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Rational extensions of $C(X)$ and semicontinuous functions

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CONTENTS

0. Introduction

The set $C(X)$ of all real-valued continuous functions defined on a topological space X may be equipped with various algebraic structures. Most prominently, $C(X)$ is a commutative ring with unit as a subdirect product of copies of the field $\boldsymbol{R}$. As such, it has been explored very profoundly (see, e.g., [GJ]); however, it seems that the algebraic structure of $C(X)$ prevalently was used as a tool to investigate the topological structure of X and related spaces. There is one notable exception: In [FGL], a purely ring-theoretic concept, that of a maximal ring of quotients (alias maximal rational extension) is studied for its own sake in the case of rings $C(X)$ (yielding topological benefits also, naturally). $C(X)$ is also a distributive lattice as a subdirect product of copies of the chain $\boldsymbol{R}$. Interest in the lattices $C(X)$ centered on completeness properties; see, e.g., [St] and [Dl]. The latter paper, on the MacNeille completion of $C^*(X)$, highlights the role of semicontinuous functions in this context.

The present paper tries, to some extent, to blend ring-theoretic and lattice-theoretic aspects of $C(X)$. The unifying concept is that of a rational extension of a semigroup. Given two commutative semigroups S, T such that $S \subseteq T$, call T a *rational extension* of S provided that for any triple t, t_1, $t_2 \in T$ with $t_1 \neq t_2$ there exists $s \in S$ satisfying $st \in S$, $st_1 \neq st_2$. This concept obviously generalizes to the noncommutative case, but this will not be needed for our purposes. Terminology goes back to Findlay and Lambek [FL]; T is commonly called a *semigroup of quotients* of S today, but we stick to "rational extension" in order to avoid conflicting uses of "quotient" in the case S, T are lattices. The concept of rational extensions of semigroups is mainly due to McMorris and Berthiaume (see, e.g., [Be], [MM 1], [MM 2]), generalizing work by Findlay and Lambek (see [FL], [BL]) on the ring case. We refer the reader to Lambek's book [La] and to Weinert's survey article [We].

The key fact in the theory of rational extensions is the existence – under mild assumptions on S – of a *maximal rational extension* $Q(S)$ of a given semigroup S (called commonly *maximal ring of quotients*, respectively *maximal semigroup of quotients*). The present paper considers $C(X)$ endowed with one or more of the operations of pointwise sum, product, infimum or supremum of functions and computes $Q(C(X))$ in each of these cases. It turns out that all these extensions – including the case of the ring $C(X)$ –

fit nicely in the framework of semicontinuous extended-real-valued functions defined on X, that they are rather closely interrelated and that they all turn up as sublattices of the MacNeille completion of the lattice $\bar{C}(X)$ of all continuous extended-real-valued functions defined on X.

The paper is organized as follows: Section 1 sets up terminology and notation and gives a few relevant facts on spaces, extended-real-valued functions and rational extensions of semigroups. In Section 2 we collect the necessary material on semicontinuous functions; this seemed necessary since the facts we need are rather scattered through the literature and sometimes stated, we feel, under unnecessarily restrictive conditions. Section 3 contains (the more algebraic) properties of certain well-behaved semicontinuous functions. Most of the material presented goes essentially back to Dilworth [Dl], he considered such functions – calling them normal – which were, additionally, finite-valued and bounded. Section 4 describes the maximal rational extension of the semigroup $C(X)$ with pointwise infimum as operation. This turns out to be a distributive lattice contained in the lattice of all normal (lower) semicontinuous functions on X, and related to the MacNeille completions of $C(X)$ and $\bar{C}(X)$. In Section 5, the effects of forming maximal rational extensions – successively – with respect to pointwise infimum and supremum as semigroup operations are studied. The key ingredient here is that of a normal semicontinuous function which is continuous on a dense open set. Finally, Section 6 deals with $C(X)$ as a multiplicative semigroup and as a ring. The maximal rational extension is seen to be a reduct of the maximal ring of quotients of $C(X)$, and both in their natural order form a sublattice of the MacNeille completion of $\bar{C}(X)$. We conclude by locating the MacNeille completions of the various maximal rational extensions obtained in the preceding sections.

1. Notations and basic facts

X always denotes a topological space, which is completely regular and Hausdorff unless otherwise stated. $\boldsymbol{R}$ stands for the reals in their usual topology. The set of *extended reals* is $\bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\}$, with additional order relations $-\infty < x < +\infty$ for all $x \in \boldsymbol{R}$. $\bar{\boldsymbol{R}}$ is topologized by taking all sets of the form $[-\infty, \alpha)$, (α, β), $(\beta, +\infty]$ with $\alpha, \beta \in \boldsymbol{R}$ as an open base. For any space X and point $x \in X$, $U(x)$ denotes the neighbourhood filter of x.

We will be concerned mainly with maps $f\colon X \to \boldsymbol{R}$ or $f\colon X \to \bar{\boldsymbol{R}}$ for some topological space X. The latter are called extended-real-valued or *numeric*, according to Bourbaki jargon. A map will be called *function* iff it satisfies one of the continuity properties considered below. We put

$$C(X) = \{f\colon X \to \boldsymbol{R};\ f \text{ continuous}\}$$

and

$$\bar{C}(X) = \{f\colon X \to \bar{\boldsymbol{R}};\ f \text{ continuous}\}.$$

$C(X)$ and $\bar{C}(X)$ denote just sets since different algebraic structures on them will be considered.

A map $f\colon X \to \bar{\boldsymbol{R}}$ is *lower semicontinuous* (lsc) iff $f^{-1}(\alpha, +\infty]$ is open for all $\alpha \in \boldsymbol{R}$; dually, *upper semicontinuous* (usc) iff $f^{-1}[-\infty, \alpha)$ is such for all $\alpha \in \boldsymbol{R}$. As usual,

$$Z(f) = \{x \in X;\ f(x) = 0\} \quad \text{and} \quad \operatorname{coz}(f) = \{x \in X;\ f(x) \neq 0\};$$

furthermore,

$$\operatorname{cont}(f) = \{x \in X;\ f \text{ is continuous at } x\} \quad \text{and} \quad \operatorname{disc}(f) = X \setminus \operatorname{cont}(f).$$

For any $\alpha \in \bar{\boldsymbol{R}}$ $\alpha \cdot 1$ stands for the constant function with value a where $1(x) = 1$ and $0(x) = 0$ for each $x \in X$.

Let $f\colon X \to \bar{\boldsymbol{R}}$ be any map, and suppose P is any local property which f may possess at any given point $x \in X$ (e.g., continuity). We will say that f *is almost-P* iff there exists a *dense open* set $U \subseteq X$ such that f has P at every point $x \in U$.

$C(X)$ is a ring under pointwise addition and multiplication, likewise, it is a lattice under pointwise order. $\bar{C}(X)$ is a lattice, too, under pointwise order and contains $C(X)$ as a sublattice, but ring operations do not extend from $C(X)$ to $\bar{C}(X)$ since, e.g., $0 \cdot (+\infty)$ is not defined. More generally, $\bar{\boldsymbol{R}}^X$ is a lattice under pointwise order; we will be concerned mainly with lattices L consisting of certain semicontinuous functions such that $C(X) \subseteq L \subseteq \bar{\boldsymbol{R}}^X$. If $f_i\colon X \to \bar{\boldsymbol{R}}$ are any maps ($i \in I$), then $\sup_i f_i$ and $\inf_i f_i$ will always mean the pointwise sup and inf taken in $\bar{\boldsymbol{R}}^X$ (which always exists since $\bar{\boldsymbol{R}}$ is a complete lattice); if the sup or inf is taken within some sublattice $L \subseteq \bar{\boldsymbol{R}}^X$ we will write $\sup_L$ and $\inf_L$. Given two lattices $L_1 \subseteq L_2$, L_1 is a *regular* sublattice of L_2 iff

$$\sup_{L_1,\, i} x_i = \sup_{L_2,\, i} x_i \qquad (x_i \in L_1,\ i \in I)$$

whenever the left-hand side exists and analogously for inf. $C(X)$ is clearly a regular sublattice of $\bar{C}(X)$, while $\bar{C}(X)$ is not a regular sublattice of $\bar{\boldsymbol{R}}^X$. For any lattice L and $a \in L$ we put

$$(a] = \{x \in L;\ x \leqslant a\} \quad \text{and} \quad [a) = \{x \in L;\ x \geqslant a\}.$$

$L^\wedge$ stands for the MacNeille completion of L.

Let L be any lattice. L is said to satisfy *join infinite distributivity* (*JID*) iff

$$x \wedge \sup_i y_i = \sup_i (x \wedge y_i)$$

for every set $\{y_i \in L;\ i \in I\}$ whose sup happens to exist in L. The dual condition is *meet infinite distributivity* (*MID*):

$$x \vee \inf_i y_i = \inf_i (x \vee y_i)$$

whenever the left-hand side exists.

Turning to $C(X)$, we note that for *any* topological space X the lattice $C(X)$ satisfies both *JID* and *MID*. This is a direct consequence of the fact that $C(X)$ equipped with pointwise addition is a l-group (see [Bi, XIII, § 14]). We proceed to show the analogous result for the lattices $\bar{C}(X)$ restricted to the class of spaces which is of interest here, namely, that of completely regular spaces. This must be done differently since $\bar{C}(X)$ is not a group under addition. We isolate the crucial step of the proof since it will be used repeatedly.

LEMMA 1.1. *Let $A \subseteq X$ be closed, and $p \notin A$. Then there exists $h \in \bar{C}(X)$ such that $h \geqslant 1$, $h(p) = 1$ and $h(x) = +\infty$ for $x \in A$.*

Proof. By complete regularity, there is $f \in C(X)$ such that $0 \leqslant f \leqslant 1$, $f(p) = 1$ and $f(x) = 0$ for $x \in A$. Define

$$h(x) = \begin{cases} 1/f(x), & x \in \operatorname{coz}(f), \\ +\infty, & x \in Z(f). \end{cases}$$

Since $p \in \operatorname{coz}(f)$ and $A \subseteq Z(f)$, h has the required properties and is obviously continuous. ∎

PROPOSITION 1.2. *$\bar{C}(X)$ satisfies both JID and MID.*

Proof. Assume that $\{s_\lambda\}_{\lambda \in \Lambda} \subseteq \bar{C}(X)$ is such that $s = \sup_{\bar{C}(X)} s_\lambda$ exists, and consider any $f \in \bar{C}(X)$. In order to obtain a contradiction to *JID*, we suppose that there is a $g \in \bar{C}(X)$ such that $f \wedge s > g \geqslant f \wedge s_\lambda$ for all $\lambda \in \Lambda$. $f \wedge s$ and g being continuous, there exists $\emptyset \neq U \subseteq X$ open and $\alpha, \beta \in \boldsymbol{R}$ satisfying $\alpha > \beta$ and $(f \wedge s)(x) > \alpha > \beta > g(x)$ for all $x \in U$. Adding a suitable constant function $\gamma \cdot 1$ to all functions involved, we may restrict ourselves to the case where $\beta = 0$. Select $p \in U$ and find, by Lemma 1.1, $h \in \bar{C}(X)$ satisfying $h \geqslant 1$, $h(p) = 1$ and $h(x) = +\infty$ for $x \in X \setminus U$. Now, for $x \in U$, we have $s_\lambda(x) \leqslant g(x)$; thus $s \wedge \frac{1}{2}(\alpha \cdot 1) h$ is a new upper bound for $\{s_\lambda\}_{\lambda \in \Lambda}$ strictly less than s. This is the desired contradiction and establishes *JID*. *MID* now follows from the fact that $\inf_\lambda s_\lambda = -\sup_\lambda(-s_\lambda)$ whenever it exists. ∎

Remark. We have not determined for what class of spaces $\bar{C}(X)$ satisfies *JID* and *MID*. Complete regularity is certainly not a necessary condition: For any indiscrete space X we have $\bar{C}(X) \cong \bar{\boldsymbol{R}}$, and $\bar{\boldsymbol{R}}$ has both *JID* and *MID*.

In the remainder of this section we list some basics on rational extensions. Let S and T be two commutative semigroups, $S \subseteq T$. T is a *rational* *extension* of S — or equivalently, S is *rationally dense* in T — iff for all t, $t_1 \neq t_2 \in T$ there exists $s \in S$ such that $st \in S$, $st_1 \neq st_2$. Under mild assumptions — always satisfied in this paper — S is a rational extension of itself, and then there exists a semigroup $Q(S)$ such that (i) $Q(S)$ is a rational extension of S and (ii) whenever T is any rational extension of S, then $T \subseteq Q(S)$. $Q(S)$ is called the *maximal rational extension* of S or the *rational completion* of S. Generally, any semigroup S with $S = Q(S)$ is called *rationally complete*, since $Q(S) = Q(Q(S))$ our use of "completion" is justified. Finally, if S is rationally dense in T and T is rationally complete, then $Q(S) = T$. The semigroups we are concerned with here are always certain sets of numeric functions endowed with one of the operations of pointwise sum, product, sup or inf, written $+$, $\cdot$, $\vee$ and $\wedge$. The rational completions of say, $C(X)$, with respect to one of these operations will be written as

$$Q(+)C(X), \quad Q(\cdot)C(X), \quad Q(\vee)C(X), \quad Q(\wedge)C(X),$$

respectively, and similarly for other semigroups of functions. In order to indicate unambiguously the semigroup operation in question, we shall also write (say) $(\wedge)$-rational extension or completion, and $(\wedge)$-rationally dense (and the like). There is one exception to this rule: The maximal ring of quotients of the ring $(C(X), +, \cdot)$ will be denoted by $Q(X)$ in order to stay consistent with [FGL].

We conclude with a brief sketch of the construction of $Q(S)$ from S. $D \subseteq S$ is a *dense ideal* iff $DS \subseteq S$ and S is a rational extension of D. A map $f\colon D \to S$ is a *S-homomorphism* iff $f(ds) = f(d)s$ for all $d \in D$, $s \in S$. Write $\mathrm{Hom}_S(D, S)$ for the collection of all S-homomorphisms from D into S and put

$$H_S = \bigcup \{\mathrm{Hom}_S(D, S);\ D \subseteq S \text{ is a dense ideal}\}.$$

For $f, g \in H_S$ define fg by $(fg)(x) = f(g(x))$ for all $x \in \mathrm{dom}(g)$ such that $g(x) \in \mathrm{dom}(f)$ (it must be shown, of course, that this set is a dense ideal in S). Moreover, put $f \equiv g$ iff $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ includes a dense ideal. $\equiv$ is a congruence on H_S with respect to the operation defined above, and $Q(S) \cong H_S/\equiv$. S embeds canonically into $Q(S)$ by assigning to each $x \in S$ the S-homomorphism $f_x\colon S \to S$ given by $f_x(s) = xs$ for all $s \in S$.

Finally, for any unexplained notions the reader is referred to [Bi] and [Gr] for lattices and to [La], [MM 1] and [We] for rational extensions. [GJ] and [Au] are the standard references for functions.

2. Semicontinuous functions

This section lists, in loose form, some of the pertinent properties of semicontinuous functions. Most of the facts given below are well known, at least in similar forms. Our standard reference is [Au].

2.1. We start by defining the *upper* and *lower limit functions* f^* and f_* of any given map $f\colon D\to\bar{R}$ where $D\subseteq X$ is dense (more generally, we might consider maps $f\colon S\to\bar{R}$ with S any subset of X, but this will not be needed for our purposes). For $x\in X$, define

$$f^*(x)=\inf_{U\in U(x)}\ \sup_{y\in U\cap D} f(y) \quad \text{and} \quad f_*(x)=\sup_{U\in U(x)}\ \inf_{y\in U\cap D} f(y).$$

Obviously, f^* and f_* are well defined maps from X to $\bar{R}$.

2.2. For any map $f\colon D\to\bar{R}$ ($D\subseteq X$ dense) $f^*\colon X\to\bar{R}$ is usc and $f_*\colon X\to\bar{R}$ is lsc (which justifies our use of "function" in 2.1). Moreover, for any map $g\colon X\to\bar{R}$, g is usc iff $g=g^*$ and lsc iff $g=g_*$. Hence, g is continuous iff $g_*=g=g^*$.

For proofs of these facts see [Au], 5.4.1.3, 5.4.3, 5.4.6.

2.3. Properties of the operations * and $_*$. Let $f, g\colon D\to\bar{R}$ be any maps, $D\subseteq X$ dense. Then:

(i) $f_*\leqslant f^*$ and for all $x\in D$, $f_*(x)\leqslant f(x)\leqslant f^*(x)$;

(ii) $f\leqslant g$ implies $f_*\leqslant g_*$ and $f^*\leqslant g^*$;

(iii) $(f^*)^*=f^*$, $(f_*)_*=f_*$;

(iv) $\big(((f^*)_*)^*\big)_*=(f^*)_*$, $\big(((f_*)^*)_*\big)^*=(f_*)^*$.

Proofs of these properties may be found in [Au], 5.4.1, 5.4.9.

2.4. Let $f\colon X\to\bar{R}$ be usc. Then f is the (pointwise) inf of $\{g\in\bar{C}(X);\ g\geqslant f\}$. Dually, any lsc function $f\colon X\to\bar{R}$ is the (pointwise) sup of $\{g\in\bar{C}(x);\ g\leqslant f\}$. In fact, each of these relations characterizes completely regular spaces . Proof. See [TG], IX § 1.

It follows that if f is usc and there exists $g\in C(X)$ such that $g\geqslant f$, then $f=\inf\{g\in C(X);\ g\geqslant f\}$. Dually, any lsc function f satisfies $f=\sup\{g\in C(X);\ g\leqslant f\}$ whenever this latter set is nonempty. More generally, the sup of any family and the inf of any finite family of lsc functions is again lsc. Dually, the inf of any family and the sup of any finite family of usc functions is usc.

Proofs, see [Au], 5.4.6.1.

2.5. The following is the central definition of this section. Assume $f\colon X\to\bar{R}$ is semicontinuous, that is, $f=f^*$ or $f=f_*$. Then f is called *normal* iff even $f=(f_*)^*$, respectively $f=(f^*)_*$. So the normal usc (resp. lsc) functions form subclasses of the class of all semicontinuous functions from X to $\bar{R}$, each containing all continuous functions. We have the following characterization of normal usc functions:

(*) *Let $f\colon X\to\bar{R}$ be usc. Then f is normal iff for every $\varepsilon>0$, every $x\in X$ and every $U\in U(x)$ there exists V open, $\emptyset\neq V\subseteq U$ such that $f(y)>f(x)-\varepsilon$ for all $y\in V$.*

Proof. See [Dl], 3.1.

The dual condition for lsc functions requires, of course, under the same hypotheses the existence of $\emptyset \neq V \subseteq U$ open such that $f(y) < f(x)+\varepsilon$ for all $y \in V$.

Normal usc, respectively lsc functions may be thought of as being "minimally discontinuous". This may be made precise in different ways, cf. [Au] 5.5.3.4, 5.5.8 for one possibility. Another is given in [Dl], 3.2:

(**) *An usc function f is normal iff for every $\alpha \in \mathbf{R}$ $f^{-1}(\alpha, +\infty]$ is a union if not of open sets — which would make f continuous — but at least of closures of open sets (for f lsc, $f^{-1}[-\infty, \alpha)$ must have this property).*

2.6. We pause to illustrate these concepts by an example, hereby showing that normal semicontinuous functions may still be pretty discontinuous. Let $X = \mathbf{R}$ and enumerate the rationals in a sequence $\{q_i\}_{i \in \mathbf{N}}$. Define $f\colon \mathbf{R} \to \bar{\mathbf{R}}$ by

$$f(\alpha) = \sum_{q_i < \alpha} 2^{-i}.$$

Clearly, $0 \leqslant f \leqslant 1$ and f is strictly increasing. f is continuous at each irrational α: $|f(\alpha)-f(\alpha')|$ may be made arbitrarily small by choosing α' such that the open interval between α and α' does not contain any member of a certain finite list of rational numbers. On the other hand, for any $q_i \in \mathbf{Q}$ and $q_i < \alpha \in \mathbf{R}$ we have $f(\alpha)-f(q_i) \geqslant 2^{-i}$, so f is discontinuous at each rational. It follows that disc(f) is dense in $\mathbf{R}$. f is, however, normal lsc: If $f(\alpha) > \beta$, then by the same arguments we may find $\alpha' < \alpha$ such that still $f(\alpha') > \beta$, hence $f^{-1}(\beta, +\infty]$ is open. For normality, use monotonicity of f and select, applying (*), any open interval contained in U to the left of a given point α ($U \in U(\alpha)$), or applying (**), observe that $f(\alpha) < \beta$ implies $f[-\infty, \alpha] \subseteq [-\infty, \beta)$.

2.7. cont(f), disc(f) **for** f **semicontinuous.** If $f\colon X \to \bar{\mathbf{R}}$ is semicontinuous, X an arbitrary space (not necessarily completely regular), the only thing that can be said about disc(f) in general is that this set is of first category, see [Če], 22 B 5. Additional hypotheses on the function or the space will produce the property we are interested in: (i) If X is arbitrary, but f is *normal*, then cont(f) is dense in X; (ii) if X is a *Baire* space, then *every* semicontinuous f has cont(f) dense in X, see [Au], 5.5.

2.8. Extending continuous functions. Let $D \subseteq X$ be dense, $f\colon D \to \bar{\mathbf{R}}$ continuous. We examine the possibilities of extending f to a semicontinuous function $\hat{f}\colon X \to \bar{\mathbf{R}}$. Without loss of generality — the usc case being entirely analogous — we restrict ourselves to lsc extensions.

SEPARATION LEMMA. *Assume g, h: $X \to \bar{\mathbf{R}}$ are lsc, g normal and $g(x_0) < h(x_0)$ for some $x_0 \in X$. Then there exist $\emptyset \neq V \subseteq X$ open and $\alpha, \beta \in \mathbf{R}$ satisfying $g(x_0) < \alpha < \beta < h(x_0)$ such that $g(y) < \alpha$ and $\beta < h(y)$ for all $y \in V$.*

Proof. Assume $g(x_0)$, $h(x_0)$ are finite (the cases where $h(x_0) = +\infty$ and/or $g(x_0) = -\infty$ are quite obvious). Put $\varepsilon = 3^{-1}(h(x_0) - g(x_0))$. Since h is lsc, we find $U \in U(x_0)$ such that $h(y) > h(x_0) - \varepsilon$ for all $y \in U$. g being normal we find, applying (*), V open, $\emptyset \neq V \subseteq U$ such that $g(y) < g(x_0) + \varepsilon$ for all $y \in V$. V together with $\alpha = g(x_0) + \varepsilon$, $\beta = h(x_0) - \varepsilon$ will do the job. ∎

Now let f: $D \to \bar{\mathbf{R}}$ be continuous, D dense. Consider $g = (f^*)_*$. g is normal lsc, and for $d \in D$ we have $g(d) = f(d)$. Suppose g' has the same properties as g. If $g(x_0) \neq g'(x_0)$ for some $x_0 \in X$, then $g(y) \neq g'(y)$ for all $y \in V$, V some nonempty open set by the preceding lemma. But this is impossible since g and g' agree on D which is dense. If h: $X \to \bar{\mathbf{R}}$ is lsc and extends f, then the same argument rules out that $g(x_0) < h(x_0)$ for some $x_0 \in X$. Summing up, we have:

EXTENSION LEMMA. *Let f: $D \to \bar{\mathbf{R}}$ be continuous, $D \subseteq X$ dense. Then there exists a unique normal lsc function g: $X \to \bar{\mathbf{R}}$ extending f. g equals $(f^*)_*$ and is the maximal — in the pointwise order — lsc function extending f. Dually, $h = (f_*)^*$ is the unique normal usc function extending f, and it is the minimal usc extension of f to X.*

3. Lattices of normal semicontinuous functions

We denote by $NLSC(X)$ and $NUSC(X)$, respectively, the collections of all normal lsc, respectively usc, functions f: $X \to \bar{\mathbf{R}}$ defined on a (completely regular) space X. If clear from the context, (X) will be dropped. Equipped with the pointwise order, these sets become lattices which we will denote by the same symbols.

PROPOSITION 3.1 (Dilworth). *$NLSC(X)$ and $NUSC(X)$ are complete lattices. Lattice operations are given as follows:*
For $A \subseteq NLSC$,

$$\sup_{NLSC} A = ((\sup A)^*)_* \quad \text{and} \quad \inf_{NLSC} A = (\inf A)_*;$$

for $A \subseteq NUSC$,

$$\sup_{NUSC} A = (\sup A)^* \quad \text{and} \quad \inf_{NUSC} A = ((\inf A)_*)^*.$$

Proof. As in [Dl], 4.2. ∎

By 2.3, *: $NLSC \to NUSC$ and $_*$: $NUSC \to NLSC$ are order-preserving maps, and since their arguments are restricted here to normal semicontinuous functions, $^*{}_* = 1_{NLSC}$, $_*{}^* = 1_{NUSC}$. Let $A \subseteq NLSC$, $x = \sup_{NLSC} A$. Hence $x^* \geqslant a_\lambda^*$ for all $a_\lambda \in A$. If $z \in NUSC$, $z \geqslant a_\lambda^*$ for all $a_\lambda \in A$, then

$z_* \geqslant (a_\lambda^*)_* = a_\lambda$, whence $z_* \geqslant x$ and thus $(z_*)^* \geqslant x^*$. But $(z_*)^* = z$, so $x^* = \sup_{NUSC} \{a_\lambda^*;\ a_\lambda \in A\}$. This essentially proves

PROPOSITION 3.2. *NLSC and NUSC are isomorphic lattices, and* $*$: $NLSC \to NUSC$, $_*$: $NUSC \to NLSC$ *are inverse complete lattice isomorphisms.*

Instead of investigating directly the algebraic properties of the lattices *NLSC* and *NUSC*, we will establish a representation theorem which essentially goes back to Dilworth ([DI], Section 6). Our setting is somewhat more general (an arbitrary completely regular space instead of a compact Hausdorff space, numeric functions instead of real-valued bounded functions). The proof is facilitated by the use of a particular construction of the projective cover of a completely regular space, due to Papert–Strauss [PS]. We summarize a few pertinent facts about this latter construction.

Let X be a (completely regular) space. An *open filter* on X is a filter in the lattice of all open subsets of X. An *open ultrafilter* is a maximal proper open filter. An open ultrafilter F on X *converges* to a point $x \in X$ iff F contains every open neighbourhood of x; obviously, F converges to at most one point. The *projective cover PX of X* is the space with carrier set consisting of all convergent open ultrafilters on X and with a (completely regular) topology defined by taking all sets of the form

$$W(U) = \{F \in PX;\ U \in F\} \qquad \text{for } U \subseteq X \text{ open}$$

as an open base. The natural projection p: $PX \to X$ is given by assigning to every $F \in PX$ its unique limit point. p is onto, continuous, closed, *compact* (i.e., $p^{-1}(x)$ is compact for every $x \in X$) and *proper* (that is, p maps any closed proper subset of PX onto a closed proper subset of X). Finally, PX is an *extremally disconnected* space, that is, the closure of any open subset of PX is open.

THEOREM 3.3 (essentially Dilworth). *For any completely regular space X, the lattice $NLSC(X)$ is isomorphic with the lattice $\bar{C}(PX)$.*

Proof. For any map f: $X \to \bar{\boldsymbol{R}}$ define $\hat{f}$: $PX \to \bar{\boldsymbol{R}}$ by $\hat{f} = f \circ p$.

(i) *If f is lsc, then $\hat{f}^*$ is continuous.*

Indeed, for any $\alpha \in \boldsymbol{R}$, $\hat{f}^{-1}(\alpha, +\infty] = p^{-1}(f^{-1}(\alpha, +\infty])$ is open, since p is continuous and f is lsc. Hence $\hat{f}$ is lsc. Now $\hat{f}^* = (\hat{f}_*)^* = \hat{f}_*{}^*{}_*{}^* = (\hat{f}^*)_*{}^*$, that is, $\hat{f}^*$ is normal usc. Hence $\hat{f}^{*-1}[-\infty, \alpha)$ is open for any $\alpha \in \boldsymbol{R}$, and $\hat{f}^{*-1}(\alpha, +\infty]$ is a union of closures of open sets by criterion (**) in 2.5. But PX is extremally disconnected, so the latter set is actually open and $\hat{f}^*$ is continuous.

(ii) *If f, g: $X \to \bar{\boldsymbol{R}}$ are normal lsc functions, then $f \leqslant g$ iff $\hat{f}^* \leqslant \hat{g}^*$. Hence, $f \neq g$ implies $\hat{f}^* \neq \hat{g}^*$.*

Indeed, $f \leqslant g$ obviously implies $\hat{f} \leqslant \hat{g}$ and thus $\hat{f}^* \leqslant \hat{g}^*$. For the converse, assume $f \nleqslant g$, that is, $f(x_0) > g(x_0)$ for some $x_0 \in X$. By the Separation Lemma (2.8) we find $\emptyset \neq V \subseteq X$ open and $\alpha, \beta \in \boldsymbol{R}$ such that $g(y) < \alpha < \beta < f(y)$ whenever $y = x_0$ or $y \in V$. X is a regular space, so we find $\emptyset \neq V_1$ open satisfying $V_1 \subseteq \operatorname{cl} V_1 \subseteq V$. Consider any $F \in PX$ with $V_1 \in F$. Hence $p(F) \in \operatorname{cl} V_1$, and so $\hat{g}(F) < \alpha$, $\hat{f}(F) > \beta$. Select some $F_1 \in PX$ with $V_1 \in F_1$. $W_1 = \{F \in PX;\ V_1 \in F\}$ is then an open neighbourhood of F_1. We obtain

$$\hat{g}^*(F_1) = \inf_{W \in U(F_1)} \sup_{F \in W} \hat{g}(F) = \inf_{W \in U(F_1)} \sup_{F \in W \cap W_1} \hat{g}(F) < \alpha$$

(since $\sup_{F \in W \cap W_1} \hat{g}(F) \leqslant \sup_{F \in W} \hat{g}(F)$ and $W \cap W_1 \in U(F_1)$) and

$$\hat{f}^*(F_1) = \inf_{W \in U(F_1)} \sup_{F \in W} \hat{f}(F) = \inf_{W \in U(F_1)} \sup_{F \in W \cap W_1} \hat{f}(F) \geqslant \beta.$$

Hence $\hat{g}^*(F_1) < \hat{f}^*(F_1)$, that is $\hat{f}^* \nleqslant \hat{g}^*$.

(iii) *Let* $h \in \bar{C}(PX)$. *For* $x \in X$, *define* $f(x) = \inf\{h(F);\ p(F) = x\}$. *Then* $f \in NLSC(X)$ *and* $\hat{f}^* = h$.

Indeed, given $x_0 \in X$ and $\varepsilon > 0$, we will find $U \in U(x_0)$ such that $f(y) > f(x_0) - \varepsilon$ for all $y \in U$, thus proving that f is lsc. Since p is compact, h attains its minimum on $p^{-1}(x_0)$. Pick $F_0 \in PX$ with $h(F_0) = f(x_0)$. Since h is continuous,

$$W_0 = \{F \in PX;\ h(F) > f(x_0) - \varepsilon\}$$

is an open neighbourhood of F_0. Since p is closed, $U := X \setminus p[PX \setminus W_0] \subseteq X$ is open and $x_0 \in U$. Consider any $y \in U$: For an arbitrary $F \in p^{-1}(y)$ we have $F \in W_0$, thus $h(F) > f(x_0) - \varepsilon$; h attains its minimum on $p^{-1}(y)$, so we obtain that $f(y) > f(x_0) - \varepsilon$.

It remains to prove the normality of f. We will find, for any given $U \in U(x_0)$ open and $\varepsilon > 0$, some $\emptyset \neq V \subseteq U$ open such that $f(y) < f(x_0) + \varepsilon$ for all $y \in V$. Let $W = p^{-1}[U]$ and put $W_1 = \{F \in W;\ h(F) < f(x_0) + \varepsilon\}$. W_1 is open since h is continuous, and nonempty since $F_0 \in W_1$. So $PX \setminus W_1$ is a closed proper subset of PX, and because p is closed and proper, $U_1 := X \setminus p[PX \setminus W_1] \subseteq X$ is open and nonempty. Since $W_1 \subseteq W$,

$$U_1 \subseteq X \setminus p[PX \setminus W] = X \setminus p[PX \setminus p^{-1}[U]] = U.$$

Consider any $y \in U_1$: For an arbitrary $F \in p^{-1}(y)$ we have $h(F) < f(x_0) + \varepsilon$, whence $f(y) < f(x_0) + \varepsilon$. In order to obtain $\hat{f}^* = h$, observe first that for any given $F_0 \in PX$ we have

$$\hat{f}(F_0) = (pF_0) = \inf\{h(F);\ pF = pF_0\} \leqslant h(F_0).$$

Hence $\hat{f} \leqslant h$ and consequently $\hat{f}^* \leqslant h^* = h$. Suppose there is $F_0 \in PX$ such that $\hat{f}^*(F_0) < h(F_0)$. $\hat{f}^*$ and h both being continuous, there exists

$\emptyset \neq W_0 \subseteq PX$ open such that $\hat{f}^*(F) < \alpha < \beta < h(F)$ for all $F \in W_0$ with suitable $\alpha, \beta \in \boldsymbol{R}$. Using closedness and properness of p, we see that $W_1 = p^{-1}[X \setminus p[PX \setminus W_0]]$ is a nonempty open subset of W such that $F \in W_1$, $F' \in PX$ and $p(F) = p(F')$ imply that $F' \in W_1$. Select $F_1 \in W_1$:

$$\hat{f}^*(F_1) = \inf_{W \in U(F_1)} \sup_{F \in W} \hat{f}(F) = \inf_{W \in U(F_1)} \sup_{F \in W \cap W_1} \hat{f}(F)$$

(as in part (ii) above), but for any $F \in W_1$,

$$\hat{f}(F) = \inf\{h(F');\ p(F') = p(F)\} \geqslant \beta,$$

since $p(F') = p(F)$ implies $F' \in W_1$. It follows that $\hat{f}^*(F_1) \geqslant \beta$, contradicting $\hat{f}^*(F) < \alpha$.

Summing up, the assignment $f \mapsto \hat{f}^*$ maps $NLSC(X)$ bijectively onto $\bar{C}(PX)$ and preserves order in both directions, so these lattices are isomorphic. ∎

Apart from its intrinsic interest, Theorem 3.3 yields immediately.

COROLLARY 3.4. *$NLSC(X)$ and $NUSC(X)$ are distributive lattices satisfying both JID and MID.*

Proof: This is a direct consequence of Proposition 1.2.

We shall use Corollary 3.4 in order to obtain information on certain sublattices of $NLSC$, respectively $NUSC$. We concentrate on normal lsc functions, understanding that analogous results hold in the usc case.

Let $f \in NLSC$. $\mathrm{Cont}(f)$ is then dense in X (2.7), but this does not exclude the possibility that $\mathrm{disc}(f)$ is also dense – see the example given in 2.6. According to our general terminology, we call any map $f\colon X \to \bar{\boldsymbol{R}}$ *almost continuous* (abbreviated a.c.) iff $\mathrm{cont}(f)$ contains a dense open set. This is equivalent to $\mathrm{disc}(f)$ being nowhere dense, the case which we are interested in.

LEMMA 3.5. *Let f, g be a.c. Then $f \wedge g$, $f \vee g$, f^* and f_* are also a.c.*

Proof. $\mathrm{cont}(f \vee g) \supseteq \mathrm{cont}(f) \cap \mathrm{cont}(g)$, and since the intersection of two dense open sets is dense open, $f \vee g$ must be a.c.; similarly for $f \wedge g$. Moreover, $\mathrm{cont}(f) \subseteq \mathrm{cont}(f^*) \cap \mathrm{cont}(f_*)$ which proves the rest. ∎

Define

$$AC(X) = \{f \in NLSC(X);\ f \text{ is almost continuous}\}.$$

COROLLARY. 3.6. *$AC(X)$ is a sublattice of $NLSC(X)$ containing $C(X)$. The natural embedding $AC(X) \hookrightarrow NLSC(X)$ is sup-regular; hence, $AC(X)$ satisfies JID.*

Proof. Only the last assertions need checking. Consider any subset $B \subseteq AC(X)$ such that $s = \sup_{AC} B$ exists. Suppose there is $f \in NLSC(X)$ such that $B \leqslant f < s$. The Separation Lemma (2.8) provides $\emptyset \neq V \subseteq X$ open and

$\alpha, \beta \in \boldsymbol{R}$ with $f(y) < \alpha < \beta < s(y)$ for all $y \in V$. Use Lemma 1.1 to produce $f_0 \in \bar{C}(X)$ such that $f_0 \geqslant \alpha \cdot 1$, $f \equiv +\infty$ on $X \setminus V$ and $f(p) = \alpha$ for at least one point $p \in V$. Then $(s \wedge f_0)_* \in AC$, $(s \wedge f_0)_* \geqslant f \geqslant B$ but $(s \wedge f_0)_* < s$, contradicting $s = \sup_{AC} B$. Hence $s = \sup_{NLSC} B$. Now any instance violating JID in AC would via embedding in $NLSC$ violate JID in $NLSC$ which is impossible by Corollary 3.4. ∎

4. $Q(\wedge)C(X)$

Our description of $Q(\wedge)C(X)$ will be based on [Ba–Sch], [Sch 1] and [Sch 2], where for an arbitrary distributive lattice L its maximal $\wedge$-rational extension $Q(\wedge)L$ is constructed in terms of certain ideals of L and shown to be a distributive lattice. We review, therefore, a few basic notions on lattice ideals relevant in our context.

Let L be any lattice, $A \subseteq L$ any subset of L. We write

$$A^l = \{x \in L;\ x \leqslant a \text{ for all } a \in A\}$$

and dually

$$A^u = \{x \in L;\ x \geqslant a \text{ for all } a \in A\}.$$

An ideal $J \subseteq L$ is called *normal* iff $J = J^{ul}$. J is called *complete* provided $\sup_L G \in J$ whenever $G \subseteq J$ is such that $\sup_L G$ exists. It is immediate from these definitions that every normal ideal is complete. We aim to characterize complete ideals in the lattices $C(X)$ and $\bar{C}(X)$ by means of

LEMMA 4.1. *Let $f \in NLSC(X)$ and put $\bar{J}_f = \{h \in \bar{C}(X);\ h \leqslant f\}$. Then $\bar{J}_f \subseteq \bar{C}(X)$ is a complete ideal and $\sup \bar{J}_f$ (pointwise!) equals f.*

Proof. Let $h_i \in \bar{J}_f$ $(i \in I)$ and suppose $h = \sup_{\bar{C}(X)} h_i$ exists. If $h(x_0) > f(x_0)$ for some $x_0 \in X$, then the Separation Lemma (2.8) provides us with $\emptyset \neq V \subseteq X$ open and $\alpha, \beta \in \boldsymbol{R}$ such that $f(y) < \alpha < \beta < h(y)$ for all $y \in V$. Pick $p \in V$. Using Lemma 1.1, we construct $g \in \bar{C}(X)$ such that $g \geqslant \alpha \cdot 1$, $g(p) = \alpha$ and $g \equiv +\infty$ on $X \setminus V$. Consider $h \wedge g$: $h \wedge g \geqslant h_i$ (all $i \in I$) but $(h \wedge g)(p) = \alpha < h(p)$, hence h cannot be the sup of the h_i taken in $\bar{C}(X)$. This contradiction proves $h \leqslant f$, that is, $h \in \bar{J}_f$ and $\bar{J}_f$ is thus complete. The second part of the assertion is clear by 2.4. ∎

The converse to Lemma 4.1 is also true:

LEMMA 4.2. *Suppose $J \subseteq \bar{C}(X)$ is a complete ideal. Then $f = \sup J$ (pointwise!) is normal lsc and $J = \bar{J}_f = \{h \in \bar{C}(X);\ h \leqslant f\}$.*

Proof. f is lsc (2.4). Put $g = (f^*)_*$. Then $f \leqslant g$ and g is normal lsc. In view of the preceding lemma, it will suffice to show that $J = \bar{J}_g$. Consider $h \in \bar{J}_g$. Put $J_0 = \{j \in J;\ j \leqslant h\}$. Suppose there exists $h_0 \in \bar{C}(X)$ such that

$h_0 < h$ but $h_0 \geqslant J_0$. Then for some $\emptyset \neq V \subseteq X$ open and $\alpha, \beta \in \boldsymbol{R}$, $h_0(y) < \alpha < \beta < h(y)$ for all $y \in V$. Since $f = \sup J$, we conclude $f(y) \leqslant h_0(y)$ for all $y \in V$. Hence $f(y) < \alpha < \beta < g(y) = (f^*)_*(y)$ for $y \in V$ which is impossible by the definitions of * and $_*$. Hence, h_0 as considered will not exist and h is the sup – in $\bar{C}(X)$ – of $J_0 \subseteq J$. J being complete, this implies $h \in J$. But $h \in \bar{J}_g$ was arbitrary, thus $\bar{J}_g \subseteq J$ and consequently $J = \bar{J}_g$ as claimed. ∎

PROPOSITION 4.3. *For a completely regular space X, the lattice of all complete ideals in $\bar{C}(X)$ – with set inclusion as order – is isomorphic with $NLSC(X)$ and $NUSC(X)$.*

Proof. Lemmata 4.1, 4.2 and Proposition 3.2. Note that the empty ideal is not complete in $\bar{C}(X)$. ∎

Remark. With the obvious adaptations, Lemmata 4.1 and 4.2 in [DI] show that normal lsc functions $X \to \bar{\boldsymbol{R}}$ are in bijective correspondence with *normal* ideals in $\bar{C}(X)$ – whence the nomenclature. We infer that in $\bar{C}(X)$ normal and complete ideals coincide. Further, it follows that $NLSC(X)$ and $NUSC(X)$ provide isomorphic copies of the MacNeille completion $\bar{C}(X)^\wedge$ of $\bar{C}(X)$. Indeed, a "symmetric" realization of $\bar{C}(X)^\wedge$ may be obtained as follows: Call a pair (f, g) of semicontinuous functions *conjugate* iff $f \in NLSC(X)$, $g \in NUSC(X)$ and $f^* = g$, $g_* = f$. Then $\bar{C}(X)$ is isomorphic to the set of all conjugate pairs endowed with the following operations $\curlywedge$ and $\curlyvee$:

$$(f_1, g_1) \curlywedge (f_2, g_2) = \big((f_1 \wedge f_2)_*, ((f_1 \wedge f_2)_*)^*\big)$$
$$(f_1, g_1) \curlyvee (f_2, g_2) = \big(((g_1 \vee g_2)^*)_*, (g_1 \vee g_2)^*\big).$$

Turning to $C(X)$, we note that the collections of normal, respectively complete, ideals do not coincide, since $C(X)$ lacks universal bounds. Indeed, consider any continuous function $f\colon X \to \bar{\boldsymbol{R}}$ such that $f(x) > -\infty$ for all $x \in X$ and $f(x_0) = +\infty$ for at least one $x_0 \in X$. The principal ideal $(f]$ in $\bar{C}(X)$ is clearly complete, and since the natural embedding $C(X) \hookrightarrow \bar{C}(X)$ is regular, $J = (f] \cap C(X)$ is also complete, and nonempty (containing, e.g., $f \wedge \alpha \cdot 1$ for any $\alpha \in \boldsymbol{R}$). But J^u – taken in $C(X)$ – is empty, whence the last normal ideal – taken in $C(X)$ – which contains J equals $C(X)$. But if $f(x) \neq +\infty$ for some x in X, $J \neq C(X)$ and J is not normal then. The next lemma shows every complete ideal in $C(X)$ is, at least, the trace of a complete ideal in $\bar{C}(X)$.

LEMMA 4.4. *Let $J \subseteq C(X)$ be a complete ideal, and J' the complete ideal generated by J in $\bar{C}(X)$. Then $J = J' \cap C(X)$.*

Proof. Since $\bar{C}(X)$ satisfies JID (1.2), J' consists of all suprema existing in $\bar{C}(X)$ of functions in $\bar{C}(X)$ which are majorized by some member of J. Consider $f \in J' \cap C(X)$. Since $f = \sup_{\bar{C}(X)} f_i$ and $f_i \leqslant g_i$ with $g_i \in J$ $(i \in I)$, we have

$$f = \sup_i{}_{\bar{C}(X)}(f \wedge g_i) = \sup_i{}_{C(X)}(f \wedge g_i)$$

(recall the embedding $C(X) \hookrightarrow \bar{C}(X)$ is regular), whence $f \in J$ by the completeness of J. Thus $J' \cap C(X) \subseteq J$. The converse inclusion is immediate by regularity: $J' \cap C(X)$ is a complete ideal in $C(X)$ whenever J' is such in $\bar{C}(X)$. Note that the empty ideal is complete in $C(X)$. ∎

LEMMA 4.5. *An ideal $J \subseteq C(X)$ is complete iff it is of the form $J = J_f = \{g \in C(X);\ g \leqslant f\}$ for some $f \in NLSC(X)$. Moreover, $\emptyset \neq J_{f_1} = J_{f_2}$ implies $f_1 = f_2$.*

Proof. In view of the preceding lemmata, only the last assertion needs verification. Consider $g \in C(X)$ such that $g \leqslant f_i \in NLSC$ for $i = 1, 2$. If $f_1 \neq f_2$, then w.l.o.g. $f_1(x_0) < f_2(x_0)$ for some $x_0 \in X$. The Separation Lemma yields $\emptyset \neq V \subseteq X$ open and $\alpha, \beta \in \boldsymbol{R}$ with $f_1(y) < \alpha < \beta < f_2(y)$ for all $y \in V$. Hence

$$\alpha_0 = \sup_{y \in V} g(y) \leqslant \alpha.$$

Select $p \in V$ such that $\alpha_0 - g(p) < (\beta - \alpha)/2$ and use complete regularity of X to obtain $h \in C(X)$ satisfying $0 \leqslant h \leqslant h(\beta - \alpha_0) \cdot 1$, $h \equiv 0$ on $X \setminus V$ and $h(p) = \beta - \alpha_0$. Then $g + h \in C(X)$, $g + h \leqslant f_2$ but $g + h \nleqslant f_1$ whence $J_{f_1} \neq J_{f_2}$. ∎

We agree to call any semicontinuous function $f\colon X \to \bar{\boldsymbol{R}}$ *lower C-bounded* (resp. *upper C-bounded*) iff there exists $g \in C(X)$ such that $g \leqslant f$ (resp. $g \geqslant f$), *C-bounded* has the natural meaning. It is clear from these definitions that if f, g are lower C-bounded, then also $f \vee g$, $f \wedge g$, f^* and f_* are such (and analogously for upper C-boundedness). In particular, lattice operations in either $NLSC$ or $NUSC$ take C-bounded functions of either type again to such functions.

The following proposition sums up our preceding discussion of complete ideals in $C(X)$:

PROPOSITION 4.6. *For a completely regular space X, the lattice of all nonempty complete ideals in $C(X)$ — with set inclusion as order — is isomorphic to the sublattice of $NLSC(X)$ consisting of all lower C-bounded functions in $NLSC(X)$.*

Remark. The standard application of the Separation Lemma (2.8) will easily show at this point that an ideal $J \subseteq C(X)$ is normal iff $J = \emptyset$, $J = C(X)$ or J is a complete ideal satisfying $J^u \neq \emptyset$. This gives a description of the MacNeille completion $C(X)^\wedge$ of $C(X)$ analogous to that established for $\bar{C}(X)^\wedge$ in the remark following Proposition 4.3: $C(X)^\wedge$ is isomorphic to the set of all conjugate pairs (f, g) of C-bounded normal semicontinuous functions, together with pairs $(-\infty\cdot 1, -\infty\cdot 1)$ and $(+\infty\cdot 1, +\infty\cdot 1)$, endowed with the same operations $\wedge$, $\vee$ (if one does not insist on "symmetric" lattice operations – with respect to their definition in terms of pointwise inf and sup, * and $_*$ – a simpler realization of $C(X)^\wedge$ is provided by the set of all C-bounded functions in $NLSC$ together with $+\infty\cdot 1$, $-\infty\cdot 1$ in their natural order).

To complete the groundwork for our description of $Q(\wedge)C(X)$, we need one last definition. If L is any lattice and $J\subseteq L$ any ideal, the *principal extension* PJ of J is defined by

$$PJ = \{x\in L;\ (x]\cap J \text{ is a principal ideal}\}.$$

If L is distributive, PJ clearly is an ideal in L containing J. For $L=C(X)$, principal extensions of *complete* ideals are easy to describe:

LEMMA 4.7. *Let $J\subseteq C(X)$ be a nonempty complete ideal, and $f\in C(X)$. Then $f\in PJ$ iff $f\wedge \sup J\in C(X)$.*

Proof. Put $g=\sup J$. By 4.5, $g\in NLSC$ and $J=J_g=\{h\in C(X);\ h\leqslant g\}$. Now for any $f\in C(X)$, $J\cap(f]=J_{f\wedge g}$. This is a *principal* ideal in $C(X)$ iff $f\wedge g\in C(X)$; indeed, since g and thus $f\wedge g$ are lower C-bounded, $f\wedge g$ is the pointwise sup of all functions in $C(X)$ contained in $f\wedge g$ (2.4). ∎

By [Ba–Sch], pp. 341 ff., $Q(\wedge)L$ is a distributive lattice whenever L is such. The structure of $Q(\wedge)L$ for L a distributive lattice has been elaborated on in [Sch 1] and [Sch 2]. We cite [Sch 1]: Theorem 6.5 together with Lemma 6.2 there show that in the presence of JID $Q(\wedge)L$ actually is isomorphic to the set of all complete ideals $J\subseteq L$ having the additional property that PJ is join-dense in L (under set inclusion as order). (For a slightly different approach, see [Sch 2], pp. 683 ff.). Since $C(X)$ has JID, this covers our situation, and by 4.6 we are left with the task of formulating the property "PJ is join-dense in $C(X)$" in terms of semicontinuous functions. Recall that in any lattice L a subset $D\subseteq L$ is *join-dense* (in L) provided for each $x\in L$,

$$x=\sup_L\{d\in D;\ d\leqslant x\}.$$

LEMMA 4.8. *An ideal $D\subseteq C(X)$ is join-dense in $C(X)$ iff for any $\alpha\in\mathbf{R}$ the set $D_\alpha=\{x\in X;\ d(x)>\alpha$ for some $d\in D\}$ is dense in X.*

Proof. Assume the condition stated is satisfied, and consider any $f\in C(X)$. Choose $x_0\in X$ and $\alpha>f(x_0)$. Since $f\in C(X)$, there is an open neighbourhood V of x_0 such that $f(y)<\alpha$ for all $y\in V$. For each $z\in D_\alpha\cap V$, pick $d_z\in D$ satisfying $d_z(z)>\alpha$. It follows that $f\wedge d_z\in D$ and $(f\wedge d_z)(z)=f(z)$ for all $z\in D_\alpha\cap V$. Now if $h\in C(X)$ and $h\geqslant(f]\cap D$, then $h(z)\geqslant f(z)$

for all $z \in D_\alpha \cap V$. But $D_\alpha \cap V$ is dense in V, whence $h(x_0) \geqslant f(x_0)$. Since x_0 was arbitrary, we conclude that $h \geqslant f$ and thus

$$f = \sup_{C(X)} ((f] \cap D).$$

Conversely, assume D_α is not dense for some $\alpha \in \boldsymbol{R}$. We find $\emptyset \neq V \subseteq X$ open such that $d(y) \leqslant \alpha$ for all $d \in D$ and $y \in V$. By complete regularity there is $h \in C(X)$ such that $0 \geqslant h \geqslant -1$, $h \equiv 0$ on $X \setminus V$ and $h(y_0) = -1$ for some point $y_0 \in V$. Denote $g = (\alpha+1) \cdot 1$. Then $(g+h] \cap D = (g] \cap D$, so g is not the sup – in $C(X)$ – of its predecessors in D. ■

LEMMA 4.9. *Let $J \subseteq C(X)$ be a complete ideal. Then PJ is join-dense in $C(X)$ iff* $\sup J$ *is lower C-bounded and almost continuous.*

Proof. Since $P\emptyset = \emptyset$ and $\emptyset$ is not join-dense in $C(X)$, J must be nonempty. Equivalently (4.6), $g = \sup J$ is normal lsc and lower C-bounded (by $g_0 \in C(X)$, say). Assume first that g is almost continuous. So there exists $V \subseteq X$ dense open with $V \subseteq \operatorname{cont}(g)$. Select any point $p \in V$ and any $\alpha \in \boldsymbol{R}$. We will construct $f \in PJ$ with $f(p) > \alpha$; by 4.8 then, PJ will be join-dense in $C(X)$. If $g_0(p) > \alpha$, put $f = g_0$ and we are done since $g_0 \in J \subseteq PJ$. If not, we may assume $g_0(p) < \alpha$ w.l.o.g. since α was arbitrary, especially, arbitrarily large. Put $\beta = \alpha - g_0(p)$ and choose $h \in C(X)$ with $0 \leqslant h \leqslant 2\beta \cdot 1$, $h \equiv 0$ on $X \setminus V$ and $h(p) = 2\beta$. Put $f = g_0 + h$. Thus $f \in C(X)$ and $f(p) > \alpha$. Consider $f \wedge g$: On $X \setminus V$, $f \wedge g$ agrees with g_0 which is continuous, and on V, $f \wedge g$ is continuous since $V \subseteq \operatorname{cont}(g)$. Thus $f \wedge g \in C(X)$ and $f \in PJ$ by 4.7.

For the converse assume that $g = \sup J$ is not almost continuous. We find $\emptyset \neq U \subseteq X$ open such that $\operatorname{disc}(g) \cap U$ is dense in U. Consider an arbitrary $f \in C(X)$. If for some $p \in U$ we have $f(p) > g(p)$, the Separation Lemma (2.8) produces $\emptyset \neq V \subseteq U$ open and $\alpha, \beta \in \boldsymbol{R}$ with $g(y) < \alpha < \beta < f(y)$ for all $y \in V$. Hence, $g \wedge f$ and g agree on V and $g \wedge f$ is thus not continuous. But now Lemma 4.7 implies that $f(p) \leqslant g(p)$ for all $p \in U$ whenever $f \in PJ$. Pick $s \in \operatorname{disc}(g) \cap U$. It follows that $g(s) =: \beta < +\infty$. So for any $\varepsilon > 0$, there exists $\emptyset \neq W_\varepsilon \subseteq U$ open such that $g(p) < \beta + \varepsilon$ for all $p \in W_\varepsilon$, since g is normal lsc. It follows that for any $f \in PJ$ we must have $f \leqslant (\beta+\varepsilon) \cdot 1$ on W_ε; but then $D_\alpha \cap W_\varepsilon = \emptyset$ whenever $\alpha > \beta + \varepsilon$. By 4.8 then PJ is seen not to be join-dense in $C(X)$ which completes the proof. ■

We are now ready to state the main result of this section:

THEOREM 4.10. *$Q(\wedge)C(X)$ is isomorphic with the sublattice of $AC(X)$ consisting of all lower C-bounded functions in $AC(X)$.*

Proof. Combine 3.6, 4.6 and 4.9. ■

5. $Q(\wedge)C(X)$ versus $Q(\vee)C(X)$

Since $f \mapsto -f$ defines a lattice antiisomorphism of $C(X)$ onto itself, $C(X)$ is a selfdual lattice (and so is $\bar{C}(X)$). Hence, in order to obtain $Q(\vee)C(X)$, we may dualize easily all notions and constructions used so far; naturally, lsc functions will be replaced by usc functions, lower C-bounded ones by upper C-bounded ones and so on. As expected, this results in a description of $Q(\vee)C(X)$ as the lattice of all normal usc, upper C-bounded almost continuous functions, in complete analogy with Theorem 4.10. For a better comparison with $Q(\wedge)C(X)$, it is however desirable to have a description of $Q(\vee)C(X)$ based on lsc functions. Now observe that $_*$ preserves normality, almost continuity and upper C-boundedness. We obtain

THEOREM 5.1. *$Q(\vee)C(X)$ is the dual of $Q(\wedge)C(X)$; explicitly, $Q(\vee)C(X)$ is the sublattice of $AC(X)$ consisting of all upper C-bounded functions in $AC(X)$.*

The obvious question at this point is: What happens if we start out with $C(X)$ and apply alternatively the $Q(\wedge)$ – and $Q(\vee)$ – operators? As we shall see, this process will become stationary after two steps; moreover, it doesn't matter whether we start with $Q(\wedge)$ or with $Q(\vee)$.

We consider $Q(\vee)C(X)$ in its lsc version. To ease notation, put

$$L = \{f \in AC(X);\, f \text{ upper } C\text{-bounded}\}.$$

We aim to describe $Q(\wedge)L$. Now it is immediate from 3.6 that L satisfies JID, so – see the discussion after Lemma 4.7 – our task reduces again to a characterization of those complete ideals J in L which have PJ join-dense in L. Fortunately, the desired result as well as the proofs are almost identical with the material developed in the previous section.

We claim that an ideal $J \subseteq L$ is complete iff $J = \bar{J}_f = \{h \in L,\, h \leqslant f\}$ for some normal lsc function $f\colon X \to \bar{\mathbf{R}}$. Indeed, the proofs of Lemmata 4.1 and 4.2 work with only trivial changes (in 4.2, comparing h_1 and h now requires the Separation Lemma. Also, since the members of L are upper C-bounded, the proof of Lemma 4.8 works literally and gives the same characterization of join-dense (in L) ideals. As for (the analogue of) Lemma 4.9, the first half of the proof carries over to our new setting almost unchanged (for g_0 just take any member of J), while in the second half the reasoning is that $g \wedge f$ cannot be a.c.; it follows that a complete ideal $J \subseteq L$ has PJ join-dense in L iff $\sup J$ is a.c. Summing up, we obtain

PROPOSITION 5.2. *$Q(\wedge)Q(\vee)C(X) \cong AC(X)$ (= the lattice of all almost continuous normal lsc functions $X \to \bar{\mathbf{R}}$ in the pointwise order).*

COROLLARY 5.3. *$Q(\vee)Q(\wedge)C(X) \cong Q(\wedge)(Q)(\vee)C(X)$. Hence, $AC(X)$ is $\wedge$-rationally complete as well as $\vee$-rationally complete.*

Proof. $_*$ provides an isomorphism from the left to the right. ∎

Remark. Among the lattices L which (i) contain $C(X)$ as a sublattice and (ii) are both $\wedge$-rationally *and* $\vee$-rationally complete, $AC(X)$ is, in an obvious sense, the most "natural" one — this will be substantiated in Corollary 5.4 below. However, if we look beyond function lattices, there are much simpler candidates, e.g., take the set $L = C(X) \cup \{z_0, z_1, u_1, u_0\}$ ordered pointwise inside $C(X)$ and otherwise by $z_0 < z_1 < f < u_1 < u_0$ for any $f \in C(X)$. Obviously, L is a distributive selfdual lattice, satisfying both JID and MID. Now observe that exactly the *principal* ideals $J \subseteq L$ have PJ join-dense in L: Indeed, PJ is join-dense iff $u_0 \in PJ$ iff $(u_0] \cap J = J$ is principal. Hence, $Q(\wedge)L = L$, and analogously $Q(\vee)L = L$.

COROLLARY 5.4. $Q(\wedge)\bar{C}(X) \cong Q(\vee)\bar{C}(X) \cong AC(X)$.

Proof. Since $AC(X)$ is $\wedge$- and $\vee$-rationally complete, it will suffice to show that $\bar{C}(X)$ is $\wedge$- and $\vee$-rationally dense in $AC(X)$. In view of the obvious duality, we will only consider $\wedge$-density. Select functions h, h_1, $h_2 \in AC$ with $h_1 \neq h_2$.

By the Separation Lemma, we find $V \subseteq X$ nonvoid open and $\alpha, \beta \in \mathbf{R}$ such that without loss of generality, $h_1(y) < \alpha < \beta < h_2(y)$ for all $y \in V$. Now $\operatorname{cont}(h)$ contains a dense open set, so we find $\emptyset \neq U$ open, $U \subseteq V \cap \operatorname{cont}(h)$. Pick $p \in U$ arbitrarily. Use Lemma 1.1 to produce $g \in \bar{C}(X)$ with $g \geqslant 1$, $g \equiv +\infty$ on $X \setminus U$, $g(p) = 1$, Put $f = -g + 1 + \beta \cdot 1$. Then $f \leqslant \beta \cdot 1$, $f \equiv -\infty$ on $X \setminus U$ and $f(p) = \beta$. Hence $f \wedge h \in \bar{C}(X)$ ($f \wedge h \equiv -\infty$ on $X \setminus U$ and on U, $f \wedge h$ is continuous since $U \subseteq \operatorname{cont}(h)$); moreover, $f \wedge h_1 \neq f \wedge h_2$ since $(f \wedge h_2)(p) = \beta$ while $(f \wedge h_1)(p) = h_1(p) < \alpha$. ∎

6. $Q(\cdot)$ versus $Q(\wedge)$ and $Q(\vee)$

The other natural semigroup operations — besides $\wedge$ and $\vee$ — on $C(X)$ are $+$ and $\cdot$. In our context, $+$ is not of much interest: Quite obviously, $Q(+)C(X) \cong C(X)$, so any $+$-rational extension of $C(X)$ coincides with $C(X)$. So we turn to the semigroup $(C(X), \cdot)$ in order to determine $Q(\cdot)C(X)$.

According to the general procedure outlined in Section 1, $Q(\cdot)C(X)$ may be constructed as the direct limit of sets $\mathrm{Hom}_{C(X)}(D, C(X))$, where D ranges over dense semigroup ideals of $C(X)$. Explicitly: $DC(X) \subseteq D$, for any $f_1 \neq f_2$ in $C(X)$ there is $d \in D$ such that $df_1 \neq df_2$, and $\tau \in \mathrm{Hom}_{C(X)}(D, C(X))$ iff $\tau\colon D \to C(X)$ is such that $\tau(df) = \tau(d)f$ for all $d \in D$, $f \in C(X)$. $Q(\cdot)C(X)$ is then obtained from $\bigcup\{\mathrm{Hom}_{C(X)}(D, C(X));\ D \text{ dense}\}$ by identifying

$\tau_1: D_1 \to C(X)$ and $\tau_2: D_2 \to C(X)$ iff $\tau_1 \equiv \tau_2$ on $D_1 \cap D_2$. The semigroup operation on $Q(\cdot)C(X)$ is defined by composition of maps.

The procedure described on pp. 11 ff. of [FGL] in order to obtain the maximal ring of quotients $Q(X)$ of the *ring* $C(X)$ carries over to our setting and shows in fact, that $Q(\cdot)C(X)$ is isomorphic to the multiplicative semigroup of $Q(X)$. We summarize the main points:

If $D \subseteq C(X)$ is a semigroup ideal with $df_1 \neq df_2$ for some $d \in D$ whenever $f_1 \neq f_2$ in $C(X)$, then the set

$$V(D) = \bigcup \{\mathrm{coz}(d);\ d \in D\} \subseteq X$$

is open and dense.

Suppose we are given $\tau: D \to C(X)$ satisfying $\tau(df) = \tau(d)f$ for all $d \in D$, $f \in C(X)$. Consider any point $p \in V(D)$: There exists $d \in D$ with $d(p) \neq 0$. Put $h(p) := (\tau d)(p)/d(p)$. For any $d_1 \in D$, we have $\tau(dd_1) = \tau(d)d_1 = \tau(d_1)d$; hence if also $d_1(p) \neq 0$, then $(\tau d)(p)/d(p) = (\tau d_1)(p)/d_1(p)$, so the value of $h(p)$ does not depend on the particular $d \in D$ (subject to $d(p) \neq 0$) chosen for its computation. In other words, $h: V(D) \to \mathbf{R}$ is a well-defined real-valued function. We claim that h is continuous: Indeed, for any $p \in V(D)$ and $d \in D$ with $d(p) \neq 0$, $h(x) = (\tau d)(x)/d(x)$ on the open neighbourhood $\mathrm{coz}(d)$ of p; so h agrees with a continuous function on a neighbourhood of each point in $V(D)$.

Consider a fixed $d \in D$ and select, for each $p \in V(D)$, some $d_p \in D$ with $d_p(p) \neq 0$. Then

$$\begin{aligned}(\tau d)(p) &= (\tau d)(p)\,d_p(p)/d_p(p) = \big(\tau(dd_p)\big)(p)/d_p(p)\\ &= d(p)(\tau d_p)(p)/d_p(p) = d(p)\,h(p).\end{aligned}$$

It follows that $\tau d = \overline{h \cdot d}$, the (unique) continuous extension of $h \cdot d$ from $V(D)$ to all of X. Put $D' = \{$all finite sums of functions in $D\}$. Evidently, D' is a semigroup ideal and $D' \supseteq D$. If $g \in D'$, $g = d_1 + \ldots + d_r$, define $\tau' g := \overline{h \cdot d_1} + \ldots + \overline{h \cdot d_r}$. τ' is clearly well defined, $\tau'(gf) = \tau'(g)f$ for all $f \in C(X)$ and τ' extends τ. Hence, in forming $Q(\cdot)C(X)$, τ and τ' will be identified. But τ' is also additive (obviously, $\tau'(g_1 + g_2) = \tau' g_1 + \tau' g_2$ for all $g_1, g_2 \in D'$); and D' is a ring ideal in the ring $C(X)$. In view of the construction presented in [FGL], § 2, we have:

LEMMA 6.1. *The carrier sets of $Q(\cdot)C(X)$ and of $Q(X)$ coincide both with* $\varinjlim C(V)$, *V ranging over all dense open sets of X. Operations $+$ and $\cdot$ are defined pointwise, hence, $Q(\cdot)C(X)$ coincides with the multiplicative semigroup of $Q(X)$.*

The obvious question is now the relate this description of $Q(\cdot)C(X)$ with the results on $Q(\wedge)C(X)$ obtained in Sections 4 and 5. The key is given by the Extension Lemma in 2.8 which yields a description of $Q(\cdot)C(X)$ in terms of semicontinuous functions.

Consider a dense open set $V \subseteq X$ and $f \in C(V)$. The Extension Lemma tells us that there is a unique normal lsc function $\bar{f}$: $X \to \bar{\mathbf{R}}$ such that $\bar{f}|V = f$; $\bar{f}$ is given as $(f^*)_*$. If f_1, f_2 are in $C(V)$ and $f_1(x) \neq f_2(x)$ for some $x \in V$, then clearly $\overline{f_1} \neq \overline{f_2}$. On the other hand, if $f_1 \in C(V_1)$, $f_2 \in C(V_2)$ (where $V_i \subseteq X$ dense open for $i = 1, 2$) and $f_1(x) = f_2(x)$ for $x \in V_1 \cap V_2$, then $\overline{f_1} = \overline{f_2}$ since extensions to *normal* lsc functions are unique. Consequently, the assignement $f \mapsto (f^*)_*$ sets up an embedding of $\varinjlim C(V)$ into $NLSC(X)$. Conversely, if $f \in NLSC(X)$ happens to be continuous *and* finite on some dense open set $V \subseteq X$, then clearly $f = \overline{f|V}$. According to our use of "almost", we call such functions *almost continuous-finite* and put

$$ACF(X) = \{f \in NLSC(X);\ f \text{ is almost continuous-finite}\}.$$

Operations $\oplus$ and $\odot$ are readily introduced to $ACF(X)$ by

$$f_1 \oplus f_2 := \overline{f_1 + f_2}, \quad f_1 \odot f_2 := \overline{f_1 \cdot f_2} \quad \text{for any } f_1, f_2 \in ACF$$

where $f_1 + f_2$, $f_1 \cdot f_2$ are defined pointwise on the dense open set on which f_1, f_2 are jointly continuous and finite. We obtain

Proposition 6.2. $Q(\cdot)C(X) \cong (ACF(X), \odot)$ *and* $Q(X) \cong (ACF(X), \oplus, \odot)$.

ACF under its pointwise order is a lattice, easily seen to be a sublattice of $NLSC(X)$. $Q(\cdot)C(X)$ and $Q(X)$ are lattices as direct limits of the lattices $C(V)$, V dense open in X.

Corollary 6.3. *The lattices* $Q(\cdot)C(X)$, $Q(X)$ *and* $ACF(X)$ *are isomorphic sublattices of* $NLSC(X)$.

All the rational completions obtained so far – viz. $Q(\wedge)C(X)$, $Q(\vee)C(X)$, $Q(\wedge)\bar{C}(X)$, $Q(\vee)\bar{C}(X)$ and $Q(\cdot)C(X)$ – turned up as sublattices of $NLSC(X)$ which is isomorphic with the MacNeille completion of $\bar{C}(X)$, see the remark after Proposition 4.3. It seems thus natural to look at the MacNeille completions of these lattices. We do this in some detail for $Q(\cdot)C(X) \cong ACF(X)$ and just state the results for the others.

Consider a normal ideal $\emptyset \neq J \subseteq ACF$ with J^u (taken in ACF) nonempty. Let $g = \sup J$. f is clearly normal lsc, moreover, there exists $f_1 \in ACF$ with $g \leqslant f_1$. Hence, there is a dense open set $V_1 \subseteq X$ and $h_1 \in C(V_1)$ – viz. $f_1|V_1$ – such that $g \leqslant h_1$ on V_1. Naturally, since $g \geqslant J$ and $J \neq \emptyset$, there is V_2 dense open and $h_2 \in C(V_2)$ such that $g \geqslant h_2$ on V_2. Consequently, $h_2 \leqslant g \leqslant h_1$ on the dense open set $V_1 \cap V_2$, or in other words, g is *almost C-bounded*.

Conversely, let $g \in NLSC$ be almost C-bounded and put $J = \{f \in ACF; f \leqslant g\}$. We claim that $J \subseteq ACF$ is a normal ideal, $J \neq \emptyset$ and $J^u \neq \emptyset$. Let $V \subseteq X$ the dense open set on which g is C-bounded, and put $J_V = \{h \in C(V); h \leqslant g|V\}$. Clearly, $J_V \neq \emptyset$ and J_V^u (taken in $C(V)$) is nonvoid, too. Hence, J_V is a normal ideal in $C(V)$ (see the remark after Proposition 4.6). Our assertions about J and J^u now follow by observing that the members of J and J^u are just the normal lsc extensions to all of X of the members of J_V and J_V^u, respectively (Extension Lemma).

Let $\emptyset \neq J \subseteq ACF$ be normal, $J^u \neq \emptyset$, $g = \sup J$, $f \in NLSC$ almost C-bounded, $f \leqslant g$. If $f \notin J$, then $f \nleqslant J^u$ since J is normal; but then also $f \nleqslant g$ since clearly $g \leqslant J^u$. We conclude $f \in J$; this shows that the assignement $J \mapsto \sup J$ between nontrivial normal ideals in ACF and almost C-bounded functions in $NLSC$ is bijective; it clearly preserves order in both directions. Hence:

COROLLARY 6.4. *The MacNeille completion of $Q(\cdot)C(X)$ is isomorphic with the lattice of all almost C-bounded functions in $NLSC(X)$ together with $-\infty \cdot 1$ and $+\infty \cdot 1$; moreover,*

$$[Q(\cdot)C(X)]^\wedge \cong \varinjlim C(V)^\wedge,$$

V ranging over the dense open sets $V \subseteq X$.

Proof. As for the last assertion, observe that any $f \in NLSC(X)$ is determined by its restriction $f|V$ on any dense open $V \subseteq X$ by virtue of the Extension Lemma (2.8); if f is C-bounded on V, then $f|V$ may be identified with a member of $C(V)^\wedge$ since clearly $f|V \in NLSC(V)$. ∎

In a similar way, we may derive that

$$[Q(\wedge)C(X)]^\wedge \cong \{f \in NLSC(X); f \text{ lower } C\text{-bounded}\} \cup \{-\infty \cdot 1\}$$

and

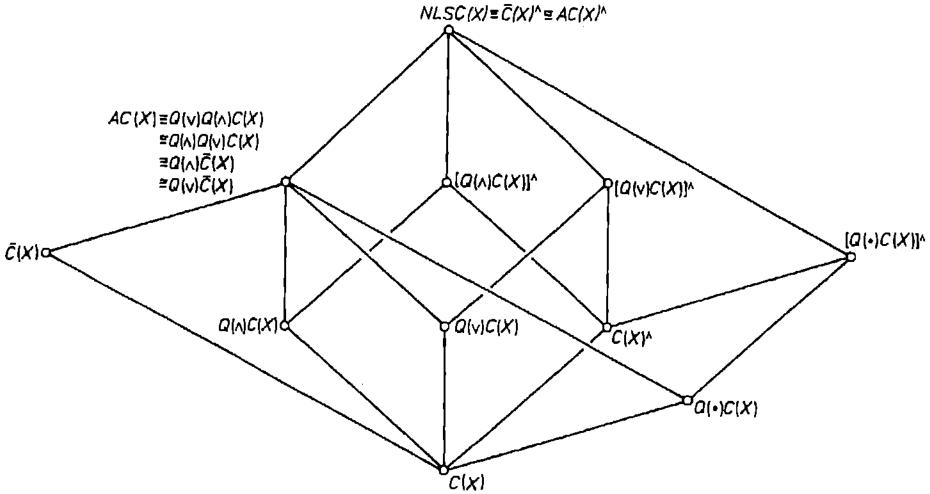
$$[Q(\vee)C(X)]^\wedge \cong \{f \in NLSC(X); f \text{ upper } C\text{-bounded}\} \cup \{+\infty \cdot 1\},$$

the right-hand sides as lattices under the pointwise order.

The sublattices of $NLSC(X)$ which turned up in this paper may be ordered by inclusion, the resulting finite poset is a lattice whose diagram we give below (note it is *not* sublattice of the lattice of all sublattices of $NLSC(X)$!). We recall here, for convenience, that any $f \in NLSC(X)$ belongs to

– $Q(\wedge)C(X)$, iff it is almost continuous and lower C-bounded;
– $Q(\vee)C(X)$, iff it is almost continuous and upper C-bounded;

- $AC(X)$, iff it is almost continuous;
- $Q(\cdot)C(X)$, iff it is almost continuous-finite;
- $C(X)^{\wedge}$, iff it is C-bounded or equal to $\pm\infty \cdot 1$.



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