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DEFINING ORBIT SPACES BY INEQUALITIES

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We discuss the authors' recent results [7] concerning the description of orbit spaces of representations of compact Lie groups.

0. Introduction

We begin by discussing two problems:

0.1. Let $p\colon \mathbf{R}^n \to \mathbf{R}^m$ be a polynomial mapping. Then the image $\operatorname{Im} p$ of p is a semialgebraic subset of $\mathbf{R}^m$. How can one find "simply" or "explicitly" the inequalities defining $\operatorname{Im} p$?

0.2. Let K be a compact Lie group and W a real representation space for K. Can one find a nice description of the orbit space W/K?

In Section 1 we will see that Problem 0.2 is a special case of Problem 0.1, and we describe the solution to 0.2. In Section 2 we present the details for the case of a finite group. In Section 3 we discuss some connections with Hilbert's 17th problem.

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1. Orbit spaces

1.0. Let W and K be as in 0.2. Then the graded algebra $\mathbf{R}[W]^K$ of K-invariant polynomial functions is finitely generated ([9], p. 274). Let $p_1, \ldots, p_m$ be homogeneous generators of $\mathbf{R}[W]^K$ and let $p = (p_1, \ldots, p_m)$ be the associated mapping from W to $\mathbf{R}^m$. Then p is proper and constant on the orbits of K, hence it induces a homeomorphism of W/K (quotient topology) with $X = \operatorname{Im} p \subseteq \mathbf{R}^m$ ([8]).

Let I denote the ideal of relations of the p_i in $\mathbf{R}[y_1, \ldots, y_m]$, and let Z denote the corresponding algebraic subset of $\mathbf{R}^m$. Then p induces an isomorphism $p^*\colon \mathbf{R}[Z] \to \mathbf{R}[W]^K$, and $X \subseteq Z$. Note that Z is determined by $\mathbf{R}[W]^K$ and our choice of generators, while to describe X we need some extra information.

1.1. Example. Let $K = \{\pm 1\}$ act by multiplication on $W = \mathbf{R}^2$. Then $\mathbf{R}[W]^K$ is generated by polynomials $p_1 = x^2+y^2$, $p_2 = x^2-y^2$ and $p_3 = 2xy$. Their ideal of relations is generated by the single polynomial $y_1^2-y_2^2-y_3^2$. Thus $Z = \{(y_1, y_2, y_3) \in \mathbf{R}^3\colon y_1^2 = y_2^2+y_3^2\}$. Since p_1 is non-negative, we must have that $X \subseteq \{(y_1, y_2, y_3) \in Z\colon y_1 \geqslant 0\}$. We will see that, in fact, there is equality.

1.2. Example. Let $W = \mathbf{R}^2$ and K the group of rotations by angles 0, $2\pi/3$, and $4\pi/3$. Then $\mathbf{R}[W]^K$ is generated by $p_1 = x^2+y^2$, $p_2 = x^3-3xy^2$ and $p_3 = y^3-3x^2y$. Their ideal of relations is generated by $y_1^3-y_2^2-y_3^2$, so $Z = \{(y_1, y_2, y_3)\colon y_1^3 = y_2^2+y_3^2\}$, and one can show that $X = \operatorname{Im} p = Z$ in this case.

1.3. Example. Let W be the space of $n \times q$ real matrices, and let $K = O(n) = O(n, \mathbf{R})$ act by left multiplication. Then W is just q copies of the standard representation of K on $\mathbf{R}^n$. By classical invariant theory, the K-invariants are generated by the inner products of the various copies of $\mathbf{R}^n$, i.e., of the columns of our $n \times q$ matrices. Thus we can define

$$p\colon W \to \operatorname{Sym}_q, \qquad A \mapsto A^t A,$$

where Sym_q denotes the space of real symmetric $q \times q$ matrices. It is an easy exercise to show that X is the set of all matrices $B \in \operatorname{Sym}_q$ such that:

(1.3.1) $$\operatorname{rank} B \leqslant n.$$

(1.3.2) B is positive semidefinite.

Then Z is defined by (1.3.1), i.e., by the condition that the determinants of all $(n+1) \times (n+1)$ minors of B are zero. The inequalities defining X come from the following:

1.4. *Remark.* Let C be a real symmetric matrix. Then C is positive semidefinite (we write $C \geqslant 0$) if and only if $C_\alpha \geqslant 0$ for all α, where $\{C_\alpha\}$ is the set of determinants of principal (i.e. symmetric) minors of C.

We now show how to find the inequalities describing X in general. The description was, essentially, conjectured by the physicists Abud and Sartori ([1], [2]): Let (,) denote a K-invariant inner product on W as well as the dual inner product on W^*. The differentials dp_i: $W \to W^*$ are K-equivariant, and the functions $w \mapsto (dp_i(w), dp_j(w))$ give an $m \times m$ symmetric matrix valued function $\mathrm{Grad}^\sim(w)$ with entries in $\boldsymbol{R}[W]^K$. There is a unique matrix valued function Grad on Z such that $\mathrm{Grad}^\sim(w) = \mathrm{Grad}(p(w))$ for all $w \in W$.

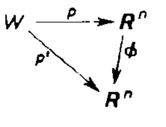
1.5. *Remark.* One may choose orthonormal co-ordinates $x_1, \ldots, x_n$ on W relative to (,). Then $\mathrm{Grad}^\sim(w) = J(w)J(w)^t$ where $J(w)$ is the Jacobian matrix of p at w. This shows that $\mathrm{Grad}^\sim(w) \geqslant 0$, or, in other words, $\mathrm{Grad}(x) \geqslant 0$ for all $x \in X$.

1.6. THEOREM. $X = \{z \in Z\colon \mathrm{Grad}(z) \geqslant 0\}$.

In Example 1.1, the theorem gives inequalities $y_1 \geqslant 0$; $y_1^2 - y_2^2 \geqslant 0$ and $y_1^2 - y_3^2 \geqslant 0$. But the last two inequalities are automatically satisfied on Z (since $y_1^2 = y_2^2 + y_3^2$), so $X = \{y \in Z\colon y_1 \geqslant 0\}$. In 1.2 one similarly gets that the inequality $y_1 \geqslant 0$ defines X, but this is already forced by the equality $y_1^3 = y_2^2 + y_3^2$, hence $X = Z$. (In [7] we show that $X = Z$ if and only if K (assumed acting effectively on W) is a finite group of odd order.) In 1.3 the theorem gives a redundant set of inequalities. One gets the condition in (1.3.2) exactly by applying a variant of Theorem 1.6 (see [7]).

1.7. Let $f(x) = x^n - b_1 x^{n-1} + \ldots + (-1)b_n$ be a real polynomial. When does f have only real roots? We use Theorem 1.6 to recover the classical criterion of Sylvester: Let $K = S_n$ denote the symmetric group which acts as usual on $W = \boldsymbol{R}^n$. Let $\sigma_1, \ldots, \sigma_n$ denote the elementary symmetric functions on the co-ordinates $x_1, \ldots, x_n$ of $\boldsymbol{R}^n$. Then $\boldsymbol{R}[W]^K = \boldsymbol{R}[\sigma_1, \ldots, \sigma_n]$. Let $p = (\sigma_1, \ldots, \sigma_n)$: $W \to \boldsymbol{R}^n$. Then f has real roots $a_1, \ldots, a_n$ if and only if $p(a) = b$, where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. In other words, f has only real roots if and only if $b \in \operatorname{Im} p$.

We apply Theorem 1.6: Let $\tau_i = \sum_{j=1}^{n} x_j^i$, $i \geqslant 0$. Then $\tau_1, \ldots, \tau_n$ generate $\boldsymbol{R}[W]^K$. Let $p' = (\tau_1, \ldots, \tau_n)$: $W \to \boldsymbol{R}^n$. The classical Newton formulae: $\tau_1 = \sigma_1$, $\tau_2 = \sigma_1^2 - 2\sigma_2$, $\tau_3 = \sigma_1^3 - 3\sigma_1\sigma_3 + 3\sigma_3$, etc. give a polynomial isomorphism ϕ: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ so that the following diagram commutes.



Thus $b \in \operatorname{Im} p$ if and only if $\phi(b) \in \operatorname{Im} p'$. Now $(d\tau_i, d\tau_j) = ij\tau_{i+j-2}$, and by Theorem 1.6 and our remarks above we see that $b \in \operatorname{Im} p$ if and only if $B(b) \geqslant 0$, where $B = (B_{ij})$ and $B_{ij}(\sigma_1, \ldots, \sigma_n) = ij\tau_{i+j-2}$. It does not affect positive semidefiniteness if we replace B_{ij} by $\frac{1}{ij} B_{ij}$ and in this way we arrive at the "Bezoutiant" matrix Bez of Sylvester. We have shown:

1.8. Corollary (Sylvester, see [6])). *Let $f(x) = x^n - b_1 x^{n-1} + \ldots$ $\ldots + (-1)^n b_n$ be a real polynomial. Then f has only real roots if and only if* $\operatorname{Bez}(b) \geqslant 0$.

For $n = 2$, one can compute that

$$\operatorname{Bez}(b) = \begin{bmatrix} 2 & b_1 \\ b_1 & b_1^2 - 2b_2 \end{bmatrix}$$

and for $n = 3$ one has

$$\operatorname{Bez}(b) = \begin{bmatrix} 3 & b_1 & b_1^2 - 2b_2 \\ b_1 & b_1^2 - 2b_2 & b_1^3 - 3b_1 b_2 + 3b_3 \\ b_1^2 - 2b_2 & b_1^3 - 3b_1 b_2 + 3b_3 & b_1^4 - 4b_1^2 b_2 + 2b_2^2 + 4b_1 b_3 \end{bmatrix}.$$

2. Finite groups

We give a proof of Theorem 1.6 in the case that K is finite: Let W, k, p and $X \subseteq Z \subseteq \boldsymbol{R}^m$ be as before. Recall that we have K-invariant inner products $(\ ,\ )$ on W and W^*, and $\widetilde{\operatorname{Grad}}(w) = (dp_i(w), dp_j(w))$. We have a point $z \in Z$ with the property that $\operatorname{Grad}(z) \geqslant 0$, and we want to show that $z \in X$.

Let $V = W \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Our K-invariant inner products extend to K-invariant non-degenerate symmetric bilinear forms on V and V^*, denoted as usual by $(\ ,\ )$. We identify $\boldsymbol{R}[W]^K$ with the elements of $\boldsymbol{C}[V]^K$ which are real on W, and then $p_1, \ldots, p_m$ generate $\boldsymbol{C}[V]^K$. Our mapping $p\colon W \to \boldsymbol{R}^n$ extends to $p\colon V \to \boldsymbol{C}^n$, and the image $p(V)$ lies in the set of complex zeroes $Z_{\boldsymbol{C}}$ of the ideal of relations of the p_i (see 1.0).

2.1. Lemma. (1) $p(V) = Z_{\boldsymbol{C}}$.

(2) *The fibers of p are (set-theoretically) the orbits of K.*

Proof. Let $\varrho\colon \boldsymbol{C}[V] \to \boldsymbol{C}[V]^K$ be the Reynold's operator (averaging over the group). Let $z \in Z$, let I_z be the corresponding maximal ideal of $\boldsymbol{C}[V]^K$ and set $J_z = I_z \boldsymbol{C}[V]$. Then $\varrho(J_z) = I_z$, so J_z is a proper ideal of $\boldsymbol{C}[V]$ and $p(x) = z$ for any zero x of J_z. Thus (1) holds, and another averaging over the group argument shows that $\boldsymbol{C}[V]^K$ separates distinct K-orbits, proving (2). □

It follows from Lemma 2.1(1) that:

2.2. *There is a point $v \in V$ such that $p(v) = z$.*

Write $v = w_1 + iw_2$ where $w_1, w_2 \in W$. Then $\bar{v} = w_1 - iw_2$. Assume the following:

2.3. PROPOSITION. *Define $\lambda \in V^*$ by $\lambda(x) = (x, iw_2)$, $x \in V$. Then there are $a_1, \ldots, a_m \in \boldsymbol{R}$ such that $\lambda = \sum a_i \, dp_i(v)$.*

Proof of Theorem 1.6. Consider the value of (λ, λ) where λ is as above. On the one hand

$$(\lambda, \lambda) = \left(\sum a_i \, dp_i(v), \sum a_j \, dp_j(v)\right) = \sum a_i a_j \operatorname{Grad}(p(v))_{ij} \geqslant 0$$

since $p(v) = z$ and $\operatorname{Grad}(z) \geqslant 0$. On the other hand

$$(\lambda, \lambda) = (iw_2, iw_2) = -(w_2, w_2) \leqslant 0.$$

Hence $(w_2, w_2) = 0$ and $v = w_1 \in W$. Hence $z = p(w_1) \in X$. □

We now establish Proposition 2.3: Since p is real on W, it follows that $p(\bar{v}) = \overline{p(v)} = z$. Hence, by Lemma 2.1(2).

2.4. *There is a $k_0 \in K$ such that $k_0 v = \bar{v}$.*
Set

$$\Delta(v) = (V^*)^{K_v},$$

where K_v is the isotropy group of K at v, and set

$$D(v) = \{df(v) \colon f \in \boldsymbol{C}[V]^K\}.$$

2.5. *Remarks.* (1) If $f \in \boldsymbol{C}[V]^K$, then

$$df(v) = d(f \circ k)(v) = df(kv) \circ k$$

for all $k \in K$, hence $df(v)$ is K_v-invariant. Thus $D(v) \subseteq \Delta(v)$.

(2) Since the p_i generate $\boldsymbol{C}[V]^K$, the complex span of the $dp_i(v)$ is $D(v)$.

2.6. LEMMA. $D(v) = \Delta(v)$.

We establish Lemma 2.6 below. Now set

$$\Delta_{\boldsymbol{R}}(v) = \{\mu \in \Delta(v) \colon \mu \circ k_0 = \bar{\mu}\},$$

where $\bar{\mu}(x) = \overline{\mu(\bar{x})}$. Note that each $dp_i(v)$ is $\Delta_{\boldsymbol{R}}(v)$, since

$$dp_i(v) \circ k_0 = dp_i(k_0^{-1} v) = dp_i(\bar{v}) = \overline{dp_i(v)}.$$

Now by Lemma 2.6, each μ in $\Delta(v)$ is a sum $\sum a_i \, dp_i(v)$, and if $\mu \in \Delta_{\boldsymbol{R}}(v)$ one easily sees, using our computation above, that one may assume that the a_i are real. Hence

2.7. *$\Delta_{\mathbf{R}}(v)$ is the real span of the $dp_i(v)$.*

Proof of Proposition 2.3. Let $f(x) = \frac{1}{2}(x, x)$, $x \in V$. Then f is real on W and $\lambda_1 := df(v) \in \Delta_{\mathbf{R}}(v)$, where $\lambda_1(x) = (x, v)$. Define $\lambda_2 \in V^*$ by $\lambda_2(x) = (x, \bar{v})$. Using 2.4 and the fact that $K_v = K_{\bar{v}} = K_{w_1} \cap K_{w_2}$, one easily establishes that $\lambda_2 \in \Delta_{\mathbf{R}}(v)$. Hence $\lambda = \frac{1}{2}(\lambda_1 - \lambda_2) \in \Delta_{\mathbf{R}}(v)$, where $\lambda(x) = (x, iw_2)$. □

Proof of Lemma 2.6. Let B_v be a small ball containing v so that, for any $k \in K$, either $kB_v = B_v$ or $B_v \cap kB_v = \emptyset$. Let $\mathscr{H}(U)$ denote the holomorphic functions on U, for U an open subset of V. Then $\mathscr{H}(KB_v)^K \simeq \mathscr{H}(B_v)^{K_v}$. If $\mu \in (V^*)^{K_v}$, then the function $f(x) := \mu(x-v)$, $x \in B_v$, lies in $\mathscr{H}(B_v)^{K_v}$ and has differential μ at v. Now $\mathbf{C}[V]$ is dense in $\mathscr{H}(B_v)$, hence $\mathbf{C}[V]^K$ is dense in $\mathscr{H}(KB_v)^K$, and it follows that $\Delta(v) = D(v)$. □

2.8. *Remark.* In case K is not finite, one has to consider the action of the complexification $K_{\mathbf{C}}$ of K on V. Not all orbits of $K_{\mathbf{C}}$ are closed, which presents complications. The new ingredients needed for the proof of Theorem 1.6 are Luna's slice theorem [5] (to prove the appropriate analogue of Lemma 2.6) and some results of Kempf and Ness [4] (to establish 2.4).

3. Hilbert's seventeenth problem

We give some applications of Theorem 1.6 to a version of Hilbert's 17th problem: The solution to Hilbert's 17th reads as follows:

3.1. THEOREM. *Let $f \in \mathbf{R}(x_1, \ldots, x_n)$ be positive,* i.e., *f is non-negative wherever it is defined. Then there are $g_1, \ldots, g_d \in \mathbf{R}(x_1, \ldots, x_n)$ such that $f = g_1^2 + \ldots + g_d^2$.*

Let K and W be as in Introduction. Does Theorem 3.1 remain true if we replace $\mathbf{R}(W)$ by $\mathbf{R}(W)^K$? The answer is:

3.2. EXAMPLE. Let $K = \{\pm 1\}$ act by multiplication on $W = \mathbf{R}$. Then $\mathbf{R}(W)^K$ consists of rational functions of x^2, and $f(x) = x^2$ is positive. If $f(x) = g_1(x^2)^2 + \ldots + g_d(x^2)^2$, then $x = g_1(x)^2 + \ldots + g_d(x)^2$, a contradiction. However, one can show that f is, in some sense, the only problem. In other words, if $g(x) \in \mathbf{R}(W)^K$ is positive, then

$$g(x) = g_0(x^2) + g_1(x^2) f(x),$$

where g_0 and g_1 are sums of squares.

3.3. Let F be a subfield of $\mathbf{R}(x_1, \ldots, x_n)$. We say that F *has property* (H) if there are positive elements $h_1, \ldots, h_q \in F$ such that every $f \in F$ which is positive can be written in the form

$$f = \sum s_i h_i \tag{3.3.1}$$

where the s_i are sums of squares in F.

3.4. THEOREM. *Let K and W be as in* 0.2. *Then $\boldsymbol{R}(W)^K$ has property* (H).

Procesi [6] established Theorem 3.4 and found the polynomials h_i of 3.3 in case $K = S_n$ acting standardly on $\boldsymbol{R}^n$. Bochnak and Efroymson [3] first conjectured Theorem 3.4. We now show how to obtain Theorem 3.4 from Theorem 1.6.

Let P be a closed semialgebraic subset of $\boldsymbol{R}^m$. We assume that the Zariski closure T of P is irreducible. We say that P is *elementary* if there are $f_1, \ldots, f_d \in \boldsymbol{R}[T]$ such that $P = \{t \in T: f_i(t) \geqslant 0, i = 1, \ldots, d\}$. We say that P is *quasi-elementary* if there is an algebraic subset Y of T such that $\dim Y < \dim T$ and $P \cup Y$ is elementary.

3.5. PROPOSITION ([3]). *Let P and T be as above.*

(1) *If P is quasi-elementary, choose $f_1, \ldots, f_d \in \boldsymbol{R}[T]$ so that $\{t \in T: f_i(t) \geqslant 0, i = 1, \ldots, d\} = P \cup Y$, where Y is algebraic and* $\dim Y < \dim T$. *Then every $f \in \boldsymbol{R}(T)$ which is positive on P can be written in the form* (3.3.1), *where the h_i are all possible products $f_{i_1} \ldots f_{i_r}$, $1 \leqslant i_1 < \ldots < i_r \leqslant d$, $0 \leqslant r \leqslant d$.*

(2) *If every $f \in \boldsymbol{R}(T)$ which is positive on P can be written in the form* (3.3.1) *for some h_i, then P is quasi-elementary.*

Proof of Theorem 3.4. Let X, Z and Grad be as in Theorem 1.6. Then one can show that Z is irreducible, that Z is the Zariski closure of X and that p^* induces an isomorphism of $\boldsymbol{R}(Z)$ with $\boldsymbol{R}(W)^K$. Let $f_1, \ldots, f_d$ be the determinants of the principal minors of Grad. Then $X = \{z \in Z: f_i(z) \geqslant 0, i = 1, \ldots, d\}$. Hence X is elementary, and Theorem 3.4 follows. □

If we drop the assumption that K is compact, then $\boldsymbol{R}(W)^K$ may fail to have property (H) ([7]). The problem is that the corresponding orbit space $X \subseteq Z$ may fail to be quasi-elementary!

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