SEQUENTIAL METHODS IN STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 16
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

A SURVEY OF SEQUENTIAL ESTIMATION IN PROCESSES WITH INDEPENDENT INCREMENTS

WOLFGANG WINKLER

Section of Mathematics, Technical University of Dresden, Dresden, G.D.R.

1. Introduction

Sequential analysis was founded by A. Wald in 1947 and his famous book ([14]) has proved basic in its further development. Wald's sequential probability ratio test originally introduced for sequences of independent and identically distributed random variables has caused a variety of modifications and generalizations, particularly with regard to Markov chains as well as to processes with independent increments. For references see, for instance, [15]. Moreover, sequential estimation problems have been investigated for special random processes such as the Bernoulli process, and the Poisson and Wiener process (see [1], [6], [11], [12], [13], [16]). Random processes with independent increments whose one-dimensional distributions belong to an exponential family of distributions have been treated systematically since 1974 by Magiera ([10]), by Franz and Winkler ([4], [5]), by Franz and Magiera ([3]) and also by I. Küchler and U. Küchler ([7], [8], [9]).

In this paper we summarize a series of lectures on sequential estimation problems in processes with independent increments given at the Banach Center in the autumn of 1981. For the sake of brevity the proofs and the examples explained in the lectures are omitted.

2. The exponential class of processes with independent increments

Let $X_T = \{X(t), t \in T\}$ be a random process defined on the probability space $[\Omega, \mathfrak{F}, P]$, where T is either the set of all nonnegative integers or the set of all nonnegative real numbers. The one-dimensional distributions induced by P on the σ-field $\mathfrak{Q}^1$ of real Borel sets are denoted by $\{P^{(t)}, t \in T\}$, i.e., we put

$$P^{(t)}(B) = P\{X(t) \in B\}, \qquad B \in \mathfrak{Q}^1.$$

Moreover, let $\mathfrak{F}_t$, $t \in T$, denote the σ-field generated by the random variables $X(s)$, $s \leqslant t$.

In the following we suppose that X_T is a stochastically continuous homogeneous process with independent increments satisfying $P\{X(0) = 0\} = 1$ and the set of these processes is denoted by $\mathscr{S}$. It is sometimes appropriate to identify the elements $X_T \in \mathscr{S}$ with the corresponding probability measures P on $[\Omega, \mathfrak{F}]$. This can be justified by assuming $[\Omega, \mathfrak{F}]$ to be given in canonical form, since $X_T \in \mathscr{S}$ is a Fellerian Markov process. Subsequently we shall use both notations, $X_T \in \mathscr{S}$ and $P \in \mathscr{S}$. We remark that every process $P \in \mathscr{S}$ is determined by its one-dimensional distributions $\{P^{(t)}, t \in T\}$.

With regard to statistical investigations we assume that the probability measure of the process under consideration depends on an unknown real parameter $\theta \in \Theta \subseteq R^1$ and we write $P = P_\theta$. Now, the exponential class of processes with independent increments can be defined as follows:

DEFINITION 1. A process $X_T \in \mathscr{S}$ belongs to the *exponential class* if its probability distributions $P_\theta^{(t)}$, $t \in T$, $\theta \in \Theta$, are dominated by a σ-finite measure μ and the corresponding densities (Radon–Nikodym derivatives) can be represented in the form

$$f(x, t, \theta) = g(x, t) \exp\{a(\theta) x + b_0(\theta) t\}, \tag{1}$$

where $x \in R^1$, $g(x, t) \geqslant 0$ and $a(\theta)$, $b_0(\theta)$ are nonconstant functions of the unknown parameter $\theta \in \Theta$.

An interesting problem consisted in describing the exponential class in terms of the Lévy–Khinchine representation for the characteristic function of processes with independent increments. This problem was solved by I. Küchler and U. Küchler in [8], and we shall give a short outline of their approach to the exponential class. For this purpose an equivalence relation is introduced in $\mathscr{S}$: Two processes $P, Q \in \mathscr{S}$ are said to be *equivalent* if their one-dimensional distributions $P^{(t)}$ and $Q^{(t)}$ are mutually absolutely continuous with densities

$$\frac{dQ^{(t)}}{dP^{(t)}} = \exp\{ux - vt\}, \tag{2}$$

where u and v are real numbers not depending on x and t. The equivalence class of an element $P \in \mathscr{S}$ is denoted by $I(P)$. Now, the exponential class of processes with independent increments can be defined in the following way:

DEFINITION 2. A nonempty equivalence class $I(P) \neq \{P\}$ is called an *exponential family of processes* and the union $\mathscr{S}_0$ of all exponential families is called the *exponential class of processes in* $\mathscr{S}$.

Next, we consider the Lévy–Khinchine representation of the characteristic function

$$\psi_P(t, \lambda) = E_P\{\exp(i\lambda X(t))\}, \qquad \lambda \in R^1,\ t \in R^1,$$

which is given by

$$\psi_P(t, \lambda) = \exp t\left\{i\gamma_P\lambda - \sigma_P^2\frac{\lambda^2}{2} + \int_{R^1\setminus\{0\}}\left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right)\nu_P(dy)\right\}, \tag{3}$$

where $\gamma_P \in R^1$, $\sigma_P^2 \in R_+^1$ and ν_P is a Borel measure satisfying

$$\int_{R^1\setminus\{0\}} \frac{y^2}{1+y^2}\nu_P(dy) < +\infty.$$

Note that the process $P \in \mathscr{S}$ is uniquely determined by its Lévy characteristics $(\gamma_P, \sigma_P^2, \nu_P)$. Now, let the set C_P be defined by

$$C_P = \left\{u \in R^1 : \int_{R^1\setminus\{0\}} \frac{y^2}{1+y^2} e^{uy}\nu_P(dy) < +\infty\right\}.$$

This set turns out to be a finite or infinite interval with $0 \in C_P$. If P is not equal to a deterministic motion, it can be proved that

$$C_P = \{u \in R^1 : \int_{R^1} e^{ux} P^{(t)}(dx) < +\infty \text{ for all } t \in T\}$$

and that $P \in \mathscr{S}_0$ holds if and only if $C_P \neq \{0\}$. We are now able to describe the exponential families $I(P)$ in terms of the Lévy characteristics of P. For this purpose we assume $P \in \mathscr{S}$ and $C_P \neq \{0\}$. For every $u \in C_P$ we define a process $P_u \in \mathscr{S}$ by

$$P_u^{(t)}(dx) = \exp\{ux - v(u)t\}\, P^{(t)}(dx),$$

where

$$v(u) = \gamma_P u + \sigma_P^2\frac{u^2}{2} + \int_{R^1\setminus\{0\}}\left(e^{uy} - 1 - \frac{uy}{1+y^2}\right)\nu_P(dy). \tag{4}$$

THEOREM 1. *Suppose $P \in \mathscr{S}$ and $C_P \neq \{0\}$. Then we have $I(P) = \{P_u, u \in C_P\}$ and the Lévy characteristics of $P_u \in I(P)$ are given by*

$$\gamma_u = \gamma_P + \sigma_P^2 u + \int_{R^1\setminus\{0\}} (1 - e^{uy})\frac{y}{1+y^2}\nu_P(dy),$$

$$\sigma_u^2 = \sigma_P^2 \qquad \textit{and} \qquad \nu_u(dx) = e^{ux}\nu_P(dx).$$

The relationship between Definitions 1 and 2 of the exponential class can be described by a simple parameter transformation. If we put $\theta(u) = \frac{dv(u)}{du}$, where $v(u)$ is given by (4), the function $u \to \theta(u)$ is a one-to-one mapping from C_P onto the interval $\Theta_P = \{\theta(u), u \in C_P\}$, so that the inverse function $\theta \to u(\theta)$ exists. Now, the parameter functions $a(\theta)$ and $b_0(\theta)$ occurring in (1) can be obtained if we put

$$a(\theta) := u(\theta), \qquad b_0(\theta) := -v(u(\theta)). \tag{5}$$

These parameter functions have the following properties: The function $a(\theta)$ is strictly increasing in θ, the functions $a(\theta)$ and $b_0(\theta)$ are both holomorphic at the inner points of Θ_P and, if $P \in \mathscr{S}_0$ and 0 is an inner point of Θ_P, all moments of the corresponding process X_T exist with respect to P.

3. Sequential estimation for processes of the exponential class

In this section we deal with sequential estimation problems for m-dimensional processes $\boldsymbol{X}(t) = (X_1(t), \ldots, X_m(t))^T$ with independent increments. The exponential class of m-dimensional processes with independent increments can be defined in the same way as in the one-dimensional case. When Definition 1 is used representation (1) of the densities should be replaced by

$$f(\boldsymbol{x}, t, \boldsymbol{\theta}) = g(\boldsymbol{x}, t) \exp\{\boldsymbol{a}^T(\boldsymbol{\theta})\boldsymbol{x} + b_0(\boldsymbol{\theta})t\}, \tag{6}$$

where $\boldsymbol{x} = (x_1, \ldots, x_m)^T \in R^m$, $g(\boldsymbol{x}, t) \geqslant 0$ and $\boldsymbol{a}(\boldsymbol{\theta}) = (a_1(\boldsymbol{\theta}), \ldots, a_m(\boldsymbol{\theta}))^T$, $b_0(\boldsymbol{\theta})$ are nonconstant functions of the unknown parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq R^k$. For the sake of simplicity we assume that $k = m$ and we make use of the notations

$$\boldsymbol{b} := \operatorname{grad}_{\boldsymbol{\theta}}(b_0) = \left(\frac{\partial b_0}{\partial \theta_1}, \ldots, \frac{\partial b_0}{\partial \theta_m}\right)^T,$$

$$\boldsymbol{A} := \operatorname{grad}_{\boldsymbol{\theta}}(\boldsymbol{a}^T) = \left(\frac{\partial a_i}{\partial \theta_j}\right)_{i,j=1,2,\ldots,m}.$$

Moreover, we shall assume that $\boldsymbol{A}^{-1}$ exists. The expectation vector of $\boldsymbol{X}(t)$ is then given by $E_{\boldsymbol{\theta}}\boldsymbol{X}(t) = -\boldsymbol{A}^{-1}\boldsymbol{b}t$ and for the covariance matrix of $\boldsymbol{X}(t)$ we obtain

$$\boldsymbol{K}_t = -\boldsymbol{A}^{-1} \operatorname{grad}_{\boldsymbol{\theta}}(\boldsymbol{A}^{-1}\boldsymbol{b})^T t.$$

Let us now consider a stopping time τ, i.e., a random variable with values in $T \cup \{\infty\}$ satisfying $\{\tau \leqslant t\} \in \mathfrak{F}_t$ for every $t \in T$. The σ-field of the τ-past of the process $\boldsymbol{X}(t)$ is denoted by

$$\mathfrak{F}_\tau = \{F \in \mathfrak{F}\colon F \cap \{\tau \leqslant t\} \in \mathfrak{F}_t \text{ for every } t \in T\}.$$

In order to be brief we summarize the main results on sequential estimation in the following items: Suppose that $X(t)$ is a process of the exponential class and let τ be a finite stopping time, i.e., a stopping time with the property $P\{\tau < +\infty\} = 1$. Then:

(i) For every fixed $\theta_0 \in \Theta$ there exists a probability measure P_{θ_0} which does not depend on the unknown parameter θ and is such that we have

$$P_\theta(A) = \int_A \exp\{\alpha^T X(\tau) + \beta\tau\}\, dP_{\theta_0} \tag{7}$$

for every $A \in \mathfrak{F}_\tau$, where $\alpha(\theta) = a(\theta) - a(\theta_0)$, $\beta(\theta) = b_0(\theta) - b_0(\theta_0)$.

(ii) Suppose that $\mathfrak{F}^*$ is the smallest σ-field in $\mathfrak{F}_\tau$ with respect to which $(\tau, X(\tau))$ is measurable. Then, the statistic $(\tau, X(\tau))$ is sufficient in the sense that the σ-field $\mathfrak{F}^*$ is sufficient for $\theta \in \Theta$.

(iii) Let $\varphi(\tau, X(\tau), \theta)$ be a measurable vector-valued function defined on $T \times R^m \times \Theta$. If suitable regularity conditions are fulfilled (see [5]), the equation

$$E_\theta\big[(AX(\tau) + b\tau)\varphi^T\big] = \operatorname{grad}_\theta(E_\theta \varphi^T) - E_\theta(\operatorname{grad}_\theta \varphi^T) \tag{8}$$

holds.

We remark that (8) may be regarded as a generalization of Wald's equations in sequential analysis. For example, if we put $\varphi = 1$ in (8), we obtain Wald's first equation

$$AE_\theta X(\tau) + bE_\theta \tau = 0$$

in the case of processes of the exponential class. If φ does not depend on θ and we have $E_\theta \varphi = h$, so that φ may serve as an unbiased estimate of the function $h(\theta)$, (8) specializes to the equation

$$E_\theta(AX(\tau) + b\tau)\varphi^T = \operatorname{grad}_\theta(h^T).$$

Next, we suppose that we are given a process $X(t)$ of the exponential class and a vector $h(\theta) = (h_1(\theta), \ldots, h_p(\theta))^T$ whose components are nonconstant differentiable functions of θ. The problem consists in finding a sequential procedure (τ, φ), where τ is a finite stopping time and φ is an unbiased estimator of h, which should be optimal in the sense of minimal variances. In the following we shall assume that φ has finite second order moments. Moreover, we put

$$H := \operatorname{grad}_\theta(h^T), \qquad \Gamma := E_\theta(AX(\tau) + b\tau)(AX(\tau) + b\tau)^T$$

and we suppose that the matrix Γ^{-1} exists. Then the following Cramér–Rao inequality can be proved: For every vector $z = (z_1, \ldots, z_p)^T$ we have

$$z^T E_\theta(\varphi - h)(\varphi - h)^T z \geqslant z^T H\Gamma^{-1} Hz. \tag{9}$$

The sign of equality holds at $\theta = \theta^*$ in (9) if and only if

$$\varphi = H^T \Gamma^{-1}(AX(\tau) + b\tau) + h|_{\theta=\theta^*}. \tag{10}$$

The Cramér–Rao inequality gives rise to the consideration of the usual concept of efficiency in the case of sequential estimation procedures for processes of the exponential class. A sequential procedure (τ, φ) is called efficient for $h(\theta)$ if the sign of equality holds in (9) for all $\theta \in \Theta$. In this case the estimator φ is also called efficient and the function $h(\theta)$ is said to be efficiently estimable if there exists an efficient estimator.

In the case of an efficient sequential procedure (τ, φ) the stopping time τ appears to be the random time at which the process will first attain a certain m-dimensional hyperplane. More precisely, we have

THEOREM 2. *If the sequential procedure (τ, φ) is efficient, then there exist coefficients $c_0, c_1, \ldots, c_m$ with $\sum_{i=0}^{m} c_i^2 > 0$ and $d \neq 0$ such that*

$$c_0 \tau + c^T X(\tau) = d \tag{11}$$

holds almost surely, where $c = (c_1, \ldots, c_m)^T$.

Since the hyperplane (11) determines the stopping time in an efficient procedure, equation (11) is called the *characteristic equation of the efficient sequential procedure* (τ, φ). Taking into account Wald's first equation, we find from Theorem 2 that the expectation of the stopping time τ in an efficient procedure is given by

$$E_\theta \tau = \frac{d}{c_0 - c^T(A^{-1} b)}.$$

We are now interested in describing efficiently estimable functions $h(\theta)$ in terms of the characteristic equation (11) and we shall improve the results given in [5]. For this purpose we introduce the expectation parameter $\gamma = E_\theta X(1) = -A^{-1} b$ of the process $X(t)$ as a new parameter. Moreover, we use the notation

$$\gamma^{(0)} = \gamma = (\gamma_1, \ldots, \gamma_m)^T,$$

$$\gamma^{(i)} = (1, \gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_m)^T \qquad \text{for} \quad i = 1, 2, \ldots, m$$

and

$$X^{(0)}(\tau) = X(\tau) = (X_1(\tau), \ldots, X_m(\tau))^T,$$

$$X^{(i)}(\tau) = (\tau, X_1(\tau), \ldots, X_{i-1}(\tau), X_{i+1}(\tau), \ldots, X_m(\tau))^T$$

$$\text{for} \quad i = 1, 2, \ldots, m.$$

From the results given in [5] we are now able to derive

THEOREM 3. *A function $\boldsymbol{h}(\boldsymbol{\theta})$ is efficiently estimable if and only if it may be represented as*

$$\boldsymbol{h}(\boldsymbol{\theta}) = \boldsymbol{k} + \frac{d}{c_0 + \boldsymbol{c}^T \boldsymbol{\gamma}^{(0)}} \boldsymbol{K}\boldsymbol{\gamma}^{(i)}, \qquad i = 0, 1, 2, \ldots, m, \tag{12}$$

where the elements of the vector $\boldsymbol{k}$ and of the matrix $\boldsymbol{K}$ are real constants. The corresponding efficient estimators are given by

$$\boldsymbol{\varphi}(\tau, \boldsymbol{X}(\tau)) = \boldsymbol{k} + \boldsymbol{K}\boldsymbol{X}^{(i)}(\tau). \tag{13}$$

Finally we remark that the methods, though originally developed for processes of the exponential class, have proved applicable also to more general cases, for instance, to birth-and-death processes, to special Markov processes as well as to semi-Markov processes. Concerning these more general cases we refer to the contributions given by Franz, Magiera and Różański and contained in this volume.

4. Sequential maximum likelihood estimation and asymptotic properties

In this section we show the existence of maximum likelihood estimates in the sequential case of processes of the exponential class. If there exists an efficient sequential estimation procedure the efficient estimator coincides with the maximum likelihood estimator and can be shown to be asymptotically normally distributed and strongly consistent. The results concerning maximum likelihood estimation have been obtained in collaboration with Kowtzsch and a special paper with full proofs is in preparation, whereas the results concerning strong consistency of efficient estimators are due to Franz [2].

Sequential maximum likelihood estimation problems are appropriately based on the statistical space $[U, \mathfrak{A}, \{Q_\theta, \theta \in \Theta\}]$, where $U = T \times R^m$, $\mathfrak{A}$ denotes the σ-field of Borel sets in U and the probability measures Q_θ are defined by

$$Q_\theta(B) = P_\theta\{(\tau, \boldsymbol{X}(\tau)) \in B\}, \qquad B \in \mathfrak{A}. \tag{14}$$

According to (7) we choose a fixed $\theta_0 \in \Theta$ and we put $Q^* = Q_{\theta_0}$. The family $\{Q_\theta, \theta \in \Theta\}$ of probability distributions is then dominated by the probability measure Q^* with the densities

$$f^*(u, \theta) = \frac{dQ_\theta}{dQ^*}(u) = \exp\{\boldsymbol{\alpha}^T(\theta)\boldsymbol{x}(u) + \beta(\theta)t(u)\}, \tag{15}$$

where $\boldsymbol{x}(u) = (x_1(u), \ldots, x_m(u))^T$ and $t(u)$, $x_1(u), \ldots, x_m(u)$ denote the coordinate functions of a point $u = (t, x_1, \ldots, x_m) \in U$. When the density $f^*(u, \theta)$

is looked upon as a likelihood function, we shall easily find the likelihood equation by differentiating $\ln f^*(u, \boldsymbol{\theta})$ with respect to θ_j, $j = 1, 2, \ldots, m$:

$$\boldsymbol{A}\boldsymbol{x}(u) + \boldsymbol{b}t(u) = 0. \tag{16}$$

We remark that (16) is a system of linear partial differential equations with respect to the unknown parameter $\boldsymbol{\theta} \in \Theta$.

However, in special cases such as multinomial and Wiener processes, in which the unknown parameter coincides with the expectation parameter of the process in question, the likelihood equation (16) provides a system of linear equations. Therefore, it seems reasonable to introduce the expectation parameter $\boldsymbol{\gamma} = \boldsymbol{w}(\boldsymbol{\theta}) = -\boldsymbol{A}^{-1}\boldsymbol{b}$. We shall assume that the mapping $\boldsymbol{\theta} \to \boldsymbol{w}(\boldsymbol{\theta})$ is one-to-one, so that $\boldsymbol{w}^{-1}(\boldsymbol{\gamma}) = \boldsymbol{\theta}$ exists. Using the invariance principle of maximum likelihood estimators (see [17]), we obtain

THEOREM 4. *The maximum likelihood estimator of $\boldsymbol{\theta}$ is given by $\hat{\boldsymbol{\theta}} = \boldsymbol{w}^{-1}(\hat{\boldsymbol{\gamma}})$, where*

$$\hat{\boldsymbol{\gamma}} = \frac{\boldsymbol{X}(\tau)}{\tau} \tag{17}$$

is the maximum likelihood estimator of the expectation parameter $\boldsymbol{\gamma}$.

Next we deal with the asymptotic properties of efficient maximum likelihood estimators. For this purpose we consider a family of stopping times $\{\tau_s, s \in S\}$, where S is either the interval $[0, +\infty)$ or the set $\{0, 1, 2, \ldots\}$. In special cases the elements of S can be thought of as prescribed levels and the stopping times τ_s are then the level crossing times. We shall make use of the following

LEMMA. *Let $\{\boldsymbol{X}(t), t \in T\}$ be an m-dimensional homogeneous process with independent increments and let $\{\tau_s, s \in S\}$ be a family of finite stopping times with the property $\tau_s/s \xrightarrow[P]{} a > 0$. We put*

$$Z(t) = \tilde{c}_0 t + \tilde{\boldsymbol{c}}^T \boldsymbol{X}(t), \qquad \tilde{\boldsymbol{c}} = (\tilde{c}_1, \ldots, \tilde{c}_m)^T, \tag{18}$$

with given constants $\tilde{c}_0, \tilde{c}_1, \ldots, \tilde{c}_m$. Then the random variables

$$\sqrt{\frac{as}{D_\theta^2 Z(1)}}\left(\frac{Z(\tau_s)}{\tau_s} - E_\theta Z(1)\right)$$

are asymptotically normally distributed with mean 0 and variance 1, if $s \to \infty$.

We remark that

$$E_\theta Z(t) = (\tilde{c}_0 - \tilde{\boldsymbol{c}}^T \boldsymbol{A}^{-1}\boldsymbol{b})t \qquad \text{and} \qquad D_\theta^2 Z(t) = \tilde{\boldsymbol{c}}^T \boldsymbol{K}_t \tilde{\boldsymbol{c}}.$$

Since $Z(t)$ has independent increments, $t^{-1}Z(t)$ is asymptotically normally distributed and the assertion of the lemma follows from the well-known Anscombe principle if the time t is replaced by the random time τ_s.

Let us now suppose that (τ_s, φ_s) is an efficient sequential procedure. Then, by Theorem 2 the characteristic equation (11) holds and, on the other hand, representation (10) holds for all $\theta \in \Theta$. Therefore, the efficient estimator φ_s can be written as

$$\varphi_s = \frac{1}{s} D(\tau_s, X^T(\tau_s))^T + h, \tag{19}$$

with $D = (c^T A^{-1} b - c_0)\, H^T G^{-1}\, (b|A)$ and $G = \operatorname{grad}_\theta (A^{-1} b)^T A^T$. We put

$$D = \begin{pmatrix} d_{10} & d_{11} & \dots & d_{1m} \\ \dots & \dots & \dots & \dots \\ d_{p0} & d_{p1} & \dots & d_{pm} \end{pmatrix}$$

and $d_k = (d_{k1}, \dots, d_{km})^T$, $k = 1, 2, \dots, p$. Now we are able to state the following

THEOREM 5. *Let $\{X(t), t \in T\}$ be a process of the exponential class. Then:*

(i) *If there exists an efficient procedure (τ, φ) for $h(\theta)$, the efficient estimator φ coincides with the maximum likelihood estimator $\hat{h}$ of h almost surely.*

(ii) *Suppose that $\{\tau_s, s \in S\}$ is a family of finite stopping times corresponding to efficient procedures (τ_s, φ_s). If $\tau_s/s \xrightarrow{P} a > 0$, then for every component $\varphi_s^{(k)}$ of φ_s the random variables*

$$\sqrt{\frac{s}{a d_k^T K_1 d_k}} \left(\varphi_s^{(k)}(\tau_s, X(\tau_s)) - h_k(\theta)\right)$$

are asymptotically normally distributed with mean 0 and variance 1 as $s \to +\infty$.

(iii) *If the family $\{\tau_s, s \in S\}$ of stopping times satisfies the condition*

$$\frac{\tau_s}{s} \xrightarrow[a.s.]{} a = E_\theta \tau_1 = (c_0 - c^T A^{-1} b)^{-1}, \tag{20}$$

the efficient estimator φ_s is strongly consistent, i.e., $\varphi_s \xrightarrow[a.s.]{} h$ *if* $s \to +\infty$.

The first assertion of Theorem 5 is a consequence of Theorems 3 and 4. The second assertion can be obtained by applying the lemma to the process $Z_k(t) = d_{k0} t + d_k^T X(t)$ and using the equation

$$\varphi_s^{(k)}(\tau_s, X(\tau_s)) = \frac{1}{s} Z_k(\tau_s) + h_k(\theta).$$

Regarding the third part of Theorem 5, we refer to [2], Theorem 2. Note that in the case of level crossing times considered in [2] the family $\{\tau_s, s \in S\}$ is also a homogeneous process with independent increments, and hence condition (20) is fulfilled by the strong law of large numbers.

References

[1] M. H. De Groot, *Unbiased sequential estimation for binomial populations*, Ann. Math. Statist. **30** (1959), 60–101.

[2] J. Franz, *Consistency and completeness in sequential estimation problems*, Math. Operationsforsch. Statist. Ser. Statist. **10** (1979), 127–139.

[3] J. Franz and R. Magiera, *On sequential plans for the exponential class of processes*, Zastos. Mat. **16** (1978), 153–165.

[4] J. Franz und W. Winkler, *Über Stoppzeiten bei statistischen Problemen für homogene Prozesse mit unabhängigen Zuwächsen*, Math. Nachr. **70** (1976), 37–53.

[5] —, —, *Sequential estimation problems for the exponential class of processes with independent increments*, Scand. J. Statist. **6** (1979), 129–139.

[6] M. A. Girshik, F. Mosteller and L. J. Savage, *Unbiased estimates for certain binomial sampling problems with applications*, Ann. Math. Statist. **17** (1946), 13–23.

[7] I. Küchler and U. Küchler, *Analytical aspects of exponential families of distribution functions*, Math. Nachr. **101** (1981), 153–164.

[8] —, —, *An analytical treatment of exponential families of stochastic processes with independent stationary increments*, Math. Nachr. **103** (1981), 21–30.

[9] —, —, *A generalized fundamental identity for processes with independent stationary increments*, Preprint der TU Dresden, 07-33-80, Dresden 1980.

[10] R. Magiera, *On the inequality of Cramér–Rao type in sequential estimation theory*, Zastos. Mat. **14** (1974), 227–235.

[11] —, *Sequential plans for the negative-binomial process*, Prace Nauk. Inst. Mat. Politech. Wrocław. Ser. Stud. Materiały **10** (1975), 3–5.

[12] R. Magiera and S. Trybuła, *Plany ukośne dla procesu dwumianowego*, Mat. Stos. **6** (1976), 41–47.

[13] S. Trybuła, *Sequential estimation in processes with independent increments*, Dissertationes Math. (Rozprawy Mat.) **60** (1968).

[14] A. Wald, *Sequential Analysis*, Wiley, New York 1947.

[15] W. Winkler, J. Franz, and I. Küchler, *Sequential statistical procedures for processes of the exponential class with independent increments*, Math. Operationsforsch. Statist. Ser. Statist. **13** (1982), 105–119.

[16] R. A. Zaidman, Y. V. Linnik and V. N. Sudakov, *Sequential estimation plans and Markov stopping times*, Dokl. Akad. Nauk SSSR **185** (1969), 1225–1225, in Russian.

[17] P. W. Zehna, *Invariance of maximum likelihood estimators*, Ann. Math. Statist. **37** (1966), 744.

Presented to the semester
Sequential Methods in Statistics
September 7–December 11, 1981