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Approximately finite-dimensional C^*-algebras

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CONTENTS

Introduction*

In one of their classic papers [21] von Neumann and Murray consider a class of factors A in which the union of an ascending sequence of full matrix algebras containing the identity is ultraweakly dense; they call these factors approximately finite. Subsequent terminology preferred the term "hyperfinite". The main concern in [21] was the construction of type II_1 factors, i.e. infinite-dimensional factors with an everywhere ultraweakly continuous trace. Von Neumann and Murray showed that there was only one such hyperfinite factor A which has the following simple description: Let P be the set of all bijections of a countably infinite set N whose fixed point set has finite complement. For $p \in P$ let U_p the unitary operator on $l^2(N)$ given by $U_p(f) = f \circ p^{-1}$. The A is isomorphic to the ultraweak closure of the linear span of the U_p, $p \in P$. One notes in passing that there are other infinite-dimensional hyperfinite factors such as $L(H)$ for an infinite-dimensional Hilbert space; this example, of course, has no faithful trace.

A considerably greater variety of C^*-algebras arise if, instead of considering ultraweak closures of unions of matrix algebras one considers norm closures. The first systematic investigation of this set-up appears to be Glimm's work [16]. He calls a C^*-algebra A with identity *uniformly hyperfinite* (or shortly UHF) provided $A = (\bigcup A_n)^-$, where $\{A_n\}$ is an ascending sequence of full matrix algebras containing the identity of A. The closure is now taken in the operator norm. Glimm shows that UHF-algebras are simple and gives a complete classification. Several authors observed that a suitable pro-cyclic group is the most appropriate invariant for this classification. Dixmier in [12] removes the requirements on the identity. His extended class of so-called *matroid C^*-algebras* included the algebra of compact operators on separable Hilbert space. All matroid C^*-algebras are simple; their classification, however, is more involved than that of UHF-algebras. We mention that one of the interesting applications of UHF-algebras was Powers' construction in [22] of a continuum of non-isomorphic hyperfinite type III factors, thereby settling a long outstanding question about these factors.

The final generalization of the idea of approximating C^*-algebras from within by finite-dimensional subalgebras was developed by Bratteli in a series of papers [7], [8], [9], [10]. He defines an almost finite-dimensional C^*-algebra A by requiring $A = (\bigcup A_n)^-$, where $\{A_n\}$ is a sequence of finite dimensional C^*-algebras (and no longer necessarily full matrix algebras). In the earlier cases the inclusion maps $A_n \to A_{n+1}$ were comparatively simple due to the simplicity of A_k; now these maps are relatively complicated. Bratteli successfully overcomes the difficulty by associating with the sequence of the A_n a combinatorial object which carries virtually all the necessary information. For the ideal theory of Bratteli's algebras he deduces from his combinatorial invariants a partially ordered set which is simpler and which still carries all the information necessary to describe the ideal theory of the algebra. This aspect of the theory was discussed in greater detail by Dooley [13].

One must also refer the reader in this context to a series of studies by Behncke and Leptin (and collaborators) on the structure theory of C^*-algebras with finite duals [3], [4], [5], [6]. Beginning with two element duals they finally succeed in giving a full characterization of separable C^*-algebras with finite dual space (i.e. space of unitary equivalence classes of irreducible representations). Their classification also heavily uses devices from the theory of partially ordered sets. A common super-theory for the Behncke–Leptin theory and the Bratteli-theory is not yet available; such a theory would have to concern itself with colimits of C^*-algebras with finite duals, and one is entitled to the speculation that combinatorial and order theoretical devices would have to play a significant role in such a theory. Certain generalizations of their original methods are indicated by Behncke and Leptin in [6].

The purpose of the present exposition is to give a coherent essentially self-contained account of the fundamentals of the theory of almost finite-dimensional C^*-algebras, their ideal theory and their spectral theory. Not everything we present is merely expository. Our definition of an AFC^*-algebra is phrased without separability assumption, and large portion of the theory, although by no means its entirety is carried through on this basis. In particular, we are unable to show that in the absence of separability the almost finite dimensionality property is equivalent to the hypothesis that every finite subset of the algebra is within ε-distance from some finite-dimensional subalgebra for arbitrarily small $\varepsilon > 0$. We reprove by different methods, using bundle theoretical techniques, Bratteli's key lemma which answers the separable case in the affirmative. We describe the ideal theory of the algebra in terms of that of a suitably constructed partially ordered set more systematically than it was done in earlier approaches. In particular, we utilize relatively recent lattice theoretical techniques to settle in the affirmative a question which remained

open in this generality: Is every quasicompact T_0 space with a countable basis of quasicompact open sets in which all irreducible closed sets are singleton closures the primitive spectrum of a separable almost-finite dimensional C^*-algebra? Such spaces are called *spectral*. It was proved by Bratteli and Dooley that indeed every separable AFC^*-algebra had a quasicompact spectral space as primitive ideal spectrum. Our approach renders unnecessary certain technical complications arising in earlier discussions (Bratteli's constructions in [8], [9], [10]; Dooley's acyclic covers).

We proceed as follows: In Section 1 we give a systematic account of finite-dimensional C^*-algebras and their morphisms; in other words, we analyse the category of finite-dimensional C^*-algebras and show that it is essentially equivalent to a slightly beefed-up category of rectangular matrices of non-negative integers. The matrices are the key for the combinatorial invariants used in the later characterization of AFC^*-algebras.

In Section 2 we introduce AFC^*-algebras, i.e. almost finite dimensional C^*-algebras, and discuss basic aspects of equivalent forms of their definition and some examples. We then introduce the Bratteli-diagram and show how it carries all the essential information on an AFC^*-algebra.

In Section 3 we establish the important bijection between closed two-sided ideals and an AFC^*-algebra and certain substructures of the associated Bratteli diagram.

Section 4 proceeds by introducing what we call an *augmented poset* (partially ordered set); every Bratteli diagram (and hence every AFC^*-algebra) gives rise to one of these; conversely, at least every countable augmented poset of a particular type arises in this fashion (which is enough to study separable AFC^*-algebras); we do not know whether this converse holds generally. It is shown that the ideal theory of AFC^*-algebras is that of its augmented poset, where ideals are appropriately introduced for posets. We prove that the lattice of closed two-sided ideals of an AFC^*-algebra is a complete algebraic Brouwerian lattice.

In Section 5 we turn to the spectral theory of AFC^*-algebras. We identify the prime ideals in an augmented poset and introduce the hull kernel topology in the spectrum of an augmented poset. From the results of Section 4 we derive that the space of closed two-sided prime ideals of an AFC^*-algebra is homeomorphic to the spectrum of its poset; in the separable case one knows closed prime ideals to be primitive; nothing tangible appears to be known in this direction in the absence of separability. Finally we recall some general facts on Brouwerian algebraic lattices, their duality and their spectral theory. We then show that for every Brouwerian algebraic lattice L there is an augmented poset whose ideal lattice is isomorphic to L. This is precisely what is needed to prove that every spectral quasicompact space occurs as the spectrum of an AFC^*-

algebra if it is second countable. Given some known facts in the spectral theory of lattices and the Dauns–Hofmann theorem for C^*-algebras, it is then easy to derive directly Bratteli's theorem saying that every commutative separable C^*-algebra is the center of some separable AFC^*-algebra.

We test our methods by answering a question posed by Leptin to everyone who has worked in the area to probe the strength of available methods. Suppose that two separable AFC^*-algebras have equivalent spectra consisting of an ascending sequence of primitive ideals and a unique maximal ideal; do these algebras have to be isomorphic? On the basis of a result of Bratteli's it is in fact possible to answer this question in the negative on the level of the associated augmented posets. Uncountably many non-isomorphic AFC^*-algebras allow such a spectrum and a full classification of them is nowhere in sight even after our observations in this direction.

After our manuscript was completed a paper by G. A. Elliot appeared in print [26] in which a new invariant is introduced for an inductive limit of a sequence of semisimple finite-dimensional algebras over any algebraically closed field of scalars, the invariant being a partial algebra with a partially defined commutative and associative operation. We might point out in this context that the invariant "augmented poset" which we treat in the present discussion also applies to the situation of inductive limits of finite-dimensional separable algebras over an algebraically closed field.

Our notes developed from a seminar on C^*-algebras at Tulane University during 1974–1975. We gratefully acknowledge the participation of M. Dupré and J. R. Liukkonen.

1. Finite-dimensional C^*-algebras

The objects

1.1. NOTATION. We denote by FC_0^* the category of finite-dimensional C^*-algebras and *-preserving morphisms, and by FC^* the subcategory of finite-dimensional C^*-algebras (which automatically have an identity!) and identity preserving morphisms. Finite-dimensional C^*-algebras will also be called FC^*-algebras.

We regard some standard facts which essentially are due to the semisimplicity of all C^*-algebras as applied to the finite-dimensional case.

1.2. PROPOSITION. (i) *Let A be an FC^*-algebra. Then for every (two-sided) ideal I there is a unique annihilator ideal $I^\perp$ such that $A = I \oplus I^\perp$ (with the C^*-product written as an internal direct sum; specifically: the function $I \times I^\perp \mapsto A$ given by $(x, y) \mapsto x + y$ is a C^*-isomorphism).*

(ii) *Let* $\mathrm{Id}(A)$ *be the lattice of two-sided ideals of* A, *further* $\mathrm{Min}\,A$ *and* $\mathrm{Max}\,A$ *the subsets of minimal, respectively maximal elements in* $\mathrm{Id}(A)$. *Then the function* $X \mapsto \oplus X$: $2^{\mathrm{Min}A} \to \mathrm{Id}\,A$ *is an isomorphism of Boolean algebras. In particular,* $I \mapsto I^{\perp}$: $\mathrm{Min}\,A \to \mathrm{Max}\,A$ *is a bijection. Since every primitive ideal is maximal for an* FC^*-*algebra, we have* $\mathrm{Prim}\,A = \mathrm{Max}\,A$. *Moreover, any irreducible representation is determined by its kernel; hence* $\mathrm{Spec}\,A = \mathrm{Prim}\,A$.

(iii) *An ideal* $I \in \mathrm{Id}\,A$ *is simple iff* $I \in \mathrm{Min}\,A$. *A* FC^*-*algebra* A *is simple iff it is isomorphic to a* C^*-*algebra* M_n *of all complex* $n \times n$ *matrices. It is, therefore, uniquely determined up to isomorphy by* $n = \deg A$ *such that* $\dim A = n^2$. *As a consequence, any* FC^*-*algebra* A *is, up to isomorphy, uniquely determined by the function* $\deg$: $\mathrm{Min}\,A \to N = \{1, 2, ...\}$. ∎

For an arbitrary FC^*-algebra we write $\deg A = \sum \{\deg I: I \in \mathrm{Min}\,A\}$.

The morphisms

We now have to discuss background information on the morphisms of FC^* Since in the process we have to allow for zero morphisms, we are forced to leave the category FC^* and to discuss FC_0^*-morphisms in general.

1.3. DEFINITION. Let $A \in \mathrm{ob}\mathrm{FC}^*$. For each ideal I of A we write pr_I: $A \to I$ for the unique projection onto I with kernel $I^{\perp}$. Then $\mathrm{pr}_I \in \mathrm{morph}\mathrm{FC}^*$. For the embedding $I \to A$ write copr_I; note that copr_I is always in FC_0^*, but is in FC^* if and only if $I = A$. ∎

1.4. REMARK. *If* φ: $A \to B$ *is an* FC_0^*-*morphism, then* $\varphi = \sum \{\mathrm{copr}_I\,\mathrm{pr}_I\varphi$: $I \in \mathrm{Min}\,B\}$ *in the sense that* $\varphi(a) = \sum \{\mathrm{pr}_I\varphi(a): I \in \mathrm{Min}\,B\}$ *for all* $a \in A$.

If we let $e_K' = \varphi(e_K)$, *where* e_K *is the identity of* $K \in \mathrm{Min}\,A$, *then* $e_K'e_L' = (\delta_{KL})e_L'$ *and* $\mathrm{im}\,\varphi \subseteq \oplus \{e_K'\varphi(A)e_K': K \in \mathrm{Min}\,A\}$ *and* $\varphi = \sum \{\varphi\,\mathrm{copr}_K\mathrm{pr}_K$: $K \in \mathrm{Min}\,A\}$ *in the sense that* $\varphi(a)$ *is the orthogonal sum of the* $\varphi(\mathrm{pr}_K a)$, $K \in \mathrm{Min}\,A$. ∎

Note that normally a sum of FC_0^*-morphisms will at best be an involution preserving linear operator, but not in general an algebra map.

1.5. DEFINITION. If φ: $A \to B$ is in FC_0^*, we associate with φ a matrix $S(\varphi) = (\varphi_{IK})_{(I,K)\in\mathrm{Min}B\times\mathrm{Min}A}$ of FC_0^*-morphisms φ_{IK}: $K \to J$ between simple FC^*-objects given by

$$(1) \qquad \varphi_{IK} = \mathrm{pr}_I\varphi\,\mathrm{copr}_K, \qquad K \in \mathrm{Min}\,A, \quad I \in \mathrm{Min}\,B.$$

This matrix $S(\varphi)$ will be called the *matrix associated with* φ. ∎

The following remark is then immediate from 1.4:

1.6. REMARK. (1) $(\mathrm{im}\varphi_{IK})(\mathrm{im}\varphi_{IL}) = \{0\}$ *for* $K \neq L$ *in* $\mathrm{Min}\,A$.

$$(2) \qquad \varphi = \sum \{\mathrm{copr}_I\varphi_{IK}\mathrm{pr}_K: (I, K) \in \mathrm{Min}\,B \times \mathrm{Min}\,A\}.$$

Thus $\varphi \mapsto S(\varphi)$ *is a bijective correspondence between* $\mathrm{FC}_0^*(A, B)$ *and the set of* $\mathrm{Min}\,B \times \mathrm{Min}\,A$-*matrices of morphisms* $\varphi_{JK}\colon K \to I$ *satisfying* (1). ∎

We now express various properties of morphisms in terms of their associated matrices.

1.7. PROPOSITION. *Let* $\varphi\colon A \to B$ *be a* FC^*-*morphism.*

(i) φ *is injective iff for each* $K \in \mathrm{Min}\,A$ *there is a* $J \in \mathrm{Min}\,B$ *with* $\varphi_{JK} \neq 0$ (*i.e. if the associated matrix has no zero-columns*).

(ii) $\mathrm{im}\,\varphi$ *is not contained in a proper ideal of* B *iff for each* $J \in \mathrm{Min}\,B$ *there is a* $K \in \mathrm{Min}\,A$ *with* $\varphi_{JK} \neq 0$ (*i.e. iff the associated matrix has no zero rows*). *If* φ *is a* FC^*-*morphism, this is always the case.*

Proof. (i) φ is injective iff $\ker \varphi = 0$ iff $\ker(\varphi \operatorname{copr}_K) = 0$ for all $K \in \mathrm{Min}\,A$ iff for each $K \in \mathrm{Min}\,A$ there is a $J \in \mathrm{Min}\,B$ with $\varphi_{JK} = \mathrm{pr}_J \varphi \operatorname{copr}_K \neq 0$.

(ii) $\mathrm{im}\,\varphi$ is contained in a proper ideal I of B iff there is some $J \in \mathrm{Min}\,B$ with $\mathrm{pr}_J \mathrm{im}\,\varphi = 0$, i.e. with $\mathrm{pr}_J \varphi = 0$ iff there is a $J \in \mathrm{Min}\,B$ such that for all $K \in \mathrm{Min}\,A$ one has $\varphi_{JK} = \mathrm{pr}_J \varphi \operatorname{copr}_K = 0$. ∎

We have seen that morphisms can be completely described in terms of morphisms between simple FC^*-algebras. It remains to characterize these.

Recall that any simple FC^*-algebra A is of the form $L(H)$ with a finite-dimensional Hilbert space H with $\dim H = \deg A$. If $e \in A$ is a projection, then eAe is identified with $L(eH)$, hence is simple.

1.8. PROPOSITION. *Let* $\varphi\colon A \to B$ *be an* FC_0^*-*morphism between simple objects. Let* $e \in B$ *be the projection* $\varphi(1)$. *Then* φ *decomposes into the composition*

$$A \xrightarrow{\bar{\varphi}} eBe \xrightarrow{j_e} B \qquad \text{with the inclusion } j_e.$$

The map φ *is* 0 *iff* $e = 0$ *iff* φ *is not injective. We have* $\varphi \in \mathrm{morph}\,\mathrm{FC}^*$ *iff* $e = 1$. *Suppose* $\varphi \neq 0$. *Then* $\bar{\varphi}\colon A \to eBe$ *is an* FC^*-*isometry onto the image* $\varphi(A)$. *If* eBe *is identified with* $L(H)$, *then* H *decomposes into a uniquely determined Hilbert space sum* $H_1 \oplus \ldots \oplus H_m$ *such that each* K_k *is* $\varphi(A)$ *invariant and* $\varphi(A)|H_k = L(H_k) = A$; *in particular,* $\dim H_k = \deg A$ *for all* k. *One deduces that the commutant* $\varphi(A)'$ *in* eBe *is simple and isomorphic to* $L(C^m)$ *so that* $\varphi(A) \otimes \varphi(A)'$ *is isomorphic to* eBe *under the map sending* $x \otimes y$ *to* xy, *and that every minimal projection in* $\varphi(A)$ *has rank* m. *The unique natural number* m *is called the* "*multiplicity*" *of* φ, *and is written* $\mathrm{mult}\,\varphi$. *We set* $\mathrm{mult}\,\varphi = 0$ *if* $\varphi = 0$. *One has*

$$(1) \qquad\qquad \deg eBe = \mathrm{mult}\,\varphi \deg A \leqslant \deg B.$$

In particular, if $\varphi \in \mathrm{morph}\,\mathrm{FC}^*$, *then*

$$(1') \qquad\qquad \deg B = \mathrm{mult}\,\varphi \deg A. \quad\blacksquare$$

If φ, φ': $A \to B$ are FC_0^*-morphisms between simple objects and φ, φ' have the same multiplicity, then there is an automorphism β of B such that

$$
\begin{array}{ccc}
A & \xrightarrow{\ \varphi\ } & B \\
\| & & \downarrow{\scriptstyle\beta} \\
A & \xrightarrow{\ \varphi'\ } & B
\end{array}
$$

commutes, i.e. $\beta\varphi = \varphi'$ holds. Recall that an automorphism β of a simple FC^*-algebra is an inner automorphism effected by some unitary element $u \in B$ according to $\beta(b) = ubu^*$ for $b \in B$. For any unital C^*-algebra B the function $u \mapsto (b \mapsto ubu^*)$ is a homomorphism from the unitary group of B into the automorphism group of B whose kernel is the center of the unitary group (i.e. the scalar multiplications by e^{ir}, $r \in R$). In place of $\beta\varphi$ we write $u \cdot \varphi$. Thus $(u \cdot \varphi)(a) = u\varphi(a)u^*$. We say that two morphisms φ, ψ: $A \to B$ are *equivalent* and write $\varphi \sim \psi$ iff there is a unitary $u \in B$ with $\psi = u \cdot \varphi$. Evidently, $\sim$ is an equivalence relation. If ϱ: $B \to C$ is a morphism, then $\varrho(u: \varphi) = \varrho(u) \cdot \varphi$. It follows that $\sim$ is a congruence relation on the categories FC_0^* and FC^*; this means that χ: $E \to A$, φ, ψ: $A \to B$, ϱ: $B \to C$ and $\varphi \sim \psi$ implies $\varphi\chi \sim \psi\chi$ and $\varrho\varphi \sim \varrho\psi$. We denote by $\widetilde{FC}^*$ and $\widetilde{FC}^*$ the quotient categories (whose objects are those of FC_0^* and FC_0^*, respectively, and whose morphisms are the equivalence classes of morphisms with the obvious law of composition).

The multiplicity of a morphism between two simple objects uniquely determines the equivalence class of this morphism.

1.9. PROPOSITION. *Let* $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ *be* FC_0^*-*morphisms with* A *and* C *simple. Then*

(1) $$\text{mult}\,\varphi\psi = \sum\{\text{mult}\,\varphi_{CJ}\,\text{mult}\,\psi_{JA}: J \in \text{Min}\,B\}.$$

Proof. Step (i). We may assume w.l.g. that φ is injective: Indeed, let $P = \ker\varphi$. Let $\psi' = \text{pr}_P\psi$ and $\varphi' = \varphi\text{copr}_P$. Then $\varphi\psi = \varphi'\psi'$ with φ injective, and $\varphi_{CJ} = \varphi\text{copr}_J = 0$ for $J \in \text{Min}\,P$. It therefore suffices to prove the formula for φ' and ψ' in place of φ and ψ.

Step (ii). We may assume w.l.g. that ψ is injective: If not then ψ is 0 since A is simple and so the formula is correct in this case.

Step (iii). We may assume that $\varphi, \psi \in \text{morph}\,FC^*$: replace B by $B_1 = \psi(1)B\psi(1)$, further C by $C_1 = \varphi\psi(1)C\varphi\psi(1)$, and finally, ψ and φ by the restrictions and corestrictions $\varphi_1 = \varphi$: $A \to B_1$ and $\psi_1 = \psi|B_1$: $B_1 \to C_1$.

From now on we may assume $1 \in A \subseteq B \subseteq C$ with φ and ψ being inclusions. For each $J \in \text{Min}\,B$ let e_J be the identity of J and $C_J = e_J C e_J$; then $J \subseteq C_J$ and C_J is simple (since $pL(H)p = L(pH)$ for a projection p on the Hilbert space H). Notice that $\varphi_{C_J J}$ is the corestriction of $\varphi\text{copr}_J = \varphi_{CJ}$

to C_J, whence $\mathrm{mult}\,\varphi_{C_JJ} = \mathrm{mult}\,\varphi_{CJ}$. Now we use 1.8.(1') and observe

(a) $\deg J = \mathrm{mult}\,\psi_{JA}\deg A$ for all $J \in \mathrm{Min}\,B$.

(b) $\deg C_J = \mathrm{mult}\,\varphi_{CJ}\deg J$ for all $J \in \mathrm{Min}\,C$.

(c) $\sum\{\deg C_J\colon J \in \mathrm{Min}\,B\} = \deg C = \mathrm{mult}\,\varphi\psi\deg A$.

If we substitute (a) in (b) and sum over $J \in \mathrm{Min}\,B$, and divide the resulting equation by $\deg A$, we obtain

(c') $\sum\{\deg C_J\colon J \in \mathrm{Min}\,B\} = \sum\{\mathrm{mult}\,\varphi_{CJ}\mathrm{mult}\,\psi_{JA}\colon J \in \mathrm{Min}\,B\}$.

Comparison of (c) and (c') yields the desired result (1). ∎

1.10. COROLLARY. *Let* $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ *be two* FC_0^**-morphisms. Then for each* $(I, K) \in \mathrm{Min}\,C \times \mathrm{Min}\,A$ *we have*

$$(8)\quad \mathrm{mult}\,(\varphi\psi)_{IK} = \sum\{\mathrm{mult}\,\varphi_{IJ}\mathrm{mult}\,\psi_{JK}\colon J \in \mathrm{Min}\,B\}.$$

Proof. 1.5 and 1.9. ∎

Before we proceed we establish the following lemmas which we will use in Section 3, but which belong to the present context.

1.11. LEMMA. *Let* $1 \in A \subseteq B$ *in* FC^* *with the inclusion morphism* $\varphi\colon A \to B$. *If* I *is a two-sided ideal in* B, *and if* $K \in \mathrm{MIN}\,B$, *then we have the following two statements:*

(1) *If* $K \subseteq I$ *and* $\mathrm{mult}\,\varphi_{K'K} \neq 0$ *for* $K' \in \mathrm{Min}\,B$, *then* $K' \subseteq I$.

(2) *If* $K' \subseteq I$ *for all* $K' \in \mathrm{Min}\,B$ *with* $\mathrm{mult}\,\varphi_{K'K} \neq 0$, *then* $K \subseteq I$.

Proof. First we note that we have $I = \sum\{K' \in \mathrm{Min}\,B\colon K' \subseteq I\}$. In particular, if $\mathrm{pr}_{K'}$ is the projection of B onto K', then $I \cap K' \neq \{0\}$ iff $K' \subseteq I$ iff $\mathrm{pr}_{K'}I \neq \{0\}$. Now we prove (1): If $\mathrm{mult}\,\varphi_{K'K} \neq 0$, then $\{0\} \neq \varphi_{K'K}(K) = \mathrm{pr}_{K'}K$, and if $K \subseteq I$, then $\mathrm{pr}_{K'}I \neq \{0\}$, whence $K' \subseteq I$ after the preceding remark.

Proof of (2). We have $\mathrm{mult}\,\varphi_{K'K} \neq 0$ precisely when $\mathrm{pr}_{K'}K \neq \{0\}$. If $K' \subseteq I$ for all of these $K' \in \mathrm{Min}\,B$, then $\mathrm{pr}_{K'}K \subseteq I$ for all $K' \in \mathrm{Min}\,B$, which implies $K \subseteq I$. ∎

1.12. LEMMA. *Let* $1 \in A \subseteq B$ *in* FC^* *with the inclusion morphism* $\varphi\colon A \to B$. *Let* $Y(A) \subseteq \mathrm{Min}\,A$ *and* $Y(B) \subseteq \mathrm{Min}\,B$ *be subsets such that*

(1) *If* $K \in Y(A)$ *and* $\mathrm{mult}\,\varphi_{K'K} \neq 0$, *then* $K' \in Y(B)$.

(2) *If* $K' \in Y(B)$ *for all* $K' \in \mathrm{Min}\,B$ *with* $\mathrm{mult}\,\varphi_{K'K} \neq 0$, *then* $K \in Y(A)$.

Set $I = \sum\{K' \in Y(B)\}$; *then* $\sum\{K\colon K \in Y(A)\} = I \cap A$.

Proof. By (1) and (2) we know that $K \in Y(A)$ iff for all $K' \in \mathrm{Min}\,B$ with $\mathrm{mult}\,\varphi_{K'K} \neq 0$ one has $K' \in Y(B)$. But $\mathrm{mult}\,\varphi_{K'K} \neq 0$ is equivalent to $\mathrm{pr}_{K'}K \neq \{0\}$. Similarly to the proof of (2) in 1.11 we conclude that for these K' one has $\mathrm{pr}_{K'}K \subseteq K' \subseteq I$, and for all other K' one concludes $\mathrm{pr}_{K'}K = \{0\} \subseteq I$, thus $K \subseteq I$. Conversely, if $K \subseteq I$, then $\mathrm{pr}_{K'}K \neq \{0\}$ precisely for the $K' \in Y(B)$. Thus we have $K \in Y(A)$ iff $K \subseteq I$ which is equivalent to the assertion. ∎

The matrix calculus

At this point we depart momentarily from C^*-algebras and survey the algebra of matrix calculus. We recall that the set of all $m \times n$ matrices over any semiring R form a category. Specifically:

1.13. DEFINITION. Let R be a semiring (i.e. an algebra with an abelian addition which is a monoid with identity 0 and a multiplication relative to which is a monoid with identity 1, and in which both distributive laws hold). Then the following definitions yield a small category $M_0(R)$:

(i) $\mathrm{ob}\,M_0(R) = \{R, R^2, R^3, \ldots\}$,

(ii) $\mathrm{morph}\,M_0(R) = \{(r_{ik})_{\substack{i=1,\ldots,m \\ k=1,\ldots,n}}: m, n = 1, 2, \ldots, r_{ik} \in R\}$.

The elements of $M_0(R)(R^n, R^m)$ are precisely the $m \times n$ matrices, and composition of morphisms is matrix multiplication. We call $M_0(R)$ the *special matrix category over* R. ∎

It is more convenient for our purposes to replace it by an equivalent category $M(R)$ which is no longer small (and of which $M_0(R)$ is a skeleton):

1.14. LEMMA. *Let R be a semiring; then the following definition yields a category:*

(i) $\mathrm{ob}\,M(R) = R^X$: *X a finite set,*

(ii) $\mathrm{morph}\,M(R) = \{(r_{xy})_{(x,y)\in X \times Y}: X, Y$ *finite sets,* $r_{xy} \in R\}$.

The elements of $M(R)(R^Y, R^X)$ are precisely the $X \times Y$-matrices, and composition of morphisms is matrix multiplication. ∎

The category $M(R)$ will be called the *matrix category over* R. ∎

We note that the emphasis in these matrix categories rests on the morphisms, i.e. the matrices, whereas the objects play a subordinate role as the more or less natural domains and codomains of matrices when these are considered as morphisms. This emphasis becomes particularly apparent if one considers functors from a given category into a matrix category; practically all information is carried by the assignment of matrices to morphisms of the domain category. The ancillary role of the objects in such situations is well illustrated in the main application of matrix categories in our present context which we now proceed to present. Later (1.14) we will discuss a modification of the concept of a matrix category which is a bit more technical, but has actual advantages for our present application.

Recall that a functor is called *representative* if every object in its codomain category is isomorphic to an image object. The following theorem will be crucial in all that follows.

1.15. THEOREM. *Let Z^+ be the semiring of non-negative integers. We define a function $M\colon \mathrm{FC}_0^* \to M(Z^+)$ by*

$$M(A) = (Z^+)^{\mathrm{Min}\,A} \quad for \quad A \in \mathrm{ob}\,\mathrm{FC}_0^*$$

and by

$$M(\varphi) = (\mathrm{mult}\,\varphi_{J,K})_{(J,K)\in\mathrm{Min}B\times\mathrm{Min}A} \quad for \quad \varphi\colon A \to B$$

in morph FC_0^*. *Then we have the following conclusions*:

(I) *M is a representative functor.*

(II) *For $A \in \mathrm{ob}\,\mathrm{FC}^*$ let $\bar{A}$ denote the column matrix $(\deg I)_{I\in\mathrm{Min}A}$. Then for each $\varphi\colon A \to B$ in FC^* we have*

(1) $\bar{B} = M(\varphi)\bar{A}.$

(III) *If $A, B \in \mathrm{ob}\,\mathrm{FC}^*$ and if D is a matrix in $M(Z^+)\big(M(A), M(B)\big)$ with $\bar{A} = D\bar{B}$, then there is a $\varphi\colon A \to B$ in FC^* with $M(\varphi) = D$.*

(IV) *If $\varphi, \psi\colon A \to B$ are FC_0^*-morphisms with $M(\varphi) = M(\psi)$, then $\varphi \sim \psi$.*

(V) *If φ is an FC^*-morphism, then $M(\varphi)$ has no zero row. An FC_0^*-morphism φ is injective iff $M(\varphi)$ has no zero column.*

Proof. (I) The fact that M is a functor follows form 1.10. The functor M is representative: Indeed, let $(Z^+)^X$ be an object in $M(Z^+)$; find any object A of FC^* so that X and $\mathrm{Min}A$ have the same cardinality. Then $M(A) \cong (Z^+)^X$.

(II) We have to show that $\deg J = \sum \{\mathrm{mult}\,\varphi_{JK}\,\deg K\colon K \in \mathrm{Min}A\}$ for all $J \in \mathrm{Min}B$, provided that φ preserves identities: For $K \in \mathrm{Min}A$ let e_K be the identity of K and set $e'_K = \mathrm{pr}_J\varphi(e_K) = \mathrm{pr}_J\varphi\,\mathrm{copr}_K(e_K)$ $= \varphi_{JK}(e_K)$; define $J_K = e'_K J e'_K$. If $K \neq L$ in $\mathrm{Min}A$, then $e_K e_L = 0$, hence $e'_K e'_L = \mathrm{pr}_J(e_K e_L) = 0$, and $1 = \sum\{e_K\colon K \in \mathrm{Min}A\}$ implies $\sum\{e'_K\colon K \in \mathrm{Min}A\} = \mathrm{pr}_J\varphi(1) = 1$ since φ and pr_J are in FC^*. It follows that

(a) $\deg J = \sum\{\deg J_K\colon K \in \mathrm{Min}A\}.$

But

(b) $\deg J_K = \mathrm{mult}\,\varphi_{JK}\deg K \quad$ for all $K \in \mathrm{Min}A$

by 1.8 (1'), for if $\varphi_{JK} \neq 0$, then its corestriction $\varphi_{JK}\colon K \to J_K$ is in FC^*, and if $\varphi_{JK} = 0$, then both $\mathrm{mult}\,\varphi_{JK} = 0$ and $J_K = 0$. The assertion follows from (a) and (b).

(III) Suppose that $D = (d_{JK})_{(J,K)\in\mathrm{Min}B\times\mathrm{Min}A}$ is a matrix such that

(1') $\deg J = \sum\{d_{JK}\,\deg K\colon K \in \mathrm{Min}A\} \quad$ for all $J \in \mathrm{Min}B.$

By 1.3, we have to find a matrix $(\varphi_{JK})_{(J,K)\in\mathrm{Min}B\times\mathrm{Min}A}$ with $(\mathrm{im}\,\varphi_{JK})(\mathrm{im}\,\varphi_{IL}) = 0$ for $K \neq L$ in $\mathrm{Min}(A)$ and with $\mathrm{mult}\,\varphi_{JK} = d_{JK}.$

Now fix $J \in \mathrm{Min}B$. Since $\deg J = \sum\{d_{JK}\deg K\colon K \in \mathrm{Min}A\}$ we may decompose the identity f of J as $f = \sum\{e_K\colon K \in \mathrm{Min}A\}$ with orthogonal projections e_K of rank $d_{JK}\deg K$. Then we may define $\bar{\varphi}_{JK}\colon K \to e_K J e_K$ to be 0 if d_{JK}, hence e_K is zero, and to be injective of multiplicity d_{JK} otherwise, since K and $e_K J e_K$ are both simple hence $K = L(C^{\deg K})$ and

$e_K J e_K = L(C^d J K^{\deg K})$. Let φ_{JK}: $K \to J$ be the coextension of $\bar{\varphi}_{JK}$. The matrix $(\varphi_{JK})_{(J,K) \in \text{Min}B \times \text{Min}A}$ is the required one.

(IV) Suppose $M(\varphi) = M(\psi)$ for two FC_0^*-morphisms φ, ψ: $A \to B$. This means that

(a) $$\text{mult}\,\varphi_{JK} = \text{mult}\,\psi_{JK}, \qquad \varphi_{JK}, \psi_{JK}: K \to J$$

$$\text{for all} \quad J \in \text{Min}B, \ K \in \text{Min}A.$$

Suppose momentarily that we find a unitary element u_J of J such that $u_J \cdot \text{pr}_J \varphi = \text{pr}_J \psi_J$; then we are finished, since $\varphi = \sum \text{pr}_J \varphi$, $\psi = \sum \text{pr}_J \psi$ and we may define the unitary element u componentwise by u_J on J. We thus may assume that B is simple and $M(\varphi)$ and $M(\psi)$ consist of a single row. For each $K \in \text{Min}A$ let e_K, respectively, f_K be the image in B of the identity of K under φ, respectively ψ. Then $e = \sum e_K$, resp. $f = \sum f_K$ is the image of the identity 1 of A under φ and ψ, respectively. We then have $\text{rank}\, e_K = \deg e_K B e_K = \text{mult}\,\varphi_{BK} \deg A = \text{mult}\,\psi_{BK} \deg A = \deg f_K B f_K = \text{rank}\, f_K$. By the remarks following 1.8 we find an isometry β_K: $e_K B e_K \to f_K B f_K$ with $\beta_K \varphi_{BK} = \psi_{BK}$. Since the e_K as well as the f_K are orthogonal there is a unitary element u of B such that $b \mapsto ubu^*$ induces on $e_K B e_K$ the isomorphism β_K onto $f_K B f_K$. (The most concrete way to verify these assertions is to identify B with $L(H)$ for some Hilbert space H of dimension $\deg B$.) Now e_K (resp. f_K) is an orthogonal projection onto a $\text{rank}\, e_K = \text{rank}\, f_K$ dimensional subspace H_K (resp. H_K'), and β_J is given by conjugation with an isometry u_K: $H_K \to H_K'$. Since the H_K (resp. the K_K') are orthogonal, there is a unitary operator u on H which induces u_K on H_K. Now

$$\psi = \sum \{\psi_{BK} \text{pr}_K: K \in \text{Min}A\} = \sum \{u_K \cdot \varphi_{BK} \text{pr}_K: K \in \text{Min}B\}$$

$$= \sum \{u \cdot \varphi_{BK} \text{pr}_K: K \in \text{Min}B\} = u \cdot \varphi.$$

(V) follows from 1.7. ∎

We observe that property (III) describes to which extent the functor M is surjective on morphisms and (IV) says that M induces a faithful functor on $\widetilde{\text{FC}}_0^*$ and $\widetilde{\text{FC}}^*$, i.e. the matrix associated with a morphism characterizes it up to equivalence. We finally note that (II) and (III) have an FC_0^* counterpart. Each $(\mathbf{Z}^+)^n$ has a natural partial order which is given by $p \leqslant q$ iff there is a w with $q = p + w$. With this partial order we have

(II$_0$) *For each* φ: $A \to B$ *in* FC_0^* *we have*

(1$_0$) $$M(\varphi)\bar{A} \leqslant \bar{B}.$$

(III$_0$) *If* D *is a matrix in* $M(\mathbf{Z}^+)[M(A), M(B)]$ *with* $D\bar{B} \leqslant \bar{A}$, *then there is a* φ: $A \to B$ *in* FC_0^* *with* $M(\varphi) = D$.

The proof of (II$_0$) uses 1.8. (1) in place of 1.8 (1'), and the proof given for (III) works for (III$_0$). Note that the relation $\bar{B} = M(\varphi)\bar{A}$ for an FC_0^*-morphism φ: $A \to B$ implies $\varphi \in \text{morph}\,\text{FC}^*$.

For the purposes of describing FC^* and its morphisms, Theorem 1.15 is perfectly adequate. From the functorial point of view a slight variation of the same theme is a bit more satisfactory. Let us make the following general observation:

If $M(R)$ is the matrix category over a semiring, we may produce a new category $\tilde{M}(R)$ as follows:

(i) The objects of $\tilde{M}(R)$ are the elements of R^X *with non-zero components* (considered as column matrices), where X is any finite set; the morphisms of $\tilde{M}(R)$ are triples (x, D, y) with objects x, y and a matrix D over R such that $y = Dx$; the domain of (x, D, y) is y, the codomain is x. Two morphisms (x, D, y) and (x', D', y') compose iff $y = x'$, and then $(x, D, y)(y, D', y') = (x, DD', y)$. We call $\tilde{M}(R)$ the *localized matrix category* over R. There is a functor $P = P_R$: $\tilde{M}(R) \to M(R)$ given by $P(x) = R^X$, where $x \in R^X$ and $P(x, D, y) = D$; then P is a faithful and representative functor which is clearly not full. We apply this, in particular, to the case $R = Z^+$ and factor the functor M of 1.10 through $P = P_{Z^+}$:

1.16. PROPOSITION. *Define a function* $\tilde{M}$: $FC^* \to \tilde{M}(Z^+)$ *by* $\tilde{M}(A)$ $= \bar{A} \in (Z^+)^{\mathrm{Min}A}$ *(the column matrix* $(\deg K)_{K \in \mathrm{Min}A}$*) and* $\tilde{M}(\mathrm{class}\,\varphi)$ $= (\bar{B}, M(\varphi), \bar{A})$ *for a morphism* φ: $A \to B$. *Then*

(i) *the functor* $\tilde{M}$ *establishes an equivalence from* FC^* *onto the localized matrix category over* Z^+,

(ii) *the following diagram commutes*

$$
\begin{array}{ccc}
FC^* & \xrightarrow{\text{quotient functor}} & \tilde{F C}^* \\
{\scriptstyle M}\downarrow & \cong & \downarrow{\scriptstyle \tilde{M}} \\
M(Z^+) & \xleftarrow[\;\;P_{Z^+}\;\;]{} & \tilde{M}(Z^+).
\end{array}
$$

Proof. The proof is clear from Theorem 1.15 and the preceding remarks. ∎

The graphic representation

We have seen that the localized matrix category over Z^+ completely describes the category of finite-dimensional C^*-algebras with identity (up to equivalence of morphisms). What remains to be done is to introduce a convenient graphical way to represent morphisms in $\tilde{F C}^*$ (i.e. in $\tilde{M}(Z^+)$). A morphism

$$
\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \; (d_{jk})_{\substack{j=1,\ldots,m \\ k=1,\ldots,n}}, \; \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) \quad \text{in} \quad \tilde{M}(Z^+)
$$

is represented by a weighted graph with $m+n$ vertices preferably depicted in two layers, one layer containing m and the other n vertices. The vertices are indexed by $j = 1, \ldots, m$ and $k = 1, \ldots, n$ and carry weights x_j resp. y_k. The vertex j is connected with a vertex k by an edge of multiplicity d_{jk} (and multiplicity 0 means the absence of an edge).
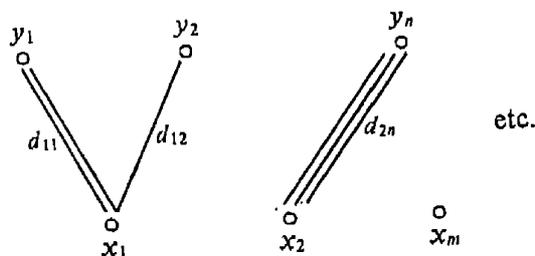
By 1.15 (V), every one of the lower (j-vertices) is connected with at least one of the upper (k-vertices), and if the matrix $D = (d_{jk})$ represents an injective morphism of FC^*, then every one of the upper (k-indices) is connected with at least one of the lower (j-indices).

Furthermore, the weight x_j is equal to $\sum_{k=1}^{n} d_{jk} y_k$. Clearly, every morphism of $\tilde{M}(Z^+)$ and hence every morphism of $\tilde{FC}^*$ is described in an essentially unique fashion by precisely one of the weighted graphs with the properties described above.

Matrix units

We return to C^*-algebras and conclude the section with an appendix in which we record the definition of a set of matrix units of a (finite-dimensional) C^*-algebra.

We consider first a finite-dimensional C^*-algebra A. Then $A = \oplus \operatorname{Min} A$. Each $I \in \operatorname{Min} A$ is isomorphic to a matrix algebra $M_{r(I)}$ over C. Let $u_{Ik} \in I$ be the element corresponding to the matrix with a 1 in the first column and kth row and zeros elsewhere. Then the family

$$\{u_{Ik}\colon I \in \operatorname{Min} A, \ 0 \leqslant k \leqslant r(I)\}$$

is a set of matrix units for A. We can generalize this situation to apply to arbitrary C^*-algebras with identity:

1.17. DEFINITION. Let A be a C^*-algebra with identity, and let $r\colon \{1, \ldots, n\} \to N$ be a function. Then a *set of matrix units* (*of type* r) is a family $\{u_{ik} \in A\colon 1 \leqslant i \leqslant n, \ 1 \leqslant k \leqslant r(i)\}$ such that the following conditions are satisfied:

(1) Each u_{ik} is a partial isometry (i.e. $u_{ik} u_{ik}^*$ is a projection).

(2) $\sum \{u_{ik} u_{ik}^* \colon 1 \leqslant i \leqslant n, \; 1 \leqslant k \leqslant r(i)\} = 1.$

(3) $u_{ik}^* u_{jl} = \delta_{ij} \delta_{kl} u_{il}$ for all i, j, k, l.

Note that in particular u_{il} is a projection for all $i = 1, \ldots, n$. ∎

The subalgebra generated by such a set of matrix units in A is isomorphic to $M_{r(1)} \oplus \ldots \oplus M_{r(n)}$. One should remark that often the family of elements $\{u_{ij}^* u_{ik}^* \colon 1 \leqslant i \leqslant n, \; 1 \leqslant j, k \leqslant r(i)\}$ are also called *matrix units*; these evidently can be recovered from the family we introduced in 1.13.

2. Almost finite-dimensional C^*-algebras

The definitions

Let C_0^* be the category of C^*-algebras and C^* the subcategory of C^*-algebras with identity and identity preserving morphisms.

2.1. PROPOSITION. *Let A be a C^*-algebra. Then the following statements* (1) *and* (2) *are equivalent:*

(1) *There is a direct system $\{A_j; f_{jk} , j, k \in J\}$ $f_{jk} \colon A_k \to A_j$ of finite-dimensional C^*-algebras and A is (isomorphic to) the colimit of this system in the category of C^*-algebras.*

(2) *A contains an upwards directed family $\{A_j \colon j \in J\}$ of finite dimensional subalgebras such that $\bigcup \{A_j \colon j \in J\}$ is dense in A.*

These statements imply

(3) *For every finite subset F of A and every $\varepsilon > 0$ there is an injective function $t \colon F \to A$ and a finite dimensional subalgebra $B = B(F, \varepsilon)$ such that $t(F) \subseteq B$ and $\|t(a) - a\| < \varepsilon$ for all $a \in F$.*

Proof. $(1) \Rightarrow (2)$: If a direct system $\{A_j \colon f_{jk}, j, k \in J\}$ is given for a directed set J and if $A = \operatorname{colim} A_j$ with colimit maps $f_j \colon A_j \to A$, then $\{f_j(A_j) \colon j \in J\}$ is an upward directed family of finite-dimensional subalgebras. Let B be the closure of the union of this family. If $f_j' \colon A_j \to B$ denotes the corestriction of f, then by the colimit property there is a unique morphism $\varphi \colon A \to B$ with $f_j' = \varphi f_j$ for all $j \in J$. If $\psi \colon B \to A$ is the inclusion, then the following infinite diagram commutes:

By the uniqueness property of the colimit it follows that $\psi\varphi = \operatorname{id}_A$. Thus ψ is surjective, i.e. $B = A$.

(2) ⇒ (3) is trivial.

(2) ⇒ (1) will be proved by the following slightly more general lemma:

2.2. LEMMA. *If A is a C^*-algebra and $\{A_j\colon j \in J\}$ an up-directed family of C^*-subalgebras such that $\bigcup A_j$ is dense in A, then $A = \operatorname{colim} A_j$ (relative to the system of inclusion maps).*

Proof. Let $f_{jk}\colon A_k \to A_j$ be the inclusion if $k \leqslant j$. We define $f_j\colon A_j \to A$ to be the inclusion and show that $A = \operatorname{colim} A_j$. Let B be a C^*-algebra and $g_j\colon A_j \to B$ a family of C^*-morphisms such that $g_j f_{jk} = g_k$ for $k \leqslant j$, i.e. $g_j | A_k = g_k$. We then have a unique involutive linear map $g\colon \bigcup \{A_j\colon j \in J\} \to B$ such that $g | A_j = g_j$. Since all g_k are contractions as C^*-morphisms, g is a contraction. Hence it extends uniquely to a C^*-morphism $g\colon A \to B$ with $g f_j = g | A_j = g_j$. ∎

2.3. DEFINITION. A C^*-algebra A is called *almost finite-dimensional* or an *AFC_0^*-algebra* iff it satisfies the equivalent conditions of (1) and (2) of 2.1. It is called an *AFC^*-algebra* if in addition it has an identity. The full subcategory in C_0^* of AFC_0^*-algebras is called AFC_0^*. The subcategory of AFC_0^*-objects with identity preserving morphisms is called AFC^*. A C^*-algebra is *weakly almost finite-dimensional* iff it satisfies (3) of 2.1 and a *weak AFC^*-algebra* if, in addition, it has an identity.

2.4. PROPOSITION. *Let A be an AFC^*-algebra. Then the following statements are equivalent:*

(1) *A is separable.*

(2) *There is a sequence of finite-dimensional subalgebras $A_1 \subseteq A_2 \subseteq$ with $A = \bigcup \{A_n\colon n \in N\}^-$.*

Proof. (2) ⇒ (1) is trivial.

(1) ⇒ (2): Let $D = \{a_1, a_2, \ldots\}$ be a countable dense subset of A and $\{A_j\colon j \in J\}$ an up-directed family of finite-dimensional subalgebras with dense union. We define a sequence $j(n) \in J$ as follows: Pick $j(1)$ so that $\operatorname{dist}(a_1, A_{j(1)}) < 1/2$. Suppose that $j(1), \ldots, j(n)$ is an increasing set of indices such that $\operatorname{dist}(a_k, A_{j(n)}) < 1/n$, for $k = 1, \ldots, n$. Now pick an index $j' \in J$ such that $\operatorname{dist}(a_k, A_{j'}) < 1/n+1$ for $k = 1, \ldots, n+1$ and pick $j(n+1)$ arbitrarily above $j(n)$ and j'. In this fashion we obtain an ascending sequence $B_n = A_{j(n)}$ of subalgebras such that $\bigcup \{B_n\colon n \in N\}^-$ contains D, hence A. ∎

In the separable case, the situation was further clarified by Bratteli [7].

2.5. PROPOSITION. *Let A be a separable C^*-algebra. Then A is an AFC_0^*-algebra iff it satisfies condition (3) of 2.1.*

The proof we are now going to present is different from that of Bratteli. We use the concept of a Banach space bundle with variable fibers and a simple bundle theoretical lemma due to Fell. A Banach space bundle is a continuous open surjective function $p\colon E \to B$ such that all fibers

$E_x = p^{-1}(x)$ are Banach spaces in the induced topology and that the following conditions are satisfied: (1) Addition is a continuous function $E \times_B E \to E$ (where $E \times_B E = \{(x, y) \in E \times E \colon p(x) = p(y)\}$ is the pullback over B); (2) scalar multiplication $C \times E \to E$ is a continuous function; (3) the zero section $x \mapsto O(x) \colon B \to E$ is continuous; (4) the norm $x \mapsto \|x\| \colon E \to R^+$ is continuous; (5) through each point $x \in E$ there is a local section $s \colon U \to E$, $s(p(x)) = x$, U open in B containing $p(x)$, and x has arbitrarily small neighborhoods of the form $\{y \in E \colon p(y) \in V$, $\|y - s(p(y))\| < \varepsilon\}$, V open in U containing x, $0 < \varepsilon$. The definition of a C^*-algebra bundle is now clear.

We recall the definition of matrix units in a C^*-algebra given in 1.17 and are ready for the formulation of

FELL'S LEMMA. *Let* $p \colon E \to B$ *be a* C^*-*algebra bundle and let* $\{u_{ik} \colon 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant r(i)\}$ *be a set of matrix units of type* r *in the fiber* $E_x = p^{-1}(x)$. *Then there is a neighborhood* U *of* $p(x)$ *in* B *and there are bounded continuous sections* $\beta_{ik} \colon U \to E$, $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant r(i)$ *such that*

(i) $\beta_{ik}(x) = u_{jk}$ *for all* i *and* k.

/   (ii) $\{\beta_{ik}(y) \colon 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant r(i)\}$ *is a set of matrix units in* E_y *for* $y \in U$.

Moreover, if $\{\beta'_{ik} \colon 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant r(i)\}$ *is another such set of local sections defined on an open neighborhood* U' *of* x, *then there is an open neighborhood* W *of* x *in* $U \cap U'$ *and a local section* $\gamma \colon W \to E$ *such that*

(i) $\gamma(x) = 1$,

(ii) $\gamma(y)$ *is unitary for all* $y \in W$,

(iii) $\gamma(y)^* \beta'_{ik}(y) \gamma(y) = \beta_{ik}(y)$.

Proof. The first assertion is just a restatement of Fell's lemma. To show the second assertion, let $\{e_{ij}(y) \colon 1 \leqslant i \leqslant n; 1 \leqslant j \leqslant r(i)\}$ be the final projection of $\beta_{ij}(y)$ for $y \in U \cup U'$. The e_{ij} are of course a set of local sections defined on the neighborhood $U \cap U'$ of x. Similarly let $e'_{ij}(y)$ be the final projection of $\beta'_{ij}(y)$. By Fell's lemma there is a set of local sections $\{\mu_i \colon 1 \leqslant i \leqslant n\}$ on some neighborhood $V \subseteq U \cap U'$ of x with the properties

(a) $\mu_i(x) = u_{i1}(x)$;

(b) $\mu_i(y)$ is a partial isometry in E_y with initial projection $e_{i1}(y)$ and final projection $e'_{ij}(y)$.

Now $\gamma_{ik}(y) = \beta'_{ik}(y) \mu_i(y) \beta^*_{ik}(y)$ is a partial isometry with initial projection $e_{ik}(y)$ and final projection $e'_{ik}(y)$. As the sum of partial isometries with orthogonal initial and orthogonal final projections is itself a partial isometry, it follows that $\gamma(y) = \sum_{ik} \gamma_{ik}(y)$ is a partial isometry. Condition (ii) of Definition 1.13 assures that $\gamma(y)$ is in fact unitary.

Now

$$\gamma(x) \;=\; \sum_{i,k} \beta'_{ik}(x)\,\mu_i(x)\,\beta^*_{ik}(x) \sum u_{ik}\,u_{il}\,u^*_{ik},$$

$$\sum u_{ik}\,u^*_{ik} \;=\; 1$$

by (ii) of 1.13.

Also if $y \in V$,

$$\beta'_{ik}(y)\gamma(y) \;=\; \beta'_{ik}(y)\gamma_{il}(y) \;=\; \gamma_{ik}(y)\beta_{ik}(y)\gamma(y)\beta_{ik}(y),$$

so that $\gamma(y)^* \beta'_{ik}(y)\gamma(y) = \beta_{ik}(y)$. ∎

We will use Fell's Lemma in the proof of the following result:

2.6. PROPOSITION. *Let A be a weak AFC^*-algebra with identity. Let B be a finite-dimensional subalgebra, $\varepsilon > 0$ and $t \in A$. Then there is a finite-dimensional subalgebra C containing B such that $\|t - C\| \leqslant \varepsilon$.*

Proof. For each n, let T^n be the trivial bundle $p_1\colon A^n \times A \to A$ and let B^n be the subbundle of T^n consisting of (ξ, x) such that x is in the C^*-algebra generated by ξ and 1. It is straightforward to verify that B^n is a C^*-bundle.

Now let $\{u_{ik}\colon 1 \leqslant i \leqslant n;\ 1 \leqslant k \leqslant r(i)\}$ be a system of matrix units which generates the C^*-algebra B. Let $x_0 = (\{u_{ik}\colon 1 \leqslant i \leqslant n;\ 1 \leqslant k \leqslant r(i)\}, t) \in A^{q+1}$, where q is the degree of B. By Fell's lemma there is a neighborhood U of x_0 in A^{q+1} and a family $\{\beta_{ik}\}$ of local cross sections of B^{q+1} on U such that $\beta_{ik}(x_0) = u_{ik}$. Let $\{\beta'_{ik}\}$ be global cross sections of the bundle T^{q+1} which are identically u_{ik}. By the second part of Fell's lemma there is a neighborhood V of x_0 in T^{q+1} and a local cross section on V consisting of unitary elements such that

$$\gamma(y)^* u_{ik}\gamma(y) \;=\; \gamma(y)^* \beta'_{ik}(y)\gamma(y) \;=\; \beta_{ik}(y).$$

Now approximate finiteness of A implies that for each q, $\{y \in A^q\colon B^q_y$ is finite dimensional$\}$ is dense in A^q. On the other hand, by continuity of γ there is a $y \in V$ such that (i) $\|y - x_0\| \leqslant \varepsilon/2$, (ii) $\|\gamma(y) - 1\| \leqslant \tfrac{1}{2}\varepsilon(1 + \|t\| + \varepsilon)$, (iii) B^{q+1}_y is finite dimensional.

(i) implies $\|t - B^{q+1}_y\| \leqslant \varepsilon/2$. As

$$u_{ik} \;=\; \gamma(y)\beta_{ik}(y)\gamma^*(y),$$

ti follows that $B \subseteq \gamma(y)B^{q+1}_y\gamma^*(y)$ and clearly

$$\|t - \gamma(y)B^{q+1}_y\gamma^*_\varepsilon(y)\| \leqslant \varepsilon. \quad ∎$$

The following now strengthens Proposition 2.4 and proves Proposition 2.5:

2.7. COROLLARY. *Let A be a separable C^*-algebra. Then the following statements are equivalent:*

(1) *A is a weak AFC^*-algebra;*

(2) *There is a sequence of finite-dimensional subalgebras* $A_1 \subseteq A_2$ $\subseteq$ *with* $A = \bigcup \{A_n : n = 1, 2, \ldots\}^-$;

(3) A *is an* $\mathrm{AF}C^*$-*algebra.*

Proof. (2) $\Rightarrow$ (3) is trivial and by 1.2 we have (3) $\Rightarrow$ (1).

In order to prove (1) $\Rightarrow$ (2) we let $\{x_i\}$ be dense in A. By Proposition 2.6 applied inductively, there is an increasing sequence of finite-dimensional C^*-subalgebras of A such that $\|x_n - A_n\| \leqslant 1/n$. ∎

We do not understand the situation in the absence of separability.

2.8. REMARK. *Let* A *be an* C_0^*-*algebra and* $\tilde{A}$ *the algebra obtained by adjoining an identity, Then* $\tilde{A}$ *is an* $\mathrm{AF}C^*$-*algebra if* A *is an* $\mathrm{AF}C_0^*$-*algebra.*

Proof. Straightforward. ∎

After this remark it is no great loss to assume that we operate in $\mathrm{AF}C^*$.

Examples

In order to become familiar with the concept of almost finite-dimensional C^*-algebras we consider the abelian case. We first note a simple lemma whose proof we leave to the reader (as an easy application of Gelfand–Naimark duality):

2.9. LEMMA. *Let* X *be a compact Hausdorff space. Then a subalgebra* A *of* $C(X)$ *is finite dimensional if and only if there is a finite cover of* X *by compact open sets* $U_1, \ldots, U_n$ *such that the functions of* A *are constant on each* U_k. ∎

Since the collection of all finite compact open covers of X is directed w.r.t. the relation of refinement of covers, we have the following remark as an immediate consequence, using the theorem of Gelfand and Naimark on commutative C^*-algebras:

2.10. PROPOSITION. *In any commutative* C^*-*algebra* A *with identity the family* $\mathscr{F}(A)$ *of finite-dimensional subalgebras is up-directed w.r.t. inclusion.* ∎

For the next step recall that a function on a topological space is locally constant iff the inverse image of each range point is open.

2.11. COROLLARY. *If* X *is a compact Hausdorff space and* $L \subseteq C(X)$ *the subalgebra of all locally constant functions, then* L *is the smallest closed subalgebra containing all finite-dimensional subalgebras.*

Proof. By Lemma 2.9 and Proposition 2.10, L is the union of all finite-dimensional subalgebras, whence the assertion. ∎

The following result then is a parallel to 2.5:

2.12. PROPOSITION. *Let* A *be a commutative* C^*-*algebra with identity. Then the three conditions of* 2.1 *are equivalent.*

Proof. We may assume $A = C(X)$. We have to show that 2.1 (3) implies 2.1 (2). Now if condition 2.1 (3) is satisfied, then every function

of $C(X)$ can be uniformly approximated by locally constant functions in view of Lemma 2.9. Hence $A = L$ (with L as in 2.11). Hence 2.1 (2) follows and 2.11. ∎

This allows the following characterization of commutative AFC*-algebras:

2.13. PROPOSITION. *The category* AFC_{ab}^{*} *of commutative almost finite dimensional* C^{*}*-algebras with identity and identity preserving maps is dual to the category of Boolean spaces (compact zero-dimensional spaces).*

The functors implementing the duality are Spec: $\mathrm{AFC}_{ab}^{*} \to$ Boole, $\mathrm{Spec}\,A = C^{*}(A, C)$ *(with the* $weak^{*}$*-topology) and* C: Boole $\to \mathrm{AFC}_{ab}$, $C(X) = \mathrm{Top}(X, C)$ *(with the pointwise operation and sup-norm.)*

Proof. By the theorem of Gelfand and Naimark on commutative C^{*}-algebras the functors Spec: $C_{ab}^{*} \to$ Comp and Comp $\to C_{ab}^{*}$ implement a duality. By 2.10, the full subcategory of Comp which under this duality corresponds to the full subcategory AFC_{ab}^{*} in C_{ab}^{*} is that of all compact spaces X which are profinite, i.e. are projective limits of an inverse system of finite discrete spaces and (surjective) continuous maps. These spaces are precisely the Boolean spaces. ∎

Note that the separable algebras among C_{ab}^{*}-algebras are characterized by the fact that their spectrum is a separable Boolean space.

Let us also consider a typical non-abelian example:

2.14. EXAMPLE. Let H be a Hilbert space and $A = LC(H)$ the C^{*}-algebra of compact operators. Then A is an AFC_{0}^{*}-algebra.

Proof. For each orthogonal projection e of H of finite rank we obtain a finite-dimensional subalgebra eAe whose degree is the rank of e. Conversely, every finite-dimensional subalgebra B of A is of the form eAe with the identity e of B. The finite rank orthogonal projections form a (modular) lattice isomorphic to the lattice of finite-dimensional subspaces. If $e \leqslant f$, then $eAe \subseteq fAf$. Thus the family of all $\{eAe$: e a finite rank projection of $H\}$ is the collection $\mathcal{F}(A)$ of all finite-dimensional subalgebras and is upwards directed. Moreover, every hermitian compact operator can be approximated by a finite rank hermitian operator. It follows that $\left(\bigcup \mathcal{F}(A) \right)^{-}$ is dense in A. Thus A is an AFC_{0}^{*}-algebra. ∎

By an earlier remark, $C \cdot 1 + A$ is an AFC*-algebra.

In the abelian case and the case $LC(H)$, the collection of finite subalgebras in A is (upwards) directed under inclusion. We do not know whether this is the case for all AFC*-algebras.

Bratteli's scheme

We now utilize the results of Section 1 to give a description of an AFC*-algebra together in terms of a system of FC*-algebras whose colimit it is. Our presentation is an elaboration of Bratteli's ideas.

To each system $\{A; A_j, j \in J\}$ of an AFC^*-algebra A together with an up-directed family of FC^*-algebras A_j with $A = (\bigcup A_j)^-$ we associate firstly the inclusion maps $\varphi_{jk}: A_j \to A_k$ for $j \leqslant k$. From Section 1 we obtain a diagram $\{\tilde{M}(A_j) = \bar{A}_j,\ M(\varphi_{kj}),\ j, k \in J\}$ in the localized matrix category over $\mathbf{Z}^+$.

We would like to discard some inessential complications in the possible index sets J. First, we note that it is no loss in generality to assume that the following finite predecessor condition holds

(FPD) $\{k \in J: k \leqslant j\}$ is finite for all j.

Indeed, we may assume that the family $\{A_j: j \in J\}$ is self-indexed, i.e. that J is a set of finite-dimensional subalgebras in A which is up-directed relative to $\subseteq$ and whose union is dense in A, so that $j = A_j$ and $j \leqslant k$ means $j \subseteq k$.

Secondly, it turns out to be somewhat inconvenient at a later point to deal with finite index sets which may occur if A is finite dimensional and in the self-indexed situation occurs precisely then. If J is finite, then it has a maximal element j. We then replace J by $J' = J \cup \{j_1, j_2, ...\}$ with $i \leqslant j_1 \leqslant j_2 \leqslant ...$ for all $i \in J$ and set $A_{j_k} = A_j = A$, defining

$$\varphi_{j_n j_m} = 1_A \text{ for } m \leqslant n \quad \text{and} \quad \varphi_{j_n i} = \varphi_{ji} \text{ for } i \in J.$$

Note that $\{A; A_{j'}, j' \in J'\}$ now gives rise to an infinite diagram such that J' still satisfies (FPD) if J does.

Finally we may assume that J has a minimal element 1 such that $A_1 = C \cdot 1$.

In order to have a convenient way of talking about such diagrams we introduce the following notation:

2.15. DEFINITION. Let J be an infinite directed set with minimal element 1 satisfying the finite predecessor condition. A family

$$\{(z_x^j)_{x \in X(j)};\ (d_{xy}^{jk})_{(x,y) \in X(j) \times X(k)}:\ j, k \in J\}$$

s called a *Bratteli diagram* if it satisfies the following conditions:

(1) All $X(j)$ are finite sets, and $X(1)$ is a singleton.

(2) $z_x^j \in \mathbf{Z}^+ \setminus \{0\}$ for all $j \in J,\ x \in X(j)$; and $z_x^1 = 1$.

(3) $d_{xy}^{jk} \in \mathbf{Z}^+$ for all $j, k \in J$ with $k < j$,

$$(x, y) \in X(j) \times X(k),$$

such that for each $j, k \in J$ with $k \leqslant j$ the following condition holds:

(i) for each $x \in X(j)$ there is a $y \in X(k)$ with $d_{xy}^{jk} \neq 0$.

(4) $d_{xu}^{ik} = \sum_{y \in X(j)} d_{xy}^{ij} d_{yu}^{jk}$ for all $i, j, k \in J$ with $k \leqslant j \leqslant i$.

(5) $z_x^j = \sum_{y \in X(k)} d_{xy}^{jk} z_y^k$ for all $j, k \in J$ with $k \leqslant j$.

We note that, as a consequence of (4) (upon considering $k = 1$) we know that for each j, $k \in J$ with $k \leqslant j$ also the following condition holds

(ii) for each $y \in X(k)$ there is an $x \in X(j)$ with $d_{xy}^{jk} \neq 0$. ∎

We can visualize a Bratteli diagram as a weighted graph whose vertices are the points of $X = \bigcup \{X(j): j \in J\}$ (where we consider the sets $X(j)$ to be disjoint) such that to the vertex $x \in X(j)$ one associates the weight z_x^j, and in which each vertex $x \in X(k)$ is connected with a vertex $y \in X(j)$ with $j \geqslant k$ with an edge of weight d_{xy}^{jk} (where $d_{xy}^{jk} = 0$ is understood to signify the absence of an edge).

The following proposition expresses the fact that Bratteli diagram carries much of the information needed to reconstruct an AFC*-algebra from approximating FC*-subalgebras; only in the separable case, however, will we be able to show that every Bratteli diagram arises in this way.

2.16. PROPOSITION. *For each AFC*-algebra A together with an infinite up-directed system $\{A_j: j \in J\}$ of finite-dimensional sub-algebras with $A_1 = C \cdot 1$ the family $\{\overline{A}_j; M(\varphi_{jk}): j, k \in J\}$ with $\varphi_{jk}: A_k \to A_j$ the inclusion for $j \geqslant k$, is a Bratteli diagram.*

Proof. This follows from the results of Section 1. ∎

Let us look for a moment at the obstructions we encounter if we seek to construct an AFC*-algebra from a given Bratteli diagram $\{(z_x^j)_{x \in X(j)}; (d_{xy}^{jk})_{(x,y) \in X(j) \times X(k)}: j, k \in J\}$. We find a family $\{A_j: j \in J\}$ of FC*-algebras A_j with $\overline{A}_j = (z_x^j)_{x \in X(j)}$ for all $j \in J$. By 1.15 there are morphisms $\varphi_{kj}: A_j \to A_k$ such that $M(\varphi_{kj}) = (d_{xy}^{jk})_{(x,y) \in X(j) \times X(k)}$, and φ_{kj} is unique up to unitary equivalence. By 1.15. (V), φ_{kj} is injective. The difficulty, however, is that $\{\varphi_{kj}: j, k \in J\}$ need not be a direct system of maps, since we have no guarantee that with the random selection of φ_{kj} within its unitary equivalence class we have $\varphi_{kj}\varphi_{ji} = \varphi_{ki}$ for $i \leqslant j \leqslant k$ in J. For very complicated index sets J we see no way how this difficulty may be overcome so that $\operatorname{colim}\{A_j; \varphi_{kj}, j, k \in J\}$ becomes a direct limit.

The separable case

The situation is more fortunate in the separable case. Here we may assume that $J = N$, and if we start as in 2.16, the system $\{\overline{A}_j; M(\varphi_{jk}): j, k \in J\}$ is uniquely determined by the sequence of matrices $M(\varphi_{k+1,k})$, $k \in N$ with $M(\varphi_{21})$ being a row matrix. Conversely, if a sequence $B^j = (b_{xy}^j)_{(x,y) \in X(j+1) \times X(j)}$, $j = 1, 2, \ldots$, of matrices $(b_{xy}^j)_{(x,y) \in X(j+1) \times X(j)}$, $j = 1, 2, \ldots$, with no zero rows and columns is given for a disjoint sequence $X(j)$ of finite sets, then we may use condition (4) in 2.15 to construct inductively a Bratteli diagram

$$\{(z_x^j)_{x \in X(j)}; D^{jk} = (d_{xy}^{jk})_{(x,y \in X(j) \times X()}^k, j, k \in N\},$$

where $D^{jk} = B^j B^{j-1} \ldots B^{k+1} B^k$. Using 1.15 (III), we may, at the same time, construct a sequence of inclusion morphisms $\varphi_j \colon A_j \to A_{j+1}$ of FC^*-algebras such that $\bar{A}_j = (z_x^j)_{x \in X(j)}$ and $M(\varphi_j) = (b_{xy}^j)_{(x,y) \in X(j+1) \times X(j)}$. The family of maps $\varphi_{kj} = \varphi_{k,k-1} \ldots \varphi_j, \, j \leqslant k$ is an inverse system with a colimit A, and the colimit maps A_j are isometries onto their image in A. If we identify the A_j with their images in A we have an AFC^*-algebra and an approximating system of finite-dimensional subalgebras which together lead to the given Bratteli diagram in the canonical fashion. Thus we have:

2.17. COROLLARY. *Every separable AFC^*-algebra A together with a infinite sequence $A_1 \subseteq A_2 \subseteq \ldots$ of subalgebras with a dense union and $A_1 = C \cdot 1$ determines a sequence of matrices with no zero rows and columns*

$$(*) \qquad \vert B^j = (b_{xy}^j)_{(x,y) \in X(j+1) \times X(j)}, \qquad j = 1, \ldots,$$

with $X(1)$ singleton such that $M(\varphi_j) = B_j$ for the inclusion $\varphi_j \colon A_j \to A_{j+1}$, $j = 1, 2, \ldots$ Every sequence of matrices $(*)$ is obtained in this fashion. There is a bijection between sequences of matrices $(*)$ with no zero rows and columns and Bratteli diagrams

$$(**) \qquad \{(z_x^j)_{x \in X(j)}; \, (d_{xy}^{jk})_{(x,y) \in X(j) \times X(k)} \colon j, k \in N\}$$

(see 2.14). ∎

More examples

As an illustration we draw the weighted graphs of some of the examples of AFC^*-algebras which we discussed earlier.

2.18. EXAMPLES. (1) The graph of the Bratteli diagram associated with $C(X)$, where X is the Cantor set in $[0, 1]$: Set $A = C(X)$, $A_0 = $ all constant functions, A_n all locally constant functions which are constant on the set of numbers $x \in X$ whose normalized triadic expansion agrees on the first n digits.

The associated Bratteli diagram has the following graph
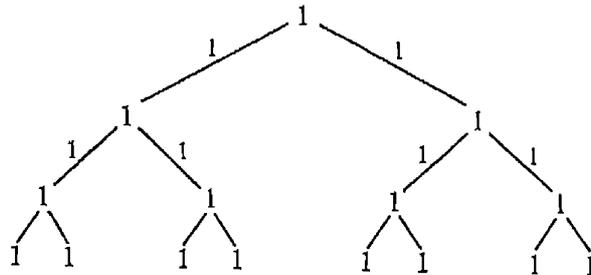
(2) Consider $A = LC(H) + C \cdot 1$ for a separable Hilbert space H. Let ξ_n, $n = 1, 2, \ldots$, be an orthonormal basis of H and $e_n \in LC(H)$ the orthogonal projection of H onto the span of $\xi_1, \ldots, \xi_n$. We define $A_0 = C \cdot 1$ and $A_n = e_n A e_n + C \cdot 1$. Then A_n decomposes into the direct sum $e_n A e_n \oplus I_n$ of minimal ideals with $e_n A e_n \cong M_n(C)$, $I_n = C(1 - e_n) = C$. Thus $X(n) = \{e_n A e_n, I_n\}$.

The matrix elements of the inclusion $\varphi \colon A_n \to A_{n+1}$ are as follows:

(1) $\varphi_{e_{n+1} A e_{n+1}, e_n A e_n}$ is the natural inclusion, multiplicity 1;

(2) $\varphi_{e_{n+1} A e_{n+1}, I_n}$ is the natural map of $C(1 - e_n)$ to $C(e_{n+1} - e_n)$, multiplicity 1;

(3) $\varphi_{I_{n+1}, e_n A e_n}$ is zero;

(4) φ_{I_{n+1}, I_n} is the isomorphism from $C(1 - e_n)$ to $C(1 - e_{n+1})$, multiplicity 1.

The associated multiplicity matrix therefore is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We obtain the following graph for the associated Bratteli diagram:
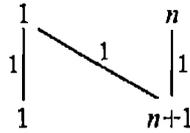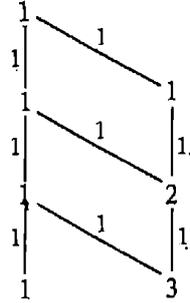
(3) Let $B_1, B_2, \ldots$ be a sequence of full matrix algebras of rank $r_1, r_2, \ldots$, respectively, and set $A_0 = C \cdot 1$, $A_{n+1} = A_n \otimes B_{n+1}$ and let $\varphi_n \colon A_n \to A_{n+1}$ be given by $a \mapsto a \otimes 1$. Define $A = \operatorname{colim} A_n$ in C^*. Then each A_n is simple, and φ_n has multiplicity r_{n+1}. If we set $d_n = r_1 + \ldots + r_n$ we have the associated Bratteli diagram.

$$1$$
$$r_1 \,\big|$$
$$d_1 \,\big|$$
$$r_2 \,\big|$$
$$d_2$$

A is a UHF algebra (see Glimm [16]).

3. Ideals in AFC*-algebras

3.1. PROPOSITION. *Let A be a C^*-algebra with an up-directed family $\{A_j\colon j \in J\}$ of C^*-subalgebras with $\overline{\bigcup A_j} = A$. Let $\mathrm{Id}(A)$ denote the lattice of all closed two-sided ideals of A and $\mathrm{Id}_J A$ the collection of all families $\{I_j\colon j \in J\}$ such that*

(a) $I_j \in \mathrm{Id}(A_j)$,

(b) $j \leqslant k$ in J implies $I_j = I_k \cap A_j$.

Define

$$\varphi\colon \mathrm{Id}(A) \to \mathrm{Id}_J(A) \quad by \quad \varphi(I) = \{I \cap A_j\colon j \in J\},$$

$$\psi\colon \mathrm{Id}_J(A) \to \mathrm{Id}(A) \quad by \quad \psi(\{I_j\colon j \in J\}) = \big(\bigcup\{I_j\colon j \in J\}\big)^-$$

Then φ and ψ are inverses of each other.

Proof. First we note that φ and ψ are well defined; in the case of ψ this follows from the fact that the family of the I_j is up-directed by (b) and that $\bigcup I_j$ is an ideal in $\bigcup A_j$ by (a). We now show that φ and ψ are inverses of each other.

Step (i). To prove that $\psi\varphi = 1$, we must show that $I = \big(\bigcup\{I \cap A_j\colon j \in J\}\big)^-$. The inclusion $\supseteq$ is clear. To check the converse, we consider the infinite diagram

$$
\begin{array}{ccccc}
I \cap A_j & \to & A_j & \to & A_j/(I \cap A_j) \cong (A_j + I)/I \\
\downarrow & & \downarrow{\scriptstyle \Phi_{kj}} & & \downarrow{\scriptstyle (\Phi_{kj} + I)/I} \\
I \cap A_k & \to & A_k & \to & A_k/(I \cap A_k) \cong (A_k + I)/I
\end{array}
$$

(see Dixmier [11], p. 18). By 2.2. we have $\mathrm{colim}\, I \cap A_j = \big(\bigcup (I \cap A_j)\big)^-$, $\mathrm{colim}\, A_j = \big(\bigcup A_j\big)^- = A$ and $\mathrm{colim}\,(A_j + I)/I = A/I$. Since the colimit functor preserves colimits (see e.g. Mitchell [20], p. 67, 12.2), and since

quotients are co-equalizers, hence colimits, the sequence

$$\left(\bigcup (I \cap A_k) \right)^{-} \to A \to A/I$$

is exact. Hence $\left(\bigcup (I \cap A_k) \right)^{-} = I$.

Step (ii). In order to prove that $\varphi\psi = 1$, we have to show that $\left(\bigcup I_k \right)^{-} \cap A_j = I_j$ for all j. If $j \leqslant k$, then the map $x + I_j \mapsto x + I_k$: $A_j/I_j \to A_k/I_k$ is injective by (b), hence is an isometry onto its image (Dixmier [11], p. 18). Hence $\|a + I_j\| = \|a + I_k\|$ for all $k \geqslant j$ and for all $a \in A_j$. It follows that $\|a + I_j\| = \| a + (\bigcup I_k)^{-} \|$, for $a \in A_j$, whence $\left(\bigcup I_k \right)^{-} \cap A^j = I_j$. ∎

We now describe the ideals of an AFC*-algebra in terms of an associated Bratteli diagram. Here we refer to Lemmas 1.11 and 1.12. Suppose that A is an AFC*-algebra and $\{A_j: j \in J\}$ an up-directed family of finite-dimensional subalgebras such that $\bigcup A_j$ is dense. We denote $\operatorname{Min} A_j$ by $X(j)$. Let us now assume that I is a closed two-sided ideal of A and let $I_j = I \cap A_j$. Then we set $Y(j) = \{K \in X(j): K \subseteq I_j\}$. Then $I_j = \sum \{K: K \in Y(j)\}$. The essential properties of the sequence of sets $Y(j)$ are described in the following lemma, which is an immediate consequence of Lemma 1.12.

3.2. LEMMA. *Suppose that* $j \leqslant k$ *in* J. *Then the following statements hold*:

(1) *If* $K \in Y(j)$ *and* $K' \in X(k)$ *such that* $\operatorname{mult}(\varphi^{kj})_{K'K} \neq 0$, *then* $K' \in Y(k)$.

(2) *If* $K \in X(j)$ *and* $K' \in Y(k)$ *for all* $K' \in X(k)$ *for which* $\operatorname{mult}(\varphi^{kj})_{K'K} \neq 0$, *then* $K \in Y(j)$. ∎

We also have a converse of this lemma:

3.3. LEMMA. *If* A *is an* AFC*-*algebra with up-directed family* $\{A_j; j \in J\}$ *of finite-dimensional subalgebras such that* $\bigcup A_j$ *is dense in* A. *Suppose that* $Y(j) \subseteq X(j) = \operatorname{Min} A_j$ *denotes a family of subsets which satisfies* (1) *and* (2) *of* 1.11. *We set* $I_j = \sum \{K: K \in Y(j)\}$ *and define* $I = \left(\bigcup \{I_j: j \in J\} \right)^{-}$. *Then* I *is a closed two-sided ideal of* A, *and* $I \cap A_j = I_j$.

Proof. It suffices to observe that the family $\{I_j: j \in I\}$ satisfies conditions (a) and (b) of 3.1. But (a) is trivial and Lemma 1.11 secures (b). ∎

We may now reformulate these results as follows:

3.4. PROPOSITION. *Let* A *be an* AFC*-*algebra with an up-directed family* $\{A_j: j \in J\}$ *of finite-dimensional subalgebras with* $A = \left(\bigcup A_j \right)^{-}$. *Then there is a bijection between the set* $\operatorname{Id}(A)$ *of closed two-sided ideals of* A *and the set of families* $\{Y(j): j \in J\}$ *of subsets* $Y(j) \subseteq X(j) = \operatorname{Min} A_j$ *satisfying the following conditions for all pairs of indices* j, k *with* $j \leqslant k$:

(1) *For all* $(K', K) \in X(k) \times X(j)$ *with* $\operatorname{mult}(\varphi^{kj})_{K'K} \neq 0$ *the relation* $K \in Y(j)$ *implies* $K' \in Y(k)$.

(2) *For all* $K \in X(j)$ *the relation* $K' \in Y(k)$ *for all* $K' \in X(k)$ *with* $\mathrm{mult}(\varphi^{kj})_{K'K} \neq 0$ *implies* $K \in Y(j)$.

The bijection is given by $I \mapsto \{\mathrm{Min}(I \cap A_j) \colon j \in J\}$. ∎

4. Bratteli diagrams and partially ordered sets

The information carried along in the Bratteli diagrams is too complex for most applications. It is fortuitous that for the ideal theory of AFC*-algebras the essential information is carried by the order structure of these diagrams. The weights can be neglected for this purpose. We carry out the details in the following section.

Augmented posets

A *partially ordered set* or *poset* is a set P together with a partial order $\leqslant$ (i.e. a reflexive, transitive, antisymmetric binary relation). The *opposite poset* P^{op} of a poset P is the same underlying set with the relation $\leqslant'$ given by $x \leqslant' y$ iff $y \leqslant x$. A function $f\colon P \to Q$ between posets is a poset morphism iff $x \leqslant y$ in P implies $f(x) \leqslant f(y)$. We use the notation $\uparrow x = \{y \in P \colon x \leqslant y\}$ and $\downarrow x = \{y \in P \colon y \leqslant x\}$, and if $X \subseteq P$, then $\uparrow X = \bigcup \{\uparrow x \colon x \in X\}, \downarrow X = \bigcup \{\downarrow x \colon x \in X\}$.

4.1. DEFINITION. An *augmented poset* is a triple (P, δ, J) consisting of two posets P and J and a surjective poset morphism $\delta\colon P \to J^{\mathrm{op}}$ such that the following conditions are satisfied:

(i) P has a largest and J has a smallest, but no largest element, and δ preserves maximal elements (i.e. $\delta(\mathrm{max}P) = \mathrm{min}J$).

(ii) J is directed.

(iii) J satisfies the finite predecessor condition (i.e. every subset $\downarrow j, j \in J$ is finite.

(iv) All sets $P^j = \delta^{-1}(j), j \in J$, are finite.

(v) If $j \leqslant k$ in J and if $x \in P^j$, then there is an $x' \in P^k$ with $x' \leqslant x$.

(vi) If $i \leqslant j \leqslant k$ in J and $x \in P^i$ with $z \in \downarrow x \cap P^k$, then there is a $y \in \downarrow z \cap \downarrow x \cap P^j$.

Frequently we will just speak of an *augmented poset* P. The poset J is called the *degree set*, and δ is called an *augmentation* or a *degree function*. The set $\delta^{-1}(j)$ is called a *layer* (*of degree* j). ∎

The order reversing property of d is a matter of convenience: We will see that for our applications P should have a largest element, whereas a frequent degree set is the set $N = \{1, 2, \ldots\}$ of natural numbers with its natural order. Conditions (v) and notable (vi) turn out to be quite useful.

4.2. PROPOSITION. *Let* $\{(z_x^j)_{x \in X(j)};\ (d_{xy}^{jk})_{(x,y) \in X(j) \times X(k)}: j,\ k \in J\}$ *be a Bratteli diagram (see* 2.15*). We may assume that the* $X(j)$ *are disjoint. Set* $P = \bigcup \{X(j): j \in J\}$ *and define* $\delta: P \to J$ *by* $\delta(x) = j$ *iff* $x \in X(j)$. *If* $x, y \in P$ *define* $x \leqslant y$ *iff* $\delta(y) \leqslant \delta(x)$ *and* $d_{xy}^{\delta(x)\delta(y)} \neq 0$. *Then* (P, δ, J) *is an augmented poset.*

The proof is straightforward from the data. ∎

Geometrically we visualize (P, δ, J) in the same fashion as was described following 2.15 but we now omit all weight data. Thus, the augmented poset carries the geometric or combinatorial information contained in the Bratteli diagram.

4.3. DEFINITION. We will say that (P, δ, J) is the *augmented poset associated with the Bratteli diagram in* 4.2. If this Bratteli diagram is in particular of the form $\{\bar{A}_j;\ M(\varphi_{jk}): j,\ k \in J\}$ for an AFC*-algebra A with a system $\{A_j: j \in J\}$ of finite-dimensional subalgebras with $A = (\bigcup_J A_j)^-$, then we say that (P, δ, J) is the *augmented poset associated with* $\{A; A_j: j \in J\}$.

It would be very desirable to know whether in fact *any* augmented poset (P, δ, J) is associated with some Bratteli diagram, or better still, with a system $\{A, A_j, j \in J\}$ of C^*-algebras with $A = (\bigcup_J A_j)^-$, $A_j \in \text{FC}^*$. This appears to be difficult to decide; one is inclined to believe that the answer is negative in general.

However, if $J = N$, a simple inductive process provides an affirmative answer.

4.4. PROPOSITION. *Every augmented poset* (P, δ, N) *is associated with some AFC*-algebra* A *and some sequence* $A_1 = C \cdot 1 \subseteq A_2 \subseteq A_3 \subseteq$ *of finite-dimensional subalgebras with dense union.*

Proof. By Corollary 2.17 it will suffice to construct a sequence of matrices

(*) $$B^j = (b_{xy}^j)_{(x,y) \in X(j+1) \times X(j)}, \qquad j = 1, 2, \ldots,$$

where $X(1)$ is singleton, $X(j) = \delta^{-1}(j)$ such that $b_{xy}^j > 0$ iff $x \leqslant y$, $x \in X(j+1)$, $y \in X(j)$. By 4.1 (v) and (vi) this will imply that B^j has no zero column, respectively, row.

There are many ways to construct the B^j. For example, we can take

$$b_{xy}^j = \begin{cases} 1 & \text{if} \quad x \leqslant y, \\ 0 & \text{otherwise.} \end{cases} \quad \blacksquare$$

Note that we do not obtain any uniqueness in the existence proof for 4.4. We want to point out further that at this point we have made no claim that every *countable* augmented poset (P, δ, J) is associated with an AFC*-algebra or a Bratteli diagram even though in this case J contains a cofinal copy of N.

Ideals of augmented posets

We now proceed to the definition of an ideal in an augmented poset:

4.5. DEFINITION. Let (P, δ, J) be an augmented poset. Then $I \subseteq P$ is an *ideal* iff the following two conditions are satisfied:

(i) $\downarrow I \subseteq I$.

(ii) If $x \in P$, $\delta(x) \leqslant j$ and $\downarrow x \cap P_u^j \subseteq I$, then $x \in I$.

4.6. PROPOSITION. *The set* $\mathrm{Id}(P)$ *of all ideals of an augmented poset is a complete lattice with respect to intersection as the* inf *operation.*

Proof. The empty set is an ideal, and P is an ideal. If $\{I_s \colon s \in S\}$ is a family of ideals, then $\bigcap \{I_s \colon s \in S\}$ is an ideal since it satisfies (i) and (ii).

4.7. LEMMA. *Let* $X \subseteq P$ *in an augmented poset. Define* $[X] = \{x \in P \colon$ *there is a* $j \geqslant \delta(x)$ *such that* $\downarrow x \cap P^j \subseteq \downarrow X\}$; *then* $[X]$ *is an ideal, in fact the smallest ideal containing* X.

Proof. Since $[X] = [\downarrow X]$, we may assume that $\downarrow X = X$. In order to verify 4.4 (i), let $x \in [X]$ and $y \leqslant x$. We must show that $y \in [X]$. By definition of $[X]$ there is a $j \in J$ with $\delta(x) \leqslant j$ such that $\downarrow x \cap P^j \subseteq X$. We pick a $k \in J$ with $\delta(y), j \leqslant k$. Now take any $z \in \downarrow y \cap P^k$. By 4.1 (vi), since $\delta(x) \leqslant j \leqslant k$, there is a $u \in P$ with $z \leqslant u \leqslant x$. Then $u \in \downarrow x \cap P^j \subseteq X$, and thus $z \in \downarrow X \subseteq X$.

Next we show 4.4 (ii): Suppose $x \in P$, $\delta(x) \leqslant j$ and $\downarrow x \cap P^j \subseteq [X]$. We must show that $x \in [X]$. Now $\downarrow x \cap P^j = \{x_1, \ldots, x_n\}$ by 4.1 (iv). By the definition of $[X]$, for each $k \in \{1, \ldots, n\}$ there is a $j_k \in J$ such that $\downarrow x_k \cap P^{j_k} \subseteq X$. Now we pick an index $j \geqslant j_1, \ldots, j_n$ and take an arbitrary $y \in \downarrow x \cap P^j$; if we can show that $y \in X$ we are done, for then $\downarrow x \cap P^j \subseteq X$, whence $x \in [X]$. But by 4.1(vi) there is an $x' \in \downarrow x \cap P^j$ with $y \leqslant x'$. Hence there is a $k \in \{1, \ldots, n\}$ with $x' = x_k$. By 4.1 (vi) once more there is an $x'' \in \downarrow x_k \cap P^{j_k} \subseteq X$ with $y \leqslant x''$. But then $y \subseteq \downarrow X \subseteq X$ as had to be shown.

4.8. LEMMA. *If* $\{I_s \colon s \in S\}$ *is a family of ideals of* P, *then* $\bigvee \{I_s \colon s \in S\}$
$= \left[\bigcup \{I_{s_j} \colon s \in S\} \right] = \{x \in P \colon$ *There is a* $j \geqslant \delta(x)$ *such that* $\downarrow x \cap P^j$
$\subseteq I_{s_1} \cup \ldots \cup I_{s_n}$ *for some suitable* $s_1, \ldots, s_n \in S\}$.

Proof. The first equality is clear from 4.6 and the second follows from the fact that $\downarrow x \cap P^j$ is finite by 4.1 (iv). ∎

4.9. LEMMA. $\mathrm{Id}(P)$ *is a complete Brouwerian lattice, i.e. for any family* $\{I_s \colon s \in S\}$ *of ideals and any ideal* J *we have* $J \cap \bigvee \{I_s \colon s \in S\} = \bigvee \{J \cap I_s \colon s \in S\}$.

Proof. The inclusion $\supseteq$ is trivial; we have to show the converse. Let $x \in J \cap \bigvee_S I_s = J \cap [\bigcup_S I_s]$ (4.8); then (a) $x \in J$, and (b) there is

a $j \geqslant \delta(x)$ with $\downarrow x \cap P^j \subseteq \bigcup_S I_s$. By (a), 4.1 and 4.5 (i) we also have $\downarrow x \cap P^j \subseteq \downarrow x \subseteq J$. Thus $\downarrow x \cap P^j \subseteq J \cap \bigcup_S I_s = \bigcup_S (J \cap I_s)$, whence $x \in \left[\bigcup_S (J \cap I_s) \right] = \bigvee \{ J \cap I_s : s \in S \}$ by 4.8. This finishes the proof. ∎

The lattice of ideals of an augmented poset

We recall at this point the lattice theoretical concepts of a compact element and a prime element in a lattice.

4.10. DEFINITION. Let L be a lattice.

(i) An element $k \in L$ is called *compact* iff for each subset $S \subseteq L$ with $k \leqslant \sup S$ there is a finite subset $F \subseteq S$ with $k \leqslant \sup S$. We write $K(L) = \{ k \in L : k$ is compact$\}$.

(ii) An element $p \in L$ is called *prime* iff for each finite set $F \subseteq L$ with $\inf F \leqslant p$ there is an $x \in F$ with $x \leqslant F$. We write $\mathrm{Prime}(L) = \{ p \in L : p$ is prime$\}$. One says that $p \in L$ is *meet irreducible* if for every finite set F with $\inf F = p$ there is an x with $x = p$. ∎

Evidently, each prime element is meet irreducible, but not necessarily conversely. We remark that in a distributive lattice L an element is prime exactly if it is meet irreducible. For the purposes of the lattices $\mathrm{Id}(P)$, by 4.9 one therefore need not distinguish between the two concepts. We know now what we mean by compact or prime ideals of an augmented poset.

We next identify the compact elements in $\mathrm{Id}(P)$. The result we encounter is typical for situations similar to ours.

4.11. LEMMA. *An ideal* I *of* $\mathrm{Id}(P)$ (*for an augmented poset*) *is compact iff it is finitely generated, i.e. iff* $I = [x_1, \ldots, x_n]$ *for suitable elements* $x_m \in P$.

Proof. (i) Suppose $I = [x_1, \ldots, x_n]$, and assume that $I \subseteq \bigvee \{ I_s : s \in S \}$ for a family of ideals I_s. For each finite subset $F \subseteq S$ we set $I_F = \bigvee \{ I_s : s \in S \} = \left[\bigcup \{ I_s : s \in S \} \right]$ (4.8). Then $\bigvee \{ I_s : s \in S \} = \bigcup \{ I_F : F$ finite in $S \}$ and the latter union is upwards directed. Thus, in particular, $\{ x_1, \ldots, x_n \} \subseteq \bigcup \{ I_F : F$ finite in $S \}$. Hence there is a finite $F \subseteq S$ with $x_m \in I_F$ for all $m = 1, \ldots, n$, whence $I \subseteq I_F$. This proves that I is a compact element of $\mathrm{Id}(P)$.

(ii) Now suppose that I is a compact element of $\mathrm{Id}(P)$. Then $I \subseteq \bigvee \{ [x] : x \in J \}$. Hence, by compactness, there are elements $x_1, \ldots, x_n \in P$ such that

$$ I \subseteq \bigvee \{ [x_m] : m = 1, \ldots, n \} = [x_1, \ldots, x_n]. ∎ $$

It is now clear that every ideal of $\mathrm{Id}(P)$ is the least upper bound of the compact (i.e. finitely generated) ideals it contains. Lattices with this property are well known in lattice theory:

4.12. DEFINITION. Let L bo a lattice. Then L is called *algebraic* if (i) it is complete, (ii) every $x \in L$ satisfies $x = \sup(\downarrow x \cap K(L))$. ∎

We note that completeness is not always included in the definition but the relevant concept arises from the conjunction of (i) and (ii). In view of 4.9 we now have

4.13. THEOREM. *For every augmented poset* $(P\ \delta, J)$, *the lattice* $\mathrm{Id}(P)$ *is an algebraic Brouwerian lattice.* ∎

As we will see later (5.24), every algebraic Brouwerian lattice arises in this fashion. The following lemma illustrates why Theorem 4.13 is relevant:

4.14. LEMMA. *An algebraic lattice* L *is Brouwerian if and only if every* $x \in L$ *satisfies* $x = \inf(\uparrow x \cap \mathrm{Prime}(L))$ (*i.e. every element is the* inf *of primes*). ∎

For references see e.g. [18], notably p. 39, Corollary 3.6, p. 41, Theorem 3.10, and p. 71, Theorem 1.37. The second property in 4.14 is expressed in this source by saying that L is *primally generated*. Other sources call L *semiprime* under these circumstances. In any case we have the following result:

4.15. THEOREM. *Let* (P, δ, J) *be an augmented poset, and* I *an ideal of* P. *Then for any* $x \notin I$ *there is a prime ideal* I' *containing* I *with* $x \notin I'$.

Proof. The assertion is evidently equivalent to the claim that I is the intersection of prime ideals; it then follows from 4.13 and 4.14. ∎

AFC*-algebras and augmented posets have equivalent ideal theories

The significance of the augmented posets in the context of AFC*-algebras is that they fully reflect the ideal theory of the algebras. Specifically, we have the following result:

4.16. THEOREM. *Let* A *be an* AFC*-*algebra and* $\{A_j: j \in J\}$ *an infinite up-directed family of finite-dimensional subalgebras such that* $A = (\bigcup A_j)^-$. *Let* (P, δ, J) *be the augmented poset associated with* $\{A; A_j: j \in J\}$. *For each member* I *in the lattice* $\mathrm{Id}(A)$ *of closed two-sided ideals of* A *we obtain a unique family* $\{Y(j): j \in J\}$ *of subsets of* P^j (*according to 3.6*) *and* $\varphi(I) = \bigcup_J Y(j)$ *is an ideal of* φ. *The function*

$$\varphi: \mathrm{Id}(A) \to \mathrm{Id}(P)$$

is a lattice isomorphism.

Proof. By 3.6 the assignment $I \mapsto \{Y(j): j \in J\}$ is a bijection from the set of closed two-sided ideals of A to the set of families satisfying conditions (1) and (2) of 3.4. In view of Definition 4.5 of an ideal in P these conditions say precisely that $\bigcup_J Y(j)$ is an ideal in P. This shows that φ is a bijection. Since it is evidently isotone, it is an isomorphism of partially

ordered sets with respect to $\subseteq$, which then implies that φ is in fact a lattice isomorphism. ∎

This theorem allows us to investigate the spectral theory of AFC*-algebras by studying the spectral theory of augmented posets. Since in the lattice $\mathrm{Id}A$ of a C^*-algebra we have $IJ = I \cap J$, the closed prime ideals of A are precisely the prime elements of the lattice $\mathrm{Id}(A)$ and these are precisely the meet irreducible elements since $\mathrm{Id}(A)$ is distributive by 4.9. The only shortcoming is that the primitive ideals in $\mathrm{Id}(A)$ are not intrinsically characterized in the lattice. The spectrum which we will characterize therefore is the prime ideal spectrum of A. We note that for separable C^*-algebras the concepts of prime and primitive agree:

4.17. PROPOSITION. *A closed two-sided ideal in a separable C^*-algebra is prime if and only if it is primitive.*

Proof. See [11], p. 79: 3.9.1. ∎

It seems to be unknown whether this situation prevails in the non-separable case. A more recent study by Archbold [1] which characterizes prime C^*-algebras in terms of the order does not shed more light on this question.

5. The spectral theory of augmented posets

Filters and prime ideals

We now embark into a more detailed study of the complete Brouwerian lattice $\mathrm{Id}P$.

5.1. LEMMA. *Let (P, δ, J) be an augmented poset and take $x, y \in P$. Then $[x] \cap [y] = [\downarrow x \cap \downarrow y]$.*

Proof. The inclusion $\supseteq$ is clear. Thus assume that $z \in [x] \cap [y]$. Then by the definition of $[x]$, $[y]$ (see 4.6) there are indices $j, k \in J$ with $\downarrow z \cap P^j \subseteq \downarrow x$ and $\downarrow z \cap P^k \subseteq \downarrow y$. Let $m \in J$ be such that $j, k \leqslant m$. Then $\downarrow z \cap P^m \subseteq \downarrow(\downarrow z \cap P^j) \cup \downarrow(\downarrow z \cap P^k)$ by 4.1 (vi). It follows that $\downarrow z \cap P^m \subseteq \downarrow x \cap \downarrow y$, whence $z \in [\downarrow x \cap \downarrow y]$. ∎

5.2. DEFINITION. A *filter* F on a poset P is any subset satisfying the following two conditions:

(i) F is downwards directed, i.e. for all $x, y \in F$ there is a $z \in F$ with $z \leqslant x, y$.

(ii) $\uparrow F \subseteq F$, i.e. if $x \in F$ and $x \leqslant y$, then $y \in F$.

A filter F on an augmented poset P is *cofinal*, if $\delta(F)$ is cofinal in J.

5.3. PROPOSITION. *Let P be an augmented poset and I an ideal. Then the following statements are equivalent:*

(1) *I is a prime ideal.*

(2) *For all* $x, y \in P$ *the relation* $[x] \cap [y] \subseteq I$ *implies* $x \in I$ *or* $y \in I$.

(3) $P \setminus I$ *is a filter.*

(4) $P \setminus I$ *is a cofinal filter.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (1): Suppose that $K_1 \nsubseteq I$ and $K_2 \nsubseteq I$ for two ideals K_1, K_2. Then there are elements $x \in K_1 \setminus I$ and $y \in K_2 \setminus I$, whence $[x] \cap [y] \nsubseteq I$ by (2). But since $[x] \subseteq K_1$ and $[y] \subseteq K_2$ this implies $K_1 \cap K_2 \nsubseteq I$.

(3) $\Rightarrow$ (2): If $x, y \notin I$, then by (3) there is a $z \notin I$ with $z \leqslant x, y$. But then $z \in [x] \cap [y]$, whence $[x] \cap [y] \nsubseteq I$.

(2) $\Rightarrow$ (3): Let $x, y \notin I$; then, by Lemma 5.1 and (2), we have $[\downarrow x \cap \downarrow y]$ $= [x] \cap [y] \nsubseteq I$. Thus there is a $z' \notin I$ with

(*) $\downarrow z' \cap P^j \subseteq \downarrow x \cap \downarrow y$ for some j

by definition of $[X]$ in 4.6. Since I is an ideal and $z' \notin I$ we have $\downarrow z' \cap P^j \nsubseteq I$. Thus there is a $z \in (\downarrow z' \cap P^j) \setminus I$. But then also $z \leqslant x, y$ by (*). The fact that $\uparrow (P \setminus I) \subseteq P \setminus I$ is trivial from 4.5 (i).

(4) $\Rightarrow$ (3) is trivial. We finish the proof by showing that $P \setminus I$ is cofinal for any ideal I. Assume not; then there is a $j \in J$ such that $\delta^{-1}(\uparrow j) \cap (P \setminus I)$ $= \emptyset$, i.e. that $\delta^{-1}(\uparrow j) \subseteq I$. Take an $x \in P \setminus I$. Since J is directed, there is a k in J with $\delta(x), j \leqslant k$. Then $\downarrow x \cap P^k \neq \emptyset$ by 4.1 (v), hence $\delta(\downarrow x \cap P^k)$ $= \{k\} \subseteq \uparrow j$, thus $\downarrow x \cap P^k \subseteq \delta^{-1}(\uparrow j) \subseteq I$. But then, by 4.5 (ii), we find $x \in I$ which is a contradiction. ∎

5.4. Lemma. *For a filter in* (P, δ, J) *the following statements are equivalent:*

(1) *F has a minimal element.*

(2) *F has a minimum.*

(3) *F* $= \uparrow x$ *for some* x.

(4) *F is finite.*

If these conditions hold, F cannot be cofinal. But there may be infinite filters which are not cofinal.

Proof. (1) $\Rightarrow$ (2) since F is downwards directed; (2) $\Rightarrow$ (3) by 5.2 (ii); (3) $\Rightarrow$ (4) follows from 4.1. (iii) and (iv), (4) $\Rightarrow$ (1) is trivial. Since J has no smallest element, cofinal filters must be infinite. If (P, δ, J) is such that $J = P^{op}$ the same set with the opposite order, with $\delta = 1_P$, then P is the only cofinal filter. It is easy to find examples with proper infinite filters. ∎

We recall that for various reasons the case $J = N$ is particularly relevant: In this case, 5.3 is complemented by the following:

5.5. Remark. *If (P, δ, N) is an augmented poset, then the following statements are equivalent for any filter F:*

(1) *F is cofinal.*

(2) *F is infinite.*

(3) *$F \neq \uparrow x$ for any $x \in P$.* ∎

We now prove a converse of 5.3.

5.6. Proposition. *Let (P, δ, J) be an augmented poset. If F is a cofinal filter, then the complement $P \setminus F$ is a prime ideal.*

Proof. From 5.2 (ii) it is evident that $\downarrow(P \setminus F) = P \setminus F$. Let $x \in P$ and suppose that for some $j \in J$ we have $\downarrow x \cap P^j \cap F = \emptyset$. We will show that $x \notin F$. By way of contradiction, assume that $x \in F$. Since F is cofinal, there is a $z \in \downarrow x \cap F$ with $j \leqslant \delta(z)$. From $\delta(x) \leqslant j \leqslant \delta(z)$ and 4.1 (vi) there is a $y \in \downarrow x \cap P^j \cap \uparrow z$. But $z \leqslant y$ and $z \in F$ implies $y \in F$, so $y \in \downarrow x \cap P^j \cap \uparrow z$, a contradiction. ∎

5.7. Corollary. *For any augmented poset P, the function $I \mapsto P \setminus I$ induces a bijection from the set $\mathrm{Prime}(\mathrm{Id}(P))$ of prime ideals of P onto the set of cofinal filters of P.* ∎

For brevity, we will write $\mathrm{Pr}(P)$ in place of $\mathrm{Prime}(\mathrm{Id}(P))$ and denote the set of cofinal filters on P by $\mathrm{Spec}\,P$.

With 5.7, Theorem 4.15 translates as follows:

5.8. Corollary. *If I is an ideal in an augmented poset and $x \notin I$, then there is a cofinal filter F with $x \in F$ and $I \cap F = \emptyset$.* ∎

Hull-kernel topologies

We now turn to the introduction of the hull-kernel topology on $\mathrm{Pr}(P)$ and $\mathrm{Spec}\,P$. Since this is a general process applicable to all algebraic Brouwerian lattices, we record a few general facts. (For a general reference we refer e.g. to [2], [17], and [18].)

Let L be an algebraic Brouwerian lattice. For $x \in L$ we write $S(x) = (L \setminus \uparrow x) \cap \mathrm{Prime}\,L = \{p \in \mathrm{Prime}\,L : x \nleqslant p\}$. Then $\{S(x) : x \in L\}$ is a topology $O(\mathrm{Prime}\,L)$ on $\mathrm{Prime}\,L$, called the *hull-kernel topology*, and the map $x \mapsto S(x) : L \to O(\mathrm{Prime}\,L)$ is a lattice isomorphism whose inverse is given by $U \mapsto \inf((\mathrm{Prime}\,L) \setminus U)$. An element $k \in L$ is compact iff $S(x)$ is quasicompact (i.e. satisfies the Heine–Borel property). In particular, $\mathrm{Prime}\,L = S(1)$ is quasicompact iff $1 \in K(L)$. Note that $\{S(k) : k \in K(S)\}$, the set of quasicompact open subsets of $O(\mathrm{Prime}\,L)$ is a basis for the topology.

5.9. Definition. A subset E of a topological space X is called *irreducible* iff it is closed and is not the union of two proper closed subsets.

Note that every singleton closure $\overline{\{x\}}$ is irreducible. If X is an infinite space with the cofinite topology, then X is irreducible but has no dense point.

5.10. DEFINITION. A topological space X is called a *spectral space* iff,

(i) X is a T_0-space.

(ii) Every irreducible set has a dense point.

(iii) The topology has a basis of quasicompact sets.

We quote the following result (see e.g. [17]).

5.11. PROPOSITION. *The assignment* $L \mapsto \mathrm{Prime} L$ *establishes a bijection* (*up to isomorphism*) *between the class of algebraic Brouwerian lattices and spectral spaces; the inverse operation is given up to natural equivalence by* $X \mapsto O(X)$, *where* $O(X)$ *is the lattice of open sets of the space* X. *The isomorphism* $L \to O(\mathrm{Prime} L)$ *is given by* $x \mapsto S(x)$, *and the isomorphism* $X \to \mathrm{Prime} O(X)$ *by* $x \mapsto X \setminus \overline{\{x\}}$. *The bijection induces a bijection between the class of algebraic Brouwerian lattices with compact* ($=$ *isolated*) *identity and quasicompact spectral spaces.* ∎

We now translate this general set-up into our concrete situation of augmented posets. Let (P, δ, J) be an augmented poset, $L = \mathrm{Id}(P)$ its Brouwerian algebraic lattice of ideals. For $X \subseteq P$ we write $S(X)$ $= \{I \in \mathrm{Pr}(P): X \nsubseteq I\}$. Then clearly $S(X) = S([X])$ which makes the notation compatible with the one used in the general situation. We will also write $s(X) = \{F \in \mathrm{Spec} P: X \setminus F \ne \varnothing\}$. Then $P \setminus S(X) = s(X)$, and the bijection $I \mapsto P \setminus I$: $\mathrm{Pr}(P) \to \mathrm{Spec} P$ maps $S(X)$ onto $s(X)$. We record the following formulae:

5.12. LEMMA. (1) $S\left(\bigcup X_j\right) = \bigcup S(X_j)$ and $s\left(\bigcup X_j\right) = \bigcap s(X_j)$.

(2) $S([X] \cap [Y]) = S(X) \cap S(Y)$, and $s([X] \cap [Y]) = s(X) \cup s(Y)$.

Proof. This follows immediately from $S(X) = S([X])$ and the fact that $I \mapsto S(I)$: $\mathrm{Id}(P) \to O(\mathrm{Pr}(P))$ is a lattice isomorphism. ∎

Since the maximal element P of $\mathrm{Id}(P)$ is the finitely generated ideal [1], it is a compact element of $\mathrm{Id}(P)$ by 4.11. Hence $\mathrm{Pr}(P)$ is quasicompact. By 5.12 (1) we have $S(X) = \bigcup \{S(x): x \in X\}$, so that $\{S(x): x \in P\}$ is a basis of the hull-kernel topology consisting of quasicompact sets. We summarize:

5.13. PROPOSITION. *Let* (P, δ, J) *be an augmented poset. Then the set of all* $S(X)$, $X \subseteq P$ *is a topology on* $\mathrm{Pr}(P)$ *making* $\mathrm{Pr}(P)$ *into a quasicompact spectral space. A similar statement is true for the set of all* $s(X)$, $X \subseteq P$ *and* $\mathrm{Spec} P$. *The collections* $\{S(x): x \in P\}$, *respectively,* $\{s(x): x \in P\}$ *are bases of the respective topologies consisting of quasicompact open sets.* ∎

Henceforth we will always assume that $\mathrm{Pr}(P)$ and $\mathrm{Spec} P$ are equipped with these topologies.

It should be pointed out that in a Brouwerian algebraic lattice L the subset $K(L)$ is a sup-semilattice but not in general a sublattice (even if it is a lattice in its own right). The following example of an augmented poset is due to Dooley and illustrates this situation:
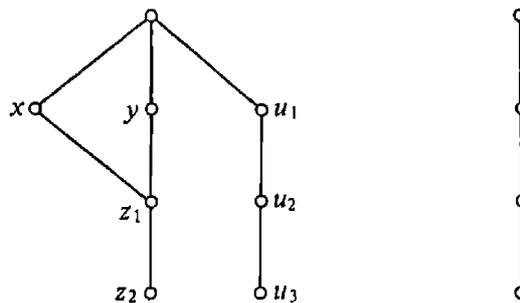
**5.14. EXAMPLE (Dooley).** Define (P, δ, N) as follows:
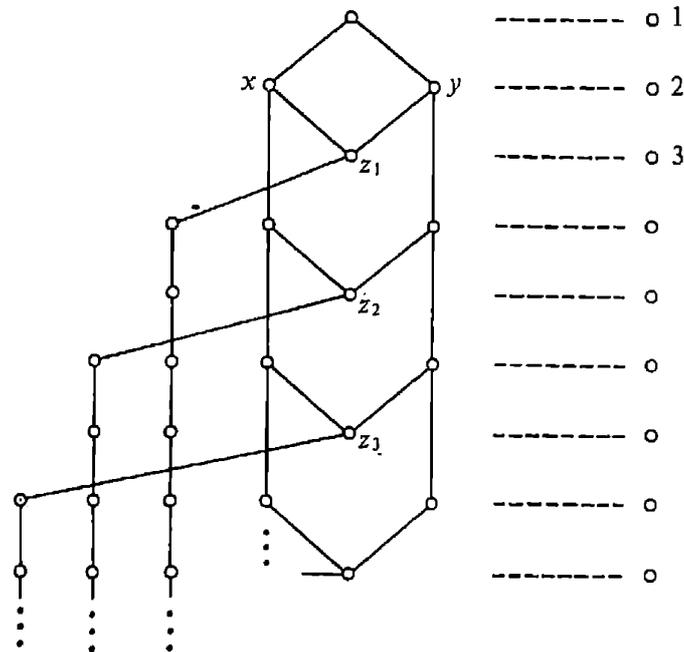
Then $S(x) \cap S(y) = S(z_1) \cup S(z_2) \cup S(z_3) \cup$ and no finite union of the $S(z_n)$ equals $S(x) \cap S(y)$. ∎

While $I \mapsto s(I)$: $\mathrm{Id}(P) \to O(\mathrm{Spec}P)$ is a lattice isomorphism, the reader should be warned that the function $x \mapsto s(x)$: $P \to O(\mathrm{Spec}P)$ need not even be injective. This remark is illustrated in the following example:

5.15. EXAMPLE. Define (P, δ, N) as follows:

Then $[x] = [y] = [z_n]$, $n = 1, 2, \ldots$, and the poset $\{s(z): z \in P\}$ relative to $\subseteq$ is given by
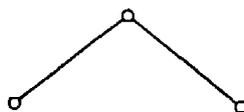
The example shows that points in the same layer such as x, y may generate the same ideal. ∎

AFC*-algebras and augmented posets
have equivalent spectral theories

We will now observe that the natural lattice isomorphism from the lattice of ideals of an AFC*-algebra to the lattice of ideals of the associated augmented poset P induces a homeomorphism from the prime spectrum of A to PrP (and thus to Spec$P$). This will allow us to discuss all topological questions of the prime spectrum of A in terms of the spectral theory of augmented posets.

5.16. THEOREM. *Let* A *be an* AFC*-*algebra and* $\{A_j: \ j \in J\}$ *an up-directed family of finite-dimensional subalgebras such that* $A = \left(\bigcup A_j\right)^-$. *Let* (P, δ, J) *be the augmented poset associated with* $\{A; A_j: \ j \in J\}$. *Then the lattice isomorphism*

$$\varphi: \ \mathrm{Id}(A) \to \mathrm{Id}(P)$$

(given in 4.9) induces homeomorphisms

$$\mathrm{Prime}A \xrightarrow{\ \varphi|\mathrm{Prime}A\ } \mathrm{Pr}P \xrightarrow{\ I \to P\backslash I\ } \mathrm{Spec}P,$$

where Prime$A$ *is the space of closed two-sided prime ideals of* A *with the hull-kernel topology. In particular,* Prime$A$ *is a quasicompact spectral space.*

Proof. Since φ is a lattice isomorphism, it follows that it induces a bijection between the respective sets of prime elements in the lattices Id(A) and Id(P). In order to prove that the restriction and corestriction of φ is a homeomorphism we have to observe that the hull-kernel topology has precisely the open sets $S(G) = \{I \in \mathrm{Prime}A: \ G \nsubseteq I\}$, $G \in \mathrm{Id}A$; and indeed $S(G) = \mathrm{Prime}A \backslash h(G)$, $h(G) = \{I \in \mathrm{Prime}A: \ G \subseteq I\}$. The remainder is clear from 5.7 and 5.13. ∎

It is not hard to verify that the spectrum of Dooley's poset 5.9 is a sequence of isolated points p_n, $n = 1, 2, \ldots$, converging to two distinct points p, p'. Specifically, Spec$P$ is homeomorphic to $N \cup \{p, p'\}$ with a basis of open sets consisting of all $\{n\}$, $n \in N$, $\uparrow n \cup \{p\}$, $n = 1, 2, \ldots$ This is also the spectrum of the AFC*-algebra of sequences $(a_n)_{n=1,2,\ldots}$ of 2 by 2 complex matrices a_n converging to a diagonal matrix.
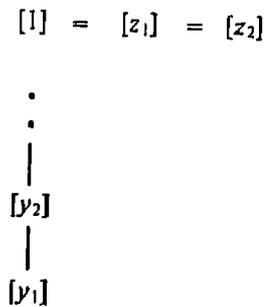
Example 5.14 shows that $\mathscr{B}$ need not be closed under intersections, where $\mathscr{B}$ is the set of quasicompact sets.

The following example will illustrate that $\mathscr{B}$ need not satisfy the finite predecessor condition. ∎

5.17. EXAMPLE. We consider the augmented poset (P, δ, N) given in the following diagram:
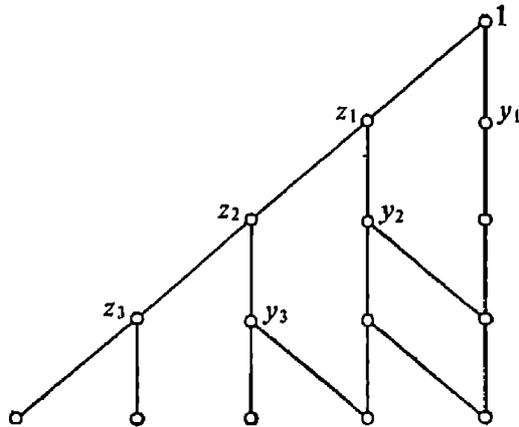
The poset $\mathrm{Id}\,P$ is given in the following diagram

$$[1] = [z_1] = [z_2]$$

All ideals are principal and prime. The basis $\mathscr{B} = \{s(x)\colon x \in P\}$ consists of an ascending sequence of sets and their union. The poset is associated with an AFC*-algebra (4.4). ∎

At this point we summarize what has been achieved: With each AFC*-algebra A we were able to associate an augmented poset (P, δ, J) such that the lattice $\mathrm{Id}(A)$ of closed two-sided ideals of A was isomorphic to the lattice $\mathrm{Id}(P)$ of ideals of P; the latter was recognized as an algebraic Brouwerian lattice. Since the spectral theory of prime elements of Brouwerian algebraic lattices is known, we know the spectral theory of closed two-sided prime ideals of A. (Recall that, unfortunately, we know only in the separable case that each closed two-sided prime ideal of A is primitive.) The algebraic Brouwerian lattices correspond bijectively to the spectral spaces (5.11). Thus we know by Theorem 5.16 that each AFC*-algebra has a quasicompact spectral space as prime spectrum $\mathrm{Prime}\,A$.

Now the following question poses itself naturally: Does every quasi-compact spectral space arise as the prime spectrum of some AFC*-algebra of at the least of some augmented poset? We now address ourselves to this question.

We deal with this question by developing a theory for semilattices which parallels in some features that which we discussed for AFC*-algebras in Sections 2 and 3.

5.18. DEFINITION. A sup-semilattice K is called *distributive* if for all $x, y, z \in K$ the relation $x \leqslant y \vee z$ implies the existence of two elements $y' \leqslant y$ and $z' \leqslant z$ such that $x = y' \vee z'$. In the Appendix, we prove a result of Gaskill's saying that this condition is equivalent to the following: Every finite set $X \subseteq K$ is contained in some finite distributive subsemilattice.

Notice that every finite semilattice with identity is a lattice. For any lattice, the usual concept of distributivity is equivalent to that introduced in 5.18. Hence Gaskill's condition says that K is the union of finite sup-subsemilattices each of which is a distributive lattice in its own right.

We now assume that K is an infinite distributive sup-semilattice with identity 0 and maximum 1. If J is the collection of all finite distributive subsemilattices containing $0, 1$, then

(i) J is up-directed with respect to $\subseteq$,

(ii) $K = \bigcup J$,

(iii) J has the smallest element $\{0, 1\}$,

(iv) J satisfies the finite predecessor condition.

We will now assume that we are given a directed set J with minimal element 0 which satisfies the finite predecessor condition and an up-directed family $\{K_j : j \in J\}$ of finite distributive subsemilattices containing 0 such that $K_0 = \{0, 1\}$, and that $K = \bigcup \{K_j : j \in J\}$.

An element p in a sup-semilattice S is *co-prime* iff the relation $p \leqslant x \vee y$ implies $p \leqslant x$ or $p \leqslant y$. In a finite distributive sup-semilattice every element is a sup of co-primes (and this property is characteristic for distributivity in finite lattices, see 4.13).

For $j \in J$ we set $P_j = \{p \in K_j : p$ co-prime in $K_j, p \neq 0\}$. Now we define $P = \{(p, j) \in K \times J : p \in P_j\}$ and let $\delta : P \to J$ be the projection onto J. We partially order P as follows:

$$(p', j') \leqslant (p, j) \text{ iff either } p' = p \text{ and } j' = j \text{ or } p' \quad p \text{ and } j' > j.$$

Evidently, $\delta : P \to J^{\mathrm{op}}$ is an order morphism, and P has the smallest element $(1, \{0, 1\})$. Clearly, $P^j = \delta^{-1}(j) = P_j \times \{j\}$ hence is finite.

5.19. LEMMA. (P, δ, J) *is an augmented poset.*

Proof. If remains to verify conditions (v), (vi) of 4.1 (v): Let $(p, j) \in P^j$ and suppose that $j' \geqslant j$. Since $K_j \subseteq K_{j'}$, we have $p \in K_{j'}$, hence $p = \sup(\downarrow p \cap P_{j'})$ (since $K_{j'}$ is a distributive lattice). In particular, there is a $q \in K_{j'}$ with $q \leqslant p$. Then $(q, j') \in P^{j'}$ and $(q, j') \leqslant (p, j)$.

(vi) Suppose $j_1 \leqslant j_2 \leqslant j_3$ and assume $(p_3, j_3) \quad (p_1, j_1)$ in P. Then $K_{j_1} \subseteq K_{j_2}$ implies $p_1 = \sup(\downarrow p_1 \cap P_{j_2})$, since K_{j_2} is distributive. But p_3

is coprime in K_{j_3} and the finite sup p_1 dominates p_3, hence there is at least one $p_2 \in \downarrow p_1 \cap P_{j_2}$ with $p_3 \leqslant p_2$. Then $(p_3, j_3) \leqslant (p_2, j_2) \leqslant (p_1, j_1)$, and $(p_2, j_2) \in P^{j_2}$. ∎

A co-filter (or ideal) in K is a (sup-) subsemilattice F with $\downarrow F = F$. The set of ideals of K is written $\mathrm{Id}(K)$. For $F \in \mathrm{Id}(K)$ we set $\varphi(F) = \{(p, j) \in P : p \in F\}$.

5.20. LEMMA. $\varphi(F) \in \mathrm{Id}(P)$ (see 4.5, 4.6).

Proof. (i) Let $(p', j') \leqslant (p, j) \in \varphi(F)$. Then $p' \leqslant p \in F = \downarrow F$, whence $p' \in F$, thus $(p', j') \in \varphi(F)$.

(ii) Let $(p, j) \in P$ be such that $\downarrow (p, j) \cap P^{j'} \subseteq \varphi(F)$ for some $j' \geqslant j$. This means precisely that $\downarrow p \cap P_{j'} \subseteq F$. But once again we have $p = \sup(\downarrow p \cap P_{j'})$ since $K_{j'}$ is distributive. Then $p \in F$ by the definition of an ideal. Hence $(p, j) \in \varphi(F)$. ∎

For $I \in \mathrm{Id}(P)$ we set $\psi(I) =$ sup-subsemilattice generated in K by all $p \in K$ with $(p, j) \in I$.

5.21. LEMMA. Let $k \in \psi(I)$ and $j_0 \in J$. Then there is a $j \geqslant j_0$ and there are $p_1, \ldots, p_n \in P_j$ such that $k = p_1 \vee \quad \vee p_n$, and $(p_i, j) \in I$ for $i = 1, \ldots, n$.

Proof. By the definition of $\psi(I)$ we have $k = \sup X$ for a finite subset $X \in K$ with $(x, j_x) \in I$ for each $x \in X$ with a suitable $j_x \in J$. We pick any $j \geqslant j_0, j_x, x \in X$. Then $X \subseteq K_j$ and for each $x \in X$ we have $x = \sup(\downarrow x \cap P_j)$. Thus $k = \sup X = \sup \bigcup\{\downarrow x \cap P_j : x \in X\}$. We enumerate the elements of $\bigcup\{\downarrow x \cap P_j : x \in X\}$ as $p_1, \ldots, p_n$; then $(p_i, j) \in P^j$. Since each p_i is below some x_i and $(p_i, j) \leqslant (x_i, j_{x_i}) \in I$, we conclude that $(p_i, j) \in I$. ∎

5.22. LEMMA. $\psi(I) \in \mathrm{Id}(K)$.

Proof. By definition, $\psi(I)$ is a sup-semilattice. We have to show that $k' \leqslant k \in \psi(I)$ implies $k' \in \psi(I)$. By 5.21, $k = p_1 \vee \quad \vee p_n$ with $(p_i, j) \in I$ for suitable p_i and $j \in J$: Let $j' > j$ and such that $k' \in K_{j'}$. Since $K_{j'}$ is distributive, we have $k' = (k' \wedge p_1) \vee \quad \wedge (k' \vee p_n)$ (with $\wedge$ in $K_{j'}$). For each $i \in \{1, \ldots, n\}$ we have $k' \wedge p_i = \sup X_i$, $X_i = \downarrow (k' \wedge p_i) \cap P_{j'}$. Since $x \in X_i$ implies $x \leqslant p_i$, hence $(x, j') \leqslant (p_i, j) \in I$, whence $(x, j') \in I$. Thus $k' = \sup Y$ with $Y = X_1 \cup \ldots \cup X_n$ such that $(y, j') \in I$. Then $k' \in \psi(I)$ by definition. ∎

5.23. LEMMA. For each $I \in \mathrm{Id}(P)$ we have $\varphi\psi(I) = I$.

Proof. Trivially, $I \subseteq \varphi\psi(I)$. Let $(p, j) \in \varphi\psi(I)$. This means that $p \in \psi(I)$, i.e. that $p = p_1 \vee \quad \vee p_n$ for suitable p_i with $(p_i, j') \in I$, and we may assume that $j' > j$. Let $j'' > j'$ and suppose $(q, j'') \in \downarrow (p, j) \cap P^{j''}$. Then $q \leqslant p = p_1 \vee \quad \vee p_n$, and since q is co-prime, $q \leqslant p_i$ for some i, whence $(q, j'') \leqslant (p_i, j') \in I$. Thus $\downarrow (P, j) \cap P^{j''} \subseteq I$. Hence $(p, j) \in I$ since I is an ideal in P. ∎

5.24. LEMMA. *For each* $F \in \mathrm{Id}(K)$ *we have* $\psi\varphi(F) = F$.

Proof. Trivially, $F \subseteq \psi\varphi(F)$. Let $k \in \psi\varphi(F)$. Then $k = p_1 \vee \quad \vee p_n$ for some $(p_i, j) \in \varphi(F)$, $i = 1, \ldots, n$. This means that $p_i \in F$, and so $k = p_1 \vee \quad \vee p_n \in F$. ∎

Since the maps φ and ψ are clearly monotone, we have shown:

5.25. LEMMA. $\varphi \colon \mathrm{Id}(K) \to \mathrm{Id}(P)$ *is an isomorphism of lattices with inverse* ψ. ∎

We recall from [18] that there is a complete duality between sup-semilattices K and complete algebraic lattices L; the duality may be described by associating with a sup-semilattice K the lattice $\mathrm{Id}(K)$ and with an algebraic lattice L the sup-semilattice $K(L)$ of compact elements.

Now we have the following theorem.

5.26. THEOREM. *Let* K *be a sup-semilattice with a maximal element* 1, *and* L *its dual algebraic lattice. Then the following statements are equivalent.*

(1) K *is distributive* (5.18).

(2) L *is Brouwerian.*

(3) *There is an augmented poset* (P, δ, J) *such that* $L \cong \mathrm{Id}P$.

(4) *There is a quasicompact spectral space* (5.10) *such that* $L \cong O(X)$ *and* $X \cong \mathrm{Prime}\,L$.

If these conditions are satisfied, then $\mathrm{Pr}P$ *and* $\mathrm{Spec}P$ *are homeomorphic to* X, *and* K *is isomorphic to the* $\cup$-*semilattice of quasicompact open sets of* X.

Proof. By the preceding remarks, (1) ⇔ (2).

We have proved in 5.25 that (1) ⇒ (3); and (3) ⇒ (2) was observed in 4.13. The equivalence (2) ⇔ (4) was noted in 5.11. ∎

For the record we remark how P arises concretely when a quasicompact spectral space is given. From a space X we obtain a complete Brouwerian lattice $O(X)$ and if X is spectral, then $O(X)$ is algebraic. The $\cup$-semilattice $KO(X)$ of quasicompact open subsets is a distributive semilattice in the sense of 5.18. From $KO(X)$ we obtain the up directed set J of finite distributive subsemilattices containing $\emptyset$ and X. This allows the formation of an augmented poset (P, δ, J) according to 5.19 and $O(X) \cong \mathrm{Id}P$; thus $\mathrm{Prim}O(X) \cong \mathrm{Prime}\,\mathrm{Id}P = \mathrm{Pr}P \cong \mathrm{Spec}P$. But $\mathrm{Prim}O(X) \cong X$ since X is spectral. Since we may work with any up directed family $\{K_j \colon j \in J\}$ of finite distributive subsemilattices covering $KO(X)$, the augmented poset (P, δ, J) is not uniquely determined by the condition that $L \cong \mathrm{Id}P$. In fact, this degree of freedom comes in very handy for the conclusive result on the spectra of separable AFC*-algebras which we discuss now. Indeed, as a consequence of the construction carried out in 5.18–5.25 we have the following

5.27. COROLLARY. *Let K be a countable sup-semilattice with 0 and 1. Then there is an augmented poset* (P, δ, N) *with* $\mathrm{Id}(K) \cong \mathrm{Id}(P)$.

Proof. The collection J of all finite distributive subsemilattices containing 0, 1 is countable and directed with respect to $\subseteq$. By induction we find a cofinal subfamily $\{K_n : n \in N\}$ with $m \leqslant n$ implying $K_m \subseteq K_n$. The construction then shows $\mathrm{Id}\,K \cong \mathrm{Id}\,P$. ∎

Prim A characterized for separable AFC*-algebras

As a consequence we derive the following theorem on AFC*-algebras which was sought after for some time:

5.28. THEOREM. *Every quasicompact spectral space with a countable basis is the primitive ideal spectrum of a separable AFC*-algebra.*

Proof. Let X be a quasicompact 2nd countable spectral space. Let $L = O(X)$, $K = KO(X)$. Then by 5.26 there is an augmented poset (P, δ, N) with $\mathrm{Id}(K) = \mathrm{Id}(P)$. By 5.25, $X \cong \mathrm{Prim}\,L \cong \mathrm{Prim}\,\mathrm{Id}(K) \cong \mathrm{Pr}\,P$. By 4.4 there is a separable AFC*-algebra A and a sequence $A_1 \subseteq A_2 \subseteq$ of FC*-algebras with $A = (\bigcup A_n)^-$ and (P, δ, N) being associated with A and the sequence $A_1 \subseteq A_2 \subseteq \ldots$ By 5.16, Prime $A \cong \mathrm{Pr}\,P$. By 4.17, Prime $A = \mathrm{Prim}\,A$ is the primitive ideal spectrum. ∎

This complements 5.16 and shows

5.29. COROLLARY. *The quasicompact spectral spaces* (5.10) *with a countable basis are precisely* (*up to homeomorphism*) *the primitive spectra of separable AFC*-algebras.* ∎

Under the additional hypothesis that the collection of quasicompact sets is closed under intersection, 5.28 was proved directly by Bratteli [5]; Example 5.14 shows that this need not be the case.

On the center of AFC*-algebras

We illustrate the viability of the lattice theoretical approach by the derivation of Bratteli's result that every commutative separable C^*-algebra occurs as the center of a suitable separable AFC*-algebra.

First we record a lattice theoretical fact which is utilized:

5.30. LEMMA. *Let L be a complete Brouwerian lattice with a compact maximal ideal spectrum. Then there is a continuous retraction m:* $\mathrm{Prime}\,\mathrm{Id}\,L$ $\rightarrow \mathrm{Max}\,\mathrm{Id}\,L$ *from the prime ideal spectrum to the maximal ideal spectrum of L with $I \subseteq m(I)$ for all $I \in \mathrm{Prime}\,\mathrm{Id}\,L$. Moreover, m is in fact the universal Hausdorffisation map* (*or Čech compactification map*). *Finally,* $\mathrm{Prime}\,\mathrm{Id}\,L$ *is a quasicompact spectral space.*

Proof. For these facts we refer to the literature. For example, the first two assertions are described in [17], 6.17 on pp. 70, 71 (in the language of characters rather than ideals). The remark that $\operatorname{Prim} L$ is a spectral space is given in [17], 5.11, p. 54. ∎

For $X \subseteq L$ we write $S(X) = \{I \in \operatorname{Id} L : X \not\subseteq I\}$ and abbreviate $S(\{x\})$ by $S(x)$. We set $S_M(X) = \operatorname{Max} \operatorname{Id} L \cap S(X)$. We note $S(X) = m^{-1}(S_M(X))$, for all $X \subseteq L$.

**5.31. REMARK.** *Let L be a complete Brouwerian lattice with $1 \in K(L)$. Then for every subset $X \subseteq L$ we have*

$$S_M(\sup X) = \bigcup \{S_M(x) : x \in X\}.$$

Proof. Trivially, $S(x) \subseteq S(\sup X)$ for all $x \in X$. Now let $I \in S_M(\sup X)$. Since L is distributive and $\sup I \neq 1$ (as $1 \in K(L)$), there is a prime element $p \in L$, $p \neq 1$ with $\sup I \leqslant p$, whence $I \subseteq \downarrow p$. But $\downarrow p$ is a prime ideal and I a maximal ideal, whence $I = \downarrow p$, i.e. $p = \max I$. From $\sup X \notin I$ we now deduce $\sup X \not\leqslant p$; so there is some $x \in X$ with $x \not\leqslant p$, i.e. $x \notin I$. Hence $I \in S(x)$. ∎

Thus, if L contains a countable subset C with $x = \sup(\downarrow x \cap C)$ for all x, then $\{S_M(c) : c \in C\}$ is a countable basis for $\operatorname{Max} \operatorname{Id} L$. There is no reason to believe, however, that $\{S(c) : c \in C\}$ is a basis for the topology of $\operatorname{Prime} \operatorname{Id} L$. To get around this difficulty we note a simple point set topological fact:

5.32. LEMMA. *Let $m \colon P \to X$ be the Hausdorffization map of the quasi-compact space P. Then we have the following conclusions:*

(a) *If*
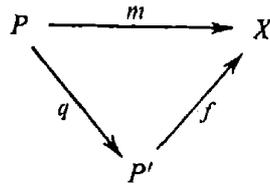
is a factorization with a surjective q, then f is a Hausdorffization map.

(b) *Let $\mathscr{U}$ be any family of open sets in P containing all $m^{-1}(V)$, $V \in \mathscr{V}$ for some basis $\mathscr{V}$ of the topology of X. Let the relation R be given by $x R x'$ iff for all $U \in \mathscr{U}$ one has $x \in U \Leftrightarrow x' \in U$. Then R is an equivalence relation such that there is a factorization*
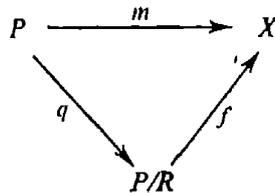
and P/R *is* T_0 *relative to the topology generated by* $q(\mathscr{U})$. *If* $\mathscr{U}$ *consists of quasicompact sets, then so does* $q(\mathscr{U})$. *If every irreducible set of* P *has a dense point, the same is true for* P/R.

Proof. (a) Let $g\colon P' \to Z$ be a continuous function into a compact space. Then by hypothesis on m there is a unique continuous function $h\colon X \to Z$ such that $hfq = hm = gq$. Since q is surjective, $hf = g$, and h is uniquely determined by this condition.

(b) Let $\mathscr{T}$ be the topology generated by $\mathscr{U}$. Then $m\colon (P, \mathscr{T}) \to X$ is continuous since $m^{-1}(V) \in \mathscr{T}$ for a collection of basis open sets $V \in \mathscr{V}$ of X. The quotient map $q\colon (P, \mathscr{T}) \to P/R$ is the left reflection of the space $(P, \mathscr{T})$ into the subcategory of T_0-spaces, and since X is Hausdorff, the factorization $m = fq$ follows. If all $U \in \mathscr{U}$ are quasicompact in P, then all $q(U)$ are quasicompact as continuous images. If $J \subseteq P/R$ is an irreducible set, then $I = q^{-1}(J)$ is irreducible in $(P, \mathscr{T})$, since all open and closed sets S in $(P, \mathscr{T})$ are q-saturated (i.e. satisfy $S = q^{-1}q(S)$). Since $\mathscr{T}$ is coarser than the given topology, then I is irreducible in P. Hence if some x is a dense point in I for the given topology on P, then x is dense in I with respect to $\mathscr{T}$. ∎

All of this only served the purpose to prove the following fact of general topology.

5.33. PROPOSITION. *Let* X *be a compact Hausdorff space. Then there is a quasicompact spectral space* Q *and a continuous map* $f\colon Q \to X$ *which is the universal Hausdorffization map (or Čech compactification map). If* X *has a countable basis one may choose* Q *so as to have a countable basis, too.*

Proof. First we apply 5.30 with $L = O(X)$, the lattice of open subsets of X. We may identify X with $\mathrm{MaxId}\, O(X)$ under $x \mapsto \{U \in O(X)\colon x \notin U\}$ (see [17], notably p. 71), and we obtain a Hausdorffization map $m\colon P \to X$ of a quasicompact spectral space P.

Now suppose that the topology $O(X)$ has a countable basis C. Then define a collection $\mathscr{U}$ of open subsets of $P = \mathrm{Prime\,Id}\, O(X)$ by $\mathscr{U} = \{S(c)\colon c \in C\}$. Then $\mathscr{V} = \{S_M(c)\colon c \in C\}$ is a basis for the topology of $\mathrm{MaxId}\, O(X) = X$. Since $m^{-1}(S_M(c)) = S(c)$ for all c we may apply Lemma 5.32 and define Q by P/R. Then Q is a T_0-space with a countable basis $q(\mathscr{U})$ of quasicompact sets. By 5.32, since P is spectral, so is Q. Also, 5.32 (a) shows that $f\colon Q \to X$ in the factorization $m = fq$ is the Čech compactification map of Q. ∎

At this point we recall a known fact on unital C^*-algebras.

5.34. LEMMA (The Dauns–Hofmann Theorem.) *Let* A *be a* C^*-*algebra with identity,* Z *its center. Then* $Z \cong C(\mathrm{Prim}\,A, C)$ *under a map which associates with* $z \in Z$ *a well-defined function* $f\colon \mathrm{Prim}\,A \to C$ *such that* $f(I)\cdot 1 = z \bmod I$. *The function* $I \mapsto I \cap Z\colon \mathrm{Prim}\,A \to \mathrm{Max}\,Z$ *is the Čech compactification map.* ∎

Now we have the following result which was proved by Bratteli by frontal assault:

5.35. Theorem (Bratteli). *Every commutative separable C^*-algebra occurs as the center of some AFC^*-algebra.*

Proof. Let Z be a commutative separable C^*-algebra and $X = \operatorname{Max} Z$. By Proposition 5.33 there is a spectral space P with a countable basis for its topology and a continuous function $f\colon P \to X$ which is in fact the separable Čech compactification map. By Theorem 5.29 there is a separable AFC^*-algebra with $\operatorname{Prim} A = P$. Then, by the Dauns–Hofmann Theorem 5.34, the center of A is isomorphic to Z. ∎

The isomorphy of separable AFC*-algebras reflected in their augmented posets

In order to be able to associate with a given AFC^*-algebra A, firstly a Bratteli diagram and then an augmented poset, we actually needed to specify a given system $\{A_j\colon j \in J\}$ of finite-dimensional subalgebras whose union was dense in A. Even in the separable case, one and the same AFC^*-algebra may be approximated by different systems of finite-dimensional subalgebras and may in this fashion give rise to different Bratteli diagrams and even augmented posets; only after one has finally passed to the spectrum, the ambiguity vanishes. How such different approximating systems are related was carefully analyzed by Bratteli in his key paper [7]. This analysis was needed before he could give necessary and sufficient conditions for two separable AFC^*-algebras A and B to be isomorphic. These conditions were expressed in terms of arbitrary approximating systems of finite dimensional subalgebras $\{A_j\colon j \in N\}$ and $\{B_j\colon j \in N\}$, respectively. Specifically, he established the following result ([7], 2.7, p. 208):

5.36. Theorem (Bratteli). *For two separable AFC^*-algebras A and B and two ascending sequences $\{A_j\colon i \in N\}$ and $\{B_j\colon j \in N\}$ of finite-dimensional subalgebras with $A = \left(\bigcup_N A_j\right)^-$ and $B = \left(\bigcup_N B_j\right)^-$ the following statements are equivalent:*

(1) $A \cong B$.

(2) *There is an ascending sequence $j(n)$ of natural numbers and a C^* subalgebra $B'_n \subseteq A_{j(n)}$ for all $n = 0, 1, \ldots$ containing $1 \in A$ such that the following two conditions are satisfied:*

(i) *$B'_n \subseteq B'_{n+1}$ for all n and there is an isomorphism of $*$-algebras $f\colon \bigcup_N B_n \to \bigcup_N B'_n$ with $f(B_n) = B'_n$ for all n.*

(ii) *For each n there is a k such that $A_n \subseteq B'_k$.* ∎

We will now use this theorem to find out in what way two different augmented posets of two isomorphic separable AFC^*-algebras have to

be related. In the converse direction, we will then sometimes be in the position to conclude that two AFC*-algebras with given posets cannot be isomorphic on the basis of given properties of the posets.

It is clear that in 5.36 (2), in view of (i), condition (ii) is equivalent to

(ii') *There is an ascending sequence* $k(n)$ *of natural numbers such that* $A_n \subseteq B'_{k(n)}$.

We form the following new family of finite dimensional subalgebras:

$$A_1 \subseteq B'_{k(1)} \subseteq A_{j(k(1))} \subseteq B'_{k(j(k(1)))} \subseteq$$

Thus there is an ascending sequence $p(n)$ of natural numbers with $p(1) = 1$ such that

$$A_1 \subseteq B'_{p(2)} \subseteq A_{p(3)} \subseteq B'_{p(4)} \subseteq$$

and that the *-algebra isomorphism

$$f\colon \; \bigcup_N B_{p(2n)} = \bigcup_N B_n \to \bigcup_N B'_n = \bigcup_N B'_{p(2n)}$$

induces isomorphisms $B_{p(2n)} \to B'_{p(2n)}$ for all n.

Conversely, if these data are given, the isomorphism f extends to an isomorphism $A \to B$ of C^*-algebras. We therefore note a slight reformulation of Bratteli's result which is convenient for our purposes.

5.37. REMARK. *Under the circumstances of Theorem* 5.35, (1) *and* (2) *are also equivalent to the following condition:*

(3) *There is an ascending sequence* $p(n)$ *of natural numbers and a sequence of algebras*

$$A_1 \subseteq B'_{p(2)} \subseteq A_{p(3)} \subseteq B'_{p(4)} \subseteq \dots,$$

*where one has a *-isomorphism of algebras* $f\colon \; \bigcup_N B_{p(2n)} \to \bigcup_N B'_{p(2n)}$ *inducing isomorphisms* $B_{p(2n)} \to B'_{p(2n)}$. ∎

In order to translate this condition into the language of the associated augmented posets it is convenient to introduce a few simple concepts.

5.38. DEFINITION. If (P, δ, J) is an augmented poset we say that an augmented poset (Q, ε, K) is *cofinal in* (P, δ, J) iff $K \subseteq J$ is a cofinal subset, $Q = \delta^{-1}(K)$ and $\varepsilon = \delta|Q$. ∎

The proof of the following easy observation is left to the reader; it shows that passage to a cofinal sub-poset leaves the ideal and spectral theory intact:

5.39. REMARK. *Let* (Q, ε, K) *be cofinal in* (P, δ, J). *Then for each ideal* I *of* P *in the intersection* $I \cap Q$ *is an ideal of* Q *and the function* $I \mapsto I \cap Q$: $\mathrm{Id}(P) \to \mathrm{Id}(Q)$ *is an isomorphism with inverse* $I \to [I]$, *where* $[I]$ *is the ideal generated in* P *by* $I \in \mathrm{Id}Q$. *In particular,* $\mathrm{Pr}(P)$ *[resp., Spec$P$] is homeomorphic to* $\mathrm{Pr}(Q)$ *[resp. Spec$Q$].*

5.40. DEFINITION. Let (P, δ, J) and (Q, ε, K) be augmented posets. We say that P and Q can be spliced if P resp., Q has a cofinal sub-poset

(P', δ', J'), resp., (Q', ε', K') and if P' and Q' are cofinal in some augmented poset (R, ϱ, L) (up to isomorphism of augmented posets).

After Remark 5.39 it is clear that for two posets P and Q which can be spliced we have $\mathrm{Id}(P) \cong \mathrm{Id}(Q)$ and $\mathrm{Pr}(P) \cong \mathrm{Pr}(Q)$.

5.41. LEMMA. *Let (P, δ, N) and (Q, ε, N) be two augmented posets over N. Then the following statements are equivalent:*

(1) *P and Q can be spliced.*

(2) *There is an increasing sequence $p(n)$ of natural numbers with $p(n) = 1$ and an augmented poset (R, ϱ, N) such that*

$$R^{2j-1} = P^{p(2j-1)} \quad and \quad R^{2j} = Q^{p(2j)}, \quad j = 1, \ldots$$

such that the partial orders induced on $R_1 = \bigcup R^{2j-1}$ [resp., on $R_2 = \bigcup R^{2j}$] by P and R [resp. by Q and R] agree.

Proof. (2) $\Rightarrow$ (1): R_1 is cofinal both in R and in P and R_2 is cofinal both in R and in Q.

(1) $\Rightarrow$ (2): (Indication only; the details may be safely left to the reader.) Let (P', δ', J') be cofinal in P and (O', ε', K') cofinal in Q, and suppose that we have an augmented poset (M, μ, A) and a commutative diagram of order preserving functions with F, f, G and g injective with cofinal images:

$$
\begin{array}{ccccc}
P' & \xrightarrow{\ F\ } & M & \xleftarrow{\ G\ } & Q' \\
\Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\varepsilon} \\
J' & \xrightarrow[\ f\ ]{} & A & \xleftarrow[\ g\ ]{} & K'
\end{array}
$$

Let $p(1) = 1$, $p(2) = \min K' \setminus \{1\}$, $p(3) \in J'$ with $1 < p(3)$ and $f(p(3)) > g(p(2))$, $p(4) \in K'$ with $p(2) < p(4)$ and $g(p(4)) > f(p(3))$ and so on. Let $A' = \{f(p(2j-1)), g(p(2j)): j = 1, \ldots\}$, $M' = \mu^{-1}(A')$, $\mu' = \mu|M'$. Then (M', μ', A') is a cofinal augmented poset in M. If we let $R^{2i-1} = P^{p(2i-1)}$, $R^{2j} = Q^{p(2j)}$, we have isomorphisms $R^{2j-1} \to M^{f(p(2j-1))}$ induced by F and $R^{2j} \to M^{g(p(2j))}$ induced by G. If we set $R = \bigcup_N R^{2j-1} \cup R^{2j}$ we can piece together a bijection $R \to M'$. If we equip R with the structure of an augmented poset by pulling back the one we have on M' under this bijection we obtain precisely the required augmented poset R. ∎

Now we have the following proposition:

5.42. PROPOSITION. *Let A and B be two isomorphic AFC*-algebras. If (P, δ, N) and (Q, ε, N) are two augmented posets associated with A [resp. B] (and some ascending chain of finite-dimensional subalgebras in A [resp. B] with dense union), then P and Q can be spliced.*

Proof. We go back to 5.37 and the method of associating with an AFC*-algebra A together with an ascending system of approximating

finite-dimensional subalgebras an augmented poset (4.2, 4.3). Let (P, δ, N) be associated with A and $\{A_j: j \in N\}$ and (Q, ε, N) with B and $\{B_j: j \in N\}$. Let (R, ϱ, N) be the augmented poset associated with A and the system $\{C_j: j \in N\}$ with $C_1 = A_1 \subseteq C_2 = B'_{p(2)} \subseteq C_3 = A_{p(3)} \subseteq C_4 = B'_{p(4)} \subseteq \ldots$ Then $R^{2j-1} = P^{p(2j-1)}$ and there is an order injection $\bigcup \{Q^{p(2j)}: j \in N\} \to R$ mapping $Q^{p(2j)}$ bijectively onto the layer R^{2j} which is induced by the isomorphism $f: \bigcup_N B_{p(2j)} \to \bigcup_N B'_{p(2j)}$. Thus condition (2) of 5.41 is satisfied and so (by the easy implication in 5.41) P and Q can be spliced. ∎

Proposition 5.41 can be useful in deciding on the non-isomorphy of given AFC*-algebras A and B. We illustrate this by an example. The following problem was raised by Leptin:

PROBLEM (Leptin). *Suppose that A and B are separable* AFC*-*algebras whose spectrum in both cases is of the form*
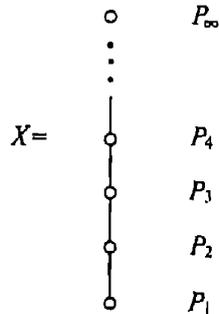
with a sequence $P_1 \subsetneq P_2 \subsetneq \ldots$ of primitive ideals ascending to a maximal ideal P_∞. Are A and B necessarily isomorphic?
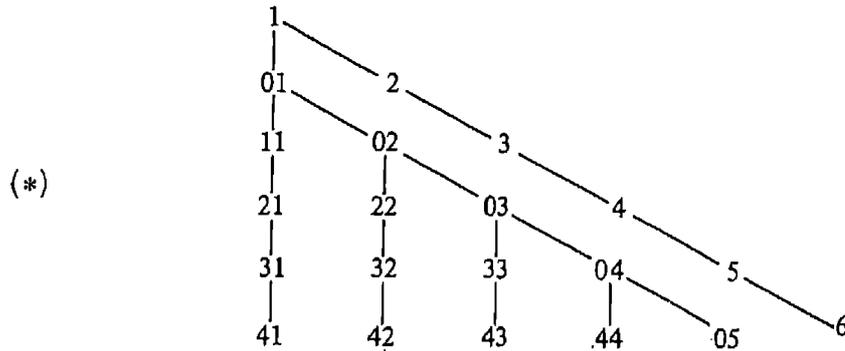
We make a few observations. Since for every augmented poset (P, δ, N) the poset $\mathrm{Id}(P)$ is a Brouwerian algebraic lattice (4.12), every $I \in \mathrm{Id}(P)$ is the intersection of primes (4.13). Thus if $\mathrm{Pr}(P) = X$, where X is as in Leptin's problem, then $\mathrm{Id}(P) = \mathrm{Pr}(P)$, i.e. every ideal is prime. Thus for each $I \in \mathrm{Pr}(P)$, $P \setminus I$ is directed (a filter) (5.3). If we write $\mathrm{Pr}(P) = \mathrm{Id}(P) = \{I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_\infty\}$ with $I_n \neq I_{n+1}$, we conclude that

(1) for all $n = 2, 3$, and all $x \in P$ one has $\downarrow x \cap I_n \neq \emptyset$.

For if we had $\downarrow x \cap I_n = \emptyset$ with $n > 1$, then also $[x] \cap I_n = \emptyset$ (as follows easily from 4.7 and 4.5 (ii)). Since I_1 is prime, $[x] \cap I_n = \emptyset \subseteq I_1$ implied $[x] \subseteq I_1$ or $I_n \subseteq I_1$; since $n > 1$ and $[x] \cap I_1 \subseteq [x] \cap I_n = \emptyset$, this is a contradiction.

We now construct an augmented poset (P, δ, N) which has the required ideal structure. We consider the set of points in which the following relations hold once and for all: $1 > 2 > 3 > \ldots; 1 > 01 > 02 > 03 > \ldots; mp > np$ for all $m > n$ and $p = 1, 2, 3, \ldots$

In order to satisfy condition (1) above we have to impose further relations.

(*)

```
1
|
01        2
|          \
11    02      3
|      |       \
21    22    03     4
|      |      |      \
31    32    33    04     5
|      |      |      \     \
41    42    43    44    05    6
```

For each $n = 2, 3, \ldots$ we fix an arbitrary subset C_n of $\{n, n+1, n+2, \ldots\}$ which is cofinal for $n = 2, 3, \ldots$ and include all the relations $(n+1, c-1) < (n, c)$, $n = 2, 3, \ldots$, $c \in C_n$, and all those which follow by transitivity. Then we set $I_n = [(n, n)] = \{(p, q): q = 1, \ldots, n, \text{ and } p = q, q+1, q+2, \ldots\}$ and verify that all I_n are ideals. The only other non-empty proper ideal is $I_\infty = [(0, 1)] = \downarrow(0, 1)$. We note that condition (1) is satisfied as it should be. The augmented poset P depends on the family C_n; we therefore denote it by $P(\{C_n: n = 2, 3, \ldots\})$.

The construction we suggest does not give a classification of all splice classes of augmented posets with spectrum $\{I_1 \subseteq I_2, \ldots \subseteq I_\infty\}$. But a modification (and slight complication) of the construction above will yield an uncountable number of augmented posets with the same ideal structure such that no two of them can be spliced: For $m = 1, 2, 3, \ldots$ let $(P_m, d_m, \{m, m+1, \ldots\})$ be a simple augmented poset (i.e. one with no proper ideals except $\emptyset$ and P_m); then construct in place of (*) a new augmented poset (P, δ, N) as follows:

(**)

```
1
|
01        2
|          \
P₁    02      3
|      |       \
P₂    03        4
|
P₃
```
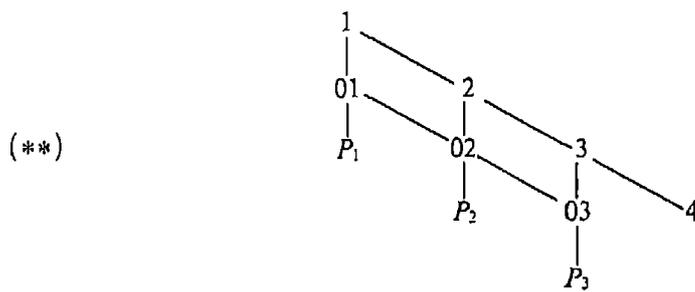
with the obvious layering given for $x \in P_m$ by $\delta(x) = d_m(x)$. For $m = 2, 3, \ldots$, we let $C_m \subseteq P_m$ be a cofinal totally ordered subset (which exists since P_m is simple, thus $\emptyset$ is prime, and since P_m is countable); for each $c \in P_m$ we pick arbitrarily an element $c' \in P_{m-1}$ with $\delta(c') = \delta(c) + 1$ and introduce the relation $c' < c$, and then add all relations following from transitivity. The sets $I_m = P_1 \cup \ldots \cup P_m$ and the set $\downarrow(01)$ are the only non-empty proper ideals. If an augmented poset Q is constructed from

a sequence of simple augmented posets Q_m in the same manner as P was constructed from the posets P_m, then we observe that whenever P and Q can be spliced, then P_m and Q_m can be spliced for all $m = 1, 2, \ldots$; conversely, if P_m and Q_m cannot be spliced for some m, then P and Q cannot be spliced and any pair of AFC*-algebras A and B associated with P, respectively Q cannot be isomorphic by 5.42. Since there exist simple posets P' and Q' which cannot be spliced (e.g. take two simple non-isomorphic AFC*-algebras S and T and choose any associated augmented poset P', respectively, Q') we have in fact uncountably many non-isomorphic separable AFC*-algebras whose spectrum consists of an ascending sequence of ideals and a maximal ideal.

Added in proof. George A. Elliot pointed out to us (letter 8. 7. 77) that a postliminary separable C^*-algebra with spectrum N is unique.

6. Problems

In this very short section we list some problems we ran into in the course of our discussion and which we were unable to settle.

6.1. **Problem** (See 2.1 ff.). Is every weakly almost finite-dimensional C^*-algebra with identity an AFC*-algebra?

This was shown in the separable and the abelian case. We were unable to settle this question in general. In a related vein one observes, that in general in a C^*-algebra two finite-dimensional subalgebras may not be contained in finite-dimensional subalgebra; consider e.g. the case of an infinite-dimensional C^*-algebra generated by two non-commuting projections. However, can this happen in an AFC*-algebra?

6.2. **Problem.** Let A be an AFC*-algebra. Is the collection |of finite-dimensional subalgebras up-directed with respect to $\subseteq$?

We verified this for abelian AFC*-algebras and $LC(H) + C \cdot 1$. We do not appear to know the situation even in the separable case.

6.3. **Problem** (See 2.16). Is every Bratteli diagram associated with an AFC*-algebra?

6.4. **Problem** (See 4.4). Is every augmented poset associated with a Bratteli diagram, hence (by 2.16) with an AFC*-algebra?

We do not know the answer even for countable augmented posets in general. We do know the affirmative answer in the case that the index poset J is the set of natural numbers. This suffices to cover the spectral theory of separable AFC*-algebras, but it would be desirable to understand the general situation.

6.5. **Problem** (See 4.10). What is known on closed two-sided prime ideals in non-separable unital C^*-algebras? Are they primitive?

In view of the semisimplicity of C^*-algebras this is tantamount to asking whether all irreducible (closed) subsets of the primitive ideal spectrum Prim A of a unital C^*-algebra A have a dense point. Or, to put it once again in an equivalent fashion: If A is a prime C^*-algebra (i.e. if $\{0\}$ is a prime ideal) is A primitive?

Can these questions be decided for (non-separable) AFC^*-algebras?

6.6. PROBLEM. Is there a common super-theory which subsumes the theory of AFC^*-algebras as it was developed here and Behncke's and Leptin's C^*-algebras with finite dual?

Appendix

Some remarks on the distributivity of semilattices

The concept of distributivity, very familiar for lattices, is somewhat elusive for semilattices. There are several definitions (see e.g. [18], [23], [24] and the references given there). The most natural concept for many purposes is the following: An (inf-) semilattice is distributive iff $ab \leqslant x$ implies $a'b' = x$ for suitable elements $a' \geqslant a$, $b' \geqslant b$. This concept entails that a semilattice is distributive if and only if its filter lattice is a distributive lattice. All concepts of distributivity for semilattices, evidently have to share the property that for lattices the concept reduces to the ordinary one.

It was first shown by Gaskill in [15] that there is more evidence that the concept of distributivity described above is "the right one". He shows, in fact, that a semi-lattice with identity is distributive if and only every finite subset thereof is contained in a finite distributive subsemilattice with identity (which by finiteness, of course, is a distributive lattice in its own right). In order to be fairly self-contained we offer an independent (although related) proof.

The word "semilattice" means semilattice with identity. The category $\mathbf{Z}$ is the category of compact zero dimensional semilattices or, equivalently, complete algebraic lattices; for these matters see [18].

If $T \in \mathbf{Z}$, then $K(T)$ denotes the sup-semilattice of local minima (resp. compact elements).

A.1. LEMMA. *Let $T \in \mathbf{Z}$ and S a finite subsemilattice of T. Then there is a function $f \colon T \to K(T)$ with the following properties:*

(1) *f is monotone (i.e. $t \leqslant t'$ implies $f(t) \leqslant f(t')$).*

(2) *$f(t) \leqslant t$.*

(3) *$f|S$ is injective; more accurately,*

$$[f(s), s] \cap [f(s'), s'] = \varnothing \quad for \quad s \neq s' \ in \ S.$$

(4) $f(k) = k$ for all $k \in S \cap K(T)$.

Moreover, f may be constructed with arbitrarily small displacement, i.e.:

(5) *If U is a given entourage of the uniform structure of T, then $(f(t), t)$ $\in U$ for all $t \in T$.*

Proof. Let $\mathscr{R}$ be the collection of all open closed congruences $R \subseteq T \times T$. Then every point $t \in T$ has a neighborhood basis of the form $\{R(t) \colon R \in \mathscr{R}\}$ Numakura's theorem, see e.g. [19], p. 52. It follows that we find an $R \in \mathscr{R}$ such that

(a) sRs' and $s, s' \in S$ implies $s = s'$,

(b) $k \in S \cap K(T)$ implies $k = \min R(k)$,

(c) $R \subseteq U$;

indeed for any $k \in K(T)$ the set $\uparrow k$ is a neighborhood of k. Now we define $f(t) = \min R(t)$. Then $f(t) \in K(T)$ and $f \colon S \to K(T)$ is well-defined. Condition (a) implies (3) and (b) implies (4). Condition (2) is obvious. If $t \leqslant t'$ in T, then $tt' = t$, thus $f(t)f(t') \in R(t)R(t') \subseteq R(tt')$, whence $f(t) = f(tt')$ $\leqslant f(t)f(t') \leqslant f(t)$, i.e. $f(t)f(t') = f(t)$, whence $f(t) \leqslant f(t')$. Thus (1) is satisfied.

We note that for any neighborhood U of the diagonal in $T \times T$ we can find f so that $(t, f(t)) \in U$ for all $t \in T$. ∎

A.2. LEMMA. *Let $T \in \mathbf{Z}$ and let S be a finite-distributive sublattice of T. Suppose that $F \colon S \to K(T)$ is a function satisfying the following conditions:*

(i) *F is monotone.*

(ii) *$F(s) \leqslant s$.*

Then there is a sup-morphism $G \colon S \to K(S)$ with $F(s) \leqslant G(s) \leqslant s$.

Proof. We denote with Q the set of co-primes in S. Since S is distributive, every element s is a sup of co-primes. For each $p \in Q$ we pick a $p' \in K(T) \cap \downarrow p$ in such a fashion that $F(p_1 \vee \quad \vee p_n) \leqslant (p'_1 \vee \quad \vee p'_n)$ for all n-tuples $(p_1, \ldots, p_n) \in Q^n$, $n = 1, \ldots, \mathrm{card}\, Q$; this is possible since the sup-operation is continuous from below.

Next we define $p'' = \sup \{q' \colon q \in Q \cap \downarrow p\}$. Then $p'' \in K(T)$ with $p' \leqslant p''$ $\leqslant p$, and $p_1 \leqslant p_2$ in Q implies $Q \cap \downarrow p_1 \subseteq Q \cap \downarrow p_2$ hence $p''_1 \leqslant p''_2$. Now suppose that $s \in S$ and assume that $s = p_1 \vee \quad \vee p_m = q_1 \vee \quad \vee q_n$ for suitable elements $p_j, q_k \in Q$, $j = 1, \ldots, m$, $k = 1, \ldots, n$. Since each p_j is coprime and satisfies $p_j \leqslant q_1 \leqslant \ldots \leqslant q_n$, there is at least one $q_{w(j)}$, $w(j) \in \{1, \ldots, n\}$ with $p_j \leqslant q_{w(j)}$. This implies that $p''_1 \vee \ldots \vee p''_m$ $\leqslant q''_{w(1)} \vee \quad \vee q''_{w(m)} \leqslant q''_1 \vee \quad \vee q''_n$. By symmetry we also have $q''_1 \vee$ $\vee q''_n \vee \quad \vee p''_m$. Hence

$$p_1 \vee \quad \vee p_m = q_1 \vee \quad \vee q_n \quad \text{for } p_j, q_k \in Q \text{ implies}$$

$$p''_1 \vee \quad \vee p''_m = q''_1 \vee \quad \vee q''_n.$$

Now for $s \in S$ we write $s = p_1 \vee \quad \vee p_m$ and define $G(s) = p''_1 \vee$ $\vee p''_n$. Then $G(s) \in K(T)$ is defined independently of the representation

of s as a sup of co-primes. If $s = p_1 \vee \quad \vee p_m$ and $t = q_1 \vee \quad \vee q_m$ we have $G(s) \vee G(t) = p_1'' \vee \quad \vee p_m'' \vee q_1'' \vee \quad \vee q_n'' = G(p_1 \vee \quad \vee p_m \vee q_1 \vee \quad \vee q_n) = G(s \vee t)$. ∎

Thus G is a sup-morphism. If $s = p_1 \vee \ldots \vee p_m$, then $F(s) = F(p_1 \vee \quad \vee p_m) \leqslant p_1' \vee \quad \vee p_m' \leqslant p_1'' \vee \quad \vee p_m'' = G(s) \leqslant p_1 \vee \quad \vee p_m = s$. ∎

A.3. LEMMA. *Let* $T \in \mathbf{Z}$ *and let* S *be a finite distributive sub-semilattice of* T. *Then there is an injective* sup-*morphism* $G\colon S \to K(T)$ *such that* $G(k) = k$ *for* $k \in S \cap K(T)$. *In particular,* $G(s)$ *is a* sup-*subsemilattice of* $K(T)$ *isomorphic to* S *as a* sup-*semilattice*.

Proof. Construct $f\colon T \to K(T)$ according to Lemma 1, let $F = f|S\colon S \to K(T)$, and determine $G\colon S \to K(T)$ according to Lemma 2. Then (4) of Lemma 1 implies $f(k) \leqslant G(k) \leqslant k$ for $k \in S \cap K(T)$. Finally, since $G(s) \in [f(s), s]$ for all $s \in S$, condition (3) of Lemma 1 implies that G is injective. ∎

We recall that every sup-semilattice K can be represented as $K(T)$ for a $\mathbf{Z}$-object T (which is, in fact, the ideal semilattice of K or, alternatively the character semilattice of K). The definition of distributivity was given in 5.18.

A.4. LEMMA. *Let* $T \in \mathbf{Z}$. *Then the following statements are equivalent*:

(a) *The* sup-*semilattice* $K(T)$ *is distributive.*

(b) *The underlying lattice of* T *is distributive.*

(c) *The underlying lattice is a complete algebraic Brouwerian lattice.*

Proof. See [18], notably 1.37 on p. 71. ∎

A.5. MAIN LEMMA. *Let* K *be a distributive* (sup-) *semilattice. Then every finite subset is contained in a finite distributive subsemilattice (which then is a distributive lattice in its own right).*

Proof. Every finite subset of K is clearly contained in a (sup-) subsemilattice L of n elements, say. (Every finitely-generated semilattice is finite!) We consider K as the sup-subsemilattice $K(T)$ of the unique dual $\mathbf{Z}$ object T of K. Then we let $S = L^n$ be the (inf-) subsemilattice of T generated by $L \subseteq K \subseteq T$. If $a_i, b_j \in L$ for $i, j = 1, \ldots, n$, then by the distributivity of T (see Lemma 5) we calculate $x = a_1 \ldots a_n \vee b_1 \ldots b_n = \inf\{a_i \vee b_j\colon i, j = 1, \ldots, n\}$. Since $a_i \vee b_j \in L$ since L is a sup-semilattice, we conclude that $x \in S$. Thus S is a sup-semilattice and thus a sublattice of T, hence it is a distributive sublattice. Let $G\colon S \to K(T)$ be as in Lemma 3. Then $k \in L$ implies $k \in S \cap K$, hence $G(k) = k$. Thus $G(S)$ is a sup-subsemilattice of K containing L and, being isomorphic to S, is a distributive lattice in its own right. ∎

We have in fact shown the following result:

A.6. THEOREM. *Let* K *be an arbitrary semilattice with identity and* T *its dual. Then the following statements are equivalent*:

(1) *K is distributive.*

(2) *K is a union of an up-directed family of finite subsemilattices of K each of which is a distributive lattice in its own right.*

(3) *K is an injective limit of finite-distributive lattices in the category of semilattices with identity and identity preserving semilattice morphisms.*

(4) *T is distributive.*

(5) *The underlying lattice of T is a Brouwerian algebraic lattice.*

(6) *T has enough finite distributive semilattice quotients to separate the points.*

(7) *T is the projective limit of finite distributive lattices in the category Z of compact 0-dimensional semilattices and identity preserving continuous semilattice morphisms.*

Proof. $(1) \Leftrightarrow (4) \Leftrightarrow (5)$ by Lemma 4. Trivially, (2) and (3) are equivalent, as are (6) and (7). By duality, $(3) \Leftrightarrow (7)$ (see [18], 4.2, p. 21).

$(2) \Rightarrow (1)$: If $x \leqslant a \vee b$ in K, find a finite distributive subsemilattice $D \subseteq K$ with $x, a, b \in D$. Then set $a' = a \wedge_D x$ and $b' = b \wedge_D x$ and observe $a' \vee b' = (a \wedge_D x) \wedge (b \wedge_D c) = (a \vee b) \wedge_D x = x$.

$(1) \Rightarrow (2)$ is an immediate consequence of Main Lemma 5. ∎

One way of expressing the Theorem is the following: The category of finite distributive semilattices is prodense in the full category of distributive Z objects and co-pro-dense in the category of discrete distributive semilattices with identity. A consequence of Theorem is that the direct limit functor in the category of semilattices preserves distributivity.

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