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Introduction

Many categories encountered in various branches of mathematics have the following common feature: each object A has an *underlying set* of points, and each morphism $f\colon A \to B$ is determined by a function from the underlying set of A to the underlying set of B. Such categories are called *concrete*. In the pioneer papers by S. Eilenberg and S. MacLane [1942], [1945] much attention was paid to categories of this type. It is a rather curious phenomenon that as yet there is no general theory of concrete categories. Investigating concrete categories, we are often confronted with strange or pathological facts (cf., e.g., Isbell [1963], Calenko [1969], Freyd [1970]). Even the precise definition of a concrete category offers some problems and there are some non-equivalent definitions accepted by various authors.

Nevertheless, the notion of a concrete category seems to be especially useful if we wish to use categorical language, e.g., in general topology or functional analysis. Concrete categories constitute a bridge between general category theory and its non-homological applications. There are many useful constructions which cannot be carried out in arbitrary categories but can be conveniently defined in concrete categories (cf., e.g., Semadeni [1971], pp. 201–202, and this paper, Section 10.9).

The present paper is devoted to the discussion of some general problems concerning concrete categories (Chapters II and III). Additionally, a detailed description of the category of logical kits is given in Chapter IV.

Chapter I contains preliminaries. We recall well-known definitions and introduce some notions which are needed in the sequel. Set-theoretical lemmas stated in Section 3 are needed in Chapter II, the category of equivalence relations described in Section 4 is used in Chapter IV, and the notion of a pseudo-reflection introduced in Section 5 is applied in Chapter III.

In Chapter II two main topics are discussed. In Section 7 the notion of a *concrete duality* is examined. In Section 9 generalized images, coimages, embeddings and quotients are defined and investigated. These notions are defined in an arbitrary category (not necessarily concrete) with the aid of some preorders (called *D-preorders* and *C-preorders*) in the class of all morphisms of the category. General properties and examples

of D-preorders and C-preorders are considered in Section 8. It is shown in Section 9 that every C-preorder determines a class of *generalized embeddings* and every D-preorder determines a class of *generalized quotient morphisms*. In every category there are two extremal C-preorders; the generalized embeddings determined by the greatest C-preorder are identical with isomorphisms, and the generalized embeddings determined by the least C-preorder are identical with monomorphisms. Similarly, the generalized quotient morphisms determined by the greatest D-preorder are isomorphisms, and the generalized quotient morphisms determined by the least D-preorder are epimorphisms. Other intermediate D-preorders [C-preorders] give classes of generalized quotient morphisms [generalized embeddings] contained in the class of all epimorphisms [monomorphisms] and containing the class of all isomorphisms. Most important is the case when the category is concrete. The duality properties of generalized quotient morphisms and generalized embeddings in concrete categories are discussed.

The notions of an embedding and of a quotient morphism were considered by many authors from different points of view. J. R. Isbell [1964], H. Herrlich [1968], H. Herrlich and G. E. Strecker [1973] employ, following the idea of A. Grothendieck [1957], the notion of an *extremal monomorphism* as a categorical analogue of the topological notion of an embedding. Another categorical notion of an embedding was applied by H.-B. Brinkmann and D. Puppe [1966] (under the name *Einbettung*) and by M. Barr, P. A. Grillet and D. H. Osdol [1971] (under the name *subregular monomorphism*). Yet another notion of an embedding in a concrete category was introduced by Z. Semadeni [1971]. Embeddings and quotient morphisms can also be introduced axiomatically. This leads to the notion of a bicategory in the sense of S. MacLane [1950] and J. R. Isbell [1957], [1958], cf. also Z. Semadeni [1963]. It is shown in the present paper that embeddings in the sense of Z. Semadeni [1971], as well as subregular monomorphisms, are particular cases of generalized embeddings for suitably chosen C-preorders.

In Chapter III the conditions of unicity and of transfer are investigated. A concrete category $\mathfrak{A}$ is said to satisfy the *condition of unicity* iff there are no two different objects A, B in $\mathfrak{A}$ having the same underlying set and such that the identity function *is* an isomorphism from A to B. E.g., the category of Banach spaces and linear contractions satisfies this condition, while the category of Banach spaces and bounded linear operators — does not. A concrete category $\mathfrak{A}$ is said to satisfy the *condition of transfer* iff every set equipollent with the underlying set of an object A is the underlying set of some object B such that the bijection from the underlying set of A to the underlying set of B *is* an isomorphism from A to B. It is shown that the conditions of transfer and

of unicity determine some functors on the category of *all* concrete categories and covariant functors commuting with the forgetful functors. It will be shown in Section 13 that for every concrete category $\mathfrak{A}$ there exists a concrete category $\mathfrak{A}_{tu}$ which satisfies the conditions of transfer and of unicity and *best approximates* $\mathfrak{A}$. In Section 14 the connexion with the structures in the sense of C. Ehresmann [1957] is discussed.

Some problems discussed in Chapters II and III are partially known in special cases or in different, though somewhat related, context (cf., e.g., V. S. Garvackiĭ and B. M. Šaĭn [1970], L. Kučera and A. Pultr [1972]).

Chapter IV has a different character. It is devoted to the study of the category of logical kits. The notion of a logical kit was introduced by Z. Semadeni [1974] and is closely related to the notion of an automaton. The reader interested in applications of categories to computer science can read Chapter IV independently from Chapters II and III. Only the knowledge of Sections 1, 2 and 4 from Chapter I is required.

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Chapter I

Preliminaries

1. Notation. Categories and functors

The following logical symbols will be used: implication $\Rightarrow$, equivalence $\Leftrightarrow$, existential quantifier $\exists$, and universal quentifier $\forall$. The symbol $\{x\colon \Phi(x)\}$ will denote the class of all x such that $\Phi(x)$. The symbol $\operatorname{card} A$ will denote the cardinal number of a set A. The empty set is denoted by $\emptyset$.

The symbols $0, 1, 2, \ldots$ will denote natural numbers in the sense of von Neumann, i.e.

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \ldots$$

The question-mark designates a variable, e.g., $? + a$ is the symbol of a function of the form $f(x) = x + a$; using the barred arrow we shall also denote this function by $x \mapsto x + a$.

If $(A_t)_{t \in T}$ is an indexed family of sets, then $\mathop{\mathsf{P}}\limits_{t \in T} A_t$ will denote the Cartesian product, pr_s will denote the canonical projection $\mathop{\mathsf{P}}\limits_{t \in T} A_t \to A_s$, and $\mathop{\mathsf{S}}\limits_{t \in T} A_t$ will denote the disjoint sum, i.e. the set

$$\bigcup_{t \in T} (A_t \times \{t\}).$$

Instead of $\mathop{\mathsf{S}}\limits_{t \in 2} A_t$ we shall also write $A_0 + A_1$.

If A is a set, then $\mathscr{P}(A)$ will denote the power set of A, i.e. the set of all subsets of A.

By a *relation* we mean a binary relation, i.e. a set of ordered pairs. If R is a relation, then the condition $(a, b) \in R$ is also written in the form aRb. The sets

$$\operatorname{Dom} R = \{a\colon \exists_b\, aRb\} \quad \text{and} \quad \operatorname{Im} R = \{b\colon \exists_a\, aRb\}$$

are called the *domain* and the *image* of R, respectively. If X is a set, then the sets

$$\{b\colon \exists_{x \in X}\, xRb\} \quad \text{and} \quad \{a\colon \exists_{x \in X}\, aRx\}$$

will be denoted by $R^{\rightarrow}(X)$ and $R^{\leftarrow}(X)$, respectively. R^{-1} will denote the inverse relation $R^{-1} = \{(a, b)\colon bRa\}$. If R and S are relations, then $S \circ R$ will denote the composition of R and S, i.e.

$$S \circ R = \{(a, c)\colon \exists_b (aRb \text{ and } bRc)\}.$$

If A is a set, then id_A will denote the identity relation on A, i.e. $\mathrm{id}_A = \{(a, a)\colon a \in A\}$.

A relation R is *symmetric* iff aRb implies bRa; R is *transitive* iff aRb and bRc imply aRc; R is *reflexive* iff $a \in \mathrm{Dom}\, R$ implies aRa.

An *equivalence relation* on a set A is a reflexive, symmetric and transitive relation R such that $\mathrm{Dom}\, R = \mathrm{Im}\, R = A$. A *preorder* on a set A is a reflexive and transitive relation R such that $\mathrm{Dom}\, R = \mathrm{Im}\, R = A$.

We shall distinguish between a function $f\colon A \to B$ and a map $f\colon A \to B$. If A and B are sets, then by a function from A to B we shall mean a relation $f = \{(a, f(a))\colon a \in A\}$ such that $\mathrm{Im} f \subset B$. By a map $f\colon A \to B$ we shall mean a triple (f, A, B), where f is a function from A to B. We may also use a single letter f for a map $f\colon A \to B$ if this is not confusing.

If $C \subset A$ and f is a function from A to B, then $f|C$ denotes the restriction of f to C, i.e., the function $f \cap (C \times B)$.

A function f will be called *injective* iff it is one-to-one. If f is an injective function, then the inverse relation f^{-1} is a function defined on $\mathrm{Im} f$. A map $f\colon A \to B$ is called an *injection* iff f is an injective function; it is called a *surjection* iff $B = \mathrm{Im} f$; and it is called a *bijection* iff it is both an injection and a surjection.

If A is a set and $a \in A$, then

$$\bar{a}\colon 1 \to A$$

will denote the map defined by $\bar{a}(0) = a$.

The arguments will be based on the Tarski–Grothendieck–Sonner system of set theory with universes (for the definition of a universe cf., e.g., Schubert [1972], MacLane [1972]).

By the Tarski–Grothendieck–Sonner system of set theory we shall mean the Zermelo–Fraenkel system of set theory with the following additional axiom:

For every set A there is a universe U such that $A \in U$.

In particular, for every universe U there is a universe V such that $U \in V$. We shall say that the universe V is larger than U.

In the sequel U will denote a fixed universe, and V will denote a fixed universe larger than U. We shall assume that U contains as an element the set N of all positive integers.

All considerations carried out within U can be easily interpreted in the Gödel–Bernays system of set theory if we agree that "a set" means "an element of U" and "a class" means "a subset of U".

We shall assume that for every universe U we are given a fixed function $c_U\colon \mathscr{P}(U)\to U$ such that $c_U(A)\in A$ for all $A\subset U$, $A\neq\emptyset$. The function c_U is called the *choice function* for U.

The reader is supposed to be familiar with basic notions of category theory. However, we recall some definitions for the purpose of fixing terminology and notation.

By a *category* we shall mean a 6-tuple

$$\mathfrak{A} = (\mathfrak{A}^{\mathrm{o}}, \mathfrak{A}^{\mathrm{m}}, \mathrm{Dom}, \mathrm{Cod}, \iota, v), \tag{1}$$

where $\mathfrak{A}^{\mathrm{o}}$ is the set of objects, $\mathfrak{A}^{\mathrm{m}}$ is the set of morphisms, Dom and Cod are the functions from $\mathfrak{A}^{\mathrm{m}}$ to $\mathfrak{A}^{\mathrm{o}}$ assigning to each morphism α its domain $\mathrm{Dom}\,\alpha$ and codomain $\mathrm{Cod}\,\alpha$, ι is the function from $\mathfrak{A}^{\mathrm{o}}$ to $\mathfrak{A}^{\mathrm{m}}$ assigning to each object A the identity morphism $\iota_A\colon A\to A$, and v is the function from the set

$$\mathfrak{A}^{\mathrm{c}} = \{(\alpha,\beta)\in\mathfrak{A}^{\mathrm{m}}\times\mathfrak{A}^{\mathrm{m}}\colon \mathrm{Dom}\,\beta = \mathrm{Cod}\,\alpha\}$$

to $\mathfrak{A}^{\mathrm{m}}$ assigning to each pair (α,β) in $\mathfrak{A}^{\mathrm{c}}$ the composition $\beta\alpha$.

These data are supposed to satisfy the following axioms:

$$\begin{aligned}
\mathrm{Dom}\,\iota_A &= \mathrm{Cod}\,\iota_A = A,\\
\mathrm{Dom}(\beta\alpha) &= \mathrm{Dom}\,\alpha, \qquad \mathrm{Cod}(\beta\alpha) = \mathrm{Cod}\,\beta,\\
\gamma(\beta\alpha) &= (\gamma\beta)\alpha,\\
\alpha\iota_{\mathrm{Dom}\,\alpha} &= \alpha, \qquad\qquad \iota_{\mathrm{Cod}\,\alpha}\alpha = \alpha
\end{aligned}$$

for A in $\mathfrak{A}^{\mathrm{o}}$, and α,β,γ in $\mathfrak{A}^{\mathrm{m}}$, (α,β) in $\mathfrak{A}^{\mathrm{c}}$, (β,γ) in $\mathfrak{A}^{\mathrm{c}}$.

The set $\{\alpha\in\mathfrak{A}^{\mathrm{m}}\colon \mathrm{Dom}\,\alpha = A \text{ and } \mathrm{Cod}\,\alpha = B\}$ of all morphisms from A to B will be denoted by $\langle A, B\rangle_{\mathfrak{A}}$ or $\langle\mathfrak{A},\mathfrak{B}\rangle$.

The *dual category* of category (1) is the category $\mathfrak{A}^* = (\mathfrak{A}^{\mathrm{o}}, \mathfrak{A}^{\mathrm{m}}, \mathrm{Cod}, \mathrm{Dom}, \iota, v^*)$, where v^* is defined by $v^*(\alpha,\beta) = v(\beta,\alpha)$.

By $\mathrm{mono}(\mathfrak{A})$, $\mathrm{epi}(\mathfrak{A})$, $\mathrm{iso}(\mathfrak{A})$ we shall denote the subcategories of $\mathfrak{A}$ such that $\mathrm{mono}(\mathfrak{A})^{\mathrm{o}} = \mathrm{epi}(\mathfrak{A})^{\mathrm{o}} = \mathrm{iso}(\mathfrak{A})^{\mathrm{o}} = \mathfrak{A}^{\mathrm{o}}$ and $\mathrm{mono}(\mathfrak{A})^{\mathrm{m}}$, $\mathrm{epi}(\mathfrak{A})^{\mathrm{m}}$, $\mathrm{iso}(\mathfrak{A})^{\mathrm{m}}$ are the classes of all monomorphisms, epimorphisms, and isomorphisms in $\mathfrak{A}$ respectively.

By U-Ens we shall denote the category whose objects are those sets which are elements of U, and morphisms from A to B are maps $f\colon A\to B$. The functions Dom, Cod, ι, v are defined in an obvious way. In particular, $\iota_A = (\mathrm{id}_A, A, A)$.

By U-Binr we shall denote the category whose objects are elements of U and morphisms are triples (R, A, B), where $A, B\in U$ and $R\subset A\times B$. The functions Dom, Cod, ι, v are defined by $\mathrm{Dom}(R, A, B) = A$,

$\mathrm{Cod}(R, A, B) = B$, $\iota_A = (\mathrm{id}_A, A, A)$, $v\big((R, A, B), (S, B, C)\big) = (S \circ R, A, C)$. It should be noted that, in general, $\mathrm{Dom}(R, A, B) = A \neq \mathrm{Dom}\, R$. The category U-**Ens** is a subcategory of the category U-**Binr**.

A category $\mathfrak{A}$ is said to be a *U-category* iff $\mathfrak{A}^{\mathrm{o}} \subset U$ and $\langle A, B\rangle_{\mathfrak{A}} \in U$ for all A, B in $\mathfrak{A}^{\mathrm{o}}$ (these conditions imply $\mathfrak{A}^{\mathrm{m}} \subset U$). A category $\mathfrak{A}$ is said to be *U-small* iff it is a U-category and $\mathfrak{A}^{\mathrm{o}} \in U$ (this implies $\mathfrak{A}^{\mathrm{m}} \in U$). The categories U-**Ens** and U-**Binr** are U-categories, but they are not U-small. Every U-category is V-small for a larger universe V.

If $\mathfrak{A}$ and $\mathfrak{B}$ are categories, then by a covariant [contravariant] functor $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ we shall mean a pair $(\Phi^{\mathrm{o}}, \Phi^{\mathrm{m}})$, where

$$\Phi^{\mathrm{o}}\colon \mathfrak{A}^{\mathrm{o}} \to \mathfrak{B}^{\mathrm{o}}, \qquad \Phi^{\mathrm{m}}\colon \mathfrak{A}^{\mathrm{m}} \to \mathfrak{B}^{\mathrm{m}}$$

are functions satisfying the conditions

$$\begin{aligned}
\Phi^{\mathrm{o}}(\mathrm{Dom}\, a) &= \mathrm{Dom}\, \Phi^{\mathrm{m}}(a) \qquad [\Phi^{\mathrm{o}}(\mathrm{Dom}\, a) = \mathrm{Cod}\, \Phi^{\mathrm{m}}(a)],\\
\Phi^{\mathrm{o}}(\mathrm{Cod}\, a) &= \mathrm{Cod}\, \Phi^{\mathrm{m}}(a) \qquad [\Phi^{\mathrm{o}}(\mathrm{Cod}\, a) = \mathrm{Dom}\, \Phi^{\mathrm{m}}(a)],\\
\Phi^{\mathrm{m}}(\beta a) &= \Phi^{\mathrm{m}}(\beta)\Phi^{\mathrm{m}}(a) \qquad [\Phi^{\mathrm{m}}(\beta a) = \Phi^{\mathrm{m}}(a)\Phi^{\mathrm{m}}(\beta)],\\
\Phi^{\mathrm{m}}(\iota_A) &= \iota_{\Phi^{\mathrm{o}}(A)}
\end{aligned}$$

for all a in $\mathfrak{A}^{\mathrm{m}}$, A in $\mathfrak{A}^{\mathrm{o}}$, and (a, β) in $\mathfrak{A}^{\mathrm{c}}$.

Φ^{o} and Φ^{m} are called the *object transformation* and the *morphism transformation* of Φ respectively.

In the sequel we shall denote both functions Φ^{o} and Φ^{m} by the single letter Φ.

By a functor we shall mean a covariant functor if not explicitly stated otherwise. If $\mathfrak{B}$ is a subcategory of $\mathfrak{A}$, then the inclusion functor $\mathfrak{B} \to \mathfrak{A}$ will be denoted by $I_{\mathfrak{B}}^{\mathfrak{A}}$. Instead of $I_{\mathfrak{A}}^{\mathfrak{A}}$ we shall write $I_{\mathfrak{A}}$.

We shall say that categories $\mathfrak{A}$ and $\mathfrak{B}$ are *quasi-isomorphic* (equivalent in the terminology of Freyd [1964]) iff there are functors $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ and $\Psi\colon \mathfrak{B} \to \mathfrak{A}$ such that $\Psi\Phi$ and $\Phi\Psi$ are naturally equivalent to $I_{\mathfrak{A}}$ and $I_{\mathfrak{B}}$ respectively.

If $\mathfrak{A}$ is a category and $A \in \mathfrak{A}^{\mathrm{o}}$, then the category $\mathfrak{A}/A$ of objects above A (denoted also by $\mathrm{Morph}^{A}\mathfrak{A}$) is defined as follows: objects of $\mathfrak{A}/A$ are morphisms of $\mathfrak{A}$ with codomain A; morphisms of $\mathfrak{A}/A$ from $\beta\colon B \to A$ to $\gamma\colon C \to A$ are triples (φ, β, γ), where $\varphi\colon B \to C$ is a morphism in $\mathfrak{A}$ such that $\gamma\varphi = \beta$. The composition rule is

$$(\varphi', \gamma, \delta)(\varphi, \beta, \gamma) = (\varphi'\varphi, \beta, \delta).$$

Let $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ and $\Psi\colon \mathfrak{B} \to \mathfrak{A}$ be functors. Φ is a *left adjoint* of Ψ and Ψ is a *right adjoint* of Φ iff the following equivalent conditions are satisfied:

(i) there exists a natural equivalence

$$\omega_{A,B}\colon \langle \Phi(A), B\rangle_{\mathfrak{B}} \to \langle A, \Psi(B)\rangle_{\mathfrak{A}}$$

$(A \in \mathfrak{A}^{\circ}, B \in \mathfrak{B}^{\circ})$,

(ii) there exists a natural transformation

$$\eta\colon I_{\mathfrak{A}} \to \Psi\Phi$$

such that for every morphism $\xi\colon A \to \Psi(B)$ in $\mathfrak{A}^{\mathrm{m}}$ there is a unique morphism $\vartheta\colon \Phi(A) \to B$ in $\mathfrak{B}^{\mathrm{m}}$ such that $\Psi(\vartheta)\eta_A = \xi$,

(iii) there exists a natural transformation

$$\nu\colon \Phi\Psi \to I_{\mathfrak{B}}$$

such that for every morphism $\xi\colon \Phi(A) \to B$ in $\mathfrak{B}^{\mathrm{m}}$ there is a unique morphism $\vartheta\colon A \to \Psi(B)$ in $\mathfrak{A}^{\mathrm{m}}$ such that $\nu_B\Phi(\vartheta) = \xi$,

(iv) there exist natural transformations

$$\eta\colon I_{\mathfrak{A}} \to \Psi\Phi, \qquad \nu\colon \Phi\Psi \to I_{\mathfrak{B}}$$

satisfying the identities

$$\nu_{\Phi(A)}\Phi(\eta_A) = \iota_{\Phi(A)}, \qquad \Psi(\nu_B)\eta_{\Psi(B)} = \iota_{\Psi(B)} \qquad (A \in \mathfrak{A}^{\circ}, B \in \mathfrak{B}^{\circ}).$$

Each of conditions (i)–(iv) may be assumed to be an equivalent definition of adjointness. The natural transformations η and ν will be called *canonical natural transformations*.

By U-**Cat** we shall denote the category whose objects are U-categories and morphisms are functors. More precisely, U-$\mathbf{Cat}^{\circ}$ is the set of all U-categories, U-$\mathbf{Cat}^{\mathrm{m}}$ is the set of all triples $(\Phi, \mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}, \mathfrak{B}$ are U-categories and $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a functor; the functions Dom, Cod, ι, υ are defined in an obvious way.

The category U-**Cat** is not a U-category, but is a V-category for a larger universe V.

It is convenient to say that the elements of U-$\mathbf{Cat}^{\mathrm{m}}$ are functors. Therefore in this paper we shall mean by a "functor $\Phi\colon \mathfrak{A} \to \mathfrak{B}$" either the pair $(\Phi^{\circ}, \Phi^{\mathrm{m}})$ or the triple $((\Phi^{\circ}, \Phi^{\mathrm{m}}), \mathfrak{A}, \mathfrak{B})$. The actual meaning will be clear in each case from the context.

A covariant [contravariant] functor $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ meant as the triple $(\Phi, \mathfrak{A}, \mathfrak{B})$ is called an *injector* [*surjector*, *bijector*] iff both maps Φ°, Φ^{m} are injections [surjections, bijections, respectively].

2. Concrete categories

There are various definitions of a concrete category. The definition "a concrete category is a subcategory of the category of sets" is obviously inadequate, because such categories as the category of groups, the category of topological spaces, the category of vector spaces etc. are not

concrete in this sense. According to Kuroš, Livšic, and Šulgeĭfer [1960] a category $\mathfrak{A}$ is called *concrete* iff

(A) $\mathfrak{A}$ is isomorphic to a subcategory of the category of sets.

It is well known (as well as easily provable) that condition (A) is equivalent to the following one:

(B) There exists a faithful functor from $\mathfrak{A}$ to the category of sets.

However, this definition does not express in a satisfactory manner the intuitive concept of a "concrete" category whose objects are sets equipped with some structures and whose morphisms are some maps. In fact, Eilenberg and MacLane [1945] have shown that every small category $\mathfrak{A}$ satisfies conditions (A), (B), even if the morphisms of $\mathfrak{A}$ have nothing to do with usual maps.

Therefore it seems reasonable to call a category satisfying conditions (A), (B) a concretizable category. To be more specific, we shall say that $\mathfrak{A}$ is a *U-concretizable category* iff $\mathfrak{A}$ is a U-category and there exists a faithful functor $\mathfrak{A} \to U$-Ens. Every U-category is V-concretizable for a larger universe V.

J. R. Isbell [1963] and P. Freyd [1964] have given examples of categories which are not concretizable (within a fixed universe). P. Freyd [1970] proved that the category of topological spaces and homotopy classes of continuous maps is not concretizable.

A concrete category is defined by many authors (cf., e. g., Kučera and Pultr [1972], Wyler [1965]) as a pair $(\mathfrak{A}, \square)$ consisting of a category $\mathfrak{A}$ and a faithful functor $\square$ from $\mathfrak{A}$ to the category of sets. This definition has also some disadvantages: a concrete category is not a category but a pair consisting of a category and a functor; hence the category of all small concrete categories is not a subcategory of the category of all small categories.

We shall accept a different definition of a concrete category (cf. MacLane and Birkhoff [1967], Herrlich and Strecker [1973]).

2.1. Definition. A category $\mathfrak{A}$ will be called *U-concrete* iff $\mathfrak{A}$ is a U-category and the following conditions are satisfied:

(i) every morphism a in $\mathfrak{A}^{\mathrm{m}}$ is a triple $(f, \mathrm{Dom}\, a, \mathrm{Cod}\, a)$, where f is a function and $f \in U$; the function f will be denoted by $|a|$,

(ii) if a is an identity, i.e. $a = \iota_A$ for some $A \in \mathfrak{A}^{\mathrm{o}}$, then $|a| = \mathrm{id}_S$ for some $S \in U$; the set S (determined uniquely by A) will be denoted by $|A|$,

(iii) for every $a \in \mathfrak{A}^{\mathrm{m}}$ the set $|\mathrm{Dom}\, a|$ is the domain of $|a|$, and the set $|\mathrm{Cod}\, a|$ contains the image of $|a|$,

(iv) for all morphisms $a, \beta \in \mathfrak{A}^{\mathrm{m}}$ the condition $\mathrm{Cod}\, a = \mathrm{Dom}\, \beta$ implies $|\beta a| = |\beta| \circ |a|$.

The set $|A|$ is called the *underlying set* of A and the function $|a|$ is called the *underlying function* of a.

If $\mathfrak{A}$ is a U-concrete category, then the formulae

$$\begin{aligned} \square_{\mathfrak{A}}(A) &= |A|, \\ \square_{\mathfrak{A}}(a) &= (|a|, |\mathrm{Dom}\, a|, |\mathrm{Cod}\, a|) \\ &\qquad (A \in \mathfrak{A}^{\circ}, a \in \mathfrak{A}^{\mathrm{m}}), \end{aligned}$$

define a faithful functor $\square_{\mathfrak{A}}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}$ called the *forgetful functor*.

2.2. Suppose that we are given: 1° the set $\mathfrak{A}^{\circ} \subset U$, 2° "the underlying set function" $A \mapsto |A|$ defined on $\mathfrak{A}^{\circ}$ with values in U, 3° the set $\mathfrak{A}^{\mathrm{m}}$ whose elements are triples of the form (f, A, B), where $A, B \in \mathfrak{A}^{\circ}$ and f is a function from $|A|$ into $|B|$. These data determine a U-concrete category iff the following two conditions are satisfied:

(c_1) $(\mathrm{id}_{|A|}, A, A) \in \mathfrak{A}^{\mathrm{m}}$ for every $A \in \mathfrak{A}^{\circ}$,

(c_2) if $(f, A, B) \in \mathfrak{A}^{\mathrm{m}}$ and $(g, B, C) \in \mathfrak{A}^{\mathrm{m}}$, then $(g \circ f, A, C) \in \mathfrak{A}^{\mathrm{m}}$.

2.3. By U-**Concat** we shall denote the full subcategory of U-**Cat** whose objects are U-concrete categories. We shall show that the collection of all functors $\square_{\mathfrak{A}}$ may be regarded as a single functor defined on a suitably chosen subcategory of U-**Concat**.

We apply the following more general schema. Let $\mathfrak{M}$ be a subcategory of U-**Cat** and let $\mathfrak{C}$ be a fixed U-category. Suppose that for every $\mathfrak{A} \in \mathfrak{M}^{\circ}$ there is a functor $\Phi_{\mathfrak{A}}$ from $\mathfrak{A}$ to $\mathfrak{C}$. A functor $\Psi\colon \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A} \in \mathfrak{M}^{\circ}$ and $\mathfrak{B} \in \mathfrak{M}^{\circ}$, will be called *$\Phi$-commuting* iff $\Phi_{\mathfrak{B}}\Psi = \Phi_{\mathfrak{A}}$. Let $\mathfrak{M}_{\Phi}$ be the subcategory of $\mathfrak{M}$ defined in the following way: $\mathfrak{M}_{\Phi}^{\circ} = \mathfrak{M}^{\circ}$ and $\mathfrak{M}_{\Phi}^{\mathrm{m}}$ is the class of all Φ-commuting functors in $\mathfrak{M}^{\mathrm{m}}$. Then the family $(\Phi_{\mathfrak{A}})_{\mathfrak{A} \in \mathfrak{M}^{\circ}}$ induces the functor $\Phi\colon \mathfrak{M}_{\Phi} \to \mathfrak{M}/\mathfrak{C}$.

In particular, if $\mathfrak{M} = U$-**Concat**, $\Phi_{\mathfrak{A}} = \square_{\mathfrak{A}}$, and $\mathfrak{C} = U$-**Ens**, we obtain the functor

$$\square\colon U\text{-}\mathbf{Concat}_{\square} \to U\text{-}\mathbf{Concat}/U\text{-}\mathbf{Ens}.$$

The category U-$\mathbf{Concat}_{\square}$ is a subcategory of U-**Concat**; the objects of U-$\mathbf{Concat}_{\square}$ are U-concrete categories and the morphisms are *$\square$-commuting* functors, i.e. functors $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ such that $\square_{\mathfrak{B}}\Phi = \square_{\mathfrak{A}}$.

Let us note that $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is $\square$-commuting iff

$$|\Phi(A)| = |A| \quad \text{and} \quad |\Phi(a)| = |a|$$

for all $A \in \mathfrak{A}^{\circ}$, $a \in \mathfrak{A}^{\mathrm{m}}$. In other words, $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is $\square$-commuting if

$$\Phi(f, A, A') = \big(f, \Phi(A), \Phi(A')\big)$$

for all $(f, A, A') \in \mathfrak{A}^{\mathrm{m}}$.

Every $\square$-commuting functor is determined uniquely by its object transformation. Every map $\Phi^{\circ}\colon \mathfrak{A}^{\circ}\to\mathfrak{B}^{\circ}$ such that

$$(f, A, A')\in\mathfrak{A}^{\mathrm{m}} \Rightarrow \big(f, \Phi^{\circ}(A), \Phi^{\circ}(A')\big)\in\mathfrak{B}^{\mathrm{m}}$$

is the object transformation of the $\square$-commuting functor $\Phi\colon \mathfrak{A}\to\mathfrak{B}$ defined by

$$\Phi(a) = \big(|a|, \Phi^{\circ}(\mathrm{Dom}\, a), \Phi^{\circ}(\mathrm{Cod}\, a)\big).$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be U-concrete categories. We shall say that $\mathfrak{A}$ is *$\square$-isomorphic* with $\mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as objects of the category U-$\mathbf{Concat}_{\square}$, i.e. iff there exists a $\square$-commuting bijector $J\colon \mathfrak{A}\to\mathfrak{B}$.

2.4. Examples. (A) Let **Top** be the category of topological spaces and continuous maps. For every topological space A let $\Phi_d^{\circ}(A)$ $[\Phi_a^{\circ}(A)]$ be the same space with the discrete [antidiscrete] topology. The maps $\Phi_d^{\circ}\colon \mathbf{Top}^{\circ}\to\mathbf{Top}^{\circ}$ and $\Phi_a^{\circ}\colon \mathbf{Top}^{\circ}\to\mathbf{Top}^{\circ}$ are object transformations of the $\square$-commuting functors $\Phi_d\colon \mathbf{Top}\to\mathbf{Top}$ and $\Phi_a\colon \mathbf{Top}\to\mathbf{Top}$.

(B) Various "forgetful" functors such as the forgetful functor from the category of Banach spaces and bounded linear operations to the category of topological spaces and continuous maps are $\square$-commuting.

(C) The functors $F_w\colon \mathbf{Lchs}\to\mathbf{Lchs}$ and $F_m\colon \mathbf{Lchs}\to\mathbf{Lchs}$ considered by Wiweger [1966] are $\square$-commuting. The functor l defined by Porta [1972] is $\square$-commuting.

Definition 2.1 is in many cases too restrictive and can be generalized as follows.

2.5. Definition. Let n be a positive integer. A category $\mathfrak{A}$ will be called *n-U-concrete* iff $\mathfrak{A}$ is a U-category and the following conditions are satisfied:

(i) every morphism a in $\mathfrak{A}^{\mathrm{m}}$ is a triple $(f, \mathrm{Dom}\, a, \mathrm{Cod}\, a)$, where $f = (f_0, \ldots, f_{n-1})$ and $f_0, \ldots, f_{n-1}$ are functions belonging to U; we shall denote $|a| = f$, $|a|_i = f_i$ for $i = 0, \ldots, n-1$,

(ii) if a is an identity, i.e. $a = \iota_A$ for some $A\in\mathfrak{A}^{\circ}$, then $|a|_i = \mathrm{id}_{S_i}$ for some $S_i\in U$, $i = 0, \ldots, n-1$; we shall denote $|A| = (S_0, \ldots, S_{n-1})$, $|A|_i = S_i$, $i = 0, \ldots, n-1$,

(iii) for every $a\in\mathfrak{A}^{\mathrm{m}}$ and $i = 0, \ldots, n-1$ the set $|\mathrm{Dom}\, a|_i$ is the domain of $|a|_i$, and the set $|\mathrm{Cod}\, a|_i$ contains the image of $|a|_i$,

(iv) for all morphisms $a, \beta\in\mathfrak{A}^{\mathrm{m}}$ the condition $\mathrm{Cod}\, a = \mathrm{Dom}\, \beta$ implies $|\beta a|_i = |\beta|_i \circ |a|_i$ for $i = 0, \ldots, n-1$.

2.6. Examples. (A) The category U-**Cat** is 2-*V-concrete* for a larger universe V. If $\mathfrak{A}\in U$-$\mathbf{Cat}^{\circ}$ and $\Phi\in U$-$\mathbf{Cat}^{\mathrm{m}}$, then $|\mathfrak{A}|_0 = \mathfrak{A}^{\circ}$, $|\mathfrak{A}|_1 = \mathfrak{A}^{\mathrm{m}}$, $|\Phi|_0 = \Phi^{\circ}$, $|\Phi|_1 = \Phi^{\mathrm{m}}$.

(B) The category of deterministic Mealy automata (cf. Ehrig and Pfender [1972]) within a fixed universe U is 3-U-concrete.

Another example of a 3-U-concrete category is the category U-**Kt** discussed in Chapter IV.

2.7. Remarks. It is easy to verify that every U-category isomorphic to an n-U-concrete category is a U-concretizable category. Definition 2.1 has an unpleasant feature that a U-category isomorphic to a U-concrete category need not be U-concrete, the dual of a U-concrete category is not a U-concrete category, and the product of U-concrete categories is not a U-concrete category. However, the product of n U-concrete categories can be identified in a natural way with a n-U-concrete category, and the notion of the dual category of a U-concrete category can be modified (cf. Section 7) in such a way that the "concrete dual" is still a U-concrete category.

3. Power functors. Set-theoretical lemmas

3.1. Let f be a function with $\mathrm{Dom} f = A$ and $\mathrm{Im} f \subset B$. It is obvious that for all X, X' in $\mathscr{P}(A)$ and Y, Y' in $\mathscr{P}(B)$

$$X \subset X' \text{ implies } f^{\rightarrow}(X) \subset f^{\rightarrow}(X'),$$

$$Y \subset Y' \text{ implies } f^{\leftarrow}(Y) \subset f^{\leftarrow}(Y'),$$

$$f^{\leftarrow}f^{\rightarrow}(X) = \{a \in A\colon \underset{x \in X}{\exists}\, f(a) = f(x)\} \supset X,$$

$$f^{\rightarrow}f^{\leftarrow}(Y) = Y \cap f^{\rightarrow}(A) \subset Y,$$

$$f^{\rightarrow}f^{\leftarrow}f^{\rightarrow}(X) = f^{\rightarrow}(X), \quad f^{\leftarrow}f^{\rightarrow}f^{\leftarrow}(Y) = f^{\leftarrow}(Y).$$

3.2. Assigning to any map $f\colon A \to B$ the maps

$$\mathscr{P}_+(f)\colon \mathscr{P}(A) \to \mathscr{P}(B), \quad \mathscr{P}_-(f)\colon \mathscr{P}(B) \to \mathscr{P}(A)$$

defined by

$$\mathscr{P}_+(f)(X) = f^{\rightarrow}(X), \quad \mathscr{P}_-(f)(Y) = f^{\leftarrow}(Y),$$

we obtain the *covariant power functor*

$$\mathscr{P}_+\colon U\text{-}\mathbf{Ens} \to U\text{-}\mathbf{Ens},$$

and the *contravariant power functor*

$$\mathscr{P}_-\colon U\text{-}\mathbf{Ens} \to U\text{-}\mathbf{Ens}.$$

The functors $\mathscr{P}_+$ and $\mathscr{P}_-$ are injectors.

It is well known that if the object map of a (covariant or contravariant) functor $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is an injection, then Φ determines a subcategory of $\mathfrak{B}$ with $\{\Phi(A)\colon A \in \mathfrak{A}^{\mathrm{o}}\}$ as the class of objects and $\{\Phi(a)\colon a \in \mathfrak{A}^{\mathrm{m}}\}$ as the class of morphisms. This subcategory is denoted by $\Phi(\mathfrak{A})$.

In particular, the functors $\mathscr{P}_+$: U-Ens$\to U$-Ens and $\mathscr{P}_-$: U-Ens $\to U$-Ens determine the subcategories $\mathscr{P}_+(U\text{-Ens})$ and $\mathscr{P}_-(U\text{-Ens})$ of U-Ens. These subcategories can be characterized in the following way:

3.3. PROPOSTION. *A map f: $A\to B$ is a morphism in $\mathscr{P}_+(U\text{-Ens})$ if and only if the following conditions are satisfied:*

(i) *A and B are power sets, i.e. there exist sets M and N such that $A = \mathscr{P}(M)$ and $B = \mathscr{P}(N)$,*

(ii) *f is union-preserving, i.e. for every indexed family $(X_t)_{t\in T}$ of elements of A*

$$f(\bigcup_{t\in T} X_t) = \bigcup_{t\in T} f(X_t),$$

(iii) *for every X in A the condition* $\operatorname{card} X = 1$ *implies* $\operatorname{card} f(X) = 1$.

Proof. If there exists a map g: $M\to N$ such that $f = \mathscr{P}_+(g)$, then conditions (i)–(iii) are obviously satisfied.

If f satisfies (i)–(iii), then $f = \mathscr{P}_+(g)$, where

$$g = \{(m, n)\in M\times N\colon \{n\} = f(\{m\})\}.$$

3.4. PROPOSITION. *A map f: $A\to B$ is a morphism in $\mathscr{P}_-(U\text{-Ens})$ if and only if the following conditions are satisfied:*

(i) *A and B are power sets,*

(ii) *f is union-preserving,*

(iii) *f is disjointness-preserving, i.e. for all X, Y belonging to A the condition $X\cap Y = \emptyset$ implies $f(X)\cap f(Y) = \emptyset$,*

(iv) $f(\bigcup_{X\in A} X) = \bigcup_{Y\in B} Y$.

Proof. If there exists a map g: $N\to M$ such that $A = \mathscr{P}(M)$, $B = \mathscr{P}(N)$ and $f = \mathscr{P}_-(g)$, then conditions (i)–(iv) are obviously satisfied.

Conversely, suppose that f: $A\to B$ is a map satisfying (i)–(iv). Let $M = \bigcup_{X\in A} X$, $N = \bigcup_{Y\in B} Y$, and let

$$g = \{(n, m)\in N\times M\colon n\in f(\{m\})\}.$$

It follows from (ii) and (iv) that

$$\bigcup_{m\in M} f(\{m\}) = f(M) = N,$$

i.e. for every n in N there is m in M such that $(n, m)\in g$. If $(n, m)\in g$ and $(n, m')\in g$, then $n\in f(\{m\})$ and $n\in f(\{m'\})$, i.e. $f(\{m\})\cap f(\{m'\}) \neq \emptyset$. Consequently, by (iii), $\{m\}\cap\{m'\} \neq \emptyset$, i.e. $m = m'$. Hence g is a function from N to M. It remains to show that $\mathscr{P}_-(g) = f$. For every X in A

$$g^{\leftarrow}(X) = g^{\leftarrow}(\bigcup_{m\in X}\{m\}) = \bigcup_{m\in X} g^{\leftarrow}(\{m\}) = \bigcup_{m\in X} f(\{m\}) = f(\bigcup_{m\in X}\{m\}) = f(X).$$

The following two lemmas can be easily derived from properties 3.1 and will be needed in the sequel (cf. Propositions 8.12 and 8.13).

3.5. LEMMA. *Let $f\colon A\to B$ and $g\colon A\to C$ be two maps. The following conditions are equivalent:*

(i) $f(a)=f(a')$ *implies* $g(a)=g(a')$ *for all* $a, a'\in A$,

(ii) $f^{\leftarrow}f^{\rightarrow}g^{\leftarrow}(Z)=g^{\leftarrow}(Z)$ *for every* Z *in* $\mathscr{P}(C)$,

(iii) $g^{\rightarrow}f^{\leftarrow}f^{\rightarrow}(X)=g^{\rightarrow}(X)$ *for every* X *in* $\mathscr{P}(A)$,

(iv) $\mathscr{P}_+(f)(X)=\mathscr{P}_+(f)(X')$ *implies* $\mathscr{P}_+(g)(X)=\mathscr{P}_+(g)(X')$ *for all* $X, X'\in\mathscr{P}(A)$,

(v) $\mathscr{P}_-(g)^{\rightarrow}(\mathscr{P}(C))\subset\mathscr{P}_-(f)^{\rightarrow}(\mathscr{P}(B))$.

3.6. LEMMA. *Let $f\colon B\to A$ and $g\colon C\to A$ be two maps. The following conditions are equivalent:*

(i) $g^{\rightarrow}(C)\subset f^{\rightarrow}(B)$,

(ii) $g^{\leftarrow}f^{\rightarrow}f^{\leftarrow}(X)=g^{\leftarrow}(X)$ *for every* X *in* $\mathscr{P}(A)$,

(iii) $f^{\rightarrow}f^{\leftarrow}g^{\rightarrow}(Z)=g^{\rightarrow}(Z)$ *for every* Z *in* $\mathscr{P}(C)$,

(iv) $\mathscr{P}_+(g)^{\rightarrow}(\mathscr{P}(C))\subset\mathscr{P}_+(f)^{\rightarrow}(\mathscr{P}(B))$,

(v) $\mathscr{P}_-(f)(X)=\mathscr{P}_-(f)(X')$ *implies* $\mathscr{P}_-(g)(X)=\mathscr{P}_-(g)(X')$ *for all* $X, X'\in\mathscr{P}(A)$.

For $g=\mathrm{id}_A$ we obtain from Lemmas 3.5 and 3.6 the following corollaries:

3.7. COROLLARY. *Let $f\colon A\to B$ be a map. The following conditions are equivalent:*

(i) $f\colon A\to B$ *is an injection,*

(ii) $\mathscr{P}_+(f)\colon\mathscr{P}(A)\to\mathscr{P}(B)$ *is an injection,*

(iii) $\mathscr{P}_-(f)\colon\mathscr{P}(B)\to\mathscr{P}(A)$ *is a surjection.*

3.8. COROLLARY. *Let $f\colon B\to A$ be a map. The following conditions are equivalent:*

(i) $f\colon B\to A$ *is a surjection,*

(ii) $\mathscr{P}_+(f)\colon\mathscr{P}(B)\to\mathscr{P}(A)$ *is a surjection,*

(iii) $\mathscr{P}_-(f)\colon\mathscr{P}(A)\to\mathscr{P}(B)$ *is an injection.*

3.9. LEMMA. *Let M, N be sets, and let $f\colon\mathscr{P}(M)\to\mathscr{P}(N)$ be a union-preserving map. If f is a bijection, then f is a morphism in $\mathscr{P}_+(U\text{-}\mathrm{Ens})$ and in $\mathscr{P}_-(U\text{-}\mathrm{Ens})$.*

Proof. It is easy to verify that f satisfies conditions 3.3 (i)–(iii) and 3.4 (i)–(iv).

3.10. Remarks. The power functors defined in 6.2 are restrictions to U-Ens of the power functors

$$\mathscr{P}'_+ \colon U\text{-}\mathbf{Binr} \to U\text{-}\mathbf{Ens}, \qquad \mathscr{P}'_- \colon U\text{-}\mathbf{Binr} \to U\text{-}\mathbf{Ens}$$

defined in a similar way. The functors $\mathscr{P}'_+$ and $\mathscr{P}'_-$ are injectors and it is well known that the morphisms of the category $\mathscr{P}'_+(U\text{-}\mathbf{Binr})$ are precisely those maps of the form $f\colon \mathscr{P}(M) \to \mathscr{P}(N)$ which satisfy condition 3.3 (ii). The functor $\mathscr{P}'_+\colon U\text{-}\mathbf{Binr} \to U\text{-}\mathbf{Ens}$ is a right adjoint of the inclusion functor $I^{U\text{-}\mathbf{Binr}}_{U\text{-}\mathbf{Ens}}$. Moreover, $\mathscr{P}'_- = \mathscr{P}'_+ \mathscr{A}$, where $\mathscr{A}\colon U\text{-}\mathbf{Binr} \to U\text{-}\mathbf{Binr}$ is a contravariant bijector defined by $\mathscr{A}(R, A, B) = (R^{-1}, B, A)$.

4. The category of equivalence relations

The category U-**Eq** of equivalence relations within U is a U-concrete category defined as follows: the objects are equivalence relations R satisfying the condition $R \in U$; the underlying set of an equivalence relation R is defined as

$$|R| = \{a\colon (a, a) \in R\};$$

the set of morphisms from an equivalence relation R to an equivalence relation R' is the set of all triples (f, R, R'), where f is a function from $|R|$ to $|R'|$ such that $(a, b) \in R$ implies $\big(f(a), f(b)\big) \in R'$. It is clear that conditions (c_1), (c_2) from Section 2.2 are satisfied.

The forgetful functor

$$\square\colon U\text{-}\mathbf{Eq} \to U\text{-}\mathbf{Ens}$$

with the object transformation $\square(R) = |R|$ and the morphism transformation $\square(f, R, R') = \big(f, \square(R), \square(R')\big)$ has a left adjoint and a right adjoint. It can easily be verified that the functor

$$\mathscr{D}\colon U\text{-}\mathbf{Ens} \to U\text{-}\mathbf{Eq}$$

defined by

$$\mathscr{D}(A) = \{(a, a)\colon a \in A\}, \qquad \mathscr{D}(f, A, A') = \big(f, \mathscr{D}(A), \mathscr{D}(A')\big)$$

for every morphism (f, A, A') in U-**Ens**, is a left adjoint of $\square$, while the functor

$$?^2\colon U\text{-}\mathbf{Ens} \to U\text{-}\mathbf{Eq}$$

defined by

$$A^2 = A \times A, \qquad (f, A, A')^2 = (f, A^2, A'^2)$$

is a right adjoint of $\square$.

The functor $\mathscr{D}$ has also a left adjoint. Let

$$Q\colon U\text{-}\mathbf{Eq} \to U\text{-}\mathbf{Ens}$$

be the functor defined in the following way: if R is an equivalence relation, then

$$Q(R) = \square(R)/R;$$

if (f, R, R') is a morphism in U-Eq, then

$$Q(f, R, R')(a/R) = f(a)/R', \quad \text{where } a \in \square(R).$$

(It follows from the definition of morphisms in U-Eq that $f(a)/R'$ does not depend on the choice of a in the equivalence class a/R.)

It is easy to verify that Q is a left adjoint of $\mathscr{D}$.

Let us note that the functors $\mathscr{D}$ and $?^2$ are $\square$-commuting, but Q is not $\square$-commuting.

5. Pseudo-reflections and reflections

Let $\mathfrak{A}$ be a category, and let R be an equivalence relation on $\mathfrak{A}^{\mathrm{m}}$ such that

$$\alpha R \beta \text{ implies } \mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta \text{ and } \mathrm{Cod}\,\alpha = \mathrm{Cod}\,\beta.$$

In other words, R is an equivalence relation defined on each of the sets $\langle A, B\rangle_{\mathfrak{A}}$ separately.

5.1. Definitions. Let $\mathfrak{B}$ be a subcategory of $\mathfrak{A}$. A morphism $\tau\colon A \to B$ in $\mathfrak{A}^{\mathrm{m}}$ is called an *R-pseudo-reflection* from $\mathfrak{A}$ into $\mathfrak{B}$ iff $B \in \mathfrak{B}^{\mathrm{o}}$ and the following two conditions are satisfied:

(i) for every $X \in \mathfrak{B}^{\mathrm{o}}$ and every $\xi\colon A \to X$ in $\mathfrak{A}^{\mathrm{m}}$ there is $\vartheta\colon B \to X$ in $\mathfrak{B}^{\mathrm{m}}$ such that $\xi = \vartheta\tau$,

(ii) for all $\vartheta\colon B \to X$, $\vartheta'\colon B \to X$ in $\mathfrak{B}^{\mathrm{m}}$ the equality $\vartheta\tau = \vartheta'\tau$ implies $\vartheta R \vartheta'$.

If R is the identity relation, then an R-pseudo-reflection is called a *reflection* (cf. Freyd [1964]).

If $\tau\colon A \to B$ is a reflection from $\mathfrak{A}$ into $\mathfrak{B}$ then we say that B is a reflection of A in $\mathfrak{B}$. If every object in $\mathfrak{A}^{\mathrm{o}}$ has a reflection in $\mathfrak{B}$ we say that $\mathfrak{B}$ is a *reflective* subcategory of $\mathfrak{A}$. In this case the inclusion functor $I_{\mathfrak{B}}^{\mathfrak{A}}$ has a left adjoint $\Phi\colon \mathfrak{A} \to \mathfrak{B}$; the functor Φ assigns to each object in $\mathfrak{A}^{\mathrm{o}}$ its reflection in $\mathfrak{B}$ and is called a *reflector*.

A morphism $\alpha\colon A \to A'$ in $\mathfrak{A}^{\mathrm{m}}$ is called an *R-pseudo-isomorphism* in $\mathfrak{A}$ iff there exists $\alpha' \in \mathfrak{A}^{\mathrm{m}}$ such that $(\alpha'\alpha) R \iota_A$ and $(\alpha\alpha') R \iota_{A'}$. It may be shown by the well-known argument that if $\tau\colon A \to B$ and $\tau'\colon A \to B'$ are R-pseudo-reflections from $\mathfrak{A}$ into $\mathfrak{B}$, then $\tau = \beta\tau'$, where β is an R-pseudo-isomorphism in $\mathfrak{B}$.

5.2. Examples. (A) Let $\mathfrak{A}$ be a subcategory of U-Cat and let $\alpha R \beta$ mean that the functors α and β are naturally equivalent. If a functor $\alpha\colon A \to B$ is an R-pseudo-isomorphism in $\mathfrak{A}$, then α is a quasi-isomorphism of categories A and B; the converse is true if $\mathfrak{A}$ is a full subcategory of U-Cat. An example of an R-pseudo-reflection in this case will be given in Section 11.

(B) Let $\mathfrak{A}$ be an arbitrary category. For all $\alpha, \beta \in \langle A, B\rangle_{\mathfrak{A}}$ let $\alpha R_0 \beta$ mean that there exists an isomorphism $\eta\colon B \to B$ such that $\alpha = \eta\beta$. It is easy to verify that every R_0-pseudo-isomorphism is an isomorphism. If $\mathfrak{B}$ is a reflective subcategory of $\mathfrak{A}$, then every R_0-pseudo-reflection from $\mathfrak{A}$ into $\mathfrak{B}$ is a reflection.

(C) Let $\mathfrak{A} = \mathrm{mono}(U\text{-}\mathbf{Ens})$. Let $\mathfrak{B}$ be the full subcategory of $\mathfrak{A}$ such that $\mathfrak{B}^\circ$ is the class of all infinite sets in U-$\mathbf{Ens}^\circ$. For every $A \in U$ the canonical injection of A into $A + N$ (where N is the set of all positive integers and $+$ denotes the disjoint sum, i.e. the coproduct in U-$\mathbf{Ens}$) is an R_0-pseudo-reflection from $\mathfrak{A}$ to $\mathfrak{B}$ (where R_0 is the relation defined in (B)). Because $\mathfrak{B}$ is not a reflective subcategory of $\mathfrak{A}$, this example shows that there are R_0-pseudo-reflections which are not reflections.

6. Universal points. Generators and cogenerators

Let $\mathfrak{A}$ be a U-category and let $\Lambda\colon \mathfrak{A} \to U$-$\mathbf{Ens}$ be a covariant [contravariant] functor. A *universal* [*couniversal*] *point* of Λ is a pair (A, a) such that

(i) $A \in \mathfrak{A}^\circ$ and $a \in \Lambda(A)$,

(ii) for every X in $\mathfrak{A}^\circ$ and every x in $\Lambda(X)$ there exists a unique ϑ in $\langle A, X\rangle$ [in $\langle X, A\rangle$] such that the diagram

$$\begin{tikzcd} 1 \arrow[r, "\bar{a}"] \arrow[dr, "\bar{x}"'] & \Lambda(A) \arrow[d, "\Lambda(\vartheta)"] \\ & \Lambda(X) \end{tikzcd}$$

is commutative.

It is well known that many categorical notions (e.g., limits of diagrams and reflections) can be defined as universal points of some functors.

A *generator* in $\mathfrak{A}$ is an object G such that the principal covariant functor

$$\langle G, ?\rangle_{\mathfrak{A}}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}$$

is faithful, i.e. the conditions $\varphi, \psi \in \langle A, B\rangle_{\mathfrak{A}}$ and $\varphi\xi = \psi\xi$ for all $\xi \in \langle G, A\rangle_{\mathfrak{A}}$ imply $\varphi = \psi$.

A *cogenerator* in $\mathfrak{A}$ is an object C such that the principal contravariant functor

$$\langle ?, C\rangle_{\mathfrak{A}}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}$$

is faithful, i.e. the conditions $\varphi, \psi \in \langle A, B\rangle_{\mathfrak{A}}$ and $\xi\varphi = \xi\psi$ for all $\xi \in \langle B, C\rangle_{\mathfrak{A}}$ imply $\varphi = \psi$.

Let $\mathfrak{A}$ be a U-concrete category, and let $\mathfrak{m}$ be a cardinal number in the sense of von Neumann such that $\mathfrak{m} \in U$. An $\mathfrak{m}$-*free* object in $\mathfrak{A}$ is an object F together with a map $\sigma: \mathfrak{m} \to |F|$ satisfying the following condition: for any object X and any map $\xi: \mathfrak{m} \to |X|$ there exists a unique ϑ in $\langle F, X\rangle_{\mathfrak{A}}$ such that $(\square\vartheta)\sigma = \xi$. We say that F is *freely* $\mathfrak{m}$-*generated* by σ. An $\mathfrak{m}$-*cofree* object in $\mathfrak{A}$ is an object H together with a map $\pi: |H| \to \mathfrak{m}$ satisfying the following condition: for any object X and any map $\xi: |X| \to \mathfrak{m}$ there exists a unique ϑ in $\langle X, H\rangle_{\mathfrak{A}}$ such that $\pi(\square\vartheta) = \xi$. We shall say that H is *freely* $\mathfrak{m}$-*cogenerated* by π.

Note that F is an $\mathfrak{m}$-free object freely $\mathfrak{m}$-generated by σ if and only if (F, σ) is a universal point of the covariant functor

$$\langle \mathfrak{m}, \square_{\mathfrak{A}}(?)\rangle_{U\text{-Ens}}: \mathfrak{A} \to U\text{-Ens}.$$

Similarly, H is an $\mathfrak{m}$-cofree object freely $\mathfrak{m}$-cogenerated by π if and only if (H, π) is a couniversal point of the contravariant functor

$$\langle \square_{\mathfrak{A}}(?), \mathfrak{m}\rangle_{U\text{-Ens}}: \mathfrak{A} \to U\text{-Ens}.$$

6.1. Proposition. *If $\mathfrak{A}$ is a concrete category, then every 1-free object in $\mathfrak{A}$ is a generator in $\mathfrak{A}$, and every 2-cofree object in $\mathfrak{A}$ is a cogenerator in $\mathfrak{A}$.*

Proof. Let H be a 2-cofree object in $\mathfrak{A}$. Suppose that $\varphi, \psi \in \langle A, B\rangle$ and $\xi\varphi = \xi\psi$ for all $\xi \in \langle B, H\rangle$. If $\pi: |H| \to 2$ is a map 2-cogenerating H, then for every map $\zeta: |B| \to 2$ there exists a unique $\xi \in \langle B, H\rangle$ such that $\pi(\square\xi) = \zeta$, and hence $\zeta(\square\psi) = \pi\square(\xi\psi) = \pi\square(\xi\varphi) = \zeta(\square\varphi)$. Since 2 is a cogenerator in U-Ens and the forgetful functor $\square: \mathfrak{A} \to U$-Ens is faithful, it follows that $\square\psi = \square\varphi$ and $\psi = \varphi$.

The proof of the first part of the proposition is similar.

Chapter II

Duality. Generalized embeddings and quotients

7. Dual notions in concrete categories

It is well known that the dual of a U-concretizable category is a U-concretizable category. J. R. Isbell [1964] has shown that under some assumptions the dual of a category of algebras is isomorphic with a category of algebras. On the other hand, the dual of a U-concrete category need not be a U-concrete category. For that reason we shall introduce the notion of a "concrete-dual" category $\mathfrak{A}^{\diamond}$ of a given U-concrete category $\mathfrak{A}$. This notion seems to be helpful in investigating dual notions in concrete categories.

7.1. Definition. Let $\mathfrak{A}$ be a U-concrete category. The *concrete-dual category* of $\mathfrak{A}$ is the U-concrete category $\mathfrak{A}^{\diamond}$ defined as follows. The objects of $\mathfrak{A}^{\diamond}$ are the same as the objects of $\mathfrak{A}$, i.e. $(\mathfrak{A}^{\diamond})^{\mathrm{o}} = \mathfrak{A}^{\mathrm{o}}$; if A is an object of $\mathfrak{A}$ and $|A|$ is the underlying set of A in $\mathfrak{A}$, then the underlying set of A in $\mathfrak{A}^{\diamond}$ is the set

$$|A|^{\diamond} = \mathscr{P}(|A|);$$

the morphisms of $\mathfrak{A}^{\diamond}$ are all triples of the form

$$\big(\mathscr{P}_{-}(f), A, B\big),$$

where (f, B, A) is a morphism in $\mathfrak{A}^{\mathrm{m}}$, i.e.

$$(\mathfrak{A}^{\diamond})^{\mathrm{m}} = \big\{\big(\mathscr{P}_{-}(f), A, B\big) \colon (f, B, A) \in \mathfrak{A}^{\mathrm{m}}\big\}.$$

It follows immediately from the definition that $\mathfrak{A}^{\diamond}$ is isomorphic with the (ordinary) dual category $\mathfrak{A}^*$ and that the functor

$$(1) \qquad ?^{\diamond} \colon \mathfrak{A} \to \mathfrak{A}^{\diamond}$$

defined as

$$A^{\diamond} = A \quad \text{for } A \in \mathfrak{A}^{\mathrm{o}},$$

$$a^{\diamond} = \big(\mathscr{P}_{-}(|a|), \operatorname{Cod} a, \operatorname{Dom} a\big) \quad \text{for } a \in \mathfrak{A}^{\mathrm{m}},$$

is a contravariant bijector.

Moreover, the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[d, "?^{\circ}"'] \arrow[rr, "\mathscr{P}_{-}\square_{\mathfrak{A}}"] & & U\text{-}\mathbf{Ens} \\ \mathfrak{A}^{\circ} \arrow[rr, "\square_{\mathfrak{A}^{\circ}}"'] & & \end{tikzcd}$$

is commutative.

Let us note that $\mathfrak{A}^{\circ\circ} \neq \mathfrak{A}$ for every U-concrete category $\mathfrak{A}$.

It is easy to see that the category U-$\mathbf{Ens}^{\circ}$ is $\square$-isomorphic with the category $\mathscr{P}_{-}(U\text{-}\mathbf{Ens})$.

7.2. If $\mathfrak{A}$ and $\mathfrak{B}$ are U-concrete categories, and

$$\Phi\colon \mathfrak{A} \to \mathfrak{B}$$

is a covariant [contravariant] functor, then there exists a unique covariant [contravariant] functor

$$\Phi^{\circ}\colon \mathfrak{A}^{\circ} \to \mathfrak{B}^{\circ}$$

such that the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\Phi"] \arrow[d, "?^{\circ}"'] & \mathfrak{B} \arrow[d, "?^{\circ}"] \\ \mathfrak{A}^{\circ} \arrow[r, "\Phi^{\circ}"'] & \mathfrak{B}^{\circ} \end{tikzcd}$$

is commutative. If Φ is $\square$-commuting, then Φ° is also $\square$-commuting. If Φ is an injector [a surjector, a bijector], then Φ° is also an injector [a surjector, a bijector, respectively].

Let us write $\mathscr{S}(\mathfrak{A}) = \mathfrak{A}^{\circ}$ for every U-concrete category $\mathfrak{A}$, and $\mathscr{S}(\Phi) = \mathfrak{A}^{\circ}$ for every covariant functor Φ between U-concrete categories. This yields the covariant functors

$$\mathscr{S}\colon U\text{-}\mathbf{Concat} \to U\text{-}\mathbf{Concat}$$

and

$$\mathscr{S}\colon U\text{-}\mathbf{Concat}_{\square} \to U\text{-}\mathbf{Concat}_{\square}.$$

7.3. Definitions. Let i and j be non-negative integers. A *c-predicate* [*cc-predicate*] of type (i, j) is a function

$$p\colon U\text{-}\mathbf{Cat}^{\circ} \to U \qquad [p\colon U\text{-}\mathbf{Concat}^{\circ} \to U]$$

such that for every [U-concrete] U-category $\mathfrak{A}$ the set $p(\mathfrak{A})$ is a subset of the set

$$\underbrace{\mathfrak{A}^{\mathrm{m}} \times \ldots \times \mathfrak{A}^{\mathrm{m}}}_{i \text{ times}} \times \underbrace{\mathfrak{A}^{\mathrm{o}} \times \ldots \times \mathfrak{A}^{\mathrm{o}}}_{j \text{ times}}$$

and for every [$\square$-commuting] covariant bijector $\Phi\colon \mathfrak{A}\to\mathfrak{B}$ the condition

$$(a_1, \ldots, a_i, A_1, \ldots, A_j)\in p(\mathfrak{A})$$

implies

$$\big(\Phi(a_1), \ldots, \Phi(a_i), \Phi(A_1), \ldots, \Phi(A_j)\big)\in p(\mathfrak{B}).$$

A c-predicate [cc-predicate] q will be called *dual* [*c-dual*] to the c-predicate [cc-predicate] p if q is of the same type as p and for every [U-concrete] U-category $\mathfrak{A}$ and all $a_1, \ldots, a_i\in\mathfrak{A}^{\mathrm{m}}$, $A_1, \ldots, A_j\in\mathfrak{A}^{\mathrm{o}}$ the condition

$$(a_1, \ldots, a_i, A_1, \ldots, A_j)\in q(\mathfrak{A})$$

is equivalent to the condition

$$(a_1^*, \ldots, a_i^*, A_1^*, \ldots, A_j^*)\in p(\mathfrak{A}^*) \qquad [(a_1^\circ, \ldots, a_i^\circ, A_1^\circ, \ldots, A_j^\circ)\in p(\mathfrak{A}^\circ)].$$

It is obvious that if a c-predicate q is dual to a c-predicate p, then p is dual to q (cf., however, Example 7.5(B)). The following proposition is also obvious.

7.4. Proposition. *Let p be a c-predicate* [*cc-predicate*]. *Then there exists a dual* [*c-dual*] *c-predicate* [*cc-predicate*] *q defined by*

$$q(\mathfrak{A}) = \{(a_1, \ldots, A_1, \ldots)\colon (a_1^\circ, \ldots, A_1^\circ, \ldots)\in p(\mathfrak{A}^\circ)\}.$$

7.5. Examples. (A) A morphism a in a U-concrete category is called an *injection* [a *surjection*, a *bijection*] if $\square a$ is an injection [a surjection, a bijection, respectively].

For every U-concrete category $\mathfrak{A}$ let $\mathrm{in}(\mathfrak{A})$, $\mathrm{sur}(\mathfrak{A})$, and $\mathrm{bij}(\mathfrak{A})$ be subcategories of $\mathfrak{A}$ such that

$$\mathrm{in}(\mathfrak{A})^{\mathrm{o}} = \mathrm{sur}(\mathfrak{A})^{\mathrm{o}} = \mathrm{bij}(\mathfrak{A})^{\mathrm{o}} = \mathfrak{A}^{\mathrm{o}},$$

and $\mathrm{in}(\mathfrak{A})^{\mathrm{m}}$ is the class of all injections in $\mathfrak{A}$, $\mathrm{sur}(\mathfrak{A})^{\mathrm{m}}$ is the class of all surjections in $\mathfrak{A}$, and $\mathrm{bij}(\mathfrak{A})^{\mathrm{m}}$ is the class of all bijections in $\mathfrak{A}$.

Let

$$p_i(\mathfrak{A}) = \mathrm{in}(\mathfrak{A})^{\mathrm{m}}, \qquad p_s(\mathfrak{A}) = \mathrm{sur}(\mathfrak{A})^{\mathrm{m}}, \qquad p_b(\mathfrak{A}) = \mathrm{bij}(\mathfrak{A})^{\mathrm{m}}.$$

The functions p_i, p_s, and p_b are cc-predicates of the type (1, 0). By Corollary 3.7 p_i is c-dual to p_s, and by Corollary 3.8 p_s is c-dual to p_i. The cc-predicate p_b is c-dual to itself. (Another form of a duality theorem for the notions of an injection and a surjection was proved by Garvackiĭ and Šaĭn [1970].)

(B) Let $\mathfrak{m}$ be a cardinal number, and let

$$c_{\mathfrak{m}}(\mathfrak{A}) = \{A\in\mathfrak{A}^{\mathrm{o}}\colon \mathrm{card}\,|A| = \mathfrak{m}\}.$$

The function $c_{\mathfrak{m}}$ is a cc-predicate of the type (0, 1). It is easy to see that $c_{\mathfrak{m}}$ is c-dual to $c_{2\mathfrak{m}}$. This example shows that, contrary to the previous example, it may happen that p is c-dual to q, but q is not c-dual to p.

Other examples of c-duality will be considered in Section 8.

7.6. Remark. Let us note that c-predicates and cc-predicates of type $(n, 0)$, where n is a positive integer, can be described in the following alternative form:

Let U-$\mathbf{Rel}_n$ be a U-concrete category defined in the following way: the objects are pairs (A, R), where $A \in U$ and R is an n-ary relation on A; the morphisms from (A, R) to (A', R') are triples $\big(f, (A, R), (A', R')\big)$, where $f\colon A \to A'$ is a function such that for all $a_1, \ldots, a_n \in A$ the condition $(a_1, \ldots, a_n) \in R$ implies $\big(f(a_1), \ldots, f(a_n)\big) \in R'$.

It is obvious that c-predicates of type $(n, 0)$ can be identified with covariant $\square$-commuting functors

$$p\colon \operatorname{bij}(U\text{-}\mathbf{Cat}) \to U\text{-}\mathbf{Rel}_n,$$

and cc-predicates of type $(n, 0)$ can be identified with covariant $\square$-commuting functors

$$q\colon \operatorname{bij}(U\text{-}\mathbf{Concat}_\square) \to U\text{-}\mathbf{Rel}_n.$$

7.7. The dualization functor (1) does not preserve underlying sets of objects, i.e. $|A^\circ| \neq |A|$. Another dualization functor $\Diamond_{\mathfrak{A}}$ preserving underlying sets can be constructed in the following way:

Let $\mathfrak{A}$ be a U-concrete category. By $\mathfrak{A}^\#$ we denote the U-concrete category defined as follows: $(\mathfrak{A}^\#)^\circ = \mathfrak{A}^\circ$; the underlying set of A in $\mathfrak{A}^\#$, denoted by $|A|^\#$, is $\mathscr{P}(|A|)$, where $|A|$ is the underlying set of A in $\mathfrak{A}$; the class of morphisms is

$$(\mathfrak{A}^\#)^{\mathrm{m}} = \big\{\big(\mathscr{P}_+(f), A, B\big)\colon (f, A, B) \in \mathfrak{A}^{\mathrm{m}}\big\}.$$

It is easy to see that $\mathfrak{A}^\#$ is isomorphic with $\mathfrak{A}$ and the isomorphism

$$?^\#\colon \mathfrak{A} \to \mathfrak{A}^\#$$

is defined by

$$A^\# = A \quad \text{for } A \in \mathfrak{A}^\circ,$$

$$a^\# = \big(\mathscr{P}_+(|a|), \operatorname{Dom} a, \operatorname{Cod} a\big) \quad \text{for } a \in \mathfrak{A}^{\mathrm{m}}.$$

Moreover, there exists a unique contravariant bijector $\Diamond_{\mathfrak{A}}\colon \mathfrak{A}^\# \to {}^\circ\mathfrak{A}$ such that the diagram

$$\begin{tikzcd}
 & \mathfrak{A}^\# \arrow[d, "\Diamond_{\mathfrak{A}}"] \\
\mathfrak{A} \arrow[ur, "?^\#"] \arrow[r, "?^\circ"'] & {}^\circ\mathfrak{A}
\end{tikzcd}$$

is commutative. We have $|\Diamond_{\mathfrak{A}} A| = |A|$ for every A in $(\mathfrak{A}^\#)^\circ$, i.e. the functor $\Diamond_{\mathfrak{A}}$ preserves underlying sets.

It is easy to see that the category U-Ens$^{\#}$ is $\square$-isomorphic with the category $\mathscr{P}_+(U\text{-Ens})$.

7.8. Definition. A *cc*-predicate p will be called *$\#$-invariant* if for every U-concrete category $\mathfrak{A}$ and for all $a_1, \ldots, a_i \in \mathfrak{A}^{\mathrm{m}}$, $A_1, \ldots, A_j \in \mathfrak{A}^{\mathrm{o}}$ the condition

$$(a_1, \ldots, a_i, A_1, \ldots, A_j) \in p(\mathfrak{A})$$

is equivalent to the condition

$$(a_1^{\#}, \ldots, a_i^{\#}, A_1^{\#}, \ldots, A_j^{\#}) \in p(\mathfrak{A}^{\#}).$$

It follows from Corollaries 3.7 and 3.8 that the *cc*-predicates p_i, p_s, and p_b defined in Example 7.5(A) are $\#$-invariant.

7.9. According to Proposition 7.4 every "concrete-categorical" notion has a dual. For example, the *c*-dual to the notion of a 1-free object can be described as an object F together with a subset S of $|F|$ satisfying the following condition:

For every object X in $\mathfrak{A}^{\mathrm{o}}$ and every subset T of $|X|$ there is a unique morphism $\vartheta\colon X \to F$ such that

$$|\vartheta|^{\leftarrow}(S) = T.$$

On the other hand, the notion of a 1-free object is "almost" *c*-dual to the notion of a 2-cofree object in the following sense:

A pair (F, σ) is a universal point of the functor

$$\langle 1, \square_{\mathfrak{A}}(?)\rangle_{U\text{-Ens}}\colon \mathfrak{A} \to U\text{-Ens}$$

iff $(F, \mathscr{P}_-(\sigma))$ is a couniversal point of the functor

$$\langle \square_{\mathfrak{A}^{\mathrm{o}}}(?), 2\rangle_{\mathscr{P}_-(U\text{-Ens})}\colon \mathfrak{A}^{\mathrm{o}} \to U\text{-Ens}.$$

In fact, the condition $\sigma \in \langle 1, \square_{\mathfrak{A}}(F)\rangle_{U\text{-Ens}}$ is equivalent to the condition $\mathscr{P}_-(\sigma) \in \langle \square_{\mathfrak{A}^{\mathrm{o}}}(F), 2\rangle_{\mathscr{P}_-(U\text{-Ens})}$. For every X in $\mathfrak{A}^{\mathrm{o}}$ there is a one-to-one correspondence between elements $\xi \in \langle 1, \square_{\mathfrak{A}}(X)\rangle_{U\text{-Ens}}$ and elements $\mathscr{P}_-(\sigma) \in \langle \square_{\mathfrak{A}^{\mathrm{o}}}(X), 2\rangle_{\mathscr{P}_-(U\text{-Ens})}$. Moreover, for every ϑ in $\langle F, X\rangle_{\mathfrak{A}}$ the conditions

$$\square_{\mathfrak{A}}(\vartheta)\sigma = \xi, \quad \text{and} \quad \mathscr{P}_-(\sigma)\square_{\mathfrak{A}^{\mathrm{o}}}(\vartheta) = \mathscr{P}_-(\xi)$$

are equivalent.

8. D-preorders and C-preorders

If R and R' are preorders on a set A such that

$$aRb \text{ implies } aR'b \text{ for all } a, b \in A,$$

then we shall say that R' is *greater* than R and R is *smaller* than R'.

8.1. DEFINITIONS. Let $\mathfrak{A}$ be a category. A *D-preorder* on $\mathfrak{A}$ is any preorder R on $\mathfrak{A}^{\mathrm{m}}$ satisfying the following two conditions:

(D_1) $\alpha R\beta$ implies $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$,

(D_2) $\alpha R\beta$ and $\mathrm{Cod}\,\alpha = \mathrm{Dom}\,\gamma$ imply $(\gamma\alpha)R\beta$.

A *C-preorder* on $\mathfrak{A}$ is a D-preorder on the dual category $\mathfrak{A}^*$; in other words, a C-preorder S is a preorder on $\mathfrak{A}^{\mathrm{m}}$ satisfying the following two conditions:

(C_1) $\alpha S\beta$ implies $\mathrm{Cod}\,\alpha = \mathrm{Cod}\,\beta$,

(C_2) $\alpha S\beta$ and $\mathrm{Dom}\,\alpha = \mathrm{Cod}\,\gamma$ imply $(\alpha\gamma)S\beta$.

The D-preorders considered in examples below will satisfy in most cases the following additional condition:

(D_3) $\alpha R\beta$ and $\mathrm{Cod}\,\delta = \mathrm{Dom}\,\alpha$ imply $(\alpha\delta)R(\beta\delta)$.

The dual condition for C-preorders is:

(C_3) $\alpha S\beta$ and $\mathrm{Dom}\,\delta = \mathrm{Cod}\,\alpha$ imply $(\delta\alpha)S(\delta\beta)$.

Let $\mathfrak{M}$ be a subset of $\mathfrak{A}^{\mathrm{m}}$. A D-preorder R will be called *regular* with respect to $\mathfrak{M}$ if the conditions $\lambda \in \mathfrak{M}$ and $(\alpha\lambda)R(\beta\lambda)$ imply $\alpha R\beta$. A C-preorder S will be called *regular* with respect to $\mathfrak{M}$ if the conditions $\lambda \in \mathfrak{M}$ and $(\lambda\alpha)S(\lambda\beta)$ imply $\alpha S\beta$.

8.2. EXAMPLES. (A) Let

$$R_0(\mathfrak{A}) = \{(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}} \colon \mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta\},$$
$$S_0(\mathfrak{A}) = \{(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}} \colon \mathrm{Cod}\,\alpha = \mathrm{Cod}\,\beta\}.$$

Instead of $R_0(\mathfrak{A})$ and $S_0(\mathfrak{A})$ we shall also write shortly R_0 and S_0. It is obvious that R_0 is a D-preorder on $\mathfrak{A}$ satisfying (D_3), and S_0 is a C-preorder on $\mathfrak{A}$ satisfying (C_3). Moreover, R_0 is the greatest D-preorder on $\mathfrak{A}$, and S_0 is the greatest C-preorder on $\mathfrak{A}$. R_0 and S_0 are regular with respect to the set of all morphisms in $\mathfrak{A}$.

(B) Let

$$R_1(\mathfrak{A}) = R_1 = \{(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}} \colon \alpha = \varphi\beta \text{ for some } \varphi \text{ in } \mathfrak{A}^{\mathrm{m}}\},$$
$$S_1(\mathfrak{A}) = S_1 = \{(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}} \colon \alpha = \beta\varphi \text{ for some } \varphi \text{ in } \mathfrak{A}^{\mathrm{m}}\}.$$

It is easy to see that R_1 is a D-preorder on $\mathfrak{A}$ satisfying (D_3), and S_1 is a C-preorder on $\mathfrak{A}$ satisfying (C_3). Moreover, R_1 is the least D-preorder on $\mathfrak{A}$ and S_1 is the least C-preorder on $\mathfrak{A}$. In fact, if $\alpha R_1\beta$ and R is a D-preorder on $\mathfrak{A}$, then $\alpha = \varphi\beta$ for some φ in $\mathfrak{A}^{\mathrm{m}}$, and it follows from ($D_2$) that $(\varphi\beta)R\beta$, i.e. $\alpha R\beta$; similarly, $\alpha S_1\beta$ implies $\alpha S\beta$ for any C-preorder S. R_1 is regular with respect to the set of all epimorphisms in $\mathfrak{A}$, and S_1 is regular with respect to the set of all monomorphisms in $\mathfrak{A}$. In fact, if λ is an epimorphism and $(\alpha\lambda)R_1(\beta\lambda)$, then there exists φ such that $\alpha\lambda = \varphi\beta\lambda$; since λ is right-cancellable, it follows that $\alpha = \varphi\beta$, i.e. $\alpha R_1\beta$.

(C) Let A be a fixed object in $\mathfrak{A}^{\mathrm{o}}$, and let

$$R_{a,A}(\mathfrak{A}) = R_{a,A}$$
$$= \{(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}}\colon \beta\varphi = \beta\psi \text{ implies } \alpha\varphi = \alpha\psi \text{ for all } \varphi, \psi \in \langle A, \mathrm{Dom}\,\beta\rangle\},$$
$$S_{a,A}(\mathfrak{A}) = S_{a,A}$$
$$= \{(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}}\colon \varphi\beta = \psi\beta \text{ implies } \varphi\alpha = \psi\alpha \text{ for all } \varphi, \psi \in \langle \mathrm{Cod}\,\beta, A\rangle\}.$$

It is easy to verify that $R_{a,A}$ is a D-preorder on $\mathfrak{A}$ satisfying (D_3), and $S_{a,A}$ is a C-preorder satisfying (C_3). $R_{a,A}$ is regular with respect to the set of all retractions in $\mathfrak{A}$. In fact, if $\lambda\lambda' = \iota_{\mathrm{Cod}\,\lambda}$, $(\alpha\lambda)R_{a,A}(\beta\lambda)$, and $\beta\varphi = \beta\psi$, then it follows from $\beta\lambda\lambda'\varphi = \beta\lambda\lambda'\psi$ that $\alpha\lambda\lambda'\varphi = \alpha\lambda\lambda'\psi$, i.e. $\alpha\varphi = \alpha\psi$. Dually, $S_{a,A}$ is regular with respect to the set of all coretractions in $\mathfrak{A}$.

(D) Let

$$R_a(\mathfrak{A}) = R_a = \bigcap_{A \in \mathfrak{A}^{\mathrm{o}}} R_{a,A}, \qquad S_a(\mathfrak{A}) = S_a = \bigcap_{A \in \mathfrak{A}^{\mathrm{o}}} S_{a,A}$$

It is easy to verify that R_a is a D-preorder satisfying (D_3), and S_a is a C-preorder satisfying (C_3). For all $\alpha, \beta \in \mathfrak{A}^{\mathrm{m}}$ we have

$$\alpha R_a \beta \Leftrightarrow (\beta\varphi = \beta\psi \text{ implies } \alpha\varphi = \alpha\psi \text{ for all } \varphi, \psi \in \mathfrak{A}^{\mathrm{m}}),$$
$$\alpha S_a \beta \Leftrightarrow (\varphi\beta = \psi\beta \text{ implies } \varphi\alpha = \psi\alpha \text{ for all } \varphi, \psi \in \mathfrak{A}^{\mathrm{m}}).$$

R_a is regular with respect to the set of all retractions, and S_a is regular with respect to the set all coretractions.

(E) Let R_b be the set of all pairs $(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}}$ satisfying the following condition:

For every factorization $\beta = \beta''\beta'$ such that β' is an epimorphism there exists a factorization $\alpha = \alpha''\alpha'$ such that α' is an epimorphism and $\alpha' R_1 \beta'$.

Dually, let S_b be the set of all pairs $(\alpha, \beta) \in \mathfrak{A}^{\mathrm{m}} \times \mathfrak{A}^{\mathrm{m}}$ satisfying the following condition:

For every factorization $\beta = \beta''\beta'$ such that β'' is a monomorphism there exists a factorization $\alpha = \alpha''\alpha'$ such that α'' is a monomorphism and $\alpha'' S_1 \beta''$.

It is easy to see that R_b is a D-preorder, and S_b is a C-preorder.

(F) Let $\mathfrak{A}$ be a U-concrete category, and let $R_c(\mathfrak{A}) = R_c$ be the set of all pairs $(\alpha, \beta) \in R_0(\mathfrak{A})$ satisfying the following condition:

If $a, b \in |\mathrm{Dom}\,\alpha|$ and $|\beta|(a) = |\beta|(b)$, then $|\alpha|(a) = |\alpha|(b)$.

Let $S_c(\mathfrak{A}) = S_c$ be the set of all pairs $(\alpha, \beta) \in S_0(\mathfrak{A})$ satisfying the following condition:

$$|\alpha|^{\rightarrow}(|\mathrm{Dom}\,\alpha|) \subset |\beta|^{\rightarrow}(|\mathrm{Dom}\,\beta|).$$

It is easy to verify that R_c is a D-preorder, and S_c is a C-preorder.

R_c is regular with respect to the set of all surjections in $\mathfrak{A}$. In fact, let λ be a surjection, and let $(\alpha\lambda)R_c(\beta\lambda)$. If $a, b \in |\mathrm{Dom}\,\alpha|$ and $|\beta|(a) = |\beta|(b)$, then there exist x, y in $|\mathrm{Dom}\,\lambda|$ such that $a = |\lambda|(x)$, $b = |\lambda|(y)$. We have

$$|\beta\lambda|(x) = |\beta|(a) = |\beta|(b) = |\beta\lambda|(y),$$

and consequently

$$|\alpha|(a) = |\alpha\lambda|(x) = |\alpha\lambda|(y) = |\alpha|(b),$$

which proves that $\alpha R_c\beta$.

S_c is regular with respect to the set of all injections in $\mathfrak{A}$. In fact, if λ is an injection, $(\lambda\alpha)S_c(\lambda\beta)$, and $a \in |\alpha|^{\rightarrow}(|\mathrm{Dom}\,\alpha|)$, then we have $|\lambda|(a) \in |\lambda\alpha|^{\rightarrow} \times (|\mathrm{Dom}\,\alpha|) \subset |\lambda\beta|^{\rightarrow}(|\mathrm{Dom}\,\beta|)$, i.e. $|\lambda|(a) = |\lambda\beta|(b)$ for some b in $|\mathrm{Dom}\,\beta|$. Hence $a = |\beta|(b)$, i.e. $a \in |\beta|^{\rightarrow}(|\mathrm{Dom}\,\beta|)$.

We shall now examine the relationship between the preorders defined in this section.

It is obvious that for every A in $\mathfrak{A}^{\circ}$

$$R_a(\mathfrak{A}) \subset R_{a,A}(\mathfrak{A}) \quad \text{and} \quad S_a(\mathfrak{A}) \subset S_{a,A}(\mathfrak{A}). \tag{2}$$

8.3. Proposition. *If A is a generator in $\mathfrak{A}$, then $R_{a,A}(\mathfrak{A}) = R_a^{\mathrm{l}}(\mathfrak{A})$. If A is a cogenerator in $\mathfrak{A}$, then $S_{a,A}(\mathfrak{A}) = S_a(\mathfrak{A})$.*

Proof. Let A be a generator in $\mathfrak{A}$. Suppose that $\alpha R_{a,A}\beta$ and $\beta\varphi = \beta\psi$. If $\varphi \neq \psi$, then it follows from the definition of a generator that $\langle A, \mathrm{Dom}\,\varphi\rangle \neq 0$. The conditions $\alpha R_{a,A}\beta$ and $\beta\varphi = \beta\psi$ imply $\alpha\varphi\chi = \alpha\psi\chi$ for all χ in $\langle A, \mathrm{Dom}\,\varphi\rangle$. Since A is a generator, it follows that $\alpha\varphi = \alpha\psi$, i.e. $\alpha R_a\beta$. Thus $R_{a,A}(\mathfrak{A}) \subset R_a(\mathfrak{A})$, and, by (2), $R_{a,A}(\mathfrak{A}) = R_a(\mathfrak{A})$. The second part of the proposition follows by duality.

8.4. Proposition. *If $\mathfrak{A}$ is a concrete category, then $R_c(\mathfrak{A}) \subset R_a(\mathfrak{A})$ and $S_c(\mathfrak{A}) \subset S_a(\mathfrak{A})$.*

Proof. Let $\alpha R_c\beta$ and $\beta\varphi = \beta\psi$ for some φ, ψ in $\mathfrak{A}^{\mathrm{m}}$. For every $x \in |\mathrm{Dom}\,\varphi|$ we have $|\beta|\big(|\varphi|(x)\big) = |\beta|\big(|\psi|(x)\big)$, hence $|\alpha|\big(|\varphi|(x)\big) = |\alpha|\big(|\psi|(x)\big)$, i.e. $\alpha\varphi = \alpha\psi$. Thus $\alpha R_a\beta$.

Let $\alpha S_a\beta$ and $\varphi\beta = \psi\beta$ for some φ, ψ in $\mathfrak{A}^{\mathrm{m}}$. Then for every a in $|\mathrm{Dom}\,\alpha|$ there exists a b in $|\mathrm{Dom}\,\beta|$ such that $|\alpha|(a) = |\beta|(b)$. Consequently,

$$|\varphi|\big(|\alpha|(a)\big) = |\varphi|\big(|\beta|(b)\big) = |\psi|\big(|\beta|(b)\big) = |\psi|\big(|\alpha|(a)\big),$$

and this means that $\varphi\alpha = \varphi\beta$. Thus $\alpha S_a\beta$.

8.5. Example. Let Z be a set such that $\mathrm{card}\,Z \geqslant 2$, and let $\mathfrak{A}$ be the subcategory of **Ens** defined as follows: $\mathfrak{A}^{\circ} = U\text{-}\mathbf{Ens}^{\circ}$, $\langle A, B\rangle_{\mathfrak{A}} = \langle A, B\rangle_{U\text{-}\mathbf{Ens}}$ if $B \neq Z$, $\langle Z, Z\rangle_{\mathfrak{A}} = \{\iota_Z\}$, $\langle A, Z\rangle_{\mathfrak{A}} = \emptyset$ if $A \neq Z$.

It is easy to see that $\alpha R_a\beta$ for all α, β in $\mathfrak{A}^{\mathrm{m}}$ such that $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta = Z$. On the other hand, there exist α and β in $\mathfrak{A}^{\mathrm{m}}$ with $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta = Z$ such that $\alpha R_c\beta$ is not true.

This example shows that $R_a(\mathfrak{A}) \neq R_c(\mathfrak{A})$, in general.

8.6. PROPOSITION. *Let $\mathfrak{A}$ be a U-concrete category. If there exists a 1-free object F in $\mathfrak{A}$, then*

$$R_c(\mathfrak{A}) = R_a(\mathfrak{A}) = R_{a,F}(\mathfrak{A}).$$

If there exists a 2-cofree object H in $\mathfrak{A}$, then

$$S_c(\mathfrak{A}) = S_a(\mathfrak{A}) = S_{a,H}(\mathfrak{A}).$$

Proof. By (2) and Proposition 8.4 it suffices to prove that $R_{a,F}(\mathfrak{A}) \subset R_c(\mathfrak{A})$ and $S_{a,H}(\mathfrak{A}) \subset S_c(\mathfrak{A})$. Let $\alpha R_{a,F}\beta$ and let $|\beta|(a) = |\beta|(b)$ for some a, b in $|\mathrm{Dom}\,\beta|$. If $\sigma\colon 1 \to |F|$ is a map freely 1-generating F, then there are (uniquely determined) morphisms φ, ψ in $\langle F, \mathrm{Dom}\,\beta\rangle$ such that $(\Box\varphi)\sigma = \bar{a}$ and $(\Box\psi)\sigma = \bar{b}$. There is also a unique ϑ in $\langle F, \mathrm{Cod}\,\beta\rangle$ such that $(\Box\vartheta)\sigma = \overline{|\beta|(a)} = \overline{|\beta|(b)}$. Since $\big(\Box(\beta\varphi)\big)\sigma = (\Box\beta)\bar{a} = \overline{|\beta|(a)}$ and $\big(\Box(\beta\psi)\big)\sigma = \overline{|\beta|(b)}$, it follows from the uniqueness of ϑ that $\beta\varphi = \beta\psi$. Hence $\alpha\varphi = \alpha\psi$, and $\Box(\alpha\varphi)\sigma = \Box(\alpha\psi)\sigma$, i.e.

$$\overline{|\alpha|(a)} = \big(\Box(\alpha\varphi)\big)\sigma = \big(\Box(\alpha\psi)\big)\sigma = \overline{|\alpha|(b)}.$$

Thus $\alpha R_c\beta$.

Let $\alpha S_{a,H}\beta$ and let $\eta, \zeta\colon |\mathrm{Cod}\,\beta| \to 2$ be the maps defined as follows:

$$\eta(x) = 0 \quad \text{for every } x \text{ in } |\mathrm{Cod}\,\beta|,$$

$$\zeta(x) = \begin{cases} 0 & \text{for every } x \text{ in } |\beta|^{\to}(|\mathrm{Dom}\,\beta|), \\ 1 & \text{for every } x \text{ in } |\mathrm{Cod}\,\beta| \setminus |\beta|^{\to}(|\mathrm{Dom}\,\beta|). \end{cases}$$

It is easy to see that $\eta\Box\beta = \zeta\Box\beta$ and for every map $\varkappa$

$$\eta\varkappa = \zeta\varkappa \quad \text{implies} \quad \varkappa^{\to}(\mathrm{Dom}\,\varkappa) \subset |\beta|^{\to}(|\mathrm{Dom}\,\beta|). \tag{3}$$

If $\pi\colon |H| \to 2$ is a map freely 2-cogenerating H, then there are (uniquely determined) morphisms φ, ψ in $\langle \mathrm{Cod}\,\beta, H\rangle$ such that $\pi(\Box\varphi) = \eta$ and $\pi(\Box\psi) = \zeta$. Moreover, there exists a unique morphism ϑ in $\langle \mathrm{Dom}\,\beta, H\rangle$ such that $\pi(\Box\vartheta) = \eta\Box\beta = \zeta\Box\beta$. We have $\pi\big(\Box(\varphi\beta)\big) = \pi(\Box\varphi)(\Box\beta) = \eta\Box\beta$, and $\pi\big(\Box(\psi\beta)\big) = \zeta\Box\beta$, hence $\vartheta = \varphi\beta = \psi\beta$. Consequently, $\varphi\alpha = \psi\alpha$, and $\zeta\Box\alpha = \pi\Box(\psi\alpha) = \pi\Box(\varphi\alpha) = \eta\Box\alpha$. By (3) we obtain $|\alpha|^{\to}(|\mathrm{Dom}\,\alpha|) \subset |\beta|^{\to}(|\mathrm{Dom}\,\beta|)$. Thus $\alpha S_c\beta$.

8.7. COROLLARY. $R_c(U\text{-Ens}) = R_a(U\text{-Ens}) = R_{a,1}(U\text{-Ens})$ *and* $S_c(U\text{-Ens}) = S_a(U\text{-Ens}) = S_{a,2}(U\text{-Ens})$.

In fact, 1 is a 1-free object in U-Ens, and 2 is a 2-cofree object in U-Ens.

8.8. COROLLARY. *If $\mathfrak{A}$ is a concrete category, then for all α, β in $\mathfrak{A}^{\mathrm{m}}$*

$$\alpha R_c(\mathfrak{A})\beta \Leftrightarrow (\Box\alpha)R_a(U\text{-Ens})(\Box\beta),$$

$$\alpha S_c(\mathfrak{A})\beta \Leftrightarrow (\Box\alpha)S_a(U\text{-Ens})(\Box\beta).$$

The D-preorders and C-preorders considered in 8.2(A), (B), (D), (E) are defined for all U-categories simultaneously. The D-preorder R_c and the C-preorder S_c are defined for all U-concrete categories simultaneously.

8.9. Definitions. A *D-c-predicate* [*C-c-predicate*] is a c-predicate p of the type $(2,0)$ such that $p(\mathfrak{A})$ is a D-preorder [C-preorder] on $\mathfrak{A}$ for every U-category $\mathfrak{A}$. A *D-cc-predicate* [*C-cc-predicate*] is a cc-predicate q of the type $(2,0)$ such that $q(\mathfrak{A})$ is a D-preorder [C-preorder] on $\mathfrak{A}$ for every U-concrete category $\mathfrak{A}$.

8.10. Proposition. *The functions*

$$R_0(?),\ R_1(?),\ R_a(?),\ R_b(?)$$

*(from U-**Cat**$^\circ$ to U) are D-c-predicates, and the functions*

$$S_0(?),\ S_1(?),\ S_a(?),\ S_b(?)$$

*(from U-**Cat**$^\circ$ to U) are C-c-predicates. The function*

$$R_c(?)\colon\ U\text{-}\mathbf{Concat}^\circ \to U$$

is a D-cc-predicate, and the function

$$S_c(?)\colon\ U\text{-}\mathbf{Concat}^\circ \to U$$

is a C-cc-predicate.

Proof. It is easy to see that if $\Phi\colon \mathfrak{A}\to\mathfrak{B}$ is a covariant bijector, and x denotes one of the symbols 0, 1, a, b, then

$$\alpha R_x(\mathfrak{A})\beta \Rightarrow \Phi(\alpha)R_x(\mathfrak{B})\Phi(\beta),$$

and

$$\alpha S_x(\mathfrak{A})\beta \Rightarrow \Phi(\alpha)S_x(\mathfrak{B})\Phi(\beta).$$

If, in addition, the categories $\mathfrak{A}, \mathfrak{B}$ are U-concrete, and the functor Φ is $\square$-commuting, then these conditions are satisfied also for $x = c$.

8.11. Proposition. *The D-c-predicates $R_0(?)$, $R_1(?)$, $R_a(?)$, and $R_b(?)$ are dual to the C-c-predicates $S_0(?)$, $S_1(?)$, $S_a(?)$, and $S_b(?)$, respectively.*

Proof. Proposition follows immediately from definitions.

The following two propositions are easy consequences of Lemmas 3.5 and 3.6.

8.12. Proposition. *The D-cc-predicate $R_c(?)$ is c-dual to the C-cc-predicate $S_c(?)$, and conversely, $S_c(?)$ is c-dual to $R_c(?)$, i.e., the condition $\alpha R_c(\mathfrak{A})\beta$ is equivalent to the condition $\alpha^\diamond S_c(\mathfrak{A}^\diamond)\beta^\diamond$, and the condition $\alpha S_c(\mathfrak{A})\beta$ is equivalent to the condition $\alpha^\diamond R_c(\mathfrak{A}^\diamond)\beta^\diamond$.*

8.13. Proposition. *The cc-predicates $R_c(?)$ and $S_c(?)$ are $\#$-invariant, i.e., the condition $\alpha R_c(\mathfrak{A})\beta$ is equivalent to the condition $\alpha^\# R_c(\mathfrak{A}^\#)\beta^\#$, and the condition $\alpha S_c(\mathfrak{A})\beta$ is equivalent to the condition $\alpha^\# S_c(\mathfrak{A}^\#)\beta^\#$.*

9. Generalized images, coimages, embeddings, and quotients

9.1. Let $\mathfrak{A}$ be a U-category, and let a be a fixed morphism in $\mathfrak{A}^{\mathrm{m}}$. With every D-preorder R on $\mathfrak{A}$ we can associate a covariant functor

$$\Delta_{R,a}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}$$

defined in the following way:

If $X \in \mathfrak{A}^{\mathrm{o}}$, then

$$\Delta_{R,a}(X) = \{\xi \in \mathfrak{A}^{\mathrm{m}}\colon \xi R a \text{ and } \operatorname{Cod}\xi = X\}.$$

If $\varphi \in \mathfrak{A}^{\mathrm{m}}$, then

$$\Delta_{R,a}(\varphi)\colon \Delta_{R,a}(\operatorname{Dom}\varphi) \to \Delta_{R,a}(\operatorname{Cod}\varphi)$$

is the map

$$\Delta_{R,a}(\varphi)(\xi) = \varphi\xi, \quad \xi \in \Delta_{R,a}(\operatorname{Dom}\varphi).$$

In virtue of condition 8.1(D_1),

$$\Delta_{R,a}(X) \subset \langle \operatorname{Dom} a, X\rangle_{\mathfrak{A}} \quad \text{for every } X \text{ in } \mathfrak{A}^{\mathrm{o}}.$$

Condition 8.1(D_2) implies that the map $\Delta_{R,a}(\varphi)$ is well defined for every φ in $\mathfrak{A}^{\mathrm{m}}$. In fact, if $\xi \in \Delta_{R,a}(\operatorname{Dom}\varphi)$, then $\xi R a$ and $\operatorname{Cod}\xi = \operatorname{Dom}\varphi$; consequently, $(\varphi\xi) R a$ and $\operatorname{Cod}(\varphi\xi) = \operatorname{Cod}\varphi$, i.e., $\varphi\xi \in \Delta_{R,a}(\operatorname{Cod}\varphi)$. The functor $\Delta_{R,a}$ is a subfunctor of the principal covariant functor

$$\langle \operatorname{Dom} a, ?\rangle_{\mathfrak{A}}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}$$

(i.e. $\Delta_{R,a}$ is a covariant ideal in the sense of Isbell [1964]).

Similarly, with every C-preorder S on $\mathfrak{A}$ we can associate a contravariant functor

$$\Gamma_{S,a}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}$$

defined in the following way:

If $X \in \mathfrak{A}^{\mathrm{o}}$, then

$$\Gamma_{S,a}(X) = \{\xi \in \mathfrak{A}^{\mathrm{m}}\colon \xi S a \text{ and } \operatorname{Dom}\xi = X\}.$$

If $\varphi \in \mathfrak{A}^{\mathrm{m}}$, then

$$\Gamma_{S,a}(\varphi)\colon \Gamma_{S,a}(\operatorname{Cod}\varphi) \to \Gamma_{S,a}(\operatorname{Dom}\varphi)$$

is the map

$$\Gamma_{S,a}(\varphi)(\xi) = \xi\varphi, \quad \xi \in \Gamma_{S,a}(\operatorname{Cod}\varphi).$$

We have

$$\Gamma_{S,a}(X) \subset \langle X, \operatorname{Cod} a\rangle_{\mathfrak{A}} \quad \text{for every } X \text{ in } \mathfrak{A}^{\mathrm{o}},$$

and the functor $\Gamma_{S,a}$ is a subfunctor of the principal contravariant functor

$$\langle ?, \operatorname{Cod} a\rangle_{\mathfrak{A}}\colon \mathfrak{A} \to U\text{-}\mathbf{Ens}.$$

(For a special case $S = S_a$ see Brinkmann and Puppe [1966], p. 52.)

9.2. Definitions. An *R-coimage* of a morphism a is a morphism π such that $(\operatorname{Cod}\pi, \pi)$ is a universal point of the functor $\Delta_{R,a}$.

An *S-image* of a is a morphism σ such that $(\mathrm{Dom}\,\sigma, \sigma)$ is a couniversal point of the functor $\Gamma_{S,a}$.

An S-image of a will be denoted by S-im a, and an R-coimage of a will be denoted by R-coim a.

In other words, an R-coimage of a is a morphism π satisfying the following conditions:

(c_1) $\pi R a$,

(c_2) for every morphism ξ such that $\xi R a$ there is a unique morphism ϑ: $\mathrm{Cod}\,\pi \to \mathrm{Cod}\,\xi$ such that the diagram

$$\begin{tikzcd}
\mathrm{Dom}\,a \arrow[r, "a"] \arrow[d, "\pi"'] \arrow[dr, "\xi"] & \mathrm{Cod}\,a \\
\mathrm{Cod}\,\pi \arrow[r, "\vartheta"'] & \mathrm{Cod}\,\xi
\end{tikzcd}$$

is commutative.

An S-image of a is a morphism σ satisfying the following conditions:

(i_1) $\sigma S a$,

(i_2) for every morphism ξ such that $\xi S a$ there is a unique morphism ϑ: $\mathrm{Dom}\,\xi \to \mathrm{Dom}\,\sigma$ such that the diagram

$$\begin{tikzcd}
\mathrm{Dom}\,a \arrow[r, "a"] & \mathrm{Cod}\,a \\
\mathrm{Dom}\,\xi \arrow[r, "\vartheta"'] \arrow[ur, "\xi"] & \mathrm{Dom}\,\sigma \arrow[u, "\sigma"']
\end{tikzcd}$$

is commutative.

S-im a and R-coim a are determined by a uniquely up to an isomorphism. More precisely:

9.3. Proposition. *If* $\sigma = S$-im a *and* $\sigma' = S$-im a, *then there exists an isomorphism* ζ: $\mathrm{Dom}\,\sigma \to \mathrm{Dom}\,\sigma'$ *such that* $\sigma = \sigma'\zeta$. *If* $\pi = R$-coim a *and* $\pi' = R$-coim a, *then there exists an isomorphism* η: $\mathrm{Cod}\,\pi' \to \mathrm{Cod}\,\pi$ *such that* $\pi = \eta\pi'$.

Proof. The proposition follows from the uniqueness of an universal [couniversal] point of a functor.

Let $a =_R \beta$ denote that $aR\beta$ and $\beta R a$. The symbol $a =_S \beta$ is defined similarly.

9.4. Proposition. *If* R-coim a *exists, then* $a =_R R$-coim a. *If* S-im a *exists, then* $a =_S S$-im a.

Proof. If $\pi = R\text{-coim}\,a$, then $\pi R a$ by condition 9.2(c_1). It follows from condition 9.2(c_2) (for $\xi = a$) that $a = \vartheta\pi$ for some ϑ, i.e. $aR\pi$ by 8.1(D_2). The second part of the proposition follows by duality.

9.5. Proposition. *Every R-coimage is an epimorphism. Every S-image is a monomorphism.*

Proof. Let $\pi = R\text{-coim}\,a$, and let φ, ψ be morphisms such that $\varphi\pi = \psi\pi$. Since $\pi R a$, it follows from 8.1(D_2) that $(\varphi\pi)Ra$. By 9.2(c_2) there is a unique ϑ such that $\vartheta\pi = \varphi\pi = \psi\pi$. Hence $\vartheta = \varphi = \psi$. The second part of the proposition follows by duality.

9.6. Definitions. An *R-quotient morphism* is a morphism π such that $\pi = R\text{-coim}\,\pi$. An *S-embedding* is a morphism σ such that $\sigma = S\text{-im}\,\sigma$.

Note that $R\text{-coim}\,a$ is an R-quotient morphism, and $S\text{-im}\,a$ is an S-embedding for every a.

We shall now consider some particular cases of S-images, R-coimages, R-quotient morphisms, and S-embeddings.

9.7. Proposition. *A morphism β is an R_0-coimage [S_0-image] of a morphism a if and only if* $\mathrm{Dom}\,\beta = \mathrm{Dom}\,a$ [$\mathrm{Cod}\,\beta = \mathrm{Cod}\,a$] *and β is an isomorphism.*

Proof. It is obvious that $\iota_{\mathrm{Dom}\,a} = R_0\text{-coim}\,a$. If $\pi = R_0\text{-coim}\,a$, then by Proposition 9.3 there exists an isomorphism η such that $\pi = \eta\iota_{\mathrm{Dom}\,a} = \eta$. The second part follows by duality.

9.8. Corollary. *A morphism β is an R_0-quotient morphism [an S_0-embedding] if and only if β is an isomorphism.*

9.9. Proposition. *A morphism π is an R_1-coimage of a morphism a if and only if the following conditions are satisfied:*

(i) *π is an epimorphism and* $\mathrm{Dom}\,\pi = \mathrm{Dom}\,a$,

(ii) *there exist morphisms π' and π'' such that $a = \pi'\pi$ and $\pi = \pi''a$.*

A morphism π is an R_1-quotient morphism if and only if π is an epimorphism.

Proof. If $\pi = R_1\text{-coim}\,a$, then conditions (i) and (ii) are satisfied in virtue of Propositions 9.5 and 9.4.

Suppose that π satisfies conditions (i) and (ii). Then obviously $\pi R_1 a$. Let ξ be a morphism such that $\xi R_1 a$, i.e. $\xi = \xi''a$ for some morphism ξ''. We have $\xi''\pi'\pi = \xi''a = \xi$, i.e. $\xi = \vartheta\pi$ for $\vartheta = \xi''\pi$. Since π is an epimorphism, ϑ is determined uniquely by the condition $\xi = \vartheta\pi$. Hence $\pi = R_1\text{-coim}\,a$.

The dual result is:

9.10. Proposition. *A morphism σ is an S_1-image of a morphism a if and only if the following conditions are satisfied:*

(i) *σ is a monomorphism and* $\operatorname{Cod}\sigma = \operatorname{Cod} a$,

(ii) *there exist morphisms σ' and σ'' such that $a = \sigma\sigma'$ and $\sigma = a\sigma''$.*

A morphism σ is an S_1-embedding if and only if σ is a monomorphism.

9.11. We shall now show that the notions of an S-image and an R-coimage are generalizations of the notions of an image and a coimage of a morphism.

Recall (cf. Mitchell [1965]) that an *image* of a morphism a is a monomorphism σ, denoted by $\operatorname{im} a$, satisfying the following conditions:

(im_1) $aS_1\sigma$,

(im_2) if μ is a monomorphism such that $aS_1\mu$, then $\sigma S_1\mu$.

A *coimage* of a morphism a is an epimorphism π, denoted by $\operatorname{coim} a$, satisfying the following conditions:

(coim_1) $aR_1\pi$,

(coim_2) if λ is an epimorphism such that $aR_1\lambda$, then $\pi R_1\lambda$.

9.12. Proposition. *A morphism π is a coimage of a if and only if π is an R_b-coimage of a. A morphism σ is an image of a if and only if σ is an S_b-image of a.*

Proof. Suppose that $\pi = \operatorname{coim} a$. Let $a = a''a'$ be a factorization such that a' is an epimorphism. Then aR_1a', and $\pi R_1 a'$. Since π is an epimorphism, it follows that $\pi R_b a$. Let $\xi R_b a$. By (coim_1) we have a factorization $a = \pi'\pi$ for some morphism π'. Hence there exists a factorization $\xi = \xi''\xi'$ such that ξ' is an epimorphism and $\xi' R_1 \pi$. In other words, there exists a morphism ζ such that $\xi' = \zeta\pi$. Let $\vartheta = \xi''\zeta$. We have $\vartheta\pi = \xi''\zeta\pi = \xi''\xi' = \xi$. Since π is an epimorphism, the morphism ϑ is determined by the condition $\vartheta\pi = \xi$ uniquely. Thus $\pi = R_b\text{-}\operatorname{coim} a$.

Conversely, suppose that $\pi = R_b\text{-}\operatorname{coim} a$. By Proposition 9.5 the morphism π is an epimorphism. By condition 9.2(c_2) (with $\xi = a$) there exists a factorization $a = \pi'\pi$, i.e. $aR_1\pi$. Let λ be an epimorphism such that $aR_1\lambda$, i.e. $a = \lambda'\lambda$ for some λ'. Since $\pi R_b a$, there exists a factorization $\pi = \pi_2\pi_1$ such that π_1 is an epimorphism and $\pi_1 R_1 \lambda$. We have $(\pi_2\pi_1)R_1\lambda$. Thus conditions (coim_1) and (coim_2) are satisfied.

The second part of the proposition follows by duality.

9.13. The notion of an R_a-quotient morphism coincides with the notion of a subregular epimorphism in the terminology of Barr, Grillet and Osdol [1971] ("Identifizierung" in the terminology of Brinkmann–Puppe [1966]). The notion of an S_a-embedding coincides with the notion of a subregular monomorphism ("Einbettung").

Let $\mathfrak{A}$ be a U-concrete category. It follows from Proposition 8.6 that if there exists a 1-free object in $\mathfrak{A}$, then the notion of an R_a-quotient morphism in $\mathfrak{A}$ coincides with the notion of an R_c-quotient morphism in $\mathfrak{A}$, and if there exists a 2-cofree object in $\mathfrak{A}$, then the notion of an S_a-embedding in $\mathfrak{A}$ coincides with the notion of an S_c-embedding in $\mathfrak{A}$.

Surjective R_c-quotient morphisms and injective S_c-embeddings can be characterized in the following way:

9.14. PROPOSITION. *A surjection π in a U-concrete category $\mathfrak{A}$ is an R_c-quotient morphism if and only if the following condition is satisfied:*

(q) *for every object X in $\mathfrak{A}^{\mathrm{o}}$ and every map f: $|\mathrm{Cod}\,\pi| \to |X|$ it follows from $(f \circ |\pi|, \mathrm{Dom}\,\pi, X) \in \mathfrak{A}^{\mathrm{m}}$ that $(f, \mathrm{Cod}\,\pi, X) \in \mathfrak{A}^{\mathrm{m}}$.*

Proof. Let π be a surjection in $\mathfrak{A}$. Suppose that $\pi = R_c\text{-coim}\,\pi$. If $X \in \mathfrak{A}^{\mathrm{o}}$ and f: $|\mathrm{Cod}\,\pi| \to |X|$ is a map such that $\xi = (f \circ |\pi|, \mathrm{Dom}\,\pi, X) \in \mathfrak{A}^{\mathrm{m}}$, then obviously $\xi R_c \pi$. Consequently, by condition 9.2(c_2), there exists a unique morphism ϑ: $\mathrm{Cod}\,\pi \to X$ such that $\vartheta\pi = \xi$. For every b in $|\mathrm{Dom}\,\pi|$ we have $f(|\pi|(b)) = |\xi|(b) = |\vartheta|(|\pi|(b))$. Since π is a surjection, we obtain $|\vartheta| = f$, i.e. $(f, \mathrm{Cod}\,\pi, X) = \vartheta$ is a morphism in $\mathfrak{A}$. Thus condition (q) is satisfied.

Conversely, suppose that (q) is satisfied and that ξ: $\mathrm{Dom}\,\pi \to X$ is a morphism such that $\xi R_c \pi$. Let $b \in |\mathrm{Cod}\,\pi|$. Since π is a surjection, there exists an a in $|\mathrm{Dom}\,\pi|$ such that $|\pi|(a) = b$. Let $f(b) = |\xi|(a)$. The element $f(b)$ does not depend on the choice of a; if a' is such that $|\pi|(a') = b$, then it follows from $\xi R_c \pi$ that $|\xi|(a') = |\xi|(a)$. Hence there is a function f from $|\mathrm{Cod}\,\pi|$ to $|X|$ such that $f \circ |\pi| = |\xi|$. By condition (q) the triple $\vartheta = (f, \mathrm{Cod}\,\pi, X)$ is a morphism in $\mathfrak{A}$; obviously $\xi = \vartheta\pi$ and ϑ is determined uniquely by this condition. Thus $\pi = R_c\text{-coim}\,\pi$.

9.15. PROPOSITION. *An injection σ in a U-concrete category $\mathfrak{A}$ is an S_c-embedding if and only if the following condition is satisfied:*

(e) *for every object X in $\mathfrak{A}^{\mathrm{o}}$ and every map f: $|X| \to |\mathrm{Dom}\,\sigma|$ it follows from $(|\sigma| \circ f, X, \mathrm{Cod}\,\sigma) \in \mathfrak{A}^{\mathrm{m}}$ that $(f, X, \mathrm{Dom}\,\sigma) \in \mathfrak{A}^{\mathrm{m}}$.*

Proof. Let σ be an injection. Suppose that $\sigma = S_c\text{-im}\,\sigma$. If $X \in \mathfrak{A}^{\mathrm{o}}$ and f: $|X| \to |\mathrm{Dom}\,\sigma|$ is a map such that $\xi = (|\sigma| \circ f, X, \mathrm{Cod}\,\sigma) \in \mathfrak{A}^{\mathrm{m}}$, then obviously $\xi S_c \sigma$. Consequently, by condition 9.2(i_2), there exists a unique morphism ϑ: $X \to \mathrm{Dom}\,\sigma$ such that $\sigma\vartheta = \xi$. Since $|\sigma| \circ |\vartheta| = |\sigma| \circ f$ and σ is injective, it follows that $|\vartheta| = f$, i.e. $(f, X, \mathrm{Dom}\,\sigma) = \vartheta$ is a morphism in $\mathfrak{A}$. Thus condition (e) is satisfied.

Conversely, suppose that (e) is satisfied. Let ξ: $X \to \mathrm{Cod}\,\sigma$ be a morphism such that $\xi S_c \sigma$, and let $x \in |X|$. Since $|\xi|(x) \in |\sigma|^{\to}(|\mathrm{Dom}\,\sigma|)$ and σ is injective, there exists a unique $y = f(x)$ in $|\mathrm{Dom}\,\sigma|$ such that $|\sigma|(y) = |\xi|(x)$. The function f: $|X| \to |\mathrm{Dom}\,\sigma|$ satisfies the condition $|\sigma| \circ f = |\xi|$. By condition (e) the triple $\vartheta = (f, X\ \mathrm{Dom}\,\sigma)$ is a (unique) morphism satisfying $\sigma\vartheta = \xi$. Thus $\sigma = S_c\text{-im}\,\sigma$.

Proposition 9.14 shows that the notion of a surjective R_c-quotient morphism coincides with the notion of a quotient map in the terminology of Semadeni [1971].

Some additional properties of injective S_c-embeddings and surjective R_c-quotient morphisms will be discussed in Section 10 (cf. Propositions 10.7 and 10.8).

Chapter III

The conditions of transfer and of unicity

10. Definitions, examples, and basic properties

Let X be a topological space regarded as a pair $X = (|X|, \mathfrak{G})$, where $|X|$ is the underlying set of X and $\mathfrak{G}$ is the set of all open sets in X. For every set S such that $\operatorname{card} S = \operatorname{card}|X|$ and every bijection $f\colon |X| \to S$ there is a topological space Y such that $|Y| = S$ and $f\colon X \to Y$ is a homeomorphism, i.e. an isomorphism in the category **Top** of topological spaces and continuous maps. In other words, the structure of the space X can be transferred from $|X|$ onto S. Therefore we say that the category **Top** has the property of *transfer*.

Let $X_1 = (|X_1|, \mathfrak{G}_1)$ and $X_2 = (|X_2|, \mathfrak{G}_2)$ be topological spaces. If $|X_1| = |X_2|$ and the identical map $X_1 \to X_2$ is an isomorphism in **Top**, then necessarily $\mathfrak{G}_1 = \mathfrak{G}_2$, and consequently $X_1 = X_2$. We express this fact by saying that the category **Top** has the property of *unicity*.

In this chapter we shall deal with U-concrete categories which have the property of transfer and/or the property of unicity. We shall show that these properties determine some functors on the category U-**Concat**$_\square$ defined in Section 2.

10.1. Definitions. Let $\mathfrak{A}$ be a U-concrete category. We say that $\mathfrak{A}$ has the property of *U-transfer* (cf., e.g., Pultr [1968]) iff for every $B \in \mathfrak{A}^o$ and every bijection $f\colon |B| \to S$ onto an arbitrary set $S \in U$ there exists $C \in \mathfrak{A}^o$ such that $|C| = S$, $(f, B, C) \in \mathfrak{A}^m$, and $(f^{-1}, C, B) \in \mathfrak{A}^m$ (i.e. (f, B, C) is an isomorphism in $\mathfrak{A}$).

If $\mathfrak{A}$ has the property of U-transfer, then we say that $\mathfrak{A}$ is a *U-t-concrete* category. In other words, $\mathfrak{A}$ is U-t-concrete iff the forgetful functor $\square_{\mathfrak{A}}\colon \mathfrak{A} \to U$-**Ens** reduced to $\operatorname{iso}(\mathfrak{A})$ is a full surjector onto the full subcategory $\mathfrak{B}$ of $\operatorname{iso}(U$-**Ens**$)$ with the object class

$$\mathfrak{B}^o = \{X \in U\colon \underset{A \in \mathfrak{A}^o}{\exists} \operatorname{card}|A| = \operatorname{card} X\}.$$

We say that $\mathfrak{A}$ has the property of *unicity* (or that $\mathfrak{A}$ is *U-u-concrete*) iff every isomorphism η in $\mathfrak{A}$ such that $\square_{\mathfrak{A}}(\eta)$ is an identity in U-**Ens**,

is an identity in $\mathfrak{A}$. In other words, $\mathfrak{A}$ is U-u-concrete iff the conditions $(f, A, B) \in \mathrm{iso}(\mathfrak{A})^{\mathrm{m}}$, $(f, A, C) \in \mathrm{iso}(\mathfrak{A})^{\mathrm{m}}$ imply $B = C$.

We say that $\mathfrak{A}$ is *U-regular* (or that $\mathfrak{A}$ is *U-r-concrete*) iff for every set S the set

$$\mathrm{Ob}(S) = \{B \in \mathfrak{A}^{\mathrm{o}}\colon |B| = B\}$$

is a member of U.

A category which is both U-t-concrete and U-u-concrete will be called *U-tu-concrete*. Similarly we define *U-rt-concrete*, *U-ru-concrete*, and *U-rtu-concrete* categories.

Let us note that a category $\mathfrak{A}$ is U-tu-concrete iff for every $B \in \mathfrak{A}^{\mathrm{o}}$ and every bijection $f\colon |B| \to S$ onto an arbitrary set $S \in U$ there exists a *unique* $C \in \mathfrak{A}^{\mathrm{o}}$ such that $|C| = S$, $(f, B, C) \in \mathfrak{A}^{\mathrm{m}}$, and $(f^{-1}, C, B) \in \mathfrak{A}^{\mathrm{m}}$. We shall call this condition the property of *unique transfer*.

It is easy to verify that every category $\square$-isomorphic with a U-t-concrete [U-u-concrete, U-r-concrete] category is U-t-concrete [U-u-concrete, U-r-concrete, respectively].

By U-t-$\mathbf{Concat}_{\square}$ we shall denote the full subcategory of the category U-$\mathbf{Concat}_{\square}$ whose objects are all U-t-concrete categories. The categories U-r-$\mathbf{Concat}_{\square}$, U-u-$\mathbf{Concat}_{\square}$, U-rt-$\mathbf{Concat}_{\square}$, U-ru-$\mathbf{Concat}_{\square}$, U-tu-$\mathbf{Concat}_{\square}$, and U-rtu-$\mathbf{Concat}_{\square}$ are defined similarly.

10.2. Examples. (A) Let U-$\mathbf{Ban}$ be the category of Banach spaces belonging to U and bounded linear transformations. The category U-$\mathbf{Ban}$ is U-rt-concrete.

On the other hand, it is easy to see that the category U-$\mathbf{Ban}$ is not U-u-concrete. In fact, if $\|\cdot\|$ and $\|\cdot\|'$ are two different equivalent Banach norms on a vector space X, then $\mathrm{id}_X\colon (X, \|\cdot\|) \to (X, \|\cdot\|')$ is a non-identical isomorphism in U-$\mathbf{Ban}$. However, the category U-$\mathbf{Ban}_1$ of Banach spaces belonging to U and linear contractions (i.e., linear operators a such that $\|a\| \leqslant 1$) is U-rtu-concrete.

(B) Let $\mathrm{id}(U$-$\mathbf{Ens})$ be the subcategory of U-$\mathbf{Ens}$ such that the objects are any sets and morphisms are identities. The category $\mathrm{id}(U$-$\mathbf{Ens})$ is U-ru-concrete, but not U-t-concrete.

(C) Let $\mathfrak{A}$ be the concrete category defined as follows. The objects are pairs $(A, M) \in U \times U$, and $|(A, M)| = A$. For $(A, M) \in \mathfrak{A}^{\mathrm{o}}$, $(B, N) \in \mathfrak{A}^{\mathrm{o}}$ let $\langle (A, M), (B, N) \rangle_{\mathfrak{A}} = \emptyset$ if $M \neq N$, and let $\langle (A, M), (B, M) \rangle_{\mathfrak{A}}$ be the set of all triples $\big(f, (A, M), (B, N)\big)$, where f is a function from A to B. The category $\mathfrak{A}$ is U-tu-concrete, but not U-r-concrete.

If $\mathfrak{A}$ is a U-t-concrete category, then neither $\mathfrak{A}^{\diamond}$ nor $\mathfrak{A}^{\#}$ satisfies the condition of transfer. In fact, the underlying set of an object in $\mathfrak{A}^{\diamond}$ or $\mathfrak{A}^{\#}$ is a power set, but a set equipollent with a power set need not be a power set. This motivates the following definition:

10.3. Definition. A category $\mathfrak{A}$ will be called *U-pt-concrete* iff $\mathfrak{A}$ is U-concrete and for every B in $\mathfrak{A}^{\circ}$ such that $|B|$ is a power set, every set N in U, and every union-preserving bijection $f\colon |B|\to\mathscr{P}(N)$, there exists a C in $\mathfrak{A}^{\circ}$ such that $|C| = \mathscr{P}(N)$, $(f, B, C)\in\mathfrak{A}^{\mathrm{m}}$, and $(f^{-1}, C, B)\in\mathfrak{A}^{\mathrm{m}}$.

10.4. Proposition. *Let $\mathfrak{A}$ be a concrete category. The following conditions are equivalent*:

(i) $\mathfrak{A}$ *is U-t-concrete*,

(ii) $\mathfrak{A}^{\diamond}$ *is U-pt-concrete*,

(iii) $\mathfrak{A}^{\#}$ *is U-pt-concrete*.

Proof. (i)$\Rightarrow$(ii). Let $B\in(\mathfrak{A}^{\diamond})^{\circ}$, $|B|^{\diamond} = \mathscr{P}(M)$, $N\in U$, and let f: be a union-preserving bijection from $\mathscr{P}(M)$ to $\mathscr{P}(N)$. By Lemma 3.9 there exists a map $g\colon N\to M$ such that $f = \mathscr{P}_{-}(g)$. By Corollaries 3.7 and 3.8 the map g is a bijection. Since $|B| = M$ and $\mathfrak{A}$ is U-t-concrete, there exists a C in $\mathfrak{A}^{\circ}$ such that $|C| = N$ and (g, C, B) is an isomorphism in $\mathfrak{A}$. Consequently, $\big(\mathscr{P}_{-}(g), B, C\big)$ is an isomorphism in $\mathfrak{A}^{\diamond}$ such that $|C|^{\diamond} = \mathscr{P}(N)$.

(ii)$\Rightarrow$(iii). Suppose that $B\in(\mathfrak{A}^{\#})^{\circ}$, $|B|^{\#} = \mathscr{P}(M)$, $N\in U$, and f: is a union-preserving bijection from $\mathscr{P}(M)$ to $\mathscr{P}(N)$. By Lemma 3.9 there exists a map $g\colon M\to N$ such that $f = \mathscr{P}_{+}(g)$. By Corollaries 3.7 and 3.8 the maps $g\colon M\to N$ and $\mathscr{P}_{-}(g)\colon \mathscr{P}(N)\to\mathscr{P}(M)$ are bijections. Since $\mathfrak{A}^{\diamond}$ is U-*pt*-concrete, there exists a C in $(\mathfrak{A}^{\diamond})^{\circ}$ such that $\big(\mathscr{P}_{-}(g), C, B\big)$ is an isomorphism in $\mathfrak{A}^{\diamond}$. Hence $\big(\mathscr{P}_{+}(g), B, C\big)$ is an isomorphism in $\mathfrak{A}^{\#}$.

(iii)$\Rightarrow$(i). Let $B\in\mathfrak{A}^{\circ}$ and let $f\colon |B|\to N$ be a bijection with $N\in U$. Since $\mathscr{P}_{+}(f)\colon |B|^{\#}\to\mathscr{P}(N)$ is union-preserving and $\mathfrak{A}^{\#}$ is U-*pt*-concrete, there exists a C in $(\mathfrak{A}^{\#})^{\circ}$ such that $|C|^{\#} = \mathscr{P}(N)$ and $\big(\mathscr{P}_{+}(f), B, C\big)$ is an isomorphism in $\mathfrak{A}^{\#}$. Then $|C| = N$ and (f, B, C) is an isomorphism in $\mathfrak{A}$.

10.5. Proposition. *Let $\mathfrak{A}$ be a concrete category. The following conditions are equivalent*:

(i) $\mathfrak{A}$ *is U-u-concrete*,

(ii) $\mathfrak{A}^{\diamond}$ *is U-u-concrete*,

(iii) $\mathfrak{A}^{\#}$ *is U-u-concrete*.

Proof. (i)$\Rightarrow$(ii). If $\big(\mathscr{P}_{-}(f), A, B\big)$ is an isomorphism in $\mathfrak{A}^{\diamond}$ such that $|A|^{\diamond} = |B|^{\diamond}$ and $\mathscr{P}_{-}(f) = \mathrm{id}_{|A|^{\diamond}}$, then $f = \mathrm{id}_{|A|}$ and $(\mathrm{id}_{|A|}, B, A)$ is an isomorphism in $\mathfrak{A}$, i.e. $B = A$. The proofs of implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are similar.

10.6. Proposition. *Let $\mathfrak{A}$ be a concrete category. The following conditions are equivalent*:

(i) $\mathfrak{A}$ *is U-r-concrete*,

(ii) $\mathfrak{A}^{\diamond}$ *is U-r-concrete*,

(iii) $\mathfrak{A}^{\#}$ *is U-r-concrete*.

Proof. The condition

$$\mathrm{Ob}_{\mathfrak{A}}(S) = \{B \in \mathfrak{A}^{\circ}\colon |B| = S\} \in U \quad \text{for every set } S$$

is equivalent to the condition

$$\mathrm{Ob}_{\mathfrak{A}^{\circ}}(T) = \mathrm{Ob}_{\mathfrak{A}^{\#}}(T) \in U \quad \text{for every set } T.$$

In fact, $\mathrm{Ob}_{\mathfrak{A}^{\circ}}(T) = 0$ if T is not a power set, and $\mathrm{Ob}_{\mathfrak{A}^{\circ}}(T) = \mathrm{Ob}_{\mathfrak{A}}(S)$ if $T = \mathscr{P}(S)$.

The following two propositions deal with injective S_c-embeddings and surjective R_c-quotient morphisms (cf. Section 9) in U-t-concrete categories.

10.7. Proposition. *Let $\mathfrak{A}$ be a U-t-concrete category. If σ is an injective S_c-embedding in $\mathfrak{A}$, then there exist morphisms σ_0 and η in $\mathfrak{A}$ such that*

(i) $\sigma = \sigma_0\eta$,

(ii) *η is an isomorphism,*

(iii) *σ_0 is an S_c-embedding such that $|\mathrm{Dom}\,\sigma_0| \subset |\mathrm{Cod}\,\sigma_0|$ and $|\sigma_0|(a) = a$ for every a in $|\mathrm{Dom}\,\sigma_0|$.*

Proof. The map $|\sigma|\colon |\mathrm{Dom}\,\sigma| \to |\sigma|^{\rightarrow}(|\mathrm{Dom}\,\sigma|)$ is a bijection. By the condition of transfer there exists an object A in $\mathfrak{A}^{\circ}$ such that $|A| = |\sigma|^{\rightarrow}(|\mathrm{Dom}\,\sigma|)$ and $\eta = (|\sigma|, \mathrm{Dom}\,\sigma, A)$ is an isomorphism in $\mathfrak{A}$. Let $\sigma_0 = \sigma\eta^{-1}$. It suffices to show that σ_0 is an S_c-embedding. Let $f\colon |X| \to |\mathrm{Dom}\,\sigma_0|$ be a map such that $(|\sigma_0| \circ f, X, \mathrm{Cod}\,\sigma_0) \in \mathfrak{A}^{\mathrm{m}}$. Since $|\sigma_0| \circ f = |\sigma| \circ |\eta| \circ f$ and $\mathrm{Cod}\,\sigma_0 = \mathrm{Cod}\,\sigma$, it follows from Proposition 9.15 that $(|\eta| \circ f, X, \mathrm{Dom}\,\sigma) \in \mathfrak{A}^{\mathrm{m}}$. Consequently, $(f, X, \mathrm{Dom}\,\sigma_0) = \eta^{-1}(|\eta| \circ f, X, \mathrm{Dom}\,\sigma) \in \mathfrak{A}^{\mathrm{m}}$, i.e. σ_0 satisfies condition 9.15(e).

10.8. Proposition. *Let $\mathfrak{A}$ be a U-t-concrete category. If π is a surjective R_c-quotient morphism in $\mathfrak{A}$, then there exist morphisms π_0 and ζ in $\mathfrak{A}$ such that*

(i) $\pi = \zeta\pi_0$,

(ii) *ζ is an isomorphism,*

(iii) *π_0 is an R_c-quotient morphism such that*

$$|\mathrm{Cod}\,\pi_0| = (|\mathrm{Dom}\,\pi_0|)/\sim,$$

where $\sim$ is an equivalence relation on $|\mathrm{Dom}\,\pi_0|$ and $|\pi_0|$ is the canonical surjection.

The proof is similar to the proof of Proposition 10.7.

10.9. An S_c-embedding σ_0 satisfying condition 10.7 (iii) could be called an *ordinary embedding* in $\mathfrak{A}$. Similarly, an R_c-quotient morphism π_0 satisfying condition 10.8 (iii) could be called an *ordinary quotient morphism* in $\mathfrak{A}$.

For example, it is easy to see that ordinary embeddings [ordinary quotient morphisms] in the category Top of topological spaces and continuous maps are embeddings [canonical quotient maps] in the usual topological sense (cf., e.g., Engelking [1968]).

11. The "transfer" functor

With every U-concrete category $\mathfrak{A}$ we can associate a U-concrete category $\mathfrak{A}_t$ defined as follows. The objects of $\mathfrak{A}_t$ are all pairs (A, f), where $A \in \mathfrak{A}^{\mathrm{o}}$ and f is a function defined on $|A|$ such that $\mathrm{Im} f \in U$ and the map $f\colon |A| \to \mathrm{Im} f$ is a bijection. The underlying set of (A, f) is $\mathrm{Im} f$. The morphisms from (A, f) to (B, g) are all triples of the form

$$t(a, f, g) = \big(g \circ |a| \circ f^{-1}, (A, f), (B, g)\big), \tag{1}$$

where $a \in \langle A, B\rangle_{\mathfrak{A}}$. In other words, $\big(h, (A, f), (B, g)\big)$ is a morphism in $\mathfrak{A}_t^{\mathrm{m}}$ iff h is a function from $\mathrm{Im} f$ to $\mathrm{Im} g$ such that $(g^{-1} \circ h \circ f, A, B)$ is a morphism in $\mathfrak{A}^{\mathrm{m}}$.

It is obvious that these data satisfy conditions (c_1) and (c_2) in 2.2.

Let us note that for every morphism $a\colon A \to B$ of $\mathfrak{A}$ we have the following decomposition in $\mathfrak{A}_t$:

$$t(a, t, g) = t(\iota_B, \mathrm{id}_{|B|}, g)\, t(a, \mathrm{id}_{|A|}, \mathrm{id}_{|B|})\, t(\iota_A, f, \mathrm{id}_{|A|}). \tag{2}$$

11.1. Proposition. *For every U-concrete category $\mathfrak{A}$ the category $\mathfrak{A}_t$ is U-t-concrete.*

Proof. Let $(A, f) \in \mathfrak{A}_t^{\mathrm{o}}$, $S \in U$, and let $j\colon \mathrm{Im} f \to S$ be a bijection. It is obvious that $(A, j \circ f) \in \mathfrak{A}_t^{\mathrm{o}}$ and $|(A, j \circ f)| = S$. Moreover,

$$\big(j, (A, f), (A, j \circ f)\big) = t(\iota_A, f, j \circ f) \in \mathfrak{A}_t^{\mathrm{m}},$$

and

$$\big(j^{-1}, (A, j \circ f), (A, f)\big) = t(\iota_A, j \circ f, f) \in \mathfrak{A}_t^{\mathrm{m}}.$$

Hence $\mathfrak{A}_t$ has the property of transfer.

11.2. Let us note that if the category $\mathfrak{A}$ is not skeletal, i.e. if $\mathfrak{A}$ has at least two different isomorphic objects, then $\mathfrak{A}_t$ has too many objects to satisfy the condition of unicity. In fact, if $A, B \in \mathfrak{A}^{\mathrm{o}}$, $A \neq B$, and $\eta\colon A \to B$ is an isomorphism in $\mathfrak{A}$, then $(A, \mathrm{id}_{|A|})$ and $(B, |\eta^{-1}|)$ are two different objects of $\mathfrak{A}_t$ such that $\big(\mathrm{id}_{|A|}, (A, \mathrm{id}_{|A|}), (B, |\eta^{-1}|)\big)$ is an isomorphism in $\mathfrak{A}_t$.

11.3. Let $T_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{A}_t$ be the functor defined as follows

$$T_{\mathfrak{A}}(A) = (A, \mathrm{id}_{|A|}) \quad \text{for all } A \in \mathfrak{A}^{\mathrm{o}},$$

$$T_{\mathfrak{A}}(a) = \big(|a|, (\mathrm{Dom}\, a, \mathrm{id}_{|\mathrm{Dom}\, a|}), (\mathrm{Cod}\, a, \mathrm{id}_{|\mathrm{Cod}\, a|})\big) \quad \text{for all } a \in \mathfrak{A}^{\mathrm{m}}.$$

It is easy to see that $T_{\mathfrak{A}}$ is a $\square$-commuting full injector. Hence $\mathfrak{A}$ is $\square$-isomorphic with the full subcategory $T_{\mathfrak{A}}(\mathfrak{A})$ of the category $\mathfrak{A}_t$.

11.4. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be U-concrete categories and let $\Phi\colon \mathfrak{A}\to\mathfrak{A}'$ be a $\square$-commuting functor. For every $(A,f)\in\mathfrak{A}_t$ let

$$\Phi_t(A,f) = \big(\Phi(A),f\big).$$

It follows from $|\Phi(A)| = |A|$ that $\big(\Phi(A),f\big)\in(\mathfrak{A}')_t^{\circ}$. For every triple $\big(h,(A,f),(B,g)\big)\in\mathfrak{A}_t$ let

$$\Phi_t\big(h,(A,f),(B,g)\big) = \big(h,\big(\Phi(A),f\big),\big(\Phi(B),g\big)\big).$$

Since $(g^{-1}\circ h\circ f, A, B)\in\mathfrak{A}^{\mathrm{m}}$ and Φ is $\square$-commuting, it follows that $\big(g^{-1}\circ h\circ f, \Phi(A), \Phi(B)\big)\in(\mathfrak{A}')^{\mathrm{m}}$. Hence Φ_t transforms $\mathfrak{A}_t$ into $\mathfrak{A}_t'$. It is easy to verify that $\Phi_t\colon \mathfrak{A}_t\to\mathfrak{A}_t'$ is a $\square$-commuting functor.

It follows from Proposition 11.1. that the functions

$$\mathfrak{A}\mapsto\mathfrak{A}_t, \qquad \Phi\mapsto\Phi_t$$

define the "transfer" functor

$$?_t\colon\ U\text{-}\mathbf{Concat}_{\square}\to U\text{-}t\text{-}\mathbf{Concat}_{\square}.$$

It is easy to see that the family $(T_{\mathfrak{A}})$ defined in 11.3 is a natural transformation of the identical functor $I_{U\text{-}\mathbf{Concat}_{\square}}$ into the functor $?^t\colon U\text{-}\mathbf{Concat}_{\square}\to U\text{-}\mathbf{Concat}_{\square}$ defined as the composition

$$U\text{-}\mathbf{Concat}_{\square}\xrightarrow[?_t]{}U\text{-}t\text{-}\mathbf{Concat}_{\square}\xrightarrow[\text{inclusion}]{}U\text{-}\mathbf{Concat}_{\square}.$$

11.5. Proposition. *For every U-concrete category $\mathfrak{A}$ the functor $T_{\mathfrak{A}}\colon\mathfrak{A}\to\mathfrak{A}_t$ is an R-pseudo-reflection from U-$\mathbf{Concat}_{\square}$ into U-t-$\mathbf{Concat}_{\square}$, where R is the relation defined in* 5.2(A).

In other words, for every U-t-concrete category $\mathfrak{X}$ and for every $\square$-commuting functor $\Xi\colon\mathfrak{A}\to\mathfrak{X}$ there exists a $\square$-commuting functor $\Xi\colon\mathfrak{A}_t\to\mathfrak{X}$ such that $\Xi = \Theta T_{\mathfrak{A}}$; if, moreover, $\Theta'\colon\mathfrak{A}_t\to\mathfrak{X}$ is another $\square$-commuting functor such that $\Xi = \Theta' T_{\mathfrak{A}}$, then Θ and Θ' are naturally equivalent.

Proof. Suppose that $\mathfrak{X}$ is a U-t-concrete category and $\Xi\colon\mathfrak{A}\to\mathfrak{X}$ is a $\square$-commuting functor. Let $(A,f)\in\mathfrak{A}_t^{\circ}$. Because $\mathfrak{X}$ has the property of transfer and $|\Xi(A)| = |A|$, there exists $X\in\mathfrak{X}^{\circ}$ such that $\big(f,\Xi(A),X\big)$ is an isomorphism in $\mathfrak{X}$. Let

$$\Theta(A,f) = c_U\big\{X\in\mathfrak{X}^{\circ}\colon\ \big(f,\Xi(A),X\big)\in\mathrm{iso}(\mathfrak{X})^{\mathrm{m}}\big\},$$

and for $t(a,f,g)\in\mathfrak{A}_t^{\mathrm{m}}$, $a\in\langle A,B\rangle_{\mathfrak{A}}$, let

$$\Theta\big(t(a,f,g)\big) = \big(g,\Xi(B),\Theta(B,g)\big)\Xi(a)\big(f^{-1},\Theta(A,f),\Xi(A)\big).$$

It is easy to verify that $\Theta\colon\mathfrak{A}_t\to\mathfrak{X}$ is a $\square$-commuting functor satisfying $\Xi = \Theta T_{\mathfrak{A}}$.

Let $\Theta' : \mathfrak{A}_t \to \mathfrak{X}$ be another $\square$-commuting functor such that $\Xi = \Theta' T_{\mathfrak{A}}$. Let

$$\begin{aligned}\lambda_{(A,f)} &= [\Theta' t(\iota_A, f, \mathrm{id}_{|A|})]^{-1} \Theta t(\iota_A, f, \mathrm{id}_{|A|}) \\ &= \Theta' t(\iota_A, \mathrm{id}_{|A|}, f) \Theta t(\iota_A, f, \mathrm{id}_{|A|}).\end{aligned}$$

Since $t(\iota_A, \mathrm{id}_{|A|}, f)$ is an isomorphism in $\mathfrak{A}_t$, it follows that $\lambda_{(A,f)}$ is an isomorphism from $\Theta(A, f)$ to $\Theta'(A, f)$ in $\mathfrak{X}$. We shall show that the family $(\lambda_{(A,f)})$, where $(A, f) \in \mathfrak{A}_t^{\mathrm{o}}$, is a natural equivalence of functors Θ and Θ'. In other words, we shall show that for every morphism

$$t(a, f, g) \colon (A, f) \to (B, g)$$

in $\mathfrak{A}_t^{\mathrm{m}}$ the following diagram

$$\begin{tikzcd} \Theta(A, f) \arrow[r, "\lambda_{(A,f)}"] \arrow[d, "\Theta t(a,f,g)"'] & \Theta'(A, f) \arrow[d, "\Theta t(a,f,g)"] \\ \Theta(B, g) \arrow[r, "\lambda_{(B,g)}"'] & \Theta'(B, g) \end{tikzcd}$$

commutes. In fact, using (2) we have

$$\begin{aligned}&\Theta' t(a, f, g) \lambda_{(A,f)} \\ &\quad = \Theta' t(\iota_B, \mathrm{id}_{|B|}, g) \Xi(a) \Theta' t(\iota_A, f, \mathrm{id}_{|A|}) \Theta' t(\iota_A, \mathrm{id}_{|A|}, f) \Theta t(\iota_A, f, \mathrm{id}_{|A|}) \\ &\quad = \Theta' t(\iota_B, \mathrm{id}_{|B|}, g) \Xi(a) \Theta t(\iota_A, f, \mathrm{id}_{|A|}) \\ &\quad = \Theta' t(\iota_B, \mathrm{id}_{|B|}, g) \Theta t(\iota_B, g, \mathrm{id}_{|B|}) \Theta t(\iota_B, \mathrm{id}_{|B|}, g) \Xi(a) \Theta t(\iota_A, f, \mathrm{id}_{|A|}) \\ &\quad = \lambda_{(B,g)} \Theta t(a, f, g).\end{aligned}$$

12. The "unicity" functor

Let $\mathfrak{A}$ be a U-concrete category. Write $A \underset{\mathfrak{A}}{\approx} B$ (or simply $A \approx B$) iff $A, B \in \mathfrak{A}^{\mathrm{o}}$, $|A| = |B|$, and $(\mathrm{id}_{|A|}, A, B)$ is an isomorphism in $\mathfrak{A}$. It is easy to see that $\approx$ is an equivalence relation on $\mathfrak{A}^{\mathrm{o}}$. For every $A \in \mathfrak{A}^{\mathrm{o}}$ let $A/\approx$ denote the equivalence class containing A. Let $\mathfrak{A}_u$ be the full subcategory of $\mathfrak{A}$ with the object class

$$\mathfrak{A}_u^{\mathrm{o}} = \{c_U(A/\approx) \colon A \in \mathfrak{A}^{\mathrm{o}}\}.$$

It might appear to be more natural to define $\mathfrak{A}_u$ without the use of the choice function c_U as a category whose objects are equivalence classes $A/\approx$. But $A/\approx$ need not be a member of U. If $\mathfrak{A}$ is U-r-concrete, then $(A/\approx) \in U$ for all $A \in \mathfrak{A}^{\mathrm{o}}$.

12.1. Proposition. *For every U-concrete category $\mathfrak{A}$ the category $\mathfrak{A}_u$ is U-u-concrete.*

Proof. If $(\mathrm{id}_{|A|}, A, B)$ is an isomorphism in $\mathfrak{A}_u$, then $A \underset{\mathfrak{A}}{\approx} B$, i.e. $A = c_U(A/\approx) = c_U(B/\approx) = B$. It means that $\mathfrak{A}_u$ has he property of unicity.

12.2. Remarks. 1) A U-concrete category $\mathfrak{A}$ has the property of unicity iff $A \approx B$ implies $A = B$ for all $A, B \in \mathfrak{A}^{\mathrm{o}}$.

2) $A \underset{\mathfrak{A}}{\approx} B \Rightarrow \Phi(A) \underset{\mathfrak{B}}{\approx} \Phi(B)$ for every $\square$-commuting functor $\Phi\colon \mathfrak{A} \to \mathfrak{B}$.

12.3. For every U-concrete category $\mathfrak{A}$ let

$$W_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{A}_u$$

be the functor defined as follows

$$W_{\mathfrak{A}}(A) = c_U(A/\approx) \qquad \text{for all } A \in \mathfrak{A}^{\mathrm{o}},$$

$$W_{\mathfrak{A}}(f, A, B) = \big(f, W_{\mathfrak{A}}(A), W_{\mathfrak{A}}(B)\big) \quad \text{for all } (f, A, B) \in \mathfrak{A}^{\mathrm{m}}.$$

It is easy to see that $W_{\mathfrak{A}}$ is a $\square$-commuting surjector.

12.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be U-concrete categories and let $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ be a $\square$-commuting functor. Let

$$\Phi_u(A) = c_U\big(\Phi(A)/\underset{\mathfrak{B}}{\approx}\big) \qquad \text{for all } A \in \mathfrak{A}_u^{\mathrm{o}},$$

$$\Phi_u(f, A, B) = \big(f, \Phi_u(A), \Phi_u(B)\big) \quad \text{for all } (f, A, B) \in \mathfrak{A}_u^{\mathrm{m}}. \tag{3}$$

Let us note that $(f, A, B) \in \mathfrak{A}_u^{\mathrm{m}}$ implies $\big(f, \Phi(A), \Phi(B)\big) \in \mathfrak{B}^{\mathrm{m}}$. Since $\Phi(A) \underset{\mathfrak{B}}{\approx} \Phi_u(A)$ and $\Phi(B) \underset{\mathfrak{B}}{\approx} \Phi_u(B)$, it follows that $\big(f, \Phi_u(A), \Phi_u(B)\big) \in \mathfrak{B}_u^{\mathrm{m}}$. Hence formulae (3) define a $\square$-commuting functor $\Phi_u\colon \mathfrak{A}_u \to \mathfrak{B}_u$.

The functions

$$\mathfrak{A} \mapsto \mathfrak{A}_u, \qquad \Phi \mapsto \Phi_u$$

define the "unicity" functor

$$?_u\colon U\text{-}\mathbf{Concat}_{\square} \to U\text{-}u\text{-}\mathbf{Concat}_{\square}. \tag{4}$$

The family $(W_{\mathfrak{A}})$ defined in 12.3 is a natural transformation of the identical functor $I_{U\text{-}\mathbf{Concat}_{\square}}$ into the functor $?_u\colon U\text{-}\mathbf{Concat}_{\square} \to U\text{-}\mathbf{Concat}_{\square}$ (defined as the composition of functor (4) with the inclusion functor).

12.5. Proposition. *For every U-concrete category $\mathfrak{A}$ the functor $W_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{A}_u$ is a reflection from U-$\mathbf{Concat}_{\square}$ into U-u-$\mathbf{Concat}_{\square}$. In other words, U-u-$\mathbf{Concat}_{\square}$ is a reflective subcategory of U-$\mathbf{Concat}_{\square}$, and functor* (4) *is a reflector.*

Proof. We shall show that for every U-u-concrete category $\mathfrak{X}$ and every $\square$-commuting functor $\Xi\colon \mathfrak{A} \to \mathfrak{X}$ there exists a unique $\square$-commuting functor $\Theta\colon \mathfrak{A}_u \to \mathfrak{X}$ such that $\Xi = \Theta W_{\mathfrak{A}}$.

Let

$$\Theta = \Xi I_{\mathfrak{A}_u}^{\mathfrak{A}}.$$

We shall show that $\Theta W_{\mathfrak{A}} = \Xi$. Since Ξ is $\square$-commuting, it suffices to prove that $\Xi(A) = \Theta W_{\mathfrak{A}}(A) = \Xi W_{\mathfrak{A}}(A)$ for all $A \in \mathfrak{A}^{\circ}$. In other words, it suffices to prove that

$$\Xi(A) = \Xi\big(c_U(A/\approx)\big) \tag{5}$$

for all $A \in \mathfrak{A}^{\circ}$. We have

$$A \underset{\mathfrak{A}}{\approx} c_U(A/\approx),$$

hence

$$\Xi(A) \underset{\mathfrak{X}}{\approx} \Xi\big(c_U(A/\approx)\big).$$

Since $\mathfrak{X}$ has the property of unicity, this implies (5). To prove the uniqueness of Θ observe that

$$W_{\mathfrak{A}} I^{\mathfrak{A}}_{\mathfrak{A}_u} = I_{\mathfrak{A}_u}. \tag{6}$$

Hence, if $\Theta'\colon \mathfrak{A}_u \to \mathfrak{X}$ is a $\square$-commuting functor such that $\Theta' W_{\mathfrak{A}} = \Xi$, then by (6)

$$\Theta' = \Theta' W_{\mathfrak{A}} I^{\mathfrak{A}}_{\mathfrak{A}_u} = \Xi I^{\mathfrak{A}}_{\mathfrak{A}_u} = \Theta.$$

12.6. It is easy to see that if $\mathfrak{A}$ is U-u-concrete, then $\mathfrak{A}_u = \mathfrak{A}$ and $W_{\mathfrak{A}}$ is an identity. For every U-concrete category $\mathfrak{A}$ the category $\mathfrak{A}_u$ is quasi-isomorphic with $\mathfrak{A}$ and $W_{\mathfrak{A}}$ is a quasi-isomorphism; in fact, $W_{\mathfrak{A}} I^{\mathfrak{A}}_{\mathfrak{A}_u}$ is an identity and $I^{\mathfrak{A}}_{\mathfrak{A}_u} W_{\mathfrak{A}}$ is naturally equivalent to $I_{\mathfrak{A}}$. The family $(\lambda_A)_{A \in \mathfrak{A}^{\circ}}$, where $\lambda_A = \big(\mathrm{id}_{|A|}, A, c_U(A/\approx)\big)$, is a natural equivalence of the functors $I_{\mathfrak{A}}$ and $I^{\mathfrak{A}}_{\mathfrak{A}_u} W_{\mathfrak{A}}$.

12.7. Proposition. *If $\mathfrak{A}$ is U-r-concrete, then $\mathfrak{A}_U$ is U-ru-concrete. If $\mathfrak{A}$ is U-t-concrete, then $\mathfrak{A}_U$ is U-tu-concrete.*

Proof. The first part of the proposition is obvious. Suppose that $\mathfrak{A}$ has the property of U-transfer. Let $A \in \mathfrak{A}^{\circ}_u$ and let $f\colon |A| \to S$ be a bijection with $S \in U$. There exists $X \in \mathfrak{A}^{\circ}$ such that $|X| = S$, $(f, A, X) \in \mathfrak{A}^{\mathrm{m}}$, and $(f^{-1}, X, A) \in \mathfrak{A}^{\mathrm{m}}$. Let $Y = c_U(X/\approx)$. We have $Y \in \mathfrak{A}^{\circ}_u$, $|Y| = S$, $(f, A, Y) \in \mathfrak{A}^{\mathrm{m}}_u$, and $(f^{-1}, Y, A) \in \mathfrak{A}^{\mathrm{m}}_u$. Hence $\mathfrak{A}_u$ has the property of U-transfer.

12.8. Example. The category U-$\mathbf{Ban}_u$ is $\square$-isomorphic with the category of locally bounded locally convex Hausdorff spaces and linear continuous transformations.

13. The "unique transfer" functor

By Proposition 12.7 the functor $?_u$ transforms U-t-$\mathbf{Concat}_{\square}$ into U-tu-$\mathbf{Concat}_{\square}$. The "unique transfer" functor

$$?_{tu}\colon\ U\text{-}\mathbf{Concat}_{\square} \to U\text{-}tu\text{-}\mathbf{Concat}_{\square} \tag{7}$$

is defined as the composition

$$U\text{-}\mathbf{Concat}_{\square} \xrightarrow[?_t]{} U\text{-}t\text{-}\mathbf{Concat}_{\square} \xrightarrow[?_u]{} U\text{-}tu\text{-}\mathbf{Concat}_{\square}.$$

The category $\mathfrak{A}_{tu} = (\mathfrak{A}_t)_u$ is a full subcategory of $\mathfrak{A}_t$. The objects of $\mathfrak{A}_{tu}$ are elements

$$c_U\big((A,f)/\underset{\mathfrak{A}_t}{\approx}\big),$$

where $(A,f)\in\mathfrak{A}_t^o$ and $\underset{\mathfrak{A}_t}{\approx}$ is an equivalence relation on $\mathfrak{A}_t^o$ defined by the condition

$$(A,f)\approx(B,g)\Leftrightarrow(\mathrm{Im}f=\mathrm{Im}g) \text{ and } (g^{-1}\circ f, A, B)\in\mathfrak{A}^m \text{ and } (f^{-1}\circ g; B, A)\in\mathfrak{A}^m.$$

The morphisms of $\mathfrak{A}_{tu}$ from $M = c_U\big((A,f)/\approx\big)$ to $N = c_U\big((B,g)/\approx\big)$ are triples

$$s(a,f,g) = (g\circ|a|\circ f^{-1}, M, N),$$

where $a\in\langle A, B\rangle_{\mathfrak{A}}$.

For every morphism $a\colon A\to B$ in $\mathfrak{A}^m$ we have the following decomposition similar to (2):

$$s(a,f,g) = s(\iota_B, \mathrm{id}_{|B|}, g)s(a, \mathrm{id}_{|A|}, \mathrm{id}_{|B|})s(\iota_A, f, \mathrm{id}_{|A|}).$$

Let $S_{\mathfrak{A}}\colon \mathfrak{A}\to\mathfrak{A}_{tu}$ be the functor defined as the composition

$$\mathfrak{A}\xrightarrow[T_{\mathfrak{A}}]{}\mathfrak{A}_t\xrightarrow[W_{\mathfrak{A}_t}]{}\mathfrak{A}_{tu}.$$

It is easy to see that the family $(S_{\mathfrak{A}})$ is a natural transformation from $I_{U\text{-Concat}_\square}$ into the functor $I^{U\text{-Concat}_\square}_{U\text{-}tu\text{-Concat}_\square}(\mathfrak{A}_{tu})$.

13.1. Proposition. *For every U-concrete category $\mathfrak{A}$ the functor $S_{\mathfrak{A}}\colon \mathfrak{A}\to\mathfrak{A}_{tu}$ is a reflection from U-Concat$_\square$ into U-tu-Concat$_\square$. In other words, U-tu-Concat$_\square$ is a reflective subcategory of U-Concat$_\square$, and functor* (7) *is a reflector.*

Proof. We shall show that for every U-tu-concrete category $\mathfrak{X}$ and for every $\square$-commuting functor $\Xi\colon \mathfrak{A}\to\mathfrak{X}$ there exists a unique $\square$-commuting functor $\Theta\colon \mathfrak{A}_{tu}\to\mathfrak{X}$ such that $\Xi = \Theta S_{\mathfrak{A}}$. First we shall prove the existence of Θ. By Proposition 11.5. there exists a $\square$-commuting functor $\Gamma\colon \mathfrak{A}_t\to\mathfrak{X}$ such that $\Xi = \Gamma T_{\mathfrak{A}}$. By Proposition 12.5 there exists a $\square$-commuting functor $\Theta = \mathfrak{A}_{tu}\to\mathfrak{X}$ such that $\Gamma = \Theta W_{\mathfrak{A}_t}$. Hence

$$\Xi = \Gamma T_{\mathfrak{A}} = \Theta W_{\mathfrak{A}_t} T_{\mathfrak{A}} = \Theta S_{\mathfrak{A}}.$$

Suppose now that $\Theta'\colon \mathfrak{A}_{tu}\to\mathfrak{X}$ is a $\square$-commuting functor such that $\Xi = \Theta' S_{\mathfrak{A}}$. For every $(A,f)\in\mathfrak{A}_t^o$ consider the following isomorphisms in $\mathfrak{X}$:

$$\begin{aligned}
\mu &= \Theta s(\iota_A, \mathrm{id}_{|A|}, f)\\
&= \Theta\big(f, c_U\big((A, \mathrm{id}_{|A|})/\approx\big), c_U\big((A,f)/\approx\big)\big)\\
&= \big(f, \Theta c_U\big((A, \mathrm{id}_{|A|})/\approx\big), \Theta c_U\big((A,f)/\approx\big)\big)\\
&= \big(f, \Theta S_{\mathfrak{A}}(A), \Theta c_U\big((A,f)/\approx\big)\big),\\
\nu &= \Theta' s(\iota_A, \mathrm{id}_{|A|}, f) = \big(f, \Theta' S_{\mathfrak{A}}(A), \Theta' c_U\big((A,f)/\approx\big)\big).
\end{aligned}$$

Since $\Theta' S_{\mathfrak{A}} = \Xi = \Theta S_{\mathfrak{A}}$, the composition

$$\nu\mu^{-1}\colon \Theta c_U\big((A,f)/\approx\big) \to \Theta' c_U\big((A,f)/\approx\big)$$

is defined and is an isomorphism in $\mathfrak{X}$. Moreover,

$$|\nu\mu^{-1}| = \mathrm{id}_{|c_U((A,f)/\approx)|} = \mathrm{id}_{\mathrm{Im} f}.$$

The category $\mathfrak{X}$ has the property of unicity, hence

$$\Theta' c_U\big((A,f)/\approx\big) = \Theta c_U\big((A,f)/\approx\big).$$

Since Θ' and Θ are $\square$-commuting, it follows that $\Theta' = \Theta$.

13.2. Corollary. *If $\mathfrak{A}$ is U-tu-concrete, then $\mathfrak{A}_{tu}$ is $\square$-isomorphic with $\mathfrak{A}$.*

13.3. Examples. (A) Let $\mathfrak{A} = \mathrm{id}(U\text{-Ens})$ (cf. 10.2.(B)). The objects of $\mathfrak{A}_t$ are pairs (A,f), where $A \in U$, $f \in U$, and f is an injective function on A. The morphisms of $\mathfrak{A}_t$ are triples $\big(h,(A,f),(A,g)\big)$, where h is a function from $\mathrm{Im} f$ to $\mathrm{Im} g$ such that $h \circ f = g$. If $A, B \in U$ and $A \neq B$, then $\langle (A,f),(B,g)\rangle_{\mathfrak{A}_t} = \emptyset$. Obviously

$$(A,f) \underset{\mathfrak{A}_t}{\approx} (B,g) \Leftrightarrow (A,f) = (B,g),$$

and consequently $\mathfrak{A}_{tu} = \mathfrak{A}_t$. This example shows that if $\mathfrak{A}$ is U-regular, then $\mathfrak{A}_{tu}$ need not be U-regular.

(B) Let F be the field of real or complex numbers. For any set T let $F^{[T]}$ be the vector space of all functions x from T to F such that $\{t\colon x(t) \neq 0\}$ is finite, with the usual operations of addition and multiplication by scalars.

Let U-Vect be the category of vector spaces over F belonging to U and F-linear transformations. Let $\mathfrak{A}$ be the full subcategory of U-Vect whose objects are spaces $F^{[T]}$ with $T \in U$.

It may be shown that $\mathfrak{A}_{tu}$ is $\square$-isomorphic with U-Vect.

14. Structures and concrete categories

The following notions are due to Ehresmann [1957] (cf. Bucur and Deleanu [1968], p. 84, Kučera and Pultr [1972]).

By a *species of structure in U* we mean a covariant functor

$$\Theta\colon \mathrm{iso}(U\text{-Ens}) \to U\text{-Ens}.$$

If $M \in U$ and $s \in \Theta(M)$, then s is said to be a *structure of species Θ* on the set M.

Let Θ be a species of structure in U and let $\mathfrak{A}$ be a category. We shall say that $\mathfrak{A}$ is a *category of species Θ* iff the following conditions are

satisfied:

(i) $\mathfrak{A}$ is a U-concrete category,

(ii) $\mathfrak{A}^{\circ} = \{(S, s)\colon S \in U \text{ and } s \in \Theta(S)\}$,

(iii) the underlying set of (S, s) is S,

(iv) if $f\colon S \to S'$ is a bijection in U-Ens and $s \in \Theta(S)$, then the triple $\big(f, (S, s), (S', \Theta(f)(s))\big)$ belongs to $\mathfrak{A}^{\mathrm{m}}$.

Note that a category of species Θ is not determined uniquely by Θ.

14.1. PROPOSITION. *Every category $\mathfrak{A}$ of species Θ in U is U-rt-concrete.*

Proof. For every $S \in U$ we have

$$\mathrm{Ob}(S) = \{(S, s)\colon s \in \Theta(S)\} \in U,$$

i.e. $\mathfrak{A}$ is U-regular.

Let $B = (S, s) \in \mathfrak{A}^{\circ}$ and let $f\colon S \to S'$ be a bijection. By (iv) the triple (f, B, C), where $C = \big(S', \Theta(f)(s)\big)$, is a morphism in $\mathfrak{A}^{\mathrm{m}}$. Since $\Theta(f^{-1}) = \big(\Theta(f)\big)^{-1}$, it follows from (iv) that $(f^{-1}, C, B) \in \mathfrak{A}^{\mathrm{m}}$. Hence $\mathfrak{A}$ has the property of U-transfer.

14.2. Let us note that a category of some species Θ in U need not be U-u-concrete, as the example of the category U-Ban shows.

It is easy to see that a category $\mathfrak{A}$ of species Θ in U is an *U-u-category* iff for every set $S \in U$ the following condition is satisfied:

$$\big(\mathrm{id}_S, (S, s), (S, s')\big) \in \mathfrak{A}^{\mathrm{m}} \Rightarrow s = s'.$$

14.3. PROPOSITION. *For every U-rtu-concrete category $\mathfrak{A}$ there exists a species of structure $\Theta_{\mathfrak{A}}$ in U such that $\mathfrak{A}$ is $\square$-isomorphic with a category of species $\Theta_{\mathfrak{A}}$.*

Proof. For every $M \in U$ let

$$\Theta_{\mathfrak{A}}(M) = \mathrm{Ob}(M).$$

It follows from the regularity of $\mathfrak{A}$ that $\Theta_{\mathfrak{A}}(M) \in U$. Let $M, N \in U$ and let $f\colon M \to N$ be a bijection. Since $\mathfrak{A}$ has the property of unique transfer, for every $A \in \mathfrak{A}^{\circ}$ such that $|A| = M$ there is a unique $A_f \in \mathfrak{A}^{\circ}$ such that $A_f| = N$ and (f, A, A_f) is an isomorphism in $\mathfrak{A}$. Let

$$\Theta_{\mathfrak{A}}(f)\colon \Theta_{\mathfrak{A}}(M) \to \Theta_{\mathfrak{A}}(N)$$

be the transformation defined by the assignment $A \mapsto A_f$. It is easy to see that $f \mapsto \Theta_{\mathfrak{A}}(f)$ defines a functor from iso(U-Ens) into U-Ens. In fact, if $f = \iota_M$, then $A_f = A$ for all $A \in \Theta_{\mathfrak{A}}(M)$, and $\Theta_{\mathfrak{A}}(f) = \iota_{\Theta_{\mathfrak{A}}(M)}$. If $M, N, S \in U$, and $f\colon M \to N$, $g\colon N \to S$ are bijections, then for every $A \in \Theta_{\mathfrak{A}}(M)$

the triples (f, A, A_f) and $\big(g, A_f, (A_f)_g\big)$ are isomorphisms in $\mathfrak{A}$. Hence

$$\big(g\circ f, A, (A_f)_g\big) = \big(g, A_f, (A_f)_g\big)(f, A, A_f)$$

is an isomorphism in $\mathfrak{A}$. Since $(g\circ f, A, A_{(g\circ f)})$ is also an isomorphism in $\mathfrak{A}$, and $|(A_f)_g| = |A_{(g\circ f)}|$, it follows from the property of unicity that $(A_f)_g = A_{(g\circ f)}$. Consequently, $\Theta_{\mathfrak{A}}(g\circ f) = \Theta_{\mathfrak{A}}(g)\Theta_{\mathfrak{A}}(f)$, which means that $\Theta_{\mathfrak{A}}$ is a covariant functor, i.e. a species of structure in U.

Let $\mathfrak{B}$ be a U-concrete category defined as follows: The class $\mathfrak{B}^{\mathrm{o}}$ is the class of all pairs $(|A|, A)$, where $A \in \mathfrak{A}^{\mathrm{o}}$. The underlying set of $(|A|, A)$ is $|A|$. The class $\mathfrak{B}^{\mathrm{m}}$ is the class of all triples $\big(f, (|A|, A), (|B|, B)\big)$ such that $(f, A, B) \in \mathfrak{A}^{\mathrm{m}}$. It is easy to see that $\mathfrak{B}$ is a category of species $\Theta_{\mathfrak{A}}$ and that the functor $\Phi\colon \mathfrak{A}\to\mathfrak{B}$ defined by

$$\Phi(A) = (|A|, A),$$
$$\Phi\big((f, A, B)\big) = \big(f, (|A|, A), (|B|, B)\big)$$

is an $\square$-isomorphism.

14.4. We shall show that not every U-rt-concrete category is $\square$-isomorphic with a category of some species Θ. If a category $\mathfrak{A}$ is $\square$-isomorphic with a category of some species Θ, then obviously

$$\operatorname{card} M = \operatorname{card} N \Rightarrow \operatorname{card}\operatorname{Ob}(M) = \operatorname{card}\operatorname{Ob}(N) \tag{8}$$

for all $M, N \in U$.

Let $\mathfrak{A}$ be a concrete category defined as follows:

$$\mathfrak{A}^{\mathrm{o}} = \{(M, \emptyset)\colon M \in U \text{ and } \operatorname{card} M = 1\} \cup \big\{(\{\emptyset\}, \{\emptyset\})\big\};$$

the underlying set of (M, S) is M. The morphisms are all triples

$$\big(f, (M, S), (N, T)\big),$$

where $(M, S), (N, T) \in \mathfrak{A}^{\mathrm{o}}$ and $f\colon M\to N$ is the (unique) transformation from M to N. The category $\mathfrak{A}$ is U-rt-concrete, but does not satisfy condition (8). In fact, for $M = \{\emptyset\}$, $N = \big\{\{\emptyset\}\big\}$ we have $2 = \operatorname{card}\operatorname{Ob}(M) \neq \operatorname{card}\operatorname{Ob}(N) = 1$.

Chapter IV

The category of logical kits

15. Definitions and basic properties

The notion of a logical kit introduced by Z. Semadeni [1974] is motivated by some problems of classifying things according to their features and, in particular, by kits used in teaching logic and set theory at a primary school level. Logical kits are closely related to automata and to some mathematical models of classical linguistic structures.

The arguments applied in this chapter do not require changing a universe. For that reason we assume once for all that all considered sets are members of a fixed universe U. Accordingly, the categories U-**Ens** and U-**Eq** (cf. Section 4) will be denoted shortly by **Ens** and **Eq** respectively. The category of logical kits in U will be denoted by **Kt** (instead of U-**Kt**).

Z. Semadeni [1974] has shown that the canonical forgetful functor from **Kt** into **Ens** $\times$ **Ens** $\times$ **Ens** has a left adjoint. In Section 16 we shall construct adjoints of canonical forgetful functors from **Kt** into **Ens** $\times$ **Eq** and into **Ens** $\times$ **Ens**. In Sections 17 and 18 the constructions of coproducts and coequalizers in the category **Kt** are described.

15.1. Definition. The category **Kt** is defined as follows. The objects are *logical kits*, called shortly *kits*, i.e. quintuples

$$K = (K_0, K_1, K_2, f_K, p_K), \tag{1}$$

where K_i $(i = 0, 1, 2)$ are sets and

$$f_K\colon K_0 \times K_1 \to K_2, \quad p_K\colon K_2 \to K_1 \tag{2}$$

are functions such that

$$p_K f_K(x, a) = a \quad \text{for all } x \in K_0,\ a \in K_1, \tag{3}$$

i.e. such that the diagram

$$\begin{tikzcd} K_0 \times K_1 \arrow[rr, "f_K"] \arrow[dr, "\mathrm{pr}_2"'] & & K_2 \arrow[dl, "p_K"] \\ & K_1 & \end{tikzcd}$$

is commutative (pr_2 denotes the canonical projection on the second axis). The morphisms from (1) to a kit

$$L = (L_0, L_1, L_2, f_L, p_L)$$

are triples

$$(a, K, L),$$

where $a = (a_0, a_1, a_2)$ is a triple consisting of three functions $a_i\colon K \to L_i$ $(i = 0, 1, 2)$ such that

$$a_2\big(f_K(x, a)\big) = f_L\big(a_0(x), a_1(a)\big) \tag{4}$$

and

$$a_1\big(p_K(v)\big) = p_L\big(a_2(v)\big) \tag{5}$$

for all $x \in K_0$, $a \in K_1$, $v \in K_2$, i.e. such that the diagram

$$\begin{tikzcd} K_0 \times K_1 \arrow[rr, "f_K"] \arrow[dd, "a_0 \times a_1"'] \arrow[dr, "\mathrm{pr}_2"] & & K_2 \arrow[dl, "p_K"'] \arrow[dd, "a_2"] \\ & K_1 \arrow[dd, "a_1"] & \\ L_0 \times L_1 \arrow[rr, "f_L"] \arrow[dr, "\mathrm{pr}_2"'] & & L_2 \arrow[dl, "p_L"] \\ & L_1 & \end{tikzcd}$$

is commutative. The composition of morphisms is defined in an obvious way.

15.2. Let us note that there is an empty kit K with $K_0 = K_1 = K_2 = f_K = p_K = \emptyset$.

If K is a kit such that $K_0 \times K_1 \neq \emptyset$, then the map $p_K\colon K_2 \to K_1$ is a surjection. In fact, if $a \in K_1$, then there exists an x in K_0, and we have $a = p_K(v)$, where $v = f_K(x, a)$.

If K is a kit such that $K_0 \times K_1 = \emptyset$, then either $K_0 = \emptyset$ or $K_1 = \emptyset$. If $K_1 = \emptyset$, then $p_K = K_2 = \emptyset$ and the map $p_K\colon K_2 \to K_1$ is still a surjection. If $K_0 = \emptyset$ and $K_1 \neq \emptyset$, then the map $p_K\colon K_2 \to K_1$ need not be a surjection.

If (a, K, L) is a morphism in **Kt** and $K_0 \neq \emptyset$, then condition (5) shows that a_1 is completely determined by a_2; if, in addition, the map $f_K\colon K_0 \times K_1 \to K_2$ is a surjection, then condition (4) shows that a_2 is completely determined by a_0 and a_1.

15.3. In the definition of a kit K, the set K_1 and the function p_K are of secondary interest; what we really need is the fact that K_1 and p_K determine an equivalence relation on K_2. This relation is defined by

$$R_K = \{(v, v') \in K_2 \times K_2\colon\ p_K(v) = p_K(v')\}. \tag{6}$$

By a *regular kit* we shall mean a kit K satisfying the conditions $K_1 = K_2/R_K$ and $p_K(v) = v/R_K$ for $v \in K_2$.

It is easy to see that every kit K such that $K_0 \neq \emptyset$ is **Kt**-isomorphic with a regular kit $\bar{K}$ defined in the following way:

$$\bar{K}_0 = K_0, \quad \bar{K}_1 = K_2/R_K, \quad \bar{K}_2 = K_2,$$

$$f_{\bar{K}}(x, v/R_K) = f_K\big(x, p_K(v)\big), \quad p_{\bar{K}}(v) = v/R_K \quad (x \in K_0,\ v \in K_2).$$

The isomorphism $\eta\colon \bar{K} \to K$ is defined by (cf. Section 4)

$$\eta_0 = \mathrm{id}_{K_0}, \quad \eta_1 = Q\big(p_K, R_K, \mathscr{D}(K_1)\big), \quad \eta_2 = \mathrm{id}_{K_2}.$$

15.4. Let K and L be arbitrary kits and let (a, K, L) be an arbitrary morphism in **Kt**. Then (a_2, R_K, R_L) is a morphism in **Eq**; in fact, by (6) and (5) we have:

$$(v, w) \in R_K \Rightarrow p_K(v) = p_K(w) \Rightarrow p_L a_2(v) = p_L a_2(w)$$
$$\Rightarrow \big(a_2(v), a_2(w)\big) \in R_L.$$

Hence we have the forgetful functor

$$\square_{\mathbf{Kt}}^{\mathbf{Ens} \times \mathbf{Eq}}\colon \mathbf{Kt} \to \mathbf{Ens} \times \mathbf{Eq} \tag{7}$$

defined as follows: if K is a kit, then

$$\square_{\mathbf{Kt}}^{\mathbf{Ens} \times \mathbf{Eq}}(K) = (K_0, R_K),$$

if (a, K, L) is a morphism of kits, then

$$\square_{\mathbf{Kt}}^{\mathbf{Ens} \times \mathbf{Eq}}(a, K, L) = \big((a_0, K_0, L_0), (a_2, R_K, R_L)\big).$$

The full subcategory of regular kits can be described in the following alternative form:

A regular kit K can be identified with a triple (K_0, R_K, f_K), where K_0 is a set, R_K is an equivalence relation, and

$$f_K\colon K_0 \times Q(R_K) \to \square(R_K)$$

is a function such that $f(x, a) \in a$ for all $x \in K_0$, $a \in Q(R_K)$ (then $K_1 = Q(R_K)$, $K_2 = \square(R_K)$, and p_K is the canonical surjection).

If $K = (K_0, R_K, f_K)$ and $L = (L_0, R_L, f_L)$ are regular kits, then a morphism from K to L can be identified with a triple

$$\big((a_0, a_2), K, L\big),$$

where

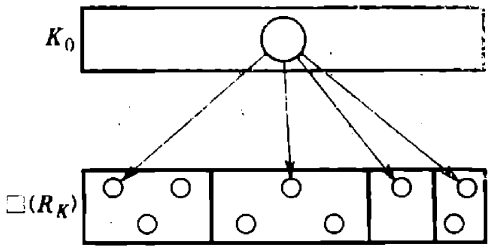
$$a_0\colon K_0 \to L_0 \quad \text{and} \quad a_2\colon \square(R_K) \to \square(R_L)$$

are functions such that (a_2, R_K, R_L) is a morphism in **Eq** and the diagram

$$\begin{tikzcd}
K_0 \times Q(R_K) \arrow[r, "f_K"] \arrow[d, "a_0 \times Q(a_2)"'] & \square(R_K) \arrow[d, "a_2"] \\
L_0 \times Q(R_L) \arrow[r, "f_L"'] & \square(R_L)
\end{tikzcd}$$

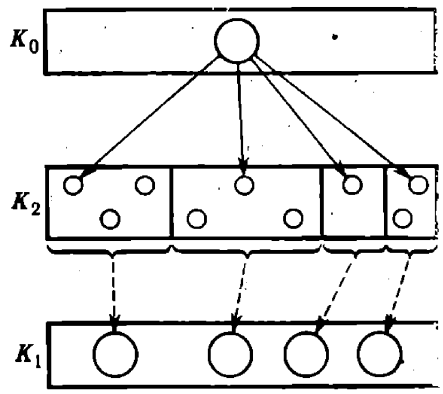
is commutative (then $a_1 = Q(a_2)$).

A regular kit (K_0, R_K, f_K) may be visualized by the following scheme:



The elements of the sets K_0 and $\square(R_K)$ are represented by circles in the upper and the lower rectangle, respectively. The relation R_K is represented by the partition of the set $\square(R_K)$. If $x \in K_0$, $a \in \square(R_K)/R_K$, and $v \in a$, then there is an arrow from the circle representing x to the circle representing v if and only if $v = f_K(x, a)$.

An arbitrary kit K may be visualized by the scheme:



If $q \in \square(R_K)/R_K$ and $a = p_K(v)$ for $v \in q$, then there is a dashed arrow from the rectangle representing q to the circle representing a.

16. Adjoints of some forgetful functors

We shall now construct a left adjoint

$$F^{\mathbf{Kt}}_{\mathbf{Ens}\times\mathbf{Eq}}\colon \mathbf{Ens}\times\mathbf{Eq}\to\mathbf{Kt} \tag{8}$$

of the functor (7).

Let S be a set and let R be an equivalence relation. By $F(S, R)$ we shall denote the kit defined as follows:

$$F(S, R)_0 = S, \quad F(S, R)_1 = Q(R),$$
$$F(S, R)_2 = \square(R) + [S\times Q(R)].$$

(Recall that by $+$ we denote the disjoint sum; to avoid cumbersome notation we shall identify considered sets with their images in a disjoint sum.)

The function $f_{F(S,R)}\colon S\times Q(R)\to\square(R)+[S\times Q(R)]$ is the canonical injection into the second component, i.e.

$$f_{F(S,R)}(s, q) = (s, q) \quad \text{for } s \text{ is } S \text{ and } q \text{ in } Q(R). \tag{9}$$

The function $p_{F(S,R)}\colon \square(R)+[S\times Q(R)]\to Q(R)$ is defined by the conditions: $p_{F(S,R)}|\square(R)$ is the canonical surjection, $p_{F(S,R)}|S\times Q(R) = \mathrm{pr}_2$; in other words,

$$p_{F(S,R)}(v) = \begin{cases} v/R & \text{for } v \text{ in } \square(R), \\ q & \text{for } v = (s, q) \text{ in } S\times Q(R). \end{cases} \tag{10}$$

Let $\tau_0 = \mathrm{id}_S$ and let τ_2 be the canonical injection of $\square(R)$ into $F(S, R)_2$.

16.1. Proposition. *The kit $F(S, R)$ together with the maps*

$$\tau_0\colon S\to F(S, R)_0, \quad \tau_2\colon \square(R)\to F(S, R)_2 \tag{11}$$

has the following property:

For every kit K, every map $\xi_0\colon S\to K_0$, and every morphism $\xi_2\colon R\to R_K$ in Eq, *there exists a unique morphism $\vartheta\colon F(S, R)\to K$ in* Kt *such that*

$$\vartheta_0\tau_0 = \xi_0 \quad \textit{and} \quad \vartheta_2\tau_2 = \xi_2. \tag{12}$$

In other words, $F(S, R)$ can be regarded as a free kit generated by the set S and the equivalence relation R.

Proof. Let $\vartheta_i\colon F(S, R)_i\to K_i$ $(i = 0, 1, 2)$ be functions defined in the following way:

$$\vartheta_0 = \xi_0, \tag{13}$$

$$\vartheta_1(q) = p_K\xi_2(v) \quad \text{for } q \text{ in } Q(R) \text{ and } v \text{ in } q. \tag{14}$$

(If $v, v' \in q$, then $(\xi_2(v), \xi_2(v')) \in R_K$ and $p_K \xi_2(v) = p_K \xi_2(v')$, hence $\vartheta_1(q)$ does not depend on the choice of an element v in q.)

(15) $$\vartheta_2 | \square(R) = \xi_2,$$

(16) $$\vartheta_2(s, q) = f_K(\xi_0(s), \vartheta_1(q)) \quad \text{for } (s, q) \text{ in } S \times Q(R).$$

We shall show that the triple $(\vartheta, F(S, R), K)$, where $\vartheta = (\vartheta_0, \vartheta_1, \vartheta_2)$, is a morphism in **Kt**, i.e. that conditions (4) and (5) are satisfied with K replaced by $F(S, R)$, L replaced by K, and a replaced by ϑ. Since (4) follows from (9), (13), and (16), it suffices to show that

(17) $$\vartheta_1 p_{F(S,R)}(v) = p_K \vartheta_2(v)$$

for every v in $F(S, R)_2$. If $v \in \square(R)$, then by (10), (14), and (15) we have

$$\vartheta_1 p_{F(S,R)}(v) = \vartheta_1(v/R) = p_K \xi_2(v) = p_K \vartheta_2(v).$$

If $v \in S \times Q(R)$, i.e. $v = (s, q)$, then by (3), (10), and (16)

$$\vartheta_1 p_{F(S,R)}(s, q) = \vartheta_1(q) = p_K f_K(\vartheta_0(s), \vartheta_1(q)) = p_K \vartheta_2(s, q).$$

Hence (17) is satisfied for every v in $F(S, R)_2$.

We shall now show that ϑ is determined by conditions (12) uniquely. Suppose that ζ: $F(S, R) \to K$ is another morphism in **Kt** such that

$$\zeta_0 \tau_0 = \xi_0 \quad \text{and} \quad \zeta_2 \tau_2 = \xi_2.$$

Then obviously

(18) $$\zeta_0 = \xi_0 = \vartheta_0,$$

and

(19) $$\zeta_2 | \square(R) = \xi_2 | \square(R) = \vartheta_2 | \square(R).$$

Let $q \in Q(R)$, i.e. $q = v/R$ for some v in $\square(R)$. By (5), (10), (14), and (19) we have

$$\zeta_1(q) = \zeta_1 p_{F(S,R)}(v) = p_K \zeta_2(v) = p_K \vartheta_2(v) = \vartheta_1(q),$$

i.e.

$$\zeta_1 = \vartheta_1.$$

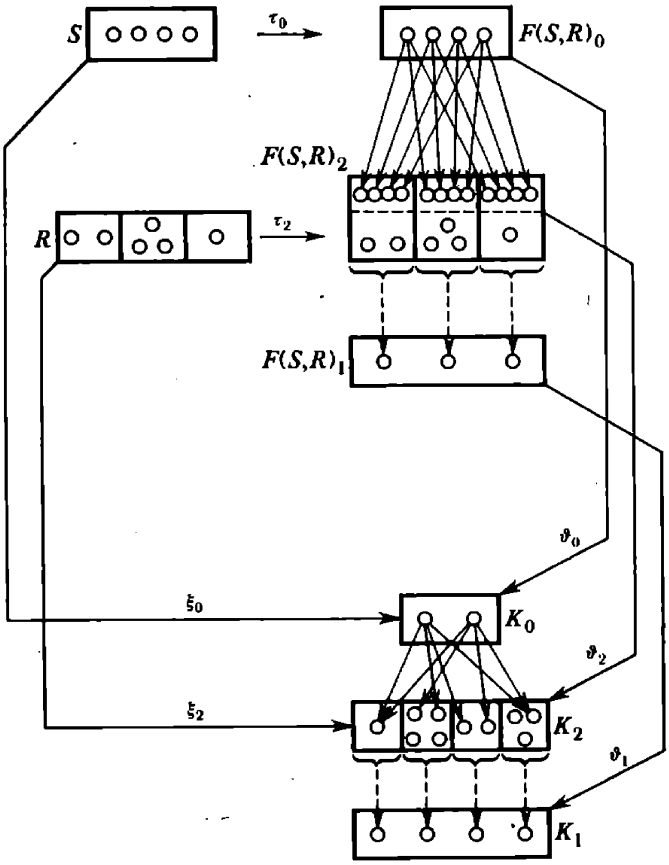
Let $(s, q) \in S \times Q(R)$. By (4), (9), (18) we have

$$\begin{aligned} \zeta_2(s, q) &= \zeta_2 f_{F(S,R)}(s, q) = f_K(\zeta_0(s), \zeta_1(q)) \\ &= f_K(\xi_0(s), \vartheta_1(q)) = \vartheta_2(s, q). \end{aligned}$$

This completes the proof.

The construction of the kit $F(S, R)$ and the proof of Proposition 16.1 may be illustrated by the scheme on the next page.

By Proposition 16.1 functor (8) with the object transformation $F^{\mathbf{Kt}}_{\mathbf{Ens} \times \mathbf{Eq}}(S, R) = F(S, R)$ can be defined in a standard way. This



functor is a left adjoint of functor (7). The maps (11) yield a natural canonical transformation

$$I_{\mathbf{Ens}\times\mathbf{Eq}} \to \square_{\mathbf{Kt}}^{\mathbf{Ens}\times\mathbf{Eq}} F_{\mathbf{Ens}\times\mathbf{Eq}}^{\mathbf{Kt}}.$$

Z. Semadeni [1974] has shown that the forgetful functor

$$\square_{\mathbf{Kt}}^{\mathbf{Ens}\times\mathbf{Ens}}\colon \mathbf{Kt}\to\mathbf{Ens}\times\mathbf{Ens} \tag{20}$$

with the object transformation $\square_{\mathbf{Kt}}^{\mathbf{Ens}\times\mathbf{Ens}}(K) = (K_0, K_1)$ has the left adjoint $F_{\mathbf{Ens}\times\mathbf{Ens}}^{\mathbf{Kt}}\colon \mathbf{Ens}\times\mathbf{Ens}\to\mathbf{Kt}$ with the object transformation

$$F_{\mathbf{Ens}\times\mathbf{Ens}}^{\mathbf{Kt}}(S_0, S_1) = (S_0, S_1, S_0\times S_1, \mathrm{id}_{S_0\times S_1}, \mathrm{pr}_2).$$

16.2. Proposition. *Functor* (20) *has a right adjoint*

$$G_{\mathbf{Ens}\times\mathbf{Ens}}^{\mathbf{Kt}}\colon \mathbf{Ens}\times\mathbf{Ens}\to\mathbf{Kt} \tag{21}$$

with the object transformation

$$G_{\mathbf{Ens}\times\mathbf{Ens}}^{\mathbf{Kt}}(S_0, S_1) = (S_0, S_1, S_1, \mathrm{pr}_2, \mathrm{id}_{S_1}). \tag{22}$$

Proof. The quintuple (22) is obviously a kit; we shall denote it by $G(S_0, S_1)$. It is easy to verify that for every kit K and every pair of

maps

$$\xi_0\colon K_0 \to S_0, \qquad \xi_1\colon K_1 \to S_1$$

the triple $\big((\xi_0, \xi_1, \xi_1 p_K), K, G(S_0, S_1)\big)$ is the unique morphism ϑ from K to $G(S_0, S_1)$ in **Kt** satisfying the conditions $\vartheta_0 = \xi_0$ and $\vartheta_1 = \xi_1$. Hence transformation (22) gives rise to the right adjoint of functor (20).

17. Coproducts of logical kits

17.1. Let $(K^t)_{t\in T}$ be an indexed family of kits. It is easy to see that the kit B defined as follows:

$$B_i = \mathop{\mathsf{P}}_{t\in T} K_i^t \qquad (i = 0, 1, 2),$$

$$f_B\big((x_t)_{t\in T}, (a_t)_{t\in T}\big) = \big(f_{K^t}(x_t, a_t)\big)_{t\in T} \qquad \text{for } x_t \text{ in } K_0^t,\ a_t \text{ in } K_1^t,$$

$$p_B\big((v_t)_{t\in T}\big) = \big(p_{K^t}(v_t)\big)_{t\in T} \qquad \text{for } v_t \text{ in } K_2^t,$$

together with obvious projections $B \to K^t$, is a product of the family $(K^t)_{t\in T}$ in **Kt**.

17.2. The construction of a coproduct is more complicated and is somewhat similar to the construction of coproducts in categories of automata (cf. Wiweger [1973]). From Proposition 16.2 it follows that "0-coordinates" and "1-coordinates" of coproducts in **Kt** are constructed in the same way as coproducts in the category **Ens**.

Let C be a kit defined as follows:

$$C_0 = \mathop{\mathsf{S}}_{t\in T} K_0^t, \qquad C_1 = \mathop{\mathsf{S}}_{t\in T} K_1^t,$$

$$C_2 = \mathop{\mathsf{S}}_{t\in T} K_2^t + \mathop{\mathsf{S}}_{\substack{t,u\in T\\ t\neq u}} (K_0^t \times K_1^u),$$

$$f_C\colon \Big(\mathop{\mathsf{S}}_{t\in T} K_0^t\Big) \times \Big(\mathop{\mathsf{S}}_{t\in T} K_1^t\Big) \to \mathop{\mathsf{S}}_{t\in T} K_2^t + \mathop{\mathsf{S}}_{\substack{t,u\in T\\ t\neq u}} (K_0^t \times K_1^u)$$

is defined by

$$\begin{aligned} f_C|(K_0^t \times K_1^t) &= f_{K^t},\\ f_C|(K_0^t \times K_1^u) &= \mathrm{id}_{K_0^t \times K_1^u} \qquad \text{for } t \neq u, \end{aligned} \tag{23}$$

$$p_C\colon \mathop{\mathsf{S}}_{t\in T} K_2^t + \mathop{\mathsf{S}}_{\substack{t,u\in T\\ t\neq u}} (K_0^t \times K_1^u) \to \mathop{\mathsf{S}}_{t\in T} K_1^t$$

is defined by

$$\begin{aligned} p_C|K_2^t &= p_{K^t},\\ p_C|(K_0^t \times K_1^u) &= \mathrm{pr}_2 \qquad \text{for } t \neq u. \end{aligned} \tag{24}$$

It is easy to see that C is a kit; in fact, if $x \in K_0^t$ and $a \in K_1^t$, then

$p_C f_C(x, a) = p_{K^t} f_{K^t}(x, a) = a$; if $x \in K_0^t$, $a \in K_1^u$, and $t \neq u$, then $p_C f_C(x, a) = p_C(x, a) = \mathrm{pr}_2(x, a) = a$.

Let

$$\sigma_i^t\colon K_i^t \to C_i \quad (t \in T;\ i = 0, 1, 2) \tag{25}$$

be the canonical injections. We shall show that the triple

$$(\sigma^t, K^t, C), \quad \text{where } \sigma^t = (\sigma_0^t, \sigma_1^t, \sigma_2^t),$$

is a morphism in **Kt**. In fact, for all x in K_0^t, a in K_1^t, and v in K_2^t we have

$$\sigma_2^t f_{K^t}(x, a) = f_{K^t}(x, a) = f_C(x, a) = f_C\big(\sigma_0^t(x), \sigma_1^t(a)\big),$$
$$\sigma_1^t p_{K^t}(v) = p_{K^t}(v) = p_C(v) = p_C \sigma_2^t(v).$$

17.3. Proposition. *The kit C together with the morphisms*

$$\sigma^t\colon K^t \to C \quad (t \in T)$$

is a coproduct of the family $(K^t)_{t \in T}$ in **Kt**.

Proof. Let K be a kit and let

$$\xi^t\colon K^t \to K \quad (t \in T)$$

be a family of morphisms in **Kt**. We shall show that there exists a unique morphism $\vartheta\colon C \to K$ such that

$$\vartheta \sigma^t = \xi^t \quad \text{for every } t \text{ in } T. \tag{26}$$

Let $\vartheta_i\colon C_i \to K_i$ $(i = 0, 1, 2)$ be defined as follows:

$$\vartheta_i | K_i^t = \xi_i^t \quad \text{for } t \text{ in } T \text{ and } i = 0, 1, 2, \tag{27}$$

$$\vartheta_2(v) = f_K\big(\xi_0^t(x_t), \xi_1^u(a_u)\big) \quad \text{for } v = (x_t, a_u) \text{ in } K_0^t \times K_1^u \ (t \neq u). \tag{28}$$

We shall show that the triple

$$(\vartheta, C, K), \quad \text{where } \vartheta = (\vartheta_0, \vartheta_1, \vartheta_2),$$

is a morphism in **Kt**.

Let $x \in C_0$ and $a \in C_1$, i.e. $x \in K_0^t$ and $a \in K_1^u$ for some t, u in T. If $t = u$, then by (4) (with a replaced by ξ^t), (23), and (27) we get

$$\vartheta_2 f_C(x, a) = \vartheta_2 f_{K^t}(x, a) = \xi_2^t f_{K^t}(x, a) = f_K\big(\xi_0^t(x), \xi_1^t(a)\big)$$
$$= f_K\big(\vartheta_0(x), \vartheta_1(a)\big).$$

If $t \neq u$, then by (23), (27), and (28)

$$\vartheta_2 f_C(x, a) = \vartheta_2(x, a) = f_K\big(\xi_0^t(x), \xi_1^u(a)\big) = f_K\big(\vartheta_0(x), \vartheta_1(a)\big).$$

Hence (ϑ, C, K) satisfies condition (4).

Let $v \in C_2$. If $v \in K_2^t$ for some t in T, then by (5) (with a replaced by ξ^t),

(24), and (27) we get

$$\vartheta_1 p_C(v) = \vartheta_1 p_{K^t}(v) = \xi_1^t p_{K^t}(v) = p_K \xi_2^t(v) = p_K \vartheta_2(v).$$

If $v \in K_0^t \times K_1^u (t, u \in T, t \neq u)$, i.e. $v = (x, a)$, then by (3), (24), (27), and (28)

$$\vartheta_1 p_C(v) = \vartheta_1(a) = \xi_1^u(a) = p_K f_K\big(\xi_0^t(x), \xi_1^u(a)\big) = p_K \vartheta_2(v).$$

Hence (ϑ, C, K) satisfies condition (5).

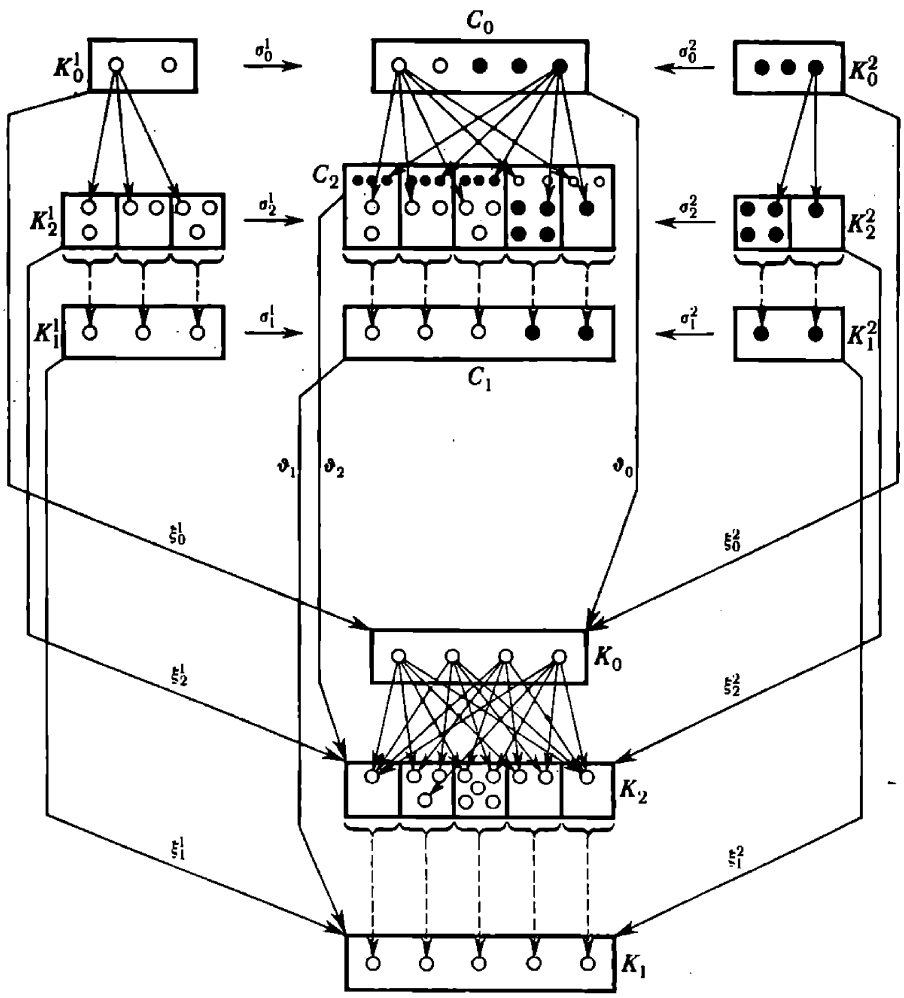
It follows from (27) that the morphism (ϑ, C, K) satisfies conditions (26). Let (ζ, C, K) be another morphism in **Kt** satisfying the conditions $\zeta\sigma^t = \xi^t$ for t in T. Then

$$\zeta_i | K_i^t = \xi_i^t = \vartheta_i | K_i^t \quad \text{for } t \text{ in } T \text{ and } i = 0, 1, 2.$$

It remains to show that ζ_2 and ϑ_2 agree on the sets

$$K_0^t \times K_1^u \quad (t, u \in T, t \neq u).$$

Let $(x_t, a_u) \in K_0^t \times K_1^u$. By (4) (with K replaced by C, L replaced by K,



and a replaced by ζ), and (23) we have

$$\begin{aligned}\zeta_2(x_t, a_u) &= \zeta_2 f_C(x_t, a_u) = f_K\big(\zeta_0(x_t), \zeta_1(a_u)\big)\\ &= f_K\big(\xi_0^t(x_t), \xi_1^u(a_u)\big) = \vartheta_2(x_t, a_u).\end{aligned}$$

Hence $\zeta = \vartheta$, and this completes the proof.

The construction of the coproduct and the proof of Proposition 17.3 are visualized by the scheme on the page 60 in the particular case $T = \{1, 2\}$.

18. Coequalizers in the category of logical kits

18.1. Let

$$a\colon K \to L, \quad \beta\colon K \to L \tag{29}$$

be morphisms in **Kt**. It is easy to see that the kit K' defined as follows:

$$\begin{aligned}K'_i &= \{k \in K_i\colon a_i(k) = \beta_i(k)\} \quad \text{for } i = 0, 1, 2,\\ f_{K'} &= f_K|(K'_0 \times K'_1),\\ p_{K'} &= p_K|K'_2,\end{aligned}$$

together with the obvious morphism $\varepsilon\colon K' \to K$, is an equalizer of the morphisms (29) in the category **Kt**.

18.2. We shall now describe the construction of a coequalizer of the morphisms (29). It follows from Proposition 16.2 that "0-coordinates" and "1-coordinates" of coequalizers in **Kt** are constructed in the same way as coequalizers in the category **Ens**.

Let

$$H_i = \big\{\big(a_i(k), \beta_i(k)\big)\colon k \in K_i\big\} \quad \text{for } i = 0, 1, 2,$$

and let E_i $(i = 0, 1)$ be the smallest equivalence relation containing the set $\mathscr{D}(L_i) \cup H_i$. Let

$$H'_2 = \big\{\big(f_L(y, b), f_L(y', b')\big)\colon (y, y') \in E_0 \text{ and } (b, b') \in E_1\big\},$$

and let E_2 be the smallest equivalence relation containing the set $\mathscr{D}(L_2) \cup \cup H_2 \cup H'_2$.

Thus, $\square(E_i) = L_i$ for $i = 0, 1, 2$.

18.3. Lemma. *If* $(w, w') \in E_2$, *then* $\big(p_L(w), p_L(w')\big) \in E_1$.

Proof. It suffices to show that the condition $(w, w') \in H_2 \cup H'_2$ implies $\big(p_L(w), p_L(w')\big) \in E_1$.

If $(w, w') \in H_2$, then there exists v in K_2 such that $w = a_2(v)$, $w' = \beta_2(v)$; hence, by (5), $a_1 p_K(v) = p_L a_2(v) = p_L(w)$, and $\beta_1 p_K(v) = p_L \beta_2(v) = p_L(w')$; this means that $\big(p_L(w), p_L(w')\big) \in H_1 \subset E_1$.

If $(w, w') \in H_2'$, then there exist (y, y') in E_0 and (b, b') in L_1 such that $w = f_L(y, b)$, $w' = f_L(y', b)$; consequently, $(p_L(w), p_L(w')) = (b, b') \in E_1$.

18.4. Let

$$M_i = L_i/E_i \quad \text{for } i = 0, 1, 2,$$

and let f_M: $M_0 \times M_1 \to M_2$ be the function defined by

$$f_M(y/E_0, b/E_1) = f_L(y, b)/E_2 \quad \text{for } y \text{ in } L_0 \text{ and } b \text{ in } L_1. \tag{30}$$

It follows from the construction that the value of the function f_M does not depend on the choice of y and b in the equivalence classes y/E_0 and b/E_1 respectively.

Let p_M: $M_2 \to M_1$ be the function defined by

$$p_M(w/E_2) = p_L(w)/E_1 \quad \text{for } w \text{ in } L_2. \tag{31}$$

It follows from Lemma 18.3 that the value of the function p_M does not depend on the choice of an element w in the equivalence class w/E_2.

The quintuple $M = (M_0, M_1, M_2, f_M, p_M)$ is a kit. In fact, if $y \in L_0$ and $b \in L_1$, then by (30) and (31)

$$p_M f_M(y/E_0, b/E_1) = p_M(f_L(y, b)/E_2) = (p_L f_L(y, b))/E_1 = b/E_1.$$

Let ϱ_i: $L_i \to M_i$ $(i = 0, 1, 2)$ be canonical surjections. It is easy to verify that the triple (ϱ, L, M), where $\varrho = (\varrho_0, \varrho_1, \varrho_2)$, is a morphism in **Kt**. In fact, for all y in L_0 and b in L_1 we have

$$\varrho_2 f_L(y, b) = f_L(y, b)/E_2 = f_M(y/E_0, b/E_1) = f_M(\varrho_0(y), \varrho_1(b)),$$

i.e. condition (4) (with letters changed) is satisfied; moreover, for every w in L_2

$$\varrho_1 p_L(w) = p_L(w)/E_1 = p_M(w/E_2) = p_M \varrho_2(w),$$

i.e. condition (5) is also satisfied.

18.5. Proposition. *The kit M together with the morphism ϱ: $L \to M$ is a coequalizer of the morphisms* (29) *in* **Kt**.

Proof. It is obvious that $\varrho\alpha = \varrho\beta$. Suppose that X is a kit and ξ: $L \to X$ is a morphism in **Kt** such that $\xi\alpha = \xi\beta$. Let ϑ_i: $M_i \to X_i$ $(i = 0, 1, 2)$ be functions defined by

$$\vartheta_i(z/E_i) = \xi_i(z) \quad \text{for } z \text{ in } L_i \text{ and } i = 0, 1, 2. \tag{32}$$

The functions ϑ_i are well defined, i.e. $\xi_i(z)$ does not depend on the choice of z in the equivalence class z/E_i. We shall prove this fact for $i = 2$ (the proof for $i = 0, 1$ is straightforward). It suffices to show that each of the conditions $(w, w') \in H_2$, $(w, w') \in H_2'$ implies $\xi_2(w) = \xi_2(w')$. If $(w', w) \in H_2$, then $w = \alpha_2(v)$, $w' = \beta_2(v)$ for some v in K_2, and $\xi_2(w) = \xi_2\alpha_2(v) = \xi_2\beta_2(v) = \xi_2(w')$. If $(w, w') \in H_2'$, then $w = f_L(y, b)$, $w' = f_L(y', b')$ for

(y, y') in E_0 and (b, b') in E_1. The conditions $(y, y') \in E_0$ and $(b, b') \in E_1$ imply $\xi_0(y) = \xi_0(y')$ and $\xi_1(b) = \xi_1(b')$. Consequently, $\xi_2(w) = \xi_2 f_L(y, b) = f_L(\xi_0(y), \xi_1(b)) = f_L(\xi_0(y'), \xi_1(b')) = \xi_2 f_L(y', b') = \xi_2(w')$.

We shall now show that the triple (ϑ, M, X), where $\vartheta = (\vartheta_0, \vartheta_1, \vartheta_2)$, is a morphism in **Kt**. By (30) and (32),

$$\vartheta_2 f_M(y/E_0, b/E_1) = \vartheta_2(f_L(y, b)/E_2) = \xi_2 f_L(y, b)$$
$$= f_X(\xi_0(y), \xi_1(b)) = f_X(\vartheta_0(y|E_0), \vartheta_1(b|E\ )),$$

i.e. (ϑ, M, X) satisfies condition (4). By (31) and (32),

$$\vartheta_1 p_M(w/E_2) = \vartheta_1(p_L(w)/E_1) = \xi_1 p_L(w) = p_X \xi_2(w) = p_X \vartheta_2(w/E_2),$$

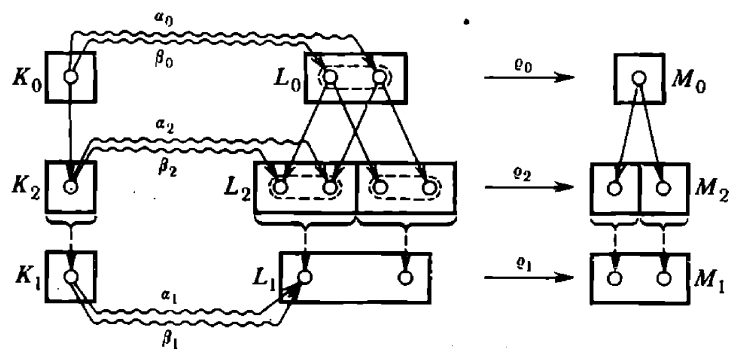
i.e. (ϑ, M, X) satisfies condition (5).

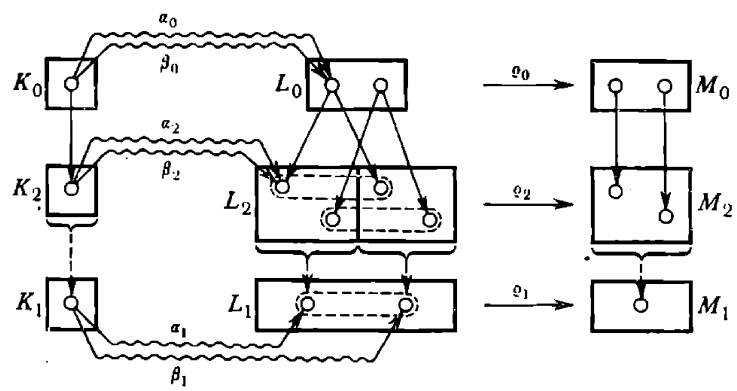
It follows from (32) that

$$\vartheta\varrho = \xi.$$

Since ϱ_i: $L_i \to M_i$ are surjections, the morphism (ϑ, M, X) is determined by this condition uniquely. This completes the proof.

Two examples of coequalizers given on schemes below show that the relation E_2 is not, in general, the smallest equivalence relation containing the set $\mathscr{D}(L_2) \cup H_2$ and only one of the sets $\{(f_L(y, b), f_L(y', b)): (y, y') \in E_0 \text{ and } b \in L_1\}$, $\{(f_L(y, b), f_L(y, b')): y \in L_0 \text{ and } (b, b') \in E_1\}$.





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