norming, we may find $\{x_k^*\}_{k=1}^\infty \subseteq \Gamma$ such that $\|x_k^*\| = 1$ and $|x_k^* \circ \mu(B_k)| > \alpha$. By hypothesis, we may find a subsequence $\{x_{k_l}^*\}_{l=1}^\infty$ that converges weak*. But then $\{x_{k_l}^* \circ \mu\}_{l=1}^\infty$ converges weak* in $B(\mathscr{F})^*$ and by (G) it is uniformly exhaustive, in contradiction to $|x_{k_l}^* \circ \mu(B_{k_l})| > \alpha$ for every l. ∎

Now we shall reduce the case of a general Banach space X to the case of c_0.

6.2. THEOREM. *Let $\mathscr{F}$ be a Boolean algebra such that there is a Banach space X and a bounded measure $\mu: \mathscr{F} \to X$ which is weakly σ-additive but not strongly σ-additive. Then there is a bounded measure $v: \mathscr{F} \to c_0$ which is weakly σ-additive but not strongly σ-additive.*

Before proving Theorem 6.2 let us note the following consequence of 6.1 and 6.2.

6.3. THEOREM. (G) *implies* (OP), *i.e. a Boolean algebra having* (G) *has* (OP). ∎

Following the advice of the referee, we want to use this occasion to formulate properties (VHS), (G), (N) and (OP) in terms of c_0. The following proposition is easily verified for (VHS), (G), (N) while for (OP) it is just the content of Theorem 6.2. In c_0 let $\tau, \tau_\sigma, \tau_s$ denote norm topology, weak topology and the topology of coordinate-wise convergence, respectively.

6.4. PROPOSITION:

(a) $\mathscr{F}$ *has* (VHS) *iff every τ_s-exhaustive measure $\mu\colon \mathscr{F} \to c_0$ is τ-exhaustive.*

(b) $\mathscr{F}$ *has* (G) *iff every τ_σ-exhaustive measure $\mu\colon \mathscr{F} \to c_0$ is τ-exhaustive.*

(c) $\mathscr{F}$ *has* (N) *iff every τ_s-exhaustive measure $\mu\colon \mathscr{F} \to c_0$ is τ_σ-exhaustive* (*i.e. τ-bounded*).

(d) $\mathscr{F}$ *has* (OP) *iff every τ_σ countably additive measure $\mu\colon \mathscr{F} \to c_0$ is τ countably additive.* ∎

In order to prove 6.2 we still need a lemma.

6.5. LEMMA. *Let $\{x_n\}_{n=1}^\infty$ in X be such that $\sum\limits_{n=1}^\infty x_n$ converges weakly to $x_0 \in X$.*

Then for $\varepsilon > 0$ and $N \in N$ there is an $M \geqslant N$ such that for each block of mutually distinct natural numbers $n_1, \ldots, n_k > M$ there are natural numbers $m_1, \ldots, m_l > N$ and scalars $\lambda_1, \ldots, \lambda_l$ with $\lambda_j \in [0, 1]$ so that $m_1, \ldots, m_l, n_1, \ldots, n_k$ are all mutually distinct and

$$\Big\| \sum_{i=1}^k x_{n_i} + \sum_{j=1}^l \lambda_j x_{m_j} \Big\| < \varepsilon.$$

Proof. Given $\varepsilon > 0$ and $N \in N$, the set of points of the form $\sum\limits_{n=1}^N x_n + \sum\limits_{n=N+1}^M \mu_n x_n$, where $M > N$ and $\mu_n \in [0, 1]$, is a convex subset of X. As the weak closure and the strong closure of a convex set coincide, for some M and $\bar{\mu}_{N+1}, \bar{\mu}_{N+2}, \ldots, \bar{\mu}_M$ we have

$$\Big\| \sum_{n=1}^N x_n + \sum_{n=N+1}^M \bar{\mu}_n x_n - x_0 \Big\| < \varepsilon/2.$$

If $n_1, \ldots, n_k > M$ are fixed, then again the set of points of the form

$$\Big[\sum_{n=1}^N x_n + \sum_{n=N+1}^M \bar{\mu}_n x_n + \sum_{i=1}^k x_{n_i} \Big] + \sum_{j=1}^l \lambda_j x_{m_j},$$

where, for each j, $m_j \notin \{1, \ldots, N, n_1, \ldots, n_k\}$, and either $\lambda_j \in [0, 1]$ or $\lambda_j \in [0, 1 - \bar{\mu}_{m_j}]$ if $m_j \in \{N+1, \ldots, M\}$, and l ranges in N, is a convex set

whose strong closure contains x_0. So again we may find $\bar{\lambda}_1, ..., \bar{\lambda}_j$ and $\bar{m}_1, ..., \bar{m}_j$ such that

$$\left\| \sum_{n=1}^{N} x_n + \sum_{n=N+1}^{M} \bar{\mu}_n x_n + \sum_{i=1}^{k} x_{n_i} + \sum_{j=1}^{l} \bar{\lambda}_j x_{m_j} - x_0 \right\| < \varepsilon/2.$$

Hence

$$\left\| \sum_{i=1}^{k} x_{n_i} + \sum_{j=1}^{l} \bar{\lambda}_j x_{m_j} \right\| < \varepsilon. \ \blacksquare$$

Proof of Theorem 6.2. By assumption there is a disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{F}$ such that $A_\infty = \bigvee_{n=1}^{\infty} A_n$ exists in $\mathscr{F}$ and such that $\sum_{n=1}^{\infty} \mu(A_n)$ converges weakly but not strongly to $\mu(A_\infty)$. Perhaps by forming blocks of the form $A_{n_m+1} \vee ... \vee A_{n_{m+1}}$ we may suppose that there is an $\alpha > 0$ such that $\|\mu(A_n)\| \geqslant \alpha$ for all n.

Clearly we may assume $\alpha = 1$.

Choose a sequence $x_n^* \in X^*$, $\|x_n^*\| = 1$ such that $|x_n^* \circ \mu(A_n)| \geqslant 1$ and call μ_n the scalar σ-additive measure $x_n^* \circ \mu$.

We now construct inductively a sequence $\{B_i\}_{i=1}^{\infty}$ of finite unions of A_n's and a subsequence $\{\mu_{n_i}\}_{i=1}^{\infty}$.

At the first step let $n_1 = 1$ and add to A_{n_1} finitely many $A_1^{(1)}, ..., A_{p_1}^{(1)}$ (taken from the sequence $\{A_n\}_{n=1}^{\infty}$), such that for $B_1 = A_{n_1} \vee A_1^{(1)} \vee ... \vee A_{p_1}^{(1)}$,

$$|\mu_{n_1}|(A_\infty \setminus B_1) < 1,$$

$|\mu_{n_1}|$ denoting the variation measure of μ_{n_1}. Define

$$\alpha_1 = \liminf_{n \to \infty} |\mu_n|(B_1)$$

and let S_1 be an infinite subset of N such that for $n \in S_1$

$$\left| |\mu_n|(B_1) - \alpha_1 \right| < 1/2.$$

At the ith step suppose that $n_1, ..., n_{i-1}$, $B_1, ..., B_{i-1}$, $\alpha_1, ..., \alpha_{i-1}$ and S_{i-1} have been defined. Choose n_i from S_{i-1} such that A_{n_i} has not been used to build one of the blocks $B_1, ..., B_{i-1}$. Choose elements $A_1^{(i)}, ..., A_{p_i}^{(i)}$ from the sequence $\{A_n\}_{n=1}^{\infty}$ such that none of them has been built into one of the blocks $B_1, ..., B_{i-1}$ and such that for

$$B_i = A_{n_i} \vee A_1^{(i)} \vee ... \vee A_{p_i}^{(i)}$$

we have

(*) $$|\mu_{n_i}|(A_\infty \setminus (B_1 \vee ... \vee B_i)) < i^{-1}.$$

To make sure that the B_i's finally exhaust the whole sequence $\{A_n\}_{n=1}^{\infty}$, we may and do assume that $A_1, ..., A_{i-1}$ have all been built into one of

the blocks $B_1, ..., B_i$. Define

$$\alpha_i = \lim_{n \in S_{i-1}} \inf |\mu_n|(B_i)$$

and let S_i be an infinite subset of S_{i-1} such that, for $n \in S_i$,

(**) $\left| |\mu_n|(B_i) - \alpha_i \right| < 2^{-i}.$

This completes the induction step.

We shall now show that for $C \in \mathcal{F}$

(***) $\lim_{i \to \infty} \mu_{n_i}(C \wedge B_i) = 0.$

Assuming (***) for the moment, we can finish the proof as follows. Define $v: \mathcal{F} \to c_0$ to be given coordinatewise by $v_i = \mu_{n_i|B_i}$. It follows from (***) that v takes its values in c_0 and clearly v is bounded and weakly σ-additive. That v is not strongly additive is easily seen from the facts

$$\bigvee_{n=1}^{\infty} A_n = A_{\infty}$$

and

$$\| v(A_{n_i}) \| \geq |\mu_{n_i}(A_{n_i} \wedge B_i)| \geq 1.$$

So let us show (***). First observe that

$$\sum_{i=1}^{\infty} \alpha_i \leq \sup_n \|\mu_n\| + 1 < \infty.$$

Indeed, if this were not the case, we could find a j such that

$$\sum_{i=1}^{j} \alpha_i > \sup_n \|\mu_n\| + 1.$$

For any $n \in S_j$ we have

$$\|\mu_n\| \geq |\mu_n|(B_1) + ... + |\mu_n|(B_j) \geq \alpha(1) - 2^{-1} + ... + \alpha_j - 2^{-j} > \sum_{i=1}^{j} \alpha_i - 1 > \|\mu_n\|,$$

a contradiction. Now fix $C \in \mathcal{F}$ and $\varepsilon > 0$. Choose $N \in N$ such that

$$\sum_{i=N+1}^{\infty} (\alpha_i + 2^{-i}) + N^{-1} < \varepsilon/2.$$

Noting that $\sum_{i=1}^{\infty} \mu(C \wedge B_i)$ converges weakly to $\mu(C \wedge A_{\infty})$, we infer from Lemma 6.4 that there is an $M \geq N$ such that for every $i > M$ we may find an element

$$\lambda_1 \cdot \mu(C \wedge B_{k_1}) + ... + \lambda_r \cdot \mu(C \wedge B_{k_r})$$

such that

$$\|\lambda_1 \cdot \mu(C \wedge B_{k_1}) + \ldots + \lambda_r \cdot \mu(C \wedge B_{k_r}) + \mu(C \wedge B_i)\| < \varepsilon/2$$

where $\lambda_j \in [0, 1]$ and the indices $\{k_j\}_{j=1}^r$ are greater than N and different from i. In particular, we have

$$|\lambda_1 \cdot \mu_{n_i}(C \wedge B_{k_1}) + \ldots + \lambda_r \cdot \mu_{n_i}(C \wedge B_{k_r}) + \mu_{n_i}(C \wedge B_i)| < \varepsilon/2.$$

On the other hand,

$$|\lambda_1 \cdot \mu_{n_1}(C \wedge B_{k_1}) + \ldots + \lambda_r \cdot \mu_{n_i}(C \wedge B_{k_r})|$$
$$\leqslant |\mu_{n_i}|((C \wedge B_{k_1}) \vee \ldots \vee (C \wedge B_{k_r}))$$
$$\leqslant |\mu_{n_i}|\left(\bigvee_{k=N+1}^{i-1} B_k \vee \bigvee_{k=i+1}^{\infty} B_k\right) \leqslant \sum_{k=N+1}^{i-1} (\alpha_k + 2^{-k}) + i^{-1} < \varepsilon/2,$$

where the last line follows from (∗) and (∗∗). So, for $i > M$,

$$|\mu_{n_i}(C \wedge B_i)| < \varepsilon,$$

which shows (∗∗∗) and completes the proof of 6.2. ∎

Remark. From 6.3 and 2.11 it actually follows that, if $\mathscr{F}$ has (G), every quotient algebra of $\mathscr{F}$ has (OP). I do not know if the converse holds true.

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ERRATA

Page, line	For	Read
6^1	barelled	barrelled
6^{12}	coordinate	coordinate)
6_7	barreled	barrelled ´
7_{11}	Bodean algebra	Boolean algebra
9^{13}	Randon-measure	Radon-measure
17^6	concides	coincides
25_1	j^*	j_*
$29^{3,3}$	x_{m_j}	$x_{\overline{m}_j}$