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Bifurcation from a saddle connection in functional differential equations: An approach with inclination lemmas

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Introduction

Consider a one-parameter family of semiflows on a Banach space with stationary point 0. Suppose 0 is a saddle point, and that there is a homoclinic trajectory at some critical value a_0, i.e. a trajectory which is defined on $\boldsymbol{R}$, not constant and tends to 0 as $t \to +\infty, -\infty$. If this connection is broken by a small change of the parameter then periodic orbits may bifurcate off — whether this actually happens depends on further conditions.

In terms of return maps, defined by translation along trajectories, this phenomenon corresponds to a family of fixed points which at the critical parameter cross into the domain of the map. The following figure for trajectories of vector fields in $\boldsymbol{R}^2$ may help to clarify this.

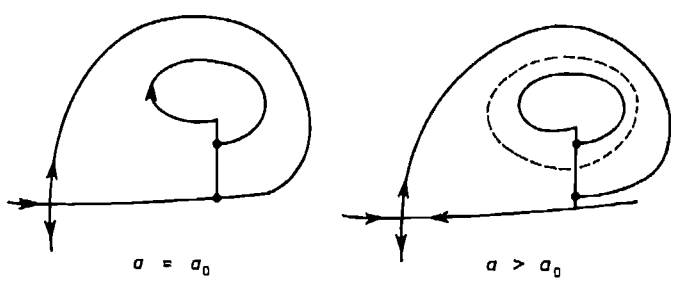

Fig. 1

Loosely speaking, sufficient conditions for fixed points at $a > a_0$ are that contraction in the stable space of the linearization of the flow at 0 is stronger than expansion in the unstable space, and that for $a > a_0$ the unstable manifold intersects with the transversal "above" the stable manifold. For details in the two-dimensional case, see e.g. [3], [6].

A similar phenomenon for parameterized vector fields on $\boldsymbol{R}^2$ which are periodic in the first component x_1 of $x \in \boldsymbol{R}^2$ with period, say, ξ_1, is bifurcation from heteroclinic to periodic solutions of the second kind, with

$$x(\cdot + \pi) = x + (\xi_1, 0)$$

for some $\pi > 0$. For an example, see e.g. Ch. VIII of [3].

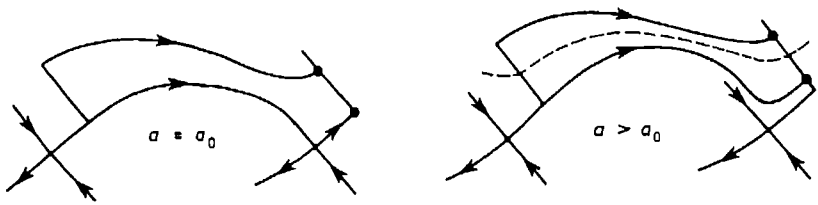

Fig. 2

The corresponding orbits of the induced vector fields on the cylinder $(\boldsymbol{R} \bmod \xi_1) \times \boldsymbol{R}$ are closed but not contractible to a point, in contrast to induced orbits of periodic solutions in $\boldsymbol{R}^2$.

Results about bifurcation from saddle loops of vector fields on $\boldsymbol{R}^n$ are due to L. P. Šilnikov ([18]). He introduced the idea of artificially continuing the return map to the transversal beyond the stable manifold, by the intersection of the unstable manifold with the transversal. For the homoclinic trajectory, one has a fixed point which can be continued for $a \neq a_0$.

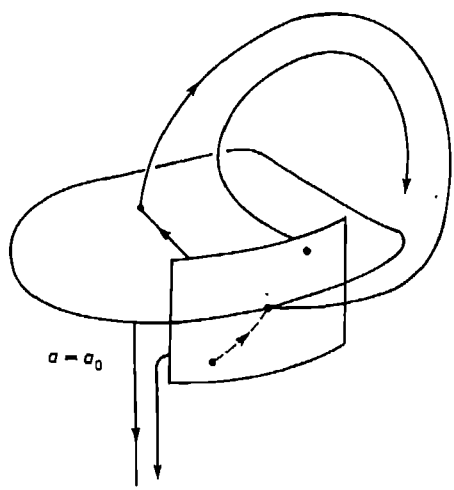

Fig. 3

The difficulty involved is to show that the extended return map is indeed smooth enough for continuation of this fixed point. Šilnikov's proof of smoothness requires that the vector field is at least of class C^2.

In a recent paper [22] we proved bifurcation from a saddle-to-saddle connection for retarded functional differential equations of the form

$$(ah) \qquad \dot{x}(t) = ah(x(t-1)), \qquad a \in (0, 1),$$

with periodic nonlinearity h: $\boldsymbol{R} \to \boldsymbol{R}$ of class C^1. Let $\xi_h > 0$ denote the minimal period of h, assume $h(0) = 0$, $h(\xi_0) = 0$ for some $\xi_0 \in (0, \xi_h)$, $0 < h$ in $(0, \xi_0)$ and $h < 0$ in (ξ_0, ξ_h).

Equation (ah) then describes a state variable on the circle $\boldsymbol{R} \bmod \xi_h$ which tries to compensate deviations from the stationary state $\xi_0 \bmod \xi_h$, after a reaction lag a.

The dynamics of these equations are surprisingly rich. In addition to bifurcation from saddle connections addressed here, periodic solutions emerge in a series of Hopf bifurcations from each equilibrium as the parameter increases from 0 to $+\infty$. Secondary bifurcation from branches of periodic orbits occurs ([21]), and chaotic invariant sets exist, due to periodic solutions with transverse homoclinic orbit ([20], [2], [12]).

Applications include models of phase-locked loops designed to control high frequency generators, see [15], [8] and Section 5 in [22].

Let $h(\cdot+\xi_0)$ also satisfy the remaining hypotheses in Theorem 2 ([22]) on the shape of its graph. It follows that there exists a critical parameter a_0 with a heteroclinic solution x^{a_0} which connects the zero solution to $t\to\xi_h$, and that for $a>a_0$ periodic solutions of the second kind, with

$$x(\cdot+\pi)=x+\xi_h \quad \text{for some } \pi>0,$$

bifurcate off.

For such solutions the point $x(t) \bmod \xi_h$ on the circle rotates periodically. The first result on existence of periodic solutions of the second kind for differential delay equations was proved by T. Furumochi ([8]). Numerical computations of Y. Ueda ([19]) indicate that equation (*ah*) may have more complicated periodic solutions of the second kind as well, with $k=k(\pi)$ in $\boldsymbol{Z}\setminus\{0,1\}$ for the minimal $\pi>0$ with

$$x(\cdot+\pi)=x+k\xi_h.$$

This means that the point on the circle completes $|k|$ full turns around during each minimal period.

Recall that on the Banach space C of continuous functions ϕ: $[-1,0]\to\boldsymbol{R}$, equation (*ah*) defines a parameterized semiflow

$$X\colon \boldsymbol{R}_0^+\times C\times\boldsymbol{R}\ni(t,\phi,a)\to x_t\in C$$

where $x_t(s)=x(t+s)$ for all $t\geqslant 0$ and $s\in[-1,0]$, with the solution $x=x(\phi,a)$: $[-1,\infty)\to\boldsymbol{R}$ of the initial value problem

$$\dot{x}(t)=ah(x(t-1)), \qquad x_0=\phi.$$

Let $\bar{\xi}_0$, $\bar{\xi}_h$ denote the functions $t\to\xi_0$, $t\to\xi_h$ in C. In [22], each initial value for a bifurcating solution was obtained as fixed point of a map $\mathscr{P}_a-\bar{\xi}_h$, $a>a_0$, with

$$\mathscr{P}_a\colon \mathscr{D}_a\to\mathscr{D}_a+\bar{\xi}_h$$

a compact translation along trajectories close to the heteroclinic connection.

The advantage of this approach is that the functional on the right hand side of the parameterized functional differential equation considered is only required to be C^1-smooth. On the other hand, it does not lead to uniqueness and stability. Both are to be expected from the two essential properties of the

linearizations $T(\cdot, \cdot, a)$ of the semiflow at the stationary points $Z\xi_h$: The unstable linear space P_a of $T(\cdot, \cdot, a)$ has dimension one, and contraction in the stable space Q_a is stronger than expansion in P_a. More exactly, the spectrum σ_a of the generator of the semigroup $T(\cdot, \cdot, a)$ contains only one point $u(a)$ in the right halfplane, $u(a) > 0$ is a simple eigenvalue, and there are bounds $\lambda_3 < 0$, $\mu_2 > 0$ such that

$$\lambda_3 < -\mu_2 \text{ and } u(a) < \mu_2, \operatorname{Re} z < \lambda_3 \tag{0.1}$$

for all $z \in \sigma_a \setminus \{u(a)\}$ and for all parameters a in a small neighborhood of a_0.

In the present paper we develop a new method for bifurcation from a saddle connection and apply it to the equations (ah) where $h(\cdot + \xi_0)$ satisfies the conditions from [22] and is in addition of class C^2.

As a result, we obtain a differentiable curve of initial values for bifurcating periodic solutions of the second kind which are unique, stable, and attractive with asymptotic phase. The precise statement is contained in Theorem 13.2.

With a view to the models for phase-locked loops from Section 5 in [22], we may say that we obtain bifurcation to attractive periodic solutions of the second kind in the presence of attractive phase-locked states. Asymptotic stability of the solutions $t \to \xi_0 + j\xi_h, j \in \boldsymbol{Z}$, follows from an inspection of the characteristic equation $z + ae^{-z} = 0, 0 < a < 1$.

The proof of Theorem 13.2 employs a map Σ of translation along trajectories, close to the hyperbolic equilibrium $0 \in C$, and its continuation Σ_2, herein following Šilnikov's idea. The crucial part is to ensure smoothness properties of Σ_2. The proof of these is based on elementary geometrical considerations and on a sharpened inclination lemma. Below we give an outline.

Incidentially it should be noted that the construction of Σ_2 and our proof of smoothness do not exploit special properties of equation (ah). Any family of retarded functional differential equations

$$\dot{x}(t) = F(a, x_t)$$

can be treated, provided that the map

$$F\colon (a_0 - \varepsilon, a_0 + \varepsilon) \times C([-r, 0], \boldsymbol{R}^n) \to \boldsymbol{R}^n, \quad r > 0, n \in \boldsymbol{N},$$

$a_0 \in \boldsymbol{R}$ and $\varepsilon > 0$, is of class C^2 and there is a saddle point $x^* \in C([-r, 0], \boldsymbol{R}^n)$ with one-dimensional unstable manifold such that (0.1) holds for the linearized semiflow. This immediately implies a more abstract theorem on bifurcation from, say, a saddle-loop for such functional differential equations, analogous to Šilnikov's result where existence of a homoclinic solution is assumed, not asserted. Details are left as an exercise.

To explain our approach, we begin with the construction of a map f

which defines a fixed point equation

$$f(\phi, a) = \phi$$

for initial values of periodic solutions of the second kind. f will be called the *Šilnikov return map.*

Let us assume for simplicity that there is a neighborhood $\mathscr{B}$ of $0 \in C$ so that the local stable and unstable manifolds of this stationary point are given by

$$S_a = Q_a \cap \mathscr{B}, \quad U_a = P_a \cap \mathscr{B}.$$

(In the body of the paper we have to introduce local coordinates and a transformed semiflow Y in order to achieve this.)

Let p_a and q_a denote the projections onto P_a and Q_a, according to the decomposition

$$C = P_a + Q_a.$$

Note $P_a = \boldsymbol{R}\Phi_a$ with $\phi_a(t) = e^{u(a)t}$ for all $t \in [-1, 0]$. In particular, if $0 \neq \phi \in P_a$ then either $\phi < 0$ or $\phi > 0$.

Embed a point $x_t^{a_0} \in U_{a_0}$ into a family of initial values $\eta_0 \Phi_a \in V_a$. We shall have $\eta_0 > 0$, from properties of the heteroclinic solution. Choose transversals

$$H_a^+ = \eta_0 \Phi_a + Q_a$$

to U_a and open parallelograms

$$E_a = E(\delta_1, \eta_1, a) = \{\phi \in C\colon\ |p_a \phi| < \eta_1, |q_a \phi| < \delta_1\}$$

with $\eta_1 > 0$, $\delta_1 > 0$, so that $E_a \cap H_a^+ = \emptyset$ for a in some neighborhhod of a_0.

Fix $\theta > 0$ with

$$x_{s+t}^{a_0} \in E_{a_0} + \bar{\xi}_h \quad \text{for all } s \geqslant \theta. \tag{0.2}$$

There are open neighborhoods $\tilde{\mathscr{B}}$ of $x_t^{a_0} = \eta_0 \Phi_{a_0}$ and $\tilde{\mathscr{A}}$ of a_0 such that $X(\theta, \cdot, a) - \bar{\xi}_h$ maps $\tilde{\mathscr{B}}$ into E_a for all $a \in \tilde{\mathscr{A}}$.

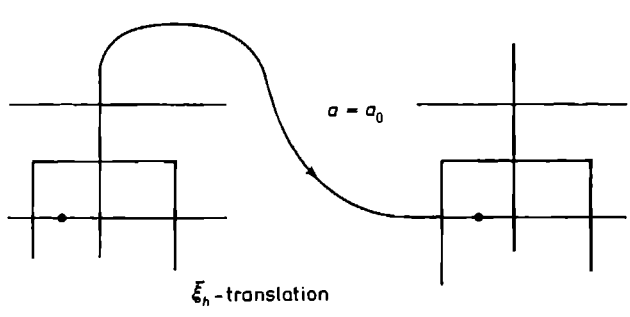

Fig. 4

For η_1, δ_1 and $\tilde{\mathscr{A}}$ sufficiently small, every trajectory $0 \leqslant t \to X(t, \phi, a)$ which starts in the open upper half

$$E_a^+ = \{\phi \in E_a\colon\ p_a\phi > 0\}$$

of E_a reaches the transversal H_a^+ at a first time

$$\sigma = \sigma(\phi, a) > 1$$

while trajectories starting in

$$S_a \cap E_a = \{\phi \in E_a\colon\ p_a\phi = 0\}$$

converge to 0. Set

$$\Sigma(\phi, a) = X\big(\sigma(\phi, a), \phi, a\big) \quad \text{if } \phi \in E_a^+,\ a \in \tilde{\mathscr{A}},$$

and continue Σ to a map Σ_2, by

$$\Sigma_2(\phi, a) = \eta_0 \Phi_a \quad \text{for } \phi \in E_a \backslash E_a^+,\ a \in \tilde{\mathscr{A}}.$$

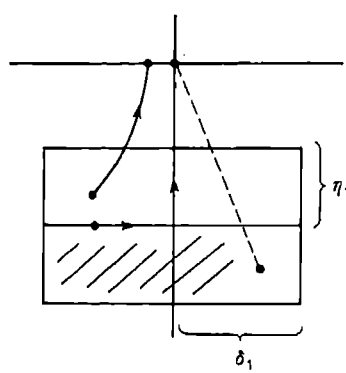

Fig. 5

The composition

$$(\phi, a) \to \Sigma_2\big(X(\theta, \phi, a) - \bar{\xi}_h, a\big)$$

defines a map

$$f\colon\ \tilde{\mathscr{B}} \times \tilde{\mathscr{A}} \to C$$

with $f(\tilde{\mathscr{B}} \times \{a\}) \subset H_a^+$ for $a \in \tilde{\mathscr{A}}$ and $f(\eta_0 \Phi_{a_0}, a_0) = \eta_0 \Phi_{a_0}$.

The last equation holds since (0.2) and ξ_h-periodicity imply

$$X\big(s, X(\theta, \eta_0 \Phi_{a_0}, a_0) - \bar{\xi}_h, a_0\big) = x_{s+\theta}^{a_0} - \bar{\xi}_h \in E_{a_0} \quad \text{for } s \geqslant 0$$

which in turn gives

$$X(\theta, \eta_0 \Phi_{a_0}, a_0) - \xi_h \in S_a,$$

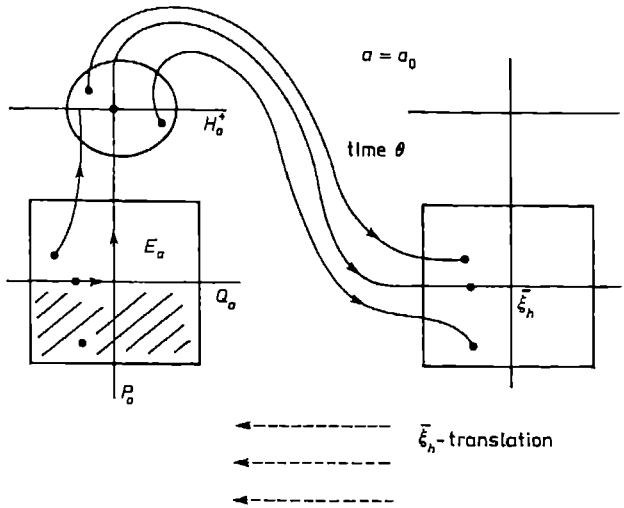

Fig. 6

hence

$$f(\eta_0 \Phi_{a_0}, a_0) \in \Sigma_2((E_{a_0} \setminus E_{a_0}^+) \times \{a_0\}) = \{\eta_0 \Phi_{a_0}\}.$$

Moreover, fixed points $\phi \in H_a^+ \cap \tilde{\mathscr{B}}$ of $f(\cdot, a)$ — with the additional property $X(\theta, \phi, a) \in E_a^+ - \bar{\xi}_h$ — define initial values of periodic solutions of the second kind.

The definition of Σ_2 shows that if $D\Sigma_2(\phi, a)$ would exist on all of its domain, not only for $\phi \in E_a^+$ or $p_a \phi < 0$, then

$$D_1 \Sigma_2(\phi, a) = 0 \quad \text{for } \phi \in E_a \cap S_a,$$

and in particular

$$D_1 f(\eta_0 \Phi_{a_0}, a_0) = 0.$$

This would allow to solve the equation $f(\phi, a) = \phi$ with a unique differentiable curve

$$a \to \phi_a^*, \quad \phi_{a_0}^* = \eta_0 \Phi_{a_0} = x_1^{a_0},$$

by means of an implicit function theorem.

The key to sufficient smoothness properties of the Šilnikov return map is Theorem 6.1 which asserts that there is $\delta_2 \in (0, \delta_1)$ such that

$$D_1 \Sigma(\psi, a) \to 0 \quad \text{as } \eta \to 0,$$

uniformly for all $a \in \tilde{\mathscr{A}}$ sufficiently close to a_0 and all ψ in open layers

$$E^+(\delta_2, \eta, a) = \{\psi \in C\colon\ 0 < p_a \psi < \eta,\ |q_a \psi| < \delta_2\}$$

of height η above the stable manifolds.

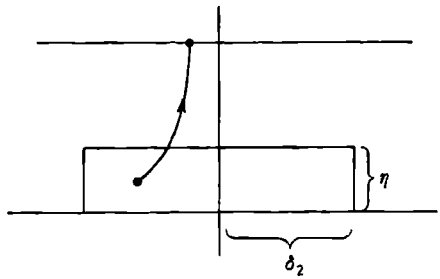

Fig. 7

The proof requires a closer look at the hyperbolic behavior of the semiflow near the stationary point $0 \in C$.

Without too much effort, it can be reduced to an estimate of $D_1 \Sigma(\cdot, a)$ at points in level sets

$$H_{ak}^+ \subset E_a^+$$

defined by

$$\sigma(\cdot, a) = kN, \qquad k \in \boldsymbol{N},$$

where N is a suitably fixed positive integer. The sets H_{ak}^+ converge to the stable space Q_a as $k \to +\infty$. On H_{ak}^+, $\Sigma(\cdot, a)$ coincides with the iterate $(\hat{g}_a)^k$ of a restricted time-N-map $\hat{g}_a$ of the semiflow $X(\cdot, \cdot, a)$.

Furthermore, we can restrict the investigation to points $\psi \in H_{ak}^+$ such that the trajectory $0 \leqslant t \to X(t, \psi, a)$ admits a tangent vector $w = w^{\psi,a}$ at $t = 0$, and where H_{ak}^+ is a submanifold of codimension 1 in some neighborhood of ψ, with w not contained in its tangent space at ψ. The decomposition

$$C = T_\psi H_{ak}^+ \oplus \boldsymbol{R}w$$

now permits an estimate of $|D_1 \Sigma(\psi, a)|$ in terms of

- $D_1 \Sigma(\psi, a) | \boldsymbol{R}w$;
- $D_1 \Sigma(\psi, a) | T_\psi H_{ak}^+ = D(\hat{g}_a)^k(\psi) | T_\psi H_{ak}^+$;
- the angle between the spaces $\boldsymbol{R}w$ and $T_\psi H_{ak}^+$.

Observe $D_1 \Sigma(\psi, a) w = 0$, Σ being constant along trajectories. We arrive at an estimate

$$|D_1 \Sigma(\psi, a)| \leqslant \text{const} \cdot |D(\hat{g}_a)^k(\psi) | T_\psi H_{ak}^+| \cdot \left(1 + \frac{|qw|}{|pw|}\right), \tag{0.3}$$

where the term $|qw|/|pw|$ expresses the angle between $\boldsymbol{R}w$ and $T_\psi H_{ak}^+$ by means of projections p onto P_a, q onto $T_\psi H_{ak}^+$, which are given by another decomposition

$$C = T_\psi H_{ak}^+ \oplus P_a.$$

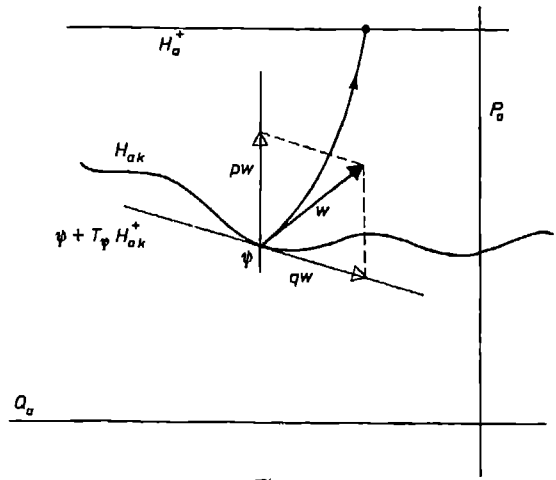

Fig. 8

The last decomposition follows from an inclination lemma for the preimages

$$\hat{H}_{ak}^+ = (\hat{g}_a)^{-k}(H_a^+) \supset H_{ak}^+.$$

In addition, we get a bound

$$\text{const} \cdot |p_a \psi|^{-\lambda_3/\mu_2}$$

for the second factor on the right hand side in (0.3) which clearly tends to 0 uniformly for $\psi \in E^+(\delta_1, \eta, a)$ as $\eta \to 0$.

With regard to (0.3) and (0.1) it suffices to derive an inequality

$$\frac{|qw|}{|pw|} \leqslant \text{const} \cdot \frac{1}{|p_a \psi|}.$$

The numerator $|qw|$ is easily seen to be bounded by a constant, so we look for an estimate

$$\text{const} \cdot |p_a \psi| \leqslant |pw| \quad \text{in case } qw \neq 0.$$

Straightforward estimates of the semiflow and relations among the projections involved only give a constant c_4 such that

$$|pw| \geqslant c_4 |p_a \psi| - |q_a w| \cdot \Lambda_a(qw)$$

where

$$\Lambda_a(qw) = \frac{|p_a qw|}{|q_a qw|}$$

is the inclination of the nonzero tangent vector $qw \in T_\psi H_{ak}^+$.

Working in a sufficiently small neighborhood of $0 \in C$, we can bound $|q_a w|$ by any prescribed constant.

So what we finally need for the completion of the proof is an a-priori estimate

$$\Lambda_a(\chi) \leqslant \text{const} \cdot |p_a \psi| \tag{0.4}$$

for tangent vectors $\chi \in T_\psi \hat{H}^+_{ak} \setminus \{0\}$ at points $\psi \in \hat{H}^+_{ak}$, for all $k \in N$ and all parameters a in some neighborhood of a_0.

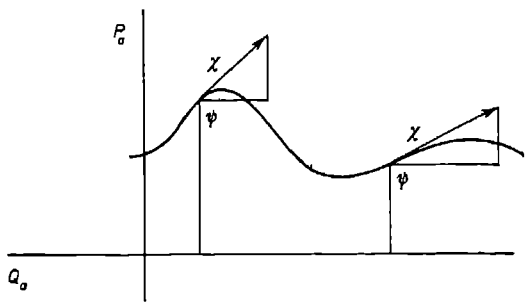

Fig. 9

Such a pointwise estimate is considerably sharper than assertions on uniform convergence in the inclination lemmas contained in [16], [17], [13], [12], [14], compare e.g. the remark following Corollary 8.1.

All results on inclinations needed for the proof of Theorem 6.1 have been combined into Lemma 2.1 ([23]) on maps g in a Banach space with hyperbolic fixed point 0. These maps g may not be invertible, of course.

The derivation of estimate (t, x) in Lemma 2.1 ([23]), which corresponds to (0.4) above, requires that they are of class C^2. In addition we have to impose a condition ((*)) which is trivially satisfied if the unstable space P of $Dg(0)$ is one-dimensional as in our application, but becomes a restriction on the distribution of the spectrum of $Dg(0)$ in the complex plane for $\dim P > 1$.

Another particular feature of Lemma 2.1 ([23]) is that the given transversal H to P and its preimages $g^{-k}(H)$ do not have to be submanifolds. (Tangent vectors to arbitrary sets are defined as usual, by means of differentiable curves with trace in the set.) This generalization simplifies the application to maps $g = \hat{g}_a$ which are taken from a semiflow as $X(\cdot, \cdot, a)$ above: Even if the initially given set H is a submanifold, one cannot use transversal intersection of trajectories with H in order to show that the preimages $g^{-k}(H)$ are submanifolds, too. The reason for this is that not every trajectory $0 \leqslant t \to (t, \phi, a)$ has a tangent vector at $t = 0$.

Organization of the paper. Chapter I contains preliminary material, including a review of results from [22] on the unstable manifolds of the zero solution to equation (ah). Section 3 examines the semigroups $T(\cdot, \cdot, a)$ and prepares a set of constants for the application of Lemma 2.1 ([23]). Section

4 specifies some easy consequences of the saddle point property for the nonlinear semiflows $X(\cdot, \cdot, a)$ close to $0 \in C$.

Section 5 in Chapter II begins with the standard transformation of X to a parameterized local semiflow Y on a neighborhood of $0 \in C$ so that the local invariant manifolds become open subsets of the spaces Q_a and P_a. We construct domains on which the time-N-maps g_a and $\hat{g}_a$ taken from the semiflow $Y(\cdot, \cdot, a)$ satisfy the hypotheses of Lemma 2.1 ([23]), and we estimate the growth of P_a- and Q_a-components for trajectories of $Y(\cdot, \cdot, a)$ and their tangent vectors. Section 6 contains the definition and basic properties of the map Σ, the statement of Theorem 6.1, and a step-by-step outline of its proof. The proof itself follows in the next 4 sections. The central part using an estimate of type (0.4) is contained in Section 9.

Chapter III starts with the construction of the Šilnikov return map. Proposition 6.2 and Theorem 6.1 imply that it is smooth enough for the application of an implicit function theorem, stated as Theorem 13.1. The main result of the paper, Theorem 13.2, includes a precise description what uniqueness, stability and attractivity with asymptotic phase mean for periodic solutions of the second kind. The detailed proof of Theorem 13.2 in the last sections is still rather long, but mainly for technical reasons. The major difficulty has been overcome with the proof of smoothness for the Šilnikov return map.

Throughout all sections, proofs are given with almost every detail, This helps to avoid vague descriptions of situations which involve both local and global behavior of the semiflows. Sometimes properties of functional differential equations are used without further comment. In these cases the reader should consult [10] and [9]. The general reference for calculus and a little functional analysis is [7]. The elementary facts about transversality needed in Section 7 are found in [1].

Parallel to the work presented here, C. M. Blazquez ([4]) and S. N. Chow and B. Deng ([5]) used other ideas to prove similar results on bifurcation from saddle connections in infinite-dimensional spaces (restricted to the case of one-dimensional unstable manifolds, as in our approach). Blazquez extends Šilnikov's method to semiflows for parabolic equations. Chow and Deng treat such semiflows, too, describe a proof for functional differential equations and give an application to the equations from [22]. The resulting theorem is slightly weaker than our Theorem 13.2, assuming that h is of class C^3 while differentiability of the bifurcating curve of initial values is not asserted.

I

1. Preliminaries. All norms are denoted by $|\ |$. $L_c(Z, Z')$ stands for the space of continuous linear maps $Z \to Z'$ between Banach spaces Z, Z' over the fields $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$. A map F: $W \to Z'$, $W \subset Z$ open, is of class C^k iff it has

continuous derivatives DF: $W \to L_c(Z, Z')$, $D(DF)$: $W \to L_c(Z, L_c(Z, Z'))$, ... up to order k. For maps $\bar{h}$: $\boldsymbol{R} \to \boldsymbol{R}$ or x: $J \to \boldsymbol{R}$, y: $J \to \boldsymbol{C}$ with $J \subset \boldsymbol{R}$, we set

$$\bar{h}'(\xi) := D\bar{h}(\xi)(1), \quad \dot{x}(t) := Dx(t)(1), \quad \dot{y}(t) := Dy(t)(1).$$

In Section 8, we consider tangents to an arbitrary subset S of a Banach space Z: A vector $\chi \in Z$ is called *tangent to* S at the point $z \in Z$ iff there is a differentiable curve γ: $(-1, 1) \to Z$ with $\gamma((-1, 1)) \subset S$, $\gamma(0) = z$, $D\gamma(0)(1) = \chi$. $T_z S$ denotes the set of all tangents to S at z. As usual, the hypotheses $S \subset W$, F: $W \to Z'$ differentiable with $F(S) \subset S'$, $z \in S$ and $\chi \in T_z S$ altogether imply $DF(z)\chi \in T_2(z)S'$. Also, if F_1: $W \to Z'$ and F_2: $W \to Z'$ are differentiable with $F_1 = F_2$ on $S \subset W$, then $DF_1(z)\chi = DF_2(z)\chi$ for all $z \in S$, $\chi \in T_z S$.

For a C^1-submanifold $S \subset Z$ (see [1]), $T_z S$ as introduced above is the usual closed subspace of Z, with a closed complement in Z.

Preimages S_k, $k \in N_0$, of a set $S \subset Z$ with respect to a map F: $W \to Z$ are defined by $S_0 := S$, $S_{k+1} := F^{-1}(S_k)$ for $k \in N_0$. This implies $S_k = (F^k)^{-1}(S_0)$ for all $k \in N_0$, with the iterates F^k: $W_k \to Z$ given by $W_0 := W := W_1$, $F^0(w) := w$ for $w \in W_0$, $F^1 := F$, $W_{k+1} := F^{-1}(W_k)$ and $F^{k+1} := F^k \circ F$ for $k \in N$.

C and C_C denote the Banach spaces of real- and complex-valued continuous functions on the initial interval $I := [-1, 0]$, with supremum norm.

We shall successively construct nonempty open balls

$$B_1 \supset B_2 \supset \ldots \supset B_7, \quad D^0 \supset D^1 \supset D^2 \supset \ldots \supset D^5$$

in C, centered at $0 \in C$, and nonempty open parameter intervals

$$A_0 \supset A_1 \supset A_2 \ldots \supset A_{24}$$

centered at a fixed critical parameter $a_0 > 0$, such that $\operatorname{cl} B_{k+1} \subset B_k$, $\operatorname{cl} D^{k+1} \subset D^k$, $\operatorname{cl} A_{k+1} \subset A_k$ for all relevant indices: Let us adopt the convention not to mention explicitly these properties once new $B_{k+1} \subset B_k$, $D^{k+1} \subset D^k$ or $A_{k+1} \subset A_k$ are chosen.

For real- or complex-valued functions x on an open interval $J \subset \boldsymbol{R}$ and for $t \in J$ with $[t-1, t] \subset J$, the translate $I \ni s \to x(t+s)$ of the restriction of x to $[t-1, t]$ is denoted by x_t, as usual in functional differential equations.

$\boldsymbol{R}^+$ stands for the positive reals, $\boldsymbol{R}_0^+ := \boldsymbol{R}^+ \cup \{0\}$.

2. Solutions of a family of differential delay equations with periodic nonlinearity. We study the equations

$$(ah) \qquad \dot{x}(t) = ah(x(t-1)),\ a > 0,$$

with nonlinearity h: $\boldsymbol{R} \to \boldsymbol{R}$ in the set $\mathscr{H}$ defined as follows: There exist a C^2-function $\bar{h}$: $\boldsymbol{R} \to \boldsymbol{R}$ and $a^+ \in (0, 1)$ with $h = \bar{h}/a^+$, and

(H.0) there are positive reals ξ_0, $\xi_h > \xi_0$, r_h, ξ_1, ξ_2 such that

$$2r_h < \xi_1 < \xi_2 < \xi_0 - r_h < \xi_0 + r_h < \xi_h - r_h;$$

(H.1) $\bar{h}$ is periodic with period ξ_h;

(H2) $\bar{h} > 0$ in $(0, \xi_0)$ and $\bar{h} < 0$ in (ξ_0, ξ_h);

(H.3) $|\bar{h}| < r_h/2$ in $(-r_h, r_h) \cup (\xi_0 - r_h, \xi_h - r_h]$;

(H.4) there exists $q \in (0, 1)$ with $|\bar{h}(\xi)| \leqslant q|\xi|$ for $|\xi| < r_h$;

(H.5) $\bar{h}'(0) = a^+$ and $\log a^+ < -u(a^+)$ for $u = u(a^+) > 0$ with $u - a^+ e^{-u} = 0$;

(H.6) $\phi \in C$ and $0 \leqslant \phi \leqslant \xi_1 = \phi(0)$ imply

$$\xi_1 + \int_I \bar{h} \circ \phi \leqslant \xi_2;$$

$\psi \in C$ and $\xi_1 \leqslant \psi \leqslant \xi_2 = \psi(0)$ imply

$$\xi_h + r_h < \xi_2 + \int_I \bar{h} \circ \psi.$$

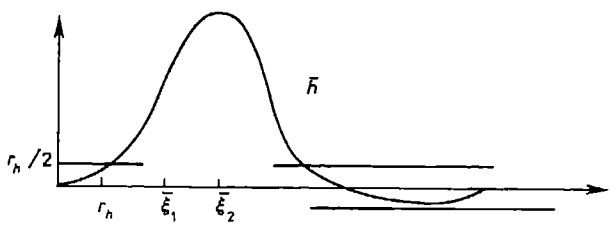

Fig. 10

It follows that $h'(0) = 1$. We fix a positive real $\delta_h < r_h/2$ with $h' > 0$ in $[-\delta_h, \delta_h]$.

Note that the nonlinearities $h(\cdot + \xi_0)$, $h \in \mathscr{H}$, are identical with the functions considered in [22]. For an example arising in applications, see Section 5 in [22].

A solution of equation (ah) is either a differentiable function x: $\boldsymbol{R} \to \boldsymbol{R}$ which satisfies equation (ah) everywhere, or a continuous function x: $[s-1, \infty) \to \boldsymbol{R}$, $s \in \boldsymbol{R}$, which is differentiable on (s, ∞) and satisfies equation (ah) for all $t > s$.

Repeated application of the formula

$$x(t) = x(n) + a \int_{n-1}^{t-1} h \circ x \tag{2.1}$$

on the intervals $[n, n+1]$, $n \in N_0$, shows that every initial value $\phi \in C$ and every parameter $a > 0$ define a unique solution $x = x(\phi, a)$ of equation (ah) on $[-1, \infty)$ with $x_0 = \phi$.

The zeros $\boldsymbol{Z}\xi_h \cup (\xi_0 + \boldsymbol{Z}\xi_h)$ of h yield constant solutions on $\boldsymbol{R}$.

We have continuous dependence of solutions on $(\phi, a) \in C \times \boldsymbol{R}^+$ with respect to the supremum-norm on compact intervals $[-1, t]$, $t \geqslant -1$. The relation

$$X(t, \phi, a) = x(\phi, a)$$

defines a parameterized continuous semiflow

$$X\colon \boldsymbol{R}_0^+ \times C \times \boldsymbol{R}^+ \to C$$

with stationary points $\boldsymbol{Z}\bar{\xi}_h$, $\bar{\xi}_0 + \boldsymbol{Z}\bar{\xi}_h$,

$$\bar{\xi}_h(t) := \xi_h, \quad \bar{\xi}_0(t) := \xi_0 \quad \text{on } I.$$

Note that all functions $x + \boldsymbol{Z}\xi_h$ are solutions whenever x is a solution of equation (ah), due to the periodicity of h. We have

$$X(t, \phi + k\bar{\xi}_h, a) = X(t, \phi, a) + k\bar{\xi}_h \tag{2.2}$$

for all $(t, \phi, a) \in \boldsymbol{R}_0^+ \times C \times \boldsymbol{R}^+$, $k \in \boldsymbol{Z}$.

The results of Sections 1 and 2 of [22] imply that there are a critical parameter $a_0 \in (0, a^+)$, an open interval $A_0 \Subset (0, a^+) \subset (0, 1)$ with center a_0 and a family of solutions $x^a\colon \boldsymbol{R} \to \boldsymbol{R}$ of equation (ah), $a \in A_0$, so that the map $a \to x_0^a$ is continuous, with

$$\left.\begin{array}{ll} 0 < x^a & \\ x^a(t) \to 0 & \text{as } t \to -\infty \end{array}\right\} \quad \text{for all } a \in A_0,$$

$$x^{a_0}(t) \to \xi_h \quad \text{as } t \to +\infty,$$

$$\liminf_{t \to +\infty} x^a > \xi_h \quad \text{for } a_0 < a \in A_0.$$

In terms of Section 4 below, we have a family of solutions in the unstable manifolds of the equilibrium solution $t \to 0$ such that for $a = a_0$, the solution is heteroclinic and connects $t \to 0$ to the equilibrium solution $t \to \xi_h$, while for $a > a_0$ the solutions overshoot ξ_h and do not return.

We need a few elementary properties of solutions.

PROPOSITION 2.1. *Let* $(\phi, a) \in C \times A_0$, $x := x(\phi, a)$.

(i) $0 < \phi < \xi_0$ *implies* $0 < \dot{x}$ *on* $(0, 1]$.

(ii) $\xi_0 \leqslant \phi(0)$ *implies* $\xi_0 - (r_h/2) < x$ *on* $\boldsymbol{R}_0^+$.

(iii) $\xi_0 < \phi < \xi_h$ *implies* $x < \xi_h$ *on* $[-1, \infty)$.

(iv) $|x(t)| \leqslant |\phi|(1 + \max|\bar{h}'|)$ *for all* $t \in [0, 1]$.

Proof. (i) follows from (H.2) and equation (ah).

(ii) Suppose $x(t) < \xi_0$, for $t > 0$. Let z denote the largest zero of $x - \xi_0$ in $[-1, t]$. Then $0 \leqslant z < t$. If $t \leqslant z + 1$, then

$$x(t) = \xi_0 + \int_z^t \dot{x} \geqslant \xi_0 + \min \bar{h} > \xi_0 - (r_h/2),$$

by (H.3) and periodicity of h. If $z + 1 < t$, then

$$0 < \xi_0 - (r_h/2) < x < \xi_0 \quad \text{in } (z, z+1],$$

as before. Using (i) we infer $0 < \dot{x}$ on $[z, t]$. Hence

$$\xi_0 - (r_h/2) < x(t).$$

(iii) $h < 0$ on (ξ_0, ξ_h) implies that either $\xi_0 < x$ and $\dot{x} < 0$ on $\boldsymbol{R}^+$, or that there exists a first zero z_1 of $x - \xi_0$ in $\boldsymbol{R}^+$, and $\dot{x} < 0$ in $(0, z_1+1)$. By (ii), $\xi_0 - (r_h/2) < x(z_1+1)$. Now, $0 < ah < \bar{h} < r_h/2$ on $(\xi_0 - r_h, \xi_0)$ yields – either $0 < \dot{x}$ and $\xi_0 - (r_h/2) < x < \xi_0$ on (z_1+1, ∞), or existence of a first zero z_2 of $x - \xi_0$ in (z_1+1, ∞) with $0 < \dot{x} < r_h/2$ on (z_1+1, z_2+1). In the last case, $0 < x < \xi_0 + (r_h/2) < \xi_h$ on (z_2, z_2+1), and we are almost in the same situation as at the beginning of the argument. Now it is easy to complete the proof.

(iv) follows from (2.1) and from the mean value theorem, with $\bar{h}(0) = 0$. ∎

COROLLARY 2.1. *For $0 < \phi < \xi_0$, either $x < \xi_0$ and $0 < \dot{x}$ on $\boldsymbol{R}^+$, or there exists $t = t(x) > 0$ with $x(t) = \xi_0$, $0 < \dot{x}$ on $(0, t+1)$, $\xi_0 - (r_h/2) < x$ on $[t+1, \infty)$.*

PROPOSITION 2.2. (i) *For every $a \in [a_0, \sup A_0)$ there exists $t(a) \in \boldsymbol{R}$ with $x^a(t(a)) = \xi_0$ and $0 < \dot{x}^a$ on $(-\infty, t(a)+1)$.*

(ii) *There exists $b_1 \in (t(a_0), t(a_0)+1)$ such that $x^{a_0} < \xi_h$ on $(-\infty, b_1)$, $\xi_h - r_h < x^{a_0} < \xi_h + r_h$ on $[b_1, \infty)$, and every interval $[t-1, t]$, $t \geqslant b_1$, contains at least one simple zero of $x^{a_0} - \xi_h$.*

(iii) *If $t \geqslant t(a_0)+1$ and $\dot{x}^{a_0}(s) < 0$, $s \in [t-1, t]$, then $0 \leqslant x^{a_0}(s')$ for some $s' \in [t-1, t]$.*

Proof. (i) follows from Corollary 2.1. For (ii), see Section 2 in [22]. Let $t \geqslant t(a_0)+1 > b_1$ and $\dot{x}^{a_0}(s) < 0$. Suppose $\dot{x}^{a_0} < 0$ on $[t-1, t]$. We have $x^{a_0}(\boldsymbol{R}) \subset (0, \xi_h + r_h)$, by (i) and (ii) and $0 < x^{a_0}$, so that equation $(a_0 h)$ and (H.2) yield $\xi_0 < x^{a_0} < \xi_h$ on $[t-2, t-1]$. But this contradicts (i) and (ii). ∎

We exclude periodic solutions (of the first kind) close to the heteroclinic solutions $x^{a_0} + \boldsymbol{Z}\xi_h$. This proves already a part of assertion (vi) in the final Theorem 13.2.

PROPOSITION 2.3. (i) *The set $\mathrm{cl}\{x_t^a\colon t \in \boldsymbol{R}\} = \{\ldots\} \cup \{\bar{\xi}_h\}$ is compact.*

(ii) *There exist an open neighborhood V_{01} of $\mathrm{cl}\{\ldots\}$ and an open interval A_1 such that for $a \in A_1$ there is no non-constant periodic solution $x\colon \boldsymbol{R} \to \boldsymbol{R}$ of equation (ah) with orbit $\{x_t\colon t \in \boldsymbol{R}\}$ contained in $V_{01} + \boldsymbol{Z}\bar{\xi}_h$.*

Proof. (i) follows from $\lim_{t \to -\infty} x^{a_0}(t) = 0$, $\lim_{t \to +\infty} x^{a_0}(t) = \xi_h$.

The proof of (ii) is rather long:

a) By Proposition 2.2, there are reals $t_- < t_+$ with $x^{a_0}(t_-) = r_h$, $x^{a_0}(t_+) = \xi_h - r_h$. Then $t(a_0) < t_+ < b_1$, $0 < x^{a_0}$ and $0 < \dot{x}^{a_0}$ on $(-\infty, t_+]$, and $\xi_h - r_h < x^{a_0}$ on (t_+, ∞). Choose $\varepsilon_0 > 0$ so small that

$$2\varepsilon_0 < \min_{[t_-, t_+]} \dot{x}^{a_0}, \tag{2.3}$$

$$\xi_n - r_h + \varepsilon_0 < x^{a_0} < \xi_h + r_h - \varepsilon_0 \quad \text{on } [b_1, \infty), \tag{2.4}$$

in particular, $\varepsilon_0 < r_h$.

Using continuous dependence and compactness of $[t_-, b_1]$, we find $\varepsilon_{01} \in (0, \varepsilon_0)$ and an interval A_1 such that $|\phi - x_t^{a_0}| < \varepsilon_{01}$ with $t \in [t_-, b_1]$ and $a \in A_1$ imply $x(s) > \xi_h$ at some $s > 0$ for the solution x: $[-1, \infty) \to \boldsymbol{R}$ of equation (ah) with $x_0 = \phi$. We define

$$V_{01} := \{\phi \in C\colon |\phi| < \varepsilon_{01} \text{ or } |\phi - \bar{\xi}_h| < \varepsilon_{01} \text{ or } |\phi - x_t^{a_0}| < \varepsilon_{01} \text{ for some } t \in \boldsymbol{R}\}.$$

Suppose y: $\boldsymbol{R} \to \boldsymbol{R}$ is a periodic solution, non-constant, of equation (ah), $a \in A_1$, with $\{y_t\colon t \in \boldsymbol{R}\} \subset V_{01} + \boldsymbol{Z}\bar{\xi}_h$. All translates $y + k\xi_h$, $k \in \boldsymbol{Z}$, are periodic solutions of equation (ah) as well. We obtain a non-constant periodic solution x: $\boldsymbol{R} \to \boldsymbol{R}$ of equation (ah) with $-r_h < \min x < \xi_h$ and with $x_t \in V_{01} + \boldsymbol{Z}\bar{\xi}_h$ for all $t \in \boldsymbol{R}$.

b) The case $x(\boldsymbol{R}) \subset (-r_h, r_h)$. Then every interval $[t-1, t]$ contains a zero of x (otherwise, equation (ah) would imply $|x(t)| \geqslant r_h$ for some $t \in \boldsymbol{R}$, see the sign condition (H.2)). Choose $t \in \boldsymbol{R}$ with $|x(t)| = \max|x| > 0$ (x is non-constant). For the largest zero $z \in [t-1, t]$,

$$x(t) = 0 + \int_z^t \dot{x} = \int_z^t ah(x(s-1))ds = \int_{z-1}^{t-1} ah(x(s))ds = \frac{a}{a_+} \int_{z-1}^{t-1} \bar{h} \circ x,$$

and (H.4) gives

$$|x(t)| \leqslant 1 \cdot \frac{a}{a_+} \cdot q \cdot \max|x| \leqslant q|x(t)|$$

which contradicts $x(t) \neq 0$.

c) It follows that $x(\boldsymbol{R}) \subset (\xi_h - r_h, \xi_h + r_h)$ is impossible, too.

d) The case $r_h \leqslant x(t) \leqslant \xi_0 - r_h$ for some $t \in \boldsymbol{R}$. (H.2), (H.3) and equation (ah) give $0 < r_h/2 < x$ on $[t, t+1]$. By (H.2) and equation (ah), there exists a first zero $z > t$ of $x - \xi_0$, or $0 < \dot{x}$ on $[t+1, \infty)$ and $x < \xi_0$ on $[t, \infty)$. In the first case, Proposition 2.1 (ii) yields $x > \xi_0 - r_h$ on $[z, \infty)$, a contradiction to periodicity and $x(t) \leqslant \xi_0 - r_h$. The second case contradicts periodicity, too.

e) The case $\xi_0 - r_h < x(t) \leqslant \xi_h - r_h$ for some $t \in \boldsymbol{R}$. We have $x_t \in V_{01} + \boldsymbol{Z}\bar{\xi}_h$, and $r_h < x(t) \leqslant \xi_h - r_h$. Therefore $|x_t - x_v^{a_0}| < \varepsilon_{01}$ with some $v \in \boldsymbol{R}$. (2.4) and $x(t) \leqslant \xi_h - r_h$ exclude $v \geqslant b_1$. The definition of t_- and $\varepsilon_0 < r_h$, in particular $r_h + \varepsilon_0 < \xi_h - r_h < x(t)$, give $t_- \leqslant v$. By the choice of ε_{01} and A_1, we now obtain $x(s) > \xi_h$ at some $s > t$. Consider $\nu \in N$ with $s < t + \nu p$ where p is the period of x. There is a largest $t^* \in (s, t + \nu p)$ with $x(t^*) = \xi_h$. On $(t^*, t + \nu p]$, $x < \xi_h$. On $[t^*, t^* + 1]$, $x \geqslant \xi_h - (r_h/2) > x(t + \nu p)$, by (H.2), (H.3) and equation (ah). Hence $t^* + 1 < t + \nu p$, and $\xi_0 < x < \xi_h$ on $(t^*, t^* + 1]$. It follows that either $0 > \dot{x}$ and $\xi_0 < x < \xi_h$ on $[t^* + 1, \infty)$, or that there exists a first zero z of $x - \xi_0$ in $(t^* + 1, \infty)$. The first possibility contradicts periodicity. The second possibility implies $x(z) = \xi_0$ and $\xi_0 < x(z-1)$. As above, we infer $|x_z - x_v^{a_0}| < \varepsilon_0$ with

$v \in [t_-, t_+]$. Using (2.3) we get

$$x(z) \geqslant x^{a_0}(v) - \varepsilon_0 > x^{a_0}(v-1) + 2\varepsilon_0 - \varepsilon_0 > x(z-1),$$

a contradiction.

f) The case $\xi_h - r_h < x$. By construction, $\min x < \xi_h$. Part c) above implies $x(t) = \xi_h + r_h$ for some $t \in \boldsymbol{R}$. Consider $\tilde{x} := x - \xi_h$. This is a periodic solution of equation (ah) with $\tilde{x}(t) = r_h$, and the arguments of part d) lead to a contradiction. ∎

Another tiny part of the proof of Theorem 13.2 is

PROPOSITION 2.4. *Equation* (ah), $a \in A_1$, *admits no solution* x: $\boldsymbol{R} \to \boldsymbol{R}$ *with* $x(\cdot + p) = x(\cdot) + k\xi_h$ *where* $p > 0$ *and* $k \in -\boldsymbol{N}$.

Proof. For a solution as above, $x(vp) \to -\infty$ as $v \to +\infty$. On the other hand, $x(t) \geqslant \xi_0 + l\xi_h$ for some $t \in \boldsymbol{R}$ and $l \in \boldsymbol{Z}$. With the periodicity of h, we obtain $x \geqslant \xi_0 + l\xi_h - r_h$ on $[t, \infty)$, see Proposition 2.1 (ii), a contradiction. ∎

3. Linearization of the semiflow. The parameterized semiflow X is of class C^1 on $(1, \infty) \times C \times \boldsymbol{R}^+$, and of class C^2 on $(2, \infty) \times C \times \boldsymbol{R}^+$. The partial derivative $D_2 X$ exists and is continuous on all of the domain of X. We have

$$D_2 X(t, \phi, a)\psi = y_t$$

with the solution y: $[-1, \infty) \to \boldsymbol{R}$ of the linear variational equation along $x = x(\phi, a)$,

$$\dot{y}(t) = ah'(x(t-1))y(t-1)$$

with $y_0 = \psi$.

COROLLARY 3.1. *For every* $(t, \phi, a) \in \boldsymbol{R}_0^+ \times C \times (0, 1]$, $\psi \in C$ *and all* $s \in [0, t]$,

$$|D_2 X(s, \phi, a)| \leqslant (1 + \max|h'|)^{t+1},$$

$$|X(s, \phi, a) - X(s, \psi, a)| \leqslant (\ldots)^{t+1}|\phi - \psi|.$$

Proof. Let $\phi \in C$, $\psi \in C$ and $a \in (0, 1]$ be given. Consider the solution y of the linear variational equation along $x = x(\phi, a)$, with $y_0 = \psi$. For $n \in \boldsymbol{N}$, $t \in [n-1, n]$ and $n-1 \leqslant s \leqslant t$, we obtain

$$y(s) = y(n) + a\int_{n-1}^{s-1} h' \circ x \cdot y,$$

$$|y(s)| \leqslant (1 + \sup|h'|)|y_n|.$$

By induction,

$$|y(s)| \leqslant (1 + \sup|h'|)^k |\psi|$$

whenever $s \in [k-1, t] \subset [k-1, k]$, $k \in \boldsymbol{N}$. With $D_2 X(s, \phi, a)\psi = y_s$ for all

$s \geqslant 0$, we obtain the first estimate. The second inequality follows by the mean value theorem. ∎

For $(t, \phi, a) \in (1, \infty) \times C \times \boldsymbol{R}^+$,

$$D_1 X(t, \phi, a)(1) = \dot{x}_t(\phi, a).$$

This shows that there is no partial derivative $D_1 X$ if, say, $0 < t < 1$ and ϕ is not differentiable at some point in $[t-1, 0]$. $\dot{x}_t(\phi, a)$ is called the *tangent vector of the trajectory* $\boldsymbol{R}_0^+ \ni s \to x_s(\phi, a) \in C$ at $s = t$.

The shifts $C \ni \phi \to X(t, \phi, a) \in C$ with $t \geqslant 1$, $a \in \boldsymbol{R}^+$, are compact as is easily seen from the construction of solutions by means of (2.1).

We linearize at $\phi = 0$ and define

$$T(t, \psi, a) := D_2 X(t, 0, a)\psi \quad \text{on } \boldsymbol{R}_0^+ \times C \times \boldsymbol{R}^+.$$

Then $T(t, \psi, a) = y_t(\psi, a)$ with the solution y: $[-1, \infty) \to \boldsymbol{R}$ of the linear equation

$$(a) \qquad \dot{y}(t) = ay(t-1)$$

with $y_0 = \psi$. Using complex-valued solutions, we extend T to a parameterized semigroup from $\boldsymbol{R}_0^+ \times C_{\boldsymbol{C}} \times \boldsymbol{R}^+$ into $C_{\boldsymbol{C}}$, with the same smoothness and compactness properties as stated for X. For each $a > 0$, $T(\cdot, \cdot, a)$ is a strongly continuous semigroup of bounded linear operators $C_{\boldsymbol{C}} \ni \phi \to T(t, \phi, a) \in C_{\boldsymbol{C}}$, $t \geqslant 0$. The spectrum σ_a of the infinitesimal generator of $T(\cdot, \cdot, a)$, $a > 0$, is given by the characteristic values, i.e. by the discrete, infinite, countable set of zeros of the analytic function

$$\Delta_a\colon\ z \to z - ae^{-z}.$$

PROPOSITION 3.1. *For every $a > 0$ there is precisely one positive zero $u = u(a)$ of Δ_a. $u(a)$ is a simple zero. The map $a \to u(a)$ is analytic and strictly increasing with $\lim_{a \to 0} u(a) = 0$. We have $\operatorname{Re}\lambda < \log a$ for every $a > 0$ and for every zero $\lambda \neq u(a)$ of Δ_a.*

Proof. See Proposition 2 in [22]. ∎

(H.5) shows that for all $a \in A_1$, $\log a < -u(a)$. Therefore we can choose an interval A_2 and reals λ, μ such that

$$(3.1) \qquad 0 < \mu \leqslant u(a) < -\lambda \leqslant -\operatorname{Re} z$$

for all $a \in A_2$, and all zeros $z \neq u(a)$ of Δ_a. In particular, $\sigma_a \cap i\boldsymbol{R} = \emptyset$ for $a \in A_2$, $\sigma_a \cap (\boldsymbol{R}^+ + i\boldsymbol{R}) = \{u(a)\}$, and $u(a)$ is a simple eigenvalue with eigenvector Φ_a: $I \ni t \to e^{u(a)t} \in \boldsymbol{R}^+$. We obtain

$$C = P_a \oplus Q_a$$

with

$$P_a = R\Phi_a = p_a C, \quad Q_a = q_a C, \quad q_a = \mathrm{id}_C - p_a,$$

$$p_a\phi = \langle \Psi_a, \phi\rangle_a \Phi_a,$$

$$\langle \Psi_a, \phi\rangle_a = \Psi_a(0)\phi(0) + a\int_I \Psi_a(t+1)\phi(t)dt \quad \text{for all } \phi \in C,$$

$$\Psi_a(t) = [1+ae^{-u(a)}]^{-1}e^{-u(a)t} \quad \text{for } t \in [0, 1].$$

p_a and q_a are projections of C onto P_a and Q_a, respectively, so that $1 \leqslant |p_a|$, $1 \leqslant |q_a|$. The spaces P_a and Q_a are invariant under the maps $T(t, \cdot, a)$. On P_a,

$$T(t, \phi, a) = e^{u(a)t}\phi \quad \text{for all } t \geqslant 0 \tag{3.2}$$

so that we have expansions on the "unstable spaces" P_a for $t > 0$.

There exist $\lambda_1 < 0$, an interval $A_3 \subset A_2$ and a constant $k_0 \geqslant 1$ such that

$$u(a) < -\lambda_1 < -\lambda \quad \text{on } A_3, \tag{3.3}$$

$$|T(t, \phi, a)| \leqslant k_0 e^{\lambda_1 t}|\phi| \quad \text{for all } a \in A_3,\ \phi \in Q_a,\ t \geqslant 0 \tag{3.4}$$

or, $T(t, \cdot, a)$ defines a contraction on the "stable space" Q_a (if $t > 0$ is sufficiently large) which is stronger than expansion in P_a. This property is crucial for the bifurcation result which we are going to prove.

For later use we prepare estimates of shifts $T(N, \cdot, a)$ with constants which are locally uniform with respect to the parameter, and which satisfy certain inequalities.

PROPOSITION 3.2. *There exist $N \in N$, $N \geqslant 3$, real numbers $\mu_0, \mu_1, \mu_2, \lambda_2, \lambda_3$ with $0 < \mu_0 < \mu_1 < \mu_2 < -\lambda_3 < -\lambda_2 < -\lambda_1$, a constant $c > 0$, an interval A_4 and constants $\alpha < 1$, $\beta > 1$, $\gamma \geqslant \beta$ such that*

$$\mu_0 < u(a) < \mu_1 \quad \textit{for all } a \in A_4, \tag{3.5}$$

$$\begin{cases} \beta|\phi| \leqslant |T(N, \phi, a)| \leqslant \gamma|\phi| & \textit{for all } \phi \in P_a,\ a \in A_4, \\ |T(N, \phi, a)| \leqslant \alpha|\phi| & \textit{for all } \phi \in Q_a,\ a \in A_4, \end{cases} \tag{3.6}$$

$$c < \alpha < \alpha + 2c < 1 < \beta - c, \tag{3.7}$$

$$\frac{(\alpha+3c)(\gamma+c)}{\beta-c} \leqslant 1, \tag{3.8}$$

$$c < \frac{((\beta-c)-1)^2}{8}, \tag{3.9}$$

$$\alpha + 4c < \frac{1}{\beta+c}, \tag{3.10}$$

$$c < \mu_0 e^{3\mu_0}, \tag{3..11}$$

$$\gamma + c = \beta + 2c < e^{\mu_2 N}, \tag{3.12}$$

$$\alpha + c < e^{\lambda_3 N}. \tag{3.13}$$

Proof. a) Choose $\mu_0, \mu_2, \lambda_3, \lambda_2$ such that $0 < \mu_0 < u(a_0) < \mu_2 < -\lambda_3 < -\lambda_2 < -\lambda_1$. Choose $N \in N$, $N \geqslant 3$ and $k_0 e^{\lambda_1 N} < e^{\lambda_2 N} < 1$. Set $\alpha := e^{\lambda_2 N}$ so that the second estimate in (3.6) holds true, by (3.4). Set $\tilde{\beta} := e^{u(a_0)N}$. Choose $\tilde{c} > 0$ so small that $\tilde{c} < \alpha < \alpha + \tilde{c} < 1 < \tilde{\beta} - \tilde{c}$,

$$\frac{(\alpha+3\tilde{c})(\tilde{\beta}+\tilde{c})}{\tilde{\beta}-\tilde{c}} \leqslant 1,$$

$$\tilde{c} < \frac{((\tilde{\beta}-\tilde{c})-1)^2}{8}, \quad \alpha+2\tilde{c} < \frac{1}{\tilde{\beta}+(\tilde{c}/2)},$$

$$\tilde{c} < \mu_0 e^{3\mu_0}, \quad \tilde{\beta}+2\tilde{c} < e^{\mu_2 N}, \quad \alpha+\tilde{c} < e^{\lambda_3 N}.$$

b) The map $\mu \to e^{\mu N}$ is continuous, and there exists $\mu_1 \in (u(a_0), \mu_2)$ with $e^{\mu_1 N} < \tilde{\beta} + (\tilde{c}/2)$. By continuity of $0 < a \to u(a)$, we may choose an interval A_4 so small that $\mu_0 < u(a) < \mu_1$ and

$$e^{\mu_1 N} < e^{u(a)N} + (\tilde{c}/2) \quad \text{for all } a \in A_4. \tag{3.14}$$

c) Set $c := \tilde{c}/2$, $\beta := e^{\mu_1 N} - c$, $\gamma := \beta + c = e^{\mu_1 N}$. Let $a \in A_4$. For (3.5), see b). The first line in (3.6) follows from (3.2), with (3.4) and $e^{u(a)N} < e^{\mu_1 N} = \gamma$. The inequalities (3.7)–(3.13) follow from

$$\tilde{\beta} - (\tilde{c}/2) = e^{u(a_0)N} - (\tilde{c}/2) < e^{\mu_1 N} - (\tilde{c}/2) = \beta < \tilde{\beta}$$

and from the inequalities in a). ∎

Frequently we shall use that p_a is monotone:

$$p_a \phi > 0 \quad \text{if } 0 \neq \phi \geqslant 0 \text{ and } a \in A_4, \tag{3.15}$$

that the maps $A_4 \ni a \to p_a \in L_c(C, C)$ and $A_4 \ni a \to q_a \in L_c(C, C)$ are analytic, and that p_a and q_a are bounded on A_4. We set

$$c_{pq} := \sup_{A_4} |p_a| + |q_a| < \infty.$$

Note $c_{pq} \geqslant 1$.

4. The saddle point property. The properties of X and its linearization T imply that there exist open balls $B_2 \subset B_1 \subset \{\phi \in C\colon |\phi| < \delta_h\}$, constants $k_1, \varkappa > 0$, an interval A_5 and maps

$$\left.\begin{aligned} u_a&\colon P_a \cap B_1 \to Q_a \subset C, \\ s_a&\colon Q_a \cap B_1 \to P_a \subset C, \end{aligned}\right\} \quad a \in A_5,$$

such that for every $a \in A_5$ the following holds true:

(4.1) $u_a(0) = 0$, $s_a(0) = 0$; u_a and s_a are Lipschitz continuous with constant $1/2$; the "graphs" $\{\phi + u_a(\phi)\colon \phi \in P_a \cap B_1\}$ and $\{\phi + s_a(\phi)\colon \phi \in Q_a \cap B_1\}$ are tangent to P_a and Q_a, respectively, at $\phi = 0$, i.e. $\lim_{\phi \to 0, \neq} |u_a(\phi)|/|\phi| = 0$ and $\lim_{\phi \to 0, \neq} |s_a(\phi)|/|\phi| = 0$.

(4.2) For every $\phi \in P_a \cap B_1$ there is a unique solution $x^* = x^*(\phi + u_a(\phi), a)$: $\boldsymbol{R} \to \boldsymbol{R}$ of equation (ah) with $x_0^* = \phi + u_a(\phi)$ and $|x_t^*| \leqslant k_1 e^{\varkappa t} |x_0^*|$ for all $t \leqslant 0$.

(4.3) If x: $\boldsymbol{R} \to \boldsymbol{R}$ is a solution to equation (ah) with $x_t \in B_2$ for all $t \leqslant 0$, then $x = x^*(\phi + u_a(\phi), a)$ for some $\phi \in P_a \cap B_1$.

(4.4) $\phi \in Q_a \cap B_1$ and $t \geqslant 0$ imply $|x_t| \leqslant k_1 e^{\varkappa t} |x_0|$ for $x = x(\phi + s_a(\phi), a)$.

(4.5) If x: $[-1, \infty) \to \boldsymbol{R}$ is a solution to equation (ah) with $x_t \in B_2$ for all $t \geqslant 0$, then $x_0 = \phi + s_a(\phi)$ for some $\phi \in Q_a \cap B_1$.

Moreover, one can achieve that the maps $B_2 \times A_5 \ni (\phi, a) \to u_a(p_a \phi)$, $s_a(q_a \phi) \in C$ are defined and of class C^2. This will be used in Section 5.

We derive some easy consequences.

PROPOSITION 4.1. *There is an open ball B_3 such that for every $a \in A_5$ we have*

(i) $u_a(P_a \cap B_3) \subset Q_a \cap B_3$, $s_a(Q_a \cap B_3) \subset P_a \cap B_3$; *i.e. the "graphs"* $U_a := \{\phi + u_a(\phi)\colon \phi \in P_a \cap B_3\}$ *and* $S_a := \{s_a(\phi) + \phi\colon \phi \in Q_a \cap B_3\}$ *are both contained in the parallelogram* $V_{a3} := (P_a \cap B_3) + (Q_a \cap B_3)$.

(ii) $|\phi| < \delta_h$ *for all* $\phi \in V_{a3}$,

(4.1′) $U_a \cap S_a = \{0\}$; U_a *is tangent to* P_a *at* $\phi = 0$; S_a *is tangent to* Q_a *at* $\phi = 0$;

(4.2′) *for every* $\phi \in U_a$ *there is a unique solution* $x^* = x^*(\phi, a)$: $\boldsymbol{R} \to \boldsymbol{R}$ *of equation* (ah) *with* $x_0^* = \phi$ *and* $|x_t^*| \leqslant k_1 e^{\varkappa t} |\phi|$ *for all* $t \leqslant 0$;

(4.3′) *if* x: $\boldsymbol{R} \to \boldsymbol{R}$ *is a solution of equation* (ah) *with* $x_t \in V_{a3}$ *for all* $t \leqslant 0$ *then* $x = x^*(\phi, a)$ *for some* $\phi \in U_a$;

(4.4′) $\phi \in S_a$ *and* $t \geqslant 0$ *imply* $|x_t| \leqslant k_1 e^{\varkappa t} |\phi|$ *for* $x = x(\phi, a)$;

(4.5′) *if* x: $[-1, \infty) \to \boldsymbol{R}$ *is a solution of equation* (ah) *with* $x_t \in V_{a3}$ *for all* $t \geqslant 0$ *then* $x_0 = \phi$ *for some* $\phi \in S_a$;

(4.6) *if* $\phi \in U_a$, $x = x(\phi, a)$, $t \geqslant 0$ *and* $x_s \in V_{a3}$ *for all* $s \in [0, t]$ *then* $x_s \in U_a$ *for all* $s \in [0, t]$.

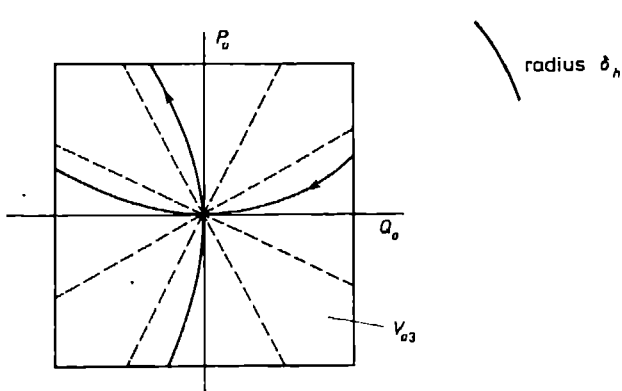

Fig. 11

Proof. Choose

$$B_3 \subset \frac{1}{2(k_1+1)} \cdot B_2 \cap \frac{1}{2} \cdot B_1.$$

Let $a \in A_5$. (i) follows from the Lipschitz condition in (4.1). If $\phi \in V_{a3}$ then $\phi = p_a\phi + q_a\phi \in 2 \cdot B_3$. Hence $V_{a3} \subset 2 \cdot B_3 \subset B_2$, which gives (ii). $U_a \cap S_a = \{0\}$ in (4.1′) is also a consequence of the Lipschitz condition in (4.1). (4.2′) and (4.4′) are obvious from (4.2) and (4.4), and from $B_3 \subset B_1$.

Proof of (4.3′). Let $x: \boldsymbol{R} \to \boldsymbol{R}$ be a solution of equation (*ah*) with $x_t \in V_{a3} \subset B_2$ for all $t \leqslant 0$. (4.3) implies $x_0 = \phi + u_a(\phi)$ for some $\phi \in P_a \cap B_1$, and $x = x^*(\ldots, a)$. $x_0 \in V_{a3}$ gives $\phi \in B_3$, or $x_0 \in U_a$.

The proof of (4.5′) is analogous.

Proof of (4.6). Let $\phi \in U_a \subset V_{a3}$. Assume $x_s \in V_{a3}$ $(\subset B_2)$ on some interval $[0, t]$, for the solution $x = x(\phi, a)$: $[-1, \infty) \to \boldsymbol{R}$. (4.2′) shows that on $[-1, \infty)$, x and $x^*(\phi, a)$ coincide. For $s \leqslant 0$, $|x_s^*| \leqslant k_1|\phi| = k_1|p_a\phi + q_a\phi|$, see (4.2). We have $k_1(p_a\phi + q_a\phi) \in k_1 \cdot V_{a3} \subset 2k_1 \cdot B_3 \subset B_2$. Therefore $x_s^* \in B_2$ for all $s \leqslant 0$.

Now let $\psi = x_s$ for some $s \in [0, t]$. Then $\psi = x_0^*(s + \cdot)$, for the solution $x^*(s + \cdot)$: $\boldsymbol{R} \to \boldsymbol{R}$ of equation (*ah*). All segments $x_v^*(s + \cdot)$, $v \leqslant 0$, are contained in B_2, and (4.3) yields $x_s = \psi = x_0^*(s + \cdot) = \tilde{\phi} + u_a(\tilde{\phi})$ for some $\tilde{\phi} \in P_a \cap B_1$. $x_s \in V_{a3}$ gives $\tilde{\phi} \in B_3$. Therefore $x_s \in U_a$. ∎

PROPOSITION 4.2. *Let $a \in A_5$. There exist $y_a > 0$ and solutions x^{a+}: $\boldsymbol{R} \to \boldsymbol{R}$, x^{a-}: $\boldsymbol{R} \to \boldsymbol{R}$ of equation (ah), and $v = v(a) \in \boldsymbol{R}$ such that*

(i) $0 < y < y_a$ *implies* $0 < y\Phi_a + u_a(y\Phi_a)$; $0 > y > -y_a$ *implies* $0 > y\Phi_a + u_a(y\Phi_a)$;

(ii) $p_a x_0^{a+} \in \partial B_3$, $\lim_{t \to -\infty} x^{a+}(t) = 0$, $x_t^{a+} \in U_a \subset V_{a3}$ *and*

$$0 < \left\{ \begin{array}{l} x^{a+} \\ \dot{x}^{a+} \\ p_a x_t^{a+} \\ p_a x_t^{a+} - s_a(q_a x_t^{a+}) \end{array} \right\} \quad \textit{for all } t < 0;$$

(iii) $\phi \in U_a$ *and* $(0 < p_a\phi$ *or* $0 < p_a\phi - s_a(q_a\phi))$ *imply* $\phi = x_t^{a+}$ *for some* $t < 0$;

(iv) $p_a x_0^{a-} \in \partial B_3$, $\lim_{t \to -\infty} x^{a-}(t) = 0$, $x_t^{a-} \in U_a \subset V_{a3}$ *and*

$$0 > \left\{ \begin{array}{l} x^{a-} \\ \dot{x}^{a-} \\ p_a x_t^{a-} \\ p_a x_t^{a-} - s_a(q_a x_t^{a-}) \end{array} \right\} \quad \textit{for all } t < 0;$$

(v) $\phi \in U_a$ *and* $(p_a\phi < 0$ *or* $p_a\phi - s_a(q_a\phi) < 0)$ *imply* $\phi = x_t^{a-}$ *for some* $t < 0$;

(vi) $x^{a+} = x^a(\cdot + v(a))$.

Proof. Let $a \in A_5$.

a) U_a is tangent to P_a at $\phi = 0$, and there exists $y_a > 0$ with

$$|[y\Phi_a + u_a(y\Phi_a)] - y\Phi_a| = |u_a(y\Phi_a)| \leqslant 2^{-1} e^{-u(a)} |y\Phi_a| = 2^{-1} e^{-u(a)} |y|$$

for $0 < |y| < y_a$, $y \in \boldsymbol{R}$. $-y\Phi_a(t) = ye^{u(a)t} \geqslant ye^{-u(a)}$ for all $t \in I$ implies $[\ldots] > 0$ for $0 < y < y_a$ and $[\ldots] < 0$ for $0 > y > -y_a$. This proves existence of y_a such that (i) holds.

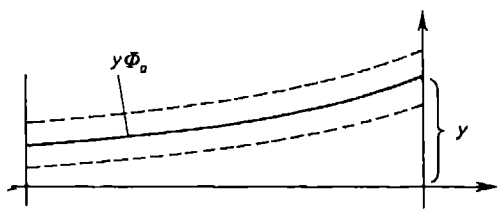

Fig. 12

b) Let r_3 denote the radius of B_3. Choose $y \in (0, y_a)$ so small that $2y < r_3$, $k_1 \cdot 2y < r_3 c_{pq}^{-1}$, $k_1 \cdot 2y < y_a/|p_a|$. Then $y\Phi_a \in P_a \cap B_3$. Consider the solution $x^* = x^*(y\Phi_a + u_a(y\Phi_a), a)$. (i) shows $0 < x_0^*$. For $t \leqslant 0$ we have

$$|x_t^*| \leqslant k_1 |x_0^*| \leqslant k_1 \cdot 2y. \tag{4.2''}$$

We prove $x_t^* \in U_a$ for all $t \leqslant 0$: (4.2″) implies $|p_a x_t^*| < r_3$ on $(-\infty, 0]$, and $|q_a x_t^*| < r_3$ on $(-\infty, 0]$. Hence $x_t^* \in V_{a3}$ on $(-\infty, 0]$. Now one can use (4.3′) in order to obtain $x_t^* \in U_a$ for all $t \leqslant 0$.

Proof of $0 < p_a x_t^*$ for all $t \leqslant 0$: Recall $p_a x_t^* = \langle \Psi_a, x_t^* \rangle_a \Phi_a$. Suppose $\langle \Psi_a, x_t^* \rangle_a \leqslant 0$ for some $t \leqslant 0$. We have $0 < x_0^*$, and consequently $0 < \langle \Psi_a, x_0^* \rangle_a$. It follows that there exists $s < 0$ with $0 = \langle \Psi_a, x_s^* \rangle_a$, or $p_a x_s^* = 0$. With $x_s^* \in U_a$, we find $x_s^* = 0$, hence $x_0^* = 0$, a contradiction.

Proof of $0 < x^*$ and $0 < \dot{x}^*$ for all $t \leqslant 0$: (4.2″) gives $|p_a x_t^*| < y_a$. With $0 < p_a x_t^* = \langle \Psi_a, x_t^* \rangle_a \Phi_a$, we find $0 < \langle \Psi_a, x_t^* \rangle_a < y_a \cdot x_t^* \in U_a$ yields $x_t^* = \langle \ldots \rangle_a \Phi_a + u_a(\langle \ldots \rangle_a \Phi_a)$. Now (i) shows $0 < x_t^*$. Therefore $0 < x^*$ on $(-\infty, 0]$. We have $x_t^* \in U_a \subset V_{a3}$ for $t \leqslant 0$, hence $|x_t^*| < \delta_h$ for all $t \leqslant 0$. $0 < x^* < \delta_h$ on $(-\infty, 0]$ and equation (ah) imply $0 < \dot{x}^*$ on $(-\infty, 0]$.

c) We consider x^* on $\boldsymbol{R}_0^+$. There exists $s > 0$ such that $0 < x^* < \delta_h$ on $(-\infty, s)$ and $x^*(s) = \delta_h$, as follows from $0 < h$ on $(0, \xi_0) \supset (0, \delta_h)$. Because of $|\phi| < \delta_h$ for $\phi \in V_{a3}$, there is a smallest $\hat{s} \in (0, s)$ with $x_t^* \in V_{a3}$ for $t < \hat{s}$, and $|p_a x_{\hat{s}}^*| = r_3$ or $|q_a x_{\hat{s}}^*| = r_3$. (4.6) implies $x_t^* \in U_a$ for all $t < \hat{s}$. The Lipschitz condition for u_a gives

$$|q_a x_{\hat{s}}^*| = |\lim_{\substack{t \to s \\ <}} q_a x_t^*| = \lim_{\cdots} |\ldots| = \lim_{\cdots} |u_a(p_a x_t^*)| \leqslant 2^{-1} \lim_{\cdots} |p_a x_t^*| \leqslant 2^{-1} r_3.$$

Therefore $|p_a x_{\hat{s}}^*| = r_3$.

d) We set $x^{a+} := x^*(\cdot + \hat{s})$. Then $0 < x^{a+}$ and $0 < \dot{x}^{a+}$ on $(-\infty, 0]$, $x_t^{a+} \in U_a$ for $t < 0$, $|p_a x_0^{a+}| = r_3$, $\lim_{t \to -\infty} x^{a+}(t) = 0$. $0 < \langle \Psi_a, x_t^{a+} \rangle_a$ and $0 < p_a x_t^{a+}$ for $t < 0$ are also obvious.

e) Proof of $0 < p_a x_t^{a+} - s_a(q_a x_t^{a+})$ for $t < 0$: Let $t < 0$, $\phi := x_t^{a+}$. Then $0 < \phi \in U_a$, and $\phi = p_a \phi + u_a(p_a \phi)$, or $q_a \phi = u_a(p_a \phi)$. Hence $|q_a \phi| \leqslant 2^{-1} |p_a \phi|$. With $0 < \phi$, $0 < p_a \phi$. Therefore $|q_a \phi| < |p_a \phi|$. Consider the points $\phi_s := p_a \phi + s \cdot q_a \phi$, $s \in [0, 1]$. None of these points is contained in S_a – $\phi_s \in S_a$ would imply $0 < p_a \phi = s_a(s \cdot q_a \phi)$, $0 < |p_a \phi| \leqslant 2^{-1} s |q_a \phi| \leqslant |q_a \phi|$, a contradiction to $|q_a \phi| < |p_a \phi|$.

We obtain $p_a \phi_s - s_a(q_a \phi_s) = y_s \Phi_a$ with $y_s \neq 0$ for all s in $[0, 1]$. Continuity of $s \to y_s = \langle \Psi_a, p_a \phi_s - s_a(q_a \phi_s) \rangle_a$ implies $y_s > 0$ on $[0, 1]$, or $0 > y_s$ on $[0, 1]$. We have $\phi_0 = p_a \phi > 0$, hence $q_a \phi_0 = 0$, and $y_0 \Phi_a = p_a \phi_0 - s_a(q_a \phi_0) = p_a \Phi > 0$. From $\Phi_a > 0$, we infer $y_0 > 0$, and consequently $y_s > 0$ on $[0, 1]$. For $s = 1$, $\phi_1 = \phi = x_t^{a+}$, and $p_a x_t^{a+} - s_a(q_a x_t^{a+}) = y_1 \Phi_a$ with $y_1 > 0$.

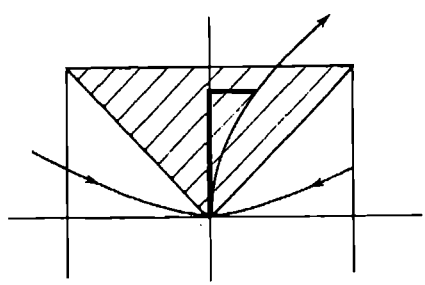

Fig. 13

f) Proof of (iii): Let $\phi \in U_a$, $0 < p_a \phi$. $r_3 > |p_a \phi|$, $r_3 = |p_a x_0^{a+}|$ and $\lim_{t \to -\infty} x^{a+}(t) = 0$ imply $|p_a \phi| = |p_a x_t^{a+}|$ for some $t < 0$. $0 < p_a \phi$ and $0 < p_a x_t^{a+}$ give $p_a \phi = p_a x_t^{a+}$. $\phi \in U_a \ni x_t^{a+}$ now yields $\phi = x_t^{a+}$.

Let $\phi \in U_a$, $p_a \phi - s_a(q_a \phi) > 0$. Then $q_a \phi = u_a(p_a \phi)$, and $|s_a(q_a \phi)| \leqslant 4^{-1} |p_a \phi|$. We have $p_a \phi = 0$ or $p_a \phi < 0$ or $p_a \phi > 0$. $p_a \phi = 0$ would imply $\phi = 0$, a contradiction to $0 < p_a \phi - s_a(q_a \phi)$. In case $p_a \phi < 0$, we would have

$$p_a \phi - s_a(q_a \phi) \leqslant p_a \phi + 4^{-1} |p_a \phi|,$$

$$0 > -|p_a \phi| = \min p_a \phi = p_a \phi(s) \quad \text{for some } s \in I,$$

hence

$$p_a \phi(s) + 4^{-1} |p_a \phi| < 0, \qquad [p_a \phi - s_a(q_a \phi)](s) < 0,$$

a contradiction.

Therefore $p_a \phi > 0$, and consequently $\phi = x_t^{a+}$ with $t < 0$.

g) Existence of x^{a-} with (iv) and (v) is proved in the same way as we obtained x^{a+} with (ii) and (iii).

h) Proof of (vi): Consider x^{a+}. We have $x_t^{a+} \in V_{a3}$ on $(-\infty, 0)$, $|x^{a+}(t)| \leqslant |x_t^{a+}| < \delta_h$, see Proposition 4.1(ii). Recall Corollary 2.1: $\delta_h < \xi_0 - r_h/2$, $0 < \dot{x}^{a+}$ and $0 < x^{a+}$ on $(-\infty, 0]$ imply $x^{a+}(0) < x^{a+}$ on $\boldsymbol{R}^+$. We infer

$$(4.7) \qquad t_1 = t_2 < 0 \quad \text{whenever} \quad x^{a+}(t_1) = x^{a+}(t_2) < x^{a+}(0).$$

$\lim_{t\to-\infty} x^a(t) = 0$ implies $x_t^a \in U_a$ for all $t \leqslant t^*$, with some $t^* \in \boldsymbol{R}$, see (4.3′). Note $p_a x_t^a > 0$. Part (iii) gives $x_t^a = x_v^{a+}$ with $v = v(t) < 0$, for all $t \leqslant t^*$. Set $v^* := v(t^*)$ and $v(a) := t^* - v^*$. Then $x^a = x^{a+}(\cdot + v^* - t^*)$ on $[t^*, \infty)$.

Let $t < t^*$. As above, $x^a = x^{a+}(\cdot + v - t)$ on $[t, \infty)$. In particular, $(x^{a+}(v^*) =)\ x^a(t^*) = x^{a+}(t^* + v - t)$. We have $0 < x^{a+}(v^*) \leqslant |x_{v^*}^{a+}| < \delta_h$, since $x_{v^*}^{a+} \in V_{a3}$ (see (ii), $v^* < 0$, and Proposition 4.2 (ii)). By (4.7), $v^* = t^* + v - t$, or $x^a(t) = x^{a+}(t + v - t) = x^{a+}(t + v^* - t^*)$.

It follows that $x^{a+}(t) = x^a(t + v(a))$ for all $t \in \boldsymbol{R}$. ∎

As another little part of Theorem 13.2, we can now derive

COROLLARY 4.1. *For $a_0 < a \in A_5$, there is no heteroclinic solution x: $\boldsymbol{R} \to \boldsymbol{R}$ of equation (ah) with $\lim_{t\to-\infty} x(t) = 0$, $\lim_{t\to+\infty} x(t) = \xi_h$.*

Proof. a) Suppose x: $\boldsymbol{R} \to \boldsymbol{R}$ is a solution of equation (*ah*), $a_0 < a \in A_5$, with $\lim_{t\to-\infty} x(t) = 0$, $\lim_{t\to+\infty} x(t) = \xi_h$. Using (4.3′), we find $x_s \in U_a$ for some $s \in \boldsymbol{R}$. We have $x_s \neq 0$, because of $x(t) \to \xi_h$ as $t \to +\infty$. By (4.1′), $x_s \notin S_a$. Hence $x_s \in U_a \subset V_{a3} = P_a \cap B_3 + Q_a \cap B_3$, $x_s = p_a x_s + q_a x_s$ with $p_a x_s \in B_3 \ni q_a x_s$ and $p_a x_s \neq s_a(q_a x_s)$. It follows that either $p_a x_s - s_a(q_a x_s) < 0$ or $p_a x_s - s_a(q_a x_s) > 0$ (recall $p_a \phi = \langle \Psi_a, \phi \rangle_a \Phi_a$ and $\Phi_a > 0$). By Proposition 4.2 (iii) and (v), either $x_s = x_u^{a-}$ with $u < 0$, or $x_s = x_u^{a+}$ with $u < 0$.

b) In the first case, $0 > x_u^{a-} = x_s > -\delta_h$, by Proposition 4.2 (iv) and Proposition 4.1 (ii). Proposition 2.1 (iii) and the periodicity of h yield $x < 0$ on $[s-1, \infty)$, a contradiction to $x(t) \to \xi_h$ as $t \to +\infty$.

c) In the second case, $x_s = x_u^{a+} = x_{u+v}^a$, by Proposition 4.2(vi). Therefore $x(t) = x^a(t + u + v - s)$ for all $t \geqslant 0$, a contradiction to $\xi_h < \liminf_{t\to+\infty} x^a(t)$.

PROPOSITION 4.3. *There is an open ball B_4 such that for all $a \in A_5$,*

(i) $B_4 \subset V_{a3}$;

(ii) $\phi \in S_a \cap B_4$ *or* ($q_a \phi \in B_4$ *and* $\phi \in S_a$) *imply* $x_t(\phi, a) \in S_a \cap B_3$ *for all* $t \geqslant 0$.

Proof. a) Let r_3 denote the radius of B_3, as before. Choose a radius $r_4' \in (0, r_3)$ with

$$(1 + k_1)\frac{3}{2} r_4' < \frac{r_3}{c_{pq}} \leqslant r_3.$$

Let $a \in A_5$. For $|\phi| < r_4'$, $|p_a \phi| \leqslant |p_a| r_4' < r_3$ and $|q_a \phi| < r_3$, or $\phi \in V_{a3}$.

b) Let $\phi \in S_a$, $|q_a \phi| < r_4'$. Set $x := x(\phi, a)$. By (4.4′),

$$|x_t| \leqslant k_1 |\phi| \leqslant k_1(|p_a \phi| + |q_a \phi|) \leqslant k_1(2^{-1} r_4' + r_4') = k_1(3r_4'/2)$$

for all $t \geqslant 0$, and

$$|p_a x_t| < r_3, \quad |q_a x_t| < r_3 \quad \text{for all } t \geqslant 0.$$

By (4.5′), $x_t \in S_a$ for all $t \geqslant 0$. $x_t \in B_3$ is obvious from $|x_t| \leqslant k_1(3r_4'/2) < r_3$.

c) Let B_4 denote the open ball in C with radius $r_4 := r_4'/(2c_{pq}) < r_4'$, centered at $\phi = 0$. In case $|\phi| < r_4$ and $\phi \in S_a$, we find $|q_a\phi| < r_4'$, and b) applies. In case $|q_a\phi| < r_4$ and $\phi \in S_a$, observe $r_4 < r_4'$ and use b).

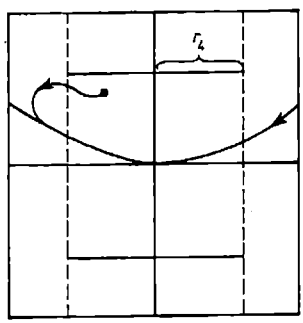

Fig. 14

PROPOSITION 4.4. *Let $a \in A_5$, $t \geqslant 0$ and assume $x_s \in V_{a3} \setminus S_a$ for all $s \in [0, t]$, with a solution x: $[-1, \infty) \to \boldsymbol{R}$ of equqtion (ah). Then $p_a x_s - s_a(q_a x_s) = y_s \Phi_a$ with either $y_s > 0$ for all s in $[0, t]$, or $0 > y_s$ for all $s \in [0, t]$.*

Proof. $y_s = \langle \Psi_a, x_s - (q_a x_s + s_a(q_a x_s)) \rangle_a$ depends continuously on $s \in [0, t]$, and $y_s = 0$ is equivalent with $x_s \in S_a$. ∎

II

5. The transformed semiflow. We introduce local coordinates at the saddle point $\phi = 0$ so that the local stable and unstable manifolds S_a and U_a become subsets of Q_a and P_a, respectively. This will be convenient for the investigation of trajectories close to $\phi = 0$.

The application of the inclination lemma from [23] in Section 9 (proof of Theorem 6.1) requires a local semiflow which is of class C^2 for $t > 2$. The original semiflow X has this property. In order to obtain from X a sufficiently smooth transformed semiflow, we make use of the fact that both maps

$$B_4 \times A_5 \ni (\phi, a) \to u_a(p_a\phi), \qquad s_a(q_a\phi) \in C$$

are of class C^2.

PROPOSITION 5.1. *There exist open balls B_5, B_6, an open interval A_6 and an open ball D^0 with the following properties.*

(i) *The* C^2*-map* $G\colon B_5 \times A_6 \to C$,

$$(\phi, a) \to [p_a\phi - s_a(q_a\phi)] + [q_a\phi - u_a(p_a\phi)],$$

satisfies $G(0, a) = 0$, $D_1 G(0, a) = \mathrm{id}_C$ *for all* $a \in A_6$,

$$|D_1 G(\phi, a) - \mathrm{id}_C| \leqslant 1/(2c_{pq}) \leqslant 1/2 \quad \text{on } B_5 \times A_6.$$

(ii) *There exists a* C^2*-map* $G^-\colon D^0 \times A_6 \to B_5$ *with*

$$G(G^-(\psi, a), a) = \psi \quad \text{on } D^0 \times A_6.$$

(iii) $G(B_6 \times A_6) \subset D^0$ *and* $G^-(G(\phi, a), a) = \phi$ *on* $B_6 \times A_6$.

(iv) *For every* $a \in A_6$, $G((U_a \cap B_5) \times \{a\}) \subset P_a$ *and* $G((S_a \cap B_5) \times \{a\}) \subset Q_a$.

(v) *For all* $a \in A_6$ *and* $\psi \in P_a \cap D^0$ *with* $\psi \left\{\begin{smallmatrix} > \\ < \end{smallmatrix}\right\} 0$, *we have*

$$U_a \ni G^-(\psi, a) = \begin{Bmatrix} x_s^{a+} = x_t^a \\ x_s^{a-} \end{Bmatrix} \quad \text{with some } s < 0,\ t \in \mathbf{R}.$$

Proof. a) Consider the C^2-map $\tilde{G}\colon B_4 \times A_5 \times C \to C$, $(\phi, a, \psi) \to G(\phi, a) - \psi$. Note $D_1\tilde{G}(0; a_0, 0) = D_1 G(0, a_0) = \mathrm{id}_C$ and $\tilde{G}(0, a_0, 0) = 0$. Apply the implicit function theorem to the restriction of $\tilde{G}$ to $B_5 \times A_{6.1} \times C$, with an open ball B_5 and an open interval $A_{6.1} \subset A_5$, $a_0 \in A_{6.1}$, so small that

$$|D_1\tilde{G}(\phi, a, 0) - \mathrm{id}_C| \leqslant 1/(2c_{pq}) \quad \text{on } B_5 \times A_{6.1},$$

and derive assertions (i)–(iv).

b) Proof of (v): Set $\phi := \tilde{G}(\psi, a) \in B_5 \subset B_4 \subset V_{a3}$ (see Proposition 4.3(i)). Then $G(\phi, a) = \psi \in P_a$, $0 = q_a\psi = q_a\phi - u_a(p_a\phi)$, $\phi \in U_a$. Furthermore,

$$0 \left\{\begin{matrix} < \\ > \end{matrix}\right\} \psi = p_a\psi = p_a\phi - s_a(q_a\phi)$$

gives

$$\phi = \begin{Bmatrix} x_s^{a+} \\ x_s^{a-} \end{Bmatrix} \quad \text{with some } s < 0,$$

see Proposition 4.2 (iii) and (v). In case $0 < \psi$, apply Proposition 4.2 (vi). ∎

In order to define the transformed parameterized semiflow Y, we choose $D^1 \subset D^0$ and $A_7 \subset A_6$ with

$$X([0, N+1] \times G^-(D^1 \times A_7) \times A_7) \subset B_6. \tag{5.1}$$

In particular,

$$G^-(D^1 \times A_7) \subset B_6. \tag{5.2}$$

We set

$$\Omega := \{(t, \psi, a) \in \mathbf{R}_0^+ \times D^1 \times A_7\colon X(s, G^-(\psi, a), a) \in B_6 \text{ for all } s \in [0, t]\},$$

$$Y(t, \psi, a) := G(X(t, G^-(\psi, a), a), a)$$

for all $(t, \psi, a) \in \Omega$. Ω is an open subset of $\boldsymbol{R}_0^+ \times D^1 \times A_7$, with

$$[0, N+1] \times D^1 \times A_7 \subset \Omega, \tag{5.3}$$

$$\boldsymbol{R}_0^+ \times \{0\} \times A_7 \subset \Omega. \tag{5.4}$$

Note

$$G^-(Y(t, \psi, a), a) = X(t, G^-(\psi, a), a) \quad \text{on } \Omega. \tag{5.5}$$

For every $a \in A_7$, Y defines a local semiflow Y_a with domain $\Omega_a := \{(t, \psi) \in \boldsymbol{R}_0^+ \times D^1 \colon (t, \psi, a) \in \Omega\}$.

We set $R := Y - T$ on Ω, and obtain

PROPOSITION 5.2. (i) *$Y(t, 0, a) = 0 = R(t, 0, a)$ on $\boldsymbol{R}_0^+ \times A_7$.*

(ii) $$(t, \psi, a) \in \Omega \quad \textit{and} \quad \psi \in \begin{Bmatrix} P_a \\ Q_a \end{Bmatrix}$$

imply

$$Y(t, \psi, a) \in \begin{Bmatrix} P_a \\ Q_a \end{Bmatrix} \ni R(t, \psi, a).$$

(iii) *The maps Y and R are of class C^1 on $(1, \infty) \times D^1 \times A_7 \cap \Omega$, of class C^2 on $(2, \infty) \times D^1 \times A_7 \cap \Omega$. The partial derivatives $D_2 Y$, $D_2 R$ exist and are continuous on all of Ω. We have $D_2 Y(t, 0, a) = T(t, \cdot, a)$ and $D_2 R(t, 0, a) = 0$ on $\boldsymbol{R}_0^+ \times A_7$.*

(iv) *Let $a \in A_7$, $(s, \psi) \in \Omega_a$, $1 < t < s$. Then $(t, \psi) \in \Omega_a$, $(t, Y_a(s-t, \psi)) \in \Omega_a$, $(s-t, Y_a(t, \psi)) \in \Omega_a$ and*

$$D_1 Y_a(s, \psi) = D_2 Y_a(s-t, Y_a(t, \psi)) \circ D_1 Y_d(t, \psi)$$
$$(= D_1 Y_a(t, Y_a(s-t, \psi)))$$

(v) *Let $a \in A_7$. The maps*

$$R_{p_a} \colon \Omega_a \to P_a \subset C, \quad (t, \psi) \to p_a R(t, \psi, a),$$

$$R_{q_a} \colon \Omega_a \to Q_a \subset C, \quad (t, \psi) \to q_a R(t, \psi, a)$$

satisfy $D_2 R_{p_a}(t, 0) = 0 = D_2 R_{q_a}(t, 0)$ on $\boldsymbol{R}_0^+$. The assignments $(t, \psi, a) \to D_2 R_{p_a}(t, \psi)$ and $(t, \psi, a) \to D_2 R_{q_a}(t, \psi)$ define continuous maps from Ω into $L_c(C, C)$. We have $D_2 R_{p_a}(t, \psi) = p_a D_2 R(t, \psi, a)$ and $D_2 R_{q_a}(t, \psi) = q_a D_2 R(t, \psi, a)$ on Ω.

(vi) *The assignments $(\psi, a) \to D_2(D_1 R_{p_a})(3, \psi)$ and $(\psi, a) \to D_2(D_1 R_{q_a})(3, \psi)$ define continuous maps from $D^1 \times A_7$ into $L_c(C, L_c(\boldsymbol{R}, C))$, with*

$$D_2(D_1 R_{p_a})(3, 0) = 0 = D_2(D_1 R_{q_a})(3, 0) \quad \textit{for all } a \in A_7.$$

For the application of Lemma 2.1 ([23]), we pick the time-N-shifts $g_a \colon D^1 \to C$, $g_a(\psi) := Y(N, \psi, a)$ for $\psi \in D^1$ and $a \in A_7$, and write $r_a := g_a - T(N, \cdot, a)$, for all $a \in A_7$. Both maps g_a, r_a are of class C^2.

The symbol D_{Q_a} in the next proposition denotes the partial derivative in the direction of the subspace Q_a.

PROPOSITION 5.3. (i) *Let* $a \in A_7$. *We have* $g_a(0) = 0 = r_a(0)$ *and* $Dg_a(0) = T(N, \cdot, a)$, $Dr_a(0) = 0$, $g_a(D^1 \cap P_a) \subset P_a$, $g_a(D^1 \cap Q_a) \subset Q_a$.

(ii) *The map* $D^1 \times A_7 \to \boldsymbol{R}$, $(\psi, a) \to |D(D_{Q_a}(p_a \circ r_a))(\psi)|$, *is continuous.*

The proof is left to the reader.

We need assertion (ii) of the preceding proposition in order to satisfy the hypotheses of Lemma 2.1 ([23]) for restrictions of the maps g_a uniformly with respect to parameters a (close to a_0); compare (5.9) below.

We begin with an interval A_8; recall $\operatorname{cl} A_8 \subset A_7$ and the constant c from Proposition 3.2. Proposition 5.2(v) and compactness of $[0, N] \times \operatorname{cl} A_8$ allow to find an open ball $D^{2.1} \subset D^1$, centered at $0 \in C$, with

$$|D_2 R_{p_a}(t, \psi)| + |D_2 R_{q_a}(t, \psi)| < c \tag{5.6}$$

for all $(t, \psi, a) \in [0, N] \times D^{2.1} \times A_8$. In particular,

$$|D(p_a \circ r_a)(\psi)| + |D(q_a \circ r_a)(\psi)| < c \tag{5.7}$$

on $D^{2.1} \times A_8$. By Proposition 5.2(vi) and $A_8 \Subset A_7$, there exists an open ball $D^{2.2} \subset D^{2.1}$ with center $0 \in C$ such that

$$|D_2(D_1 R_{q_a})(3, \psi)| + |D_2(D_1 R_{p_a})(3, \psi)| < c \tag{5.8}$$

on $D^{2.2} \times A_8$. By Proposition 5.3(ii) – and $A_8 \Subset A_7$ – there are $D^2 \subset D^{2.2}$ and a constant $m > 0$ with

$$|D(D_{Q_a}(p_a \circ r_a))(\psi)| < m \quad \text{on } D^2 \times A_8. \tag{5.9}$$

With α, β, γ from Proposition 3.2, we set

$$\alpha_c := \alpha + c, \quad \beta_c := \beta - c, \quad \gamma_c := \gamma + c,$$

$$\bar{\beta}_c := (\bar{\beta}_c + 1)/2 \in (1, \beta_c),$$

$$\bar{c} := \min\left\{c, \frac{\beta_c - 1}{\beta_c + 1}, \frac{\beta_c - 1}{\beta_c + 1}[\beta_c - \bar{\beta}_c]\right\} > 0.$$

Finally, we choose $D^3 \subset D^2$ with

$$|p_a \psi| \leqslant \bar{c}/m \quad \text{on } D^3 \times A_8 \tag{5.10}$$

and such that

$$\psi \in D^3,\ a \in A_8,\ s \in [0, 1] \text{ imply } p_a \psi + s \cdot q_a \psi \in D^2 \text{ and } s \cdot p_a \psi + q_a \psi \in D^2, \tag{5.11}$$

i.e. the straight lines from ψ to $p_a \psi$ and to $q_a \psi$ are contained in D^2. (5.11) follows if $D^3 \subset (1/c_{pq}) \cdot D^2$.

Let ϱ_3 denote the radius of D^3.

Lemma 2.1 ([23]) will be applied to the restrictions of g_a to D^2 and D^3, for a sufficiently close to a_0.

COROLLARY 5.1. *For all* $(\psi, a) \in D^3 \times A_8$,

(i) $$\beta_c |p_a \psi| \leqslant |p_a \circ g_a(\psi)| \leqslant \gamma_c |p_a \psi|,$$
$$|q_a \circ g_a(\psi)| \leqslant \alpha_c |q_a \psi|,$$

(ii) $$|Dg_a(\psi)| \leqslant |T(N, \cdot, a)| + c \leqslant c_{pq}(e^{\mu_1 N} + k_0 e^{\lambda_1 N} + c) =: c_0.$$

Proof. Let $(\psi, a) \in D^3 \times A_8$.

(i) We have $p_a \circ g_a(\psi) = T(N, p_a\psi, a) + p_a \circ r_a(\psi)$, and $p_a \circ r_a(q_a\psi) = 0$; $|p_a \circ r_a(\psi) - p_a \circ r_a(q_a\psi)| \leqslant c|p_a\psi|$, by (5.11), (5.7) and the mean value theorem. Recall (3.6). The second estimate in assertion (i) follows in the same way.

(ii) Use $Dg_a(\psi) = T(N, \cdot, a) + Dr_a(\psi)$, (3.2), $u(a) < \mu_1$, (3.4), $p_a + q_a = \mathrm{id}_C$ and (5.7), $1 \leqslant c_{pq}$. ∎

In the second part of this section, we obtain estimates for the semiflow Y, and in particular an estimate which relates the P_a-component of $Y(3, \psi, a)$ to the P_a-component of the tangent vector to the trajectory $t \to Y(t, \psi, a)$ at $t = 3$.

COROLLARY 5.2. *Set* $\varrho := (\log \beta_c)/N > 0$. *There are constants* c_1 *and* $c_2 \geqslant 1 + k_0$ *such that for every* $(t, \psi, a) \in \Omega$ *with* $a \in A_8$ *and* $Y(s, \psi, a) \in D^3$ *for all* $s \in [0, t]$,

$$c_1 e^{\varrho t} |p_a \psi| \leqslant |p_a \circ Y_a(t, \psi)| \leqslant c_2 e^{\mu_2 t} |p_a \psi|,$$
$$|q_a \circ Y_a(t, \psi)| \leqslant c_2 e^{\lambda_3 t} |q_a \psi|.$$

Proof. a) Let $(t, \psi, a) \in [0, N] \times D^3 \times A_8 \subset \Omega$ be given. Then $p_a \circ Y_a(t_a, \psi) = e^{u(a)t} p_a \psi + R_{p_a}(t, \psi)$. The straight line from ψ to $q_a\psi$ has length $|p_a\psi|$ and is contained in D^2. (5.6) and $R_{p_a}(t, q_a\psi) = 0$ and the mean value theorem yield $|R_{p_a}(t, \psi)| \leqslant c|p_a\psi|$. Therefore

$$c_1 e^{\varrho t} |p_a \psi| \leqslant (1-c)|p_a\psi| \leqslant (e^{u(a)t} - c)|p_a\psi| \leqslant |p_a \circ Y_a(t, \psi)|$$
$$\leqslant (e^{u(a)t} + c)|p_a\psi| \leqslant (1+c)e^{\mu_2 t}|p_a\psi|,$$

where $c_1 := (1-c)/e^{\varrho N}$. Set $c_2 := (1 + c + k_0)e^{-\lambda_1 N}$.

b) For $(t, \psi, a) \in \Omega$ with $a \in A_8$ and $Y(s, \psi, a) \in D^3$ on $[0, t]$, there exists $j \in N_0$ with $t \in [jN, (j+1)N]$. Then

$$Y(t, \psi, a) = Y(t - jN, Y(jN, \psi, a), a) = Y(t - jN, (g_a)^j(\psi), a).$$

By the estimate in a),

$$c_1 e^{\varrho(t - jN)} |p_a \circ (g_a)^j(\psi)| \leqslant |p_a \circ Y_a(t, \psi)| \leqslant c_2 e^{\mu_2(t - jN)} |p_a \circ (g_a)^j(\psi)|.$$

Repeated application of Corollary 5.1 (i) yields

$$e^{j\varrho N}|p_a\psi| = (\beta_c)^j|p_a\psi| \leqslant |p_a\circ(g_a)^j(\psi)| \leqslant (\gamma_c)^j|p_a\psi| \leqslant e^{j\mu_2 N}|p_a\psi|$$

with $\gamma_c = \gamma + c = \beta + 2c < e^{\mu_2 N}$, by (3.12). This proves the first estimate.

c) The second estimate follows similarly, with (3.4), $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and (3.13), and with the definition of c_2 above. ∎

COROLLARY 5.3. *There exist positive constants c_3, c_4 such that for all $(3, \psi, a)\in\Omega$ with $\psi\in D^3$, $a\in A_8$,*

$$c_3|p_a\psi| \leqslant |p_a\circ D_1 Y_a(3, \psi)|,$$

$$c_4|p_a\circ Y_a(3, \psi)| \leqslant |p_a\circ D_1 Y_a(3, \psi)|.$$

Proof. Let $(\psi, a)\in D^3\times A_8$. Then $(3, \psi, a)\in[0, N]\times D^3\times A_8 \subset \Omega$.

a) As in part a) of the proof of Corollary 5.2, we find $|p_a\circ Y_a(3, \psi)| \leqslant (c+1)e^{3\mu_2}|p_a\psi|$. We have $q_a\psi\in D^2$, $(3, q_a\psi)\in\Omega_a\cap(\boldsymbol{R}_0^+\times Q_a)$ and $\dot{R}_{p_a}(t, q_a\psi) = p_a R(t, q_a\psi, a) = 0$ on $[0, N+1]$, see Proposition 5.2(v). Therefore

$$\dot{D}_1 R_{p_a}(3, q_a\psi) = 0, \qquad |D_1 R_{p_a}(3, \psi)| = |D_1 R_{p_a}(3, \psi) - D_1 R_{p_a}(3, q_a\psi)|.$$

The straight line from ψ to $q_a\psi$ is contained in the ball D^2 so that (5.8) and the mean value theorem give

$$|D_1 R_{p_a}(3, \psi)| \leqslant c|p_a\psi|.$$

From

$$p_a\circ D_1 Y_a(3, \psi)(s) = s\cdot u(a)\cdot e^{3u(a)}p_a\psi + D_1 R_{p_a}(3, \psi)(s)$$

for all $s\in R$ we infer

$$|p_a\circ D_1 Y_a(3, \psi)| \geqslant \big(u(a)e^{3u(a)} - c\big)|p_a\psi| \geqslant (\mu_0 e^{3\mu_0} - c)|p_a\psi|,$$

see Proposition 3.2. By (3.11), $c_3 := \mu_0 e^{3\mu_0} - c > 0$. Set $c_4 := c_3/\big((c+1)e^{3\mu_2}\big)$. ∎

COROLLARY 5.4. *Let $(\psi, a)\in D^3\times A_8$ be given with $Y(t, \psi, a)\in D^3$ for all $t\in[0, 3]$. Then*

$$|q_a D_1 Y(3, \psi, a)(1)| \leqslant \big(a|T(2, \cdot, a)| + c\big)|q_a\psi|,$$

$$|p_a D_1 Y(3, \psi, a)(1)| \leqslant \big(a|T(2, \cdot, a)| + c\big)|p_a\psi|.$$

Proof. a) We have

$$q_a D_1 Y(3, \psi, a)(1) = q_a D_1 T(3, \psi, a)(1) + q_a D_1 R(3, \psi, a)(1),$$

$$D_1 T(3, \psi, a)(1) = \dot{x}_3$$

with the solution x: $[-1, \infty)\to\boldsymbol{R}$ of the linear equation (a) and $x_0 = \psi$. Hence $\dot{x}_3 = ax_2 = aT(2, \psi, a)$. It follows that

$$q_a D_1 T(3, \psi, a)(1) = a\cdot q_a T(2, \psi, a) = a\cdot T(2, q_a\psi, a).$$

b) By Proposition 5.2(v), $R_{q_a}(t, p_a\psi) = q_a R(t, p_a\psi, a) = 0$ on $[0, N+1]$. Hence $D_1 R_{q_a}(3, p_a\psi) = 0$, and $D_1 R_{q_a}(3, \psi) = D_1 R_{q_a}(3, \psi) - D_1 R_{q_a}(3, p_a\psi)$. The

straight line from ψ to $p_a\psi$ is contained in D^2 so that (5.8) and the mean value theorem imply $|D_1 R_{q_a}(3, \psi)| \leqslant c|q_a\psi|$. Now the first estimate in Corollary 5.4 becomes obvious, with $q_a \circ D_1 R(3, \psi, a) = D_1 R_{q_a}(3, \psi)$.

c) The second estimate is proved in the same way. ∎

COROLLARY 5.5. *For each $t > 0$ there exist $c(t) > 0$ such that*

$$|D_2 Y(s, \psi, a)| < c(t)$$

for all $s \in [0, t+N+3]$ and all $(\psi, a) \in D^3 \times A_8$ with $(t+N+3, \psi, a) \in \Omega$, $Y(s, \psi, a) \in D^3$ on $[0, t+N+3]$.

Proof. Let $(\psi, a) \in D^3 \times A_8$ and $t > 0$ be given with $(t+N+3, \psi, a) \in \Omega$, $Y(s, \psi, a) \in D^3$ on $[0, t+N+3]$. Let $s \in [0, t+N+3]$. There exists $j \in N_0$ with $jN \leqslant s < (j+1)N$. We have $Y(s, \psi, a) = Y(s-jN, Y(jN, \psi, a), a)$, hence

$$D_2 Y(s, \psi, a) = D_2 Y(s-jN, Y(jN, \psi, a), a) \circ D_2 Y(jN, \psi, a).$$

Corollary 5.1 (ii) and the chain rule, together with $Y(v, \psi, a) \in D^3$ on $[0, t+N+3]$ and $Y(jN, \psi, a) = (g_a)^j(\psi)$, give

$$|D_2 Y(jN, \psi, a)| \leqslant (c_0)^j.$$

Note $jN \leqslant t+N+3$, or $j \leqslant (t+3)/N+1$. Consider

$$D_2 Y(s-jN, Y(jN, \psi, a), a) = T(s-jN, \cdot, a) + D_2 R(s-jN, Y(jN, \psi, a), a).$$

For every $\chi \in C$,

$$\begin{aligned} |T(s-jN, \chi, a)| &\leqslant |T(s-jN, p_a\chi, a)| + |T(s-jN, q_a\chi, a)| \\ &\leqslant (e^{u(a)\cdot(s-jN)}|p_a| + k_0 e^{\lambda_1(s-jN)}|q_a|)|\chi| \\ &\leqslant c_{pq}(e^{\mu_1 N} + k_0)|\chi|. \end{aligned}$$

$p_a + q_a = \mathrm{id}_C$ yields

$$D_2 R(s-jN, Y(\ldots), a) = D_2 R_{p_a}(s-jN, Y(\ldots)) + D_2 R_{q_a}(s-jN, Y(\ldots)).$$

With $s-jN \in [0, N]$, $Y(\ldots) \in D^3$ and (5.6), we obtain $|D_2 R(s-jN, Y(jN, \psi, a), a)| < c$. Set

$$c(t) := (c_0)^{((t+3)/N)+1} \cdot [(e^{\mu_1 N} + k_0)c_{pq} + c].$$

6. A shift along the transformed semiflow.

PROPOSITION 6.1. *Let $a \in A_8$, $t \geqslant 0$. Suppose $Y(s, \psi, a) \in D^3$ on $[0, t]$ and $p_a\psi\{\gtrless\}0$. Then $p_a Y(s, \psi, a)\{\gtrless\}0$ on $[0, t]$.*

Proof. Recall $p_a Y(s, \psi, a) = \langle \Psi_a, Y(s, \psi, a)\rangle_a \Phi_a$ and $\Phi_a > 0$. The map $[0, t] \ni s \to \langle \Psi_a, Y(s, \psi, a)_a\rangle \in \boldsymbol{R}$ is continuous and has no zero since

$$0 < c_1 e^{\varrho s}|p_a\psi| \leqslant |p_a Y(s, \psi, a)| = |\langle \ldots \rangle_a|,$$

by Corollary 5.2 and $|\Phi_a| = 1$. ∎

For $a \in A_8$ we shall follow flow lines $0 \leqslant t \to Y_a(t, \psi)$ which start in open boxes

$$E_a^+ := E^+(\delta, \eta, a) := \{\psi \in C\colon\ 0 < p_a\psi < \eta,\ |q_a\psi| < \delta\},$$

$$E_a^- := E^-(\delta, \eta, a) := \{\psi \in C\colon\ -\eta < p_a\psi < 0,\ |q_a\psi| < \delta\},$$

with $0 < \eta$ and $0 < \delta$, above and below the stable space Q_a until they reach parallels H_a^+, H_a^- of Q_a. We fix $\eta_0 > 0$ and $\delta_0 > 0$ so small that

$$c_2 e^{\mu_2 N}\eta_0 + c_2\delta_0 < \varrho_3. \tag{6.1}$$

In particular,

$$\eta_0 + \delta_0 < \varrho_3, \tag{6.2}$$

$$\eta_0\Phi_a \in D^3 \ni -\eta_0\Phi_a \quad \text{for all } a \in A_8. \tag{6.3}$$

COROLLARY 6.1. *$a \in A_8$, $|p_a\psi| \leqslant \eta_0$ and $|q_a\psi| \leqslant \delta_0$ imply $Y(t, \psi, a) \in D^3$ for all $t \in [0, N]$.*

Proof. (6.2) gives $\psi \in D^3$ so that $Y(t, \psi, a)$ is defined on $[0, N]$. By Corollary 5.2,

$$|Y(t, \psi, a)| \leqslant |p_a \ldots| + |q_a \ldots| \leqslant c_2 e^{\mu_2 N}\eta_0 + c_2\delta_0$$

provided that $Y(s, \psi, a) \in D^3$ on $[0, t]$ and $0 \leqslant t \leqslant N$. Now one can use (6.1) in order to complete the proof. ∎

We define $H_a^+ := \eta_0\Phi_a + Q_a$ and $H_a^- := -\eta_0\Phi_a + Q_a$, for all $a \in A_8$.

PROPOSITION 6.2. *Let $a \in A_8$, $0 < \eta \leqslant \eta_0$, $0 < \delta \leqslant \delta_0$. Then $E_a^+ := E^+(\eta, \delta, a) \subset D^3$, and for every $\psi \in E_a^+$ there are $\tau = \tau(\psi, a) > 0$, $\sigma = \sigma(\psi, a) \in (0, \tau)$ with*

(i) $(\tau, \psi, a) \in \Omega$;

(ii) *$Y(t, \psi, a) \in D^3$ on $[0, \tau)$ and $Y(\tau, \psi, a) \in \partial D^3$;*

(iii) *$0 < p_a Y(t, \psi, a) < \eta_0\Phi_a$ for $0 \leqslant t < \sigma$ and $p_a Y(\sigma, \psi, a) = \eta_0\Phi_a$, i.e. $Y(\sigma, \psi, a) \in H_a^+$;*

(iv) $$\frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta} \leqslant \frac{1}{\mu_2}\log\frac{\eta_0}{c_2|p_a\psi|} \leqslant \sigma;$$

(v) $$|q_a Y(\sigma, \psi, a)| = |Y(\sigma, \psi, a) - \eta_0\Phi_a| \leqslant c_2\delta\left(\frac{\eta_0}{c_2}\right)^{\lambda_3/\mu_2}\eta^{-\lambda_3/\mu_2}.$$

Remark. An analogous result holds for all $\psi \in E_a^- = E^-(\eta, \delta, a)$, with

(iii′) $-\eta_0\Phi_a < p_a Y(t, \psi, a) < 0$ for $0 \leqslant t < \sigma$, $-\eta_0\Phi_a = p_a Y(\sigma, \psi, a)$, i.e. $Y(\sigma, \psi, a) \in H_a^-$, instead of (iii) above.

Proof. Let $a \in A_8$, $\eta \in (0, \eta_0]$, $\delta \in (0, \delta_0]$. By Corollary 6.1, $E_a^+ \subset D^3$. Let $\psi \in E_a^+$. We have $\psi \notin H_a^+$ since $0 < p_a\psi < \eta \leqslant \eta_0 = |p_a\tilde{\psi}|$ for all $\tilde{\psi} \in H_a^+$.

Case 1: $\boldsymbol{R}_0^+ \times \{\psi\} \times \{a\} \not\subset \Omega$. Then there exists $\bar{s} > 0$ with $(\bar{s}, \psi, a) \notin \Omega$, $[0, \bar{s}) \times \{\psi\} \times \{a\} \subset \Omega$. Suppose $\phi := Y(t, \psi, a) \in D^3$ for $t := \bar{s} - 2^{-1}\min\{\bar{s}, 1\}$.

Then $[0, N+1]\times\{\phi\}\times\{a\}\subset\Omega$, and consequently $[0, N+1+t]\times\{\psi\}\times\{a\}\subset\Omega$, a contradiction to $(\bar{s}, \psi, a)\notin\Omega$. It follows that there exists $\tau\in(0, \bar{s})$ with $[0, \tau]\times\{\psi\}\times\{a\}\subset\Omega$, $Y(t, \psi, a)\in D^3$ on $[0, \tau)$, $Y(\tau, \psi, a)\in\partial D^3$. We apply Corollary 5.2 to each $s\in[0, \tau)$ and find $|q_a Y(\tau, \psi, a)|\leqslant c_2|q_a\psi|\leqslant c_2\delta\leqslant c_2\delta_0$. The inequality $\eta_0\geqslant|p_a Y(\tau, \psi, a)|$ would imply $\varrho_3=|Y(\tau, \psi, a)|\leqslant\eta_0+c_2\delta_0\leqslant c_2\eta_0+c_2\delta_0$, a contradiction to (6.1). Hence $\eta_0<|p_a Y(\tau, \psi, a)|$. It follows that there is $\sigma\in(0, \tau)$ with $\eta_0=|p_a Y(\sigma, \psi, a)|$ and $|p_a Y(t, \psi, a)|<\eta_0$ on $[0, \sigma)$.

Case 2: $\boldsymbol{R}_0^+\times\{\psi\}\times\{a\}\subset\Omega$. Suppose $Y(t, \psi, a)\in D^3$ on $\boldsymbol{R}_0^+$. Then Corollary 5.2 would imply $|p_a Y(t, \psi, a)|\to+\infty$ which is a contradiction to $|p_a Y(t, \psi, a)|<\varrho_3|p_a|$ on $\boldsymbol{R}_0^+$. It follows that there exists $\tau>0$ with $Y(t, \psi, a)\in D^3$ on $[0, \tau)$, $Y(\tau, \psi, a)\in\partial D^3$. We can now continue as in case 1 and find σ as above.

$Y(t, \psi, a)\in D^3$ on $[0, \sigma]$, Proposition 6.1 and $0<p_a\psi$ give $0<p_a Y(t, \psi, a)$ on $[0, \sigma]$. Using (6.4) and $p_a Y(t, \psi, a)=\langle\Psi_a, Y(t, \psi, a)\rangle_a\Phi_a$, $0<\Phi_a$ and $|\Phi_a|=1$, we infer $p_a Y(\sigma, \psi, a)=\eta_0\Phi_a$ and $0<p_a Y(t, \psi, a)<\eta_0\Phi_a$ on $[0, \sigma)$.

With Corollary 5.2, we obtain

$$\eta_0=|p_a Y(\sigma, \psi, a)|\leqslant c_2 e^{\mu_2\sigma}|p_a\psi|,$$

or

$$\frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta}\leqslant\frac{1}{\mu_2}\log\frac{\eta_0}{c_2|p_a\psi|}\leqslant\sigma,\qquad |q_a Y(\sigma, \psi, a)|\leqslant c_2 e^{\lambda_3\sigma}\delta\leqslant c_2\delta\left(\frac{\eta_0}{c_2\eta}\right)^{(\lambda_3/\mu_2)}.$$

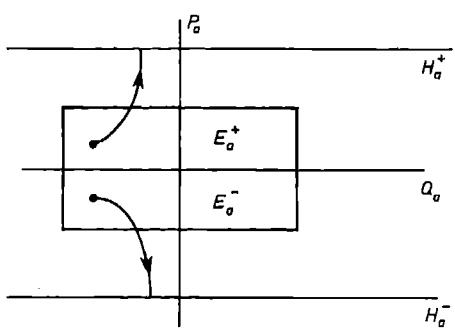

Fig. 15

PROPOSITION 6.3. *There exist an interval $A_9\subset A_8$ and positive reals $\delta_1<\delta_0$, $\eta_1<\eta_0$ such that the map σ: $E^+\to\boldsymbol{R}$ on the open subset $E^+:=\{(\psi, a)\in C\times A_9$: $\psi\in E^+(\delta_1, \eta_1, a)\}$ of $C\times\boldsymbol{R}$ is of class C^1, with*

$$\begin{aligned}\sigma(\psi, a)&>1 && \text{on } E^+,\\ \eta_0\Phi_a&<p_a Y(t, \psi, a) && \text{on } (\sigma(\psi, a), \tau(\psi, a)),\\ p_a D_1 Y(t, \psi, a)(1)&\in\boldsymbol{R}^+\cdot\Phi_a && \text{on } [\sigma(\psi, a), \tau(\psi, a))\end{aligned}$$

for each $(\psi, a)\in E^+$.

Proof. The idea is, of course, to solve the equation

$$\eta_0 = \langle \Psi_a, Y(\sigma', \psi', a')\rangle_a$$

which is equivalent to $Y(\ldots)\in H_a^+$, in a neighborhood of $(\sigma(\psi, a), \psi, a)$, $p_a\psi > 0$, by means of the implicit function theorem in the form $\sigma' = \hat{\sigma}(\psi', a')$ with a C^1-map $\hat{\sigma}$, and to show that $\hat{\sigma}$ and σ coincide close to (ψ, a).

We include the details for completeness.

I. In this part we consider $a = a_0$. We choose $\eta\in(0, \eta_0)$, $\delta\in(0, \delta_0)$ and $y\in(0, \eta)$ so small that $\psi := y\Phi_a\in E^+(\delta, \eta, a)$ and $1 < \sigma := \sigma(\psi, a)$. This is possible because of the preceding proposition. Recall $Y(\tau, \psi, a)\in\partial D^3$, for $\tau := \tau(\psi, a) > \sigma > 1$, and $Y(t, \psi, a)\in D^3$ on $[0, \tau)$. We have $Y(t, \psi, a)\in P_a$ whenever $(t, \psi, a)\in\Omega$, see Proposition 5.2. Proposition 6.1 gives $Y(\tau, \psi, a)\in \boldsymbol{R}^+\cdot\Phi_a$, hence $Y(\tau, \psi, a) = \varrho_3\Phi_a$.

a) We show $Y(t, \psi, a)\notin \operatorname{cl} D^3$ for some $t > \tau$: Recall $1 < \beta_c$. Choose $t'\in(0, \tau)$ so large that $\varrho_3 < \beta_c|Y(t', \psi, a)|$ and $\tau < t'+N$. Corollary 5.1 and $Y(t', \psi, a)\in P_a\cap D^3$ imply that

$$\varrho_3 < \beta_c|Y(t', \psi, a)| \leqslant |g_a(Y(\ldots))| = |Y(t'+N, \psi, a)|.$$

b) For $1 < s \leqslant t$, the derivative $D_1 Y(s, \psi, a)\in L_c(\boldsymbol{R}, C)$ exists. $Y(s, \psi, a)\in P_a$ on $[0, t]$ implies $\chi_s := D_1 Y(s, \psi, a)(1)\in P_a$ on $(1, t]$.

We prove $\chi_s \neq 0$ on $(1, \tau]$: Let $s\in(1, \tau]$. Suppose $\chi_s = 0$. By Proposition 5.2 (iv),

$$D_1 Y(s', \psi, a) = D_2(s'-s, Y(s, \psi, a), a)\circ D_1 Y(s, \psi, a) = 0 \quad \text{for all } s'\in[s, t].$$

Hence

$$\operatorname{cl} D^3 \not\ni Y(t, \psi, a) = Y(s, \psi, a)+\int_s^t D_1 Y(v, \psi, a)(1)\,dv = Y(s, \psi, a)$$

which contradicts $Y(v, \psi, a)\in\operatorname{cl} D^3$ on $[0, \tau]$.

c) Let ev denote the evaluation $C\ni\phi\to\phi(0)\in\boldsymbol{R}$. Set

$$b(s) := \operatorname{ev}(p_a Y(s, \psi, a)) = \langle \Psi_a, Y(v, \psi, a)\rangle_a\in\boldsymbol{R} \quad \text{for } s\in[0, t].$$

The restriction of b to the interval $(1, t)$ is of class C^1, with

$$b'(s) = Db(s)(1) = (\operatorname{ev}\circ p_a\circ D_1 Y(s, \psi, a))(1) = \chi_s(0) \quad \text{for all } s\in(1, t).$$

We show $0 < \chi_s$ on $(1, \tau]$: We know already that $\chi_s \neq 0$ on $(1, \tau]$. With $\chi_s\in\boldsymbol{R}\cdot\Phi_a$ and $0 < \Phi_a$, we obtain $\chi_s(0) \neq 0$ on $(1, \tau]$. It follows that either $b' < 0$ on $(1, \tau]$, or $b' > 0$ on $(1, \tau]$. By Proposition 6.1, $b > 0$ on $[0, \tau)$. Recall $Y(\tau, \psi, a) = \varrho_3\Phi_a$, or $b(\tau) = \varrho_3$. The relations $0 < 1 < \tau$ and $Y(1, \psi, a)\in P_a\cap D^3$ give

$$0 < b(1) = Y(1, \psi, a)(0) = |Y(1, \psi, a)| < \varrho_3.$$

Therefore $b' > 0$ on $(1, \tau]$. Next, $\chi_s(0) > 0$ on $(1, \tau]$. With $\chi_s = \langle \Psi_a, \chi_s \rangle_a \Phi_a$ and $\Phi_a(0) = 1$, we arrive at $\langle \Psi_a, \chi_a \rangle_a > 0$ on $(1, \tau]$ and $0 < \chi_s$ on $(1, \tau]$.

d) There exists $\varepsilon > 0$ with $1 < \sigma - \varepsilon < \sigma + \varepsilon < \tau < \tau + \varepsilon < t$ and $0 < \chi_s$ on $[\sigma + \varepsilon, \tau + \varepsilon]$, since the map $(1, \tau] \ni s \to \chi_s$ is continuous, and $0 < \chi_\tau$. The definition of σ implies $Y(\sigma - \varepsilon, \psi, a) < \eta_0 \Phi_a$.

We prove $\eta_0 \Phi_a < Y(\sigma + \varepsilon, \psi, a)$ and $\varrho_3 < |Y(\tau + \varepsilon, \psi, a)|$: The first inequality is a consequence of

$$Y(\sigma + \varepsilon, \psi, a) = Y(\sigma, \psi, a) + \int_\sigma^{\sigma+\varepsilon} D_1 Y(s, \psi, a)(1)\, ds = \eta_0 \Phi_a + \int_\sigma^{\sigma+\varepsilon} \chi_s\, ds$$

$$= \eta_0 \Phi_a + \int_\sigma^{\sigma+\varepsilon} \chi_s(0) \cdot \Phi_a\, ds > \eta_0 \Phi_a$$

(here we used $\chi_s = p_a \chi_s = \langle \Psi_a, \chi_s \rangle_a \Phi_a$, $\Phi_a(0) = 1$, $\chi_s > 0$ on $(1, \tau]$). The second one follows in the same way. We have

$$Y(\tau + \varepsilon, \psi, a) = Y(\tau, \psi, a) + \int_\tau^{\tau+\varepsilon} \chi_s\, ds,$$

$\chi_s = \chi_s(0) \Phi_a$ with $\chi_s(0) > 0$ on $[\tau, \tau + \varepsilon]$, and $Y(\tau, \psi, a) = \varrho_3 \Phi_a$.

II. Choice of A_9, δ_1, η_1. From now on, we write ψ_0, σ_0, τ_0, t_0 for ψ, $\sigma = \sigma(\psi, a)$, $\tau = \tau(\psi, a)$, t from part I.

a) There exist an open ball $D' \subset D^3$ with radius $\varrho' < \varrho_3$, centered at $0 \in C$, and an open interval $A_{9.1} \subset A_8$, $a_0 \in A_{9.1}$, with the following properties:

$$\psi_0 + D' \subset \{\psi \in C\colon 0 < p_a \psi < \eta_0 \Phi_a\} \quad \text{for all } a \in A_{9.1},$$

$$(\psi, a) \in (\psi_0 + D') \times A_{9.1} \quad \text{implies} \quad [0, \tau_0 + \varepsilon] \times \{\psi\} \times \{a\} \subset \Omega,$$

$$p_a Y(\sigma_0 + \varepsilon, \psi, a) > \eta_0 \Phi_a,$$

$$p_a Y(t, \psi, a) < \eta_0 \Phi_a \quad \text{on } [0, \sigma_a - \varepsilon],$$

$$\varrho_3 < |Y(\tau_0 + \varepsilon, \psi, a)|,$$

$$p_a D_1 Y(t, \psi, a)(1) > 0 \quad \text{on } [\sigma_0 - \varepsilon, \tau_0 + \varepsilon].$$

The last inequality is obtained from $p_{a_0} D_1 Y(t, \psi_0, a_0)(1) = \chi_t > 0$ on $[\sigma_0 - \varepsilon, \tau_0 + \varepsilon]$ and $1 < \sigma_0 - \varepsilon$ by continuity and compactness arguments, for D' and $A_{9.1}$ sufficiently small.

b) Choose $A_9 \subset A_{9.1}$ with $y\Phi_a \in \psi_0 + 2^{-1} D'$ for all $a \in A_9$ and positive constants $\delta_1 < \min\{\delta_0, \varrho'/(2c_2)\}$, $\eta_1 < y/(2c_2) < \eta_0$. We show that for every $(\psi, a) \in E^+ := \{(\psi, a) \in C \times A_9\colon \psi \in E^+(\delta_1, \eta_1, a)\}$ there exists $\bar{\sigma} = \bar{\sigma}(\psi, a) > 0$ with

$$\begin{cases} p_a Y(\bar{\sigma}, \psi, a) = y\Phi_a, \\ p_a Y(t, \psi, a) < y\Phi_a \text{ on } [0, \bar{\sigma}), \end{cases} \tag{6.5}$$

and with $Y(\bar{\sigma}, \psi, a) \in \psi_0 + D'$,

$$\bar{\sigma} + \sigma_0 - \varepsilon < \sigma(\psi, a) < \bar{\sigma} + \sigma_0 + \varepsilon:$$

Let $(\psi, a) \in E^+$. Take $\tau = \tau(\psi, a)$ and $\sigma = \sigma(\psi, a)$ from Proposition 6.2. The inequalities $0 < p_a\psi < \eta_1 \Phi_a < (y/(2c_2))\Phi_a \leqslant y\Phi_a < \eta_0 \Phi_a = p_a Y(\sigma, \psi, a)$ and $0 < p_a Y(t, \psi, a)$ on $[0, \sigma]$ imply the existence of $\bar{\sigma} \in (0, \sigma)$ with (6.5) –

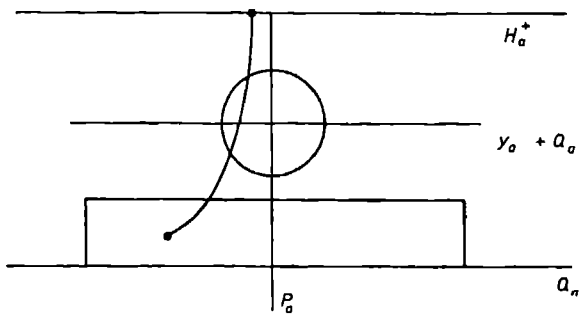

Fig. 16

recall $p_a\phi = \langle \Psi_a, \phi \rangle_a \Phi_a$ for all $\phi \in C$. By Corollary 5.2, $|q_a Y(\bar{\sigma}, \psi, a)| \leqslant \leqslant c_2 |q_a \psi| \leqslant c_2 \delta_1 < \varrho'/2$. Hence

$$|Y(\bar{\sigma}, \psi, a) - \psi_0| \leqslant |Y(\bar{\sigma}, \psi, a) - y\Phi_a| + |y\Phi_a - \psi_0| = q_a Y(\bar{\sigma}, \psi, a)| + \varrho'/2 < \varrho',$$

or

$$Y(\bar{\sigma}, \psi, a) \in \psi_0 + D'.$$

By a),

$$\eta_0 \Phi_a < p_a Y(\sigma_0 + \varepsilon, Y(\bar{\sigma}, \psi, a), a) = p_a Y(\bar{\sigma} + \sigma_0 + \varepsilon, \psi, a)$$

so that $\sigma(\psi, a) < \bar{\sigma} + \sigma_0 + \varepsilon$, compare Proposition 6.2. For $0 \leqslant t \leqslant \bar{\sigma}$, $p_a Y(t, \psi, a) \leqslant y\Phi_a < \eta_0 \Phi_a$. For $\bar{\sigma} \leqslant t \leqslant \bar{\sigma} + \sigma_0 - \varepsilon$, we have $0 \leqslant t - \bar{\sigma} \leqslant \sigma_0 - \varepsilon$ and $p_a Y(t, \psi, a) = p_a Y(t - \bar{\sigma}, Y(\bar{\sigma}, \psi, a), a) < \eta_0 \Phi_a$, as follows from Proposition 6.2 with $Y(\bar{\sigma}, \psi, a) \in \psi_0 + D'$ and with a). The minimality property of σ gives $\bar{\sigma} + \sigma_0 - \varepsilon < \sigma(\psi, a)$.

c) $(\psi, a) \in E^+$ implies $1 < \sigma(\psi, a)$ and $p_a D_1 Y(t, \psi, a)(1) \in \boldsymbol{R}^+ \Phi_a$ on $[\sigma(\psi, a), \tau(\psi, a)]$. Proof: Recall $1 < \sigma_0 - \varepsilon \leqslant \bar{\sigma} + \sigma_0 - \varepsilon < \sigma(\psi, a)$ for $(\psi, a) \in E^+$. $Y(\bar{\sigma}, \psi, a) \in \psi_0 + D'$ and $a \in A_9$ yield $(\tau_0 + \varepsilon + \bar{\sigma}, \psi, a) \in \Omega$ and $\varrho_3 < |Y(\tau_0 + \varepsilon + \bar{\sigma}, \psi, a)|$, see a). By Proposition 6.2, $\tau(\psi, a) < \tau_0 + \varepsilon + \bar{\sigma}$. Let $t \in [\sigma(\psi, a), \tau(\psi, a)]$. Then $\sigma_0 + \bar{\sigma} - \varepsilon < t < \tau_0 + \bar{\sigma} + \varepsilon$, $1 < \sigma_0 - \varepsilon < t - \bar{\sigma} < \tau_0 + \varepsilon$. We have $p_a D_1 Y(t, \psi, a)(1) = p_a D_1 Y(t - \bar{\sigma}, Y(\bar{\sigma}, \psi, a), a)(1)$, and $Y(\bar{\sigma}, \psi, a) \in \psi_0 + D'$ yields $Y(\bar{\sigma}, \psi, a) \in \boldsymbol{R}^+ \Phi_a$, see a).

III. Application of the implicit function theorem. Let $(\psi, a) \in E^+$.

a) We have $p_a Y(\sigma(\psi, a), \psi, a) - \eta_0 \Phi_a = 0$ and $0 \neq p_a D_1 Y(\sigma(\psi, a), \psi, a)$. The map $\Omega \ni (t, \hat{\psi}, \hat{a}) \to p_{\hat{a}} Y(t, \hat{\psi}, \hat{a}) - \eta_0 \Phi_{\hat{a}} \in C$ is of class C^1 in a neighborhood of $(\sigma(\psi, a), \psi, a)$. It follows that there are open neighborhoods $V \subset \Omega$ of

$(\sigma(\psi, a), \psi, a)$ and $W \subset E^+$ of (ψ, a), and a C^1-map $\hat{\sigma}$: $W \to \boldsymbol{R}$ such that

$$(\hat{\sigma}(\hat{\psi}, \hat{a}), \hat{\psi}, \hat{a}) \in V \quad \text{on } W, \tag{6.6}$$

$$p_{\hat{a}} Y(\hat{\sigma}(\hat{\psi}, \hat{a}), \hat{\psi}, \hat{a}) - \eta_0 \Phi_{\hat{a}} = 0 \quad \text{on } W, \tag{6.7}$$

$$\hat{\sigma}(\psi, a) = \sigma(\psi, a), \tag{6.8}$$

$$t = \hat{\sigma}(\hat{\psi}, \hat{a}) \quad \text{for all } (t, \hat{\psi}, \hat{a}) \in V \text{ with } p_{\hat{a}} Y(t, \hat{\psi}, \hat{a}) - \eta_0 \Phi_{\hat{a}} = 0. \tag{6.9}$$

b) Identification of $\hat{\sigma}$: (6.8) yields $\sigma \leqslant \hat{\sigma}$ on $W \subset E^+$, see the minimality property of σ in Proposition 6.2. Choose $\varepsilon' > 0$ and a neighborhood $W' \subset W$ of (ψ, a) with

$$(\sigma(\psi, a) - \varepsilon', \sigma(\psi, a) + \varepsilon') \times W' \subset V.$$

We have $0 < p_a Y(t, \psi, a) < \eta_0 \Phi_a$ on $[0, \sigma(\psi, a) - \varepsilon']$. A compactness argument shows that there is a neighborhood $W'' \subset W'$ of (ψ, a) with $0 < p_a Y(t, \hat{\psi}, \hat{a}) < \eta_0 \Phi_{\hat{a}}$ on $[0, \sigma(\psi, a) - \varepsilon'] \times W''$. This implies $\sigma(\psi, a) - \varepsilon' < \sigma(\hat{\psi}, \hat{a})$ on $W'' \subset E^+$. Recall $\sigma(\hat{\psi}, \hat{a}) \leqslant \hat{\sigma}(\hat{\psi}, \hat{a})$. By continuity of $\hat{\sigma}$ and by (6.8), there exists a third neighborhood $W''' \subset W''$ of (ψ, a) such that $\hat{\sigma}(\hat{\psi}, \hat{a}) < \sigma(\psi, a) + \varepsilon'$ on W'''. Altogether, $\sigma(\psi, a) - \varepsilon' < \sigma(\hat{\psi}, \hat{a}) \leqslant \hat{\sigma}(\hat{\psi}, \hat{a}) < \sigma(\psi, a) + \varepsilon'$ on W'''. Now $(\hat{\psi}, \hat{a}) \in W'''$ implies $(\sigma(\hat{\psi}, \hat{a}), \hat{\psi}, \hat{a}) \in (\sigma(\psi, a) - \varepsilon', \sigma(\psi, a) + \varepsilon') \times W' \subset V$, and (6.9) for $t = \sigma(\hat{\psi}, \hat{a})$ yields $\sigma = \hat{\sigma}$ on W''' so that σ is of class C^1 on W'''.

IV. Proof of $\eta_0 \Phi_a < p_a Y(t, \psi, a)$ on $(\sigma(\psi, a), \tau(\psi, a)]$, for all $(\psi, a) \in E^+$: For $\sigma(\psi, a) < t \leqslant \tau(\psi, a)$,

$$p_a Y(t, \psi, a) = p_a Y(\sigma(\psi, a), \psi, a) + \int_{\sigma(\psi, a)}^{t} p_a D_1 Y(s, \psi, a)(1)\, ds$$

$$= \eta_0 \Phi_a + \int_{\sigma(\psi, a)}^{t} p_a D_1 Y(s, \psi, a)(1)\, ds.$$

Consider the integrand for $\sigma(\psi, a) \leqslant s \leqslant t$. We have

$$0 < p_a D_1 Y(s, \psi, a)(1) = \langle \Psi_a, D_1 Y(s, \psi, a)(1) \rangle_a \Phi_a,$$

see part c) in II. With $0 < \Phi_a$, we obtain that the continuous function

$$[\sigma(\psi, a), t] \ni s \to \langle \Psi_a, D_1 Y(s, \psi, a)(1) \rangle_a \in \boldsymbol{R}$$

is positive. Now $\sigma(\psi, a) < t$ gives

$$\int_{\sigma(\psi, a)}^{t} p_a D_1 Y(s, \psi, a)(1)\, ds = \int_{\sigma(\psi, a)}^{t} \langle \Psi_a, D_1 Y(s, \psi, a)(1) \rangle_a\, ds \cdot \Phi_a > 0. \quad \blacksquare$$

COROLLARY 6.2. *The map Σ: $E^+ \ni (\psi, a) \to Y(\sigma(\psi, a), \psi, a) \in H_a^+ \subset C$ is of class C^1.*

We shall study the behavior of trajectories close to the saddle points $\phi \in \boldsymbol{Z}\xi_h^F$ in terms of the map Σ. The central result is

THEOREM 6.1. *There exist positive constants* $\delta_2 < \delta_1$, $\eta_2 < \eta_1$ *and* c_9 *such that*

$$|D_1\Sigma(\psi, a)| \leqslant c_9(|p_a\psi|^{-\lambda_3/\mu_2} + |p_a\psi|^{-(\lambda_3/\mu_2)-1})$$

for all $a \in A_9$, $\eta \in (0, \eta_2)$, $\psi \in E^+(\delta_2, \eta, a)$. *In particular,*

$$\lim_{\eta\to 0}(\sup\{|D_1\Sigma(\psi, a)|\colon a \in A_9, \psi \in E^+(\delta_2, \eta, a)\}) = 0.$$

The basic property for Theorem 6.1 to hold is that contraction in the stable direction is stronger than expansion in the unstable direction, as expressed by

$$\lambda_3 < -\mu_2$$

for the bounds $\lambda_3 > \sup\{\mathrm{Re}\lambda\colon \lambda - ae^{-\lambda} = 0, \lambda \neq u(a), a \in A_4\}$ and $\mu_2 > \sup\{u(a)\colon a \in A_4\}$, see Proposition 3.2.

The proof proceeds as follows. The first aim is to estimate $D_1\Sigma(\psi, a)$ at points $\psi \in E^+(\delta_1, \eta, a)$ in the level sets H^+_{ak} given by $\sigma(\psi, a) = kN$. On these sets, $\Sigma(\cdot, a)$ coincides with iterates $(\hat{g}_a)^k$ of restrictions $\hat{g}_a = g_a|D^3$, cf. Corollary 7.1. The strategy of proof sketched in the introduction requires a decomposition

$$C = \boldsymbol{R}w \oplus T_\psi H^+_{ak} \tag{6.10}$$

with w tangent to the trajectory $t \to Y(t, \psi, a)$ at $t = 0$. It is not guaranteed that the whole set H^+_{ak} is a submanifold of C (with codimension 1), and not every trajectory admits a tangent vector at $t = 0$. But it is possible to specify suitable points $\psi \in H^+_{ak}$ such that H^+_{ak} is locally a submanifold with (6.10) — see Propositions 7.1 and 7.3. The fact that $\Sigma(\cdot, a)$ is constant along trajectory gives $D_1\Sigma(\psi, a)w = 0$.

Section 8 contains the application of Lemma 2.1 ([23]) to preimages $\hat{H}^+_{ak} \supset H^+_{ak}$ of $H^+_a \cap D^3$ under the iterates $(\hat{g}_a)^k$. The sets $\hat{H}^+_{ak}$ converge to Q_a as $k \to \infty$, and inclinations tend to zero. Here we do not have to care whether the sets $\hat{H}^+_{ak}$ are submanifolds or not. Estimate (8.2) — which is the analogue of an estimate in the proof of Palis' λ-lemma ([16]) — yields an inequality

$$|D(\hat{g}_a)^k(\psi)\chi| \leqslant \mathrm{const}\cdot e^{\lambda_3 kN}|\chi|$$

for all $\psi \in \hat{H}^+_{ak}$, $\chi \in T_\chi\hat{H}^+_{ak}$, see Corollary 8.2. The better pointwise estimate (8.4) of inclinations, corresponding to (0.4) in the introduction, is not yet needed.

Section 9 is the core of the proof. We begin with the choice of $\delta_2 \in (0, \delta_1)$, in view of the application of (8.4) at the end of the section, and restrict the investigation to points $\psi \in H^+_{ak}$ in a certain subset of those considered in Section 7, depending on δ_2, for k sufficiently large.

The decomposition (6.10), $D_1\Sigma(\psi, a)w = 0$, equality of $\Sigma(\cdot, a)$ and $(\hat{g}_a)^k$ on H^+_{ak} and results from Section 8 lead to an estimate

$$|D_1\Sigma(\psi, a)| \leqslant \mathrm{const}\cdot e^{\lambda_3 kN}\left(1 + \frac{1}{|pw|}\right)$$

where p is the projection of C onto P_a, parallel to $T_\psi H_{ak}^+$; compare Figure 8 in the introduction.

The definition of Σ, $\sigma(\psi, a) = kN$ and Corollary 5.2 give

$$e^{\lambda_3 kN} \leqslant \text{const} \cdot |p_a \psi|^{-\lambda_3/\mu_2}$$

(part IIId) in Section 9).

Now the pointwise estimate (8.4) becomes crucial. We use it, together with the choice of δ_2, for an inequality

$$1/|pw| \leqslant \text{const} \cdot (1/|p_a \psi|)$$

(part IV of Section 9) — so that we get an estimate of $|D_1 \Sigma(\psi, a)|$ by

$$|p_a \psi|^{-\lambda_3/\mu_2} + |p_a \psi|^{-(\lambda_3/\mu_2)-1}.$$

The step in Section 10 from this to the inequality in Theorem 6.1, for arbitrary points $\psi \in E^+(\delta_2, \eta, a)$, $\eta > 0$ sufficiently small, is somewhat technical but not hard. Finally,

$$\lambda_3 < -\mu_2$$

gives the desired result on uniform convergence.

7. The level sets H_{ak}^+. We define

$$H_{ak}^+ := \{\psi \in E^+(\delta_1, \eta_1, a) \colon \sigma(\psi, a) = kN\}$$

for $(a, k) \in A_9 \times N$. Consider the iterates of the restriction $\hat{g}_a := g_a | D^3$, for $a \in A_9$. Each $(\hat{g}_a)^k$, $k \in N$, is defined on an open neighborhood $D^{3,a,k} \subset D^3$ of $\psi = 0$. We have

$$(\hat{g}_a)^k = Y(kN, \cdot, a) \quad \text{on } D^{3,a,k}$$

(but note that $Y(t, \psi, a) \notin D^3$ for some $t \in (0, kN)$, $\psi \in D^{3,a,k}$, $a \in A_9$ is possible).

COROLLARY 7.1. *For all* $(a, k) \in A_9 \times N$, $H_{ak}^+ \subset ((\hat{g}_a)^k)^{-1}(H_a^+ \cap D^3)$ *and* $\Sigma(\cdot, a) = (\hat{g}_a)^k$ *on* H_{ak}^+.

Proof. On H_{ak}^+, $\sigma(\psi, a) = kN$. Hence $\Sigma(\psi, a) = Y(kN, \psi, a)$ and $Y(t, \psi, a) \in D^3$ on $[0, kN]$. Therefore $\psi \in D^{3,a,k}$ and $(\hat{g}_a)^k(\psi) = Y(kN, \psi, a) = \Sigma(\psi, a) \in H_a^+ \cap D^3$. ∎

Next we single out points $\psi (\in H_{ak}^+)$ so that $t \to Y(t, \psi, a)$ has a tangent at $t = 0$, and H_{ak}^+ becomes a submanifold close to ψ. We say that $(\psi, a) \in D^3 \times A_9$ has the backward continuation property if and only if

(BC) there exists $\bar{\psi} \in D^3$ with $Y(t, \bar{\psi}, a) \in D^3$ on $[0, 3]$ and $\psi = Y(3, \bar{\psi}, a)$.

For $(\psi, a) \in D^3 \times A_9$ with (BC) we define the right tangent vector at $t = 0$ by

$$w^{\psi,a} := D_1 Y(3, \bar{\psi}, a)(1).$$

Note that the right hand side is independent of the choice of $\bar{\psi}$ — it is determined by values of $t \to Y(t, \psi, a)$ with $t > 0$ small.

PROPOSITION 7.1. *Let $a \in A_9$, $k \in N$ and $\psi \in H_{ak}^+$ be given with property* (BC) *for (ψ, a). Then there exists a nonempty open ball $D^\psi \subset C$, centered at $0 \in C$, such that $(\psi + D^\psi) \cap H_{ak}^+$ is a C^1-submanifold of codimension* 1. *Moreover, $D_1 \sigma(\psi, a) w^{\psi,a} = -1$.*

Proof. We show that the C^1-map $\sigma(\cdot, a)$: $E^+(\delta_1, \eta_1, a) \to \boldsymbol{R}$ and the submanifold $\{kN\} \subset \boldsymbol{R}$ are transversal at ψ. (Then one can use the standard argument on p. 45 ([1]) in order to construct a chart of C at ψ with the submanifold property.) The tangent space of $\{kN\}$ is $\{0\} \subset \boldsymbol{R}$, and the preimage under $D_1 \sigma(\psi, a)$, i.e. the kernel of a linear continuous functional, is either C or a hyperplane of codimension 1. In both cases, the preimage splits.

We have $D_1 \sigma(\psi, a) w = -1$, for $w := w^{\psi,a}$: Consider $\bar{\psi} \in D^3$ from (BC). Then $w = D_1 Y(3, \bar{\psi}, a)(1)$, and $D_1 \sigma(\psi, a) w = D\big(t \to \sigma\big(Y(t, \bar{\psi}, a), a\big)\big)(3)(1)$, with $t \in \boldsymbol{R}$ in a small neighborhood of 3 so that $Y(t, \bar{\psi}, a) \in E^+(\delta_1, \eta_1, a)$. We have $\sigma\big(Y(t, \bar{\psi}, a), a\big) = \sigma\big(Y(3, \bar{\psi}, a), a\big) - (t-3)$ for such t, see Propositions 6.2 and 6.3.

It follows that

$$T_{kN}\{kN\} \oplus D_1 \sigma(\psi, a) C = \{0\} \oplus \boldsymbol{R} = T_{kN} \boldsymbol{R},$$

and with the splitting property above, we obtain transversality. ∎

PROPOSITION 7.2. *Let $a \in A_9$, $k \in N$, $\psi \in H_{ak}^+$, and assume* (BC) *for (ψ, a). Then*

(i) $D_1 \Sigma(\psi, a) w^{\psi,a} = 0$;

(ii) $w^{\psi,a} \notin T_\psi H_{ak}^+$.

Proof. (i) We have $D_1 \Sigma(\psi, a) w^{\psi,a} = D\big(t \to \Sigma\big(Y(t, \bar{\psi}, a), a\big)\big)(3)(1)$, with t in a small neighborhood of 3, and $\bar{\psi}$ according to (BC). The properties of σ in Proposition 6.2 imply

$$\sigma\big(Y(t, \bar{\psi}, a), a\big) = \sigma\big(Y(3, \bar{\psi}, a), a\big) - (t-3) = \sigma(\psi, a) - (t-3),$$

for $t - 3 > 0$ sufficiently small. Hence

$$\begin{aligned} \Sigma\big(Y(t, \bar{\psi}, a), a\big) &= Y\big(\sigma\big(Y(t, \bar{\psi}, a), a\big), Y(t, \bar{\psi}, a), a\big) \\ &= Y\big(\sigma(\psi, a) - (t-3), Y(t-3, \psi, a), a\big) \\ &= Y\big(\sigma(\psi, a), \psi, a\big) = \Sigma(\psi, a) \quad \text{for } t-3 > 0 \text{ small.} \end{aligned}$$

(ii) Suppose $w := w^{\psi,a} \in T_\psi H_{ak}^+$. $Y(kN, \cdot, a)$ maps H_{ak}^+ into H_a^+. It follows that $D_2 Y(kN, \psi, a) w \in T_{Y(kN,\psi,a)} H_a^+ = Q_a$. On the other hand,

$$\begin{aligned} D_2 Y(kN, \psi, a) w &= D_2 Y(kN, \psi, a)[D_1 Y(3, \bar{\psi}, a)(1)] \\ &= D_1 Y(kN+3, \bar{\psi}, a)(1) = D_1 Y(kN, \psi, a)(1), \end{aligned}$$

see Proposition 5.2 (iv), and $p_a D_1 Y(kN, \psi, a)(1) \neq 0$, by Proposition 6.3. ∎

8. Inclinations of tangent vectors. We apply Lemma 2.1 ([23]) on dominated convergence for inclinations to the preimages of the sets $\hat{H}_a^+ := H_a^+ \cap D^3$ with respect to the iterates of the maps $\hat{g}_a$, for $a \in A_9$. Set $\tilde{g}_a := g_a | D^2$, for $a \in A_9$.

The inclination of a vector $\chi \in C = P_a \oplus Q_a$ with $q_a \chi \neq 0$, $a \in A_9$, is defined as

$$\Lambda_a(\chi) := |p_a \chi| / |q_a \chi|.$$

Set $p_1 := p_2 := \eta_0 > 0$ and $b := (\alpha, \beta, \gamma, c, m, p_1, p_2)$, with the constants α, β, γ, c from Proposition 3.2 and m from (5.9). (3.7) implies that the parameter vector b is contained in the set B defined at the beginning of Section 2 in [23].

COROLLARY 8.1. *There exists a constant $c_b > 0$ with*

$$\eta_0 (\gamma_c)^{-k} \leqslant |p_a \psi| \leqslant \eta_0 (\beta_c)^{-k}, \tag{8.1}$$

$$q_a \chi \neq 0 \quad \text{and} \quad \Lambda_a(\chi) \leqslant c_b (\beta_c)^{-k}, \tag{8.2}$$

in particular

$$|p_a \chi| / |\chi| \leqslant c_b c_{pq} (\beta_c)^{-k}, \tag{8.3}$$

$$(q_a \chi \neq 0 \text{ and}) \quad \Lambda_a(\chi) \leqslant c_b |p_a \psi| \tag{8.4}$$

for all $a \in A_9$, $k \in N_0$, $\psi \in \hat{H}_{ak}^+ := ((\hat{g}_a)^k)^{-1} (\hat{H}_a^+)$, $\chi \in T_\psi \hat{H}_{ak}^+ \backslash \{0\}$.

Proof. Take the constant c_b given by Lemma 2.1 ([23]). Let $a \in A_9$. We must show that $\tilde{g} := \tilde{g}_a$, $\tilde{U} := D^2$, $U := D^3$ and $H := \hat{H}_a^+$ have the properties stated in the hypotheses of Lemma 2.1 ([23]), with respect to the vector b. Note first that $0 \in D^3$, $D^3 \subset C$ open, $\tilde{g}_a(0) = 0$. This is property $(\tilde{U})$. For $D\tilde{g}_a(0) = T(N, \cdot, a)$, we have $C = P_a \oplus Q_a$ with $T(N, \cdot, a)$ -invariant closed subspaces P_a and Q_a, and (3.6), so that properties (1) and $(\alpha\beta\gamma)$ of [23] are satisfied. Property (2) ([23]) follows from Proposition 5.3 (i). (5.11) yields (U). (5.7) gives (c). (5.9) implies (m). (5.10) shows $(\bar{c}, m)$. Property (p) is a consequence of $|p_a \psi| = \eta_0$ on H_a^+. From $T_\psi(H_a^+ \cap D^3) = Q_a$ for all $\psi \in H_a^+ \cap D^3$, we infer $\Lambda_a(\chi) = 0$ whenever $0 \neq \chi \in T_\psi(H_a^+ \cap D^3)$. This gives property $(\bar{c})$. Finally, $0 < \eta_0 = p_1$ and (3.8) guarantee property $(*)$: We obtain (p, k), (t, β_c) and (t, x); corresponding to (8.1), (8.2) and (8.4) respectively. ∎

Note that the pointwise estimate (8.4), together with (8.1) on convergence of the sets $\hat{H}_{ak}^+$ to Q_a, implies (8.2) and, in particular, uniform convergence of inclinations, i.e. the typical assertion of inclination lemmas.

The sharper estimate (8.4) will not be used until part IV of Section 9 – where it becomes important.

COROLLARY 8.2. *There exists a constant $c_5 > 0$ such that*

$$|D(\hat{g}_a)^k(\psi)\chi| \leqslant c_5 e^{\lambda_3 k N} |\chi|$$

for all $a \in A_9$, $k \in N_0$, $\psi \in \hat{H}_{ak}^+$, $\chi \in T_\psi \hat{H}_{ak}^+$.

Proof. a) Choose $j \in N$ so large that for all integers $k \geqslant j$, we have $q_a\chi \neq 0 \neq q_a\tilde{\chi}$ and

$$(\alpha+c+c\Lambda_a(\chi))\frac{1+\Lambda_a(\tilde{\chi})}{1-\Lambda_a(\chi)} \leqslant e^{\lambda_3 N}$$

for all $a \in A_9$, $\psi \in \hat{H}^+_{ak}$, $\tilde{\chi} \in T_\psi \hat{H}^+_{ak} \setminus \{0\}$, $\tilde{\psi} \in \hat{H}^+_{a,k-1}$, $\tilde{\chi} \in T_{\tilde{\psi}} \hat{H}^+_{a,k-1} \setminus \{0\}$.

This is possible because of $\alpha + c < 1$ and (8.2). Now let $a \in A_9$, $k \in N$ with $k \geqslant j$, $\psi \in \hat{H}^+_{ak}$ and $\chi \in T_\psi \hat{H}^+_{ak}$ be given. We show $|D\hat{g}_a(\psi)\chi| \leqslant e^{\lambda_3 N}|\chi|$: Suppose $\tilde{\chi} := D\hat{g}_a(\psi)\chi \neq 0$. Note $\tilde{\psi} := \hat{g}_a(\psi) \in \hat{H}^+_{a,k-1}$ and $\tilde{\chi} \in T_{\tilde{\psi}} \hat{H}^+_{a,k-1}$. $\tilde{\chi} \neq 0$ implies $\chi \neq 0$. We obtain $q_a\tilde{\chi} \neq 0 \neq q_a\chi$ and

$$\frac{|D\hat{g}_a(\psi)\chi|}{|\chi|} = \frac{|p_a\tilde{\chi}+q_a\tilde{\chi}|}{|p_a\chi+q_a\chi|} = \frac{|q_a\tilde{\chi}|}{|q_a\chi|}\frac{\left|\frac{1}{|q_a\tilde{\chi}|}p_a\tilde{\chi}+\frac{1}{|q_a\tilde{\chi}|}q_a\tilde{\chi}\right|}{\left|\frac{1}{|q_a\chi|}p_a\chi+\frac{1}{|q_a\chi|}q_a\chi\right|}.$$

With (3.6) and (5.7),

$$\begin{aligned}|q_a\tilde{\chi}| = |q_a D\hat{g}_a(\psi)\chi| &= |q_a(T(N, \chi, a)+Dr_a(\psi)\chi)| \\ &\leqslant \alpha|q_a\chi|+c|\chi| \leqslant (\alpha+c)|q_a\chi|+c|p_a\chi|.\end{aligned}$$

Using this and $k \geqslant j$, we find

$$\frac{|D\hat{g}_a(\psi)\chi|}{|\chi|} \leqslant (\alpha+c+c\Lambda_a(\chi))\frac{1+\Lambda_a(\tilde{\chi})}{1-\Lambda_a(\chi)} \leqslant e^{\lambda_3 N}$$

b) There is a constant $\hat{c}_5 \geqslant 1$ with $|D\hat{g}_a(\psi)| \leqslant \hat{c}_5$ on $D^3 \times A_9$. This follows from

$$\begin{aligned}D\hat{g}_a(\psi) &= T(N, \cdot, a)+Dr_a(\psi) = \\ &= T(N, \cdot, a)\circ(p_a+q_a)+D(p_a\circ r_a)(\psi)+D(q_a\circ r_a)(\psi),\end{aligned}$$

and from (3.6) and (5.7).

Let $a \in A_9$, $k \in N_0$ with $0 \leqslant k \leqslant j$ be given and consider $\psi \in \hat{H}^+_{ak}$ and $\chi \in C$: We get $|D(\hat{g}_a)^k(\psi)\chi| \leqslant (\hat{c}_5)^k|\chi|$ since $\psi, \hat{g}_a(\psi), \ldots, (\hat{g}_a)^k(\psi) \in \hat{H}^+_a$ are all contained in D^3.

For $a \in A_9$ and $j < k \in N$, $\psi \in \hat{H}^+_{ak}$ and $\chi \in T_\psi \hat{H}^+_{ak}$, consider the finite sequences $\psi =: \psi_k, \psi_{k-1}, \ldots, \psi_j$ and $\chi =: \chi_k, \chi_{k-1}, \ldots, \chi_j$ defined by $\psi_{k-l} := \hat{g}_a(\psi_{k-l+1})$, $\chi_{k-l} := D\hat{g}_a(\psi_{k-l+1})\chi_{k-l+1}$ for $l = 1, \ldots, k-j$. Then $k-l \geqslant j$ for $l = 0, \ldots, k-j$, and $\psi_{k-l} \in \hat{H}^+_{a,k-l}$, $\chi_{k-l} \in T_{\psi_{k-l}}\hat{H}^+_{a,k-l}$. Part a) implies $|D(\hat{g}_a)^{k-j}(\psi)\chi| \leqslant e^{(k-j)\lambda_3 N}|\chi|$. Hence

$$\begin{aligned}|D(\hat{g}_a)^k(\psi)\chi| = |\{D(\hat{g}_a)^j(\psi_j)\circ D(\hat{g}_a)^{k-j}(\psi)\}\chi| &\leqslant |D(\hat{g}_a)^j(\psi_j)\{D(\hat{g}_a)^{k-j}(\psi)\chi\}| \\ &\leqslant (\hat{c}_5)^j e^{(k-j)\lambda_3 N}|\chi|.\end{aligned}$$

Set $c_5 := (\hat{c}_5)^j e^{-j\lambda_3 N}$ and note $(\hat{c}_5)^j \leqslant c_5 e^{k\lambda_3 N}$ for $k = 0, \ldots, j$. ∎

9. Estimating $D_1\Sigma(\psi, a)$ at BC-points ψ in level sets H_{ak}^+. We choose $\delta_2 \in (0, \delta_1)$ with

$$(a|T(2, \cdot, a)|+c)\cdot\delta_2 < c_4/(2c_b) \quad \text{for all } a \in A_9. \tag{9.1}$$

This is possible because of (3.2) and (3.4) and boundedness of the projections p_a, q_a on A_9. Fix $\varepsilon \in (0, 1)$ such that

$$\frac{\varepsilon}{1-\varepsilon}|q_a| < \frac{1}{2} \quad \text{on } A_9. \tag{9.2}$$

By (8.3), there exists $j \in N$ with

$$|p_a\chi|/|\chi| < \varepsilon \tag{9.3}$$

for all $a \in A_9$, $k \in N$ with $k \geqslant j$, $\psi \in \hat{H}_{ak}^+$, $\chi \in T_\psi \hat{H}_{ak}^+ \setminus \{0\}$. Let $a \in A_9$ and $k \in N$ with $k \geqslant j$ be given. Consider a point $\psi \in H_{ak}^+ \subset \hat{H}_{ak}^+$ (see Corollary 7.1) with (BC) for (ψ, a), and assume in addition that

(9.4) there exists $\bar{\psi} \in E^+(\delta_2, \eta_1, a)$ with $\psi = Y(3, \bar{\psi}, a)$ and $Y(t, \bar{\psi}, a) \in E^+(\delta_2, \eta_1, a)$ for all $t \in [0, 3]$.

This is stronger than (BC) since $E^+(\delta_2, \eta_1, a) \subset D^3$, see Corollary 6.1.

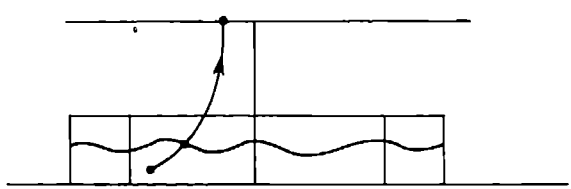

Fig. 17

We are going to estimate $|D_1\Sigma(\psi, a)|$ as indicated at the end of Section 6. This requires some preparation.

I. We have $\Phi_a \notin T_\psi H_{ak}^+$ since $T_\psi H_{ak}^+ \subset T_\psi \hat{H}_{ak}^+$, $0 \neq \Phi_a$, $|p_a\Phi_a|/|\Phi_a| = 1 > \varepsilon > |p_a\chi|/|\chi|$ for $0 \neq \chi \in T_\psi \hat{H}_{ak}^+$, by (9.3). It follows that there is a decomposition

$$C = T_\psi H_{ak}^+ \oplus \boldsymbol{R}\Phi_a = T_\psi H_{ak}^+ \oplus P_a \tag{9.5}$$

(recall that $T_\psi H_{ak}^+$ is a closed subspace of codimension 1, by Proposition 7.1). Let q and p denote the projections of C onto $T_\psi H_{ak}^+$ and P_a which are defined by (9.5). Recall Figure 8 from the Introduction.

We obtain

$$p\chi = \begin{cases} p_a\chi & \text{if } \chi \in P_a, \\ p_a\chi - (|q_a\chi|/|q_a(q\chi)|)\cdot p_a(q\chi) & \text{if } \chi \notin P_a. \end{cases}$$

Proof. Let $\chi \in C \setminus P_a$. Then $0 \neq q\chi \notin P_a$. Hence $q_a(q\chi) \neq 0$. Note $p_a\chi$

$= p_a(q\chi)+p_a(p\chi) = p_a(q\chi)+p\chi$ and $q_a\chi = q_a(q\chi)+q_a(p\chi) = q_a(q\chi)+0 \neq 0$. Therefore

$$p\chi = p_a\chi - p_a(q\chi) = p_a\chi - (|q_a\chi|/|q_a(q\chi)|)\cdot p_a(q\chi).$$

II. We define a new norm on $L_c(C, C)$ by

$$|B|_n := \sup\{|B(y\Phi_a+\chi)|:\ -1 \leqslant y \leqslant 1,\ \chi \in T_\psi H^+_{ak},\ |\chi| = 1\}.$$

Set $c_6 := (1/2)+c_{pq}$. We show $|p-p_a| < 1/2$, $|q-q_a| < 1/2$ and $|B| \leqslant c_6|B|_n$ for all $B \in L_c(C, C)$:

a) Let $\chi \in C$ be given with $|\chi| = 1$. Then $\chi = y\Phi_a+\bar{\chi}$ with $y \in \boldsymbol{R}$ and $\bar{\chi} \in T_\psi H^+_{ak}$, and $(p-p_a)\chi = y\Phi_a - p_a(y\Phi_a+\bar{\chi}) = -p_a\bar{\chi}$. $|\chi| = 1$ gives $|q_a| \geqslant |q_a\chi| = |q_a(y\Phi_a+\bar{\chi})| = |q_a\bar{\chi}|$. Recall

$$\begin{aligned} |p_a\bar{\chi}| &\leqslant \varepsilon|\bar{\chi}| \quad \text{(see (9.3))} \\ &\leqslant \varepsilon(|p_a\bar{\chi}|+|q_a\bar{\chi}|) \leqslant \varepsilon(|p_a\bar{\chi}|+|q_a|). \end{aligned}$$

Hence

$$|(p-p_a)\chi| = |p_a\bar{\chi}| \leqslant \frac{\varepsilon}{1-\varepsilon}\cdot|q_a| < \frac{1}{2},$$

by (9.2). It follows that $|p-p_a| < 1/2$, and $|q-q_a| = |\mathrm{id}_C - p - (\mathrm{id}_C - p_a)| = |p-p_a| < 1/2$.

b) Because of a), $|p| \leqslant |p_a|+1/2$ and $|q| \leqslant |q_a|+1/2$. This means that the ball with center $0 \in C$ and radius $1/c_6$ is contained in the set

$$\begin{aligned} &\{\chi \in C:\ |p\chi| \leqslant 1,\ |q\chi| \leqslant 1\} \\ &\qquad = \{\chi \in C:\ \chi = y\Phi_a+\bar{\chi} \text{ with } -1 \leqslant y \leqslant 1,\ |\bar{\chi}| \leqslant 1,\ \bar{\chi} \in T_\psi H^+_{ak}\}. \end{aligned}$$

For $B \in L_c(C, C)$, we arrive at

$$|B| = \sup\{|B\chi|:\ |\chi| \leqslant 1\} = c_6\cdot\sup\{|B\chi|:\ |\chi| \leqslant 1/c_6\} \leqslant c_6|B|_n.$$

III. Consider the right tangent vector $w := w^{\psi,a}$ at (ψ, a), defined in Section 7. By Proposition 7.2 (ii), $w \notin T_\psi H^+_{ak}$. Hence $pw \neq 0$. We prove

$$|D_1\Sigma(\psi, a)| \leqslant c_7|p_a\psi|^{-\lambda_3/\mu_2}\left(1+\frac{1}{|pw|}\right) \tag{9.6}$$

where

$$c_7 := c_6c_5(\eta_0/c_2)^{\lambda_3/\mu_2}\cdot\{1+[\tfrac{1}{2}+c_{pq}]\,[\sup_{A_9}(a|T(2,\cdot,a)|+c)]\cdot c_{pq}\cdot\varrho_3\} < \infty:$$

a) Recall $H^+_{ak} \subset \hat{H}^+_{ak}$ and $\Sigma(\cdot, a) = (\hat{g}_a)^k$ on H^+_{ak}. This implies $D_1\Sigma(\psi, a)\chi = D(\hat{g}_a)^k(\psi)\chi$ for all $\chi \in T_\psi H^+_{ak} \subset T_\psi\hat{H}^+_{ak}$. With the definition of $|\ |_n$ and with Corollary 8.2, we obtain

$$|D_1\Sigma(\psi, a)|_n \leqslant \sup\{|D_1\Sigma(\psi, a)y\Phi_a|:\ -1 \leqslant y \leqslant 1\}+c_5e^{\lambda_3kN}.$$

b) Recall $D_1\Sigma(\psi; a)w = 0$, from Proposition 7.2(i). Hence $D_1\Sigma(\psi, a)pw = -D_1\Sigma(\psi, a)qw$. With $pw \neq 0$ (see above), we get $\Phi_a \in \{(1/|pw|)\cdot pw, -(1/|pw|)\cdot pw\}$, and

$$\sup\{|D_1\Sigma(\psi, a)y\Phi_a|:\ -1 \leqslant y \leqslant 1\} \leqslant |D_1\Sigma(\psi, a)qw|/|pw|.$$

Using Corollary 8.2 as in part a), we infer

$$\sup\{\ldots\} \leqslant \frac{c_5 e^{\lambda_3 kN}|qw|}{|pw|}.$$

c) Estimate of $|qw|$: By part II, $|qw| \leqslant (\frac{1}{2}+|q_a|)|w|$. Property (BC) says that $\psi = Y(3, \bar{\psi}, a)$ with $\bar{\psi} \in D^3$ and $Y(t, \bar{\psi}, a)$ in D^3 on $[0, 3]$. This allows to apply Corollary 5.4. We get

$$\begin{aligned} |w| &\leqslant |p_a w|+|q_a w| \leqslant (a|T(2, \cdot, a)|+c)(|p_a\bar{\psi}|+|q_a\bar{\psi}|) \\ &\leqslant (a|T(2, \cdot, a)|+c)(|p_a|+|q_a|)\cdot \varrho_3, \end{aligned}$$

with the radius ϱ_3 of D^3.

d) Estimate of $e^{\lambda_3 kN}$: $\sigma(\psi, a) = kN$ and Corollary 5.2 give

$$0 < \eta_0 = |\eta_0\Phi_a| = |p_a\Sigma(\psi, a)| = |p_a Y(kN, \psi, a)| \leqslant c_2 e^{\mu_2 kN}|p_a\psi|,$$

hence

$$\frac{1}{\mu_2}\log\frac{\eta_0}{c_2|p_a\psi|} = kN.$$

With $\lambda_3 < 0$,

$$e^{\lambda_3 kN} \leqslant (\eta_0/c_2)^{\lambda_3/\mu_2}|p_a\psi|^{-\lambda_3/\mu_2}.$$

e) The results from a) – d) and the estimate $|D_1\Sigma(\psi, a)| \leqslant c_6|D_1\Sigma(\psi, a)|_n$ from part II imply (9.6).

IV. a) Consider the term $1/|pw|$. In the trivial case $w \in P_a$, $pw = p_a w$, see I. Corollary 5.3, applied to $\bar{\psi} \in D^3$ with $Y(3, \bar{\psi}, a) = \psi$ and $Y(t, \bar{\psi}, a) \in D^3$ on $[0, 3]$, yields

$$c_4|p_a\psi| \leqslant |p_a w| = |pw|. \tag{9.7}$$

In the nontrivial case $w \notin P_a$, we fully exploit the pointwise estimate (8.4) from Corollary 8.1. We have $qw \neq 0$ so that $\Lambda_a(qw)$ is defined, and

$$\begin{aligned} |pw| &\geqslant |p_a w|-|q_a w|\Lambda_a(qw) \quad \text{(see part I)} \\ &\geqslant c_4|p_a\psi|-|q_a w|\Lambda_a(qw), \end{aligned}$$

by Corollary 5.3, applied as in the trivial case. Corollary 5.4 gives

$$|q_a w| = |q_a D_1 Y(3, \bar{\psi}, a)| \leqslant (a|T(2, \cdot, a)|+c)|q_a\bar{\psi}|$$

where $\bar{\psi} \in E^+(\delta_2, \eta_1, a)$, $Y(3, \bar{\psi}, a) = \psi$, and in particular $|q_a\bar{\psi}| < \delta_2$, see (9.4).

With (9.1), we obtain $|q_a w| \leqslant c_4/(2c_b)$, and (8.4) gives

$$|pw| \geqslant c_4|p_a\psi| - (c_4/2c_b)\cdot c_b\cdot|p_a\psi| = (c_4/2)\cdot|p_a\psi|. \tag{9.8}$$

b) By (9.6), (9.7) and (9.8),

$$|D_1\Sigma(\psi, a)| \leqslant c_7\big(|p_a\psi|^{-\lambda_3/\mu_2} + (2/c_4)|p_a\psi|^{-(\lambda_3/\mu_2)-1}\big), \tag{9.9}$$

for $a\in A_9$, $k\in N$ with $k\geqslant j$, $\psi\in H_{ak}^+$ with (9.4).

10. End of the proof of Theorem 6.1. In order to apply (9.9), we consider $\psi\in E^+(\delta_2, \eta, a)$ and look for an estimate of the form $|D_1\Sigma(\psi, a)| \leqslant \text{const}\cdot|D_1\Sigma(\hat{\psi}, a)|$ with a point $\hat{\psi}$ on the trajectory $t\to Y(t, \psi, a)$ so that $\hat{\psi}\in H_{ak}^+$, $k = k(\psi, a)\geqslant j$, satisfies (9.4) and is not too far away from ψ.

a) Fix

$$t := \frac{1}{\lambda_3}\log\frac{1}{1+c_2} > 0$$

so that $c_2 e^{\lambda_3 t} < 1$. By Corollary 5.5, there exists a constant $c_8 = c(t)$ with $|D_2 Y(s, \psi, a)| < c_8$ for all $(\psi, a)\in D^3\times A_8$ such that $(t+N+3, \psi, a)\in\Omega$ and $Y(s, \psi, a)\in D^3$ on $[0, t+N+3]$.

b) Choose $\eta_2\in(0, \eta_1)$ so small that

$$t+N+3 < \frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta_2}, \tag{10.1}$$

$$c_2 e^{\mu_2(t+N+3)}\eta_2 < \eta_1, \tag{10.2}$$

$$j \leqslant \frac{1}{N}\left[\frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta_2} - (t+N+3)\right]. \tag{10.3}$$

c) Let $a\in A_9$, $\eta\in(0, \eta_2)$, $\psi\in E^+(\delta_2, \eta, a)$. We show $t+N+3 < \sigma(\psi, a)$ and $Y(s, \psi, a)\in E^+(\delta_2, \eta, a)$ on $[t, t+N+3]$: Proposition 6.2 and (10.1) yield

$$\sigma(\psi, a) \geqslant \frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta} > \frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta_2} > t+N+3.$$

In particular, $t+N+3 < \sigma(\psi, a) < \tau(\psi, a)$. By Proposition 6.2, $Y(s, \psi, a)\in D^3$ on $[0, \tau(\psi, a))$, and Corollary 5.2 gives $|q_a Y(s, \psi, a)| \leqslant c_2 e^{\lambda_3 s}|q_a\psi|$ on $[t, \infty)\cap[0, \tau(\psi, a))$. With a) and $\lambda_3 < 0$, we obtain $|q_a Y(s, \psi, a)| \leqslant |q_a\psi| < \delta_2$ on $[t, \infty)\cap[0, \tau(\psi, a))$. Moreover,

$$|p_a Y(s, \psi, a)| \leqslant c_2 e^{\mu_2 s}|p_a\psi| < \eta_1 \quad \text{on } [0, t+N+3], \tag{10.4}$$

because of Corollary 5.2 and (10.2). Again by Proposition 6.2, $0 < p_a Y(s, \psi, a) < \eta_0\Phi_a$ on $[0, \sigma(\psi, a))$. Altogether,

$$Y(s, \psi, a)\in E^+(\delta_2, \eta_1, a) \quad \text{on } [t, t+N+3].$$

d) There exists a uniquely determined $k = k(\psi, a) \in N$ with $\sigma(\psi, a) - kN \in [t+3, t+N+3]$ and $\sigma(\psi, a) - (k+1)N < t+3$. It follows that

$$k \geqslant \frac{1}{N}\{\sigma(\psi, a) - (t+N+3)\} \geqslant \frac{1}{N}\left\{\frac{1}{\mu_2}\log\frac{\eta_0}{c_2\eta} - (t+N+3)\right\} \geqslant j,$$

see Proposition 6.2 and (10.3) and $0 < \eta < \eta_2$.

e) Set $\hat{t} := \sigma(\psi, a) - kN$ and $\hat{\psi} := Y(\hat{t}, \psi, a)$. We show (9.4) for $\hat{\psi}$: Set $\bar{\psi} := Y(\bar{t}, \psi, a)$ with $\bar{t} := \sigma(\psi, a) - kN - 3 \in [t, t+N]$. Then $\hat{\psi} = Y(3, \bar{\psi}, a)$. For $s \in [0, 3]$, $Y(s, \bar{\psi}, a) = Y(s+\bar{t}, \psi, a)$ and $s+\bar{t} \in [t, t+N+3]$. Hence $Y(s, \bar{\psi}, a) \in E^+(\delta_2, \eta_1, a)$, see c).

f) Proof of $\hat{\psi} \in H^+_{ak}$: We have $\hat{\psi} \in E^+(\delta_2, \eta_1, a) \subset E^+(\delta_1, \eta_1, a)$ and

$$p_a Y(s, \bar{\psi}, a) = p_a Y(\sigma(\psi, a) - kN + s, \psi, a)\begin{Bmatrix} < \eta_0 \Phi_a \\ > 0 \end{Bmatrix} \quad \text{on } [0, kN),$$

$$p_a Y(kN, \bar{\psi}, a) = p_a Y(\sigma(\psi, a), \psi, a) = \eta_0 \Phi_a.$$

The minimality of σ in Proposition 6.2 implies $\sigma(\hat{\psi}, a) = kN$.

g) There exists an open neighborhood $V^\psi \subset E^+(\delta_2, \eta_1, a)$ of ψ such that on V^ψ, $\Sigma(\hat{\psi}, a) = \Sigma(Y(\hat{t}, \tilde{\psi}, a), a)$: Choose an open neighborhood $\hat{D} \subset E^+(\delta_2, \eta_1, a)$ of $\hat{\psi}$, and on open neighborhood V^ψ of ψ in $E^+(\delta_2, \eta_1, a)$ such that $Y(\hat{t}, \cdot, a)$ maps V^ψ into $\hat{D}$, with $0 < p_a Y(s, \tilde{\psi}, a) < \eta_0 \Phi_a$ on $[0, \hat{t}]$ for all $\tilde{\psi} \in V^\psi$ (or equivalently, $0 < \langle \Psi_a, Y(s, \tilde{\psi}, a)\rangle_a < \eta_0$ on $[0, \hat{t}] \times V^\psi$). $\hat{D}$ and V^ψ depend on ψ and on a, of course. It follows that $\sigma(\tilde{\psi}, a) > \hat{t}$ on V^ψ. We have to show that $\sigma(\tilde{\psi}, a) = \sigma(Y(\hat{t}, \tilde{\psi}, a), a) + \hat{t}$ on V^ψ. Let $\tilde{\psi} \in V^\psi$. The inequality $0 < \sigma(\tilde{\psi}, a) - \hat{t}$ and the properties of $\sigma(\tilde{\psi}, a)$ from Proposition 6.2 give $0 < p_a Y(s, Y(\hat{t}, \tilde{\psi}, a), a) < \eta_0 \Phi_a$ on $[0, \sigma(\tilde{\psi}, a) - \hat{t})$ and $p_a Y(s, Y(\hat{t}, \tilde{\psi}, a), a) = \eta_0 \Phi_a$ for $s = \sigma(\tilde{\psi}, a) - \hat{t}$. Therefore Proposition 6.2, now applied to $Y(\hat{t}, \tilde{\psi}, a) \in E^+(\delta_1, \eta_1, a)$, yields $\sigma(Y(\hat{t}, \tilde{\psi}, a), a) = \sigma(\tilde{\psi}, a) - \hat{t}$.

h) From c), we have $\tau(\psi, a) > \sigma(\psi, a) > t+N+3$ and $(t+N+3, \psi, a) \in \Omega$, $Y(s, \psi, a) \in D^3$ on $[0, t+N+3]$. The result of part a) gives, together with $0 \leqslant \hat{t} \leqslant t+N+3$, $|D_2 Y(\hat{t}, \psi, a)| \leqslant c_8$. Using part g), we infer

$$|D_1 \Sigma(\psi, a)| \leqslant |D_1 \Sigma(\hat{\psi}, a)| \cdot c_8. \tag{10.5}$$

We have $\hat{\psi} \in H^+_{ak}$, $k \geqslant j$, $a \in A_9$, and (9.4) is satisfied, too. Now (9.9) yields

$$|D_1 \Sigma(\hat{\psi}, a)| \leqslant c_7\left(1 + \frac{2}{c_4}\right)[|p_a\hat{\psi}|^{-\lambda_3/\mu_2} + |p_a\hat{\psi}|^{-(\lambda_3/\mu_2)-1}].$$

With (10.4) in c), applied to $s = \hat{t}$, and with (10.5), we arrive at

$$|D_1 \Sigma(\psi, a)| \leqslant c_9(|p_a\psi|^{-\lambda_3/\mu_2} + |p_a\psi|^{-(\lambda_3/\mu_2)-1})$$

where

$$c_9 := c_8 c_7\left(1+\frac{2}{c_4}\right)\{[c_2 e^{\mu_2(t+N+3)}]^{-\lambda_3/\mu_2}+[c_2 e^{\mu_2(t+N+3)}]^{-(\lambda_3/\mu_2)-1}\},$$

$$t = \frac{1}{\lambda_3}\log\frac{1}{1+c_2} > 0.$$

Here we have also used $\lambda_3 < -\mu_2 < 0$ which finally implies the assertion on uniform convergence since $|p_a\psi| < \eta$ for ψ in $E^+(\delta_2, \eta, a)$.

III

11. Šilnikov continuation and return map. Consider the open subset

$$E_2 := \{(\psi, a)\in C\times A_9\colon\ |p_a\psi| < \eta_2,\ |q_a\psi| < \delta_2\}$$

of $C\times \boldsymbol{R}$. We have

$$E_2 = E_2^+ \cup \{(\psi, a)\in C\times A_9\colon\ p_a\psi = 0,\ |q_a\psi| < \delta_2\} \cup E_2^-,$$

with the open upper and lower halves ,

$$E_2^+ := \{(\psi, a)\in C\times A_9\colon\ \psi\in E^+(\delta_2, \eta_2, a)\},$$

$$E_2^- := \{(\psi, a)\in C\times A_9\colon\ \psi\in E^-(\delta_2, \eta_2, a)\}.$$

Similarly to the approach in Šilnikov's paper [18], we continue the restriction $\Sigma|E_2^+$ to all of E_2 by $\Sigma_2(\psi, a) := \eta_0\Phi_a$ if $(\psi, a)\in E_2$ and $p_a\psi \leqslant 0$. It is obvious that the restrictions of Σ_2 to the sets E_2^+, E_2^- are of class C^1.

COROLLARY 11.1. (i) *Σ_2 is continuous.*
(ii) $\Sigma_2(E_2) \subset D^3$.

Proof. (i) The assertion follows by elementary arguments if we can show $\Sigma_2(\psi_n, a_n)\to\eta_0\Phi_a$ for every sequence of points (ψ_n, a_n) in E_2^+ with $(\psi_n, a_n)\to(\psi, a)\in E_2$, $p_a\psi = 0$. Let such a sequence be given. Then $0 < p_{a_n}\psi_n\to 0$, $\eta_0\Phi_{a_n}\to\eta_0\Phi_a$, and by Proposition 6.2(v),

$$\begin{aligned}|\Sigma_2(\psi_n, a_n)-\eta_0\Phi_{a_n}| &= |q_{a_n}Y(\sigma(\psi_n, a_n), \psi_n, a_n)|\\ &\leqslant c_2\delta_0(\eta_0/c_2)^{\lambda_3/\mu_2}(2|p_{a_n}\psi_n|)^{-\lambda_3/\mu_2}\end{aligned}$$

if n is so large that $2|p_{a_n}\psi_n| < \eta_0$. By $\mu_2 < -\lambda_3$, we obtain $\Sigma_2(\psi_n, a_n)\to\eta_0\Phi_a$ as $n\to\infty$.

(ii) Use Proposition 6.2 and note that $\eta_0\Phi_a\in D^3$ for $a\in A_9$, by (6.3). ∎

We want to define a parameterized return map as follows. For a close to a_0 and ψ close to $\eta_0\Phi_{a_0}$, we transform to the corresponding point in the original state space which is close to a point on the trajectory of the

heteroclinic solution x^{a_0}. Then we follow the semiflow $X(\cdot, \cdot, a)$ until we arrive in a suitably small neighborhood of the equilibrium ξ_h, come back to a neighborhood of $0 \in C$ by the translation modulo ξ_h, take local coordinates by means of $G(\cdot, a)$, and apply Σ_2 – so that $\eta_0 \Phi_{a_0}$ becomes a fixed point for $a = a_0$.

This requires some technical preparation. We set $z_t^a := X(t, G^-(\eta_0 \Phi_a, a), a)$ for $t \geqslant 0$ and $a \in A_9$, and $z_t := z_t^{a_0}$ for $t \geqslant 0$. These trajectories continue the branches of the local unstable manifolds at $0 \in C$ "above the stable manifolds":

PROPOSITION 11.1. (i) *For every $a \in A_9$, we have $z_0^a \in U_a$, $0 < p_a z_0^a - s_a(q_a z_0^a)$, and there exists $t \in \boldsymbol{R}$ with $z_s^a = x_{t+s}^a$ for all $s \geqslant 0$.*

(ii) *There exists $\theta > 1$ with $z_\theta - \xi_h \in B_6$, $(G(z_\theta - \xi_h, a_0), a_0) \in E_2$, $p_{a_0} G(z_\theta - \xi_h, a_0) = 0$, $0 \neq q_{a_0} G(z_\theta - \xi_h, a_0)$.*

(iii) *There exists an interval A_{10} with $(G(z_\theta^a - \xi_h, a), a) \in E_2$ and $0 < p_a G(z_\theta^a - \xi_h, a)$ for all $a \in A_{10}$ with $a_0 < a$.*

Proof. a) Let $a \in A_9$. Proposition 5.1 (v) gives $z_0^a = G^-(\eta_0 \Phi_a, a) \in U_a$ and $z_0^a = x_t^a$ for some $t \in \boldsymbol{R}$. Note $p_a z_0^a - s_a(q_a z_0^a) = p_a G(z_0^a, a) = p_a \eta_0 \Phi_a > 0$.

b) Proof of (ii). The properties of x^{a_0} and (i) show $z_s - \xi_h \to 0$ as $s \to +\infty$. Choose $\theta > 1$ so large that $z_s - \xi_h \in B_6$ for all $s \geqslant \theta$, and $(G(z_\theta - \xi_h, a_0), a_0) \in E_2$. (2.2) implies that $z_s - \xi_h = X(s - \theta, z_\theta - \xi_h, a_0)$ on $[\theta, \infty)$. It follows that $X(s, z_\theta - \xi_h, a_0) \in B_6 \subset V_{a_0 3}$ on $[0, \infty)$, see Proposition 4.3. (4.5′) gives $z_\theta - \xi_h \in S_{a_0}$. Therefore $p_0^a G(z_\theta - \xi_h, a_0) = 0$, by Proposition 5.1 (i) and Proposition 2.2 (ii) imply $0 \neq z_\theta - \xi_h$. By injectivity, $G(z_\theta - \xi_h, a_0) \neq 0$. Hence $q_{a_0} G(z_\theta - \xi_h, a_0) \neq 0$.

c) Choose an open interval $A_{10.1} \subset A_9$, centered at a_0, with $z_\theta^a - \xi_h \in B_6 \setminus \{0\}$ and $(G(z_\theta^a - \xi_h, a), a) \in E_2$ for all $a \in A_{10.1}$. We have $p_a G(z_\theta^a - \xi_h, a) \neq 0$ for $a \in A_{10.1}$ with $a > a_0$: Otherwise, the definition of G implies $z_\theta^a - \xi_h \in S_a$, hence

$$0 = \lim_{s \to \infty} X(s, z_\theta^a - \xi_h, a) = \lim_{s \to \infty} z_{s+\theta}^a - \xi_h$$

which is a contradiction to (i) and to the fact that x^a does not converge to ξ_h as $s \to +\infty$.

d) In order to complete the proof of (iii), observe first $\xi_0 - \xi_h < G^-(-\eta_0 \Phi_{a_0}, a_0) = x_s^{a_0 -} < 0$ with some $s < 0$. This follows from Proposition 5.1 (v) and from $G^-(-\eta_0 \Phi_{a_0}, a_0) \in B_5 \subset \{\phi \in C: |\phi| < \delta_h\}$, $\delta_h < \xi_h - \xi_0$. Suppose now that there is a sequence of parameters $a(n) \in A_{10.1} \cap (a_0, \infty)$ with $a(n) \to a_0$ as $n \to \infty$, and with $0 > p_{a(n)} G(z_\theta^{a(n)} - \xi_h, a(n))$ for all $n \in N$. Then

$$p_{a(n)} G(z_\theta^{a(n)} - \xi_h, a(n)) \to p_{a_0} G(z_\theta - \xi_h, a_0) = 0 \quad \text{as } n \to \infty. \tag{11.1}$$

We have $G(z_\theta^{a(n)} - \xi_h, a(n)) \in E^-(\delta_2, \eta_2, a(n))$. By the remark following Proposi-

tion 6.2, we obtain a sequence $(\sigma_n)_{n\in N}$ in R^+ with

$$p_{a(n)}Y\big(\sigma_n, G\big(z_\theta^{a(n)}-\bar{\xi}_h, a(n)\big), a(n)\big) = -\eta_0\Phi_{a(n)}$$

and with

$$\big|q_{a(n)}Y\big(\sigma_n, G\big(z_\theta^{a(n)}-\bar{\xi}_h, a(n)\big), a(n)\big)\big|$$
$$\leqslant c_2\delta_0(\eta_0/c_2)^{\lambda_3/\mu_2}2\big|p_{a(n)}G\big(z_\theta^{a(n)}-\bar{\xi}_h, a(n)\big)\big|^{-\lambda_3/\mu_2}$$

for all $n\in N$ so large that $2|p_{a(n)}G(z_\theta^{a(n)}-\bar{\xi}_h, a(n))| < \eta_0$. With $-\eta_0\Phi_{a(n)}\to -\eta_0\Phi_{a_0}$, (11.1) and $\mu_2 < -\lambda_2-\lambda_3$, we infer that $Y\big(\sigma_n, G\big(z_\theta^{a(n)}-\bar{\xi}_h, a(n)\big), a(n)\big)$ converges to $-\eta_0\Phi_{a_0}$. Therefore

$$\bar{\xi}_0-\bar{\xi}_h < G^-\Big(Y\big(\sigma_n, G\big(z_\theta^{a(n)}-\bar{\xi}_h, a(n)\big), a(n)\big), a(n)\Big) < 0$$

for n sufficiently large. For such n, $\bar{\xi}_0-\bar{\xi}_h < z_{\theta+\sigma_n}^{a(n)}-\bar{\xi}_h < 0$. Now (i) and Proposition 2.1 (iii) imply $x^{a(n)} < \xi_h$ on some unbounded interval in R^+ which is a contradiction to $a_0 < a(n)\in A_{10.1}$ and to the properties of $x^{a(n)}$ stated in Section 2. ∎

We are ready for the definition of the Šilnikov return map. Smoothness of the maps $X(t, \cdot, \cdot)$ for $t > 1$, of G^- and of G, and Proposition 11.1 (ii) allow to find an open neighborhood dom′ of $(\eta_0\Phi_{a_0}, a_0)$ in $C\times R$ with

$$X\big(\theta, G^-(\psi, a), a\big)-\bar{\xi}_h\in B_6 \quad \text{on dom}' \tag{11.2}$$

and such that the assignment $(\psi, a)\to G\big(X\big(\theta, G^-(\psi, a), a\big)-\bar{\xi}_h, a\big)$ defines a C^1-map $f'_{\theta 1}$: dom′ $\to C$ with

$$\big(f'_{\theta 1}(\psi, a), a\big)\in E_2, \tag{11.3}$$

$$q_a f'_{\theta 1}(\psi, a) \neq 0 \quad \text{on dom}'. \tag{11.4}$$

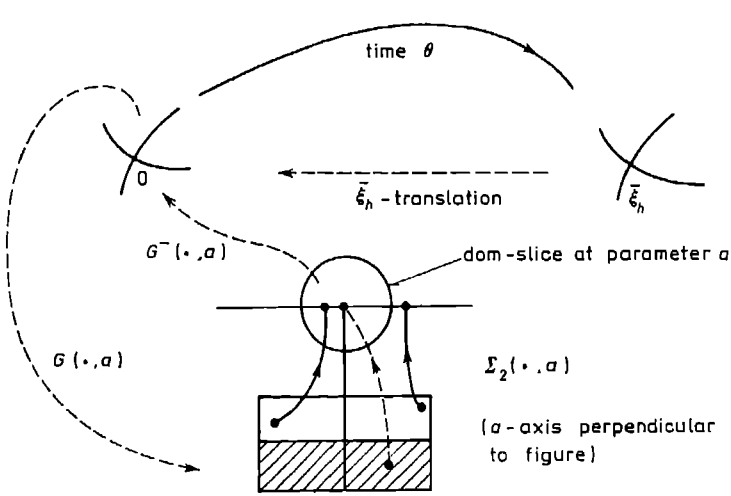

Fig. 18

We restrict $f'_{\theta 1}$ to a convex open neighborhood dom $\subset$ dom′ of $(\eta_0 \Phi_{a_0}, a_0)$ so that $f_{\theta 1} := f'_{\theta 1}|\mathrm{dom}$ and its derivatives $D_1 f_{\theta 1}, D_2 f_{\theta 1}$ are bounded on dom. Then we set $f_\theta(\psi, a) := (f_{\theta 1}(\psi, a), a)$ on dom, and $f := \Sigma_2 \circ f_\theta$. This is the desired return map. f is continuous, see Corollary 11.1. We have

$$f(\eta_0 \Phi_{a_0}, a_0) = \eta_0 \Phi_{a_0}, \tag{11.5}$$

because of $f_{\theta 1}(\eta_0 \Phi_{a_0}, a_0) = G(z_\theta - \xi_h, a_0)$, $p_{a_0} G(z_\theta - \xi_h, a_0) = 0$ (by Proposition 11.1(ii)), $\Sigma_2(\psi, a_0) = \eta_0 \Phi_{a_0}$ for all $(\psi, a) \in E_2$ with $p_{a_0}\psi = 0$.

12. Smoothness properties of f. We use Proposition 6.2 and Theorem 6.1 in order to derive

PROPOSITION 12.1. *f has partial derivatives $D_1 f$: dom $\to L_c(C, C)$ and $D_2 f$: dom $\to L_c(\boldsymbol{R}, C)$. $D_1 f$ is continuous. We have*

$$D_1 f(\psi, a) = 0, \tag{12.1}$$

$$D_2 f(\psi, a) = \eta_0 \lim_{a' \underset{\neq}{\to} a} \frac{1}{a' - a}(\Phi_{a'} - \Phi_a) \tag{12.2}$$

for all $(\psi, a) \in$ dom with $p_a f_{\theta 1}(\psi, a) \leqslant 0$. In particular,

$$D_1 f(\eta_0 \Phi_{a_0}, a_0) = 0.$$

Proof. a) The restriction of $f = \Sigma_2 \circ f_\theta$ to the open set $\{(\psi, a) \in \mathrm{dom}: p_a f_{\theta 1}(\psi, a) \neq 0\}$ is of class C^1 since Σ_2 is of class C^1 on $E_2^+ \cup E_2^-$. We have $f(\psi, a) = \Sigma_2(f_{\theta 1}(\psi, a), a) = \eta_0 \Phi_a$ for all $(\psi, a) \in$ dom with $p_a f_{\theta 1}(\psi, a) < 0$ which implies (12.1) and (12.2) for such points (ψ, a).

Let $(\psi, a) \in$ dom be given with $p_a f_{\theta 1}(\psi, a) = 0$.

b) Existence of $D_1 f(\psi, a)$ and $D_1 f(\psi, a) = 0$ follow by elementary arguments if we can show

$$d_n := \frac{|f(\psi_n, a) - f(\psi, a)|}{|\psi_n - \psi|} \to 0 \quad \text{as } n \to \infty$$

for sequences of points $(\psi_n, a) \in \mathrm{dom} \setminus \{(\psi, a)\}$ with $\lim_{n\to\infty} \psi_n = \psi$ and $p_a f_{\theta 1}(\psi_n, a) > 0$ for all $n \in N$, and also for such sequences with $\lim_{n\to\infty} \psi_n = \psi$ and with $p_a f_{\theta 1}(\psi_n, a) \leqslant 0$ for all $n \in N$. The second case is trivial since all numerators are zero, by definition of Σ_2 on $E_2 \setminus E_2^+$:

$$f(\psi_n, a) - f(\psi, a) = \Sigma_2(f_{\theta 1}(\psi_n, a), a) - \Sigma_2(f_{\theta 1}(\psi, a), a) = \eta_0 \Phi_a - \eta_0 \Phi_a.$$

Consider the first case. Note $f_{\theta 1}(\psi_n, a) - f_{\theta 1}(\psi, a) \neq 0$ for all $n \in N$ since otherwise $p_a f_{\theta 1}(\psi_n, a) = p_a f_{\theta 1}(\psi, a) = 0$. With a bound c_{10} for f, $Df_{\theta 1}$, $D_1 f_{\theta 1}$, $D_2 f_{\theta 1}$ on the convex open set dom, we obtain

$$d_n \leqslant c_{10} \frac{|\Sigma_2(f_{\theta 1}(\psi_n, a), a) - \eta_0 \Phi_a|}{|f_{\theta 1}(\psi_n, a) - f_{\theta 1}(\psi, a)|} \quad \text{for all } n \in N.$$

Observe

$$0 < |p_a f_{\theta 1}(\psi_n, a)| = |p_a f_{\theta 1}(\psi_n, a) - p_a f_{\theta 1}(\psi, a)| \leqslant |p_a|\,|f_{\theta 1}(\psi_n, a) - f_{\theta 1}(\psi, a)|.$$

Therefore

$$d_n \leqslant c_{pq} c_{10} \frac{|q_a \Sigma_2(f_{\theta 1}(\psi_n, a), a)|}{|p_a f_{\theta 1}(\psi_n, a)|}$$

(recall $p_a \Sigma_2(f_{\theta 1}(\psi_n, a), a) = \eta_0 \Phi_a$). With $p_a f_{\theta 1}(\psi_n, a) \to p_a f_{\theta 1}(\psi, a) = 0$ and Proposition 6.2(v), applied to n so large that $2|p_a f_{\theta 1}(\psi_n, a)| < \eta_0$, we arrive at

$$d_n \leqslant c_{pq} c_{10} c_2 \delta_0 (\eta_0/c_2)^{\lambda_3/\mu_2} 2^{-\lambda_3/\mu_2} |p_a f_{\theta 1}(\psi_n, a)|^{-(\lambda_3/\mu_2)-1},$$

and $\mu_2 < -\lambda_3$ shows $d_n \to 0$ as $n \to \infty$.

c) $D_1 f(\eta_0 \Phi_{a_0}, a_0) = 0$ is a consequence of b) and (11.5).

d) Proof of continuity of $D_1 f$ at (ψ, a). We have (12.1), by the results of parts a) and b). Continuity follows if in addition $D_1 f(\psi_n, a_n) \to 0$ as $n \to \infty$ for every sequence of points (ψ_n, a_n) in dom which converges to (ψ, a) as $n \to \infty$ and satisfies $p_{a_n} f_{\theta 1}(\psi_n, a_n) > 0$ for all $n \in N$. Because of a), we can compute $D_1 f(\psi_n, a_n)$ for such points from the chain rule. This gives

$$|D_1 f(\psi_n, a_n)| \leqslant |D_1 \Sigma(f_{\theta 1}(\psi_n, a_n), a_n)| \cdot c_{10} \quad \text{for all } n \in N.$$

With $0 < p_{a_n} f_{\theta 1}(\psi_n, a_n) \to p_a f_{\theta 1}(\psi, a) = 0$ and with Theorem 6.1, we obtain $D_1 f(\psi_n, a_n) \to 0$ as $n \to \infty$.

e) Existence and computation of $D_2 f(\psi, a)$. Consider a sequence of parameters a_n with $a_n \underset{\neq}{\to} a$ as $n \to \infty$, $(\psi, a_n) \in \text{dom}$ for all $n \in N$. If $p_{a_n} f_{\theta 1}(\psi, a_n) \leqslant 0$ for all $n \in N$, then

$$d_n := \frac{1}{a_n - a}(f(\psi, a_n) - f(\psi, a)) = \frac{\eta_0}{a_n - a}(\Phi_{a_n} - \Phi_a)$$

$$\to \eta_0 \lim_{a' \underset{\neq}{\to} a} \frac{1}{a' - a}(\Phi_{a'} - \Phi_a) \quad \text{as } n \to \infty.$$

If $p_{a_n} f_{\theta 1}(\psi, a_n) > 0$ for all $n \in N$, we write

$$d_n := \frac{1}{a_n - a}(\Sigma_2(f_{\theta 1}(\psi, a_n), a_n) - \eta_0 \Phi_a) = \eta_0 \frac{1}{a_n - a}(\Phi_{a_n} - \Phi_a) + d_n^*$$

with

$$d_n^* := \frac{1}{a_n - a}(\Sigma_2(f_{\theta 1}(\psi, a_n), a_n) - \eta_0 \Phi_{a_n}) \quad \text{for all } n \in N,$$

and we proceed to show $d_n^* \to 0$ as $n \to \infty$. First, we want to replace the denominator: Observe

$$0 < |p_{a_n} f_{\theta 1}(\psi, a_n) = |p_{a_n} f_{\theta 1}(\psi, a_n) + p_a f_{\theta 1}(\psi, a_n) - p_a f_{\theta 1}(\psi, a_n) - p_a f_{\theta 1}(\psi, a)|$$
$$\leqslant |p_{a_n} - p_a| c_{10} + |p_a| c_{10} |a_n - a| \leqslant c_{11} |a_n - a|,$$

where $c_{11} := c_{10}(\hat{c}_{11} + c_{pq})$ with a bound $\hat{c}_{11}$ for the derivative of the map $\hat{a} \to p_{\hat{a}}$ on A_9 (see the end of Section 3). It follows that

$$d_n^* \leqslant c_{11} \frac{|q_{a_n} \Sigma(f_{\theta 1}(\psi, a_n), a_n)|}{|p_{a_n} f_{\theta 1}(\psi, a_n)|}$$
$$\leqslant c_{11} c_2 \delta_0 (\eta_0/c_2)^{\lambda_3/\mu_2} 2^{-\lambda_3/\mu_2} |p_{a_n} f_{\theta 1}(\psi, a_n)|^{-(\lambda_3/\mu_2)-1}$$

for n so large that $2|p_{a_n} f_{\theta 1}(\psi, a_n)| < \eta_0$, because of Proposition 6.2(v) and $0 < p_{a_n} f_{\theta 1}(\psi, a_n) \to p_a f_{\theta 1}(\psi, a) = 0$. Now $\mu_2 < -\lambda_3$ yields $d_n^* \to 0$ as $n \to \infty$, or

$$d_n \to \eta_0 \lim_{a' \underset{\neq}{\to} a} \frac{1}{a' - a} (\Phi_{a'} - \Phi_a). \tag{12.3}$$

By elementary arguments, we finally obtain (12.3) for all sequences of points $(\psi, a_n) \in \mathrm{dom}$ such that $a_n \underset{\neq}{\to} a$ as $n \to \infty$. ∎

13. Bifurcation. We want to solve the equation $f(\psi, a) - \psi = 0$ in a neighborhood of $(\eta_0 \Phi_{a_0}, a_0)$ in terms of a function $a \to \psi_a^*$. An implicit function theorem which is suitable for our purpose (we did not show that f is of class C^1) is the following

THEOREM 13.1. *Let a continuous map F: $W \to Z_3$ on an open set $W \in Z_1 \times Z_2$ be given, with Banach spaces Z_1, Z_2, Z_3. Suppose $D_1 F$: $W \to L_c(Z_1, Z_3)$ is continuous and $D_2 F(w)$ exists for all $w \in W$. If $F(z_1, z_2) = 0$ and if $D_1 F(z_1, z_2)$ is an isomorphism then there exist a differentiable map ψ: $W_2 \to Z_1$ on an open neighborhood W_2 of z_2 and an open neighborhood W_1 of z_1 with the following properties: $W_1 \times W_2 \subset W$, $\psi(W_2) \subset W_1$, $F(\psi(\cdot), \cdot) = 0$ on W_1, and $w_1 = \psi(w_2)$ if $(w_1, w_2) \in W_1 \times W_2$ and $F(w_1, w_2) = 0$.*

Proof. One can modify the proof of the implicit function theorem (10.2.1) in [7] appropriately. ∎

THEOREM 13.2. *Let a function $h \in \mathscr{H}$, $h = \bar{h}/a^+$, be given. Then there are a parameter $a_0 \in (0, a^+)$, an open interval $A = A_{24} \ni a_0$, a differentiable map ϕ^*: $A \ni a \to \phi_a^* \in C$ and a family of solutions y^a: $\boldsymbol{R} \to \boldsymbol{R}$, $a_0 \leqslant a < \sup A$, of equation*

$$\dot{x}(t) = ah(x(t-1)) \tag{ah}$$

with the following properties.

(i) *$\Phi_a^* = y_0^a$ for $a_0 \leqslant a < \sup A$.*

(ii) *$y^{a_0}(t) \to 0$ as $t \to -\infty$ and $y^{a_0}(t) \to \xi_h$ as $t \to +\infty$, i.e. y^{a_0} is a heteroclinic solution which connects the equilibrium $0 \in C$ to the equilibrium $\xi_h \in C$.*

(iii) *For $a_0 < a < \sup A$, there is no heteroclinic solution connecting the equilibrium $0 \in C$ to ξ_h.*

(iv) *For $a_0 < a < \sup A$, there exists $\pi_a > 0$ with $y^a(t + \pi_a) = y^a(t) + \xi_h$ for all $t \in \boldsymbol{R}$, i.e. y^a is a periodic solution of the second kind.*

(v) *Let* $a_0 < a < \sup A$.

(v.1) *For every open set* $V \supset \{y_t^a\colon 0 \leqslant t \leqslant \pi_a\}$ *there exists an open set* $W \supset \{y_t^a\colon 0 \leqslant t \leqslant \pi_a\}$ *such that* $\phi \in W + l\xi_h$, $l \in \boldsymbol{Z}$, *implies* $X(t, \phi, a) \in V + l\xi_h + N_0\xi_h$ *on* $[0, \infty)$;

(v.2) *there is an open set* $W_a \supset \{y_t^a\colon 0 \leqslant t \leqslant \pi_a\}$ *such that for each* $\phi \in W_a + \boldsymbol{Z}\xi_h$ *there exists* $\omega \in \boldsymbol{R}$ *with* $X(t, \phi, a) - y_{t+\omega}^a \to 0$ *as* $t \to +\infty$.

(vi) *There is an open set* $V_0 \supset \mathrm{cl}\{y_t^{a_0}\colon t \in \boldsymbol{R}\}$ *so that*

(vi.1) $\{y_t^a\colon t \in \boldsymbol{R}\} \in V_0 + \boldsymbol{Z}\xi_h$ *for* $a_0 < a < \sup A$;

(vi.2) $x = y^a(\cdot + t)$ *with some* $t \in \boldsymbol{R}$ *for every* $a \in A \cap (a_0, \infty)$ *and for every nonconstant solution* $x\colon \boldsymbol{R} \to \boldsymbol{R}$ *of equation* (ah) *which satisfies*

$$(V_0) \qquad \{x_t\colon t \in \boldsymbol{R}\} \subset V_0 + \boldsymbol{Z}\xi_h,$$

$$(\pi, k) \qquad x(\cdot + \pi) = x + k\xi_h \quad \textit{with some } \pi > 0,\ k \in \boldsymbol{Z};$$

(vi.3) *there is no nonconstant solution* $x\colon \boldsymbol{R} \to \boldsymbol{R}$ *of equation* $(a_0 h)$ *with* (V_0) *and* (π, k).

Proof of Theorem 13.2 (i)–(v), for a in an interval A_{13}.

a) Consider the map $F\colon \mathrm{dom} \ni (\psi, a) \to f(\psi, a) - \psi \in C$. We have $F(\eta_0 \Phi_{a_0}, a_0) = 0$, see (11.5), and $D_1 F(\eta_0 \Phi_{a_0}, a_0) = 0 - \mathrm{id}_C$, by Proposition 12.1. Theorem 13.1 guarantees the existence of an open neighborhood D^* of $\eta_0 \Phi_{a_0}$, of an open interval A_{11} and of a differentiable map $\psi^*\colon A_{11} \ni a \to \psi_a^* \in C$ such that

$$(13.1) \qquad \psi_{a_0}^* = \eta_0 \Phi_{a_0}, \quad \psi^*(A_{11}) \subset D^*, \quad D^* \times A_{11} \subset \mathrm{dom},$$

$$(13.2) \qquad f(\psi_a^*, a) = \psi_a^* \quad \text{on } A_{11},$$

$$(13.3) \qquad \psi = \psi_a^* \quad \text{for all } (\psi, a) \in D^* \times A_{11} \text{ with } f(\psi, a) = \psi.$$

Note that (13.2) implies

$$(13.4) \qquad \psi_a^* = \Sigma_2\big(f_{\theta 1}(\psi_a^*, a), a\big) \in H_a \cap D^3,$$

see Corollary 11.1 (ii). Set $\phi_a^* := G^-(\psi_a^*, a)$, for all $a \in A_{11}$. The map $\phi^*\colon A_{11} \ni a \to \phi_a^* \in C$ is differentiable. We have

$$\begin{aligned} \phi_{a_0}^* &= z_0 \quad \text{(by the definition of } z_t,\ t \geqslant 0, \text{ in Section 11)} \\ &= x_t^{a_0} \quad \text{for some } t \in \boldsymbol{R} \quad \text{(see Proposition 11.1 (i)).} \end{aligned}$$

$y^{a_0} := x^{a_0}(t + \cdot)$ is a heteroclinic solution of equation $(a_0 h)$, with $\lim_{s \to -\infty} y^{a_0}(s) = 0$, $\lim_{s \to +\infty} y^{a_0}(s) = \xi_h$, see the properties of x^{a_0} stated in Section 2. Note $\phi_{a_0}^* = y_0^{a_0}$.

b) Assume $a_0 < a \in A_{11}$. Proof of

$$(13.5) \qquad p_a f_{\theta 1}(\psi_a^*, a) > 0:$$

Suppose $p_a f_{\theta 1}(\psi_a^*, a) \leqslant 0$. Then (13.2) and the definitions of f and Σ_2 in Section 11 yield $\psi_a^* = f(\psi_a^*, a) = \Sigma_2\big(f_{\theta 1}(\psi_a^*, a), a\big) = \eta_0 \Phi_a$, and therefore

$0 \geqslant p_a f_{\theta 1}(\eta_0 \Phi_a, a) = p_a G(z_\theta^a - \bar{\xi}_h, a)$ which is a contradiction to Proposition 11.1 (iii).

Recall (13.2) and write $\psi_a^* = f(\psi_a^*, a) = \Sigma_2(\psi, a)$, with $\psi := f_{\theta 1}(\psi_a^*, a)$. By the definitions of $f_{\theta 1}$ and Σ_2, we have

$$\psi = G\big(X(\theta, G^-(\psi_a^*, a), a) - \bar{\xi}_h, a\big) = G\big(X(\theta, \phi_a^*, a) - \bar{\xi}_h, a\big),$$

$$\Sigma_2(\psi, a) = Y\big(\sigma(\psi, a), \psi, a\big) \quad (\text{with } \Sigma_2 = \Sigma \text{ on } E_2^+, \psi \in E_2^+ \text{ by } (13.5))$$
$$= G\big(X(\sigma(\psi, a), G^-(\psi, a), a), a\big).$$

Set $\sigma := \sigma(\psi, a)$. Because of (11.2) and Proposition 5.1 (iii),

$$G^-(\psi, a) = G^-\big(G\big(X(\theta, \phi_a^*, a) - \bar{\xi}_h, a\big), a\big) = X(\theta, \phi_a^*, a) - \bar{\xi}_h.$$

Therefore

$$\psi_a^* = \Sigma_2(\psi, a) = G\big(X\big(\sigma, X(\theta, \phi_a^*, a) - \bar{\xi}_h, a\big), a\big).$$

With the definitions of Ω and Y in Section 5 and with Proposition 5.1 (iii), we obtain

$$\begin{aligned}\phi_a^* &= G^-(\psi_a^*, a) = G^-\big(G\big(X\big(\sigma, X(\theta, \phi_a^*, a) - \bar{\xi}_h, a\big), a\big), a\big)\\ &= X\big(\sigma, X(\theta, \phi_a^*, a) - \bar{\xi}_h, a\big)\\ &= X\big(\sigma, X(\theta, \phi_a^*, a), a\big) - \bar{\xi}_h \quad (\text{by } (2.2))\\ &= X(\sigma + \theta, \phi_a^*, a) - \bar{\xi}_h.\end{aligned}$$

It is now easy to see that there is a solution y^a: $\boldsymbol{R} \to \boldsymbol{R}$ of equation (ah) such that $y_0^a = \phi_a^*$ and $y^a(\cdot + \pi_a) = y^a + \xi_h$ where $\pi_a = \sigma + \theta$.

c) We proved (i), (ii) and (iv), with A replaced by A_{11}. Assertion (iii) (with A_{11} instead of A) is contained in Corollary 4.1. The assertions of (v) (for an interval A_{13} instead of A) will follow from $D_1 f(\eta_0 \Phi_{a_0}, a_0) = 0$ which implies that ψ_a^* is an attractive fixed point of $f(\cdot, a)$ for a close to a_0.

d) Let $a \in A_{11}$. Recall $\phi_a^* = G^-(\psi_a^*, a)$ and $\psi_a^* \in D^3$ (see (13.4)). By Proposition 5.1, $\psi_a^* = G(\phi_a^*, a)$. Hence

$$\phi_a^* = G^-(\psi_a^*, a) = G^-\big(f(\psi_a^*, a), a\big) = G^-\big(f\big(G(\phi_a^*, a), a\big), a\big).$$

e) By continuity, we find an open ball $\tilde{B} \subset C$, centered at $\phi_{a_0}^*$, and an interval A_{12} such that the assignment $(\phi, a) \to \big(f\big(G(\phi, a), a\big), a\big)$ defines a map on $\tilde{B} \times A_{12}$ which sends $\tilde{B} \times A_{12}$ into a neighborhood of $(\psi_{a_0}^*, a_0) \in D^3 \times A_{11}$ on which $D_1 G^-$ is bounded by a constant c_{12}.

Next, choose A_{13} so small that for $a \in A_{13}$, $\phi_a^* \in \tilde{B}$ and $c_{12}|D_1 f(\psi_a^*, a)| \times (3/2) \leqslant 1/4 < 1$. This is possible since the maps ψ^*, ϕ^* and $D_1 f$ are continuous, with $D_1 f(\psi_{a_0}^*, a_0) = D_1 f(\eta_0 \Phi_{a_0}, a) = 0$ (see Proposition 12.1). The factor 3/2 is a bound for $D_1 G^-$, compare Proposition 5.1.

f) Let $a \in A_{13}$ with $a_0 < a$ be given. Consider the map f_a: $\tilde{B} \to C$ where $f_a(\phi) := G^-\big(f\big(G(\phi, a), a\big), a\big)$ for $\phi \in \tilde{B}$. Part d) says that ϕ_a^* is a fixed point of

f_a, and e) implies that there is a nonempty open ball $B' \subset \tilde{B}$, centered at ϕ_a^* (and depending on a), with

$$|Df_a(\phi)| < 1/2 \quad \text{on } B'. \tag{13.6}$$

We look for a smaller open neighborhood of ϕ_a^* on which f_a is given by the semiflow $X(\cdot, \cdot, a)$. Observe first that $\phi_a^* = G^-(\psi_a^*, a) \in G^-(D^3 \times A_{11}) \subset B_6$ (see Section 5) and $X(\theta, \phi_a^*, a) - \xi_h = X(\theta, G^-(\psi_a^*, a), a) - \xi_h \in B_6$, by (11.2). Therefore we obtain a nonempty open ball $B^* \subset B'$, centered at ϕ_a^*, with

$$B^* \subset B_6, \tag{13.7}$$

$$X(\theta, \phi, a) - \xi_h \in B_6 \quad \text{on } B^*. \tag{13.8}$$

Recall $\psi_a^* = G(\phi_a^*, a)$, $(f_{\theta 1}(\psi_a^*, a), a) \in E_2$ and $0 < p_a f_{\theta 1}(\psi_a^*, a)$ (see b)). In other terms, $\psi \in E^+(\delta_2, \eta_2, a)$ for $\psi := f_{\theta 1}(\psi_a^*, a)$, and $f(\psi, a)$ is given by the maps σ and Σ from Section 6.

Choose a convex open neighborhood $\tilde{D}$ of ψ, $\tilde{D} \subset E^+(\delta_2, \eta_2, a)$, such that the C^1-map $\sigma(\cdot, a)$: $E^+(\delta_2, \eta_2, a) \to [1, \infty)$ and its derivative are bounded on $\tilde{D}$ by a constant c_{13}. Finally, choose a nonempty open ball $\hat{B} \subset B^*$, centered at ϕ_a^*, so that

$$f_{\theta 1}(G(\phi, a), a) \in \tilde{D} \quad \text{on } \hat{B}, \tag{13.9}$$

and with the derivative of the map $\phi \to f_{\theta 1}(G(\phi, a), a)$ bounded on $\hat{B}$. We show

$$f_a(\phi) = X(\sigma + \theta, \phi, a) - \xi_h \quad \text{on } \hat{B}, \tag{13.10}$$

where $\sigma := \sigma(f_{\theta 1}(G(\phi, a), a), a) \in [1, c_{13}]$:

Let $\phi \in \hat{B}$. $\hat{B} \subset B_6$ implies $\phi = G^-(\chi, a)$ for $\chi := G(\phi, a)$, see Proposition 5.1. Hence

$$f_{\theta 1}(\chi, a) = G(X(\theta, G^-(\chi, a), a) - \xi_h, a) = G(X(\theta, \phi, a) - \xi_h, a). \tag{13.11}$$

With (13.9) and $\tilde{D} \subset E^+(\delta_2, \eta_2, a)$ and with the definition of Σ_2, we obtain

$$\Sigma_2(f_{\theta 1}(\chi, a), a) = \Sigma(f_{\theta 1}(\chi, a), a) = Y(\bar{\sigma}, \bar{\psi}, a)$$

where

$$\bar{\psi} := f_{\theta 1}(\chi, a) \in \tilde{D}, \quad \bar{\sigma} := \sigma(\bar{\psi}, a) = \sigma. \tag{13.12}$$

Hence

$$f(G(\phi, a), a) = \Sigma_2(f_{\theta 1}(\chi, a), a) = Y(\bar{\sigma}, \bar{\psi}, a),$$

$$\begin{aligned} f_a(\phi) &= G^-(f(G(\phi, a), a), a) = G^-(Y(\sigma, \bar{\psi}, a), a) \quad \text{(see (5.5))} \\ &= X(\sigma, X(\theta, \phi, a) - \xi_h, a) \quad \text{(with (13.12), (13.11); with (13.8) applied to } \phi\text{, and with Proposition 5.1)} \\ &= X(\sigma + \theta, \phi, a) - \xi_h \quad \text{(with (2.2))}. \end{aligned}$$

g) Let an open neighborhood V of $\{y_t^a\colon 0 \leqslant t \leqslant \pi_a\}$ be given, for the

parameter a from part f). We construct an open neighborhood W of $\{y_t^a\colon 0 \leqslant t \leqslant \pi_a\}$ such that for every $\phi \in W + l\xi_h^E$, $l \in \boldsymbol{Z}$, we have $X(t, \phi, a) \in V + l\xi_h^E + \boldsymbol{N}_0 \xi_h^E$ on $[0, \infty)$, and $X(t, \phi, a) - y_{t+\omega}^a \to 0$ as $t \to +\infty$ with some $\omega = \omega(\phi) \in \boldsymbol{R}$. This will prove (v.1) and (v.2), with A replaced by A_{13} from e).

Choose $\delta > 0$ such that

$$\text{(13.13)} \qquad \operatorname{dist}(\phi, \{y_t^a\colon 0 \leqslant t \leqslant \pi_a\}) < \delta \quad \text{implies } \phi \in V.$$

Recall Corollary 3.1: There exists a nonempty open ball $B_V \subset \hat{B}$, centered at $\phi_a^* = y_0^a$, with

$$\text{(13.14)} \qquad |X(t, \phi, a) - y_t^a| < \delta \quad \text{on } [0, \theta + c_{13}] \text{ for all } \phi \in B_V.$$

For each $s \in [0, \pi_a]$ there is a nonempty open ball B^s, centered at y_s^a, with

$$\text{(13.15)} \qquad |X(t, \phi, a) - y_{t+s}^a| < \delta \quad \text{for all } t \in [0, \pi_a - s],\ \phi \in B^s,$$

and with

$$\text{(13.16)} \qquad X(\pi_a - s, \phi, a) \in B_V \quad \text{for all } \phi \in B^s.$$

Let $\phi \in W + \boldsymbol{Z}\xi_h^E$ where $W := \bigcup_{s \in [0, \pi_a]} B^s$. Then

$$\text{(13.17)} \qquad \phi = \bar{\phi} + l\xi_h^E \quad \text{with } l \in \boldsymbol{Z},\ \bar{\phi} \in B^s \text{ for some } s \in [0, \pi_a].$$

Hence

$$\text{(13.18)} \qquad X(t, \bar{\phi}, a) \in V \quad \text{on } [0, \pi_a - s],$$

by (13.15) and (13.13), and

$$\text{(13.19)} \qquad \phi_0 := X(\pi_a - s, \bar{\phi}, a) \in B_V \subset \hat{B},$$

by (13.16). For $\phi_0 \in B_V$, the iterates $\phi_k := (f_a)^k(\phi_0)$, $k \in \boldsymbol{N}_0$, are defined since $f_a(\phi_a^*) = \phi_a^*$ and $|Df_a(\cdot)| < 1/2$ on $\hat{B}$, see (13.6). In particular,

$$\text{(13.20)} \qquad |\phi_k - \phi_a^*| \leqslant 2^{-k}|\phi_0 - \phi_a^*| \quad \text{and} \quad \phi_k \in B_V \quad \text{for all } k \in \boldsymbol{N}_0.$$

Set $\sigma_k := \sigma\big(f_{\theta 1}(G(\phi_{k-1}, a), a), a\big)$, for all $k \in \boldsymbol{N}$, and $t_k := \sum_{n=1}^k \sigma_n + \theta$, $t_0 := 0$. The choice of $\hat{B}$ and $\tilde{D}$ in part f) implies that there is a Lipschitz constant c_{14} with

$$\big|\sigma_n - \sigma\big(f_{\theta 1}(G(\phi_a^*, a), a), a\big)\big| \leqslant c_{14}|\phi_{n-1} - \phi_a^*| \quad \text{for all } n \in \boldsymbol{N}.$$

Therefore

$$\begin{aligned} |\sigma_n + \theta - \pi_a| &= \big|\sigma_n + \theta - \big(\sigma f_{\theta 1}(G(\phi_a^*, a), a), a\big) + \theta\big)\big| \\ &\leqslant c_{14} 2^{-n+1}|\phi_0 - \phi_a^*| \quad \text{for all } n \in \boldsymbol{N}, \end{aligned}$$

$$\text{(13.21)} \qquad \text{the sequence } (t_k - k\pi_a)_{k \in \boldsymbol{N}} \text{ converges to } t_\infty := \sum_{n=1}^{\infty} (\sigma_n + \theta - \pi_a).$$

In particular, $t_k \to +\infty$ as $k \to +\infty$. Note also

$$\text{(13.22)} \qquad 0 < t_{k+1} - t_k < \sigma_{k+1} + \theta \leqslant \theta + c_{13} \quad \text{for all } k \in \boldsymbol{N}_0.$$

Using (13.10) and (2.2), we find

$$X(t_k, \phi_0, a) = \phi_k + k\xi_h \quad \text{for all } k \in N_0. \tag{13.22}$$

Proof of $X(t, \phi, a) \in V + l\xi_h + N_0\xi_h$ for all $t \geqslant 0$: By (13.17) and (13.18), $X(t, \phi, a) = X(t, \bar{\phi}, a) + l\xi_h \in V + l\xi_h$ on $[0, \pi_a - s]$. For $t \geqslant \pi_a - s$, $X(t, \phi, a) = X(t, \bar{\phi}, a) + l\xi_h = X(t', \phi_0, a) + l\xi_h$ with $t' := t - (\pi_a - s) \geqslant 0$. Choose $k \in N_0$ with $t_k \leqslant t' < t_{k+1}$. Then

$$\begin{aligned}
|X(t, \phi, a) - l\xi_h - y^a_{t'-t_k+k\pi_a}| &= |X(t', \phi_0, a) - y^a_{t'-t_k+k\pi_a}| \\
&= |X(t'-t_k, X(t_k, \phi_0, a), a) - [y^a_{t'-t_k} + k\xi_h]| \\
&= |X(t'-t_k, \phi_k + k\xi_h, a) - [y^a_{t'-t_k} + k\xi_h]| \\
&= |X(t'-t_k, \phi_k, a) + k\xi_h - [y^a_{t'-t_k} + k\xi_h]| \\
&= |X(t'-t_k, \phi_k, a) - y^a_{t'-t_k}| < \delta,
\end{aligned}$$

because of $\phi_k \in B_V$, (13.21) and (13.14). We have $y^a_{t'-t_k+k\pi_a} = y^a_v + m\xi_h$ for some $v \in [0, \pi_a]$, $m \in N_0$. With (13.13), we obtain $X(t, \phi, a) - l\xi_h - m\xi_h \in V$.

Proof that there exists $\omega \in \boldsymbol{R}$ with $X(t, \phi, a) - y^a_{t+\omega} \to 0$ as $t \to \infty$: We have

$$\begin{aligned}
y^a_{t_k-t_\infty} - y^a_{k\pi_a} &= y^a_{t_k-t_\infty-k\pi_a} + k\xi_h - [\phi^*_a + k\xi_h] \\
&= y^a_{t_k-k\pi_a-t_\infty} - \phi^*_a \to 0 \quad \text{as } k \to +\infty,
\end{aligned}$$

because of (13.21). Using (13.23), we infer

$$\begin{aligned}
|X(t_k, \phi_0, a) - y^a_{t_k-t_\infty}| &\leqslant |X(t_k, \phi_0, a) - y^a_{k\pi_a}| + |y^a_{k\pi_a} - y^a_{t_k-t_\infty}| \\
&= |\phi_k + k\xi_h - [\phi^*_a + k\xi_h]| + |y^a_{k\pi_a} - y^a_{t_k-t_\infty}| \to 0 \quad \text{as } k \to +\infty.
\end{aligned}$$

For all $k \in N_0$ and $t' \in [t_k, t_{k+1}] \subset [t_k, t_k + \theta + c_{13}]$,

$$\begin{aligned}
|X(t', \phi_0, a) - y^a_{t_k-t_\infty}| &= |X(t'-t_k, X(t_k, \phi_0, a), a) - X(t'-t_k, y^a_{t_k-t_\infty}, a)| \\
&\leqslant (1 + \sup|h'|)^{\theta+c_{13}+1} |X(t_k, \phi_0, a) - y^a_{t_k-t_\infty}|,
\end{aligned}$$

see Corollary 3.1. It follows that

$$X(t, \phi_0, a) - y^a_{t-t_\infty} \to 0 \quad \text{as } t \to +\infty. \tag{13.24}$$

With $\omega := -(\pi_a - s) - t_\infty + l\pi_a$, we finally obtain, for $t \geqslant \pi_a - s \geqslant 0$:

$$\begin{aligned}
X(t, \phi, a) - y^a_{t+\omega} &= X(t, \bar{\phi}, a) + l\xi_h - [y^a_{t-(\pi_a-s)-t_\infty} + l\xi_h] \\
&= X(t-(\pi_a-s), \phi_0, a) - y^a_{t-(\pi_a-s)-t_\infty},
\end{aligned}$$

and the last term tends to zero as $t \to +\infty$, by (13.24).

14. Proof of Theorem 13.2 (vi.2) and (vi.3), for a parameter interval A_{19} instead of A.

h) There exist a nonempty open ball D^*_1, centered at $\eta_0 \Phi_{a_0} = \psi^*_{a_0}$, and

A_{14} with $D_1^* \subset D^*$,

(14.1) $\quad |f(\psi, a)-f(\psi', a)| \leqslant 4^{-1}|\psi-\psi'|$

for all ψ and ψ' in D_1^*, and all $a \in A_{14}$,

(14.2) $\quad \xi_0 < X(\sigma(f_{\theta 1}(\psi, a), a)+\theta, G^-(\psi, a), a) < \xi_h$

if $\psi \in D_1^*$, $a \in A_{14}$ and $p_a f_{\theta 1}(\psi, a) < 0$,

(14.3) $$f(D_1^* \times A_{14}) \subset D_1^*.$$

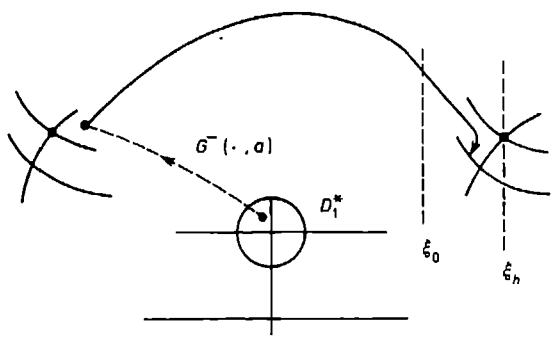

Fig. 19

Proof. Because of $D_1 f(\eta_0 \Phi_{a_0}, a_0) = 0$, there are a nonempty open ball $D_{1.1}^* \subset D^*$, centered at $\eta_0 \Phi_{a_0}$, and an open interval $A_{14.1} \subset A_{13}$ with $a_0 \in A_{14.1}$ such that

(14.4) $$|D_1 f(\psi, a)| < 1/4 \quad \text{on } D_{1.1}^* \times A_{14.1}.$$

Next, recall $G^-(-\eta_0 \Phi_{a_0}, a_0) = x_t^{a_0-}$ for some $t < 0$ (see Proposition 5.1 (v)). It follows that $-\delta_h < x_t^{a_0-} < 0$, by Proposition 4.2 (iv) and Proposition 4.1 (ii), and $\xi_0 - \xi_h < x_t^{a_0-} < 0$ (see Section 2). We obtain an open neighborhood $D_{1.2}^* \times A_{14.2}$ of $(-\eta_0 \Phi_{a_0}, a_0)$ on which

(14.5) $$\xi_0 - \xi_h < G^-(\psi, a) < 0.$$

Proposition 6.2 and the remark following it allow to choose η_3 in $(0, \eta_2)$ such that $a \in A_{14.2}$, $-\eta_3 < p_a \psi < 0$ and $|q_a \psi| < \delta_2$ altogether imply

$$|q_a Y(\sigma(\psi, a), \psi, a)| = |Y(\sigma(\psi, a), \psi, a)-(-\eta_0 \Phi_a)| < \varrho_{1.2}^*/2$$

where $\varrho_{1.2}^*/2$ is the radius of $D_{1.2}^*$. Choose an open interval $A_{14.3} \subset A_{14.2}$ with $a_0 \in A_{14.3}$ so small that $|(-\eta_0 \Phi_{a_0})-(-\eta_0 \Phi_a)| < \varrho_{1.2}^*/2$ on $A_{14.3}$. For each $a \in A_{14.3}$,

(14.6) $\quad Y(\sigma(\psi, a), \psi, a) \in D_{1.2}^* \quad$ if $-\eta_3 < p_a \psi < 0$, $|q_a \psi| < \delta_2$.

Recall $(f_{\theta 1}(\psi_{a_0}^*, a_0), a_0) \in E_2$ and $p_{a_0} f_{\theta 1}(\psi_{a_0}^*, a_0) = 0$. By continuity, there are a nonempty open ball $D_1^* \subset D_{1.2}^*$, centered at $\psi_{a_0}^*$, and an open interval

$A_{14.4} \subset A_{14.3}$ with $a_0 \in A_{14.4}$ such that on $D_1^* \times A_{14.4}$,

$$|p_a f_{\theta 1}(\psi, a)| = |p_a f_{\theta 1}(\psi, a) - p_{a_0} f_{\theta 1}(\psi_{a_0}^*, a_0)| < \eta_3. \tag{14.7}$$

Finally, choose $A_{14} \subset A_{14.4}$ so small that

$$|\psi_a^* - \psi_{a_0}^*| < 4^{-1} \varrho_1^* \quad \text{on } A_{14} \tag{14.8}$$

where ϱ_1^* denotes the radius of D_1^*.

Now (14.1) follows from (14.4). In order to derive (14.2), consider $(\psi, a) \in D_1^* \times A_{14} \subset D^* \times A_{11} \subset \operatorname{dom}$ with $p_a f_{\theta 1}(\psi, a) < 0$. Because of (14.7) and $p_a \chi = \langle \Psi_a, \chi \rangle_a \Phi_a$ for $\chi \in C$ and $|\Phi_a| = 1$, we obtain $-\eta_3 < p_a f_{\theta 1}(\psi, a) < 0$. By (14.6), $Y(\sigma(f_{\theta 1}(\psi, a), a), f_{\theta 1}(\psi, a), a) \in D_{1.2}^*$, and (14.5) gives

$$\xi_0 - \xi_h < G^-\big(Y(\sigma(f_{\theta 1}(\psi, a), a), f_{\theta 1}(\psi, a), a), a\big) < 0.$$

Therefore

$$\xi_0 - \xi_h < X\big(\sigma(f_{\theta 1}(\psi, a), a), G^-(f_{\theta 1}(\psi, a), a), a\big).$$

Recall the definition of $f_{\theta 1}$: We have $f_{\theta 1}(\psi, a) = G\big(X(\theta, G^+(\psi, a), a) - \xi_h, a\big)$ and $X(\theta, G^-(\psi, a), a) - \xi_h \in B_6$, see (11.2). By Proposition 5.1 (iii),

$$G^-(f_{\theta 1}(\psi, a), a) = X(\theta, G^-(\psi, a), a) - \xi_h.$$

It follows that

$$\begin{aligned} \xi_0 - \xi_h &< X\big(\sigma(f_{\theta 1}(\psi, a), a), X(\theta, G^-(\psi, a), a) - \xi_h, a\big) \\ &= X\big(\sigma(f_{\theta 1}(\psi, a), a), X(\theta, G^-(\psi, a), a), a\big) - \xi_h \\ &= X\big(\sigma(f_{\theta 1}(\psi, a), a) + \theta, G^-(\psi, a), a\big) - \xi_h, \end{aligned}$$

or

$$\xi_0 < X\big(\sigma(f_{\theta 1}(\psi, a), a) + \theta, G^-(\psi, a), a\big) < \xi_h.$$

Proof of (14.3): Let $(\psi, a) \in D_1^* \times A_{14}$ be given. Then

$$\begin{aligned} |f(\psi, a) - \eta_0 \Phi_{a_0}| &= |f(\psi, a) - \psi_{a_0}^*| \leqslant |f(\psi, a) - f(\psi_a^*, a)| + |\psi_a^* - \psi_{a_0}^*| \\ &\leqslant \tfrac{1}{4}|\psi - \psi_a^*| + \tfrac{1}{4}\varrho_1^* \quad \text{(by (14.1) and (14.8))} \\ &\leqslant \tfrac{1}{4}\{|\psi - \psi_{a_0}^*| + |\psi_{a_0}^* - \psi_a^*|\} + \tfrac{1}{4}\varrho_1^* \\ &\leqslant \tfrac{1}{4}\{\varrho_1^* + (\varrho_1^*/4)\} + (\varrho_1^*/4) \leqslant 3\varrho_1^*/4, \quad \text{or} \quad f(\psi, a) \in D_1^*. \end{aligned}$$

i) Recall $y_0^{a_0} = \phi_{a_0}^* = G^-(\psi_{a_0}^*, a_0) = G^-(\eta_0 \Phi_{a_0}, a_0) \in G^-(D^3 \times \{a_0\}) \subset B_6$, by (6.3). Therefore $G(y_0^{a_0}, a_0) = \psi_{a_0}^*$, and there exist a nonempty open ball $B_1^* \subset B_6$, centered at y_0^a, and an interval A_{15} such that

$$G(B_1^* \times A_{15}) \subset D_1^*. \tag{14.9}$$

j) The next aim is to construct an open neighborhood $V_{02} \subset C$ of $\operatorname{cl}\{y_t^{a_0}\colon t \in \boldsymbol{R}\}$ and an interval A_{19} such that for every solution $x\colon \boldsymbol{R} \to \boldsymbol{R}$ of equation (ah), $a \in A_{19}$, with $x(\cdot + \pi) = x + k\xi_h$ for some $\pi > 0$, $k \in \boldsymbol{N}$, and with

$\{x_t\colon t\in \boldsymbol{R}\} \subset V_{02}+\boldsymbol{Z}\xi_h$, we have $x_s+l\xi_h \in B_1^*$ for some $s\in \boldsymbol{R}$, $l\in \boldsymbol{Z}$. This will be done in parts k) and l).

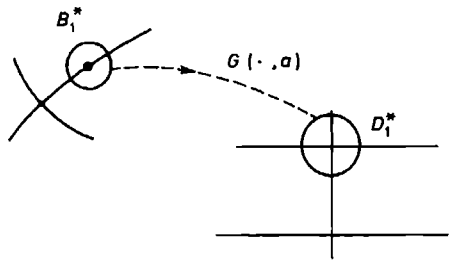

Fig. 20

k) Choose a nonempty open ball $D_2^* \subset D_1^*$, centered at $\eta_0\Phi_{a_0}$, and an interval A_{16} with

(14.10) $$G^-(D_2^* \times A_{16}^*) \subset B_1^*.$$

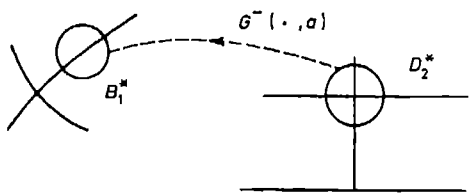

Fig. 21

Choose a ball D^4 and A_{17} with

(14.11) $$D^4 \times A_{17} \subset E_2$$

and such that

(14.12) $$(\psi, a)\in D^4\times A_{17} \quad \text{and} \quad \begin{Bmatrix} p_a\psi > 0 \\ p_a\psi < 0 \end{Bmatrix} \quad \text{implies}$$

$$\begin{cases} Y(\sigma(\psi, a), \psi, a)\in D_2^*, \\ \xi_0-\xi_h < G^-\big(Y(\sigma(\psi, a), \psi, a), a\big) < 0. \end{cases}$$

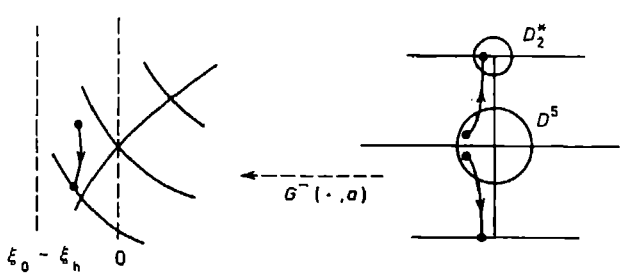

Fig. 22

This is possible because of Proposition 6.2 and the remark following it —, compare the construction in part h).

Choose B_7 and A_{18} with

$$G(B_7 \times A_{18}) \subset D^4. \tag{14.13}$$

Choose an interval $[t_-, t_+] \subset \boldsymbol{R}$ with

$$y_t^{a_0} \in B_7 \quad \text{on } (-\infty, t_-] \quad \text{and} \quad y_t^{a_0} \in B_7 + \bar{\xi}_h \quad \text{on } [t_+, \infty). \tag{14.14}$$

For every $t \in [t_-, t_+]$, there are a nonempty open ball $B^*_{2,t}$, centered at $y_t^{a_0}$, and an open interval $A_{19,t} \subset A_{18}$ with $a_0 \in A_{19,t}$ so that

$$X(t_+ - t, \phi, a) \in B_7 + \bar{\xi}_h \quad \text{on } B^*_{2,t} \times A_{19,t}. \tag{14.15}$$

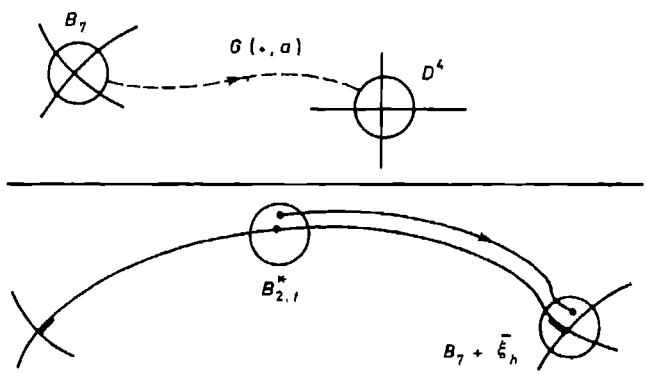

Fig. 23

Choose finitely many $t_1, \ldots, t_n \in [t_-, t_+]$ with

$$\{y_t^{a_0}\colon t_- \leqslant t \leqslant t_+\} \subset \bigcup_1^n B^*_{2,t_\nu}$$

and define

$$A_{19} := \bigcap_1^n A_{19,t_\nu}, \qquad V_{02} := B_7 \cup \Big(\bigcup_1^n B^*_{2,t_\nu}\Big) \cup (B_7 + \bar{\xi}_h).$$

V_{02} is an open neighborhood of the compact set $\operatorname{cl}\{y_t^{a_0}\colon t \in \boldsymbol{R}\} = \{0\} \cup \{y_t^{a_0}\colon t \in \boldsymbol{R}\} \cup \{\bar{\xi}_h\}$, see Proposition 2.3.

l) Let $x\colon \boldsymbol{R} \to \boldsymbol{R}$ be a solution of equation (ah), $a \in A_{19}$, with $x_t \in V_{02} + \boldsymbol{Z}\bar{\xi}_h$ for all $t \in \boldsymbol{R}$, and suppose $x(\cdot + \pi) = x + k\xi_h$ with $\pi > 0$ and $k \in \boldsymbol{N}$. Assume $x_t \in B^*_{2,t_\nu} + l\bar{\xi}_h$ with $\nu \in \{1, \ldots, n\}$ and $l \in \boldsymbol{Z}$. Then $x_t - l\bar{\xi}_h \in B^*_{2,t_\nu}$. The solution $\tilde{x}\colon \boldsymbol{R} \ni s \to x(s+t) - l\xi_h \in \boldsymbol{R}$ of equation (ah) satisfies $\tilde{x}_0 = x_t - l\bar{\xi}_h \in B^*_{2,t_\nu}$ so that (14.15) yields

$$x_{t_+ - t_\nu + t} - l\bar{\xi}_h = \tilde{x}_{t_+ - t_\nu} \in B_7 + \bar{\xi}_h,$$

and $\phi := x_{t_+ - t_\nu + t} - (l+1)\xi_h \in B_7$. By (14.13), $\psi := G(\phi, a) \in D^4$. Recall (14.12). Suppose $p_a\psi = 0$. Then, by definition of G, $\phi \in S_a$, and $X(s', \phi, a) \to 0$ as $s' \to +\infty$ (see (4.4')) which contradicts $x(\nu\pi) = x(0) + \nu k\xi_h \to +\infty$ as $N \ni \nu \to +\infty$. Suppose $p_a\psi < 0$. Then (14.12) gives $\xi_0 - \xi_h < G^-(Y(\sigma(\psi, a), \psi, a), a) < 0$, or $\xi_0 - \xi_h < X(\sigma(\psi, a), G^-(\psi, a), a) < 0$. Proposition 5.1 (iii), $\phi \in B_7 \subset B_6$, and $\psi = G(\phi, a)$ altogether give $\xi_0 - \xi_h < X(\sigma(\psi, a), \phi, a) < 0$. With (2.2) and Proposition 2.1 (iii), we obtain $X(s' + \sigma(\psi, a), \phi, a) < 0$ on $[0, \infty)$. This implies a contradiction to $x(\nu\pi) \to +\infty$ as $N \ni \nu \to +\infty$.

It follows that $p_a\psi > 0$. (14.12) gives $Y(\sigma(\psi, a), \psi, a) \in D_2^*$, therefore $X(\sigma(\psi, a), G^-(\psi, a), a) \in G^-(D_2^* \times A_{16}) \subset B_1^*$, by (14.10). With $\phi \in B_7 \subset B_6$, $X(\sigma(\psi, a), \phi, a) \in B_1^*$. Using (2.2) once more, we arrive at $x_{\sigma(\psi,a)+t_+ - t_\nu + t} - (l+1)\xi_h \in B_1^*$.

In order to complete the proof of the assertion in part j) we still have to consider the easier case $x_t \in B_7 + l\xi_h$ with $l \in \boldsymbol{Z}$. We set $\phi := x_t - l\xi_h \in B_7$ and proceed as above.

m) We define $V_0 := V_{01} \cap V_{02}$, with V_{01} from Proposition 2.3 (ii), and begin the proof of assertions (vi.2) and (vi.3), with A_{19} instead of A. Let x: $\boldsymbol{R} \to \boldsymbol{R}$ be a nonconstant solution of equation (ah), $a \in A_{19}$, with $x(\cdot + \pi) = x + k\xi_h$ for some $\pi > 0$, $k \in \boldsymbol{Z}$, and with $\{x_t\colon t \in \boldsymbol{R}\} \subset V_0 + \boldsymbol{Z}\xi_h$.

By Proposition 2.3 and 2.4, $k \in \boldsymbol{N}$. It follows that

$$(14.16) \qquad x(\nu\pi) \to +\infty \quad \text{as } \boldsymbol{Z} \ni \nu \to +\infty, \qquad x(\nu\pi) \to -\infty \quad \text{as } \boldsymbol{Z} \ni \nu \to -\infty.$$

Part j) gives $x_s + l\xi_h \in B_1^*$ for some $s \in \boldsymbol{R}$, $l \in \boldsymbol{Z}$. Consider the solution $\tilde{x}$: $\boldsymbol{R} \ni t \to x(t+s) \in \boldsymbol{R}$ of equation (ah). We have $\tilde{x}_0 \in B_1^*$, and

$$(14.17) \qquad \tilde{x}(\cdot + \pi) = x + k\xi_h,$$

$$(14.18) \qquad \tilde{x}(t) \to +\infty \quad \text{as } t \to +\infty, \qquad \tilde{x}(t) \to -\infty \quad \text{as } t \to -\infty,$$

because of (14.16) and $|x(t) - x(\nu\pi)| = \left|\int_{\nu\pi}^t \dot{x}\right| \leqslant \pi \cdot \max|h|$ on $[\nu\pi, (\nu+1)\pi]$ for all $\nu \in \boldsymbol{Z}$.

Define $\psi_0 := G(\tilde{x}_0, a) \in D_1^*$. Recall (14.9). The map f_{*a}: $D_1^* \ni \psi \to f(\psi, a) \in D_1^*$ is a strict contraction (see (14.1), (14.3)) with fixed point ψ_a^*, as follows from $D_1^* \times A_{14} \subset D^* \times A_{11}$ and (13.3). The sequence of points $\psi_\nu := (f_{*a})^\nu(\psi_0)$, $\nu \in \boldsymbol{N}_0$, converges to ψ_a^* as $\nu \to +\infty$.

n) If $p_a f_{\theta 1}(\psi_\nu, a) > 0$, $\nu \in \boldsymbol{N}_0$, then

$$G^-(\psi_{\nu+1}, a) = X(\sigma_{\nu+1} + \theta, G^-(\psi_\nu, a), a) - \xi_h \quad \text{where } \sigma_{\nu+1} := \sigma(f_{\theta 1}(\psi_\nu, a), a).$$

Proof. For ψ_ν as above,

$$\begin{aligned} \psi_{\nu+1} &= f_{*a}(\psi_\nu) = f(\psi_\nu, a) \\ &= \Sigma_2(f_{\theta 1}(\psi_\nu, a), a) = \Sigma(f_{\theta 1}(\psi_\nu, a), a) \\ &= Y(\sigma_{\nu+1}, f_{\theta 1}(\psi_\nu, a), a). \end{aligned}$$

With the definition of Y, $G^-(\psi_{\nu+1}, a) = X(\sigma_{\nu+1}, G^-(f_{\theta 1}(\psi_\nu, a), a), a)$. Recall $f_{\theta 1}(\psi_\nu, a) = G(X(\theta, G^-(\psi_\nu, a), a) - \xi_h, a)$ and $X(\theta, G^-(\psi_\nu, a), a) - \xi_h \in B_6$, by (11.2). Proposition 5.1 (iii) gives $G^-(f_{\theta 1}(\psi_\nu, a), a) = X(\theta, G^-(\psi_\nu, a), a) - \xi_h$, and the assertion follows.

o) For every $\nu \in N_0$, $p_a f_{\theta 1}(\psi_\nu, a) > 0$.

Proof. Suppose $p_a f_{\theta 1}(\psi_\nu, a) \leqslant 0$ for some $\nu \in N_0$. Let $\nu_0 \in N_0$ denote the smallest nonnegative integer with $p_a f_{\theta 1}(\psi_\nu, a) \leqslant 0$. In case $\nu_0 \geqslant 1$, part n) implies

$$G^-(\psi_{\nu_0}, a) = X(t_{\nu_0}, G^-(\psi_0, a), a) - \nu_0 \xi_h \quad \text{where } t_{\nu_0} := \sum_{\nu=0}^{\nu_0 - 1} (\sigma_{\nu+1} + \theta).$$

With $\tilde{x}_0 \in B_1^* \subset B_6$ and Proposition 5.1 (iii), we get

$$G^-(\psi_{\nu_0}, a) = X(t_{\nu_0}, \tilde{x}_0, a) - \nu_0 \xi_h. \tag{14.19}$$

In case $\nu_0 = 0$, set $t_{\nu_0} := 0$ — then (14.19) is satisfied, too. Suppose

$$0 = p_a f_{\theta 1}(\psi_{\nu_0}, a) = p_a G(X(\theta, G^-(\psi_{\nu_0}, a), a) - \xi_h, a).$$

With the definition of G, we obtain $X(\theta, G^-(\psi_{\nu_0}, a), a) - \xi_h \in S_a$, and

$$X(t, X(\theta, G^-(\psi_{\nu_0}, a), a) - \xi_h, a) \to 0 \quad \text{as } t \to +\infty,$$

see (4.4′). With (14.19) and (2.2), this results in

$$X(t + \theta + t_{\nu_0}, \tilde{x}_0, a) - (\nu_0 + 1)\xi_h \to 0 \quad \text{as } t \to +\infty$$

which contradicts (14.18).

If $0 > p_a f_{\theta 1}(\psi_{\nu_0}, a)$ then (14.2) gives

$$\xi_0 < X(\sigma(f_{\theta 1}(\psi_{\nu_0}, a), a) + \theta, G^-(\psi_{\nu_0}, a), a) < \xi_h.$$

(14.19) and (2.2) yield

$$\xi_0 < X(\sigma(f_{\theta 1}(\psi_{\nu_0}, a), a) + \theta + t_{\nu_0}, \tilde{x}_0, a) - \nu_0 \xi_h < \xi_h,$$

and Proposition 2.1 (iii) (and (2.2)) imply $X(t, \tilde{x}_0, a) - \nu_0 \xi_h < \xi_h$ for all $t \geqslant \sigma(f_{\theta 1}(\psi_{\nu_0}, a), a) + \theta + t_{\nu_0}$. This contradicts (14.18).

p) Proof of $\phi_a^* \in \{x_t\colon t \in \boldsymbol{R}\} + \boldsymbol{Z}\xi_h$:

With n) and o) and $\tilde{x}_0 \in B_1^* \subset B_6$, we obtain

$$G^-(\psi_\nu, a) = X(t_\nu, \tilde{x}_0, a) - \nu \xi_h \quad \text{for all } \nu \in N_0 \tag{14.20}$$

where $t_0 := 0$ and $t_\nu := \sum_0^{\nu-1} (\sigma_{\iota+1} + \theta)$ for all $\nu \in N$. By (14.17),

$$\tilde{x}_{t_{\nu k}} - \nu k \xi_h = \tilde{x}_{t_{\nu k} - \nu \pi} \quad \text{for all } \nu \in N_0.$$

With (14.20),

$$\tilde{x}_{t_{\nu k} - \nu \pi} = G^-(\psi_\nu, a) \to G^-(\psi_a^*, a) = \phi_a^* \quad \text{as } \nu \to +\infty.$$

Boundedness of the set $\{\tilde{x}_{t_{\nu k}-\nu\pi}\colon \nu\in N_0\}$ and (14.18) imply that the sequence $(t_{\nu k}-\nu\pi)_{\nu\in N_0}$ is bounded, too. We choose a convergent subsequence $(t_{\nu_l k-\nu_l\pi})_{l\in N}$ with limit, say, t. It follows that

$$\phi_a^* = \lim_{l\to\infty} \tilde{x}(t_{\nu_l k-\nu_l\pi}) = \tilde{x}_t - x_{t+s} + l\xi_h^F.$$

q) In case $a_0 < a\in A_{19}$, we infer from part p): $y_0^a = \phi_a^* = x_{t_x} + l\xi_h^F$ for some $t_x\in \boldsymbol{R}$, $l\in \boldsymbol{Z}$. Therefore $y^a_{-l\pi_a} = y_0^a - l\xi_h^F = x_{t_x}$, and $y^a(t-l\pi_a) = x(t+t_x)$ for all $t\geqslant -1$. In particular, $x_t\in\{y_t^a\colon t\in\boldsymbol{R}\}$ for all $t\geqslant t_x$. This implies

$$\{y_t^a\colon t\in\boldsymbol{R}\} \supset \{x_t\colon t\in\boldsymbol{R}\} \tag{14.21}$$

since for $t<t_x$ and $\nu\in N$ with $t+\nu\pi\geqslant t_x$, we have

$$x_t = x_{t+\nu\pi} - \nu k\xi_h^F = y_{\tilde{t}}^a - \nu k\xi_h^F = y^a_{\tilde{t}-\nu k\pi_a} \quad \text{for some } \tilde{t}\in\boldsymbol{R}.$$

We show

$$x = y^a(\cdot + t) \tag{14.22}$$

where $t\in\boldsymbol{R}$ satisfies $x_0 = y_t^a\colon t\in\boldsymbol{R}$ with $x_0 = y_t^a$ is uniquely determined since $y_{t^*}^a = x_0 = y_{t^{**}}^a$ with $t^*<t^{**}$ would imply $y^a(\cdot + (t^{**}-t^*)) = y^a$ on $[t^*-1, \infty)$ so that y^a would become $(t^{**}-t^*)$-periodic and bounded on $[t^*-1,\infty)$, in contradiction to $y^a(\nu\pi_a) = y^a(0)+\nu\xi_h \to +\infty$ for $\nu\to+\infty$. Let $v\in\boldsymbol{R}$. Choose $n\in N$ with $v-n<0$. By (14.21), $x_{v-n} = y_{\tilde{t}}^a$ for some $\tilde{t}\in\boldsymbol{R}$. It follows that $x_0 = y^a_{n-v+\tilde{t}}$. Therefore $n-v+\tilde{t} = t$, and $x_v = y^a_{n+\tilde{t}} = y^a_{v+t}$. In particular, $x(v) = y^a(v+t)$.

r) The case $a = a_0$. Part p) gives $y_0^{a_0} = \phi_{a_0}^* = x_{t_x} + l\xi_h^F$ for some $t_x\in\boldsymbol{R}$, $l\in\boldsymbol{Z}$. It follows that

$$x_{v+t_x} + l\xi_h^F = X(v, x_{t_x}+l\xi_h^F, a_0) = y_v^{a_0} \to \xi_h^F \quad \text{as } 0\leqslant v\to+\infty.$$

This contradicts $x(\nu\pi) = x(0)+\nu k\xi_h \to +\infty$ as $\nu\to+\infty$.

15. Proof of Theorem 13.2 (vi.1).

s) Proposition 5.1 (v) yields $G^-(\eta\Phi_{a_0}, a_0)\in\{x_t^{a_0}\colon t\in\boldsymbol{R}\} = \{y_t^{a_0}\colon t\in\boldsymbol{R}\}$ for all $\eta\in(0,\eta_0]$. Clearly $G^-(0,a_0) = 0\in \mathrm{cl}\{y_t^{a_0}\colon t\in\boldsymbol{R}\}$, and $(G^-)^{-1}(V_0)$ is an open

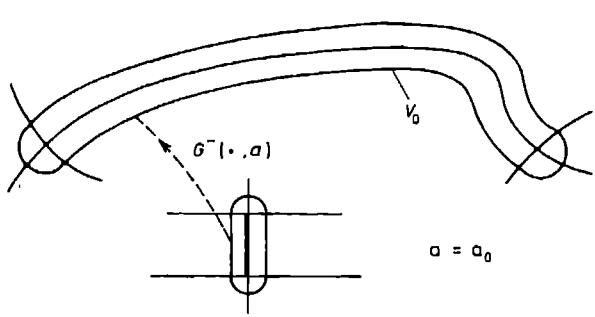

Fig. 24

neighborhood of the compact set $[0, \eta_0]\cdot\Phi_{a_0}\times\{a_0\}\subset C\times\boldsymbol{R}$. There exist $\varepsilon_1>0$ and A_{20} such that

(15.1) $\quad \operatorname{dist}(\psi, [0, \eta_0]\cdot\Phi_{a_0})<\varepsilon_1$ and $a\in A_{20}$ imply $G^-(\psi, a)\in V_0$.

Corollary 5.2 allows to choose D^5 and A_{21} with $D^5\times A_{21}\subset E_2$ and $|q_a Y(t, \psi, a)|<\varepsilon_1/2$ for all $(t, \psi, a)\in\Omega$ with $Y(s, \psi, a)\in D^3$ on $[0, t]$ and $(\psi, a)\in D^5\times A_{21}$. With Proposition 6.2, we obtain

(15.2) $\quad |q_a Y(t, \psi, a)|<\varepsilon_1/2$

on $[0, \sigma(\psi, a)]$ for all $(\psi, a)\in D^5\times A_{21}$ with $0<p_a\psi$.

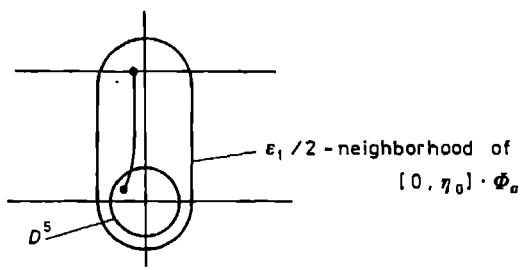

Fig. 25

Choose A_{22} with

(15.3) $\quad |\eta\Phi_a-\eta\Phi_{a_0}|<\varepsilon_1/2$ for all $\eta\in[0, \eta_0]$, $a\in A_{22}$.

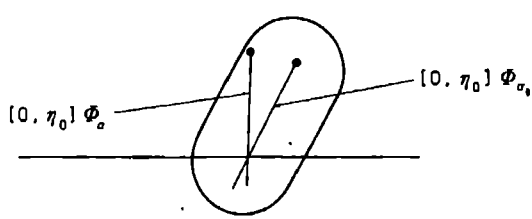

Fig. 26

Recall $\lim_{t\to\infty} y^{a_0}(t)=\xi_h$ and choose $s>\theta$ with

(15.4) $\quad G(y_s^{a_0}-\xi_h, a_0)\in D^5$ (cf. Fig. 27).

t) Recall from part b)

(15.5) $\quad 0<p_a f_{\theta 1}(\psi_a^*, a)=p_a G(y_\theta^a-\xi_h, a)$,

(15.6) $\quad \pi_a=\theta+\sigma\big(G(y_\theta^a-\xi_h, a), a\big)$,

in particular for $a_0 < a < \sup A_{22}$. By continuity,

$$p_a G(y_\theta^a - \bar{\xi}_h, a) \to p_{a_0} G(y_\theta^{a_0} - \bar{\xi}_h, a_0) = p_{a_0} G(z_\theta^{a_0} - \bar{\xi}_h, a_0) = 0$$

as $a \to a_0$, see part a) and Proposition 11.1 (ii).

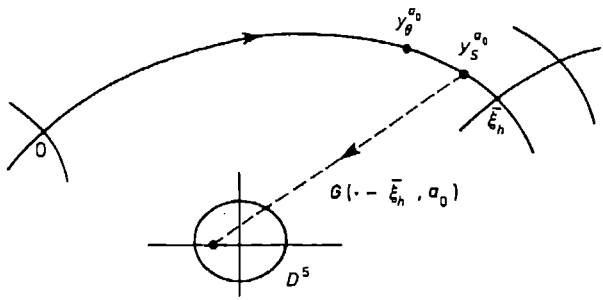

Fig. 27

Proposition 6.2 gives $\sigma(G(y_\theta^a - \bar{\xi}_h, a), a) \to +\infty$ as $a_0 < a \to a_0$, and we can choose A_{23} with

(15.7) $\quad s < \theta + \sigma(G(y_\theta^a - \bar{\xi}_h, a), a) \quad$ for all $a \in A_{23}$ with $a > a_0$.

Finally, take A_{24} so small that

(15.8) $\quad y_t^a \in V_0 \quad$ for all $t \in [0, s]$ and $a \in A_{24}$ with $a > a_0 \quad$ (cf. Fig. 28),

(15.9) $\quad G(y_s^a - \bar{\xi}_h, a) \in D^5 \quad$ for all $a \in A_{24}$ with $a > a_0 \quad$ (cf. Fig. 29).

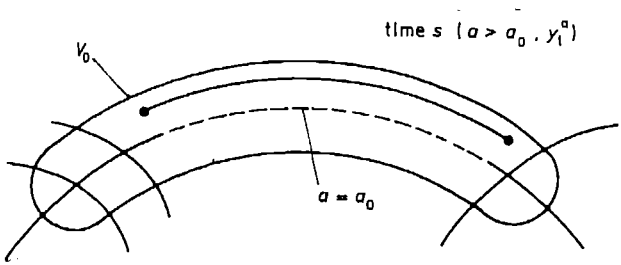

Fig. 28

u) Let $a \in A_{24}$ with $a > a_0$ be given. Set

$$\psi := G(y_s^a - \bar{\xi}_h, a) = G(X(s-\theta, y_\theta^a, a) - \bar{\xi}_h, a) = G(X(s-\theta, y_\theta^a - \bar{\xi}_h, a), a)$$
$$= G\big(X\big(s-\theta, G^-(G(y_\theta^a - \bar{\xi}_h, a), a), a\big)\big)$$

(with (11.2)). Hence

(15.10) $$\psi = Y(s-\theta, G(y_\theta^a - \bar{\xi}_h, a), a).$$

We prove $\sigma(G(y_\theta^a - \bar{\xi}_h, a), a) = s - \theta + \sigma(\psi, a)$:

Because of $\sigma(G(y_\theta^a-\xi_h, a), a) > s-\theta$, we have

$$0 < p_a Y(t, G(y_\theta^a-\xi_h, a), a) < \eta_0 \Phi_a \quad \text{on } [0, s-\theta],$$

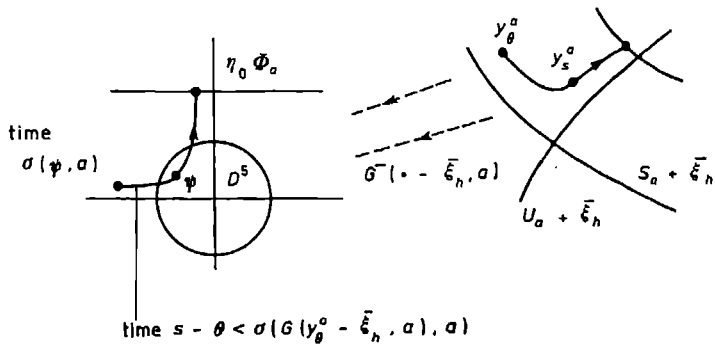

Fig. 29

see Proposition 6.2. For $t = s-\theta$, we get

$$0 < p_a \psi, \tag{15.11}$$

see (15.10). With (15.9), we obtain $(\psi, a) \in D^5 \times A_{22} \subset E_2$. We see that $\sigma(\psi, a)$ is defined, and $0 < p_a Y(t, \psi, a) < \eta_0 \Phi_a$ on $[0, \sigma(\psi, a))$, $p_a Y(t, \psi, a) = \eta_0 \Phi_a$ for $t = \sigma(\psi, a)$, by Proposition 6.2. It follows that

$$0 < p_a Y(t, G(y_\theta^a-\xi_h, a), a) < \eta_0 \Phi_a$$

on $[s-\theta, s-\theta+\sigma(\psi, a))$, and $p_a Y(t, G(y_\theta^a-\xi_h, a), a) = \eta_0 \Phi_a$ for $t = s-\theta+\sigma(\psi, a)$. The minimality property of σ in Proposition 6.2 gives $\sigma(G(y_\theta^a-\xi_h, a), a) = s-\theta+\sigma(\psi, a)$.

v) If $t \in [0, \sigma(\psi, a)]$ then $|Y(t, \psi, a)-\eta\Phi_{a_0}| < \varepsilon_1$ with some $\eta \in [0, \eta_0]$:

This follows from $(\psi, a) \in D^5 \times A_{24}$, $0 < p_a \psi$, $p_a Y(t, \psi, a) = \eta \Phi_a$ for some $\eta \in [0, \eta_0]$ (see Proposition 6.2, (15.2), (15.3)).

w) Let $t \in [s, \pi_a]$. Set $t' := t-s \in [0, \sigma(\psi, a)]$. Then

$$y_t^a-\xi_h = X(t', y_s^a, a)-\xi_h = X(t', y_s^a-\xi_h, a).$$

Using

$$\begin{aligned} y_s^a-\xi_h &= X(s-\theta, y_\theta^a, a)-\xi_h = X(s-\theta, y_\theta^a-\xi_h, a) \\ &= G^-(Y(s-\theta, G(y_\theta^a-\xi_h, a), a), a) \quad \text{(recall (11.2))} \\ &= G^-(\psi, a) \quad \text{(see (15.10))}, \end{aligned}$$

we obtain

$$y_t^a-\xi_h = X(t', y_s^a-\xi_h, a) = X(t', G^-(\psi, a), a) = G^-(Y(t', \psi, a), a).$$

Part v) and (15.1) give $y_t^a-\xi_h \in V_0$.

x) Part w) and (15.8) give $y_t^a \in V_0+Z\xi_h$ on $[0, \pi_a]$, for $a \in A_{24}$ with $a > a_0$. With assertion (iv) of Theorem 13.2, we finally obtain $y_t^a \in V_0+Z\xi_h$ for all $t \in \boldsymbol{R}$.

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