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Countable-dimensional spaces: a survey

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The terminology and notation of this survey follow the first author's *Dimension theory* book [E3], to which this text is a kind of supplement. There is, however, one important difference: in [E3] the covering dimension \dim is being considered only for normal spaces, while we consider it here in the larger class of completely regular spaces, where it is defined via functionally open covers (see [E2], p. 472). In the definitions of various kinds of countable dimensionality, we use the covering dimension (which we sometimes abbreviate: dimension); inductive dimensions ind and Ind appear here only occasionally. Let us also warn the reader that the problems are not formulated here in the greatest possible generality, it seems to us that it is reasonable to decide first how countable dimensionality behaves in hereditarily normal spaces and strong countable dimensionality – in normal spaces.

1. Definitions and characterizations

We start with the definition of countable-dimensional spaces; they will be called c. d. spaces in the sequel.

1.1. DEFINITION ([Hu], for separable metric spaces; cf. [U], p. 351). A completely regular space X is *countable-dimensional* if X can be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i < \infty$ for $i = 1, 2, \dots$

When X is a metrizable c.d. space, each of the finite-dimensional subspaces X_1, X_2, \dots can be decomposed into a finite union of subspaces having dimension not larger than 0 ([E3], Theorems 4.1.17 and 4.1.3); thus we have

1.2. PROPOSITION. *A metrizable space X is a c.d. space if and only if X can be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$*

Since in hereditarily normal spaces finite unions of finite-dimensional subspaces are finite-dimensional ([E3], Theorem 3.1.17), we have

1.3. PROPOSITION. *A hereditarily normal space X is a c.d. space if and only if X can be represented as the union of an increasing sequence $X_1 \subset X_2 \subset \dots$ of subspaces such that $\dim X_i \leq i$ for $i = 1, 2, \dots$*

Let us mention in connection with Propositions 1.2 and 1.3 the following theorem:

1.4. THEOREM ([Na2]). *A metrizable space X is a c.d. space if and only if X can be represented as the union of an increasing sequence $X_1 \subset X_2 \subset \dots$ of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$*

As observed in [dGN], this theorem follows from the universality of $K, (m)$ for c.d. metrizable spaces of weight $\leq m$ (cf. Section 6 and Remark 1.13(a)). It is also an easy consequence of (ii) in Theorem 1.16: if $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where \mathcal{B}_i 's are discrete, then we obtain the required representation by letting $X_i = X \setminus \bigcup_{j \geq i} \{Fr U : U \in \mathcal{B}_j\}$ for $i = 1, 2, \dots$ (to show that $\dim X_i \leq 0$ one applies Lemma 5.4.4 in [E2]).

Proposition 1.2 does not hold in general.

1.5. EXAMPLE ([Fe3]). *Under the assumption of the Continuum Hypothesis there exists a perfectly normal compact c.d. space X (more exactly, $\dim X = 2$) which cannot be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$*

In fact, in [Sa] and [Fe2], under the assumption of the Continuum Hypothesis, were given (rather complicated) examples of perfectly normal compact space X such that $\dim X = 2$, and yet X contains no closed subset of dimension 1 (the space X is, moreover, hereditarily separable).

Now, such a space X cannot be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$. Let us assume the contrary, i.e., that

$$X = \bigcup_{i=1}^{\infty} X_i, \quad \text{where} \quad \dim X_i \leq 0.$$

Since $\text{Ind } X \geq 2$ (see [E3], Theorem 3.1.28), and by virtue of the separation theorem for Ind (ibid., Theorem 2.1.4), there exists a closed subset F_1 of X such that $\dim F_1 \geq 1$ and $F_1 \cap X_1 = \emptyset$; by the special property of X we have $\dim F_1 = 2$. Applying induction, we obtain by a similar argument a decreasing sequence $F_1 \supset F_2 \supset \dots$ of non-empty (for that matter, two-dimensional) closed subsets of X such that $F_i \cap X_i = \emptyset$ for $i = 1, 2, \dots$

From the compactness of X it follows that $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$, and yet

$$\bigcap_{i=1}^{\infty} F_i = \left(\bigcap_{i=1}^{\infty} F_i \right) \cap \left(\bigcup_{i=1}^{\infty} X_i \right) \subset \bigcup_{i=1}^{\infty} (F_i \cap X_i) = \emptyset,$$

a contradiction.

Let us observe that in the above simple argument we applied the hereditary normality of X , rather than its perfect normality.

1.6. PROBLEM. Define, without any additional set-theoretic assumptions, a hereditarily normal (or better) c.d. space X which cannot be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$

One can also ask the following question, related to the particular situation in Example 1.5.

1.7. PROBLEM. Give an example of a hereditarily normal (or better) space X such that $\dim X = 1$ which cannot be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$

In the light of the facts established above, we introduce the class of zero-countable-dimensional spaces, called 0-c.d. spaces in the sequel, which is narrower than the class of c.d. spaces.

1.8. DEFINITION. A completely regular space X is *zero-countable-dimensional*, if X can be represented as the union of a sequence X_1, X_2, \dots of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$

From Proposition 1.2 it follows that in the realm of metrizable spaces, c.d. spaces and 0-c.d. spaces coincide; it seems that the class of c.d. spaces, rather than the class of 0-c.d. spaces, is the proper generalization to completely regular spaces. Example 1.5 shows that, under the assumption of the Continuum Hypothesis, there exist finite-dimensional perfectly normal compact spaces which are not 0-c.d. spaces. The reader should be warned that some authors use the term "countable-dimensional spaces" for what we named here 0-c.d. spaces.

1.9. PROBLEM. Give an example of a hereditarily normal (or better) 0-c.d. space which cannot be represented as the union of an increasing sequence $X_1 \subset X_2 \subset \dots$ of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$

Beside the classes of c.d. and 0-c.d. spaces one considers the class of strongly countable-dimensional spaces which shall be called s.c.d. spaces in the sequel (the reader should be warned that the spaces in this class are sometimes improperly called weakly countable-dimensional).

1.10. DEFINITION ([Hu], for separable metric spaces). A completely regular space X is *strongly countable-dimensional* if X can be represented as the union of a sequence X_1, X_2, \dots of closed subspaces such that $\dim X_i < \infty$ for $i = 1, 2, \dots$

From the countable sum theorem ([E3], Theorem 3.1.8) it follows that for s.c.d. spaces no counterpart of Proposition 1.2 holds. The countable sum theorem yields, however, the following simple

1.11. PROPOSITION. A normal space X is a s.c.d. space if and only if X can

be represented as the union of an increasing sequence $X_1 \subset X_2 \subset \dots$ of closed subspaces such that $\dim X_i \leq i$ for $i = 1, 2, \dots$

Clearly, every completely regular s.c.d. space is a c.d. space; the reverse implication does not hold.

1.12. EXAMPLE ([Sm3]; announcement [Hu], a simplified version [AP]). *There exists a compact metrizable c.d. space X which is not a s.c.d. space; that space is the union of countably many finite-dimensional cubes and a zero-dimensional set.*

In [AP] such a space is obtained by defining an appropriate upper semicontinuous decomposition of the Cantor set C ; this construction can be simplified further as follows: represent C as C^2 , arrange all vertical Cantor sets sticking out at rational points of the horizontal copy of C into a sequence, for $n = 1, 2, \dots$ decompose the n th term of the sequence into 2^{n-1} "portions" of diameter $\leq 1/n$ homeomorphic to the Cantor set and replace each "portion" by its decomposition determined by a continuous mapping of C onto I^n with fibers of cardinality $\leq n+1$; the decomposition of C^2 consisting of the decomposition of all portions and individual points on vertical Cantor sets sticking out at irrational points of the horizontal copy of C is the required upper semicontinuous decomposition – the quotient space X has the required properties.

Let us observe that the quotient mapping of C onto X has finite fibers and that the mapping of the space X onto C corresponding to the projection of C^2 onto the horizontal copy of C is open-and-closed and all its fibers are finite-dimensional.

Let us make a few additional remarks on decomposition of metrizable spaces into subspaces with \dim at most zero.

1.13. Remarks. (a) As shown in [Na2], every metrizable space X can be represented as the union of an increasing transfinite sequence $X_1 \subset X_2 \subset \dots \subset X_\alpha \subset \dots$, $\alpha < \omega_1$ of subspaces such that $\dim X_\alpha \leq 0$ for $\alpha < \omega_1$. The same result was proved earlier under the assumption of the Continuum Hypothesis in [A1] (cf. (c) below), where a proof under no additional assumptions, due to Smirnov, is announced. In the special case of separable spaces, Smirnov established the theorem earlier in [Sm2]. His proof, as observed in [dGN], can be generalized to yield the more general version (and this is of course the argument Smirnov had in mind): Since the Cartesian product $[J(m)]^{\aleph_0}$ of \aleph_0 copies of the hedgehog $J(m)$ is a universal space for the class of all metrizable spaces of weight $\leq m$ ([E2], Theorem 4.4.9), it suffices to establish the theorem for $X = [J(m)]^{\aleph_0}$. Now, the hedgehog $J(m)$ can be represented as the union of an increasing transfinite sequence $J_1 \subset J_2 \subset \dots \subset J_\alpha \subset \dots$, $\alpha < \omega_1$ of subspaces such that $\dim J_\alpha \leq 0$ for $\alpha < \omega_1$, (this can be achieved by amalgamating similar

representations of its spines); letting $X_\alpha = J_\alpha^{\aleph_0}$ we obtain the required representation of X .

(b) In [PoR1] an example is given of a metrizable separable space X such that $\dim X = 1$ which can be represented as the union of a transfinite sequence $X_1, X_2, \dots, X_\alpha, \dots, \alpha < \omega_1$ of closed subspaces such that $\dim X_\alpha \leq 0$ for $\alpha < \omega_1$. The statement that the closed unit interval I has such a decomposition is independent of the axioms of set theory (the interval I clearly has such a decomposition under the assumption of the Continuum Hypothesis, and does not have any under the assumption of the Martin Axiom (cf. [Ju], Theorem 5.2)).

(c) As shown in [A1], every metrizable space X can be represented as the union of a family $\{X_s\}_{s \in S}$ of closed subspaces such that $|S| \leq \mathfrak{c}$ and $\dim X_s \leq 0$ for $s \in S$. Again, an application of the universality of $[J(m)]^{\aleph_0}$ yields a simple proof: Let J_t — where $0 \leq t \leq 1$ be the set of all points in $J(m)$ which have distance t from the “origin” 0; the family of all Cartesian products $J_{t_1} \times J_{t_2} \times \dots$, where $0 \leq t_i \leq 1$ for $i = 1, 2, \dots$, is the required representation.

Let us also ask two questions connected with the fact that the number of “types” of compact metrizable c.d. spaces known to us is rather limited.

1.14. PROBLEM. *Is every homogeneous compact metrizable c.d. space necessarily finite-dimensional?*

1.15. PROBLEM. *Does every homogeneous compact metrizable space that contains the n -cube I^n for $n = 1, 2, \dots$ necessarily contain the Hilbert cube I^{\aleph_0} ?*

We conclude this section with some information on various characterisations of c.d. and s.c.d. spaces.

1.16. THEOREM. *For every metrizable space X the following conditions are equivalent:*

- (i) X is a c.d. space.
- (ii) X has a σ -discrete base \mathcal{B} such that the family $\{\text{Fr } U : U \in \mathcal{B}\}$ is point-finite.
- (iii) X is the image of a metrizable space X_0 such that $\dim X_0 \leq 0$ under a closed mapping with finite fibers.
- (iv) X is the image of a metrizable space X_0 such that $\dim X_0 \leq 0$ under a closed mapping with countable fibers.

The equivalence of (i)–(iii) in the above theorem was established in [Nag1]; the equivalence of (iv) to (i)–(iii) is a consequence of Corollary 7.2 below.

1.17. THEOREM ([Nag3] and [NR]). *A metrizable space X is a c.d. space if and only if for every sequence $(A_1, B_1), (A_2, B_2), \dots$ of pairs of disjoint closed subsets of X there exist closed sets L_1, L_2, \dots such that L_i is a partition between A_i and B_i and the family $\{L_i\}_{i=1}^\infty$ is point-finite.*

A related characterization was established earlier in [Nag1]. Further characterizations of metrizable c.d. spaces can be found in [Nag1] (in terms of sequences of locally finite closed coverings) and in [A1] (in terms of ranks of families of sets).

1.18. THEOREM ([A1]). *A normal space X is a s.c.d. space if and only if there exists an integer-valued function φ defined on X with the property that every finite open cover \mathcal{U} of the space X has an open finite refinement \mathcal{V} such that every point $x \in X$ belongs to at most $\varphi(x)$ members of \mathcal{V} .*

1.19. THEOREM. *For every metrizable space X the following conditions are equivalent :*

(i) X is a s.c.d. space.

(ii) X has a development $\mathcal{W}_1, \mathcal{W}_2, \dots$ such that \mathcal{W}_{i+1} is a star refinement of \mathcal{W}_i for $i = 1, 2, \dots$ and for every point $x \in X$ there exists an integer n with the property that x belongs to at most n members of \mathcal{W}_i for $i = 1, 2, \dots$

(iii) X has a strong development $\mathcal{W}_1, \mathcal{W}_2, \dots$ such that \mathcal{W}_{i+1} is a refinement of \mathcal{W}_i for $i = 1, 2, \dots$ and for every point $x \in X$ there exists an integer n with the property that x belongs to at most n members of \mathcal{W}_i for $i = 1, 2, \dots$

The equivalence of (i) and (ii) in the above theorem was established in [Nag1] (for metrizable separable spaces in [Sk2]); the equivalence of (i) and (iii) was established in [NR] (a related characterization of metrizable s.c.d. spaces can be found in [A1]). In [A1] also a characterization of metrizable s.c.d. spaces in terms of ranks is given.

2. Subspace theorems

The following proposition is a consequence of definitions and the monotonicity of \dim with respect to closed subspaces ([E3], Theorem 3.1.4).

2.1. PROPOSITION. *Every F_σ -subspace of a hereditarily normal c.d. (0-c.d.) space is a c.d. (0-c.d.) space. Every F_σ -subspace of a normal s.c.d. space is a s.c.d. space.*

From the monotonicity of \dim in strongly hereditarily normal spaces ([E3], Theorem 3.1.19) we obtain

2.2. PROPOSITION. *Every subspace of a strongly hereditarily normal c.d. (0-c.d., s.c.d.) space is a c.d. (0-c.d., s.c.d.) space.*

The monotonicity of \dim with respect to completely paracompact, in particular, strongly metrizable, subspaces (see [E3], Problems 3.2.G and 2.4.B(b) and [AP], Chapter VI, § 3) yields

2.3. PROPOSITION. *Every strongly metrizable subspace of a c.d. space is a c.d. space. Every completely paracompact subspace of a s.c.d. space is a s.c.d. space.*

Proposition 2.1 does not hold in completely regular spaces.

2.4. EXAMPLE (cf. [Sm1] or [E2], Problem 7.4.6). *There exists a completely regular space Z such that $\dim Z = 0$ which contains a closed subspace M which is not c.d.*

Let X be a normal space such that $\text{ind } X = 0$, and yet X is not c.d. (see Example 3.18) below. Consider a compactification cX of X such that $\text{ind } cX = 0$, the space $Z = (W \times cX) \setminus [\{\omega_1\} \times (cX \setminus X)]$ and its subspace $M = \{\omega_1\} \times X$, where W denotes the space of all ordinal numbers $\leq \omega_1$. Since $\beta Z = W \times cX$ (cf. [E2], Example 3.6.10 and Problem 3.12.20(d)), it follows (see *ibid.*, Corollary 7.1.18) that $\dim Z = 0$ and, clearly, M is not a c.d. space.

Let us note that a different construction (cf. [PoE2], Proposition), applied to the space X , leads to a similar example, where M is a functionally closed subspace of Z .

2.5. PROBLEM. *Does there exist a normal, or a compact, c.d. (0-c.d.) space that contains a closed subspace which is not c.d. (0-c.d.)?*

The following example shows that the word "strongly" cannot be omitted in the s.c.d. version of Proposition 2.2; Example 2.7 shows that, under the assumption of the Continuum Hypothesis, it cannot be omitted in c.d. and 0-c.d. versions, either.

2.6. EXAMPLE ([GP]). *There exists a hereditarily normal Lindelöf space X^* with $\dim X^* = 0$ that contains a subspace X which is not s.c.d.*

The space X^* is obtained by adding in an appropriate way one point to the space X defined in [GP], Example 3, and considered here in Example 3.16, below.

2.7. EXAMPLE. *Under the assumption of the Continuum Hypothesis there exists a hereditarily normal compact space X with $\dim X = 0$ that contains a subspace which is not c.d.*

Such a space can be obtained by taking the one-point compactification of the space defined in Example 3.17 below.

2.8. PROBLEM. *Define, without any additional set-theoretic assumptions a hereditarily normal c.d. (0-c.d.) space that contains a subspace which is not c.d. (0-c.d.).*

We shall establish now a simple generalization of the s.c.d. version of Proposition 2.1 to be applied in Section 3.

2.9. PROPOSITION. *If for an F_σ -subspace F of a normal space X there exists a s.c.d. subspace A of X that contains F , then F is s.c.d.*

Proof. One can assume without loss of generality that F is a closed subspace of X . Let $A = \bigcup_{i=1}^{\infty} A_i$, where A_i is closed in A and $\dim A_i < \infty$. The intersection $F \cap A_i$ is closed in A , and thus is closed both in F and in X . Since every continuous function $f: F \cap A_i \rightarrow I$ is continuously extendable over X , and thus over A_i , we have $\dim F \cap A_i \leq \dim A_i < \infty$ by virtue of Theorem 7.1.8 in [E2]. It follows that the space $F = \bigcup_{i=1}^{\infty} F \cap A_i$ is s.c.d.

The following problem is suggested by a well-known property of \dim (cf. [E3], Proposition 3.1.5).

2.10. PROBLEM. *Let X be a hereditarily normal space; is it true that if every open subspace of X is c.d. (0-c.d., s.c.d.), then every subspace of X is c.d. (0-c.d., s.c.d.)?*

Let us discuss now briefly the question of existence of finite-dimensional subspaces in infinite-dimensional c.d., 0-c.d. and s.c.d. spaces.

2.11. EXAMPLE. *Under the assumption of the Continuum Hypothesis for every integer $n \geq 1$ there exists a perfectly normal compact s.c.d. space X_n such that $\dim X_n = \infty$, and yet X_n contains no closed subspace M with $\dim M = n$.*

Indeed, in [Sa] and [Fe2], under the assumption of the Continuum Hypothesis, were given examples of perfectly normal compact spaces Y_n , where $n = 2, 3, \dots$, such that $\dim Y_n = n$ and Y_n contains no closed subset of positive dimension less than n . The one point compactification X_n of the sum $\bigoplus_{m>n} Y_m$ has, clearly, the required property.

The existence of compact (but not hereditarily normal) spaces X_n with similar properties follows – without any additional set-theoretic assumptions – from the existence of compact spaces without intermediate dimensions (see [Fe1]).

2.12. PROBLEM. *For every integer $n \geq 1$ define, without any additional set-theoretic assumptions, a hereditarily normal compact s.c.d. space X_n such that $\dim X_n = \infty$, and yet X_n contains no closed subspace M with $\dim M = n$.*

Obviously, every s.c.d. space X such that $\dim X = \infty$ contains closed subspaces of arbitrarily high dimension; we have, however.

2.13. PROBLEM. *Does there exist a hereditarily normal compact c.d. space X such that $\dim X = \infty$, and yet X contains no closed subspace of finite positive dimension?*

2.14. PROPOSITION. *Every hereditarily normal compact 0-c.d. space X such that $\dim X = \infty$ contains a closed subspace M with $\dim M = 1$.*

Proof. By the classical argument going down to [Hu] (cf. [E4], proof of Theorem 4.1) one shows that the space X has large transfinite dimension, and thus ([E4], Proposition 3.4) contains a closed subspace M with $\text{Ind } M$

= 1. The space M being compact, we have $\dim M = 1$ (see [E3], Theorems 3.1.28 and 3.1.30).

2.15. PROBLEM. *Does there exist a hereditarily normal compact 0-c.d. space X such that $\dim X = \infty$, and yet X contains no closed subspace of finite dimension > 1 ?*

2.16. THEOREM ([Hu], for compact spaces; [dGN], for separable spaces). *Every completely metrizable c.d. space such that $\dim X = \infty$ contains for $n = 1, 2, \dots$ a closed subspace M_n with $\dim M_n = n$.*

Proof. Let X be a complete space, $X = \bigcup_{i=1}^{\infty} X_i$, where $\dim X_i \leq 0$, and $\dim X = \infty$. Assume that for an integer $n_0 \geq 1$ the space X contains no closed subspace of finite dimension $\geq n_0$ (cf. [E3], Theorem 4.1.8). Since $\dim X = \infty$, there exists (cf. [E3], Theorem 4.1.3) a pair A, B of disjoint closed subsets of X such that every partition L between A and B satisfies $\dim L \geq n_0$. Consider a partition L_1 between A and B such that $L_1 \cap X_1 = \emptyset$ (see [E3], Theorem 4.1.13) and represent L_1 as the union of a locally finite family of closed sets of diameter less than 1. At least one of these sets, say F_1 , is infinite-dimensional, otherwise, by our assumption and the locally finite sum theorem ([E3], Theorem 4.1.10), we would have $\dim L_1 \leq n_0 - 1$.

Applying induction, we obtain by a similar argument a decreasing sequence $F_1 \supset F_2 \supset \dots$ of non-empty (for that matter, infinite-dimensional) closed subsets of X such that $F_i \cap X_i = \emptyset$ and $\delta(F_i) < 1/i$ for $i = 1, 2, \dots$, and this yields a contradiction.

2.17. PROBLEM. *Does Theorem 2.16 hold in arbitrary metrizable (metrizable separable) spaces?*

Obviously, every metrizable s.c.d. space X such that $\dim X = \infty$ contains closed subspaces of all finite dimensions.

3. Addition and sum theorems

The definitions yield the following two propositions.

3.1. PROPOSITION. *If a completely regular space X can be represented as a countable union of c.d. (0-c.d.) subspaces, then X is a c.d. (0-c.d) space.*

3.2. PROPOSITION. *If a completely regular space X can be represented as a countable union of closed s.c.d. subspaces, then X is a s.c.d. space.*

As shown by Example 1.12, in the last proposition the assumption that the subspaces are closed is essential, even in the case when X is represented as the union of two subspaces only. On the other hand, we clearly have

3.3. PROPOSITION. *If a perfectly normal space X can be represented as the union of two s.c.d. subspaces, one of them closed, then X is a s.c.d. space.*

The assumption of perfect normality in Proposition 3.3 cannot be relaxed to hereditary normality

3.4. EXAMPLE. *There exists a hereditarily paracompact space X which is not s.c.d., and yet can be represented as the union of a closed s.c.d. subspace and a discrete subspace.*

Let K_ω denote the subspace of the Hilbert cube I^{\aleph_0} consisting of all points in I^{\aleph_0} which have only finitely many non-zero coordinates. Clearly, K_ω is a s.c.d. space. Consider the space $X = (I^{\aleph_0})_{K_\omega}$ defined as the set I^{\aleph_0} with the new topology generated by the base consisting of all sets of the form $U \cup K$, where U is an open subset of the Hilbert cube and $K \subset I^{\aleph_0} \setminus K_\omega$. The space X is hereditarily paracompact (see [E2], Example 5.1.22) and it is the union of a s.c.d. closed subspace K_ω and the discrete subspace $X \setminus K_\omega$.

Suppose that $X = \bigcup_{i=1}^{\aleph_0} F_i$, where F_i is closed in X and $\dim X_i < \infty$ for $i = 1, 2, \dots$. Since K_ω contains no finite-dimensional open subspace, $F_i \cap K_\omega$ is a closed nowhere dense subset of K_ω for $i = 1, 2, \dots$. Take a point $x \in K_\omega \setminus F_1$; since every neighbourhood of x in X contains a neighbourhood of x in the Hilbert cube, there exists an open subset U_1 of the Hilbert cube such that $x \in U_1 \subset \text{cl}_{I^{\aleph_0}} U_1 \subset X$ and $U_1 \cap F_1 = \emptyset$. Applying induction, we define a decreasing sequence $U_1 \supset U_2 \supset \dots$ of non-empty open subsets of the Hilbert cube such that $\text{cl}_{I^{\aleph_0}} U_{i+1} \subset U_i$ and $U_i \cap F_i = \emptyset$ for $i = 1, 2, \dots$. This yields a contradiction.

Obviously, we have

3.5. PROPOSITION. *Let $\{X_s\}_{s \in S}$ be a family of completely regular spaces and let $X = \bigoplus_{s \in S} X_s$. The space X is c.d. (0-c.d., s.c.d) if and only if X_s is c.d. (0-c.d., s.c.d) for every $s \in S$.*

We pass now to locally finite and point-finite sum theorems.

3.6. THEOREM. *If a strongly hereditarily normal space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed c.d. (0-c.d) subspaces, then X is a c.d. (0-c.d) space.*

Proof. For $i = 1, 2, \dots$, denote by X_i the set of all points of the space X which belong to exactly i members of the cover $\{F_s\}_{s \in S}$ and by \mathcal{T}_i the family of all subsets of S that have exactly i elements. Then $X_i = \bigcup_{T \in \mathcal{T}_i} X_T$, where $X_T = X_i \cap \bigcap_{s \in T} F_s$. Since the family $\{X_i \cap \bigcap_{s \in T} F_s\}_{T \in \mathcal{T}_i}$ is locally finite in X_i and consists of pairwise disjoint closed subsets of X_i , we have $X_i = \bigoplus_{T \in \mathcal{T}_i} (X_i \cap \bigcap_{s \in T} F_s)$. Thus, by virtue of Propositions 2.2 and 3.5, the space X_i is c.d. (0-c.d.). To conclude the proof it suffices to apply Proposition 3.1.

3.7. THEOREM. *If a normal space X can be represented as the union of*

a locally finite family $\{F_s\}_{s \in S}$ of closed s.c.d. subspaces, then X is a s.c.d. space.

Proof. By virtue of Proposition 1.11 one can assume that for each s in S we have $F_s = \bigcup_{i=1}^{\infty} F_{s,i}$, where $F_{s,i}$ is closed in F_s , and thus in X , and $\dim F_{s,i} \leq i$ for $i = 1, 2, \dots$. The family $\{F_{s,i}\}_{s \in S}$ is locally finite, so that the subspace $X_i = \bigcup_{s \in S} F_{s,i}$ is closed in X , and thus normal, for $i = 1, 2, \dots$. From the locally finite sum theorem for \dim ([E3], Theorem 3.1.10) it follows that $\dim X_i \leq i$ for $i = 1, 2, \dots$, so that X is a s.c.d. space.

3.8. PROBLEM. Does Theorem 3.6 hold for hereditarily normal (or normal) spaces?

3.9. THEOREM. If a strongly hereditarily normal space X can be represented as the union of a point-finite family $\{U_s\}_{s \in S}$ of open c.d. (0-c.d.) subspaces, then X is a c.d. (0-c.d.) space.

Proof follows the lines of the proof of Theorem 3.6; to show that $X_i = \bigoplus_{T \in \mathcal{F}_i} (X_i \cap \bigcap_{s \in T} U_s)$ we use the fact that the family $\{X_i \cap \bigcap_{s \in T} U_s\}_{T \in \mathcal{F}_i}$ consists of pairwise disjoint open subsets of X_i .

The following counterpart of Theorem 3.9 for s.c.d. spaces was given in [A1] (where it was established in a less direct way).

3.10. THEOREM. If a normal space X can be represented as the union of a point-finite family $\{U_s\}_{s \in S}$ of open s.c.d. sets, then X is a s.c.d. space.

Proof. By virtue of Theorem 1.5.18 in [E2] and by Urysohn's Lemma, one can find an open cover $\{V_s\}_{s \in S}$ of X such that $V_s \subset U_s$ for every $s \in S$ and all V_s 's are F_σ -sets in X . From Proposition 2.9 it follows that V_s 's are s.c.d. normal spaces. Applying Proposition 1.11 to each of countably many closed subsets of X whose union is equal to V_s and taking appropriate finite unions of the sets thus obtained for each s in S we have $V_s = \bigcup_{i=1}^{\infty} F_{s,i}$, where $F_{s,1} \subset F_{s,2} \subset \dots \subset F_{s,i}$ are closed in X and $\dim F_{s,i} \leq i$ for $i = 1, 2, \dots$. The set $X_i = X \setminus \bigcup_{s \in S} (V_s \setminus F_{s,i})$ is closed for $i = 1, 2, \dots$ and $\bigcup_{i=1}^{\infty} X_i = X$.

Since $\dim X_i \cap F_{s,i} \leq \dim F_{s,i} \leq i$ and $X_i \cap F_{s,i} \subset X_i \cap V_s$ it follows from the point-finite sum theorem for \dim ([E2], Theorem 3.1.13), that $\dim X_i \leq i$ for $i = 1, 2, \dots$. Thus X is a s.c.d. space.

3.11. Remark. One can easily see that the proof of the last theorem holds under the weaker assumption that each of V_s 's is contained in a s.c.d. subspace of X .

3.12. **PROBLEM.** *Does Theorem 3.9 hold for hereditarily normal (or normal) spaces?*

From Theorem 3.9 we obtain directly

3.13. **THEOREM.** *If a strongly hereditarily normal weakly paracompact space X can be represented as the union of a family of open c.d. (0-c.d.) subspaces, then X is a c.d. (0-c.d.) space.*

The question if Theorem 3.13 holds for hereditarily normal (or normal) spaces is a variant of Problem 3.12.

Theorem 3.10 and Remark 3.11 yield

3.14. **THEOREM.** *If a normal weakly paracompact space X can be represented as the union of a family of open s.c.d. subspaces, then X is a s.c.d. space.*

The assumption of weak paracompactness in the last two theorems and the assumption of point-finiteness in Theorems 3.9 and 3.10 are essential.

3.15. **EXAMPLE ([GP]).** *There exists a perfectly normal space X such that each point of X has a neighbourhood which is c.d. (0-c.d.) and yet X is not c.d. (0-c.d.).*

3.16. **EXAMPLE ([GP]).** *There exists a perfectly normal, locally metrizable separable space X such that each point of X has a neighbourhood of dimension 0 and yet X is not s.c.d. (it is, however, a 0-c.d. space).*

3.17. **EXAMPLE (cf. [GP], Remark 2).** *Under the assumption of the Continuum Hypothesis there exists a perfectly normal locally compact and locally countable space that is not c.d.*

The space (I^{\aleph_0}, τ) obtained by Kunen's method (see [JKR], § 1) from the Hilbert cube with the natural topology (I^{\aleph_0}, ϱ) , has all the required properties. Indeed, as established in [JKR], this space is locally compact, locally countable and perfectly normal. As observed in [PoE1] and [Fe2], the dimension of the space (X, τ) obtained by Kunen's method strictly depends on the dimension of (X, ϱ) ; in particular, if (X, ϱ) is not c.d., then also (X, τ) is not c.d. Since the Hilbert cube is not c.d. (see [E3], Theorem 1.8.20), our space is not c.d., either.

3.18. **EXAMPLE.** *There exist a normal space X such that each point of X has a neighbourhood of dimension 0 and yet X is not c.d.*

As a matter of fact, the well-known modification of Dowker's example (see [E3], Example 2.2.11) due to Smirnov (see [Sm2] or [E2], Problem 7.4.7) has the required properties. We recall the construction of the space X defined by Smirnov.

Let W denote the space of all ordinals $\leq \omega_1$ and $W_0 = W \setminus \{\omega_1\}$ be the

subspace of W . Represent the Hilbert cube as the union $I^{\aleph_0} = \bigcup_{\alpha < \omega_1} S_\alpha$, where $\dim S_\alpha = 0$ and $S_\alpha \subset S_\beta$ for $\alpha < \beta < \omega_1$. Let

$$X = \bigcup_{\alpha < \omega_1} (\{\alpha\} \times S_\alpha)$$

be the subspace of the Cartesian product $W \times I^{\aleph_0}$. It is known that X is normal and each point of X has a neighbourhood of dimension 0. We shall show that X is not c.d.

We start with two auxiliary results. In the first one, the notion of a stationary subset of W appears; let us recall that a subset S of the space W is stationary if it intersects each closed cofinal subset of W .

(*) Let Y be a subspace of X and $A = \{t \in I^{\aleph_0} : \text{the set } \{\alpha < \omega_1 : (\alpha, t) \in Y\} \text{ is stationary}\}$. If for a sequence K_1, K_2, \dots of closed subsets of Y we have $\bigcap_{i=1}^{\infty} \text{cl}_Y K_i \neq \emptyset$, where cl_Y denotes the closure in the subspace $Y^* = Y \cup (\{\omega_1\} \times A)$ of $W \times I^{\aleph_0}$, then

$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset.$$

To prove (*), suppose that $(\omega_1, a) \in \bigcap_{i=1}^{\infty} \text{cl}_Y K_i$ for some $a \in A$. Then the set $L = \{\beta < \omega_1 : (\beta, a) \in \bigcap_{i=1}^{\infty} \text{cl}_X K_i\}$ is a cofinal subset of W . Indeed, consider an arbitrary $\gamma < \omega_1$ and arrange the sets K_i into a sequence K'_1, K'_2, \dots such that each K_i appears in it infinitely many times. Choose, inductively, a sequence $\alpha_1, \alpha_2, \dots$ of countable ordinals greater than γ and a sequence x_1, x_2, \dots of points of I^{\aleph_0} such that $\alpha_n < \alpha_{n+1}, (\alpha_n, x_n) \in K'_n$ and $\rho(x_n, a) < 1/n$, where ρ denotes a fixed metric on I^{\aleph_0} . Let $\alpha_0 = \lim \alpha_n$; then $(\alpha_0, a) \in \bigcap_{i=1}^{\infty} \text{cl}_X K_i$, and hence $\alpha_0 \in L \cap [\gamma, \omega_1)$. The set $S = \{\alpha < \omega_1 : (\alpha, a) \in Y\}$, being stationary, intersects the cofinal closed subset L of W ; thus there exists an $\alpha \in L \cap S$ and we have $(\alpha, a) \in Y \cap \bigcap_{i=1}^{\infty} \text{cl}_X K_i = \bigcap_{i=1}^{\infty} K_i$.

(**) If the Hilbert cube I^{\aleph_0} is represented as the union $\bigcup_{i=1}^{\infty} A_i$ of subspaces, then there exists an integer i and a sequence $(E_1, F_1), (E_2, F_2), \dots$ of pairs of disjoint closed subsets of I^{\aleph_0} with the property that if L_j is a partition between $A_i \cap E_j$ and $A_i \cap F_j$ in the space A_i for $j = 1, 2, \dots$, then

$$\bigcap_{j=1}^{\infty} L_j \neq \emptyset.$$



The proof of (**) follows an idea of Levšenko's (see [Lev], Theorem 4 or [AP], Ch. 10, § 5, Theorem 21). Let us arrange the sequence of all pairs $(\{\{x_i\} \in I^{\aleph_0} : x_n = 0\}, \{\{x_i\} \in I^{\aleph_0} : x_n = 1\})$, where $n = 1, 2, \dots$ into a double sequence $(A_{i,j}, B_{i,j})$, where $i, j = 1, 2, \dots$. It is well known (cf. [E3], proof of Theorem 1.8.20) that if $L_{i,j}$ is a partition between $A_{i,j}$ and $B_{i,j}$ in I^{\aleph_0} for $i, j = 1, 2, \dots$, then $\bigcap_{i,j=1}^{\infty} L_{i,j} \neq \emptyset$. Let $U_{i,j}, V_{i,j}$, where $i, j = 1, 2, \dots$, be open subspaces of I^{\aleph_0} such that $A_{i,j} \subset U_{i,j}, B_{i,j} \subset V_{i,j}$ and $\text{cl}_{I^{\aleph_0}} U_{i,j} \cap \text{cl}_{I^{\aleph_0}} V_{i,j} = \emptyset$. Suppose that for each i and $j = 1, 2, \dots$ there exists a partition $L_{i,j}$ in the space A_i between the sets $A_i \cap (\text{cl}_{I^{\aleph_0}} U_{i,j})$ and $A_i \cap (\text{cl}_{I^{\aleph_0}} V_{i,j})$ such that $\bigcap_{j=1}^{\infty} L_{i,j} = \emptyset$; there exists then a partition $L_{i,j}$ between $A_{i,j}$ and $B_{i,j}$ in I^{\aleph_0} such that $A_i \cap L_{i,j} \subset L_{i,j}$ (see [E3], Lemma 1.2.9). We have now $A_i \cap \bigcap_{j=1}^{\infty} L_{i,j} \subset \bigcap_{j=1}^{\infty} L_{i,j} = \emptyset$ for $i = 1, 2, \dots$, so that $\bigcap_{i,j=1}^{\infty} L_{i,j} = \emptyset$, a contradiction. Thus, there exists an integer i which satisfies (**) with $E_j = \text{cl}_{I^{\aleph_0}} U_{i,j}$ and $F_j = \text{cl}_{I^{\aleph_0}} V_{i,j}$ for $j = 1, 2, \dots$

We pass now to the proof that our space X is not c.d. Suppose that $X = \bigcup_{i=1}^{\infty} X_i$; we shall show that there exists an i such that $\dim X_i = \infty$. Let

$$A_i = \{t \in I^{\aleph_0} : \text{the set } \{\alpha : (\alpha, t) \in X_i\} \text{ is stationary}\}$$

for $i = 1, 2, \dots$. One easily checks that the union of countably many nonstationary sets is a nonstationary set (cf. [F1], Corollary 2.5), so that $I^{\aleph_0} = \bigcup_{i=1}^{\infty} A_i$. Let i and the sequence $(E_1, F_1), (E_2, F_2), \dots$ satisfy (**). Consider open subsets U_j, V_j of I^{\aleph_0} such that $E_j \subset U_j, F_j \subset V_j$ and $\text{cl}_{I^{\aleph_0}} U_j \cap \text{cl}_{I^{\aleph_0}} V_j = \emptyset$. Then the sets

$$E_j = X_i \cap (W_0 \times \text{cl}_{I^{\aleph_0}} U_j) \quad \text{and} \quad F_j = X_i \cap (W_0 \times \text{cl}_{I^{\aleph_0}} V_j)$$

are functionally closed disjoint subsets of the space X_i for $j = 1, 2, \dots$. We shall show that if L_j is a functionally closed partition between the sets E_j and F_j in the space X_i for $j = 1, 2, \dots$, then $\bigcap_{j=1}^{\infty} L_j \neq \emptyset$. Let K_j and M_j be functionally closed subsets of X_i such that

$$X_i = K_j \cup M_j, \quad E_j \cap M_j = F_j \cap K_j = \emptyset \quad \text{and} \quad M_j \cap K_j = L_j.$$

One easily checks that the set

$$L_j = \{t \in A_i : (\omega_1, t) \in \text{cl}_{X_i^*} K_j \cap \text{cl}_{X_i^*} M_j\},$$

where $X_i^* = X_i \cup (\{\omega_1\} \times A_i)$, is a partition between the sets $A_i \cap E_j$ and

$A_i \cap F_j$ in the space A_i ; hence $\bigcap_{j=1}^{\gamma} L_j \neq \emptyset$ and thus

$$\bigcap_{j=1}^{\infty} (\text{cl}_{X_i} K_j \cap \text{cl}_{X_i} M_j) \neq \emptyset.$$

Thus by (*) we have

$$\bigcap_{j=1}^{\infty} (K_j \cap M_j) = \bigcap_{j=1}^{\infty} L_j \neq \emptyset.$$

This proves that $\dim X_i = \infty$ (see [E2], Theorem 7.2.15).

3.19. PROBLEM. Define without any set-theoretic-assumptions a perfectly normal or a hereditarily normal space X such that each point of X has a neighbourhood of dimension 0 and yet X is not c.d. (or 0-c.d.).

The following theorem is a slight modification of the c.d. version of Theorem 3.13.

3.20. THEOREM. If a hereditarily normal paracompact space X can be represented as the union of a family of open c.d. subspaces, then X is a c.d. space.

Proof. Let $\{U_s\}_{s \in S}$ be an open cover of X such that U_s is a c.d. space for every $s \in S$. By virtue of the paracompactness of X there exists a locally finite closed cover $\{A_s\}_{s \in S}$ of the space X such that $A_s \subset U_s$ for every $s \in S$; it follows from Proposition 2.1 that all A_s 's are c.d. Thus, for every $s \in S$ we have $A_s = \bigcup_{i=1}^{\gamma} A_{s,i}$, where $\dim A_{s,i} < \infty$ for $i = 1, 2, \dots$. Consider an open cover $\{V_t\}_{t \in T}$ of the space X such that every V_t meets only finitely many A_s 's and a locally finite closed cover $\{F_t\}_{t \in T}$ such that $F_t \subset V_t$ for every $t \in T$. For $i = 1, 2, \dots$ the family $\{F_t \cap X_i\}_{t \in T}$, where $X_i = \bigcup_{s \in S} A_{s,i}$, is – by the addition theorem for \dim ([E2], Theorem 3.1.17) – a locally finite closed cover of the space X_i by finite-dimensional, and thus s.c.d. subspace, so that by virtue of Theorem 3.7 the space X_i is s.c.d. It follows that X is a c.d. space.

3.21. PROBLEM. Does Theorem 3.20 hold for 0-c.d. spaces?

Proposition 3.1 and Theorem 3.13 yield

3.22. PROPOSITION. If a strongly hereditarily normal weakly paracompact space X can be represented as the union of a locally countable family of c.d. (0-c.d.) subspaces, then X is a c.d. (0-c.d.) space.

From Proposition 3.2 and Theorem 3.14 easily follows

3.23. PROPOSITION. If a normal weakly paracompact space X can be represented as the union of a locally countable family of s.c.d. closed subspaces, then X is a s.c.d. space.

3.24. PROBLEM. Does Proposition 3.22 hold for hereditarily normal (normal) spaces?

The following proposition, a slight modification of the c.d. version of Proposition 3.22, follows from Proposition 3.1 and Theorem 3.20.

3.25. PROPOSITION. *If a hereditarily normal paracompact space X can be represented as the union of a locally countable family of c.d. subspaces, then X is a c.d. space.*

3.26. PROBLEM. *Does Proposition 3.25 hold for 0-c.d. spaces?*

Let us observe that, as shown by Example 3.15, the assumption of weak paracompactness in Propositions 3.22 and 3.23 cannot be omitted.

3.27. Remark. One can easily show that in Theorems 3.6 and 3.7 instead of locally finite families one can consider σ -locally finite families. Similarly, in Theorem 3.9 instead of point-finite families one can consider σ -point-finite families. On the other hand, in Theorem 3.10 we are able to replace point-finiteness by σ -point finiteness only under the additional assumption that X is countably paracompact. Thus we have

3.28. PROBLEM *Does Theorem 3.10 hold for σ -point-finite families (or countable families)?*

In the case of metrizable spaces we obtain, by the argument used in the finite-dimensional case (see [E2], Theorem 4.1.12), the following sum theorem.

3.29. THEOREM. *If a metrizable space X can be represented as the union of the transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of c.d. (s.c.d.) subspaces such that for every $\alpha < \xi$ the union $\bigcup_{\beta < \alpha} K_\beta$ is closed, then X is a c.d. (s.c.d.) space.*

The last theorem, as well as Theorem 3.7, Theorem 3.14 and Proposition 3.23 in the case of metrizable spaces, were established in [We].

Theorems 3.13, 3.14 and 3.20 can be interpreted as theorems stating conditions under which a locally c.d. (0-c.d., s.c.d.) space is c.d. (0-c.d., s.c.d.). In the case of finite-dimensional spaces there is a well-known method, due to Dowker (see [Na4], Theorem 11-17 and Remark 11-18), of obtaining hereditarily normal spaces in which the subspace theorem does not hold from hereditarily normal spaces with loc dim larger than dim , and vice versa. We conclude this section with a related problem (cf. Problem 2.9).

3.30. PROBLEM. *Is there any Dowker-type relation between locally c.d. (0-c.d., s.c.d.) spaces which are not c.d. (0-c.d., s.c.d.) and c.d. (0-c.d., s.c.d.) spaces which contain open subspaces that are not c.d. (0-c.d., s.c.d.)?*

4. Cartesian product theorems

Let us start with a counterexample which solves (under the assumption of the Continuum Hypothesis) a problem attributed to Fedorčuk (see [Prz2], Problem 20; cf. [Fe3], Problem 2).

4.1. EXAMPLE. *Under the assumption of the Continuum Hypothesis there exist a space X such that $\dim X = 0$, the Cartesian product $X \times X$ is perfectly normal, and yet $X \times X$ is not a c.d. space.*

Moreover, *the space X is locally compact and locally countable and the Cartesian product $X \times X$ is not A -weakly infinite-dimensional (see Section 8).*

The construction we present is a version of an example of Wage ([Wa2], Example 2). It is obtained by applying a general method of "factorization" described in [Wa2] and, similarly as the Wage's example, involves Kunen's technique [Kun] to achieve the perfect normality of the square (some information about Kunen's technique can be also found in Alster and Zenor [AZ]). We shall restrict ourselves to a brief description of the construction; the reader familiar with [Wa1] and [Kun] or with Przymusiński's papers [Prz2] and [Prz4] (where the ideas of [Wa1], Kunen's technique and a technique of van Douwen [vD1] are applied) will be able to complete all technical details without any difficulty.

Let \mathcal{E} denote the natural topology of the Cantor set C and let $\Delta = \{(x, x) : x \in C\} \subset C \times C$. Applying a construction of Lelek [Le1], or one of Rubin, Schori and Walsh [RSW], and a selection theorem of Kuratowski and Ryll-Nardzewski [KR] one obtains (see [PoR2]) a function $f: C \rightarrow I^{\aleph_0}$ of first Baire class such that the graph $Y = \{(x, f(x)) : x \in C\} \subset C \times I^{\aleph_0}$ of f is not A -weakly infinite-dimensional and thus is not c.d. (cf. Theorem 8.2).

Thus there exists a completely metrizable separable topology ϱ on Δ (viz. the topology of Y) finer than the natural topology of Δ such that (Δ, ϱ) is not c.d. and every open subset of (Δ, ϱ) is an F_σ -set in the natural topology of Δ .

We shall construct now a metrizable separable topology λ on the set C^2 finer than the natural topology \mathcal{E}^2 such that $\lambda|_\Delta = \varrho|_\Delta$, $\lambda|_{C^2 \setminus \Delta} = \mathcal{E}^2|_{C^2 \setminus \Delta}$ and

(*) every open subset of (C^2, λ) that contains a point $(x, x) \in \Delta$ contains also the intersection with C^2 of two disjoint discs, open in the plane, tangent to Δ at the point (x, x) .

For this purpose arrange a base for the topology ϱ on the set Δ into a sequence B_1, B_2, \dots . For each integer i we have $B_i = \bigcup_{j=1}^{\infty} F_{ij}$, where all F_{ij} 's are closed in the Euclidean topology on Δ . Let us arrange all F_{ij} into a sequence K_1, K_2, \dots .

For every positive real number r let $K_i(r) = \bigcup \{B \cap C^2 : B \text{ is a disc of radius } r, \text{ open in the plane, tangent to } \Delta \text{ in some point of } K_i\}$. Since all K_i 's are compact, we can choose inductively a sequence r_1, r_2, \dots of positive real numbers such that $r_i > r_{i+1}$ for $i = 1, 2, \dots$, $\lim_{i \rightarrow \infty} r_i = 0$ and if $K_i \cap K_j = \emptyset$, then $K_i(r_i) \cap K_j(r_j) = \emptyset$. For each i let $B_i^* = \bigcup \{K_j(2^{-i}r_j) :$

$K_j \subset B_i\}$. Let λ be the topology on C^2 generated by the family $\mathcal{B} \cup \{B_i^*: i = 1, 2, \dots\}$, where \mathcal{B} is a countable base for the space $C^2 \setminus \Delta$ with the natural topology. It is easy to verify that λ is a regular topology that satisfies all the required conditions.

For a given topology τ on a set X , let $K(\tau)$ be the set of all topologies τ^* on X such that $\tau^* \supset \tau$, the space (X, τ^*) is perfectly normal, locally compact (and thus first countable) and has the property that for each $A \subset X$, $|\text{cl}_\tau A \setminus \text{cl}_{\tau^*} A| \leq \aleph_0$.

To conclude our construction it suffices, following the factorization technique of Wage, to define two topologies σ and τ on C such that $\sigma \in K(\mathcal{E})$, $\tau \in K(\mathcal{E})$, $\sigma \times \tau$ is a perfectly normal topology finer than λ and $(\sigma \times \tau) \upharpoonright \Delta \in K(\lambda \upharpoonright \Delta) = K(\varrho)$. Indeed, from the equality $\dim(C, \mathcal{E}) = 0$ it follows that $\dim(C, \mathcal{E}^*) = 0$ for every $\mathcal{E}^* \in K(\mathcal{E})$ (see [Fe1], Lemma 7) and from the fact that (Δ, ϱ) is not c.d. it follows that (Δ, ϱ^*) is not c.d. for every $\varrho^* \in K(\varrho)$ (see Example 3.15), so that $\dim(C, \sigma) = \dim(C, \tau) = 0$ and $(\Delta, \sigma \times \tau \upharpoonright \Delta)$ is not c.d., thus $(C \times C, \sigma \times \tau)$ is not c.d. by Proposition 2.1. The sum $(C, \sigma) \oplus (C, \tau)$ is the space X we need. To obtain the perfect normality of $\sigma \times \tau$, we arrange C into a transfinite sequence $\{x_\alpha: \alpha < \omega_1\}$ by the Continuum Hypothesis and we use Kunen's technique to establish that for each $A \subset C^2$ there exists a $\beta < \omega_1$ such that

$$(\text{cl}_\lambda A) \cap \{x_\beta: \alpha > \beta\}^2 = (\text{cl}_{\sigma \times \tau} A) \cap \{x_\alpha: \alpha > \beta\}^2.$$

4.2. EXAMPLE. *There exists a separable and first-countable Lindelöf space X such that $\dim X = 0$, the Cartesian product $X \times X$ is normal, and yet $X \times X$ is not a s.c.d. space.*

Moreover, the Cartesian product $X \times X$ is not A -weakly infinite-dimensional (see Section 8) and, as pointed out in [Prz2], the space X instead of being Lindelöf can be made locally compact and locally countable.

To obtain this example the following modifications in the construction of a Lindelöf space X of dimension zero whose square $X \times X$ is normal and has the positive dimension, described in [Prz2], should be made:

1° We replace Wage's function f by the function f described in the Example 4.1 so that Δ with the f -topology is not c.d.

2° We extend the f -topology on $\Delta \subset C \times C$ to a metrizable separable topology λ on the whole of $C \times C$ as in Example 4.1.

3° We construct topologies τ_1 and τ_2 in such a way that for any sequence F_1, F_2, \dots of subsets of $C \times C$, if $|\bigcap_{i=1}^{\gamma} \text{cl}_\lambda F_i|_2 > \omega_0$, then $|\bigcap_{i=1}^{\infty} \text{cl}_{\tau_1 \times \tau_2} F_i|_2 > \omega_0$, where $|A|_2 = \min\{|S|: S \subset C \text{ and } A \subset S \times C \cup C \times S\}$ (see [Prz1]).

It is easy to verify that the space $(\Delta, (\tau_1 \times \tau_2) \upharpoonright \Delta)$ is not s.c.d. then. Now,

by Proposition 2.1 the space $(C \times C, \tau_1 \times \tau_2)$ is not s.c.d., and to obtain our space X it suffices to let $X = (C, \tau_1) \oplus (C, \tau_2)$.

We do not know whether by the method described in Example 4.2 one can obtain a space X of dimension zero such that the square $X \times X$ is not c.d., thus we have the following problem.

4.3. PROBLEM. *Define without any set-theoretic assumptions a c.d. (0-c.d.) space X such that the Cartesian product $X \times X$ is normal and yet $X \times X$ is not a c.d. (0-c.d.) space.*

A series of partial positive results follow from various Cartesian product theorems for the dimension \dim (as listed in [E3], pp. 234–236 and in [Prz3]). Let us quote some of them.

As shown in [Kd] (see [E1] for a simplification in the proof), if X is a metrizable space and the Cartesian product $X \times Y$ is normal, then $\dim(X \times Y) \leq \dim X + \dim Y$; thus we have

4.4. THEOREM. *If X is a metrizable c.d. space, Y is a c.d. (0-c.d.) space, and the Cartesian product $X \times Y$ is hereditarily normal, then $X \times Y$ is a c.d. (0-c.d.) space.*

Since the Cartesian product $X \times Y$ of a metrizable space X and a perfectly normal space Y is perfectly normal, and thus hereditarily normal (see [E2], Problem 4.5.16 (b)), Theorem 4.4 yields

4.5. COROLLARY. *If X is a metrizable c.d. space and Y is a c.d. (0-c.d.) perfectly normal space, then the Cartesian product $X \times Y$ is a c.d. (0-c.d.) space.*

From the above-mentioned result in [Kd] we obtain also

4.6. THEOREM. *If X is a metrizable s.c.d. space, Y is a s.c.d. space, and the Cartesian product $X \times Y$ is normal, then $X \times Y$ is a s.c.d. space.*

As shown in [M] (see [Prz3] for a simpler proof), if X is a paracompact, σ -locally compact space, then $\dim(X \times Y) \leq \dim X + \dim Y$ for every completely regular space Y ; thus we have

4.7. THEOREM. *If X is a paracompact, σ -locally compact s.c.d. space, then for every s.c.d. (c.d.) completely regular space Y the Cartesian product $X \times Y$ is a s.c.d. (c.d.) space.*

As announced in [Fi1] and [Pa5] (for a proof see [Fi2]), if both X and Y can be mapped by a perfect mapping onto a metrizable space, then $\dim(X \times Y) \leq \dim X + \dim Y$; thus we have

4.8. THEOREM. *If X_i , where $i = 1, 2, \dots, n$, is a s.c.d. space that can be mapped by a perfect mapping onto a metrizable space, then the Cartesian product $X_1 \times X_2 \times \dots \times X_n$ is a s.c.d. space.*

Let us note that since the Hilbert cube I^{\aleph_0} is not a c.d. space (cf. [E3], Theorem 1.8.20), an infinite Cartesian product of s.c.d. spaces need not be a c.d. space.

4.9. PROBLEM ([Fe3]). *Is the Cartesian product $X \times Y$ of two compact c.d. (0-c.d.) spaces X and Y a c.d. (0-c.d.) space? If not, which additional "assumptions of normality" on X and Y are sufficient for $X \times Y$ to be such?*

5. Compactification and completion theorems

We have the following two theorems relating the existence of c.d. and s.c.d. compactifications of metrizable separable spaces to the existence of c.d. and s.c.d. completions (by a completion of a metrizable space X we mean here a completely metrizable space \tilde{X} which contains a dense subspace homeomorphic to X).

5.1. THEOREM ([Le1], announcement [Hu]). *A metrizable separable space X has a metrizable c.d. compactification if and only if X has a c.d. completion.*

5.2. THEOREM ([Sch2], announcement [Sch1]). *A metrizable separable space X has a metrizable c.d. compactification if and only if X has a s.c.d. completion.*

In particular, we have

5.3. COROLLARY. *Every completely metrizable separable c.d. (s.c.d.) space has a metrizable c.d. (s.c.d.) compactification.*

We do not know what happens in the absence of separability:

5.4. PROBLEM. *Does every completely metrizable c.d. (s.c.d.) space have a c.d. (s.c.d.) compactification?*

5.5. EXAMPLE. *The s.c.d. space $K_\omega \subset I^{\aleph_0}$ defined in Example 3.4 has two important properties related to the subject of this section.*

(1) *K_ω has no hereditarily normal c.d. compactification and no s.c.d. compactification; in fact, K_ω has no A -weakly infinite-dimensional compactification (see Definition 8.1; cf. Theorems 8.2 and 8.3 below).*

This was established in [Sk1] (cf. [AP], Ch. 10, § 7, Theorem 31); in [Hu] it was announced that K_ω has no metrizable compactification, and in [NR] it was proved that K_ω has no c.d. completion (cf. Theorem 5.1). In the matter of fact, a slight modification of the argument in [Sk1] shows that every completion of K_ω contains a copy of I^{\aleph_0} .

Indeed, by the Lavrentieff theorem ([E2], Theorem 4.3.21) it suffices to show that if G is a G_δ -set in I^{\aleph_0} containing K_ω , then G contains a copy of the Hilbert cube. Let $G = \bigcap_{n=1}^{\infty} U_n$, where U_n is an open subset of I^{\aleph_0} = $\prod_{i=1}^{\infty} I_i$, where $I_i = I$ for $i = 1, 2, \dots$. The set $K_1 = \{\{x_i\} \in I^{\aleph_0} : x_i = 0 \text{ for } i = 2, 3, \dots\}$ is a compact subset of U_1 ; hence, by the Wallace

theorem ([E2], Theorem 3.2.10), there exist points $b_1, b_2, \dots, b_{n_1} \in I \setminus \{0\}$ such that

$$\prod_{i=1}^{n_1} [0, b_i] \times \prod_{i=n_1+1}^{\infty} I_i \subset U_1.$$

Similarly, the set

$$K_2 = \prod_{i=1}^{n_1} [0, b_i] \times \{ \{x_i\} \in \prod_{i=n_1+1}^{\infty} I_i : x_i = 0 \text{ for } i = n_1+1, n_1+2, \dots \}$$

is a compact subset of U_2 , hence there exist points $b_{n_1+1}, b_{n_1+2}, \dots, b_{n_2} \in I \setminus \{0\}$ such that

$$\prod_{i=1}^{n_2} [0, b_i] \times \prod_{i=n_2+1}^{\infty} I_i \subset U_2.$$

Applying induction, we obtain a sequence b_1, b_2, \dots of points in $I \setminus \{0\}$ such that

$$\prod_{i=1}^{\infty} [0, b_i] \subset \bigcap_{n=1}^{\infty} U_n = G.$$

(2) βK_ω is not a c.d. space.

This was established, independently, in [vD2] and [PoE3]; in fact, in [PoE3] a slightly weaker result was formulated, viz. that βK_ω is not a s.c.d. space, but the argument used there yields (2) as well. We reproduce here the proof from [PoE3] which permits to obtain (2) in a direct way.

To establish (2), we shall prove more, namely that if X is a metrizable c.d. space which contains a closed subset of the form $F = \bigoplus_{n=1}^{\infty} F_n$, where $\dim F_n \geq n$ for $n = 1, 2, \dots$, then βX is not a c.d. space. It can easily be verified by the standard kernel construction (see [St]) that F_n contains a closed subset F'_n such that for every open subspace U of F'_n we have $\dim U \geq n$; hence we can assume that F_n has the latter property for $n = 1, 2, \dots$. Assume, on the contrary, that βX is a c.d. space, i.e., that $\beta X = \bigcup_{i=1}^{\infty} X_i$, where $\dim X_i < \infty$.

In the set $Z = \text{cl}_{\beta X} F$, which is in fact the Čech–Stone compactification of F , consider for $m = 1, 2, \dots$ the set $G_m = \text{cl}_Z \left(\bigcup_{n=1}^m F_n \right)$. The sets G_m are open-and-closed in Z ; hence the set $A = Z \setminus \bigcup_{m=1}^{\infty} G_m$ is a compact functionally closed subset of Z . We shall show that (cf. Definition 8.1 below)

(*) there exists a non-empty functionally closed subset K of A which is A -weakly infinite-dimensional, i.e., for every sequence $(A_1, B_1), (A_2, B_2), \dots$ of pairs of disjoint closed subsets of K there exist in K closed sets L_1, L_2, \dots such that L_n is a partition between A_n and B_n in K and $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

To this end, we define inductively a sequence K_1, K_2, \dots of functionally closed subsets of A such that $K_1 = A$, $\text{int}_A K_i \neq \emptyset$, $K_{i+1} \subset \text{int}_A K_i$ and

(**) if the set $X_i \cap \text{int}_A K_i$ is not dense in $\text{int}_A K_i$, then K_{i+1} satisfies the equality $K_{i+1} \cap X_i = \emptyset$

for $i = 1, 2, \dots$; then we let $K = \bigcap_{i=1}^{\infty} K_i$. It suffices to check that K is weakly infinite-dimensional. Let i_1, i_2, \dots be the sequence of all integers i such that $K \cap X_i \neq \emptyset$; we have then $K \subset \bigcap_{k=1}^{\infty} X_{i_k}$ and for every integer k the set $X_{i_k} \cap \text{int}_A K_{i_k}$ is dense in $\text{int}_A K_{i_k}$. Without loss of generality we can assume that the sequence i_1, i_2, \dots is infinite.

Consider a decomposition $N = \bigcup_{k=1}^{\infty} N_k$ of the set N of positive integers such that $|N_k| = \dim X_{i_k} + 1$ and $N_k \cap N_{k'} = \emptyset$ for distinct $k, k' \in N$.

Now, let $(A_1, B_1), (A_2, B_2), \dots$ be a sequence of pairs of disjoint closed subsets of K . Since the sets A_n and B_n are compact, for $n = 1, 2, \dots$, one can find sets S_n and T_n , functionally closed in βX , such that $A_n \subset \text{int}_{\beta X} S_n$, $B_n \subset \text{int}_{\beta X} T_n$ and $S_n \cap T_n = \emptyset$. For $k = 1, 2, \dots$ and every $n \in N_k$ there exist sets S'_n and T'_n , functionally closed in X_{i_k} , such that $S_n \cap X_{i_k} \subset S'_n$, $T_n \cap X_{i_k} \subset T'_n$, $X_{i_k} = S'_n \cup T'_n$, $S_n \cap X_{i_k} \cap T'_n = \emptyset$, $T_n \cap X_{i_k} \cap S'_n = \emptyset$ and $\bigcap_{n \in N_k} (S'_n \cap T'_n) = \emptyset$ (see [E2], Theorem 7.2.15). For $n = 1, 2, \dots$ let

$$L_n = \text{cl}_{\beta X} S'_n \cap \text{cl}_{\beta X} T'_n \cap K;$$

we shall show that L_n is a partition between A_n and B_n in K . Indeed, if $n \in N_k$, then

$$\begin{aligned} K &\subset \text{int}_A K_{i_k} \subset \text{cl}_{\beta X} (X_{i_k} \cap \text{int}_A X_{i_k}) \subset \text{cl}_{\beta X} X_{i_k} \subset \text{cl}_{\beta X} S'_n \cup \text{cl}_{\beta X} T'_n, \\ A_n &\subset A \cap \text{int}_{\beta X} S_n \cap \text{int}_A K_{i_k} \subset \text{cl}_{\beta X} (A \cap \text{int}_{\beta X} S_n \cap \text{int}_A K_{i_k}) \\ &= \text{cl}_{\beta X} (A \cap \text{int}_{\beta X} S_n \cap \text{int}_A K_{i_k} \cap X_{i_k}) \subset \text{cl}_{\beta X} (A \cap \text{int}_A K_{i_k} \cap S'_n) \subset \text{cl}_{\beta X} S'_n, \end{aligned}$$

and, similarly, $B_n \subset \text{cl}_{\beta X} T'_n$. Furthermore, since

$$T'_n \cap \text{int}_{\beta X} S_n = T'_n \cap \text{int}_{\beta X} S_n \cap X_{i_k} = \emptyset,$$

then $\text{cl}_{\beta X} T'_n \cap \text{int}_{\beta X} S_n = \emptyset$ and thus $A_n \cap \text{cl}_{\beta X} T'_n = \emptyset$; similarly, $B_n \cap \text{cl}_{\beta X} S'_n = \emptyset$. We have also

$$\bigcap_{n=1}^{\infty} L_n = \bigcap_{k=1}^{\infty} \bigcap_{n \in N_k} L_n \subset \bigcup_{k=1}^{\infty} \left(\bigcap_{n \in N_k} L_n \cap X_{i_k} \right) = \bigcup_{k=1}^{\infty} \left(\bigcap_{n \in N_k} (S'_n \cap T'_n) \right) = \emptyset,$$

which concludes the proof of (*).

Now, the set K , being functionally closed in A , is functionally closed in Z ; and hence $K = \bigcap_{n=1}^{\infty} U_n$, where U_n is open in Z and $\text{cl}_Z U_{n+1} \subset U_n$ for $n = 1, 2, \dots$. For $n = 1, 2, \dots$ the set U_n intersects infinitely many sets

F_1, F_2, \dots otherwise we would have $K \subset G_m$ for some m , a contradiction. Thus for $n = 1, 2, \dots$ we can choose an $m_n \geq n$ such that $m_n > m_{n'}$, whenever $n > n'$ and $U_n \cap F_{m_n} \neq \emptyset$. For $n = 1, 2, \dots$ consider a set $Y_n \subset U_n \cap F_{m_n}$ which is closed in F and such that $\dim Y_n \geq m_n \geq n$.

The space $Y = \bigcup_{n=1}^{\infty} Y_n = \bigoplus_{n=1}^{\infty} Y_n$ is a closed subset of F . Let $Y^* = Y \cup K$ be the subspace of Z . Let us note that every pair A and B of disjoint closed subsets of Y has disjoint closures in Y^* . Indeed, A and B are closed in F and thus have disjoint closures in $\beta F = Z \supset Y^*$.

From Proposition 2.1 it follows that Y_n is a c.d. space, so that (cf. Theorem 8.2 below) Y_n is weakly infinite-dimensional for $n = 1, 2, \dots$. It is easy to check that the space Y^* is paracompact; thus

(***) Y^* is A -weakly infinite-dimensional,

since it is the union of countably many closed A -weakly infinite-dimensional subspaces K, Y_1, Y_2, \dots (see [Lev] or [AP], Ch. 10, § 5, Theorem 22).

Consider now, for $n = 1, 2, \dots$ a sequence $(A_{n,1}, B_{n,1}), (A_{n,2}, B_{n,2}), \dots, (A_{n,n}, B_{n,n})$ of pairs of disjoint closed subsets of Y_n such that if L_i is a partition between $A_{n,i}$ and $B_{n,i}$ in Y_n for $i = 1, 2, \dots, n$ then $\bigcap_{i=1}^n L_i \neq \emptyset$.

Let $A_i = \bigcup_{n=i}^{\infty} A_{n,i}$ and $B_i = \bigcup_{n=i}^{\infty} B_{n,i}$ for $i = 1, 2, \dots$; the sets A_i and B_i are disjoint and closed in Y , thus they have disjoint closures A_i^* and B_i^* in the space Y^* . Let L_i be an arbitrary partition between the sets A_i^* and B_i^* in the space Y^* for $i = 1, 2, \dots$. For each $i \leq n$ the set $L_i \cap Y_n$ is a partition between $A_{n,i}$ and $B_{n,i}$ in Y_n ; hence for $n = 1, 2, \dots$ one can find a point $x_n \in \bigcap_{i=1}^n (L_i \cap Y_n)$. The sequence x_1, x_2, \dots has an accumulation point x_0

in Z ; obviously $x_0 \in K = \bigcap_{n=1}^{\infty} \text{cl} U_n \subset Y^*$, so that $x_0 \in \bigcap_{i=1}^{\infty} L_i$, which contradicts (***) and also our original assumption that βX is a c.d. space.

5.6. PROBLEM. Does K_ω have any c.d. compactification?

Let us note that such a compactification would be a c.d. compact space which is not A -weakly infinite dimensional (cf. property (1) in Example 5.5), and thus Problem 8.4 would be answered in the negative. We expect a negative answer to Problem 5.6.

5.7. PROBLEM. Does there exist a metrizable separable c.d. space with no c.d. compactification?

In [vD2] the question when βX is c.d. (s.c.d.) was solved for metrizable spaces (in fact, for a somewhat broader class of spaces; in [PoE3] a slightly weaker result was obtained in the c.d. case (the s.c.d. case is rather simple):

5.8. THEOREM ([vD2]). For a metrizable space X the Čech–Stone compactification βX is c.d. (s.c.d.) if and only if X contains a compact c.d. (s.c.d.)

space K such that $\dim F < \infty$ whenever F is closed in X and misses K .

The question whether a similar result holds in the 0-c.d. case reduces to the following special problem:

5.9. PROBLEM. *Is the Čech–Stone compactification βX of a finite-dimensional metrizable space X a 0-c.d. space?*

To conclude, let us note that, by the proof of property (2) in Example 5.5, for the s.c.d. locally compact (and thus completely metrizable) separable metric space $X = \bigoplus_{n=1}^{\infty} I^n$ the Čech–Stone compactification βX is not a c.d. space (cf. Problem 5.4).

6. Universal space theorems

We shall tabulate positive results in this domain and then ask a few questions. First of all let us recall some standard notations: by N_{ω} we denote the subspace of the Hilbert cube I^{\aleph_0} consisting of all points which have only finitely many rational coordinates and by $K_{\omega}(m)$ the subspace of the Cartesian power $[J(m)]^{\aleph_0}$ of the hedgehog space of spininess $m \geq \aleph_0$ (cf. [E2], Example 4.1.5) consisting of all points in $[J(m)]^{\aleph_0}$ which have only finitely many rational (i.e. with rational distance from the “origin” 0) coordinates distinct from 0. The symbol $B(m)$ denotes the Baire space of weight $m \geq \aleph_0$, i.e., the Cartesian power $[D(m)]^{\aleph_0}$, where $D(m)$ is the discrete space of cardinality m , and K_{ω} is the subspace of I^{\aleph_0} consisting of all points which have only finitely many non-zero coordinates.

Countable-dimensional spaces

class of spaces	universal space	author
metrizable separable	N_{ω}	Nagata [Nag1]
strongly metrizable of weight $\leq m$	$B(m) \times N_{\omega}$	Nagata [Nag1]
metrizable of weight $\leq m$	$K_{\omega}(m)$	Nagata [Nag2]

Strongly countable-dimensional spaces

class of spaces	universal space	author
metrizable separable	K_{ω}	Smirnov [Sm3] Nagata [Nag1]
strongly metrizable of weight $\leq m$	$B(m) \times K_{\omega}$	Nagata [Nag1]
metrizable of weight $\leq m$	exists	Arhangel'skii [A4] Pasyukov [Pa3]
normal of weight $\leq m$	exists	Pasyukov [Pa4] (implicitly in [Z1]+ +[Pa1])

6.1. **PROBLEM.** *Does there exist a universal space for the class of all c.d. normal spaces of weight $\leq m$?*

6.2. **PROBLEM.** *Does there exist a universal space for the class of all 0-c.d. normal spaces of weight $\leq m$?*

Let us note that there is no universal space for the classes of all c.d. and all s.c.d. metrizable compact spaces (see [E4] Example 2.2, Proposition 3.3, and Theorem 4.1). One can state, however, the following weaker version of Problems 6.1 and 6.2.

6.3. **PROBLEM.** *Does there exist a c.d. normal space of weight $\leq m$ which topologically contains all c.d. compact spaces of weight $\leq m$?*

6.4. **PROBLEM.** *Does there exist a 0-c.d. normal space of weight $\leq m$ which topologically contains all 0-c.d. compact spaces of weight $\leq m$?*

In Nagata's and Smirnov's results tabulated above, specific universal spaces are indicated. The last two results are purely existential; they were obtained from corresponding factorization theorems, the now standard method, originated by Pasynkov in [Pa1]. The existence of universal spaces for the classes considered by Nagata and Smirnov can also be established via factorization theorems (see [AP], also for a discussion of factorization theorems), but in that way we obtain no information on how the universal space looks like and, of course, the corresponding factorization theorems have to be proved first.

7. Mapping theorems

We start with two theorems on invariance of the classes of c.d. and s.c.d. spaces under closed mappings.

7.1. **THEOREM** ([Na3]). *Let $f: X \rightarrow Y$ be a closed mapping of a c.d. metrizable space X onto a space Y ; if for every $y \in Y$ the boundary $\text{Fr } f^{-1}(y)$ is either empty or has an isolated point, then Y is a c.d. space (in fact, a 0-c.d. space).*

In the original formulation of the above theorem Y is assumed to be a metrizable space. It follows however, from Lašnev's theorem in [La], that $Y = Y_0 \cup \left(\bigcup_{i=1}^{\infty} Y_i \right)$, where Y_1, Y_2, \dots are discrete subspaces of Y and $f^{-1}(y)$ is compact for every $y \in Y_0$ (which implies that Y_0 is metrizable, cf. [E2], Theorem 4.4.15), so that the general case reduces to the case when Y is a metrizable space. Applying Lašnev's theorem and the fact that every non-empty compact metrizable space with no isolated points has cardinality \mathfrak{c} (cf. [E2], Problem 4.5.5), we deduce from the above theorem the following corollary (that was established earlier than Theorem 7.1 itself)

7.2. COROLLARY ([A2]). *Let $f: X \rightarrow Y$ be a closed mapping of a c.d. metrizable space X onto a space Y ; if $|f^{-1}(y)| < \epsilon$ for every $y \in Y$, then Y is a c.d. space (in fact, a 0-c.d. space).*

As observed in [A4], from Corollary 7.2 and Zarelua's factorization theorem (established in [Z1]) it follows that if $f: X \rightarrow Y$ is a continuous mapping of a compact c.d. space onto a metrizable compact space Y such that $|f^{-1}(y)| < \epsilon$ for every $y \in Y$, then Y is a c.d. space; let us state, after [A3], the following

7.3. PROBLEM. *Let $f: X \rightarrow Y$ be a continuous mapping of a c.d. compact space X onto a compact space Y such that $|f^{-1}(y)| < \epsilon$ for every $y \in Y$; is Y necessarily a c.d. space?*

We can also ask

7.4. PROBLEM. *Under which conditions Theorem 7.1 holds for mappings defined on non-metrizable spaces?*

As shown in [Z2], if for a closed mapping f of a normal space X onto a space Y there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $\dim Y \leq \dim X + (k - 1)$; thus we have

7.5. THEOREM. *Let $f: X \rightarrow Y$ be a closed mapping of a s.c.d. normal space X onto a space Y ; if there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then Y is a s.c.d. space.*

Let us observe that the last theorem does not hold for mappings with finite fibers (cf. Theorem 7.13):

7.6. EXAMPLE. *As shown in Example 1.12, the Cantor set can be mapped by a continuous mapping with finite fibers onto a compact metrizable space which is not a s.c.d. space.*

We have the following theorem on inverse invariance of the class of c.d. spaces under closed mappings:

7.7. THEOREM ([A2]). *Let $f: X \rightarrow Y$ be a closed mapping of a metrizable space X onto a c.d. space Y ; if $\dim f^{-1}(y) < \infty$ for every $y \in Y$, then X is a c.d. space.*

Again, we can ask

7.8. PROBLEM. *Under which conditions Theorem 7.7 holds for mappings defined on non-metrizable spaces?*

Let us observe that in Theorem 7.7 the assumption about the fibers cannot be relaxed to the assumption that $f^{-1}(y)$ is a c.d. space for every $y \in Y$. Indeed, we have the following example:

7.9. EXAMPLE ([PoR3]). *There exists a compact metrizable space X which is not a c.d. space, and yet can be mapped onto the Cantor set by an open mapping with c.d. fibers.*

There is no counterpart for Theorem 7.7 in the case of s.c.d. spaces.

7.10. EXAMPLE. As shown in Example 1.12, there exists a compact metrizable space which is not a s.c.d. space, and yet can be mapped onto the Cantor set by an open-and-closed mapping with finite-dimensional fibers (cf. Example 7.17).

We pass now to a discussion of open mappings.

7.11. THEOREM. Let $f: X \rightarrow Y$ be an open mapping of a c.d. strongly hereditarily normal space X onto a hereditarily normal space Y ; if $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then Y is a c.d. space.

Proof. Let $X = \bigcup_{i=1}^{\infty} X_i$, where $X_1 \subset X_2 \subset \dots$ and $\dim X_i \leq i$ for $i = 1, 2, \dots$ (cf. Proposition 1.3). For $i = 1, 2, \dots$ define $Y_i = Y \setminus f(X \setminus X_i)$; from the finiteness of fibers it follows that $Y = \bigcup_{i=1}^{\infty} Y_i$ so that it suffices to show that Y_i is a c.d. space for $i = 1, 2, \dots$, or else that $Y_{i,k} = Y_i \cap N_k$, where $N_k = \{y \in Y: |f^{-1}(y)| \leq k\}$ for $k = 1, 2, \dots$, is a finite-dimensional space for $i, k = 1, 2, \dots$. Fix i and k and consider the restriction $f_0 = f|_{f^{-1}(Y_{i,k})}: f^{-1}(Y_{i,k}) \rightarrow Y_{i,k}$. The mapping f_0 is open and since $f^{-1}(Y_{i,k}) \subset f^{-1}(Y) \subset X_i$, we have $\dim f^{-1}(Y_{i,k}) \leq i$ by virtue of the subspace theorem ([E3], Theorem 3.1.19). Consider an open cover $\{U_j\}_{j=1}^m$ of the space $Y_{i,k}$. The open cover $\{f_0^{-1}(U_j)\}_{j=1}^m$ of the space $f^{-1}(Y_{i,k})$ has an open shrinking $\{V_j\}_{j=1}^m$, of order $\leq i$. Now, since $y \in f_0(V_j)$ if and only if $f^{-1}(y) \cap V_j \neq \emptyset$, the order of the open shrinking $\{f_0(V_j)\}_{j=1}^m$ of the cover $\{U_j\}_{j=1}^m$ is not larger than $(i+1)k-1$, so that $\dim Y_{i,k} \leq (i+1)k-1$.

Theorem 7.11 was proved in [A1] under the assumption that X and Y are metrizable spaces; the original proof was indirect. In [A3] a direct proof was given, based on an observation applied earlier in [Na1], viz. on the fact that if $f: X \rightarrow Y$ is an open mapping with finite fibers defined on a Hausdorff space, then $Y = \bigcup_{i=1}^{\infty} Y_i$, where $f|_{f^{-1}(Y_i)}: f^{-1}(Y_i) \rightarrow Y_i$ is a local homeomorphism (the subspace Y_i consists of all points $y \in Y$ such that $|f^{-1}(y)| = i$; one easily checks that the unions $N_k = Y_1 \cup Y_2 \cup \dots \cup Y_k$ are closed). Nagami's observation yields a somewhat stronger result than the one originally formulated in [A1] (cf. Theorem 7.12 below). Our proof of Theorem 7.11 yields a still stronger result; it closely follows an argument applied in [Polk] to establish directly another result obtained in [A1] in an indirect way (Theorem 7.13 below).

From Nagami's observation stated above, Proposition 2.2 and Theorem 3.13 we obtain

7.12. THEOREM. Let $f: X \rightarrow Y$ be an open mapping of a 0-c.d. strongly hereditarily normal space X onto a strongly hereditarily normal weakly paracompact space Y ; if $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then Y is a 0-c.d. space.

Following the proof of Theorem 7.11 and introducing obvious modifications we obtain

7.13. THEOREM ([A1]). *Let $f: X \rightarrow Y$ be an open mapping of a s.c.d. normal space X onto a normal space Y ; if $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then Y is a s.c.d. space.*

In Theorems 7.11–7.13 the assumption about the fibers cannot be relaxed to countability:

7.14. EXAMPLE. *The Hilbert cube I^{\aleph_0} is the image of a finite-dimensional space under an open mapping with countable fibers.*

Indeed, as shown in [Wi], the Menger universal curve M_1^3 can be mapped onto I^{\aleph_0} by an open mapping, and – by a result in [Ro] (cf. [E3], Problem 1.12.F) – there exists a subspace $X \subset M_1^3$ such that the restriction of this mapping to X is an open mapping of X onto I^{\aleph_0} and has countable fibers.

The following theorem can be obtained by standard methods developed in the context of Alexandroff's theorem ([E3], Theorem 1.12.8 and Problem 4.3.E).

7.15. THEOREM ([A3]). *Let $f: X \rightarrow Y$ be an open-and-closed mapping of a s.c.d. metrizable space X onto a metrizable space Y ; if $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then Y is a s.c.d. space.*

Theorems on inverse invariance of classes of c.d., 0-c.d. and s.c.d. spaces under open mappings are less interesting. They are easy consequences of corresponding theorems for spaces of dimension $\leq n$ (see [Na1], [A3], [Pa2] and [E3], Section 1.12 and Problem 4.3.E for theorems of this last kind). We shall state only one such theorem which is an immediate consequence of Nagami's observation stated above, Proposition 2.2 and Theorem 3.13.

7.16. THEOREM. *Let $f: X \rightarrow Y$ be an open mapping of a strongly hereditarily normal weakly paracompact space X onto a strongly hereditarily normal c.d. (0-c.d., s.c.d.) space Y ; if $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then X is a c.d. (0-c.d., s.c.d.) space.*

We conclude this section with an example and two problems.

7.17. EXAMPLE. *There exists a completely metrizable space X which is not a c.d. space, and yet can be mapped onto the Cantor set by an open mapping with countable fibers* (cf. Example 7.10; the mapping cannot be open-and-closed, see [E3], Problem 4.3.E(e)).

Indeed, it suffices to consider the space Y defined in the Lemma in [PoR2], the projection $p: Y \rightarrow C$ onto the Cantor set and apply the procedure outlined in the hint to Problem 1.12.G(b) in [E3] (replacing K by Y).

7.18. PROBLEM. *Can every compact metrizable c.d. space be mapped onto a finite-dimensional space by a mapping with finite-dimensional fibers?*

If the answer to this question was positive then, by Theorem 7.7, it

would provide a new characterization of c.d. metrizable compact spaces. Let us note that the space K_ω (see Example 3.4) cannot be mapped by a closed mapping onto a finite-dimensional metric space containing more than one point (see [PoE4]). In connection with the above property of K_ω we state another question (see [E4] for the definition of the small transfinite dimension):

7.19. PROBLEM. *Can every σ -compact metrizable space which has the small transfinite dimension be mapped onto a finite-dimensional space by a perfect mapping?*

8. Relations to other classes of infinite-dimensional spaces

The characterization of the covering dimension in terms of partitions (see [E3], Theorem 3.2.6) suggested the following definition of class of “weakly infinite-dimensional” spaces, discussed first by Alexandroff in his 1948 preface to the Russian translation of the Hurewicz and Wallman’s *Dimension theory* book.

8.1. DEFINITION. A normal space X is *A-weakly infinite-dimensional* if for every sequence $(A_1, B_1), (A_2, B_2), \dots$ of pairs of disjoint closed subsets of X there exist closed sets L_1, L_2, \dots such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{\infty} L_i = \emptyset$.

Let us note that one also considers the class of *S-weakly infinite-dimensional* spaces introduced by Smirnov (cf. [Lev] and [Sk1]); the definition of the latter class is analogous to Definition 8.1, except that we assume that $\bigcap_{i=1}^n L_i = \emptyset$ for some integer n . In the sequel we shall call the two classes *A-w.i.d.* spaces and *S-w.i.d.* spaces, respectively. Clearly, every *S-w.i.d.* space is *A-w.i.d.*; as shown by the space $\bigoplus_{n=1}^{\infty} I^n$, the converse is not true. In the realm of compact spaces both notions coincide, and we use the abbreviation *w.i.d.*

From Levšenko’s addition theorem ([Lev], cf. [AP], Ch. 10, § 5, Theorem 21), stating that a hereditarily normal space which is the union of countably many *A-w.i.d.* subspaces is *A-w.i.d.* itself, we obtain

8.2. THEOREM. *Every hereditarily normal c.d. space is A-w.i.d.*

Similarly, from Levšenko–van Douwen’s sum theorem ([vD2]), stating that a normal space (in the original version, see [Lev] and [AP], Ch. 10, § 5, Theorem 22, one assumes moreover that the space is countably paracompact)

which is the union of countably many closed A -w.i.d. subspaces is A -w.i.d. itself, we obtain

8.3. THEOREM. *Every normal s.c.d. space is A -w.i.d.*

We have the following

8.4. PROBLEM. *Is every c.d., or every 0-c.d., compact (normal) space A -w.i.d.?*

The following example, due to R. Pol, is a recent solution of the old problem of Alexandroff's whether every w.i.d. compact metric space is c.d.

8.5. EXAMPLE ([PoR2]). *There exists a metrizable compact w.i.d. space X which is not c.d. The space X is a compactification, with the s.c.d. remainder, of a completely metrizable totally disconnected space Y which is not A -w.i.d.*

The structure of the space X described in the above example suggests the following problem.

8.6. PROBLEM. *Is every metrizable compact hereditarily A -w.i.d. space c.d.?*

An infinite-dimensional space which contains no closed c.d. subspace of dimension ≥ 1 will be called *hereditarily uncountable-dimensional* in the sequel. Some examples of non-metrizable compact w.i.d. and hereditarily uncountable dimensional spaces were given by Fedorčuk:

8.7. EXAMPLES ([Fe3]). (a) *There exists a compact w.i.d. and hereditarily uncountable-dimensional space.*

(b) *Under the assumption of the Continuum Hypothesis there exists a perfectly normal compact w.i.d. and hereditarily uncountable-dimensional space.*

(c) *Under the assumption of the Continuum Hypothesis there exists a compact w.i.d. space no infinite closed subset of which is c.d.*

(d) *Under the assumption of Jensen's principle \diamond (see [Je]) there exists a hereditarily normal, hereditarily separable and w.i.d. compact space no infinite closed subset of which is c.d. or has cardinality less than 2^{\aleph_1} .*

8.8. PROBLEM ([Fe3]). *Does there exist a metrizable compact w.i.d. space which is hereditarily uncountable-dimensional?*

8.9. Remark. Let us note that the perfectly normal compact space X in Example 1.5, defined under the assumption of the Continuum Hypothesis, is a w.i.d. space (more exactly, $\dim X = 2$) which contains no closed 0-c.d. subspace of dimension ≥ 1 .

Let us now briefly discuss the relations between countable dimensionality and the existence of transfinite dimension.

In the realm of metrizable spaces the relations are discussed in [E4]. Summarizing: every completely metrizable c.d. space has trind , every metrizable compact c.d. space has trInd , and every metrizable separable space which has trind is c.d.; there is one unsolved problem (cf. [E4], Problem 4.5, for a discussion):

8.10. **PROBLEM.** *Give an example of a metrizable space which has trind and yet is not c.d.*

All spaces considered in Examples 3.16–3.18 do have trind (equal to 0) and yet are not c.d. In this context one can consider the following weaker version of the last problem.

8.11. **PROBLEM.** *Define without any set-theoretic assumptions a perfectly normal (or hereditarily normal) space which has trind and yet is not c.d.*

In the realm of compact spaces, the relations between countable dimensionality and the existence of transfinite dimensions are discussed in [Fe3]. Summarizing again: for every compact space the existence of trind is equivalent to the existence of trInd , every compact space which has transfinite dimension is w.i.d. (more generally: every normal space which has trInd is S -w.i.d., cf. [Sm4] and [AP], Ch. 10, § 6, Theorem 28) and every hereditarily normal compact 0-c.d. space has transfinite dimension; let us note that the space in Example 1.5 is a perfectly normal compact s.c.d. space (defined under the assumption of the Continuum Hypothesis) which has no transfinite dimension.

Let us state, after [Fe3], the following three problems:

8.12. **PROBLEM.** *Define without any set-theoretic assumptions a compact, if possible perfectly or hereditarily normal, c.d. (s.c.d.) space with no transfinite dimension.*

8.13. **PROBLEM.** *Define a compact 0-c.d. space with no transfinite dimension.*

8.14. **PROBLEM.** *Is every compact space with transfinite dimension a c.d. space?*

We conclude this section with some observations on the recently introduced class of C -spaces which, in the realm of metrizable spaces, contains all c.d. spaces and is contained in the class of A -w.i.d. spaces. The class of C -spaces was introduced, to be applied in the theory of retracts, first for metrizable spaces in [Ha], and then was redefined in [AG] to include more general spaces.

8.15. **DEFINITION** ([AG]). A topological space X is a C -space if for every sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of families of open subsets of X , the union $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ of which covers X , such that for $i = 1, 2, \dots$ (i) the members of \mathcal{U}_i are pairwise disjoint and (ii) each member of \mathcal{U}_i is contained in a member of \mathcal{G}_i .

In [AG] the following two theorems were established and the question whether reverse implications hold in the realm of metrizable spaces was raised.

8.16. THEOREM ([AG]; for metrizable spaces [Ha]). *Every hereditarily paracompact c.d. space is a C-space.*

8.17. THEOREM ([AG]). *Every normal C-space is A-w.i.d.*

R. Pol's solution of Alexandroff's problem yields

8.18. EXAMPLE. *There exists a metrizable compact C-space which is not c.d.*

Indeed, the space X in Example 8.5 is a C-space. To establish this simple fact, consider a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X . Let $X \setminus Y = \bigcup_{i=2}^{\infty} Z_i$, where $\dim Z_i = 0$. For $i = 2, 3, \dots$ choose a family \mathcal{U}_i of open subsets of X such that $Z_i \subset \bigcup \mathcal{U}_i$ which satisfies conditions (i) and (ii) in Definition 8.15; (cf. [E3], Lemma 1.7.3). The subspace $Z_1 = X \setminus \bigcup_{i=2}^{\infty} (\bigcup \mathcal{U}_i)$ of X is a compact subspace of the totally disconnected space Y , and thus $\dim Z_1 = 0$. Hence, there exists a family \mathcal{U}_1 of open subsets of X such that $Z_1 \subset \bigcup \mathcal{U}_1$ which satisfies conditions (i) and (ii) in Definition 8.15; since the union $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ covers X , the proof is concluded.

The results in [PoR2] and [PoR4] solve in fact two more problems stated in [AG].

8.19. EXAMPLE. *There exists a metrizable compact C-space which contains a subspace that is not a C-space.*

This is again the space X in Example 8.5; its subspace Y is not a C-space (cf. Theorem 8.17).

8.20. EXAMPLE. *There exist separable metrizable C-spaces X_1 and X_2 such that the Cartesian product $X_1 \times X_2$ is not a C-space.*

As shown in [PoR4], if we split the subspace $X \setminus Y$ of the space X in Example 8.5 into two disjoint Bernstein sets M_0 and M_1 , then the subspace $X_k = M_k \cup Y$ of the space X is A-w.i.d. for $k = 1, 2$, and yet the Cartesian product $X_1 \times X_2$ is not A-w.i.d. By virtue of Theorem 8.17, it suffices now to show that X_k is a C-space for $k = 1, 2$. Let us consider a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X_k . For $i = 1, 2, \dots$ and for every member G of \mathcal{G}_i choose an open subset \tilde{G} of X such that $X_k \cap \tilde{G} = G$ and let $\tilde{\mathcal{G}}_i = \{\tilde{G} : G \in \mathcal{G}_i\}$ and $G_i = \bigcup \tilde{\mathcal{G}}_i$. Since the subspace $X \setminus G_i$ is compact and contained in M_{1-k} , it is countable, and thus there exists a G_δ -set $A \subset X$ such that $X \setminus \bigcap_{i=1}^{\infty} G_i \subset A$ and $\dim A = 0$ (cf. [E3], Theorem 1.2.14). Since every F_σ -set in a C-space is itself a C-space (see [AG], Corollary 2.8), $X \setminus A$ is a C-space, and there exists a sequence $\tilde{\mathcal{U}}_2, \tilde{\mathcal{U}}_3, \dots$ of families of open subsets of X , the union $\bigcup_{i=2}^{\infty} \tilde{\mathcal{U}}_i$ of which covers $X \setminus A$, such that for $i = 2, 3, \dots$ the

members of $\tilde{\mathcal{U}}_i$ are pairwise disjoint and each of them is contained in a member of $\tilde{\mathcal{G}}_i$. Since $\dim(X_k \cap A) \leq 0$ and $X_k \cap A \subset X_k \subset G_1$, there exists a family $\tilde{\mathcal{U}}_1$ of pairwise disjoint open subsets of X such that $X_k \cap A \subset \bigcup \tilde{\mathcal{U}}_1$ and each member of $\tilde{\mathcal{U}}_1$ is contained in a member of $\tilde{\mathcal{G}}_1$. The families $\mathcal{U}_i = \{X_k \cap \tilde{U} : \tilde{U} \in \tilde{\mathcal{U}}_i\}$, where $i = 1, 2, \dots$ satisfy all the conditions in Definition 8.15.

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