POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCLXXVII 277

ZYGMUNT POGORZAŁY

A characterization of representation-finite biserial algebras over a perfect field

WARSZAWA 1989
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-08944-X ISSN 0012-3862

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CONTENTS

Introduction

Almost split sequences, introduced and investigated by M. Auslander and I. Reiten [3]–[6], form a distinguished class of short exact sequences; amongst other properties, their middle terms may decompose. Then for each artin algebra A, we have two natural invariants $\alpha(A)$ and $\beta(A)$ [7]. The invariant $\alpha(A)$ records the largest possible number of indecomposable summands in the middle term of any almost split sequence (called *Auslander-Reiten sequence* sometimes), and $\beta(A)$ records the largest possible number of such summands which are neither projective nor injective. These invariants give a way of describing a complicated structure of maps between modules and it would be interesting to have a classification of artin algebras from this point of view. R. Bautista and S. Brenner have proved in [8] that, for each representation-finite algebra A, $\alpha(A) \leqslant 4$ and $\beta(A) \leqslant 3$.

The main aim of this paper is to give a rather simple description of representation-finite algebras over a perfect field with $\beta(A) \leqslant 2$ (resp. $\alpha(A) \leqslant 2$). We use many ideas and extended results from [7], [26], [33]. M. Auslander and I. Reiten proved in [7] that any representation-finite artin algebra A with $\beta(A) \leqslant 2$ is biserial, that is, the radical of any indecomposable nonuniserial projective left or right A-module is a sum of at most two uniserial submodules whose intersection is simple or zero (see [15]). In [33] A. Skowroński and J. Waschbüsch proved that the class of representation-finite algebras A over an algebraically closed field with $\beta(A) \leqslant 2$ coincides with the class of representation-finite biserial algebras. Here we shall prove that this is also true for algebras over a perfect field. The proofs of all facts are similar as in [33] and [26]. The main differences are in combinatorial parts of the proofs, because we must consider a larger class of oriented graphs with valuations instead of the ordinary quivers. Moreover, we use the more general concept of vector space categories than it was applied in [26]. The assumption that considered algebras are over a perfect field is important, because in this case we can present any algebra as a species algebra. Thus we think about modules as about representations of oriented graphs with valuations.

Well-known examples of biserial algebras are Nakayama algebras, blocks

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1980 *Mathematics Subject Classification*: Primary 16A46.

of group algebras with a cyclic defect group [18], [19], [20], algebras given by Brauer quivers and generalized tilted algebras of type A_n, B_n, C_n [1], [2].

We use the term algebra to mean finite-dimensional algebra over a perfect field K and the term module to mean finitely generated right module. We shall denote by mod-A the category of all finitely generated right A-modules. Algebras, as usual in the representation theory, are assumed to be basic. Recall that an algebra A is basic iff $A \approx \mathrm{End}_A(\bigoplus_{i=1}^{n} P_i)^{\mathrm{op}}$, where P_i are all indecomposable pairwise nonisomorphic projective left A-modules. An algebra is schurian if the endomorphism algebra of each of its indecomposable module is a division ring. Local modules are just the factors of indecomposable projective modules. Colocal modules are submodules of indecomposable injective modules.

For any algebra A and any A-module M we shall denote by $E_A(M)$ the A-injective envelope of M, by $P_A(M)$ the A-projective cover of M, by $\mathrm{top}_A(M)$ the top of M, by $\mathrm{soc}_A(M)$ the socle of M, by $\mathrm{rad}_A(M)$ the Jacobson radical of M and by $l(M)$ the length of M. Further we shall denote by ind-A the full subcategory of mod-A consisting of all indecomposable modules. We shall denote the Auslander-Reiten quiver of an algebra A [4]–[6] by Γ_A. Finally, $D\mathrm{Tr}$ will denote the Auslander-Reiten operator which maps the nonprojective indecomposables one-one to the noninjective indecomposables and has the natural inverse $\mathrm{Tr}D$.

In this paper, we shall use freely properties of the Auslander-Reiten sequences and minimal left (right) almost split morphisms [3]–[6], [16], vector space category methods [22], [23], [30], [31], [32] and covering techniques [11], [17], [35].

The construction of this paper is as follows. In Section 1 we recall basic facts from the representation theory of artin algebras. Moreover, we show that any distributive biserial algebra, satisfying the Jans' condition, has a multiplicative Cartan's basis.

In Section 2 we introduce almost multiplicity-free modules and give some sufficient condition for an algebra to be the representation-finite biserial schurian one.

In Section 3 we describe indecomposable objects of subspace categories which appear in the study of one point extensions of representation-finite biserial algebras which satisfy the (S)-condition. The description of these objects is used in Section 4.

In Section 4 we give the description of the category ind-A, where A is obtained from a known algebra by one point extension procedure. This description is used for the proof of Theorem 2, where the characterization of representation-finite biserial algebras, satisfying the (S)-condition, is given.

In Section 5 we apply the results of Section 4 to prove Theorem 3 which is the main result of this paper and was announced in [25].

The author would like to thank Andrzej Skowroński for suggesting the field of research and many helpful suggestions and comments during the preparation of this paper.

He would like to thank Professor Daniel Simson and Bogumiła Klemp for their helpful informations concerning vector space categories.

1. The structure of distributive biserial algebras

Throughout the paper K will denote a fixed perfect field. Following V. Dlab and C. M. Ringel [14] we can attach to each finite-dimensional K-algebra A a K-species with relations (Σ_A, R_A). Recall that a K-species $\Sigma = (F_i, {}_jM_i)_{j,i\in I}$ is a set of division rings $(F_i)_{i\in I}$ and bimodules $({}_jM_i)_{j,i\in I}$ such that for each j, i from I we have $F_i \supset K$, F_i and ${}_jM_i$ are finite-dimensional over K and K acts centrally on F_i and ${}_jM_i$. We can attach to a K-species Σ a K-algebra $T(\Sigma)$, called the *tensor algebra* of Σ, as follows: $\Lambda = \prod_{i\in I} F_i$ and $M = \bigoplus_{i,j\in I} {}_jM_i$. Thus the projections $\Lambda \to F_i$ define the natural Λ-Λ-bimodule structure on M. We put $M^{(0)} = \Lambda$, $M^{(1)} = M$ and inductively $M^{(t)} = M^{(t-1)} \otimes_\Lambda M$. Then

$$T(\Sigma) = \bigoplus_{t\geqslant 0} M^{(t)}$$

as a K-space and the multiplication is induced by canonical isomorphisms $M^{(r)}\otimes_\Lambda M^{(s)} \to M^{(r+s)}$.

On the other hand, one can attach the K-species Σ_A to any basic K-algebra A. We put

$$\Lambda = A/\mathrm{rad}_A(A) = \prod_{i=1}^{n} F_i, \qquad {}_\Lambda M_\Lambda = \mathrm{rad}_A(A)/\mathrm{rad}_A^2(A).$$

Moreover, $\mathrm{rad}_A(A)/\mathrm{rad}_A^2(A)$ has the uniquely determined decomposition

$$M = \bigoplus_{1\leqslant i,j\leqslant n} {}_jM_i$$

F_j-F_i-bimodules ${}_jM_i = {}_{F_j}(\mathrm{rad}_A(A)/\mathrm{rad}_A^2(A))_{F_i}$ on which K acts centrally. Thus we have

$$\Sigma_A = (F_i, {}_jM_i)_{1\leqslant i,j\leqslant n}.$$

Further we can attach an oriented graph Q_Σ with valuations to the K-species Σ. Vertices of the graph Q_Σ are elements of I. We attach a valued arrow

$$i \xrightarrow{(d_{ji},d'_{ji})} j$$

to each pair (i, j) such that ${}_jM_i \neq 0$, where $d_{ji} = \dim_{F_j}({}_jM_i)$ and $d'_{ji} = \dim({}_jM_i)_{F_i}$.

Let $\Sigma = (F_i, {}_jM_i)_{i,j\in I}$ be a K-species and let

$$w = i \xrightarrow{(d_{i_1 i}, d_{i_1 i}^{\prime})} i_1 \ldots i_t \xrightarrow{(d_{j i_t}, d_{j i_t}^{\prime})} j$$

be a path in Q_Σ. An F_j-F_i-bimodule ${}_jM_i^w = {}_jM_{i_t} \otimes \ldots \otimes {}_{i_2}M_{i_1} \otimes {}_{i_1}M_i$ is called a *bimodule of the path* w. An F_j-F_i-bimodule

$${}_jM_i^c = \bigoplus_{w\in P(i,j)} {}_jM_i^w$$

is called a *composed bimodule* from i to j, where $P(i, j)$ is the set of all paths from i to j. Let us observe that ${}_jM_i = F_jT(\Sigma)F_i$. An ideal R in $T(\Sigma)$ is called a *relation ideal* if $R \subset \mathrm{rad}^2_{T(\Sigma)}(T(\Sigma))$. Thus we say that the factor algebra $A = T(\Sigma)/R$ is a K-species algebra with relations. The ideal R defines an F_j-F_i-subbimodule ${}_jR_i \subset {}_jM_i^c$ for each pair (i, j), where ${}_jR_i = F_jRF_i$. It is not hard to see that the subbimodules ${}_jR_i \subset {}_jM_i^c$ define uniquely determined bimodule epimorphisms ${}_j\varrho_i$: ${}_jM_i^c \to {}_jM_i^c/{}_jR_i$ whose kernels determine the ideal R. From [1] we know that every finite-dimensional basic algebra over a perfect field K is isomorphic to a K-species algebra with relations.

Now we define some special types of relations for a given K-species with relations (Σ, R). A path in Q_Σ of the form

$$i \xrightarrow{(d_{i_1 i}, d_{i_1 i}^{\prime})} i_1 \to \ldots \to i_t \xrightarrow{(d_{j i_t}, d_{j i_t}^{\prime})} j$$

is called a *zero path with respect to* R if ${}_jM_{i_t} \otimes \ldots \otimes {}_{i_1}M_i \subset F_jRF_i$. A pair of *parallel paths* is a pair (v, w) of nonzero paths with the same starting point and the same ending point. Let (v, w) be a pair of parallel paths from i to j;

$$v = \alpha_n \ldots \alpha_1, \qquad w = \beta_m \ldots \beta_1,$$

$$\alpha_i = t_i \xrightarrow{(d_{t_{i+1} t_i}, d_{t_{i+1} t_i}^{\prime})} t_{i+1} \quad \text{for all } i = 1, \ldots, n,$$

$$\beta_j = l_j \xrightarrow{(d_{l_{j+1} l_j}, d_{l_{j+1} l_j}^{\prime})} l_{j+1} \quad \text{for all } j = 1, \ldots, m.$$

Then we have a *commutativity relation* with respect to R on the pair (v, w) if

$${}_{t_{n+1}}M_{t_n} \otimes \ldots \otimes {}_{t_2}M_{t_1} - {}_{l_{m+1}}M_{l_m} \otimes \ldots \otimes {}_{l_2}M_{l_1} \subset F_{t_{n+1}}RF_{t_1}.$$

Suppose now that Q_Σ has a subquiver of the form

(*) $$i \xrightarrow{(1,2)} j \xrightarrow{(2,1)} l$$

or

(**) $$i \xrightarrow{(2,1)} i_1 \xrightarrow{(1,1)} \ldots \xrightarrow{(1,1)} i_{t-1} \xrightarrow{(1,2)} l$$

and there are two finite-dimensional division K-algebras $F \subset E$ such that $\dim_F(E) = \dim(E)_F = 2$ and we have $F_i \approx F_l \approx F$, $F_j \approx E$ in the case (*) and $F_i \approx F_l \approx E$, $F_{i_1} \approx \ldots \approx F_{i_{t-1}} \approx F$ in the case (**). Thus we have two relations defined by the kernels of bimodule epimorphisms

$$\mu\colon {}_F E_E \otimes {}_E E_F \to {}_F F_F,$$

$$\pi\colon {}_E E_F \otimes {}_F F_F \otimes \ldots \otimes {}_F F_F \otimes {}_F E_E \to {}_E E_E.$$

We call them the *relation of type* μ *and* π respectively.

Let A be a finite-dimensional connected and basic algebra over a perfect field K. We can assume that $A = T(\Sigma_A)/R_A$ and let Q_{Σ_A} be the graph of Σ_A. We call A a *special algebra* (or more precisely, *special biserial algebra*) provided the following conditions are satisfied:

(SP)

(1) The numbers of arrows starting, respectively ending, at any fixed vertex of Q_{Σ_A} are bounded by 2.

(2) For any arrow α of Q_{Σ_A} there is at most one arrow β and at most one arrow γ in Q_{Σ_A} such that $\alpha\beta$ and $\gamma\alpha$ are nonzero paths with respect to R_A.

(3) The valuation of any arrow in Q_{Σ_A} is one of the form (1, 2), (2, 1) (1, 1). Moreover there is no nonzero path $\alpha_n \ldots \alpha_1$ in Q_{Σ_A} with respect to R_A such that for $i = 2, \ldots, n-1$ all α_i have the valuations (1, 1) and α_1, α_n have valuation (1, 2), or dually α_1, α_n have the valuation (2, 1).

(4) If w is a nonzero path with respect to R_A of the form

$$i \xrightarrow{(2,1)} i_1 \xrightarrow{(1,1)} \ldots \xrightarrow{(1,1)} i_t \xrightarrow{(1,2)} l \quad \text{in } Q_{\Sigma_A},$$

then there is the relation of type π on w.

(5) If v is a nonzero path with respect to R_A of the form

$$i \xrightarrow{(1,2)} j \xrightarrow{(2,1)} l \quad \text{in } Q_{\Sigma_A},$$

then there is the relation of type μ on v.

(6) If there exists a path u of the form

$$i \xrightarrow{(1,1)} j \xrightarrow{(2,1)} l \quad \text{in } Q_{\Sigma_A},$$

then u is a zero path with respect to R_A.

(7) If there exists a path u of the form

$$i \xrightarrow{(1,2)} j \xrightarrow{(1,1)} l \quad \text{in } Q_{\Sigma_A},$$

then u is a zero path with respect to R_A.

(8) Q_{Σ_A} does not contain subgraphs of the following forms:

$$\xleftarrow{(1,2)}\xrightarrow{(2,1)}, \quad \xrightarrow{(1,2)}\xleftarrow{(2,1)}, \quad \xrightarrow{(1,2)}\xleftarrow{(1,1)}, \quad \xleftarrow{(1,1)}\xrightarrow{(2,1)}.$$

(9) There is an upper bound for the length of nonzero paths in Q_{Σ_A} with respect to R_A.

Throughout this paper we shall call the valuation (1, 1) *trivial* and we shall omit it.

Keeping above notations, we can prove the following generalization of [33, Lemma 1].

LEMMA 1. *Any special algebra is biserial.*

Proof. Each arrow $\alpha = i \xrightarrow{(d_{ji}, d'_{ji})} j$ in Q_{Σ_A} determines a unique path $w = \alpha_s \ldots \alpha_2 \alpha_1$, $\alpha_1 = \alpha$, which is maximal in the set of all nonzero with respect to R_A paths u starting with α, and any such path u is a subpath of w. First suppose that w has only trivial valuations. Then the submodule $(R_A + \alpha)A$ of $\operatorname{rad}_A(e_i A)$ is uniserial and by (SP)(1) $\operatorname{rad}_A(e_i A)$ is a sum of at most two uniserial modules. Let us observe that if we have $\alpha_t = t \xrightarrow{(d_{(t+1)t}, d'_{(t+1)t})} t+1$ with $d_{(t+1)t} d'_{(t+1)t} = 2$ for some $t = 2, \ldots, s$, then w is one of the following forms: $w = \alpha_s \ldots \alpha_1$ where $\alpha_1 = \xrightarrow{(2,1)}$, $\alpha_s = \xrightarrow{(1,2)}$ and the other arrows have trivial valuations, or $w = \alpha_2 \alpha_1$ and $\alpha_1 = \xrightarrow{(1,2)}$, $\alpha_2 = \xrightarrow{(2,1)}$. Then by (SP)(4) and (5) $\operatorname{rad}_A(e_i A)$ is a sum of two uniserial modules or a uniserial module.

Now suppose we have two parallel paths $u = \alpha_n \ldots \alpha_2 \alpha_1$ and $v = \beta_m \ldots \beta_1$ starting at i, $\alpha_1 \neq \beta_1$, such that $(R_A + u)A = (R_A + v)A \neq 0$. Thus we have $\alpha_n \neq \beta_m$ by the condition (SP)(2) and moreover $(R_A + u)A \subset \operatorname{soc}_A(e_i A)$ by the condition (SP)(8). Assume that $\gamma u \notin R_A$ for an arrow γ from Q_{Σ_A}. Since $(R_A + u)A = (R_A + v)A$ is a uniserial module, so we can claim that the end of γ gives the second upper Loevy factor of $(R_A + u)A$ and thus $\gamma v \notin R_A$. But then both $\gamma\alpha_n$, $\gamma\beta_m$ do not belong to R_A and we get a contradiction to the condition (SP)(2). Since the conditions of (SP) are left-right symmetric, so A is biserial. ∎

It is shown in [33] that in general the converse of Lemma 1 is not true. We shall show that this is true for distributive biserial algebras. Recall that an algebra A is *distributive* provided the lattice of its twosided ideals is distributive. It is well known that if A is representation-finite algebra, that is, A has only finitely many nonisomorphic indecomposable left (or right) modules, then A is distributive. Let us remark that by Kupisch [20] any distributive algebra A satisfies the following conditions:

(K) For all i, j, $1 \leqslant i, j \leqslant q$, e_i – primitive idempotents in A

(1) the algebra $A_i = e_i A e_i$ is uniserial;

(2) the bimodule ${}_{A_i}(e_i A e_j)_{A_j}$ is uniserial as a left A_i-module or as a right A_j-module.

Now we are able to prove the following lemma.

LEMMA 2. *Any distributive biserial algebra A is special.*

Proof. We can assume that the algebra A is basic. Let Σ_A be a K-species of A and let ε: $T(\Sigma_A) \to A$ be an epimorphism with the kernel R_A. Moreover let Q_{Σ_A} be the graph of Σ_A and let $\boldsymbol{m} = \{m_i;\ i = 1, \ldots, r\}$ be a K-basis of $\operatorname{rad}_{T(\Sigma_A)}(T(\Sigma_A))/\operatorname{rad}^2_{T(\Sigma_A)}(T(\Sigma_A))$. We can choose $\boldsymbol{m}$ in the way that $\boldsymbol{m}$ contains all basis of ${}_j M_i$ over K, where ${}_j M_i$ corresponds to the arrow $\alpha = i \xrightarrow{(d_{ji}, d'_{ji})} j$ in Q_{Σ_A}. We shall show that there is such a choice of the K-basis of $\operatorname{rad}_{T(\Sigma_A)}(T(\Sigma_A))/\operatorname{rad}^2_{T(\Sigma_A)}(T(\Sigma_A))$ that the kernel R_1 of the induced epimorphism

ε_1: $T(\Sigma_A) \to A$ satisfies the conditions (SP). It is easy to verify, by the biseriality of A, that the conditions (1), (3), (8) are satisfied and moreover the condition (9) is satisfied by the distributivity of A. We are starting with an epimorphism ε: $T(\Sigma_A) \to A$ with the kernel R_A. We shall change elements of the K-basis of ${}_jM_i$ inductively in such a way that the other conditions will be satisfied too. The first step is to show that for any choice of a K-basis of $\mathrm{rad}_{T(\Sigma_A)}(T(\Sigma_A))/\mathrm{rad}^2_{T(\Sigma_A)}(T(\Sigma_A))$ and for any arrow $\alpha = i \xrightarrow{(d_{ji}, d'_{ji})} j$ we have that the image by ε of the K-basis of ${}_jM_i$ generates a uniserial left ideal in A and a uniserial right ideal in A. We know, by the assumption, that $\mathrm{rad}_A(e_iA)$ has a uniserial submodule U which is generated by the element $b = e_ibe_j \in \mathrm{rad}_A(e_iA) \setminus \mathrm{rad}^2_A(e_iA)$. But A is distributive, so $a = bc + u_tb$ or $a = cb + bu_j$ where $c \in F_i$ or $c \in F_j$, $c \neq 0$ and $u_t \in \mathrm{rad}_A(A_t)$. But then

$$aA = (bc + bu_j)A = b(c + u_j)A = bA \quad \text{ or } \quad aA = (c + u_i)bA = bA.$$

In both cases aA is uniserial. Similarly Aa is uniserial. Now suppose that we have in Q_{Σ_A} three arrows $\alpha = i \to j$, $\beta = j \to t$, $\gamma = j \to p$ and $\beta \neq \gamma$. Moreover let the compositions $\beta\alpha$ and $\gamma\alpha$ are nonzero paths with respect to R_A. Thus ${}_tM_j \otimes {}_jM_i$ is nonzero in $T(\Sigma_A)/R_A$ and ${}_pM_j \otimes {}_jM_i$ is nonzero in $T(\Sigma_A)/R_A$. Hence we have the nonzero elements $\varepsilon(x)$, $\varepsilon(y)$ in e_tAe_j which is uniserial as a left A_t-module or as a right A_j-module. Moreover, $\varepsilon(x) = \varepsilon(b \otimes a)$ and $\varepsilon(y) = \varepsilon(c \otimes a)$ and $a, b, c \in \boldsymbol{m}$. Thus $\varepsilon(x) = f\varepsilon(y)$ for some $f \in e_t\mathrm{rad}_A(A)e_j$. By the distributivity of A we get $f\varepsilon(c)$ is a left or right multiple of $\varepsilon(b)$ and, by Nakayama's lemma, $f\varepsilon(c) = \varepsilon(b)g$ for some $g \in e_j\mathrm{rad}_A(A)e_j$. This implies $\varepsilon(b)(1-g)\varepsilon(a) = 0$ and, if we replace the representative $\varepsilon(b)$ of b by $\varepsilon'(b) = \varepsilon(b)(1-g)$ for all K-generators of ${}_tM_j$, we get a new epimorphism ε' from $T(\Sigma_A)$ onto A with $\beta\alpha$ being a zero path with respect to $R'_A = \ker \varepsilon'$. Let us observe that we did not kill any zero path with respect to R_A changing ε by ε', because if there was a zero path $\delta\beta$ with respect to R_A then there would be a zero path $\delta\beta$ with respect to R'_A. On the other hand the above g is not a right multiple of $\varepsilon(a)$ by Nakayama's lemma, and by $\varepsilon(b)g \neq 0$ we can see that there must be another arrow α' ending at j and such that there is a zero path $\beta\alpha'$ with respect to R_A in Q_{Σ_A}. In view of this we could not kill any relations on $\beta\delta$ passing from R_A to R'_A. If the arrows α, β, γ have another valuations we shall prove in the same way that there is an epimorphism ε': $T(\Sigma_A) \to A$ which satisfies the condition (SP)(2). If $\alpha = i \to j$, $\beta = j \xrightarrow{(2,1)} t$, $j \neq t$, and $\beta\alpha$ is not a zero path with respect to R_A, then, as above, $\varepsilon(b \otimes a)$ is a nonzero element of the left uniserial module $A\varepsilon(a)$. But $\varepsilon(b) \in e_j\mathrm{rad}_A(A)e_t$ and, using Nakayama's lemma, $x = \varepsilon(b)$ for some $x \in e_tAe_i$, because $x\varepsilon(a) = \varepsilon(b)\varepsilon(a)$. But $\varepsilon(b) \in e_j\mathrm{rad}_A(A)e_t$, so we have a contradiction. Similarly we proceed in the case of the relations of type (SP)(7). If $\alpha = i \xrightarrow{(1,2)} j$, $\beta = j \xrightarrow{(2,1)} t$, then it is not hard to verify that ${}_jM_t = E$ and ${}_tM_j = E$. By the distributivity, we know that e_iAe_t is a left uniserial F-module or a right uniserial F-module, where F, E are division rings and $\dim_F E = \dim E_F = 2$, so $e_iAe_t \approx F$ and we have an epimorphism ${}_FE_E \otimes {}_EE_F \to {}_FF_F$. The case of the relations of type (SP)(4) is been proving in the similar way. Using left-right symmetric version of the above procedure, we get an epimorphism ε'' from $T(\Sigma_A)$ onto A with the kernel R''_A which satisfies the conditions (SP) and the lemma is proved.

Recall that a K-basis C of an algebra A is a *multiplicative basis*, if the product of any two elements of C belongs to C or is zero. Moreover C is the *Cartan's basis* when there is a decomposition of A in a direct sum of indecomposable left ideals Ae_i, generated by idempotents e_i, $1 \leqslant i \leqslant q$, such that for all i, j and any $t = 0, 1, 2, \ldots$, C contains a K-basis of the subspace $e_i(\operatorname{rad}^t_A(A))e_j$.

We denote by $I(A)$ the lattice of twosided ideals of A. Moreover we denote the lattice of F_j-F_i-subbimodules of the bimodule e_jAe_i by $L(e_jAe_i)$ for each two idempotents e_i, e_j of A. We say that A satisfies the Jans' condition if the images of maps from $I(A)$ into $L(e_jAe_i)$, attaching the bimodule e_jIe_i to each ideal I, form a chain (linearly ordered set) of bimodules for all $1 \leqslant i, j \leqslant q$. It is known (see [19], [20]) that every representation-finite K-alebra, satisfying the Jans' condition, satisfies the conditions (K). Hence we can prove the following theorem.

THEOREM 1. *Any distributive biserial algebra, satisfying the Jans' condition has a multiplicative Cartan's basis.*

Proof. We can assume that A is connected, i.e. Q_{Σ_A} is a connected graph. Moreover we can assume that $A = T(\Sigma_A)/R_A$ for some K-species (Σ_A, R_A) with relations. We can assume as well that A is special by Lemma 2. Thus R_A is generated by a set of paths in Q_{Σ_A}, a set of relations of types π and μ and a set of elements $u_l - c_l v_l$ where u_l and v_l are two parallel paths belonging to Q_{Σ_A}, $c_l \in K$, $c_l \neq 0$ by Lemma 1. It is enough for the proof of the theorem to prove that for the K-basis $\boldsymbol{m}$ of $\operatorname{rad}_{T(\Sigma_A)}(T(\Sigma_A))/\operatorname{rad}^2_{T(\Sigma_A)}(T(\Sigma_A))$ there exists such a choice of $\varepsilon(m)$ in A, $m \in \boldsymbol{m}$ that $c_l = 1$ for all c_l, where ε: $T(\Sigma_A) \to A$ is an epimorphism. Indeed, in this case, any set of paths in Q_{Σ_A}, representing different classes of the equivalence modulo R_A, gives us a multiplicative Cartan's basis of A. We shall prove the theorem by induction on the number n of primitive idempotents of A.

The case $n = 1$ is trivial, and we assume that A is nonlocal, because every local algebra satisfying the Jans' condition is uniserial, so it is isomorphic to $F[X]/(X^d)$ where d is a natural number different from 0. For $n > 1$ the Jans' condition implies that there is the uniserial left module Ae_i or the uniserial right module e_iA. By the symmetry we can assume that the neighborhood of the vertex i in Q_{Σ_A} is one of the following graphs:

(1) $i \xrightarrow[\alpha]{} j$;

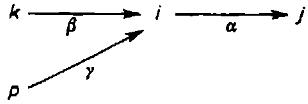
(2) $i \xrightarrow[\alpha]{(2,1)} j$;

(3) $i \xrightarrow[\alpha]{(1,2)} j$;

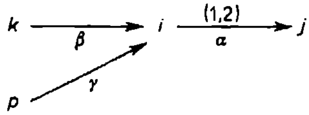
(4) $k \xrightarrow[\beta]{} i$;

(5) $k \xrightarrow[\beta]{(2,1)} i$;

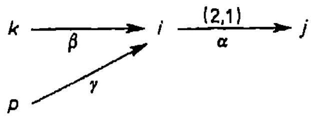
(6) $k \xrightarrow[\beta]{(1,2)} i$;

(7)


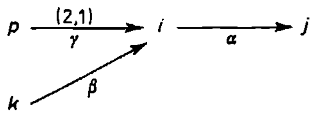
with the relations: (a) $\alpha\beta$, $\alpha\gamma$ are zero paths with respect to R_A, (b) $\alpha\gamma$ is a zero path with respect to R_A;

(8)


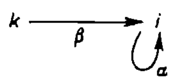
with the relations: (a) $\alpha\beta$, $\alpha\gamma$ are zero paths with respect to R_A, (b) $\alpha\gamma$ is a zero path with respect to R_A and $\alpha\beta$ is not a zero path with respect to R_A;

(9)


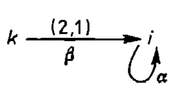
with the relations: $\alpha\beta$, $\alpha\gamma$ are zero paths with respect to R_A;

(10)


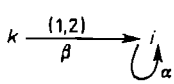
with the relations: (a) $\alpha\beta$, $\alpha\gamma$ are zero paths with respect to R_A, (b) $\alpha\gamma$ is a zero path with respect to R_A and $\alpha\beta$ is not a zero path with respect to R_A;

(11)


with the relations: (a) $\alpha\beta$ is a zero path with respect to R_A, (b) $\alpha\beta$ is not a zero path with respect to R_A;

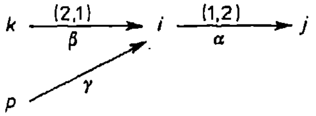
(12)


with the relations: (a) $\alpha\beta$ is a zero path with respect to R_A, (b) $\alpha\beta$ is not a zero path with respect to R_A;

(13)


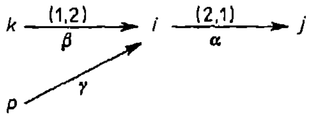
with the relation: $\alpha\beta$ is a zero path with respect to R_A;

(14)



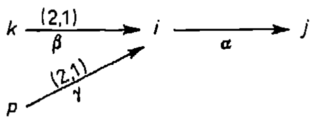
with the relation π on the path $\alpha\beta$;

(15)



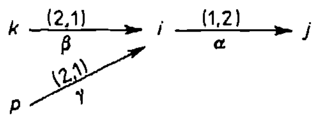
with the relation μ on the path $\alpha\beta$;

(16)



with the relations: (a) $\alpha\beta$, $\alpha\gamma$ are nonzero paths with respect to R_A, (b) $\alpha\gamma$ is a zero path with respect to R_A and $\alpha\beta$ is not a zero path with respect to R_A;

(17)



with the relations: (a) $\alpha\gamma$ is a zero path with respect to R_A and there is a relation of type π on the path $\alpha\beta$, (b) $\alpha\gamma$, $\alpha\beta$ are zero paths with respect to R_A;

(18) $k \xrightarrow[\beta]{} i \xrightarrow[\alpha]{} j$ with the relations: (a) $\alpha\beta$ is a zero path with respect to R_A, (b) $\alpha\beta$ is not a zero path with respect to R_A;

(19) $k \xrightarrow[\beta]{(2,1)} i \xrightarrow[\alpha]{} j$ with the relations: (a) $\alpha\beta$ is a zero path with respect to R_A, (b) $\alpha\beta$ is not a zero path with respect to R_A;

(20) $k \xrightarrow[\beta]{(1,2)} i \xrightarrow[\alpha]{} j$ with the relation: $\alpha\beta$ is a zero path with respect to R_A;

(21) $k \xrightarrow[\beta]{} i \xrightarrow[\alpha]{(2,1)} j$ with the relation: $\alpha\beta$ is a zero path with respect to R_A;

(22) $k \xrightarrow[\beta]{} i \xrightarrow[\alpha]{(1,2)} j$ with the relations: (a) $\alpha\beta$ is a zero path with respect to R_A, (b) $\alpha\beta$ is not a zero path with respect to R_A;

(23) $k \xrightarrow[\beta]{(2,1)} i \xrightarrow[\alpha]{(1,2)} j$ with the relation of type π on the path $\alpha\beta$;

(24) $k \xrightarrow[\beta]{(1,2)} i \xrightarrow[\alpha]{(2,1)} j$ with the relation of type μ on the path $\alpha\beta$;

(25) $k \xrightarrow[\beta]{} i \xleftarrow[\gamma]{} p$;

(26) $k \xrightarrow[\beta]{(2,1)} i \xleftarrow[\gamma]{} p$;

(27) $k \xrightarrow[\beta]{(2,1)} i \xleftarrow[\gamma]{(1,2)} p$.

The algebra $A' = (1-e_i)A(1-e_i)$ is not only biserial, but satisfies the Jans' condition and is distributive. Since A' has fewer primitive idempotents than A so we can assume inductively that induced by R_A, the Cartan's basis of A induces the multiplicative Cartan's basis of A' (by the restriction on the subset of elements contained in A'). Then $A' = T(\Sigma'_A)/R'_A$, where Σ'_A is a K-species and R'_A is generated by paths, the relations of types π and μ, and differences $u-v$, where $u \neq v$ are two parallel paths in $Q_{\Sigma'_A}$ which do not lie in R'_A. Moreover, for two parallel different paths u, v which do not lie in R'_A we have: if $u-vc \in R'_A$ for $c \in K$, then $c = 1$.

In the cases (1)–(6), (11)(a), (12)(a), (13), (16)(a), (18)(a), (19)(a), (20), (21) we get the graph $Q_{\Sigma'_A}$ from the graph Q_{Σ_A} by losing of the vertex i and the arrows α and β. Hence every nontrivial relation $u-vc$ from R_A can be realize in $Q_{\Sigma'_A}$, so $u-vc$ is in R'_A and $c = 1$.

In the case (7)(a) we get $Q_{\Sigma'_A}$ from Q_{Σ_A} by suppressing the vertex i and the arrows α, β, γ. If there is a relation $u-vc$ from R_A, where u and v end with β and γ respectively, then it is the only relation of this form and multiplying the images of generators of ${}_iM_p$ in A by c^{-1}, we get that $u-v$ lies in R_A.

We have two possibilities in the case (7)(b). At first let us assume that there are no nontrivial relations of the form $\alpha\beta-uc$ in R_A. In this case we get $Q_{\Sigma'_A}$ from Q_{Σ_A} by losing the vertex i and arrows α, β, γ and adding a new arrow from k to j corresponding to the elements $(a \otimes b)+R_A$ in A', where a is from the K-basis of ${}_jM_i$ and b is from the K-basis of ${}_iM_k$. Let $d \in {}_jM'_k$ being the element of the K-basis of ${}_jM'_k$. Thus $d = a \otimes b$, where a and b are elements of the K-basis of ${}_jM_i$ and ${}_iM_k$ respectively, because ${}_jM'_k = {}_jM_i \otimes {}_iM_k$. Any relation of the form $u-vc$, which does not end at i, is a relation in $Q_{\Sigma'_A}$, so $c = 1$.

If there is a new relation of the form $u-vc$ in R_A, where u and v end with β and γ respectively at i, then it is the unique relation with these properties and we get that $u-v$ is a relation in R_A by multiplying the K-basis of ${}_iM_p$ by c^{-1}, because γ appears in v only once and does not appear in u. Now suppose that there is a path $u \neq \alpha\beta$ in Q_{Σ_A} such that $\alpha\beta-uc$ lies in R_A. Thus we get $Q_{\Sigma'_A}$ by losing the vertex i and the arrows α, β, γ. Multiplying the elements of the K-basis of ${}_jM_i$ by c, we get $\alpha\beta-u$ lies in R_A because α cannot appear in u. Moreover, $\alpha\beta-u$ is the unique relation of this form which contains $\alpha\beta$ as a subpath. We must consider still that there is the relation $u'-v'c'$ in R_A, where u' and v' end with β and γ, respectively. But this is the only relation of this form which contains γ. Hence, we can multiply elements of the K-basis of ${}_iM_p$ by c'^{-1} and we get that $u'-v'$ lies in R_A.

Notice that we never kill the relations of types π and μ going from A to A' because A' and A are distributive. Hence, similar arguments prove the theorem in the other cases. This finishes the proof of the theorem. ∎

2. Almost multiplicity-free modules

It is shown in [26] that a finite-dimensional algebra A over an algebraically closed field being a factor of an hereditary alebra is representation-finite, schurian and $\beta(A) \leqslant 2$ if and only if every indecomposable A-module M is multiplicity-free, i.e. $\dim_K \operatorname{Hom}_A(P, M) \leqslant 1$ for each indecomposable projective

A-module P. Now we shall introduce a more general concept of an almost multiplicity-free module playing an important role in the investigations of representation-finite biserial algebras over a perfect field.

Let A be a K-algebra and let M be an indecomposable A-module. We say that M is an *almost multiplicity-free* A-module provided the following conditions are satisfied:

(1) $\dim_{\mathrm{End}_A(P)}\mathrm{Hom}_A(P, M) \leqslant 2$ for each indecomposable projective A-module P.

(2) There is at least one indecomposable projective A-module P_0 such that

$$\dim_{\mathrm{End}_A(P_0)}\mathrm{Hom}_A(P_0, M) = 1.$$

(3) If there is a projective indecomposable A-module P_0 such that $\dim_{\mathrm{End}_A(P_0)}\mathrm{Hom}_A(P_0, M) = 2$, then there are two division K-algebras E, F such that $\dim_F E = 2$ and there is an indecomposable projective A-module P_1 such that

$$\mathrm{End}_A(P_1) = E, \quad \mathrm{End}_A(P_0) = F, \quad \dim_E \mathrm{Hom}_A(P_1, M) = 1.$$

Following R. Bautista and F. Larrión [9], we say that an indecomposable projective A-module P has a *separated radical* if, for any two nonisomorphic indecomposable summands M and N of $\mathrm{rad}_A(P)$, $\mathrm{supp}(M)$ and $\mathrm{supp}(N)$ are in different weakly connected components of $\mathscr{P}_A \backslash \mathscr{D}(P)$, where $\mathscr{P}_A$ denotes the full subcategory of ind-A consisting of all projective modules,

$$\mathrm{supp}(X) = \{Q \in \mathscr{P}_A;\ \mathrm{Hom}_A(Q, X) \neq 0\}$$

and

$$\mathscr{D}(P) = \{Q \in \mathscr{P}_A;\ \text{there is a chain } P \to Q_1 \to \ldots \to Q_r \to Q \text{ of nonzero maps in } \mathscr{P}_A\};$$

and a subset G of $\mathscr{P}_A$ is *weakly connected* if, for any two modules X, Y from G, there is a chain of nonzero maps $X = X_1 - \ldots - X_r = Y$ with all X_i from G, where $X_i - X_{i+1}$ denotes $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$. Then A satisfies the (S)-*condition* if each indecomposable projective A-module has the separated radical.

Now we are able to state the following proposition.

PROPOSITION 1. *Let A be an algebra satisfying the (S)-condition and let all indecomposable A-modules be almost multiplicity-free. Then A is representation-finite, biserial and schurian.*

Proof. Let us observe that since all indecomposable modules are almost multiplicity-free, so there is an upper bound of the lengths of indecomposable A-modules and by Roiter's Theorem ([29]) A is representation-finite. Moreover, A is schurian because it satisfies the (S)-condition. Hence by the duality it is enough to prove that the radical of each nonuniserial local A-module is a sum of two uniserial submodules whose intersection is simple or zero.

We shall proceed by induction on the length of local modules. Needed condition is clear for local modules of the length 3. Let L be a nonuniserial local A-module with $l(L) \geqslant 4$, where l denotes the length. First we shall prove that $\mathrm{top}_A(\mathrm{rad}_A(L))$ is a direct sum of two simple A-modules. Suppose that $\mathrm{rad}_A(L)$ is local. Since L is nonuniserial, so $\mathrm{rad}_A(L)$ is nonuniserial and by the inductive assumption $\mathrm{rad}_A^2(L)/\mathrm{rad}_A^3(L)$ is a direct sum of two simple A-modules S and T. Let

$$X = L/\mathrm{rad}_A^3(L), \qquad Y = X/S, \qquad Z = X/T.$$

Let us observe that Y and Z are uniserial A-modules of length 3 and since L is almost multiplicity-free, so either S and T are nonisomorphic or $S \approx T$ and $\mathrm{End}_A(S) \approx \mathrm{End}_A(T) \approx F$. Let us consider two cases.

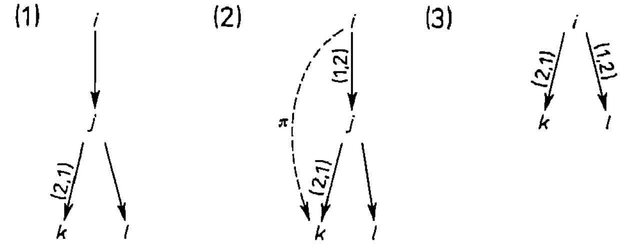
First, $S \not\approx T$. We can consider a map g: $Y \oplus Z \to \mathrm{top}_A(L)$ which is induced by the canonical epimorphisms $Y \to \mathrm{top}_A(L)$ and $Z \to \mathrm{top}_A(L)$, and we put

$$W = \ker g.$$

Thus

$$\mathrm{soc}_A\big(W/\mathrm{soc}_A(W)\big) \approx R \oplus R \quad \text{ where } R = \mathrm{top}_A\big(\mathrm{rad}_A(L)\big)$$

and W is not an almost multiplicity-free A-module because, in the opposite case, we shall have one of the following subgraphs in Q_{Σ_A}:



But thus we have a contradiction to the fact that all indecomposable A-modules are almost multiplicity-free in the cases (1) and (2). Moreover, we obtain that A is not representation-finite in the case (3) and this is a contradiction to the independently proved above fact.

Now we are going to claim that W is indecomposable. Suppose that $W = U \oplus V$ for two nonzero submodules U and V of W. Since $l(W) = 5$ and $l(\mathrm{soc}_A(W)) = 2$, so V and U are uniserial and one of them, say U, is not contained in $\mathrm{rad}_A(Y \oplus Z)$. Let $u = (y+S,\, z+T)$ be an element of U which does not belong to $\mathrm{rad}_A(Y \oplus Z)$, where $y, z \in X$. It is easy to verify that u is a generator of U, $y, z \notin \mathrm{rad}_A(X)$ and $y+z \in \mathrm{rad}_A(X)$.

Since $l(U) = 3$ and U is uniserial, so there exists the element r in $\mathrm{rad}_A^2(A)$

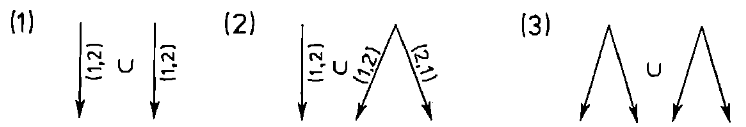
such that $ur \neq 0$. Obviously $(y+z)r = 0$ and $yr+S$ and $zr+T$ are nonzero elements of Y and Z respectively. On the other hand, U contains one of the modules $\{t+S;\ t \in T\}+0$ and $0+\{s+T;\ s \in S\}$. Thus U contains $\mathrm{soc}_A(W)$ and we get a contradiction to the fact that $U \cap V = 0$. Hence W is indecomposable but is not almost multiplicity-free and this contradiction shows that $\mathrm{rad}_A(L)$ is not a local module.

If $S \approx T$ then repeating the above proof, we get that $\mathrm{rad}_A(L)$ is not local.

Since $L/\mathrm{rad}_A^2(L)$ is a local A-module of length less than $l(L)$, so $\mathrm{top}_A(\mathrm{rad}_A(L))$ is the direct sum of two simple modules by the inductive assumption. Hence $\mathrm{rad}_A(L) = M+N$ for two proper local submodules M and N of L.

Assume that $M \cap N = 0$. Thus $l(L/M) < l(L)$, $l(L/N) < l(L)$, $\mathrm{rad}_A(L/M)$ and $\mathrm{rad}_A(L/N)$ have simple tops, so L/M and L/N are uniserial by the inductive assumption. Consequently M and N are uniserial.

Now assume that $M \cap N \neq 0$. First we shall show that $M \cap N$ is simple. Indeed, if G is a nonsimple indecomposable submodule of $M \cap N$, then, since the inductive assumption holds for M, N and $L/\mathrm{soc}_A(L)$, G is uniserial and there are two nonisomorphic uniserial submodules C and D such that $\mathrm{rad}_A(C) \approx G$ and $\mathrm{rad}_A(D) \approx G$. Let us consider the map h: $\mathrm{rad}_A(G) \to C \oplus D$ which is induced by the above isomorphisms and we put $B = \mathrm{coker}\, h$. Using dual arguments to the given ones in the first part of the proof, we shall prove that B is indecomposable, but is not almost multiplicity-free. So we get a contradiction and $M \cap N$ must be semisimple. Obviously $l(M \cap N) \leqslant 2$. Assume that $l(M \cap N) = 2$. Thus $M \cap N = \mathrm{soc}_A(N)$, $M \cap N = \mathrm{soc}_A(M)$ and $M \cap N = S' \oplus T'$ for two simple submodules S', T' of L. It is not hard to verify that $l(M) = 3$ and $l(N) = 3$ because the inductive assumption holds for L/S' and L/T'. Let i: $S' \to M \oplus N$ be the diagonal map induced by the canonical monomorphisms $S' \to M$ and $S' \to N$ and we put $Q = \mathrm{coker}\, i$. Thus Q is indecomposable, but it is not almost multiplicity-free because $\mathrm{soc}_A(Q) = S' \oplus T' \oplus T'$ and $l(M \oplus N) = 6$, so $l(Q) = 5$ and there must be a full subgraph of one of the forms in Q_{Σ_A}:



because $l(M) = 3$ and $l(N) = 3$ and A satisfies the (S)-condition. In all these cases we get a contradiction to the fact that each indecomposable A-module is almost multiplicity-free. Consequently $M \cap N$ is a simple module.

Now we are going to claim that M and N are uniserial. Suppose that one of them, say M, is not uniserial and let Z be its simple submodule nonisomorphic to $M \cap N$. Thus $l(M) = 3$ and N is uniserial because the inductive assumption holds for L/Z and $L/(M \cap N)$. Let $Y' = L/Z$, $R' = L/N$ and let e: $Y' \oplus R' \to \mathrm{top}_A(L)$ be the map induced by the canonical epimorphisms $Y' \to \mathrm{top}_A(L)$ and $R' \to \mathrm{top}_A(L)$. Thus as in the first part of the proof, we show that $W' = \ker e$ is indecomposable and $\mathrm{soc}_A(W'/\mathrm{soc}_A(W')) \supset K' \oplus K'$ where $K' = \mathrm{top}_A(M)$ and we show that W' is not almost multiplicity-free. This contradiction implies that M and N are uniserial and the proof of the proposition is completed. ■

3. One point extension

J. Waschbüsch introduced the notion of an X-sequence for a bounded quiver (Q, I) in [34]. Now we are going to generalize his concept for a K-species with relations. For two finite K-species $\Pi = (F_s, {}_sN_r)_{s,r=1,\dots,m}$ and $(\Sigma, R) = ((F_i, {}_iM_j)_{i,j=1,\dots,n}, R)$, $m \leqslant n$, such that there is neither the bimodule ${}_sN_r$ from Π with the property $\dim_{F_s}({}_sN_r) = \dim({}_sN_r)_{F_r} = 2$ nor the bimodule ${}_iM_j$ from Σ with the property $\dim_{F_i}({}_iM_j) = \dim({}_iM_j)_{F_j} = 2$ a *Π-sequence* in (Σ, R) is defined to be a K-species homomorphism $G\colon \Pi \to \Sigma$ such that for all $v = {}_{s_0}v_{s_1} \otimes {}_{s_1}v_{s_2} \otimes \dots \otimes_{s_{t-1}} v_{s_t}$, where ${}_{s_i}v_{s_{i+1}}$ is from ${}_{s_i}N_{s_{i+1}}$ and ${}_{s_i}N_{s_{i+1}}$ is a bimodule from Π for all $i = 0, \dots, t-1$, and for all elements c in K we have: if u is from $T(\Sigma)$, then the condition $cG(v) + u \in R$ implies $u = -cG(v) + u'$ for some u' from R.

A Π-sequence G in (Σ, R) is *faithful at the point s* of Π if there are injections of bimodules $G|_{{}_tN_s}\colon {}_tN_s \to {}_tM_s$ and $G|_{{}_sN_t}\colon {}_sN_t \to {}_sM_t$ for all $t = 1, \dots, m$.

A Π-sequence G in (Σ, R) is *locally faithful* when it is faithful at each point of Π.

As in [34] we can prove that if there is a locally faithful Π-sequence in (Σ_A, R_A), where Π is such that its graph is one of the extended Dynkin diagram, then A is not of finite representation type. Hence throughout this paper we shall assume that A is an algebra which has not such Π-sequences.

One of the main working tools we use in this paper is the notion of a vector space category introduced by L. A. Nazarova and A. V. Roiter in [23] and applied in the proof of the second Brauer-Thrall conjecture.

By definition, a *vector space category* $\mathscr{C}_D$ is an additive D-category together with a faithful functor

$$|-|\colon \mathscr{C}_D \to \text{mod-}D$$

to the category of finite-dimensional D-vector spaces, where D is a division ring. Then the *subspace category* $\mathscr{U}(\mathscr{C}_D)$ is defined in the following way (see [27], [28], [30]–[32]): its objects are triples of the form (U, X, φ) where U is a finite-dimensional D-vector space, X is an object of $\mathscr{C}_D$, and $\varphi\colon U \to |X|$ is a D-linear homomorphism. A morphism from (U, X, φ) to (U', X', φ') is given by a pair

$$(\alpha, \beta)\colon (U, X, \varphi) \to (U', X', \varphi')$$

where $\alpha\colon U \to U'$ is a D-linear homomorphism, $\beta\colon X \to X'$ is a map in $\mathscr{C}_D$ such that

$$|\beta|\varphi = \varphi'\alpha.$$

An important role in investigationes is played by the vector space categories with indecomposable objects whose endomorphism rings are division rings. These vector space categories are called *schurian*. Assume that $\mathscr{C}_D$ has only finitely many nonisomorphic indecomposable objects and fix their representatives $C_1, \dots, C_n$. We attach to $\mathscr{C}_D$ a valued scheme $(I_{\mathscr{C}_D}, d_{\mathscr{C}_D})$ in the following way: the points are the objects $C_1, \dots, C_n$, there is an arrow from C_i to C_j, $C_i \neq C_j$, if any only if there exists a nonzero map from C_i to C_j, and an arrow $C_i \to C_j$ has valuation (d, d') if

$$d = \dim \operatorname{Hom}_{\mathscr{C}_D}(C_i, C_j)_{\operatorname{End}_{\mathscr{C}_D}(C_j)}, \qquad d' = \dim_{\operatorname{End}_{\mathscr{C}_D}(C_i)} \operatorname{Hom}_{\mathscr{C}_D}(C_i, C_j).$$

In our applications we deal with vector space categories given by triangular matrix algebras of the form

$$R = \begin{bmatrix} D & {}_D M_S \\ 0 & S \end{bmatrix},$$

where D is a division ring and S is an algebra of finite representation type. We recall that any R-module X_R can be identified with a triple

$$X_R = (X'_D, X''_S, f)$$

where X'_D is a D-vector space, X''_S is an S-module and

$$f\colon X'_D \to \operatorname{Hom}_S({}_D M_S, X''_S)$$

is a D-linear homomorphism. We associate to R the vector space category (see [32])

$$\mathscr{C}_D(M_S) = \operatorname{Hom}_S({}_D M_S, \text{mod-}S)$$

and the full subcategory $\mathscr{X}_S(M_S)$ of ind-S formed by indecomposable modules X_S such that $\operatorname{Hom}_S({}_D M_S, X_S) \neq 0$. If $\mathscr{C}_D(M_S)$ is schurian and $L_1, \dots, L_n$ are all pairwise nonisomorphic indecomposable objects in $\mathscr{X}_S(M_S)$, we define a valued scheme $\vec{\mathscr{X}}_S(M_S)$ as follows: $L_1, \dots, L_n$ are points of $\vec{\mathscr{X}}_S(M_S)$ and $L_i \xrightarrow{(d,d')} L_j$ if and only if $\operatorname{Hom}_S(L_i, L_j) \neq 0$, where

$$d = \dim \operatorname{Hom}_S(L_i, L_j)_{\operatorname{End}_S(L_j)}, \qquad d' = \dim_{\operatorname{End}_S(L_i)} \operatorname{Hom}_S(L_i, L_j).$$

As usual we shall write simply $\to$ instead of $\xrightarrow{(1,1)}$.

We say that M_S is linear if and only if $\vec{\mathscr{X}}_S(M_S)$ has the following properties:

(a) $\vec{\mathscr{X}}_S(M_S)$ is of the form

$$X_0 \xrightarrow{(d_0, d'_0)} X_1 \xrightarrow{(d_1, d'_1)} \cdots \xrightarrow{(d_{t-1}, d'_{t-1})} X_t$$

and $d_i d'_i \leqslant 2$ for all $i = 0, \dots, t-1$.

(b) If $\dim_{\operatorname{End}_S(X_0)}(D) = 2$, then $d_i = d'_i = 1$ for all $i = 1, \dots, t-1$.

(c) There is no subscheme of $\vec{\mathscr{X}}_S(M_S)$ of the form

$$\xrightarrow{(1,2)}\rightarrow\ldots\rightarrow\xrightarrow{(1,2)} \quad \text{or} \quad \xrightarrow{(2,1)}\rightarrow\ldots\rightarrow\xrightarrow{(2,1)}.$$

Now we can prove the following proposition.

PROPOSITION 2. *Let A be an algebra satisfying the (S)-condition and $\beta(A) \leqslant 2$. Moreover let each indecomposable A-module be almost multiplicity-free. If N is an indecomposable A-module then N is linear if and only if for each noninjective factor module X of N the module $E_A(X)/X$ is uniserial and the natural morphism $E_A(X)/\mathrm{soc}_A(X) \to E_A(X)/X$ is a split epimorphism.*

Proof. First assume that N is linear. Then we have either $\mathrm{Hom}_A(U, V) \neq 0$ or $\mathrm{Hom}_A(V, U) \neq 0$ for any A-modules U, V belonging to $\mathscr{X}_A(N)$. Let X be a nonzero noninjective factor of N.

We shall prove that $E_A(X)/X$ is a uniserial module and the natural map $E_A(X)/\mathrm{soc}_A(X) \to E_A(X)/X$ is a split epimorphism.

First assume that X is a simple module. Let us observe that it is enough to prove that $E_A(X)$ is uniserial. If it is not, so then there are two uniserial submodules S and T in $E_A(X)$ whose intersection is simple, $S+T = \mathrm{rad}_A(E_A(X))$ if $E_A(X)$ is projective and $S+T = E_A(X)$ if $E_A(X)$ is nonprojective. Thus S and T belong to $\mathscr{X}_A(N)$ and $\mathrm{Hom}_A(S, T) = 0 = \mathrm{Hom}_A(T, S)$ or $S \approx T$. If $\mathrm{Hom}_A(S, T) = 0 = \mathrm{Hom}_A(T, S)$ then we have a contradiction to the fact that N is linear. If $S \approx T$ then we have that either $X \approx N$ or X is a simple injective module by the biseriality of A and our implication holds in this case too.

Now assume that X is nonsimple. Let $i\colon X \to E_A(X)$ be the canonical injection and we put

$$U = X/\mathrm{soc}_A(X), \qquad V = E_A(X).$$

It is clear that U and V belong to $\mathscr{X}_A(N)$. Let us observe that $\mathrm{Hom}_A(U, V) = 0$ because, in the opposite case, there exists a nonzero morphism $E_A(X)/\mathrm{soc}_A(X) \to E_A(X)$ and we have an oriented cycle in the Auslander–Reiten quiver Γ_A of A what is impossible because A satisfies the (S)-condition. Consequently there is a nonzero morphism $f\colon V \to U$. Moreover U is uniserial, $l(U) < l(V)$ and V is injective, so $fi \neq 0$ and f is not a monomorphism. Thus i and f induce two morphisms $g\colon U \to V/\mathrm{soc}_A(V)$ and $h\colon V/\mathrm{soc}_A(V) \to U$ such that $hg \neq 0$. But hg must be an isomorphism by the (S)-condition and g is a split epimorphism as well. Hence the natural morphism $E_A(X)/\mathrm{soc}_A(X) \to E_A(X)/X$ is a split epimorphism and $E_A(X)/X$ is uniserial.

Now suppose that $E_A(X)/X$ is a uniserial module for each nonzero noninjective factor module X of N and the natural map $E_A(X)/\mathrm{soc}_A(X) \to E_A(X)/X$ is a split epimorphism. We shall show that N is a linear module. In view of the (S)-condition it is enough to show that there is a chain of irreducible morphisms

$$N = X_0 \to X_1 \to \ldots \to X_t$$

such that $\mathscr{X}_A(N) = \{X_0, X_1, \ldots, X_t\}$ and valuations are of the form (d, d') where $dd' \leqslant 2$ because the other conditions of the linear modules are proved easy. Let us observe that if X belongs to $\mathscr{X}_A(N)$, then by [27, Lemma 2.2] there is a sequence

$$N \xrightarrow{g_1} Y_1 \to \ldots \to Y_{m-1} \xrightarrow{g_m} Y_m = X$$

of irreducible morphisms with $g_m \ldots g_1 \neq 0$ and obviously all modules Y_j belong to $\mathscr{X}_A(N)$. Thus we define $s(X)$, for every X from $\mathscr{X}_A(X)$, to be the minimal length of chains of irreducible morphisms

$$N \to X_1 \to \ldots \to X_s = X$$

with all X_i from $\mathscr{X}_A(N)$. We shall show that there is only one chain (up to isomorphism) of irreducible morphisms

$$N = X_0 \to X_1 \to \ldots \to X_{s(X)} = X$$

with X_i belonging to $\mathscr{X}_A(N)$, $i = 1, \ldots, s(X)$ such that $\mathrm{Tr}D(X_i)$ does not belong to $\mathscr{X}_A(N)$ for $i = 0, \ldots, s(X)-1$ or $\mathrm{Tr}D(X_i)$ belongs to $\mathscr{X}_A(N)$ and the short exact sequence of the form

$$0 \to X_i \to X \oplus X \to \mathrm{Tr}DX_i \to 0$$

is an Auslander–Reiten sequence. Moreover we shall show that this chain is the unique chain of irreducible morphisms between modules from $\mathscr{X}_A(N)$ which starts from N and is of length $s(X)$. We shall prove it by induction on $s(X)$. We can assume that $\alpha(A) \leqslant 2$. Indeed, let $A_A = P \oplus Q$ where Q is a direct sum of nonuniserial indecomposable projective-injective modules and P has not such direct summands. Then as in [7, Lemma 4.2] one proves that $I = \mathrm{soc}_A(Q)$ is a twosided ideal in A, $\alpha(A/I) \leqslant 2$ and any indecomposable A-module, which is not anihilated by I, is isomorphic to a direct summand in Q. We claim that any A-module from $\mathscr{X}_A(N)$ is anihilated by I so is an A/I-module.

Assume that Y is an indecomposable direct summand in Q, $f\colon N \to Y$ is a nonzero morphism and X is the image of f. Since N is uniserial, then Y is an indecomposable nonuniserial projective A-module, so f is not an epimorphism and X is a uniserial submodule in $\mathrm{rad}_A(Y)$. Thus $E_A(X) = Y$ and, by the assumption, Y/X is a uniserial module and the natural morphism $Y/\mathrm{soc}_A(Y) \to Y/X$ is a split epimorphism. On the other hand, $Y/\mathrm{soc}_A(Y)$ is an indecomposable nonuniserial A-module and we get a contradiction. Let us observe that there are no oriented cycles in the Auslander–Reiten graph $\Gamma_{A/I}$ of A/I if and only if there are no oriented cycles in the Auslander–Reiten graph Γ_A of A.

Let X be an A-module from $\mathscr{X}_A(N)$ with $s(X) = 1$. If N is injective then the natural morphism $N \to N/\mathrm{soc}_A(N)$ is the only irreducible morphism which starts at N, $X = N/\mathrm{soc}_A(N)$ and $\mathrm{Tr}D(N) = 0$. Suppose that N is noninjective.

Then X is the middle term of the Auslander–Reiten sequence

$$0 \to N \to X \to \mathrm{Tr}D(N) \to 0$$

by a dual lemma to Lemma 4.8 in [7] and $N \to X$, $X \to \mathrm{Tr}D(N)$ are the only morphisms which pass through X and whose composition starts at N and ends at $\mathrm{Tr}D(N)$. Consequently

$$N \to X \to \mathrm{Tr}D(N)$$

is the unique, starting at N and ending at $\mathrm{Tr}D(N)$, chain. This implies that $\mathrm{Hom}_A(N, \mathrm{Tr}D(N)) = 0$. Now suppose that there is the only (up to isomorphism) chain of irreducible morphisms, for some r and for every $s < r$,

$$N = X_0 \xrightarrow{f_1} X_1 \to \ldots \to X_{s-1} \xrightarrow{f_s} X_s$$

with all X_i belonging to $\mathscr{X}_A(N)$ and

$$\mathrm{Hom}_A(N, \mathrm{Tr}D(X_j)) = 0$$

or

$$0 \to X_j \to Y \oplus Y \to \mathrm{Tr}D(X_j) \to 0$$

is an Auslander–Reiten sequence. Let X be an A-module from $\mathscr{X}_A(N)$ with $s(X) = r$ and let

(*) $$N \xrightarrow{g_1} Y_1 \to \ldots \to Y_{r-1} \xrightarrow{g_r} Y_r = X$$

be a chain of irreducible morphisms with all Y_i from $\mathscr{X}_A(N)$. Since $s(Y_{r-1}) = r-1 < r$, the chain, which is formed by morphisms $g_1, \ldots, g_{r-1}$, is the unique chain of length $r-1$ which starts at N and contains modules from $\mathscr{X}_A(N)$ only and

$$\mathrm{Hom}_A(N, \mathrm{Tr}D(Y_j)) = 0$$

or

$$0 \to Y_j \to Y \oplus Y \to \mathrm{Tr}D(Y_j) \to 0$$

is an Auslander–Reiten sequence for $j < r-1$. Moreover $\alpha(A) \leqslant 2$ and

$$\mathrm{Hom}_A(N, \mathrm{Tr}D(Y_{r-2})) = 0$$

or

$$0 \to Y_{r-2} \to Y_{r-1} \oplus Y_{r-1} \to \mathrm{Tr}D(Y_{r-2}) \to 0$$

is an Auslander–Reiten sequence, the chain (*) is also the unique chain of length r, starting from N and containing only modules from $\mathscr{X}_A(N)$ as well. Now suppose that $\mathrm{Tr}D(Y_{r-1})$ belongs to $\mathscr{X}_A(N)$. Thus, in view of the above remarks, it follows that either

$$0 \to Y_{r-1} \to Y \oplus Y \to \mathrm{Tr}D(Y_{r-1}) \to 0$$

is an Auslander–Reiten sequence or there is an irreducible morphism $h\colon Y_r \to \mathrm{Tr}D(Y_{r-1})$ with $hg_r \dots g_1 \neq 0$. This implies that there is a commutative diagram of irreducible morphisms

$$\begin{array}{ccc} Y_{r-1} & \xrightarrow{g_r} & Y_r \\ \Big\downarrow u & & \Big\downarrow h \\ \mathrm{Tr}D(Y_{r-2}) & \xrightarrow{v} & \mathrm{Tr}D(Y_{r-1}) \end{array}$$

where $\mathrm{Tr}D(Y_{r-2}) \not\approx Y_r$. Hence $vug_{r-1} \dots g_1 \neq 0$ implies $hg_r \dots g_1 \neq 0$ and we get a contradiction because $\mathrm{Hom}_A(N, \mathrm{Tr}D(Y_{r-2})) = 0$.

Since A is of finite representation type, there is a chain of irreducible morphisms of the form

$$N \to X_0 \to \dots \to X_t$$

such that $\mathscr{X}_A(N) = \{N, X_0, \dots, X_t\}$. Since $\Gamma_{A/I}$ has not any oriented cycles, so $\mathrm{End}_A(X_i)$ are division algebras. Moreover, valuations are (d, d') with $dd' \leqslant 2$ in an obvious way, because A is biserial and every indecomposable A-module is almost multiplicity-free. Moreover, there is neither the subscheme of the form $\xrightarrow{(1,2)} \to \dots \to \xrightarrow{(1,2)}$ nor the subscheme of the form $\xrightarrow{(2,1)} \to \dots \to \xrightarrow{(2,1)}$ in $\tilde{\mathscr{X}}_A(N)$ because all indecomposable A-modules are almost multiplicity-free. This finishes the proof of Proposition 2. ∎

Now we are going to describe valued schemes of vector space categories appearing in one point extensions of biserial algebras satisfying the (S)-condition to biserial algebras satisfying the (S)-condition also. For our aim we can assume that the considered algebras have no locally faithful Π-sequences in their K-species with relations where Π is a K-species of an hereditary algebra of infinite representation type. Such algebras will be called *strongly biserial algebras*. Then it is not hard to verify that for any indecomposable projective module P over a strongly biserial algebra one of the conditions below is satisfied:

(i) P is uniserial.

(ii) P is noninjective nonuniserial and $\mathrm{rad}_A(P) = N \oplus T$, where $N \not\approx T$ and $\mathrm{End}_A(P) \approx \mathrm{End}_A(N) \approx \mathrm{End}_A(T)$.

(iii) P is noninjective nonuniserial and $\mathrm{rad}_A(P) = N \oplus N$, where $\dim_{\mathrm{End}_A(N)} \mathrm{End}_A(P) = 2$.

(iv) P is injective nonuniserial and $\mathrm{rad}_A(P)/\mathrm{soc}_A(P) = N \oplus T$, $N \not\approx T$, $\mathrm{End}_A(P) \approx \mathrm{End}_A(N) \approx \mathrm{End}_A(T)$.

(v) P is injective nonuniserial and $\mathrm{rad}_A(P)/\mathrm{soc}_A(P) = N \oplus N$, where $\dim_{\mathrm{End}_A(N)} \mathrm{End}_A(P) = 2$.

A B-module M_B over a strongly biserial algebra B satisfying the

(S)-condition is called *admissible* if and only if the following algebra

$$A = \begin{bmatrix} D & M_B \\ 0 & B \end{bmatrix}$$

is strongly biserial and satisfies the (S)-condition.

We are going to prove the following proposition now.

PROPOSITION 3. *Let B be a strongly biserial algebra which satisfies the (S)-condition. Moreover, let $\beta(B) \leqslant 2$ and all indecomposable B-modules be almost multiplicity-free. If M_B is an admissible B-module then the vector space category $\mathscr{C}_D(M_B)$ has the valued scheme $\vec{\mathfrak{X}}_B(M_B)$ one of the following forms:*

(a) *$\vec{\mathfrak{X}}_B(M_B)$ is linear if P is uniserial;*

(b) *$\vec{\mathfrak{X}}_B(M_B)$ is a disjoint union of two linear subschemes $\vec{\mathfrak{X}}_B(T_B)$, $\vec{\mathfrak{X}}_B(N_B)$ such that there is no arrow which connects any point from $\vec{\mathfrak{X}}_B(T_B)$ with any point from $\vec{\mathfrak{X}}_B(N_B)$, where P is noninjective nonuniserial, $\operatorname{rad}_A(P) = N \oplus T$ and $\operatorname{End}_A(P) \approx \operatorname{End}_A(T) \approx \operatorname{End}_A(N)$. Moreover, at most one of these two subschemes has nontrivial valuations;*

(c) *$\vec{\mathfrak{X}}_B(M_B)$ is a union of two linear subschemes having only one joint point M_B and, moreover, at most one of the linear subschemes has nontrivial valuations, if P is nonuniserial injective and $\operatorname{End}_A(P) \approx \operatorname{End}_A(\operatorname{rad}_A(P))$;*

(d) *$\vec{\mathfrak{X}}_B(M_B)$ is linear without nontrivial valuations, if P is nonuniserial noninjective, $\operatorname{rad}_A(P) = M_B = N \oplus N$ and $\dim_{\operatorname{End}_A(N)} \operatorname{End}_A(P) = 2$;*

(e) *$\vec{\mathfrak{X}}_B(M_B)$ is linear with only one nontrivial valuation $M_B \xrightarrow{(2,1)} X_1$, if P is nonuniserial injective, $\operatorname{rad}_A(P)/\operatorname{soc}_A(P) = N \oplus N$ and $\dim_{\operatorname{End}_A(N)} \operatorname{End}_A(P) = 2$; where P is the unique indecomposable projective A-module which is not a B-module.*

Proof. First observe that $M_B = \operatorname{rad}_A(P)$. Let V be a uniserial submodule of M_B such that $\operatorname{top}_B(V) \to \operatorname{top}_B(M)$ is nonzero. We shall prove that the B-module V is linear. It is enough to prove, by Proposition 2, that for each nonzero noninjective factor module X of V, $E_B(X)/X$ is uniserial and the natural morphism $E_B(X)/\operatorname{soc}_B(X) \to E_B(X)/X$ is a split epimorphism.

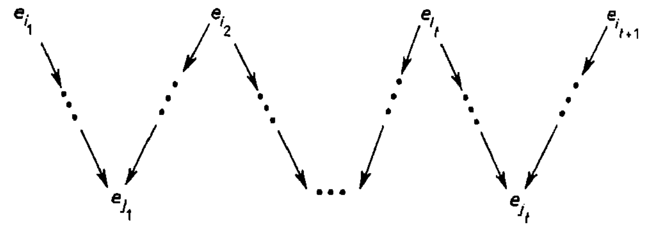
Let X be a nonzero noninjective factor module of V. Since $\operatorname{Hom}_A(P, P') = 0$ for any indecomposable projective A-module P' which is nonisomorphic to P, so we obtain that either $E_A(X)$ is uniserial and $X = \operatorname{rad}_A(E_A(X))$ or there are two uniserial submodules U and W in $E_A(X)$ such that $E_A(X) = U + W$, $X = \operatorname{rad}_A(U)$ and $\operatorname{soc}_A(X)$ is the intersection of U and W. Thus if X is simple, $E_B(X)$ is uniserial, $E_B(X)/X$ is also uniserial as well and obviously the natural morphism $E_B(X)/\operatorname{soc}_B(X) \to E_B(X)/X$ is a split epimorphism. Assume that X is nonsimple. Since X is noninjective in mod-B, $E_A(X)$ is nonuniserial, $E_B(X) = X + W$, $E_B(X)/X = N/\operatorname{soc}_B(W)$ is uniserial and the natural morphism $E_B(X)/\operatorname{soc}_B(X) \to E_B(X)/X$ is a split epimorphism. Consequently V is a linear B-module.

(a) If P is uniserial, then M_B is linear and by Proposition 2 $\vec{\mathscr{X}}_B(M_B)$ is linear.

(b) and (c). Let us observe that if P is nonuniserial and injective, then M_B is indecomposable nonuniserial injective B-module, $M/\mathrm{soc}_B(M) = N \oplus T$, N and T are two linear uniserial B-modules and there are two irreducible morphisms $M \to N$ and $M \to T$. Consequently, it is enough to prove that if P is nonuniserial noninjective and $\mathrm{rad}_A(P) = N \oplus T$, then the valued schemes $\vec{\mathscr{X}}_B(N)$ and $\vec{\mathscr{X}}_B(T)$ have no common modules.

The discussion above implies that $\vec{\mathscr{X}}_B(N)$ and $\vec{\mathscr{X}}_B(T)$ do not contain nonuniserial projective-injective B-modules. Thus, in our identification of mod-B with $\mathrm{mod}_K(Q_{\Sigma_B}, R_B)$, X is a sincere representation of a linear subgraph Q' of Q_{Σ_B} such that X has a nonzero space at each point of Q', where $\mathrm{mod}_K(Q_{\Sigma_B}, R_B)$ denotes the category of representations of the graph Q_{Σ_B} with relations R_B. If X is multiplicity-free, then Q' does not contain any relations different from μ or π. Let $f\colon N \to X$ and $g\colon T \to X$ be two nonzero morphisms and let $Y = \mathrm{im}\, f$ and $Z = \mathrm{im}\, g$. Thus since B satisfies the (S)-condition, it is not hard to see that Q' is of the form:

(∗)

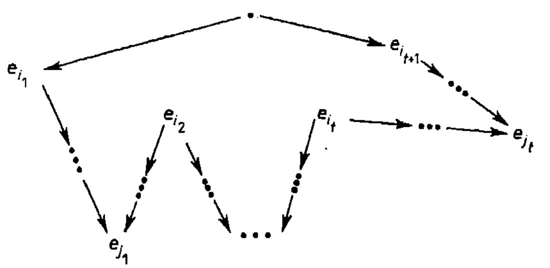


with some relations, Y and Z are the unique sincere multiplicity-free representations of graphs

(∗∗) $\quad e_{i_1} \to \dots \to e_{j_1}$ and $e_{i_{t+1}} \to \dots \to e_{j_t}$.

If X is almost multiplicity-free but not multiplicity-free, then by the assumption, Q' can contain some relations from R_B.

Let $f\colon N \to X$ and $g\colon T \to X$ be two nonzero morphisms and let $Y = \mathrm{im} f$, $Z = \mathrm{im}\, g$. Then N and T are linear, because B is strongly biserial and satisfies the (S)-condition, and it is not hard to see that there is a subgraph Q'' of the form (∗) and Y', Z' are the unique multiplicity-free sincere representations of graphs (∗∗). In both cases $\mathrm{rad}_A(P) = N \oplus T$. Hence there is a full subgraph of the form



in (Q_{Σ_A}, R_A). Thus A can neither satisfy the (S)-condition nor be strongly biserial and we have a contradiction.

Let us assume in the case (b) that each of $\vec{\mathscr{X}}_B(N)$ and $\vec{\mathscr{X}}_B(T)$ has nontrivial valuations. Let (d_i, d_i') be such that $d_i d_i' = 2$ and i is minimal with this property in $\vec{\mathscr{X}}_B(N)$. Let (d_j, d_j') be such that $d_j d_j' = 2$ and j is minimal with this property in $\vec{\mathscr{X}}_B(T)$. Then there is a full subgraph of the form

$$\xrightarrow{(d_i,d_i')} \to \dots \to \xrightarrow{(d_j,d_j')}$$

in (Q_{Σ_A}, R_A) which does not contain any relation from R_A. This implies that A is not a strongly biserial algebra and we get a contradiction. This finishes the proof of the cases (b) and (c).

The proof of the cases (d) and (e) is similar and we omit it. The proposition is also proved. ■

Now we are going to describe indecomposable objects of the subspace category $\mathscr{U}(\mathscr{C}_D(M_B))$ to apply this description to the proof of the main theorem.

Following [30], we associate to the vector space category $\mathscr{C}_D(M_B)$ the right peak ring

$$R_{\mathscr{C}} = \begin{bmatrix} G & {}_GR_D \\ 0 & D \end{bmatrix},$$

where $G = \mathrm{End}_B(\bigoplus_{j=1}^n X_j)$ and ${}_GR_D = {}_G|X_1 \oplus \dots \oplus X_n|_D$ if $X_1, \dots, X_n$ are all pairwise nonisomorphic indecomposable objects from $\mathscr{C}_D(M_B)$. It is not hard to verify that in view of Proposition 3, the right peak ring $R_{\mathscr{C}}$ is a finite--dimensional K-algebra in our considerations. Moreover $R_{\mathscr{C}}$ is artinian and schurian too, because $\mathrm{End}_B(X_i)$ are division rings. From [32] we know that $R_{\mathscr{C}}$ is of the following form

$$R_{\mathscr{C}} \approx \begin{bmatrix} F_1 & {}_1R_2 \cdots {}_1R_n & {}_1R_{n+1} \\ {}_2R_1 & F_2 \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & \\ {}_nR_1 & {}_nR_2 \cdots F_n & {}_nR_{n+1} \\ 0 & 0 \dots 0 & D \end{bmatrix}.$$

Let $R_{\mathscr{C}} = P_1 \oplus \dots \oplus P_n \oplus P_{n+1}$ be a decomposition of $R_{\mathscr{C}}$ into a direct sum of indecomposable right ideals and P_{n+1} is the simple one. Then we put

$$F_i = \mathrm{End}(P_i), \quad {}_iM_j = \mathrm{Hom}_{R_{\mathscr{C}}}(P_j, P_i).$$

We associate to $R_{\mathscr{C}}$ a *value scheme* (I_R, d_R), where $I_R = \{1, \dots, n+1\}$ and d_R is a pair of $(n+1)\times(n+1)$-matrices (d_{ij}), (d_{ij}') with

$$d_{ij} = \dim({}_iM_j)_{F_j}, \quad d_{ij}' = \dim_{F_i}({}_iM_j)$$

for $i \neq j$. We put $d_{ii}' = d_{ii} = 0$ for all i. We consider I_R with a set of valued dashed arrows

$$i \overset{(d_{ij}, d'_{ij})}{\dashrightarrow} j$$

with nonzero valuations. We write the arrow as a continuous one if there is no $t \neq i, j$ with $d_{it} \neq 0$ and $d_{tj} \neq 0$. We omit the valuations over the arrow if they are both equal one. If an arrow is uniquely determined by other ones we do not write it.

It is not hard to see that, in view of Proposition 3, the value scheme (I_R, d_R) is of the form $(I, d)^* \cup \{n+1\}$ and we have continuous arrows from the maximal incomparable points of $(I, d)^*$ to the point $n+1$ in our situation, where $(I, d)^*$ is the dual valued scheme to (I, d). Moreover we have some valuations over the additional arrows such that $d_{i(n+1)} d'_{i(n+1)} \leqslant 2$. The number of the additional arrows is equal one or two.

We call an $R_{\mathscr{C}}$-module X a *socle projective module* if $\mathrm{soc}_{R_{\mathscr{C}}}(X)$ is a projective and an essential submodule of X. We shall denote by $\mathrm{mod}_{\mathrm{sp}}\text{-}R_{\mathscr{C}}$ the full subcategory of $\mathrm{mod}\text{-}R_{\mathscr{C}}$ consisting of all socle projective modules.

Let $R_{\mathscr{C}} = P_1 \oplus \ldots \oplus P_{n+1}$ be the decomposition $R_{\mathscr{C}}$ into a direct sum of indecomposable right ideals and let $Q_j = E_{R_{\mathscr{C}}}(\mathrm{top}_{R_{\mathscr{C}}}(P_j))$, then we put

$$R_{\mathscr{C}}^{\sim} = \mathrm{End}_{R_{\mathscr{C}}}(Q_1 \oplus \ldots \oplus Q_n)$$

and by [32] we have that $(I_{R^\sim}, d_{R^\sim})$ coincides with (I_R, d_R). Following [32] we can define

$$(R_{\mathscr{C}})_* = \left((R_{\mathscr{C}}^{\mathrm{op}})^{\sim}\right)^{\mathrm{op}}$$

and we have the functor

$$G\colon \mathscr{U}(\mathscr{C}_D(M_B)) \to \mathrm{mod}_{\mathrm{sp}}\text{-}(R_{\mathscr{C}})_*$$

which induces a representation equivalence of categories $\mathscr{U}_0(\mathscr{C}_D(M_B))$ and $\mathrm{mod}_{\mathrm{sp}}\text{-}(R_{\mathscr{C}})_*$ where $\mathscr{U}_0(\mathscr{C}_D(M_B))$ is a full subcategory of $\mathscr{U}(\mathscr{C}_D(M_B))$ consisting of objects without direct summand of the form $(0, X, 0)$ where X is an object in $\mathscr{C}_D(M_B)$.

If (V, X, f) is an object of $\mathscr{U}(\mathscr{C}_D(M_B))$ we define its coordinate vector $\mathrm{cdn}(V, X, f) \in Z^{n+1}$ by the formula

$$\mathrm{cdn}(V, X, f) = (s_1, \ldots, s_n;\ s_{n+1})$$

where $s_{n+1} = \dim V_D$ and we have the following isomorphism

$$X \approx X_1^{s_1} \oplus \ldots \oplus X_n^{s_n}.$$

For any module X in $\mathrm{mod}_{\mathrm{sp}}\text{-}R_{\mathscr{C}}$ such that P_{n+1} is not a direct summand of X we put

$$\mathrm{cdn}(X) = (s_1, \ldots, s_n;\ s_{n+1})$$

where the projective cover of X has the form

$$P_{R_{\mathscr{C}}}(X) = P_1^{s_1} \oplus \ldots \oplus P_n^{s_n}$$

and $s_{n+1} = \dim(\mathrm{Hom}_{R_{\mathscr{C}}}(P_{n+1}, X))_D$. From [32, Theorem 3.3] we have the following useful lemma.

LEMMA 3. *G induces a representation equivalence of categories*

$$G: \mathscr{U}_0(\mathscr{C}_D(M_B)) \to \mathrm{mod}_{sp}\text{-}(R_{\mathscr{C}})_*$$

such that $\mathrm{cdn}(V, X, f) = \mathrm{cdn}(G(V, X, f))$ *for any object* (V, X, f) *in* $\mathscr{U}_0(\mathscr{C}_D(M_B))$.

Let (I_R, d_R) be a value scheme, $I_R = \{1, \ldots, n+1\}$. A value scheme $(I_T, \bar{d})$ is said to be an *upper subscheme* of (I_R, d_R) if $I_T \subset I_R$, the peak index of I_T is equal $n+1$ and $\bar{d}_{ij} = d_{ij}$, $\bar{d}'_{ij} = d'_{ij}$ for all $i, j \in I_T$.

Now we can prove the following proposition.

PROPOSITION 4. *Let $\mathscr{C}_D(M_B)$ be a linear vector space category such that $\dim_D \mathrm{End}_B(X) \leqslant 2$ for each indecomposable object X from $\mathscr{C}_D(M_B)$ and let $\vec{\mathscr{X}}_B(M_B)$ be its valued scheme. Then the following triples form a full list of indecomposable objects of the subspace category $\mathscr{U}(\mathscr{C}_D(M_B))$:*

(i) $Y = (0, Y, 0)$ *where Y is an indecomposable object from $\mathscr{C}_D(M_B)$.*

(ii) $(D, 0, 0)$.

(iii) $\underline{X} = (D, X, f)$ *where X is an indecomposable object from $\mathscr{C}_D(M_B)$ and f is either the identity morphism if $\mathrm{Hom}_B(M_B, X) \approx D$ or f is the diagonal morphism if $\dim_D \mathrm{Hom}_B(M_B, X) = 2$.*

(iv) $\underline{\underline{X}} = (E, X, f)$ *where X is an indecomposable object from $\mathscr{C}_D(M_B)$, $E \approx \mathrm{Hom}_B(M_B, X)$, $\dim_D E = 2$, and f is the identity morphism.*

Proof. From $\vec{\mathscr{X}}_B(M_B)$ we get (I_R, d_R) immediately. It is not hard to see that (I_R, d_R) has not any upper subscheme of the form 5(a)–5(f) from [30, Theorem 3,1] so $(R_{\mathscr{C}})_*$ is sp-representation-finite. But in our situation $(R_{\mathscr{C}})_*$ is a nice K-algebra and we can compute all indecomposable socle projective $(R_{\mathscr{C}})_*$-modules directly. Using Lemma 3 it is easy to verify that the objects of the form (i)–(iv) are all indecomposable objects of the subspace category $\mathscr{U}(\mathscr{C}_D(M_B))$. ∎

PROPOSITION 5. *Let $\mathscr{C}_D(M_B)$ be a vector space category such that $\dim_D \mathrm{End}_B(X) \leqslant 2$ for each indecomposable object X from $\mathscr{C}_D(M_B)$ and whose valued scheme $\vec{\mathscr{X}}_B(M_B)$ is a disjoint union of two linear subschemes $\vec{\mathscr{X}}_B(T_B)$ and $\vec{\mathscr{X}}_B(N_B)$ and at most one of them, say $\vec{\mathscr{X}}_B(T_B)$, has nontrivial valuations. Then the following triples form a full list of indecomposable objects in the subspace category $\mathscr{U}(\mathscr{C}_D(M_B))$:*

(i) $Y = (0, Y, 0)$ *where Y is an indecomposable object from $\mathscr{C}_D(M_B)$.*

(ii) $(D, 0, 0)$.

(iii) $\underline{X}_i = (D\ X_i, f)$ *and* f *is the identity morphism if* $\mathrm{Hom}_B(M_B, X_i) \approx D$ *or* f *is the diagonal morphism if* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, *where* X_i *is an indecomposable object from* $\mathscr{C}_D(M_B)$.

(iv) $Z_{ij} = (D, X_i \oplus Y_j, f)$ *and* f *is the diagonal morphism if* $D \approx \mathrm{Hom}_B(M_B, X_i) \approx \mathrm{Hom}_B(M_B, Y_j)$ *or* f *is the composition of the diagonal morphism with two projections if* $D \not\approx \mathrm{Hom}_B(M_B, X_i)$ *or* $D \not\approx \mathrm{Hom}_B(M_B, Y_j)$, *where* X_i *is an indecomposable object lying in* $\mathscr{X}_B(T_B)$ *and* Y_j *is an indecomposable object lying in* $\mathscr{X}_B(N_B)$.

(v) $\underline{X}_i = (E, X_i, f)$ *if* $\mathrm{Hom}_B(M_B, X_i) \approx E$, $\dim_D E = 2$, *and* f *is the identity morphism, where* X_i *is an indecomposable objects lying in* $\mathscr{X}_B(T_B)$.

(vi) $\underline{Z}_{ij} = (E, X_i \oplus Y_j, f)$ *and* f *is the diagonal morphism if* $\mathrm{Hom}_B(M_B, X_i) \approx \mathrm{Hom}_B(M_B, Y_j) \approx E$, $\dim_D E = 2$, *or* f *is the composition of the diagonal morphism with two projections if* $\mathrm{Hom}_B(M_B, X_i) \approx E$, $\mathrm{Hom}_B(M_B, Y_j) \approx D$, $\dim_D E = 2$, *where* X_i *is an indecomposable object lying in* $\mathscr{X}_B(T_B)$ *and* Y_j *is an indecomposable object lying in* $\mathscr{X}_B(N_B)$.

(vii) $\underline{Z}_{i,(j,t)} = (E, X_i \oplus Y_j \oplus Y_t, f)$ *and* f *is the composition of the diagonal morphism with three projections if* $\mathrm{Hom}_B(M_B, X_i) \approx E$, $\mathrm{Hom}_B(M_B, Y_j) \approx \mathrm{Hom}_B(M_B, Y_t) \approx D$, $\dim_D E = 2$, *where* X_i *is an indecomposable object lying in* $\mathscr{X}_B(T_B)$ *and* Y_j, Y_t *are indecomposable objects lying in* $\mathscr{X}_B(N_B)$.

Proof. Similar arguments, as in the proof of Proposition 4, prove the proposition. We left the details of the proof to the reader. ■

PROPOSITION 6. *Let* $\mathscr{C}_D(M_B)$ *be a vector space category such that* $\mathrm{End}_B(X) \approx F$ *and* $\dim_F D = 2$. *Moreover let* $\vec{\mathscr{X}}_B(M_B)$ *be a linear valued scheme of* $\mathscr{C}_D(M_B)$ *without nontrivial valuations. Then the following triples form a full list of all indecomposable objects of the subspace category* $\mathscr{U}(\mathscr{C}_D(M_B))$:

(i) $Y = (0, Y, 0)$ *where* Y *is an indecomposable object from* $\mathscr{C}_D(M_B)$.

(ii) $(D, 0, 0)$.

(iii) $\underline{X} = (D, X, f)$ *where* f *is an epimorphism from* $D \approx F \times F$ *onto* F *given by the formula* $f(x, y) = x + y$ *and* X *is an indecomposable object from* $\mathscr{C}_D(M_B)$.

(iv) $Z_{ij} = (D, X \oplus Y, f)$ *where* f *is the identity morphism and* X, Y *are indecomposable objects from* $\mathscr{C}_D(M_B)$.

Proof. The proof of the proposition is similar to the proof of Proposition 4 and we left it to the reader. ■

The description of indecomposable objects of subspace categories of the form $\mathscr{U}(\mathscr{C}_D(M_B))$ given in Propositions 4–6 will be applied in the next section for studying the category of finitely generated modules over biserial algebras which are closely related to the biserial algebras being quatients of hereditary algebras.

4. Representation-finite biserial algebras with the (S)-condition

In this section we shall describe representation-finite biserial algebras which satisfy the (S)-condition. We shall proceed by induction on the number of indecomposable projective modules, using one point extension procedures. First we shall describe the category mod-A where A is obtained by a one point extension of a known algebra B. Let B be a strongly biserial representation-finite algebra satisfying the (S)-condition, M_B an admissible B-module and let A be the algebra of the form

$$A = \begin{bmatrix} D & M_B \\ 0 & B \end{bmatrix}.$$

We denote by P the unique maximal projective A-module which is not a B-module. For the conveniance of the reader, we recall (see [31]) that there is an additive functor

$$\Phi\colon \text{mod-}A \to \mathcal{U}(\mathcal{C}_D(M_B))$$

as follows. Given a module $X_A = (X'_D, X''_B, f)$ and let $X''_B = X'''_B \oplus Y_B$ where $\mathrm{Hom}_B({}_DM_B, Y_B) = 0$ and X'''_B has no summands Z_B with $\mathrm{Hom}_B({}_DM_B, Z_B) = 0$. Then we put

$$\Phi(X_A) = (X'_D, \bar{X}'''_B, \bar{f})$$

where

$$\bar{f}\colon X'_D \to \mathrm{Hom}_B({}_DM_B, X'''_B)$$

is a D-linear morphism adjoint to $f\colon X'_D \otimes_D M_B \to X''_B$. If $\varphi\colon X \to Y$ is an A-homomorphism, then we define $\Phi(\varphi)$ as the morphism induced by φ. We have the following result (see [31], Proposition 2.1]).

Proposition 7. *The functor Φ is full and dense. Φ induces a representation equivalence*

$$\Phi'\colon \mathcal{M}_A({}_DM_B) \to \mathcal{U}(\mathcal{C}_D(M_B))$$

where $\mathcal{M}_A({}_DM_B)$ is the full subcategory of mod-A *consisting of modules having no summands* $(0, X''_B, 0)$ *with* $\mathrm{Hom}_B({}_DM_B, X''_B) = 0$.

By Proposition 7 we can use Propositions 4–6 in order to describe the category mod-A. We keep the above notations.

Proposition 8. *If P is uniserial then the following triples form a full list of nonisomorphic indecomposable objects in* mod-A.

(i) $Y = (0, Y, 0)$ *where Y is an indecomposable B-module.*
(ii) $\mathrm{top}_A(P) = (D, 0, 0)$.
(iii) $\underline{X} = (D, X, f)$ *where X is an indecomposable B-module such that*

$\mathrm{Hom}_B(M_B, X) \neq 0$ *and* f *is either the identity morphism if* $\mathrm{Hom}_B(M_B, X) \approx D$ *or* f *is the diagonal morphism if* $\dim_D \mathrm{Hom}_B(M_B, X) = 2$.

(iv) $\underline{\underline{X}} = (E, X, f)$ *where* X *is an indecomposable* B*-module with* $\mathrm{Hom}_B(M_B, X) \neq 0$, $E \approx \mathrm{Hom}_B(M_B, X)$, $\dim_D E = 2$, *and* f *is the identity morphism.*

Moreover if the set $\mathscr{X}_B(M_B)$ *consists of modules* $X_0, \ldots, X_t$ *which are ordered in such a way that the Auslander–Reiten graph* Γ_B *contains a subgraph*

$$X_0 \to \ldots \to X_t,$$

then we have the following description of the minimal left and right almost split morphisms and Auslander–Reiten sequences in mod-A.

(1) *For each indecomposable* B*-module* Y *nonisomorphic to any* X_i *(resp. nonisomorphic to any* $\mathrm{Tr}D(X_i)$*), a minimal left (right) almost split morphism starting from* Y *(ending in* Y*) in* mod-B *is also a minimal left (right) almost split morphism in* mod-A.

(2) *The natural short exact sequences*

$$0 \to X_i \to \underline{X}_i \oplus X_{i+1} \to \underline{X}_{i+1} \to 0,$$
$$0 \to X_t \to \underline{X}_t \to \mathrm{top}_A(P) \to 0$$

are Auslander–Reiten sequences in mod-A.

(3) *If* X_i *is noninjective in* mod-B *and* $D \approx \mathrm{Hom}_B(M_B, X_i)$, *then the natural short exact sequence*

$$0 \to \underline{X}_i \to \underline{X}_{i+1} \oplus \mathrm{Tr}D(X_{i-1}) \to \mathrm{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-A.

(4) *If* X_i *is a noninjective* B*-module and* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, *then the natural short exact sequence*

$$0 \to \underline{X}_i \to \underline{\underline{X}}_i \oplus \mathrm{Tr}D(X_{i-1}) \to \underline{\mathrm{Tr}D(X_{i-1})} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(5) *If* X_i *is a nonjective* B*-module and* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, *then the natural short exact sequence*

$$0 \to \underline{\underline{X}}_i \to \underline{\mathrm{Tr}D(X_{i-1})} \oplus \underline{\mathrm{Tr}D(X_{i-1})} \to \mathrm{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-A.

(6) *If* X_i *is injective in* mod-B, $D \approx \mathrm{Hom}_B(M_B, X_i)$ *and* $i < t$, *then the morphism*

$$\underline{X}_i \to \underline{X}_{i+1} \oplus \mathrm{Tr}D(X_{i-1})$$

is a minimal left almost split morphism in mod-A.

(7) *The natural morphism*

$$\underline{X}_t \to \mathrm{top}_A(P) \oplus \mathrm{Tr}D(X_{t-1})$$

is a minimal left almost split morphism in mod-A.

Proof. By Proposition 3 we know that the vector space category $\mathscr{C}_D(M_B)$ has a linear valued scheme $\vec{\mathscr{X}}_B(M_B)$. From Proposition 4 we have also the description of indecomposable objects in mod-A. Let us observe that, for a module Y from mod-B, there is an irreducible morphism of the form $\underline{X}_i \to Y$ if and only if $i > 0$ and Y is isomorphic to $\mathrm{Tr}D(X_{i-1})$. Indeed, each such a morphism is induced by a morphism $f\colon X_i \to Y$, $\mathrm{Hom}_B(M_B, Y) = 0$, so from the minimality of irreducible morphisms it is possible when $i > 0$ and $Y \approx \mathrm{Tr}D(X_{i-1})$.

Further, it is clear that each morphism of the form $Y \to \underline{X}_i$, where Y is an indecomposable B-module, factors through the module X_i and the natural morphism $\underline{X}_i \to \underline{X}_j$, $j > i$, does not factor through any indecomposable B-module. Thus the natural morphism $X_i \to \underline{X}_i$ and the natural morphism $\underline{X}_i \to \underline{X}_{i+1}$ are irreducible in mod-A provided it is not the situation $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$. If it is so then there is not an irreducible morphism $Y \to \underline{\underline{X}}_i$, because any morphism $Y \to \underline{\underline{X}}_i$ factors through the morphism $\underline{X}_i \to \underline{\underline{X}}_i$ and the morphism $\underline{X}_i \to \underline{\underline{X}}_i$ is irreducible. Moreover, as above, the morphism $\underline{X}_i \to \mathrm{Tr}D(X_{i-1})$ is an irreducible morphism too. Further, if $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, then, from the fact that A is a strongly biserial algebra, we can consider two cases:

(a) M_B is not an hereditary injective B-module. Thus $\mathrm{End}_B(M_B) \approx D$, but $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, hence either

$$0 \to X_i \to \mathrm{Tr}D(X_{i-1}) \oplus \mathrm{Tr}D(X_{i-1}) \to \mathrm{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-B or the following short exact sequence

$$0 \to X_i \to X_{i+1} \oplus \mathrm{Tr}D(X_{i-1}) \to \mathrm{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-B. In the fist case $X_{i+1} \approx \mathrm{Tr}D(X_{i-1})$ and then $\underline{X}_i \to \mathrm{Tr}D(X_{i-1})$ is an irreducible morphism and the morphism $\mathrm{Tr}D(X_{i-1}) \to \underline{\mathrm{Tr}D(X_{i-1})}$ is an irreducible morphism too. Hence

$$0 \to \underline{X}_i \to \underline{\underline{X}}_i \oplus \mathrm{Tr}D(X_{i-1}) \to \underline{\mathrm{Tr}D(X_{i-1})} \to 0$$

is an Auslander–Reiten sequence in mod-A.

In the other case if $\mathrm{End}_B(X_{i+1}) \approx D$, then $\dim_D \mathrm{Hom}_B(M_B, X_{i+1}) = 1$, and then $X_{i+1} \approx \mathrm{Tr}D(X_{i-1})$, because acting $\mathrm{Hom}_B(M_B, -)$ on the short exact sequence

$$0 \to X_i \to X_{i+1} \oplus \mathrm{Tr}D(X_{i-1}) \to \mathrm{Tr}D(X_i) \to 0$$

we get that $\mathrm{Hom}_B(M_B, \mathrm{Tr}D(X_{i-1})) \neq 0$ and from the fact that $\mathscr{C}_B(M_B)$ is a linear vector space category we know that $X_{i+1} \approx \mathrm{Tr}D(X_{i-1})$.

If $\mathrm{End}_B(X_{i+1}) \not\approx D$, then we must have $\dim_D \mathrm{Hom}_B(M_B, X_{i+1}) = 2$, but then $\mathrm{Hom}_B(M_B, X_{i+1}) \approx \mathrm{Hom}_B(M_B, X_i) \approx \mathrm{Hom}_B(X_i, X_{i+1})$. Thus the short exact sequence

$$0 \to D\mathrm{Tr}(X_{i+1}) \to X_i \oplus X_i \to X_{i+1} \to 0$$

is an Auslander–Reiten sequence in mod-B. Further the short exact sequence

$$0 \to X_{i+1} \to \mathrm{Tr}D(X_i) \oplus \mathrm{Tr}D(X_i) \to \mathrm{Tr}D(X_{i+1}) \to 0$$

is an Auslander–Reiten sequence in mod-B and then $X_{i+2} \approx \mathrm{Tr}D(X_i)$. But then we know, from the proof of Proposition 2, that $X_{i+1} \approx \mathrm{Tr}D(X_{i-1})$. Consequently, as above, the following short exact sequence

$$0 \to \underline{X}_i \to \underline{\underline{X}}_i \oplus \mathrm{Tr}D(X_{i-1}) \to \underline{\mathrm{Tr}D(X_{i-1})} \to 0$$

is an Auslander–Reiten sequence in mod-A.

Moreover observe that there is no irreducible morphism $\underline{\underline{X}}_i \to Y$ where Y is an indecomposable B-module, because such a morphism factors through $\underline{\underline{X}}_i \to \underline{\mathrm{Tr}D(X_{i-1})}$. We have also that the following short exact sequence

$$0 \to \underline{\underline{X}}_i \to \underline{\mathrm{Tr}D(X_{i-1})} \oplus \underline{\mathrm{Tr}D(X_{i-1})} \to \mathrm{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-A.

(b) M_B is an hereditary injective B-module. This case is trivial.

Moreover it is not hard to check that if X_i is an injective B-module then $\underline{X}_i$ or $\underline{\underline{X}}_i$ is an injective A-module. Hence the conditions (1)–(7) are a simple consequence of the above remarks and basic properties of Auslander–Reiten sequences and irreducible morphisms ([3]–[6]). The proof of the proposition is finished. ∎

PROPOSITION 9. *Let P be a noninjective nonuniserial A-module and* $\mathrm{rad}_A(P) = N \oplus T$, $D \approx \mathrm{End}_B(N) \approx \mathrm{End}_B(T)$. *Moreover let $\mathscr{X}_B(N_B)$ consist of modules $X_0, \ldots, X_r$ and let $\mathscr{X}_B(T_B)$ consist of modules $Y_0, \ldots, Y_l$. Let $\vec{\mathscr{X}}_B(T_B)$ be without nontrivial valuations. Then the following triples form a complete list of nonisomorphic indecomposable objects in* mod-A.

(i) $Y = (0, Y, 0)$ *where Y is an indecomposable B-module.*

(ii) $\mathrm{top}_A(P) = (D, 0, 0)$.

(iii) $\underline{X}_i = (D, X_i, f)$ *and f is the identity morphism if* $\mathrm{Hom}_B(M_B, X_i) \approx D$ *or f is the diagonal morphism if* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r$.

(iv) $\underline{Y}_j = (D, Y_j, f)$ *where f is the identity morphism,* $0 \leqslant j \leqslant l$.

(v) $Z_{ij} = (D, X_i \oplus Y_j, f)$ *and f is the diagonal morphism if* $\mathrm{Hom}_B(M_B, X_i) \approx \mathrm{Hom}_B(M_B, Y_j) \approx D$ *or f is the composition of the diagonal morphism with projections if* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $\mathrm{Hom}_B(M_B, Y_j) \approx D$, $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant l$.

(vi) $\underline{\underline{X}}_i = (E, X_i, f)$ *and f is the identity morphism if* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r$.

(vii) $\underline{Z}_{ij} = (E, X_i \oplus Y_j, f)$ *and f is the diagonal morphism if* $\mathrm{Hom}_B(M_B, X_i) \approx \mathrm{Hom}_B(M_B, Y_j) \approx E$ *and* $\dim_D E = 2$ *or f is the composition of the diagonal morphism with projections if* $E \approx \mathrm{Hom}_B(M_B, X_i)$, $\mathrm{Hom}_B(M_B, Y_j) \approx D$ *and* $\dim_D E = 2$, $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant l$.

(viii) $\underline{Z}_{i,(j,t)} = (E, X_i \oplus Y_j \oplus Y_t, f)$ *and* f *is the composition of the diagonal morphism with projections if* $\operatorname{Hom}_B(M_B, X_i) \approx E$, $\operatorname{Hom}_B(M_B, Y_j) \approx \operatorname{Hom}_B(M_B, Y_t) \approx D$ *and* $\dim_D E = 2$, $0 \leqslant i \leqslant r$, $0 \leqslant j, t \leqslant l$.

Moreover, if the modules $X_0, \dots, X_r$ *and* $Y_0, \dots, Y_l$ *are ordered in such a way that the Auslander–Reiten graph* Γ_B *contains two subgraphs*

$$X_0 \to \dots \to X_r, \qquad Y_0 \to \dots \to Y_l,$$

then we have the following description of minimal left and right almost split morphisms and Auslander–Reiten sequences in mod-A.

(1) *For each indecomposable B-module* Y *which is isomorphic neither to any* X_i *nor to any* Y_j *(resp. which is isomorphic neither to any* $\operatorname{Tr}D(X_i)$ *nor to any* $\operatorname{Tr}D(Y_j)$*), a minimal left (right) almost split morphism in* mod-B *is also a minimal left (right) almost split morphism in* mod-A.

(2) *If* Y_j *is a noninjective B-module, then the natural short exact sequence*

$$0 \to \underline{Y}_j \to \underline{Y}_{j+1} \oplus \operatorname{Tr}D(Y_{j-1}) \to \operatorname{Tr}D(Y_j) \to 0,$$

is an Auslander–Reiten sequence in mod-A.

(3) *If* X_i *is a noninjective B-module and* $D \approx \operatorname{Hom}_B(M_B, X_i)$, *then the natural short exact sequence*

$$0 \to \underline{X}_i \to \underline{X}_{i+1} \oplus \operatorname{Tr}D(X_{i-1}) \to \operatorname{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-A.

(4) *If* X_i *is a noninjective B-module and* $\dim_D \operatorname{Hom}_B(M_B, X_i) = 2$, *then the natural short exact sequences*

$$0 \to \underline{X}_i \to \underline{Z}_{i0} \oplus \operatorname{Tr}D(X_{i-1}) \to Z_{(i+1)0} \to 0,$$
$$0 \to \underline{X}_i \to \underline{\operatorname{Tr}D(X_{i-1})} \oplus \underline{\operatorname{Tr}D(X_{i-1})} \to \operatorname{Tr}D(X_i) \to 0$$

are Auslander–Reiten sequences in mod-A.

(5) *The natural short exact sequence*

$$0 \to Y_j \to Z_{0j} \oplus Y_{j+1} \to Z_{0(j+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A *for* $j = 0, \dots, l-1$.

(6) *If* $D \approx \operatorname{Hom}_B(M_B, X_i)$ *and* $0 \leqslant i \leqslant r-1$, *then the natural short exact sequence*

$$0 \to X_i \to Z_{i0} \oplus X_{i+1} \to Z_{(i+1)0} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(7) *If* $\dim_D \operatorname{Hom}_B(M_B, X_i) = 2$ *and* $0 \leqslant i \leqslant r-1$, *then the natural short exact sequence*

$$0 \to X_i \to \underline{Z}_{i0} \oplus \underline{Z}_{i0} \to \underline{Z}_{i,(0,0)} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(8) *If* $D \approx \mathrm{Hom}_B(M_B, X_i)$ *and* $0 \leqslant i \leqslant r-1$, $0 \leqslant j \leqslant l-1$, *then the following natural short exact sequence*

$$0 \to Z_{ij} \to Z_{(i+1)j} \oplus Z_{i(j+1)} \to Z_{(i+1)(j+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(9) *If* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$ *and* $0 \leqslant i \leqslant r-1$, $0 \leqslant j \leqslant l-1$, *then the natural short exact sequence*

$$0 \to Z_{ij} \to Z_{i(j+1)} \oplus \underline{\underline{Z}}_{i,(j,j)} \to \underline{\underline{Z}}_{i,(j,j+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(10) *If* $D \approx \mathrm{Hom}_B(M_B, X_i)$, $0 \leqslant i \leqslant r-1$, *then the natural short exact sequence*

$$0 \to Z_{il} \to Z_{(i+1)l} \oplus \underline{X}_i \to \underline{X}_{i+1} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(11) *If* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r-1$, *then the natural short exact sequence*

$$0 \to Z_{il} \to \underline{\underline{Z}}_{i,(0,l)} \oplus \underline{X}_i \to \underline{Z}_{i0} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(12) *If* $D \approx \mathrm{Hom}_B(M_B, X_r)$, *then the natural short exact sequence*

$$0 \to Z_{rj} \to Z_{r(j+1)} \oplus \underline{Y}_j \to \underline{Y}_{j+1} \to 0$$

is an Auslander–Reiten sequence in mod-A *for* $0 \leqslant j \leqslant l-1$.

(13) *If* $\dim_D \mathrm{Hom}_B(M_B, X_r) = 2$, *then the natural short exact sequence*

$$0 \to Z_{rj} \to \underline{\underline{Z}}_{r,(0,j)} \oplus Z_{r(j+1)} \to \underline{\underline{Z}}_{r,(0,j+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A *for* $0 \leqslant j \leqslant l-1$.

(14) *If* $D \approx \mathrm{Hom}_B(M_B, X_r)$, *then the following natural short exact sequence*

$$0 \to Z_{rl} \to \underline{X}_r \oplus \underline{Y}_l \to \underline{Z}_{r0} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(15) *If* $\dim_D \mathrm{Hom}_B(M_B, X_r) = 2$, *then the following natural short exact sequence*

$$0 \to Z_{rl} \to \underline{X}_r \oplus \underline{\underline{Z}}_{r,(0,l)} \to \underline{Z}_{r0} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(16) *If* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r-1$, $0 \leqslant j \leqslant l-1$, *then the following natural short exact sequence*

$$0 \to \underline{Z}_{ij} \to \underline{Z}_{i(j+1)} \oplus Z_{(i+1)j} \to Z_{(i+1)(j+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(17) *If* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r-1$, *then the following natural*

short exact sequence

$$0 \to \underline{Z}_{il} \to \underline{X}_i \oplus Z_{(i+1)l} \to \underline{X}_{i+1} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(18) *If* $\dim_D \mathrm{Hom}_B(M_B, X_r) = 2$, *then the following natural short exact sequence*

$$0 \to \underline{Z}_{rj} \to \underline{Z}_{r(j+1)} \oplus \underline{Y}_j \to \underline{Y}_{j+1} \to 0$$

is an Auslander–Reiten sequence in mod-A *for* $0 \leqslant j \leqslant l-1$.

(19) *If* $\dim_D \mathrm{Hom}_B(M_B, X_r) = 2$, *then the following natural short exact sequence*

$$0 \to \underline{Z}_{rl} \to \underline{X}_r \oplus \underline{Y}_l \to \mathrm{top}_A(P) \to 0$$

is an Auslander–Reiten sequence in mod-A.

(20) *If* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r-1$, $0 \leqslant j$, $t \leqslant l-1$, *then the following natural short exact sequence*

$$0 \to \underline{Z}_{i,(j,t)} \to \underline{Z}_{i,(j,t+1)} \oplus \underline{Z}_{i,(j+1,t)} \to \underline{Z}_{i,(j+1,t+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(21) *If* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, $0 \leqslant i \leqslant r-1$, $0 \leqslant j \leqslant l-1$, *then the following natural short exact sequence*

$$0 \to \underline{Z}_{i,(j,l)} \to \underline{Z}_{ij} \oplus \underline{Z}_{i,(j+1,j+1)} \to \underline{Z}_{i(j+1)} \to 0$$

is an Auslander–Reiten sequence in mod-A.

(22) *If* X_i, $i < r$, *is an injective B-module and* $D \approx \mathrm{Hom}_B(M_B\ X_i)$, *then the following natural morphism*

$$\underline{X}_i \to \underline{X}_{i+1} \oplus \mathrm{Tr}D(X_{i-1})$$

is a left minimal almost split morphism in mod-A.

(23) *If* Y_j, $j < l$, *is an injective B-module, then the following natural morphism*

$$\underline{Y}_j \to \underline{Y}_{j+1} \oplus \mathrm{Tr}D(Y_{j-1})$$

is a minimal left almost split morphism in mod-A.

(24) *If* X_i, $i < r$, *is an injective B-module and moreover* $\dim_D \mathrm{Hom}_B(M_B, X_i) = 2$, *then the following natural morphism*

$$\underline{X}_i \to \underline{\mathrm{Tr}D(X_{i-1})} \oplus \underline{\mathrm{Tr}D(X_{i-1})}$$

is a minimal left almost split morphism in mod-A.

(25) *The natural morphism*

$$\underline{Y}_l \to \mathrm{top}_A(P) \oplus \mathrm{Tr}D(Y_{l-1})$$

is a minimal right almost split morphism in mod-A.

(26) *If* $D \approx \mathrm{Hom}_B(M_B, X_r)$, *then the following natural morphism*

$$\underline{X}_r \to \mathrm{top}_A(P) \oplus \mathrm{Tr}D(X_{r-1})$$

is a minimal right almost split morphism in mod-A.

(27) *If* $\dim_D \mathrm{Hom}_B(M_B, X_r) = 2$, *then the following natural morphism*

$$\underline{\underline{X}}_r \to \mathrm{top}_A(P) \oplus \mathrm{Tr}D(X_{r-1})$$

is a minimal right almost split morphism in mod-A.

Proof. By Proposition 3, we know that the vector space category $\mathscr{C}_D(M_B)$ is a disjoint union of two linear vector space categories. Thus, by Proposition 5, we have the description of indecomposable A-module. Similarly, as in the proof of Proposition 8, we can prove the other conditions, so we left the details to the reader. ∎

PROPOSITION 10. *Let P be a noninjective* A*-module and* $M'_B = \mathrm{rad}_A(P) = N_B \oplus N_B$. *Moreover let* $\dim_{\mathrm{End}_B(N_B)} D = 2$. *Assume that* $\mathscr{X}_B(N_B)$ *consists of modules* $X_0, \ldots, X_t$. *Then the following triples form a full list of nonisomorphic indecomposable objects in* mod-A.

(i) $Y = (0, Y, 0)$ *where Y is an indecomposable B-module.*

(ii) $\mathrm{top}_A(P) = (D, 0, 0)$.

(iii) $\underline{X}_i = (D, X_i, f)$ *where f is an epimorphism from* $\mathrm{End}_B(N_B) \times \mathrm{End}_B(N_B)$ *onto* $\mathrm{End}_B(N_B)$ *given by the formula* $f(x, y) = x + y$.

(iv) $Z_{ij} = (D, X_i \oplus X_j, f)$ *where f is the identity morphism.*

Moreover if the modules $X_0, \ldots, X_t$ are ordered in such a way that the graph

$$X_0 \to \ldots \to X_t$$

is a subgraph in the Auslander–Reiten graph Γ_B, then we have the following description of minimal left and right almost split morphisms and Auslander–Reiten sequences in mod-A.

(1) *For each indecomposable B-module Y, which is not isomorphic to any X_i (resp. is not isomorphic to any* $\mathrm{Tr}D(X_i)$) *the minimal left (right) almost split morphism starting from Y (ending in Y) in* mod-B *is also a minimal left (right) almost split morphism in* mod-A *as well.*

(2) *The following short exact sequences*

$$0 \to X_i \to Z_{0i} \oplus X_{i+1} \to Z_{0(i+1)} \to 0, \quad 0 \leqslant i \leqslant t-1,$$

$$0 \to X_t \to Z_{0t} \to \underline{X}_0 \to 0,$$

$$0 \to Z_{ii} \to Z_{i(i+1)} \oplus Z_{i(i+1)} \to Z_{(i+1)(i+1)} \to 0, \quad 0 \leqslant i \leqslant t-1,$$

$$0 \to Z_{ij} \to Z_{(i+1)j} \oplus Z_{i(j+1)} \to Z_{(i+1)(j+1)} \to 0, \quad 0 \leqslant i < j \leqslant t-1,$$

$$0 \to Z_{it} \to Z_{(i+1)t} \oplus \underline{X}_i \to \underline{X}_{i+1} \to 0, \quad 0 \leqslant i \leqslant t-1,$$

$$0 \to Z_{tt} \to \underline{X}_t \oplus \underline{X}_t \to \mathrm{top}_A(P) \to 0$$

are Auslander–Reiten sequence in mod-A.

(3) *If* X_i, $i < t$, *is a noninjective B-module, then the following natural short exact sequence*

$$0 \to \underline{X}_i \to \underline{X}_{i+1} \oplus \mathrm{Tr}D(X_{i-1}) \to \mathrm{Tr}D(X_i) \to 0$$

is an Auslander–Reiten sequence in mod-A.

(4) *If* X_i, $i < t$, *is an injective B-module, then the following natural morphism*

$$\underline{X}_i \to \underline{X}_{i+1} \oplus \mathrm{Tr}D(X_{i-1})$$

is a minimal left almost split morphism in mod-A.

(5) *The natural morphism*

$$\underline{X}_t \to \mathrm{top}_A(P) \oplus \mathrm{Tr}D(X_{t-1})$$

is a minimal left almost split morphism in mod-A.

P r o o f. By Propositions 3 and 6, we have a description of indecomposable A-modules in view of Proposition 7. The proof of the other part of the proposition is similar to the proof of Proposition 8 and we left it to the reader. ■

COROLLARY. *If P is nonuniserial and injective, then there are the following indecomposable modules and minimal almost split morphisms in* mod-A.

(a) *If* $\mathrm{rad}_A(P)/\mathrm{soc}_A(P) = N \oplus T$ *and* $N \not\approx T$, *then we have the similar description of indecomposable A-modules and minimal almost split morphisms in* mod-A *as in Proposition* 9. *Moreover P is represented by the triple* $(D, \mathrm{rad}_A(P), \mathrm{id})$ *and there is an Auslander–Reiten sequence of the following form*

$$0 \to \mathrm{rad}_A(P) \to \mathrm{rad}_A(P)/\mathrm{soc}_A(P) \oplus P \to P/\mathrm{soc}_A(P) \to 0$$

in mod-A.

(b) *If* $\mathrm{rad}_A(P)/\mathrm{soc}_A(P) \approx N \oplus N$, *then we have the similar description of indecomposable A-modules and minimal almost split morphisms in* mod-A *as in Proposition* 10. *Moreover P is represented by the triple* $(D, \mathrm{rad}_A(P), \mathrm{id})$ *and there is an Auslander–Reiten sequence of the following form*

$$0 \to \mathrm{rad}_A(P) \to \mathrm{rad}_A(P)/\mathrm{soc}_A(P) \oplus P \to P/\mathrm{soc}_A(P) \to 0$$

in mod-A.

P r o o f. As in [7, Lemma 4.2] one proves that $\mathrm{soc}_A(P)$ is a twosided ideal in A and all indecomposable A-modules, nonisomorphic to P, are indecomposable A'-modules, where $A' = A/\mathrm{soc}_A(P)$. Let us observe that $\mathrm{Hom}_A(P/\mathrm{soc}_A(P), P') = 0$ for any indecomposable projective A'-module P' which is not isomorphic to $P/\mathrm{soc}_A(P)$ and the radical of $P/\mathrm{soc}_A(P)$ is a direct sum of two uniserial modules. This finishes the proof of the corollary. ■

Now we are able to prove the main theorem of this section.

THEOREM 2. *Let A be a finite-dimensional distributive K-algebra which satisfies the* (S)*-condition. The following statements are equivalent.*

(i) *Each indecomposable A-module is almost multiplicity-free.*

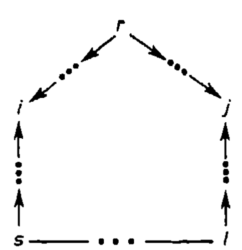
(ii) *A is representation-finite, schurian and $\beta(A) \leqslant 2$.*

(iii) *A is strongly biserial.*

Proof. First we shall show that each of the statements (i), (ii) implies the statement (iii). From [7, Theorem 4.6], it follows that if A is of finite representation type and $\beta(A) \leqslant 2$, then A is biserial. It is strongly biserial by the fact that it is of finite representation type. Similarly, if each indecomposable A-module is almost multiplicity-free, then from Proposition 1 A is representation-finite biserial, so it is strongly biserial.

Now we shall show that (iii) implies (i) and (ii). We shall proceed by induction on the number of nonisomorphic indecomposable projective A-modules. Let P be an indecomposable projective A-module such that $A_A = P \oplus Q$ and $\mathrm{Hom}_A(P, Q) = 0$. There is such a module because A satisfies the (S)-condition, so it is a factor algebra of an hereditary one. Let us put $B = \mathrm{End}_A(Q)$. Assume that A satisfies (iii) and the needed implication holds for B. Thus, from Propositions 8–10 and Corollary, it follows that A is of finite representation type and $\beta(A) \leqslant 2$. Now in order to prove the theorem it is enough to show that each indecomposable A-module is almost multiplicity-free, because by Proposition 1, A is schurian then. But this follows from the description of indecomposable objects in mod-A given in Propositions 8–10.

Indeed, it is enough to consider the objects of the form Z_{ij}, $\underline{Z}_{ij}$ and $\underline{\underline{Z}}_{i,(j,t)}$ in Proposition 9 and the objects of the form Z_{ij} in Proposition 10. Now let us consider the object of the form Z_{ij} from Proposition 9. $Z_{ij} = (D, X_i \oplus X_j, f)$ where $D \approx \mathrm{Hom}_B(M_B, X_i) \approx \mathrm{Hom}_B(M_B, Y_j)$. It is not hard to see that if Z_{ij} is not almost multiplicity-free, then the graph Q_{Σ_A} contains a subgraph Q' of the following form



with some valuations and relations, where $P(r) = P$ and $P(r)$ is the projective cover of the simple A-module S_r. But this gives a contradiction to the fact that A satisfies the (S)-condition. In the other cases, we get similar contradictions to the fact that A satisfies the (S)-condition. Consequently all indecomposable A-modules are almost multiplicity-free. This finishes the proof of Theorem 2. ■

5. Representation-finite biserial algebras

We are going to describe all biserial algebras over a perfect field which are of finite representation type. We shall apply the covering techniques developed in [11], [17], [18], [19], [35]. It is well known that if the algebra A is of the form $A \approx T(\Sigma_A)/R_A$ and its graph with relations (Q_{Σ_A}, R_A) has no commutativity relations then the Galois covering $\tilde{A}$ of A is such that the graph with relations $(Q_{\Sigma_{\tilde{A}}}, R_{\tilde{A}})$ is a valued tree with relations, so $\tilde{A}$ satisfies the (S)-condition (see [17], [22]).

Following [33], a sequence $L_1, L_2, \ldots, L_t$, $t \geqslant 1$, of local A-modules is called a *primitive V-sequence* if the following conditions hold:

(a) *A is representation-finite and $\beta(A) \leqslant 2$ if and only if A is a biserial* such that L_i and L_j are nonisomorphic.

(b) For all i, $1 \leqslant i \leqslant t$, $\operatorname{rad}_A(L_i)$ is a direct sum of two uniserial submodules $L_{i,1}$ and $L_{i,2}$ with $\operatorname{soc}_A(L_{i,1}) \approx S_i$, $\operatorname{soc}_A(L_{i,2}) \approx S_{i+1}$, where $S_1, S_2, \ldots, S_t$ are simple modules, $S_{t+1} = S_1$, and the cokernel of the diagonal embedding S_i into direct sum of $L_{i-1}/L_{i-1,1}$ and $L_i/L_{i,2}$ is colocal.

We can now prove the main theorem of this paper.

THEOREM 3. *For a distributive algebra A we have:*

(a) *A is representation-finite and $\beta(A) \leqslant 2$ if and only if A is a biserial algebra and there is no primitive V-sequence of local A-modules.*

(b) *A is representation-finite and $\alpha(A) \leqslant 2$ if and only if A is a biserial algebra, there is no primitive V-sequence of local A-modules, and any indecomposable projective-injective A-module is uniserial.*

Proof. For the proof of the theorem, it is enough to prove the condition (b). Indeed, if $A_A = P \oplus Q$ is a decomposition of A as a right module, where P is a direct sum of all indecomposable projective-injective summands of A, then $I = \operatorname{soc}_A(P)$ is a twosided ideal in A and A is of finite representation type with $\beta(A) \leqslant 2$ if and only if A/I is of finite representation type with $\alpha(A/I) \leqslant 2$ (see [7, Lemma 4.2]).

Let us assume first that A is of finite representation type and $\alpha(A) \leqslant 2$. From [7, Theorem 4.7] A is biserial and each indecomposable projective-injective A-module is uniserial. Moreover, from [34, Folgerung 1.5] the graph (Q_{Σ_A}, R_A) does not contain any locally faithful Π-sequences, where Π is an extended Dynkin graph. So A is a strongly biserial algebra. Hence there is no primitive V-sequence of local A-modules.

Now let us suppose that A is a distributive biserial algebra without primitive V-sequences of local A-modules. By Theorem 1 A is special biserial. Moreover the Galois covering $\tilde{A}$ of A satisfies the (S)-condition because $(Q_{\Sigma_{\tilde{A}}}, R_{\tilde{A}})$ is a valued tree with relations. By [17] we know that A is of finite representation type if and only if $\tilde{A}$ is locally representation-finite. $\tilde{A}$ is of course special biserial and the canonical covering functor F: mod-$\tilde{A} \to$ mod-A preserves Auslander–Reiten sequences. If in the graph with relations $(Q_{\Sigma,\Gamma}, R_{\tilde{A}})$ of $\tilde{A}$ we take a full subgraph (X, R_X), then it defines a biserial algebra by Lemma 1. Thus we have that this algebra satisfies the (S)-condition and is without primitive V-sequences of local modules, so it is strongly biserial and (X, R_X) is of finite representation type by Theorem 2. Hence $\tilde{A}$ is locally representation-finite, and A is also of finite representation type and $\alpha(A) \leqslant 2$ obviously. The theorem is also proved. ■

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