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KOMITET REDAKCYJNY
KAROL BORSUK redaktor
ANDRZEJ BIAŁYNICKI-BIRULA, BOGDAN BOJARSKI,
ZBIGNIEW CIESIELSKI, JERZY ŁOŚ, |ANDRZEJ MOSTOWSKI|,
ZBIGNIEW SEMADENI, WANDA SZMIELEW

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K. GAWĘDZKI

Fourier-like kernels in geometric quantization

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CONTENTS

I. INTRODUCTION

The problem of correspondence between classical and quantum physics has fascinated physicists since the very creation of quantum mechanics, and even before, as in building quantum mechanics Schrödinger was guided by the analogy of interrelations between "classical" geometric optics and physical optics. The procedure of passing from a classical to a quantum theory was promptly formalized to produce a procedure of the so-called canonical quantization, but we waited until 1948, when the idea of a path integral was given by Feynman, to obtain a deeper insight into connections between the classical and quantum descriptions and the optical analogy.

In recent years the canonical quantization procedure has been widely developed, mostly in the papers of B. Kostant, to embrace general mechanical systems. The Kostant procedure, called *geometric quantization*, traced back the classical system – quantum system correspondence to obtain it, for example, for any elementary Poincaré particle. This theory allows us to assign to any classical mechanical system, defined in terms of symplectic geometry, a Hilbert space of states of the corresponding quantum system given as a space of sections of a complex line bundle over the phase space of the classical sytem. To introduce the uncertainty principle, an additional geometric structure of the classical phase space, called a *polarization*, is needed. Some class of classical physical quantities is represented by symmetric operators in the Hilbert space of states. The main drawback of this procedure, at the present stage of development, is that this class is too small. There have been some attempts, not altogether satisfactory (see [4], [7]), to extend this procedure. They were closely related to the problem of independence of Kostant's construction from the choice of the polarization. This independence is believed to be assured by a generalized Fourier transform intertwining the Hilbert spaces of states obtained for different polarizations. If the existence of this transform is a very difficult problem, in many cases one can often find a kernel pretending to represent it. The present paper is devoted to the study of such kernels and of kernels arising in a slightly more general context. Basing ourselves on this notion, we also give an alternative proposition on how to extend the Kostant quantization procedure to a broader class of functions, based in fact on the idea of bringing the Kostant and Feynman procedures of quantization closer.

It is worth noting that generalized Fourier transforms appear also in studies of asymptotic oscillatory solutions of quantum equations, i.e. in a context somewhat inverse to that of quantization, namely, when we examine the classical limit behaviour of the quantum theories (see e.g. [5]).

Chapter II contains some basic geometric notions. In Chapter III we give a short account of the Kostant procedure in its full generality. Chapter IV is devoted to the study of quantization kernels and of kernel representation of geometrically quantized operators.

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II. PRELIMINARY NOTIONS

Throughout this paper the standard language of differential geometry will be used. All manifolds appearing are assumed to be Hausdorff, finite dimensional, countable at infinity, and their mappings, unless otherwise stated, to be (C^∞-) smooth. All proofs of smoothness of appearing constructions, if they follow by straightforward local considerations, will be omitted.

If $\varkappa\colon P\to Q$ is a morphism of manifolds P, Q, we shall denote by $\varkappa_*$ the tangent mapping or the induced mapping of vector fields, and by $\varkappa^*$ the induced mapping of forms.

Let M be a manifold and $\boldsymbol{W} = (W, \pi, M)$ a complex vector bundle over M (W being the bundle space, and π the bundle projection).

$\Gamma(\boldsymbol{W})$ will denote the space of all (smooth) global sections of W,
$\Gamma_{\mathrm{loc}}(\boldsymbol{W})$ — of local sections defined on open subsets of M, and
$\Gamma_U(\boldsymbol{W})$ — of sections defined on open $U \subset M$.

Let X be a generally complex vector field on M.

II.1. Definition. A mapping $D_X\colon \Gamma_{\mathrm{loc}}(\boldsymbol{W})\to\Gamma_{\mathrm{loc}}(\boldsymbol{W})$ is called an *X-derivation* of $\Gamma_{\mathrm{loc}}(\boldsymbol{W})$ if:

1. For open $U \subset M$, $D_X\colon \Gamma_U(\boldsymbol{W})\to\Gamma_U(\boldsymbol{W})$ is linear;
2. D_X commutes with restrictions, i.e. the following diagram

$$\begin{tikzcd}
\Gamma_U(\boldsymbol{W}) \arrow[r, "D_X"] \arrow[d] & \Gamma_U(\boldsymbol{W}) \arrow[d] \\
\Gamma_V(\boldsymbol{W}) \arrow[r, "D_X"] & \Gamma_V(\boldsymbol{W})
\end{tikzcd}$$

is commutative (the vertical arrows denote the operation of restriction of a section defined on U to a smaller open domain V);

3.

$$D_X(fs) = X(f)s + fD_Xs \tag{1}$$

for $s\in\Gamma_U(\boldsymbol{W})$ and $f\in C^\infty(U)$.

II.2. Example. An operation ∇_X of covariant derivation in the direction X of local sections of a bundle with a linear connection is an X-derivation.

II.3. Example. If $\boldsymbol{W}$ is a tensor bundle, then the Lie derivation $\mathscr{L}_X$ is an X-derivation.

II.4. Proposition. *Let* $\boldsymbol{W}^1 = (W^1, \pi^1, M)$ *and* $\boldsymbol{W}^2 = (W^2, \pi^2, M)$ *be two complex vector bundles over* M. D_X^i *be an* X*-derivation of* $\Gamma_{\mathrm{loc}}(\boldsymbol{W}^i)$, $i = 1, 2$. *Then there exists a unique* X*-derivation* $D_X^1 \otimes D_X^2$ *of* $\Gamma_{\mathrm{loc}}(\boldsymbol{W}^1 \otimes \boldsymbol{W}^2)$ *such that for* $s \in \Gamma_U(\boldsymbol{W}^1)$ *and* $t \in \Gamma_U(\boldsymbol{W}^2)$

$$(D_X^1 \otimes D_X^2)(s \otimes t) = (D_X^1 s) \otimes t + s \otimes (D_X^2 t). \tag{2}$$

Proof. *Uniqueness* follows immediately from the fact that locally each section from $\Gamma_{\mathrm{loc}}(\boldsymbol{W}^1 \otimes \boldsymbol{W}^2)$ is of the form $\sum_j s_j \otimes t_j$, with $s_j \in \Gamma_{\mathrm{loc}}(\boldsymbol{W}^1)$ and $t_j \in \Gamma_{\mathrm{loc}}(\boldsymbol{W}^2)$.

Existence. Let $(U_j)_{j\in J}$ be an open locally-finite covering of M such that $\boldsymbol{W}^i|_{U_j}$ is trivial, $i = 1, 2$. Let $(s_j^\alpha)_{\alpha\in A}$ be a set of sections from $\Gamma_{U_j}(\boldsymbol{W}^1)$ whose images form at each point $x \in U_j$ a basis for the fibre $W_x^1 := (\pi^1)^{-1}(\{x\})$, and let $(t_j^\beta)_{\beta\in B}$ be a subset of $\Gamma_{U_j}(\boldsymbol{W}^2)$ with a similar property. Let $(f_j)_{j\in J}$ be a partition of unity subordinated to the covering $(U_j)_{j\in J}$. Let $v \in \Gamma_U(\boldsymbol{W}^1 \otimes \boldsymbol{W}^2)$.

We have

$$f_j v(x) = \sum_{\alpha,\beta} A_j^{\alpha\beta}(x) s_j(x) \otimes t_j(x), \qquad x \in U \cap U_j,$$

where A_j are smooth functions vanishing on $U \setminus U_j$, uniquely determined by v.

Let us put

$$(D_X^1 \otimes D_X^2)v = \sum_{j,\alpha,\beta} \{X(A_j^{\alpha\beta}) \cdot s_j^\alpha \otimes t_j^\beta + A_j^{\alpha\beta}(D_X^1 s_j^\alpha) \otimes t_j^\beta + A_j^{\alpha\beta} s_j^\alpha \otimes (D_X^2 t_j^\beta)\} \tag{3}$$

(formula (3) makes sense since $A_j^{\alpha\beta}$ and $X(A_j^{\alpha\beta})$ vanish if s_j^α and t_j^β are not defined).

It can be easily seen that the right-hand side of (3) is a smoth section from $\Gamma_U(\boldsymbol{W}^1 \otimes \boldsymbol{W}^2)$ and that so defined $D_X^1 \otimes D_X^2$ is an X-derivation. (2) is also straightforward. ■

Now let $\boldsymbol{W}$ be one-dimensional. We shall consider a situation in some sense inverse to that just examined.

II.5. Definition. $(\boldsymbol{W}^{1/2}, \iota)$ is called a *square root of the one-dimensional bundle* $\boldsymbol{W}$ if $\boldsymbol{W}^{1/2}$ is a one-dimensional bundle over M and ι is an isomorphism of the bundles $\boldsymbol{W}^{1/2} \otimes \boldsymbol{W}^{1/2}$ and $\boldsymbol{W}$ (which projected to the base space of the bundles is the identity). If $y_1, y_2 \in W_x^{1/2}$ (i.e. to the fibre of $\boldsymbol{W}^{1/2}$ over $x \in M$), $y_1 \underline{\otimes} y_2$ will denote the image by ι of $y_1 \otimes y_2$ in W_x. If $s_i \in \Gamma_U(\boldsymbol{W}^{1/2})$, $i = 1, 2$, then $s_1 \underline{\otimes} s_2$ will denote a section of $\boldsymbol{W}$ such that $(s_1 \underline{\otimes} s_2)(x) = s_1(x) \underline{\otimes} s_2(x)$, $x \in U$. This convention will be extensively used throughout the paper.

II.6. **Proposition.** *Let* $(W^{1/2}, \iota)$ *be a square root of* W *and let* D_X *be an* X*-derivation of* $\Gamma_{\mathrm{loc}}(W)$. *Then there exists a unique* X*-derivation* $D_X^{1/2}$ *of* $\Gamma_{\mathrm{loc}}(W^{1/2})$ *such that if* $s \in \Gamma_U(W^{1/2})$, *then*

$$2(D_X^{1/2} s) \underline{\otimes} s = D_X(s \underline{\otimes} s). \tag{4}$$

Proof. *Uniqueness.* Let $D_X^{1/2}$ and $D_X^{1/2\prime}$ be two X-derivations of $\Gamma_{\mathrm{loc}}(W^{1/2})$ satisfying (4). Let $s \in \Gamma_U(W^{1/2})$, $x \in U$, and V be an open subset of U containing x and such that there exists a non-vanishing section t of $W^{1/2}$ over V.

Then

$$(D_X^{1/2} t) \otimes t = \frac{D_X^{1/2} t}{t} (t \otimes t) \tag{5}$$

(t is non-vanishing). Hence

$$(D_X^{1/2} t) \underline{\otimes} t = \frac{D_X^{1/2} t}{t} t \underline{\otimes} t. \tag{6}$$

But by (4),

$$(D_X^{1/2} t) \underline{\otimes} t = \frac{1}{2} D_X(t \underline{\otimes} t) = \frac{1}{2} \frac{D_X(t \underline{\otimes} t)}{(t \underline{\otimes} t)} t \underline{\otimes} t. \tag{7}$$

Comparing (6) and (7), we get

$$D_X^{1/2} t = \frac{1}{2} \frac{D_X(t \underline{\otimes} t)}{(t \underline{\otimes} t)} t. \tag{8}$$

Thus we have

$$\begin{aligned}(D_X^{1/2} s)|_V = D_X^{1/2}(s|_V) &= D_X^{1/2}(ft) = X(f)t + f D_X^{1/2} t \\ &= X(f)t + \frac{1}{2} f \frac{D_X(t \underline{\otimes} t)}{(t \underline{\otimes} t)} t = (D_X^{1/2\prime} s)|_V,\end{aligned} \tag{9}$$

which shows that $D_X^{1/2} = D_X^{1/2\prime}$.

Existence. Let now $(U_j)_{j \in J}$ be an open, locally-finite covering of M such that $W^{1/2}|_{U_j}$ is trivial, and choose for each j a non-vanishing section t_j of $W^{1/2}$ defined over U_j. Let $(f_j)_{j \in J}$ be a partition of unity subordinated to $(U_j)_{j \in J}$.

For $s \in \Gamma_U(W^{1/2})$, let us put

$$D_X^{1/2} s := \sum_j \left(X(g_j) + \frac{1}{2} g_j \frac{D_X(t_j \underline{\otimes} t_j)}{(t_j \underline{\otimes} t_j)} \right) t_j, \tag{10}$$

where $g_j \in C^\infty(U)$ satisfy

$$\begin{aligned} f_j s &= g_j t_j \quad \text{on } U \cap U_j, \\ g_j &= 0 \quad \text{on } U \setminus U_j. \end{aligned} \tag{11}$$

It can be easily seen that the right-hand side of (10) defines a (smooth) section of $W^{1/2}$ over U, and that $D_X^{1/2}$ so defined possesses the required properties. ■

II.7. Remark. Definition II.5 makes sense and an analogue of Proposition II.6 holds also for vector bundles of higher dimensions. This, however, will not be used in this paper.

Let W be again one-dimensional. Let

$$|W| := (\tilde{W} \times C)/\sim, \tag{12}$$

where $\tilde{W}$ denotes W without the image of the zero section, and

$$(y, a) \sim \left(\lambda y, \frac{a}{|\lambda|}\right) \quad \text{for } 0 \neq \lambda \in C. \tag{13}$$

Let

$$|\pi|([y, a]) := \pi(y), \tag{14}$$

where $[y, a]$ is the $\sim$-class defined by (y, a). For $t \in \Gamma_U(W)$ non-vanishing, let

$$\tau_t([t(x), a]) := (x, a), \qquad x \in U, \tag{15}$$

τ_t is a bijection of $|\pi|^{-1}(U)$ onto $U \times C$.

There exists a unique structure of a manifold on $|W|$ such that τ_t for non-vanishing $t \in \Gamma_{\text{loc}}(W)$ are diffeomorphisms.

For $x \in M$, $|\pi|^{-1}(\{x\})$ possesses a natural structure of a one-dimensional complex vector space.

Thus

$$|W| := (|W|, |\pi|, M)$$

is a one-dimensional complex vector bundle over M. If $0 \neq y \in W$, then by $|y|$ we shall denote an element $[y, 1] \in |W|$. Let $t \in \Gamma_U(W)$ be non-vanishing. Let

$$|t|(x) := |t(x)|, \qquad x \in U; \tag{16}$$

then $|t| \in \Gamma_U(|W|)$.

II.8. Remark. $|W|$ is always trivial as a hermitian structure on W, which always exists, defines a global non-vanishing section of $|W|$.

Let J be an anti-linear involutive automorphism of W (which projected to the base space is the identity). We shall denote Jy by $\bar{y}$, $y \in W$, and $J \circ t$ by $\bar{t}$ for $t \in \Gamma_{\text{loc}}(W)$.

II.9. PROPOSITION. *Let D_X be an X-derivation of $\Gamma_{\mathrm{loc}}(W)$. Then there exists a unique X-derivation $|D_X|$ of $\Gamma_{\mathrm{loc}}(|W|)$ such that*

$$|D_X|\,|t| = \frac{D_X t}{t}\,|t| \tag{17}$$

for each non-vanishing $t \in \Gamma_{\mathrm{loc}}(W)$, $t = \bar{t}$.

Proof. *Uniqueness* follows immediately from the fact that each section of $|W|$ is locally of the form $f|t|$, where f is a (complex) function defined locally on M and $t \in \Gamma_{\mathrm{loc}}(W)$ is non-vanishing, $t = \bar{t}$, and for such sections s, (1) and (17) determine $|D_X|s$ completely.

Existence. Let $(U_j)_{j\in J}$ be an open, locally finite covering of M and $(t_j)_{j\in J}$ a family of non-vanishing sections, $t_j \in \Gamma_{U_j}(W)$, $t_j = \bar{t}_j$. Let $(f_j)_{j\in J}$ be a partition of unity subordinated to $(U_j)_{j\in J}$. For $s \in \Gamma_U(|W|)$, let

$$|D_X|s := \sum_j \left\{ X(g_j)\,|t_j| + g_j \frac{D_X t_j}{t_j}\,|t_j| \right\}, \tag{18}$$

where $g_j \in C^\infty(U)$ are such that

$$\begin{aligned} f_j\, s &= g_j |t_j| && \text{on } U \cap U_j \quad \text{and} \\ g_j &= 0 && \text{on } U \setminus U_j. \end{aligned} \tag{19}$$

It is straightforward that the right-hand side of (18) defines a (smooth) section from $\Gamma_U(|W|)$, and that $|D_X|$ obtained in this way possesses the required properties. ■

II.10. EXAMPLE. Let $W = \Lambda^p T^*(M)^C$ be the bundle of complex p-covectors tangent to M, $p = \dim M$. Then $|W|$ is the bundle of absolute densities on M. If $D_X = \mathscr{L}_X$ (see Example II. 3), and J is the complex conjugation of complex p-covectors, then $|D_X|$ is the operation of the Lie derivation of fields of densities in the direction of X.

III. GEOMETRIC QUANTIZATION

B. Kostant [6], [2] (see also [11]) worked out a geometric procedure which allows representation of a certain class of functions on a symplectic manifold by self-adjoint operators acting in a Hilbert space of sections of a vector bundle over this manifold.

In the present chapter we shall give a short exposition of his results, modified according to our needs. Sections III. A, B, C will contain some basic notions and facts, mostly without proofs, and can hardly be considered as self-contained. One can find a more detailed exposition of this material in [1], [2], [6], [11], [13].

A. Elements of symplectic geometry

III.1. Definition. (M, ω) is called a *symplectic manifold* if M is a manifold and ω is a real, closed ($d\omega = 0$) non-degenerate 2-form on M.

ω defines a bundle isomorphism

$$T(M)^C \ni X_x \to X_x^b \in T^*(M)^C,$$

where

$$X_x^b := X_x \lrcorner\, \omega(x). \tag{20}$$

Let

$$T^*(M)^C \ni \eta_x \to \eta_x^\# \in T(M)^C \tag{21}$$

be the inverse isomorphism.

For $\eta_x^1, \eta_x^2 \in T_x^*(M)^C$, we shall put

$$\hat{\omega}(\eta_x^1, \eta_x^2) := \langle \eta_x^{1\#}, \eta_x^{2\#} | \omega(x) \rangle. \tag{22}$$

Diffeomorphisms of M which preserve ω will be called *canonical*. Vector fields X preserving ω (i.e. such that $\mathscr{L}_X \omega = 0$) will also be called *canonical*.

If $h \in C^\infty(M)$, then there exists a unique vector field X_h on M such that

$$X_h(x) = \big(dh(x)\big)^{\#}, \qquad x \in M. \tag{23}$$

X_h is called a *canonical vector field* generated by h. Vector fields of this form are called *hamiltonian.*

For two functions h, $k \in C^\infty(M)$, their Poisson bracket is defined by

$$\{h, k\} := X_k(h). \tag{24}$$

$C^\infty(M)$ with the Poisson bracket operation constitutes a Lie algebra.

III.2. Example (cotangent bundle). Let $\mathscr{X}$ be a manifold. Let $M = T^*(\mathscr{X})$. Let σ be the only 1-form on M such that if $s \in \Gamma\big(T^*(\mathscr{X})\big)$, then

$$s^*\sigma = s. \tag{25}$$

Then $(M, -d\sigma)$ is a symplectic manifold.

III.3. Example. Let $M = S^2 = \{(x, y, z) \in R^3 : x^2+y^2+z^2 = 1\}$. Let $\tilde{\omega}_s$, where s is a non-negative constant, be a 2-form invariant under rotations given by

$$\tilde{\omega}_s = (s+\tfrac{1}{2})\,dz \wedge d\left(\arctan\frac{y}{x}\right) \cdot \tag{26}$$

n points where the right-hand side makes sense. $(M, \tilde{\omega}_s)$ is a symplectici manifold.

III.4. **Remark.** The symplectic manifolds (M, ω) are a mathematical tool for description of the classical mechanical systems. M corresponds to the phase space of the system and $\mathscr{X}$ in Example III.2 to the configuration space. $(M, \tilde{\omega}_s)$ in Example III.3 describes an elementary classical spin system. Functions on the symplectic manifolds are, from the physical point of view, classical physical quantities of the mechanical systems.

B. Quantum bundles

Let $L = (L, \pi, M)$ be a one-dimensional bundle over M. The mapping

$$L \ni y \mapsto zy \in L$$

is for $0 \neq z \in C$ an automorphism of L, which will be denoted by m_z.

The mapping

$$\tilde{C} \ni z \mapsto zy \in \tilde{L}$$

defines for $y \in \tilde{L}$ a linear isomorphism of $\tilde{C}$ onto the fibre $\tilde{L}_{\pi(y)}$, which will be denoted by μ_y ($\tilde{C} := C \setminus \{0\}$, $\tilde{L}_x := L_x \setminus \{0_x\}$, $\tilde{L} = \bigcup_{x \in M} \tilde{L}_x$).

III.5. DEFINITION. A complex 1-form a on $\tilde{L}$ is called a *connection* on L if

$$(m_z \restriction \tilde{L})^* a = a \quad \text{for each } 0 \neq z \in C, \tag{27}$$

$$\mu_y^* a = \frac{dz}{iz} \quad \text{for each } y \in \tilde{L}. \tag{28}$$

Given a connection form a, an operation ∇_X of covariant derivation of sections of L in the direction of a vector field X can be defined as follows:

$$\nabla_X s := i\langle X \mid s^* a \rangle s \quad \text{for } s \in \Gamma_{\mathrm{loc}}(L). \tag{29}$$

It can easily be shown there exists a closed complex 2-form γ on M such that $da = \pi^* \gamma$. γ is called a *curvature form.*

Let $(\cdot, \cdot)$ be a hermitian structure on L. It is called *a-invariant* if

$$X\big((s, t)\big) = (\nabla_X s, t) + (s, \nabla_X t) \tag{30}$$

for real X and $s, t \in \Gamma_{\mathrm{loc}}(L)$.

III.6. DEFINITION. A one-dimensional complex bundle over M with a connection a and an a-invariant hermitian structure is called a *quantum bundle over the symplectic manifold* (M, ω) if the curvature form for a is equal to ω.

III.7. EXAMPLE (cotangent bundle). Let (M, ω) be as in Example III.2. Let $L := M \times C$, π: $M \times C \to M$ be the projection onto the first factor. Let

$$a := \frac{dz}{iz} - \pi^* \sigma.$$

Let the hermitian structure be defined by

$$|(m, z)|^2 := |z|^2.$$

These determine a quantum bundle over (M, ω) (which, up to isomorphism, is the only one if $\mathscr{X}$ is simply connected — see [6] for a condition for the existence and classification of quantum bundles over a given symplectic manifold).

III.8. EXAMPLE (complex projective plane). Let $(M, \tilde{\omega}_s)$ be as in Example III.3. Let $P^1(C)$ be the one-dimensional projective plane: $P^1(C) = (C^2 \setminus \{0\})/\sim$, where $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$, $0 \neq \lambda \in C$. One can use a two-chart atlas $\{w, v\}$ for $P^1(C)$:

$$w\colon 0_1 \to C, \qquad v\colon 0_2 \to C,$$
$$0_1 := \{[z_1, z_2]\colon z_1 \neq 0\}, \qquad 0_2 := \{[z_1, z_2]\colon z_2 \neq 0\}, \tag{31}$$
$$w([z_1, z_2]) := \frac{z_2}{z_1}, \qquad v([z_1, z_2]) := \frac{z_1}{z_2}.$$

Let $\varkappa\colon P^1(C) \to R^3$,

$$\big(\varkappa([z_1, z_2])\big)_i := \frac{1}{|z_1|^2 + |z_2|^2} (\bar{z}_1, \bar{z}_2)\, \sigma_i \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{32}$$

where σ_i are Pauli matrices, $i = 1, 2, 3$.

One can check that $\varkappa$ is a diffeomorphism of $P^1(C)$ onto S^2. By means of $\varkappa$, we can transport the symplectic form $\tilde{\omega}_s$ from S^2 to $P^1(C)$. Calculations yield

$$\begin{aligned} \varkappa^* \tilde{\omega}_s \restriction_{0_1} &= (2s+1)\, i \frac{dw \wedge d\bar{w}}{(1+|w|^2)^2}, \\ \varkappa^* \tilde{\omega}_s \restriction_{0_2} &= (2s+1)\, i \frac{dv \wedge d\bar{v}}{(1+|v|^2)^2}. \end{aligned} \tag{33}$$

We shall denote $\varkappa^* \tilde{\omega}_s$ by ω_s. Let L_0 be defined by

$$\tilde{L}_0 = C^2 \setminus \{0\}, \qquad \pi_0 \restriction_{\tilde{L}_0}\colon \tilde{L}_0 \to P^1(C), \qquad \pi_0(z_1, z_2) := [z_1, z_2].$$

$\boldsymbol{L}_0 := \big(L_0, \pi_0, P^1(C)\big)$ is a one-dimensional complex vector bundle over $P^1(C)$. Let

$$a_0 := \frac{1}{i} \frac{\bar{z}_1 dz_1 + \bar{z}_2 dz_2}{|z_1|^2 + |z_2|^2}, \qquad |(z_1, z_2)|_0^2 := |z_1|^2 + |z_2|^2. \tag{34}$$

It can easily be checked that $\boldsymbol{L}_0$ with the connection a_0 and the hermitian structure defined by $|\cdot|_0$ is a quantum bundle for $\big(P^1(C), \omega_0\big)$.

One can show that a quantum bundle for $\big(P^1(C), \omega_s\big)$ exists if and only if $s = m/2$, $m = 0, 1, 2, \ldots$, and for this case it is unique (up to an isomorphism) — see [13]. The quantum bundle for the case $s = m/2 > 0$ can be obtained as follows:

$$\boldsymbol{L}_{m/2} := \underbrace{\boldsymbol{L}_0 \otimes \ldots \otimes \boldsymbol{L}_0}_{(m+1)\text{ times}}.$$

The connection $a_{m/2}$ is a connection whose covariant derivative $\nabla_{m/2}$ is obtained from ∇_0 by the prescription given in Proposition II.3.

As a hermitian structure for $\boldsymbol{L}_{m/2}$ we take that one naturally induced by the hermitian structure of $\boldsymbol{L}_0$.

We shall also give an explicit description of $L_{m/2}$,

$$L_{m/2} = (\{1\}\times 0_1 \times C)\cup(\{2\}\times 0_2\times C)/\underset{m/2}{\sim},$$

where the relation $\underset{m/2}{\sim}$ is defined as follows:

$$(1,\ [1,\mathrm{w}],\ z)\underset{m/2}{\sim}(2,\ [\mathrm{v},1],\ z') \Leftrightarrow \frac{1}{\mathrm{v}} = \mathrm{w} \quad\text{and}\quad z' = w^{m+1}z.$$

The bundle projection $\pi_{m/2}$ can be defined on representing elements as the projection onto the 2-nd factor. On $\pi_{m/2}^{-1}(0_i)$, $i = 1, 2$, one can define diffeomorphic maps $F_i^{m/2}$, $F_i^{m/2}$: $\pi_{m/2}^{-1}(0_i)\to C^2$,

$$F_i^{m/2}\big([1,\ [1,\ \mathrm{w}],\ z]_{m/2}\big) := (\mathrm{w},\ z),$$
$$F_2^{m/2}\big([2,\ [\mathrm{v},\ 1],\ z]_{m/2}\big) := (\mathrm{v},\ z),$$

where $[\cdot]_{m/2}$ denotes a class of the element in between, taken according to the relation $\underset{m/2}{\sim}$. $\alpha_{m/2}$ can be given by

$$\big((F_1^{m/2})^{-1}\big)^*\alpha_{m/2} = \frac{dz}{iz} + \frac{(m+1)}{i}\,\frac{\bar{\mathrm{w}}\,d\mathrm{w}}{1+|\mathrm{w}|^2},$$
$$\big((F_2^{m/2})^{-1}\big)^*\alpha_{m/2} = \frac{dz}{iz} + \frac{(m+1)}{i}\,\frac{\bar{\mathrm{v}}\,d\mathrm{v}}{1+|\mathrm{v}|^2}.$$

For the hermitian structure,

$$|(F_1^{m/2})^{-1}(\mathrm{w},\ z)|^2_{m/2} = |z|^2(1+|\mathrm{w}|^2)^{m+1},$$
$$|(F_2^{m/2})^{-1}(\mathrm{v},\ z)|^2_{m/2} = |z|^2(1+|\mathrm{v}|^2)^{m+1}$$

is obtained.

C. Polarizations

Let $(M,\ \omega)$ be a symplectic space. M must be of even dimension. Let $n := \frac{1}{2}\dim M$.

III.9. Definition. A smooth, complex distribution F on M (i.e. a complex sub-bundle of the complex tangent bundle $T(M)^C$) is called a *polarization of* $(M,\ \omega)$ if:

1. for each $x\in M$, $\dim(F_x) = n$, and F_x is an isotropic subspace of $T_x(M)^C$, i.e. for X_x, $Y_x\in F_x$

$$\langle X_x,\ Y_x\,|\,\omega(x)\rangle = 0;$$

2. F is involutive, i.e. if X and Y are vector fields on M taking values in F, then $[X,\ Y]$ also takes values in F;

3. $F\cap\overline{F}$ is a complex distribution of constant dimension equal to k;

4. $F+\overline{F}$ is an involutive complex distribution.

We note that

$$D := \bigcup_{x\in M} F_x\cap\overline{F_x}\cap T_x(M) \tag{35}$$

and

$$E := \bigcup_{x\in M} (F_x+\overline{F_x})\cap T_x(M) \tag{36}$$

are real involutive smooth distributions of dimensions k and $(2n-k)$, respectively.

By 1, D and E are ω-ortogonal, i.e. $\langle X_x, Y_x | \omega(x)\rangle = 0$, if $X_x \in D$ and $Y_x \in E$.

F is called *positive* if

$$i\langle \overline{X_x}, X_x | \omega(x)\rangle \geqslant 0 \quad \text{for } X_x \in F. \tag{37}$$

F is called *real* if $F = \overline{F}$. If F is real, then $E = D = F\cap T(M)$.

F is called *kählerian* if $F_x\cap\overline{F_x} = \{0_x\}$ for $x\in M$. Then $D_x = \{0_x\}$ and $E = T_x(M)$.

Let K be an involutive real smooth distribution on the manifold N. K is said to be *regular* if on the quotient space N/K (with the quotient topology) of the maximal connected integral submanifolds of K there exists a structure of a manifold such that the canonical projection

$$\varrho_K\colon N\to N/K$$

is a submersion (if such a structure exists, then it is unique — see [9])

III.10. DEFINITION. We say that a polarization F is *strongly admissible* if distributions D and E are regular. Then the following submersions exist:

$$\varrho_D\colon M\to M/D,$$
$$\varrho_E\colon M\to M/E,$$
$$\varrho_{DE}\colon M/D\to M/E,$$

and

$$\varrho_{DE}\cdot\varrho_D = \varrho_E.$$

In the present paper we shall consider only strongly admissible polarizations.

III.11. EXAMPLE (cotangent bundle). Let (M, ω) be as in Example III.2. Then $F = \bigcup_{x\in M}\{X_x\in T_x(M)^C\colon \varkappa_* X_x = 0\}$, where $\varkappa\colon T^*(\mathscr{X})\to\mathscr{X}$ is the natural projection, is a real polarization. In this case $M/D = M/E$ is diffeomorphic to $\mathscr{X}$, with $\varrho_D = \varrho_E$ corresponding to $\varkappa$.

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III.12. Example (complex projective plane). Let $M = P^1(C)$ and ω_s be as in Example III.8. $P^1(C)$ is a complex manifold. $F :=$ the sub-bundle of holomorphic vectors is a positive polarization.

For a polarization F, let

$$C_F^\infty(M) := \{f \in C^\infty(M)\colon \ X(f) = 0 \quad \text{for } X \text{ taking values in } F\}.$$

III.13. Proposition. *Let F be a real polarization. Then $C_F^\infty(M)$ is a maximal commutative Lie subalgebra of $C^\infty(M)$. It is also an associative subalgebra of $C^\infty(M)$ (i.e. a subalgebra of $C^\infty(M)$ treated as an associative algebra under function multiplication).*

Proof. For $f \in C_F^\infty(M)$ and X being a vector field taking values in F, we have

$$(38) \qquad 0 = X(f) = \langle X | df\rangle = \langle X | X_f \lrcorner \omega\rangle = \langle X_f, X | \omega\rangle .$$

Thus X_f takes values in F, since for every $x \in M$, F_x is an n-dimensional, hence maximal, isotropic subspace of $T_x(M)^C$. For $f, g \in C_F^\infty(M)$, this yields

$$(39) \qquad \{f, g\} = X_g(f) = 0,$$

which proves that $C_F^\infty(M)$ is commutative.

Suppose that for $h \in C_F^\infty(M)$ and for all $f \in C_F^\infty(M)$ we have

$$(40) \qquad \{f, h\} = 0 .$$

Now $\varrho_D^*\colon C^\infty(M/D) \to C^\infty(M)$, $\varrho_D^* f = f \circ \varrho_D$, maps onto $C_F^\infty(M)$. Let $x \in M$ and $f_k \in C^\infty(M/D)$ be such that $(df_k)\big(\varrho_D(x)\big)$ form a basis in $T_{\varrho_D(x)}(M/D)$. Then, since ω is non-degenerate, $X_{f_k \cdot \varrho_D}(x)$ form a basis of F_x. By (40),

$$X_{f_k \cdot \varrho_D}(x)(h) = 0 .$$

Thus $h \in C_F^\infty(M)$, which proves the maximality of $C_F^\infty(M)$. The remaining statement of the proposition is trivial. ■

III.14. Remark. If F is not real, then, in general, the statement of Proposition III.13 fails to hold and it only holds locally (see the case of Example III.12 for which $C_F^\infty(M)$ consists of constant functions only).

In general, not all of the maximal commutative Lie subalgebras of $C^\infty(M)$ (which are automatically also associative subalgebras of $C^\infty(M)$) are of the form described in Proposition III.13 (take for example functions on S^2 constant on parallels — see Example III.3).

D. Hilbert space connected with a polarized symplectic manifold

Let F be a polarization of (M, ω). Let $D_F := \Lambda^n T^*(M)^C/F$ be the bundle of complex n-covectors tangent to M, vanishing after contracting with any vector from F.

III.15. DEFINITION. A square root $(D_F^{1/2}, \iota)$ of the bundle D_F (see Definition II.5) will be called a *square root structure* for F. Two square root structures $(D_F^{1/2}, \iota)$ and $(D_F^{1/2\prime}, \iota')$ for F will be considered equivalent if there exists a bundle isomorphism k of $D_F^{1/2}$ and $D_F^{1/2\prime}$ such that $\iota = \iota' \circ (k \otimes k)$.

III.16. Remark. The bundle $\Lambda^n T^*(M)^C/F$ defines an element $\xi_F \in H^1(M, \tilde{\mathscr{C}})$ (where $\tilde{\mathscr{C}}$ denotes the sheaf of germs of local non-vanishing complex (smooth) functions on M) in a standard way. One has a short exact sequence of sheaf homomorphisms

$$(41)\qquad 1 \to \mathscr{Z}_2 \to \tilde{\mathscr{C}} \to \tilde{\mathscr{C}} \to 1,$$

where $\mathscr{Z}_2$ is the sheaf of germs of constant functions taking values in the multiplicative group $Z_2 = \{+1, -1\}$, and the 3rd arrow from the left denoted the sheaf homomorphism induced by taking square of local $\tilde{C}$-valued functions.

From the corresponding cohomological exact sequence

$$(42)\qquad \to H^1(M, \tilde{\mathscr{C}}) \to H^1(M, \tilde{\mathscr{C}}) \xrightarrow{\delta^*} H^2(M, Z_2) \to$$

the following condition is obtained:

A square root structure for F exists if and only if $\delta^*(\xi_F) = 0$.

One can also easily show that the non-equivalent square root structures are in one-to-one correspondence with the elements of $H^1(M, Z_2)$.

For the rest of Chapter III *we shall fix a positive polarization F with a square root structure $(D_F^{1/2}, \iota)$.*

Let us consider the bundle $\Lambda^{2n-k}T^*(M)^C/(F \cap \bar{F})$ of complex $(2n-k)$-covectors vanishing after contracting with vectors from $F \cap \bar{F}$. $\Lambda^{2n-k}T^*(M)^C/(F \cap \bar{F})$ is isomorphic to $(\Lambda^{2n-k}T^*(M)/D)^C$ in an evident way.

We shall define a $1\frac{1}{2}$-linear pairing of $C^\infty(M)$-modules

$$\langle\,,\,\rangle_F\colon \Gamma(D_F^{1/2}) \times \Gamma(D_F^{1/2}) \to \Gamma\big(|\Lambda^{2n-k}T^*(M)^C/(F \cap \bar{F})|\big).$$

This will be done as follows:

Let $y \in M$ and $\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_n$ be a basis of $T_y^*(M)^C/F$ such that $\eta_1, \ldots, \eta_k$ vanish on $(F+\bar{F})$-directions. Then $\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_n}$ constitute a basis for $T_y^*(M)^C/(F \cap \bar{F})$ and $\eta_1 \wedge \ldots \wedge \eta_k \wedge \wedge \eta_{k+1} \wedge \ldots \wedge \eta_n \wedge \overline{\eta_{k+1}} \wedge \ldots \wedge \overline{\eta_n}$ is a non-zero element of $\Lambda^{2n-k}T_y^*(M)^C/(F \cap \bar{F})$.

Let $(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2}$ be an element of $(D_F^{1/2})_y$ such that

$$(43)\qquad (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} \underline{\otimes} (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} = \eta_1 \wedge \ldots \wedge \eta_n.$$

For $\varphi, \psi \in \Gamma(D_F^{1/2})$, we shall put

$$\begin{aligned}(44)\quad & \langle\varphi, \psi\rangle_F(y) \\ & := \overline{\left(\frac{\varphi(y)}{(\eta_1\wedge \ldots \wedge\eta_n)^{1/2}}\right)}\left(\frac{\psi(y)}{(\eta_1\wedge \ldots \wedge\eta_n)^{1/2}}\right)|\hat{\omega}^{n-k}(\eta_{k+1}, \ldots \\ & \qquad \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_n})|^{-1/2}|\eta_1\wedge \ldots \wedge\eta_n\wedge\overline{\eta_{k+1}}\wedge \ldots \wedge\overline{\eta_n}| \\ & \in |\Lambda^{2n-k}T_y^*(M)^C/(F\cap\overline{F})|,\end{aligned}$$

where

$$\begin{aligned}(45)\quad & \hat{\omega}^{n-k}(\eta_{k+1}, \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_k}) \\ & := \langle\eta_{k+1}^{\#}, \ldots, \eta_n^{\#}, \overline{\eta_{k+1}^{\#}}, \ldots, \overline{\eta_n^{\#}}\,|\underbrace{\omega(y)\wedge \ldots \wedge\omega(y)}_{(n-k)\text{-times}}\rangle.\end{aligned}$$

We check that the definition of $\langle\varphi, \psi\rangle_F$ is correct.

1. $\langle\varphi, \psi\rangle_F(y)$ evidently does not depend upon the choice of the "square root" $(\eta_1\wedge \ldots \wedge\eta_n)^{1/2}$ of $(\eta_1\wedge \ldots \wedge\eta_n)$, as two possible choices differ by sign.

2. We check that $\langle\varphi, \psi\rangle_F(y)$ does not depend upon the choice of the basis $\eta_1, \ldots, \eta_n$ of $T_y^*(M)^C/F$. Let $\eta_1', \ldots, \eta_n'$ be another basis. Then

$$(46)\qquad \eta_i' = \sum_{j=1}^{n} A_{ij}\eta_j, \qquad i = 1, \ldots, n,$$

where

$$(47)\qquad (A_{ij})^{n\times n} = \left(\begin{array}{c|c} a^{k\times k} & 0 \\ \hline c^{(n-k)\times k} & b^{(n-k)\times(n-k)}\end{array}\right),$$

$$(48)\qquad \eta_1'\wedge \ldots \wedge\eta_n' = (\det a)(\det b)\eta_1\wedge \ldots \wedge\eta_n,$$

$$(49)\qquad (\eta_1'\wedge \ldots \wedge\eta_n')^{1/2} = \pm(\det a)^{1/2}(\det b)^{1/2}(\eta_1\wedge \ldots \wedge\eta_n)^{1/2},$$

$$\begin{aligned}(50)\quad & \hat{\omega}^{n-k}(\eta_{k+1}', \ldots, \eta_n', \overline{\eta_{k+1}'}, \ldots, \overline{\eta_n'}) \\ & = |\det b|^2\hat{\omega}^{n-k}(\eta_{k+1}, \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_n}),\end{aligned}$$

as $\hat{\omega}(\eta_i, \eta_j) = \hat{\omega}(\eta_i, \overline{\eta_j}) = 0$ for $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$ because $\eta_i^{\#} \in F\cap\overline{F}$ and $\eta_j^{\#}, \overline{\eta_j}^{\#} \in F+\overline{F}$.

Further,

$$\begin{aligned}(51)\quad & |\eta_1'\wedge \ldots \wedge\eta_n'\wedge\overline{\eta_{k+1}'}\wedge \ldots \wedge\overline{\eta_n'}| \\ & = |\det a|\,|\det b|^2|\eta_1\wedge \ldots \wedge\eta_n\wedge\overline{\eta_{k+1}}\wedge \ldots \wedge\overline{\eta_n}|.\end{aligned}$$

Combining (49), (50) and (51), we get

$$\overline{\left(\frac{\varphi(y)}{(\eta_1' \wedge \ldots \wedge \eta_n')^{1/2}}\right)}\left(\frac{\psi(y)}{(\eta_1' \wedge \ldots \wedge \eta_n')^{1/2}}\right)|\hat{\omega}^{n-k}(\eta_{k+1}', \ldots \tag{52}$$

$$\ldots, \eta_n', \overline{\eta_{k+1}'}, \ldots, \overline{\eta_n'})|^{-1/2}|\eta_1' \wedge \ldots \wedge \eta_n' \wedge \overline{\eta_{k+n}'} \wedge \ldots \wedge \overline{\eta_n'}|$$

$$= \overline{\left(\frac{\varphi(y)}{(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2}}\right)}\left(\frac{\psi(y)}{(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2}}\right)|\hat{\omega}^{n-k}(\eta_{k+1}, \ldots$$

$$\ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_n})|^{-1/2}|\eta_1 \wedge \ldots \wedge \eta_n \wedge \overline{\eta_{k+1}} \wedge \ldots \wedge \overline{\eta_n}|,$$

which has been searched for.

3. $M \ni y \to \langle\varphi, \psi\rangle_F(y)$ is smooth, since we can locally choose the basis $\eta_1, \ldots, \eta_n$ in a smooth way (F, $F \cap \bar{F}$, and $F + \bar{F}$ are smooth distributions).

Let L with a connection α and a hermitian structure $(\cdot, \cdot)$ be a quantum bundle (see Definition III.6) over (M, ω). We shall consider the bundle $L \otimes D_F^{1/2}$. One can define a $1\frac{1}{2}$-pairing of $C^\infty(M)$-modules

$$\langle\!\langle\, , \rangle\!\rangle_F\colon \Gamma(L \otimes D_F^{1/2}) \times \Gamma(L \otimes D_F^{1/2}) \to \Gamma\big(|\Lambda^{2n-k} T^*(M)^C/(F \cap \bar{F})|\big),$$

such that

$$\langle\!\langle s \otimes \varphi, t \otimes \psi\rangle\!\rangle_F = (s, t)\langle\varphi, \psi\rangle_F \tag{53}$$

if s, $t \in \Gamma(L)$ and $\varphi, \psi \in \Gamma(D_F^{1/2})$.

The mapping $\varrho_D\colon M \to M/D$ induces an injective mapping of forms

$$\varrho_D^*\colon \Gamma_{\mathrm{loc}}\big(\Lambda^{2n-k} T^*(M/D)^C\big) \to \Gamma_{\mathrm{loc}}\big(\Lambda^{2n-k} T^*(M)^C/(F \cap \bar{F})\big)$$

and, according to a similar rule, an injective mapping

$$|\varrho_D^*|\colon \Gamma_{\mathrm{loc}}\big(|\Lambda^{2n-k} T^*(M/D)^C|\big) \to \Gamma_{\mathrm{loc}}\big(|\Lambda^{2n-k} T^*(M)^C/(F \cap \bar{F})|\big)$$

such that for non-vanishing section $\eta \in \Gamma_{\mathrm{loc}}\big(\Lambda^{2n-k} T^*(M/D)^C\big)$,

$$|\varrho_D^*|\,|\eta| = |\varrho_D^* \eta|. \tag{54}$$

For X being a vector field on M, one can obtain an X-derivation $\nabla_X \otimes \mathscr{L}_X^{1/2}$ of $\Gamma_{\mathrm{loc}}(L \otimes D_F^{1/2})$, according to the prescription given in Chapter II.

III.17. PROPOSITION. *Let* $\sigma, \varrho \in \Gamma(L \otimes D_F^{1/2})$ *be D-horizontal, i.e. let*

$$(\nabla_X \otimes \mathscr{L}_X^{1/2})\sigma = (\nabla_X \otimes \mathscr{L}_X^{1/2})\varrho = 0 \tag{55}$$

for X taking values in D.

Then there exists a unique section

$$\langle\!\langle\sigma, \varrho\rangle\!\rangle \in \Gamma\big(|\Lambda^{2n-k} T^*(M/D)^C|\big)$$

such that

$$|\varrho_D^*|\langle\!\langle\sigma, \varrho\rangle\!\rangle = \langle\!\langle\sigma, \varrho\rangle\!\rangle_F. \tag{56}$$

Proof. We shall start with the following:

LEMMA. *Let $\varkappa$; $P \to Q$ be a surjective submersion of manifolds P and Q. Let $\varkappa^{-1}(q)$ be connected for all $q \in Q$. Let G be a smooth real (complex) distribution comprising the $\varkappa$-vertical sub-bundle* $\ker \varkappa_*$ *of* $T(P)^{(C)}$, *i.e. the subbundle of vectors annihilated by $\varkappa_*$. Suppose that G satisfies*

(*) *if X, Y are vector fields on P, X being $\varkappa$-vertical and Y taking values in G, then $[X, Y]$ also takes values in G.*

Then G projects by $\varkappa_*$ to a smooth real (complex) distribution $\varkappa_*(G)$ on Q of dimension less then the dimension of G by $\dim P - \dim Q$. *If G is involutive, then so is $\varkappa_*(G)$.*

Proof of lemma. First we shall show that if X is a $\varkappa$-vertical vector field and Φ_t is its (local) flow, then Φ_t preserves G.

Let $p \in P$. Let us choose a scalar product in $T_p(P)^{(C)}$. Let E_t be an orthogonal projection onto $(\Phi_t)_*(G_{\Phi_{-t}(p)})$. E_t smoothly depends on t. Let Z be a vector field taking values in G, and let

$$Z_t := \big((\Phi_t)_* Z\big)(p).$$

We have

$$\frac{dZ_t}{dt} = \big((\Phi_t)_* \mathscr{L}_X Z\big)(p) \in E_t\big(T_p(P)^{(C)}\big)$$

as in virtue of (*), $\mathscr{L}_X Z = [X, Z]$ takes values in G. Now

$$\frac{dZ_t}{dt} = \frac{d(E_t Z_t)}{dt} = \frac{dE_t}{dt} Z_t + E_t \frac{dZ_t}{dt} = \frac{dE_t}{dt} Z_t + \frac{dZ_t}{dt}.$$

Thus

$$\frac{dE_t}{dt} Z_t = 0.$$

Since for different Z, Z_t span the range of E_t, we get

$$\frac{dE_t}{dt} E_t = 0.$$

But from $E_t = E_t^*$ and $E_t^2 = E_t$ one obtains

$$\left(\frac{dE_t}{dt}\right)^* = \frac{dE_t}{dt} \quad \text{and} \quad \frac{dE_t}{dt} E_t + E_t \frac{dE_t}{dt} = \frac{dE_t}{dt}.$$

Hence

$$\frac{dE_t}{dt} = E_t \frac{dE_t}{dt} = \left(\frac{dE_t}{dt} E_t\right)^* = 0.$$

Consequently, $E_t = E_0$, and Φ_t preserves G.

Remark. In fact, we have proved the following: if X is a vector field on a manifold N, and H is a smooth distribution on N preserved by X (i.e. such that for Y taking values in H, $[X, Y]$ also takes values in H), then the local flow of X preserves H.

Now let $q \in Q$ and $X_q \in T_q(Q)^{(C)}$. Let $p_1, p_2 \in \varkappa^{-1}(q)$, and $Y^i_{p_i} \in T_{p_i}(P)^{(C)}$ be such that $\varkappa_*(Y^i_{p_i}) = X_q$, $i = 1, 2$. Then

$$(57) \qquad (Y^1_{p_1} \in G_{p_1}) \Leftrightarrow (Y^2_{p_2} \in G_{p_2}).$$

Indeed, if we can join p_1 and p_2 by an integral curve of a $\varkappa$-vertical vector field X, that is, if $\Phi_t(p_1) = p_2$ for some t (Φ_t being the local flow of X), then $(\Phi_t)_* Y^1_{p_1} \in G_{p_2}$, but it differs from $Y^2_{p_2}$ by a $\varkappa$-vertical (and hence belonging to G_{p_2}) vector. But each pair of points in $\varkappa^{-1}(q)$ can be joined by a piece-wise smooth continuous curve, whose each smooth piece is an integral curve of a $\varkappa$-vertical vector field (in fact there always exists such a curve composed of one piece).

From (57) it follows that

$$\varkappa_*(G) = \{X \in T(Q)^{(C)}\colon \text{ there exists } Y \in G \subset T(P)^{(C)} \text{ such that } \varkappa_* Y = X\}$$

is a distribution on P of constant dimension equal to the dimension of G minus $(\dim Q - \dim P)$. The moment we have (57), smoothness of $\varkappa_*(G)$ follows trivially. Also, involutiveness of $\varkappa_*(G)$ follows easily from that of G, by taking local lifts to P of vector fields on Q taking values in $\varkappa_*(G)$. □

Corollary. *There exists on M/D a real* (*complex*) *smooth involutive distribution $E/D := \varrho_{D^*}(E)$ $(F/D^C := \varrho_{D^*}(F))$ of real* (*complex*) *dimension $2(n-k)$ $((n-k))$.*

Proof of corollary. The proof follows immediately from lemma in virtue of

$$(58) \qquad [D, F] \subset F \quad \text{and} \quad [D, E] \subset E,$$

where $[D, F]$ $([D, E])$ is a distribution spanned by commutators of fields taking values in D and F (in D and E).

Let $x \in M/D$ and $\chi_1, \ldots, \chi_n$ be 1-forms defined in a neighbourhood $U \subset M/D$ of x such that $\chi_1, \ldots, \chi_k$ vanish on vectors from E/D, $\chi_{k+1}, \ldots, \chi_n$ vanish on vectors from F/D^C, and values of $\chi_1, \ldots, \chi_n$ are linearly independent at each point.

Let

$$(59) \qquad \eta_i = \varrho^*_D \chi_i, \quad i = 1, \ldots, n.$$

By (54),

$$(60) \quad |\eta_1 \wedge \ldots \wedge \eta_n \wedge \overline{\eta_{k+1}} \wedge \ldots \wedge \overline{\eta_n}| = |\varrho^*_D|(|\chi_1 \wedge \ldots \wedge \chi_n \wedge \overline{\chi_{k+1}} \wedge \ldots \wedge \overline{\chi_n}|).$$

We shall show that on $\varrho_D^{-1}(U)$

$$(61) \qquad \langle\!\langle \sigma, \varrho \rangle\!\rangle_F = f|\varrho^*_D|(|\chi_1 \wedge \ldots \wedge \chi_n \wedge \overline{\chi_{k+1}} \wedge \ldots \wedge \overline{\chi_n}|),$$

where f is constant on fibres of ϱ_D, which will end the proof of Proposition III.17.

First we notice that $|\hat{\omega}^{n-k}(\eta_{k+1}, \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_n})|^{-1/2}$ is constant along ϱ_D-fibres. This follows from the relations

$$X\big(\hat{\omega}(\eta_i, \eta_j)\big) = X\big(\hat{\omega}(\overline{\eta_i}, \eta_j)\big) = X\big(\hat{\omega}(\overline{\eta_i}, \overline{\eta_j})\big) = 0, \tag{62}$$

where X is a vector field taking values in D. We shall prove (62). Taking the Lie derivative of both sides of the relation

$$\eta_i^{\#} \lrcorner \omega = \eta_i, \tag{63}$$

we obtain

$$[X, \eta_i^{\#}] \lrcorner \omega + \eta_i^{\#} \lrcorner \mathscr{L}_X \omega = 0, \tag{64}$$

$$[X, \eta_i^{\#}] \lrcorner \omega + \eta_i^{\#} \lrcorner d(X \lrcorner \omega) = 0. \tag{65}$$

Let Y be a vector field taking values in $F + \overline{F}$. From (65) we obtain

$$\begin{aligned} \langle Y, [X, \eta_i^{\#}] | \omega\rangle &= \langle \eta_i^{\#}, Y | d(X \lrcorner \omega)\rangle \\ &= \eta_i^{\#}(\langle X, Y|\omega\rangle) - Y(\langle X, \eta_i^{\#}|\omega\rangle) - \langle X, [\eta_i^{\#}, Y]|\omega\rangle = 0, \end{aligned} \tag{66}$$

since D and $F + \overline{F}$ are ω-orthogonal, X takes values in D, Y in $F + \overline{F}$, $\eta_i^{\#}$ also in $F + \overline{F}$, and $F + \overline{F}$ is involutive. Thus $[X, \eta_i^{\#}]$ takes values in $F \cap \overline{F}$, since $F \cap \overline{F}$ is just the distribution ω-orthogonal to $F + \overline{F}$. Hence, we have

$$\begin{aligned} 0 = \langle X, \eta_i^{\#}, \eta_j^{\#} | d\omega\rangle &= X(\langle \eta_i^{\#}, \eta_j^{\#}|\omega\rangle) - \eta_i^{\#}(\langle X, \eta_j^{\#}|\omega\rangle) + \\ &+ \eta_j^{\#}(\langle X, \eta_i^{\#}|\omega\rangle) - \langle [X, \eta_i^{\#}], \eta_j^{\#}|\omega\rangle + \langle [X, \eta_j^{\#}], \eta_i^{\#}|\omega\rangle - \\ &- \langle [\eta_i^{\#}, \eta_j^{\#}], X|\omega\rangle = X(\langle \eta_i^{\#}, \eta_j^{\#}|\omega\rangle). \end{aligned} \tag{67}$$

The other relations comprised in (62) are proved analogously. $\eta_1 \wedge \ldots \wedge \eta_n$ is a local non-vanishing section of $\Lambda^n T^*(M)^C/F$ defined on $\varrho_D^{-1}(U)$. Let $y \in M$, $\varrho_D(y) \in U$. One can choose, in a simply connected neighbourhood V of y, a (smooth) section $(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} \in \Gamma_V(D_F^{1/2})$ such that

$$(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} \underline{\otimes} (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} = (\eta_1 \wedge \ldots \wedge \eta_n)|_V. \tag{68}$$

From (4), for a vector field X taking values in D, we have

$$\big(\mathscr{L}_X^{1/2}(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2}\big) \underline{\otimes} (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} = 0. \tag{69}$$

Thus

$$\mathscr{L}_X^{1/2}(\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} = 0. \tag{70}$$

Further,

$$\sigma|_V = s \otimes (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2}, \tag{71}$$

$$\varrho|_V = t \otimes (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2}, \tag{72}$$

where $s, t \in \Gamma_V(L)$. (2), (55) and (70) yield

$$(\nabla_X s) \otimes (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} = 0, \qquad (\nabla_X t) \otimes (\eta_1 \wedge \ldots \wedge \eta_n)^{1/2} = 0,$$

and, consequently,

$$\nabla_X s = \nabla_X t = 0. \tag{73}$$

Thus

$$X\big((s, t)\big) = (\nabla_X s, t) + (s, \nabla_X t) = 0. \tag{74}$$

By (53), (73), (74) and (44), we get

$$\langle\!\langle \sigma, \varrho \rangle\!\rangle_F(y) = (s, t)(y)\, |\hat{\omega}^{n-k}(\eta_{k+1}, \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots \\ \ldots, \overline{\eta_n})|^{-\frac{1}{2}}(y)\, |\eta_1 \wedge \ldots \wedge \eta_n \wedge \overline{\eta_{k+1}} \wedge \ldots \wedge \overline{\eta_n}|(y) \tag{75}$$

for $y \in V$, and $(s, t)\, |\hat{\omega}^{n-k}(\eta_{k+1}, \ldots, \eta_n, \overline{\eta_{k+1}}, \ldots, \overline{\eta_n})^{-1/2}$ is a function constant along fibres of ϱ_D. This completes the proof of Proposition III.17. ∎

For D-horizontal sections $\sigma, \varrho \in \Gamma(L \otimes D_F^{1/2})$ $\langle\!\langle \sigma, \varrho \rangle\!\rangle$ is an absolute volume element on M/D (not necessarily non-vanishing). If $\sigma = \varrho$, then this volume element defines a positive measure.

III.18. DEFINITION. We say that a D-horizontal section $\sigma \in \Gamma(L \otimes D_F^{1/2})$ is *of finite norm* if the measure defined on M/D by $\langle\!\langle \sigma, \sigma \rangle\!\rangle$ is finite.

One easily checks that if σ and ϱ are of finite norm, then the measure defined by $\langle\!\langle \sigma, \varrho \rangle\!\rangle$ is of finite absolute variation.

Now we shall consider a subspace of $\Gamma(L \otimes D_F^{1/2})$ composed of F-horizontal sections of finite norm. We can define a scalar product on this subspace by

$$(\sigma, \varrho)_F := \int_{M/D} \langle\!\langle \sigma, \varrho \rangle\!\rangle. \tag{76}$$

The completion of this unitary space will be denoted by H_F. It is just the Hilbert space assigned to the polarized symplectic manifold with a quantum bundle by the procedure of geometric quantization. From the point of view of physics, it should be interpreted as the space of states of the quantized system.

III.19. EXAMPLE (cotangent bundle). Let (M, ω), a quantum bundle over (M, ω) and a polarization for (M, ω) be as in Examples III.2, 7 and 11. The bundle $\Lambda^n T^*(M)^C/F$ is isomorphic to the pull back $\varkappa^*\big(\Lambda^n T^*(\mathscr{X})^C\big)$ of the bundle $\Lambda^n T^*(\mathscr{X})^C$ by $\tilde{\omega}\colon M \to \mathscr{X}$. If $\xi_F \in H^1(M, \tilde{\mathscr{C}})$ is an element defined in the usual way by $\Lambda^n T^*(M)^C/F$ (see Remark III.16), then $\xi_F = \varkappa^* \zeta$, where $\zeta \in H^1(\mathscr{X}, \tilde{\mathscr{C}})$ is defined by $\Lambda^n T^*(\mathscr{X})^C$. Now from (42) and an analogous exact sequence for $\mathscr{X}$, $\delta^*(\xi_F) = \delta^*(\varkappa^* \zeta) = \varkappa^*(\delta^* \zeta) = 0$ if and only if $\delta^* \zeta = 0$, as $\varkappa^*\colon H^k(\mathscr{X}, Z_2) \to H^k(M, Z_2)$ is an isomorphism. But $\delta^* \zeta$ is the Chern class of $\Lambda^n T\ (\mathscr{X})^C$ reduced to Z_2. Thus a square root structure for F exists if and only if the Chern class of $\Lambda^n T^*(\mathfrak{X})^C$ reduced to Z_2 is zero. If it is zero, then the non-equivalent square root structures for F are in one-to-one correspondence with the elements of $H^1(\mathfrak{X}, Z_2)$. Moreover, if $(D_F^{1/2}, \iota)$ is a square root structure for F, then we can assume, at most passing to an equivalent square root structure, that $D_F^{1/2} = \varkappa^*(D_{\mathfrak{X}}^{1/2})$, and $\iota = \varkappa^* \nu$, where $(D_{\mathfrak{X}}^{1/2}, \nu)$ is a square root of $\Lambda^n T^*(\mathfrak{X})^C$. Since in our case $L = M \times C$, $L \otimes D_F^{1/2}$ is isomorphic to $D_F^{1/2}$. Using this isomorphism, we can assign to an element $\sigma \in \Gamma(L \otimes D_F^{1/2})$ an element of $\Gamma(D_F^{1/2})$. One easily checks that by this identification, F-horizontal sections are in one-to-one correspondence with F-horizontal sections of $D_F^{1/2}$ and the latter are in one-to-one correspondence with sections of $D_{\mathfrak{X}}^{1/2}$. For an F-horizontal section σ of $L \otimes D_F^{1/2}$, let $\underset{\sim}{\sigma}$ denote the corresponding section of $D_{\mathfrak{X}}^{1/2}$. We shall denote

$$\bar{\underset{\sim}{\sigma}} \underset{\sim}{\varrho} := \langle\!\langle \sigma, \varrho \rangle\!\rangle$$

due to the way in which one can produce $\langle\!\langle \sigma, \varrho \rangle\!\rangle$ directly from $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varrho}$ Thus, in the case being described, we can identify H_F with the completion $H_{\mathfrak{X}}$ of the unitary space of $\underset{\sim}{\vartheta} \in \Gamma(D_{\mathfrak{X}}^{1/2})$, such that

$$\|\vartheta\|_{\mathfrak{X}}^2 := \int_{\mathfrak{X}} \bar{\underset{\sim}{\vartheta}} \underset{\sim}{\vartheta} < \infty. \tag{77}$$

III.20. EXAMPLE (complex projective plane). Let (M, ω_s), a quantum bundle over (M, ω_s) and a polarization F be as in Examples III.8 and 12. We shall use the notation introduced there. For F, there exists a square root structure which is, up to the equivalence, unique — since $H^1(P^1(C), Z_2) = 0$. We shall describe it explicitly.

Let

$$D_F^{1/2} := (\{1\} \times 0_1 \times C) \cup (\{2\} \times 0_2 \times C) / \sim D_F^{1/2}, \tag{78}$$

where the relation $\sim D_F^{1/2}$ is defined as follows:

$$(1, [1, \mathrm{w}], z) \sim D_F^{1/2} \, (2, [\mathrm{v}, 1], z'), \qquad \frac{1}{\mathrm{v}} = \mathrm{w} \quad \text{and} \quad z' = i\bar{\mathrm{w}} z. \tag{79}$$

The bundle projection π will be defined on representing elements as the projection onto the second factor. On $\pi^{-1}(0_i)$, $i = 1, 2$, one can define diffeomorphic maps $G_i := \pi^{-1}(0_i) \to C^2$,

$$G_1\big([1, [1, \mathrm{w}], z]\big) := (\mathrm{w}, z), \tag{80}$$

$$G_2\big([2, [\mathrm{v}, 1], z]\big) := (\mathrm{v}, z). \tag{81}$$

Let us put

$$\iota\big(G_1^{-1}(\mathrm{w}, z_1) \otimes G_1^{-1}(\mathrm{w}, z_2)\big) := z_1 z_2 \, d\bar{w}([1, \mathrm{w}]), \tag{82}$$

$$i\big(G_2^{-1}(\mathrm{v}, z_1) \otimes G_2^{-1}(\mathrm{v}, z_2)\big) := z_1 z_2 \, d\bar{v}([\mathrm{v}, 1]). \tag{83}$$

We assertain that ι is a bundle isomorphism of $\boldsymbol{D}_F^{1/2} \otimes \boldsymbol{D}_F^{1/2}$ onto $\Lambda^n \boldsymbol{T}^*(M)^{\boldsymbol{C}}/F = \boldsymbol{D}_F$, and thus, $(\boldsymbol{D}_F^{1/2}, \iota)$ is a square root structure for F.

Now let us note that

$$(84) \qquad \begin{aligned} 0_1 \ni [1, \mathrm{w}] &\xrightarrow{(d\bar{w})^{1/2}} G_1^{-1}(\mathrm{w}, 1) \in D_F^{1/2}, \\ 0_2 \ni [\mathrm{v}, 1] &\xrightarrow{(d\bar{v})^{1/2}} G_2^{-1}(\mathrm{v}, 1) \in D_F^{1/2} \end{aligned}$$

are non-vanishing horizontal sections of $\boldsymbol{D}_F^{1/2}$ defined on 0_1 and 0_2, respectively, and

$$(85) \qquad (d\bar{w})^{1/2} \underline{\otimes} (d\bar{w})^{1/2} = d\bar{w}, \qquad (d\bar{v})^{1/2} \underline{\otimes} (d\bar{v})^{1/2} = d\bar{v}.$$

Similarly,

$$(86) \qquad 0_1 \ni [1, \mathrm{w}] \xrightarrow{s_1^m} (F_1^{m/2})^{-1}\left(\mathrm{w}, \frac{1}{(1+|\mathrm{w}|^2)^{m+1}}\right) \in L_{m/2},$$

$$(87) \qquad 0_2 \ni [\mathrm{v}, 1] \xrightarrow{s_2^m} (F_2^{m/2})^{-1}\left(\mathrm{v}, \frac{1}{(1+|\mathrm{v}|^2)^{m+1}}\right) \in L_{m/2}$$

are non-vanishing F-horizontal sections of $\boldsymbol{L}_{m/2}$ over 0_1 and 0_2, respectively.

If σ is an F-horizontal section of $\boldsymbol{L}_{m/2} \otimes \boldsymbol{D}_F^{1/2}$, then

$$\sigma\restriction_{0_1} = \overline{\psi_1 \circ w}\; s_1^m \otimes (d\bar{w})^{1/2} \quad \text{and} \quad \sigma\restriction_{0_2} = \overline{\psi_2 \circ v}\; s_2^m \otimes (d\bar{v})^{1/2},$$

where ψ_1 and ψ_2 are holomorphic functions on $\boldsymbol{C}$. Conditions of compatibility yield

$$\frac{\overline{\psi_2\left(\dfrac{1}{\mathrm{w}}\right)}}{\left(1 + \dfrac{1}{|\mathrm{w}|^2}\right)^{m+1}} = \frac{i\, \mathrm{w}^{m+1} \overline{\mathrm{w}}\; \overline{\psi_1(\mathrm{w})}}{(1+|\mathrm{w}|^2)^{m+1}} \qquad \text{for } \mathrm{w} \in \boldsymbol{C} \setminus \{0\}.$$

Thus,

$$(88) \qquad \psi_2\left(\frac{1}{\mathrm{w}}\right) = \frac{i}{\mathrm{w}^m}\, \psi_1(\mathrm{w}).$$

This means that function ψ_2, holomorphic on $\boldsymbol{C}$, has in ∞ a pole of degree less or equal to m. The only solutions of (88) are of the form

$$(89) \qquad \psi_1(\mathrm{w}) = \sum_{j=0}^{m} \bar{a}_j \mathrm{w}^j, \qquad \psi_2(\mathrm{v}) = i \sum_{j=0}^{m} \overline{a_{m-j}}\, \mathrm{v}^j.$$

Thus, $\boldsymbol{H}_F$ in our case is $(m+1) = (2s+1)$-dimensional. Let $\{a_j\}$ denote an element σ of $\boldsymbol{H}_F$ such that

$$\sigma\restriction_{0_1} = \overline{\psi_1 \circ w}\; s_1^m \otimes (d\bar{w})^{1/2} \quad \text{and} \quad \psi_1(\mathrm{w}) = \sum_{j=0}^{m} \bar{a}_j \mathrm{w}^j.$$

Simple calculations yield

$$
\begin{aligned}
(\{a_j\},\{b_j\})_F &= \sum_{j,k=0}^{m} (m+1)^{1/2}\overline{a_j}\,b_k \int_{C} \frac{\mathrm{w}^j\overline{\mathrm{w}}^k}{(1+|\mathrm{w}|^2)^{m+2}}\,|d\overline{\mathrm{w}}\wedge d\mathrm{w}| \\
&= \sum_{j,k=0}^{m} 2(m+1)\overline{a_j}\,b_k \int_0^{+\infty} \frac{r^{j+k+1}\,dr}{(1+r^2)^{m+2}} \int_0^{2\pi} e^{i(j-k)\varphi}\,d\varphi \\
&= \sum_{j=0}^{m} 4\pi(m+1)^{1/2}\overline{a_j}\,b_j \int_0^{+\infty} \frac{r^{2j+1}\,dr}{(1+r^2)^{m+2}} = \frac{2\pi}{(m+1)^{1/2}} \sum_{j=0}^{m} \frac{\overline{a_j}\,b_j}{\binom{m}{j}},
\end{aligned}
$$

in virtue of the following:

LEMMA.

$$\int_0^{+\infty} \frac{x^j\,dx}{(1+x^2)^{m+2}} = \frac{1}{(m+1)\binom{m}{j}}. \tag{90}$$

Proof of lemma. We evaluate

$$J := \int_0^{+\infty} \frac{(x+t)^m}{(x+1)^{m+2}}\,dx \quad \text{for } 0 < t < 1. \tag{91}$$

Substituting $u = \dfrac{x+t}{x+1}$, we get

$$J = \int_t^1 \frac{u^m\,du}{1-u} = \frac{1}{m+1}\,\frac{t^{m+1}-1}{t-1} = \frac{1}{m+1}\sum_{j=0}^{m} t^j. \tag{92}$$

Comparing coefficients at t^j in the right-hand sides of (91) and (92), we get (90). ■

E. Kostant quantization of physical quantities

We have put

$$C_F^\infty(M) = \{f \in C^\infty(M) \colon X(f) = 0 \text{ if } X \text{ takes values in } F\}.$$

Let

$$C_{FF}^\infty(M) := \{f \in C^\infty(M) \colon X_f = (df)^\# \text{ preserves } F, \text{ i.e. for any } Y$$
$$\text{taking values in } F,\ [X_f, Y] \text{ also takes values in } F\}.$$

We shall introduce in this section a mapping which assigns self-adjoint operators in H_F to some real functions from $C_{FF}^\infty(M)$.

Let $f\in C^\infty_{FF}(M)$. Then $(\nabla_{X_f}+if)$ is an X_f-derivation of $\Gamma_{\rm loc}(\boldsymbol{L})$, and $\mathscr{L}_{X_f}$ is an X_f-derivation of $\Gamma_{\rm loc}(\Lambda^n\boldsymbol{T}^*(M)^C/F)$.

III.21. LEMMA. *Let* $f\in C^\infty_{FF}(M)$, *and* $\sigma\in\Gamma_{\rm loc}(\boldsymbol{L}\otimes\boldsymbol{D}_F^{1/2})$ *be an* F*-horizontal section. Then*

$$[(\nabla_{X_f}+if)\otimes\mathscr{L}^{1/2}_{X_f}](\sigma)\quad \textit{is also } F\textit{-horizontal}.$$

Proof. We must show that if X takes values in F, then

$$A:=(\nabla_X\otimes\mathscr{L}^{1/2}_X)\big[\big((\nabla_{X_f}+if)\otimes\mathscr{L}^{1/2}_{X_f}\big)(\sigma)\big]=0. \tag{93}$$

It suffices to verify (93) locally. Let $\sigma=s\otimes\varphi$, where $s\in\Gamma_{\rm loc}(L)$ and $\varphi\in\Gamma_{\rm loc}(\boldsymbol{D}_F^{1/2})$,

$$\begin{aligned}A&=(\nabla_X\otimes\mathscr{L}^{1/2}_X)[(\nabla_{X_f}s)\otimes\varphi+ifs\otimes\varphi+s\otimes(\mathscr{L}^{1/2}_{X_f}\varphi)]\\&=(\nabla_X\nabla_{X_f}s)\otimes\varphi+(\nabla_{X_f}s)\otimes(\mathscr{L}^{1/2}_X\varphi)+iX(f)s\otimes\varphi+if(\nabla_Xs)\otimes\varphi+\\&\qquad+ifs\otimes(\mathscr{L}^{1/2}_X\varphi)+(\nabla_Xs)\otimes(\mathscr{L}^{1/2}_{X_f}\varphi)+s\otimes(\mathscr{L}^{1/2}_X\mathscr{L}^{1/2}_{X_f}\varphi).\end{aligned} \tag{94}$$

But

$$[\nabla_X,\nabla_{X_f}]s-\nabla_{[X,X_f]}s=i\langle X,X_f|\omega\rangle s=-iX(f)s, \tag{95}$$

which can be easily proved by using (29) (see also (135) below). Also,

$$\mathscr{L}_X\mathscr{L}_{X_f}-\mathscr{L}_{X_f}\mathscr{L}_X=\mathscr{L}_{[X,X_f]}, \tag{96}$$

where both sides are $[X,X_f]$-derivations of $\Gamma_{\rm loc}(\boldsymbol{D}_F)$ ($[X,X_f]$ takes values in F). But $[\mathscr{L}^{1/2}_X,\mathscr{L}^{1/2}_{X_f}]$ is a $[X,X_f]$-derivation of $\Gamma_{\rm loc}(\boldsymbol{D}_F^{1/2})$, which is connected with $[\mathscr{L}_X,\mathscr{L}_{X_f}]$ by the prescription given in Proposition II.6. Because such a $[X,X_f]$-derivation is unique, we have

$$[\mathscr{L}^{1/2}_X,\mathscr{L}^{1/2}_{X_f}]=\mathscr{L}^{1/2}_{[X,X_f]}. \tag{97}$$

Using (95) and (97), we obtain from (94)

$$\begin{aligned}A&=(\nabla_{X_f}\nabla_Xs)\otimes\varphi+(\nabla_{[X,X_f]}s)\otimes\varphi+(\nabla_{X_f}s)\otimes(\mathscr{L}^{1/2}_X\varphi)+\\&\qquad+if(\nabla_Xs)\otimes\varphi+ifs\otimes(\mathscr{L}^{1/2}_X\varphi)+(\nabla_Xs)\otimes(\mathscr{L}^{1/2}_{X_f}\varphi)+\\&\qquad+s\otimes(\mathscr{L}^{1/2}_{X_f}\mathscr{L}^{1/2}_X\varphi)+s\otimes(\mathscr{L}^{1/2}_{[X,X_f]}\varphi)\\&=(\nabla_{X_f}\otimes\mathscr{L}^{1/2}_{X_f})[(\nabla_Xs)\otimes\varphi+s\otimes(\mathscr{L}^{1/2}_X\varphi)]+(\nabla_{[X,X_f]}s)\otimes\varphi+\\&\qquad+s\otimes(\mathscr{L}^{1/2}_{[X,X_f]}\varphi)+if[(\nabla_Xs)\otimes\varphi+s\otimes(\mathscr{L}^{1/2}_X\varphi)].\end{aligned} \tag{98}$$

But $s\otimes\varphi$ is F-horizontal. Hence

$$(\nabla_Xs)\otimes\varphi+s\otimes\mathscr{L}^{1/2}_X\varphi)=0, \tag{99}$$

and

$$(\nabla_{[X,X_f]}s)\otimes\varphi+s\otimes(\mathscr{L}^{1/2}_{[X,X_f]}\varphi)=0. \tag{100}$$

Substituting (99) and (100) into (98),

$$A = 0$$

is obtained which completes the proof of Lemma III.21.

For real $f \in C^{\infty}_{FF}(M)$, let Q_f be an unbounded operator in H_F, defined on the domain

(101) $$D_{Q_f} := \{\sigma \in H_F \cap \Gamma(L \otimes D_F^{1/2}) \colon\ [(\nabla_{X_f} + if) \otimes \mathscr{L}_{X_f}^{1/2}](\sigma) \in H_F\}$$

by the formula

(102) $$Q_f \sigma := \frac{1}{i}[(\nabla_{X_f} + if) \otimes \mathscr{L}_{X_f}^{1/2}](\sigma).$$

The mapping $f \to Q_f$ is just the Kostant quantization prescription for assigning operators in H_F to real functions from $C^{\infty}_{FF}(M)$.

Now we shall suppose that real $f \in C^{\infty}_{FF}(M)$ is such that X_f is a complete vector field. Let $\tilde{\tilde{X}}_f$ denote the horizontal lift of X_f, i.e. such a (smooth) vector field on L that $\pi_* \tilde{\tilde{X}}_f = X_f$, and $\langle \tilde{\tilde{X}}_f \restriction \tilde{\tilde{L}} | a \rangle = 0$. Let m_g denote an automorphism of L,

$$m_g(s_x) := g(x) s_x \quad \text{for } s_x \in L_x,\ g \in C^{\infty}(M).$$

Let

(103) $$\tilde{\tilde{Y}}_f(s_x) := \frac{\partial}{\partial t}\Big|_{t=0} m_{e^{itf}}(s_x).$$

$\tilde{\tilde{Y}}$ is a π-vertical vector field on L. Let

(104) $$\tilde{X}_f := \tilde{\tilde{X}}_f - \tilde{\tilde{Y}}_f.$$

$\tilde{X}_f$ is a complete vector field since if $t \to x(t)$ is an integral curve of X_f, and $t \to \tilde{\tilde{x}}(t)$ its horizontal lift, then $t \to \exp\big(-itf(x(t))\big)\tilde{\tilde{x}}(t)$ is an integral curve of $\tilde{X}_f$ $\big(f(x(t)) \underset{t}{\equiv} \text{const}\big)$. Let $(\tilde{\Phi}_t)_{t \in R}$ be the one-parameter group of diffeomorphisms of L generated by $\tilde{X}_f$. $\tilde{\Phi}_t$ preserve a as $\tilde{X}_f$ does:

$$\mathscr{L}_{\tilde{X}_f} a = \tilde{X}_f \lrcorner \pi^* \omega + d(\tilde{X}_f \lrcorner a) = X_f \lrcorner \omega + d(\tilde{Y}_f \lrcorner a) = \pi^* df - d(\tilde{\tilde{Y}}_f \lrcorner a) = 0,$$

as in virtue of (28) $\tilde{\tilde{Y}}_f \lrcorner a = \pi^* f$. $\tilde{\Phi}_t$ preserve also the hermitian structure, since m_g, for g taking values in $S^1 \subset C$, and the parallel transport do so. Thus $(\widetilde{\Phi}_t)_{t \in R}$ is a one-parameter group of automorphisms of L.

(105) $$\begin{tikzcd} L \arrow[r, "\tilde{\Phi}_t"] \arrow[d, "\pi"'] & L \arrow[d, "\pi"] \\ M \arrow[r, "\Phi_t"] & M \end{tikzcd}$$

Φ_t preserve F as $f \in C^\infty_{FF}(M)$ — see remark on p. 23. Thus, there exist bundle automorphisms of D_F

(106)
$$\begin{tikzcd} D_F \arrow[r, "\underset{\sim}{\Phi}_t"] \arrow[d] & D_F \arrow[d] \\ M \arrow[r, "\Phi_t"] & M \end{tikzcd}$$

such that for $\chi \in \Gamma(D_F)$, $(\underset{\sim}{\Phi}_t)^{-1} \circ \chi \circ \Phi_t = \Phi_t^* \chi$. $(\underset{\sim}{\Phi}_t)_{t \in R}$ is a smooth one-parameter group of diffeomorphisms of D_F. Let X_f be its generator. Since $D_F^{1/2} \ni \varphi \to \varphi \underline{\otimes} \varphi \in D_F$ is a covering mapping, there exists a unique vector field $\underset{\sim}{X}_f^{1/2}$ on $D_F^{1/2}$ related to X_f by this mapping. $\underset{\sim}{X}_f^{1/2}$ is complete, as lifts of integral curves of $\underset{\sim}{X}_f$ are integral for $\underset{\sim}{X}_f^{1/2}$. The one-parameter group of diffeomorphisms of $D_F^{1/2}$, defined by $\underset{\sim}{X}_f^{1/2}$, determines a one-parameter group of automorphisms $(\underset{\sim}{\Phi}_t^{1/2})_{t \in R}$ of $D_F^{1/2}$

(107)
$$\begin{tikzcd} D_F^{1/2} \arrow[r, "\underset{\sim}{\Phi}_t^{1/2}"] \arrow[d] & D_F^{1/2} \arrow[d] \\ M \arrow[r, "\Phi_t"] & M \end{tikzcd}$$

Thus, $(\widetilde{\Phi}_t \otimes \underset{\sim}{\Phi}_t^{1/2})$ is a smooth one-parameter group of automorphisms of $L \otimes D_F^{1/1}$

(108)
$$\begin{tikzcd} L \otimes D_F^{1/2} \arrow[r, "\widetilde{\Phi}_t \otimes \underset{\sim}{\Phi}_t^{1/2}"] \arrow[d] & L \otimes D_F^{1/2} \arrow[d] \\ M \arrow[r, "\Phi_t"] & M \end{tikzcd}$$

Let us note that for $s \in \Gamma_{\text{loc}}(L)$ and X being a vector field on M,

$$\begin{aligned} (109) \qquad \nabla_X(\widetilde{\Phi_{-t}} \circ s \circ \Phi_t) \restriction_m &= i \langle (\widetilde{\Phi_{-t}} \circ s \circ \Phi_t)_* X(m) | \alpha \rangle (\widetilde{\Phi_{-t}} \circ s \circ \Phi_t)(m) \\ &= i \big\langle s_* \big((\Phi_{t*} X)(\Phi_t(m)) \big) \big| \alpha \big\rangle \widetilde{\Phi_{-t}} \circ s(\Phi_t(m)) \\ &= \widetilde{\Phi_{-t}} \circ (\nabla_{\Phi_{t*} X} s) \circ \Phi_t \restriction_m . \end{aligned}$$

If $\chi \in \Gamma_{\mathrm{loc}}(\boldsymbol{D}_F)$ and X preserves F, then

$$\mathscr{L}_X(\underset{\sim}{\Phi_{-t}} \circ \chi \circ \Phi_t) = \mathscr{L}_X(\Phi_t^* \chi) = \Phi_t^*(\mathscr{L}_{\Phi_{t*}X}\chi) = \underset{\sim}{\Phi_{-t}} \circ (\mathscr{L}_{\Phi_{t*}X}\chi) \circ \Phi_t. \tag{110}$$

If $\varphi \in \Gamma_{\mathrm{loc}}(\boldsymbol{D}_F^{1/2})$, then

$$\underset{\sim}{\Phi_t^{1/2}} \circ [\mathscr{L}_X^{1/2}(\underset{\sim}{\Phi_{-t}^{1/2}} \circ \varphi \circ \Phi_t)] \circ \Phi_{-t} = \mathscr{L}^{1/2}_{\Phi_{t*}X}\varphi, \tag{111}$$

as $\varphi \to \underset{\sim}{\Phi_t^{1/2}} \circ [\mathscr{L}_X^{1/2}(\underset{\sim}{\Phi_{-t}^{1/2}} \circ \varphi \circ \Phi_t)] \circ \Phi_{-t}$ is a $(\Phi_{t*}X)$-derivation of $\Gamma_{\mathrm{loc}}(\boldsymbol{D}_F^{1/2})$, which is related by (4) to the $(\Phi_{t*}X)$-derivation of $\Gamma_{\mathrm{loc}}(\boldsymbol{D}_F)$

$$\chi \to \underset{\sim}{\Phi_t} \circ [\mathscr{L}_X(\Phi_{-t} \circ \chi \circ \Phi_t)] \circ \Phi_{-t} = \mathscr{L}_{\Phi_{t*}X}\chi.$$

From (109) and (111) we obtain (for X preserving F and $\varrho \in \Gamma_{\mathrm{loc}}(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2})$)

$$\begin{aligned}(\nabla_X \otimes \mathscr{L}_X^{1/2})[(\overline{\Phi_{-t}} \otimes \underset{\sim}{\Phi_{-t}^{1/2}}) \circ \varrho \circ \Phi_t] \\ = (\overline{\Phi_{-t}} \otimes \underset{\sim}{\Phi_{-t}^{1/2}})[(\nabla_{\Phi_{t*}X} \otimes \mathscr{L}_{\Phi_{t*}X})\varrho] \circ \Phi_t.\end{aligned} \tag{112}$$

Thus, if ϱ is F-horizontal, then so is $(\overline{\Phi_{-t}} \otimes \underset{\sim}{\Phi_{-t}^{1/2}}) \circ \varrho \circ \Phi_t$. For $s \in \Gamma_{\mathrm{loc}}(\boldsymbol{L})$,

$$\left(\frac{d}{dt}\Big|_{t=0} \overline{\Phi_{-t}} \circ s \circ \Phi_t\right)(m) = \frac{d}{dt}\Big|_{t=0}\left(e^{itf(m)}\frac{s\big(x(t)\big)}{\tilde{\tilde{x}}(t)}s(m)\right),$$

where $t \mapsto x(t)$ is an integral curve of X_f, $x(0) = m$, and $t \mapsto \tilde{\tilde{x}}(t)$ is its α-horizontal lift passing, at $t = 0$, through $s(m)$. Thus,

$$\frac{d}{dt}\Big|_{t=0}(\tilde{\Phi}_{-t} \circ s \circ \Phi_t)\Big|_m = (\nabla_{X_f} + if)s\Big|_m. \tag{113}$$

Now $\Gamma_{\mathrm{loc}}(\boldsymbol{D}_F^{1/2}) \ni \varphi \mapsto \frac{d}{dt}\Big|_{t=0}(\underset{\sim}{\Phi_{-t}^{1/2}} \circ \varphi \circ \Phi_t)$ is an X_f-derivation of $\Gamma_{\mathrm{loc}}(\boldsymbol{D}_F^{1/2})$, related by (4) to the X_f-derivation of $\Gamma_{\mathrm{loc}}(\boldsymbol{D}_F)$

$$\Gamma_{\mathrm{loc}}(\boldsymbol{D}_F) \ni \chi \mapsto \frac{d}{dt}\Big|_{t=0}(\underset{\sim}{\Phi_{-t}} \circ \chi \circ \Phi_t) = \frac{d}{dt}\Big|_{t=0}\Phi_t^*\chi = \mathscr{L}_{X_f}\chi.$$

Hence

$$\frac{d}{dt}\Big|_{t=0}(\overline{\Phi_{-t}^{1/2}} \circ \varphi \circ \Phi_t) = \mathscr{L}_{X_f}^{1/2}\varphi. \tag{114}$$

Combining (113) and (114), for $\varrho \in \Gamma_{\mathrm{loc}}(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2})$ we obtain

$$\frac{d}{dt}\Big|_{t=0}[(\overline{\Phi_{-t}} \otimes \underset{\sim}{\Phi_{-t}^{1/2}}) \circ \varrho \circ \Phi_t] = [(\nabla_{X_f} + if) \otimes \mathscr{L}_{X_f}^{1/2}]\varrho. \tag{115}$$

Since Φ_t, $t \in \boldsymbol{R}$, preserves F, it also preserves D. Let $(\Phi_t)_D$ be a diffeomorphism of M/D, such that $(\Phi_t)_D \circ \varrho_D = \varrho_D \circ \Phi_t$. $(\Phi_t)_D$ defines a bundle automorphism $|\utilde{(\Phi_t)_D}|$

(116)
$$\begin{tikzcd}
|\Lambda^{2n-k}\boldsymbol{T}^*(M/D)^{\boldsymbol{C}}| \arrow[r, "|\utilde{(\Phi_t)_D}|"] \arrow[d] & |\Lambda^{2n-k}\boldsymbol{T}^*(M/D)^{\boldsymbol{C}}| \arrow[d] \\
M/D \arrow[r, "(\Phi_t)_D"] & M/D
\end{tikzcd}$$

such that if $\psi\in\Gamma\left(\Lambda^{2n-k}\boldsymbol{T}^*(M/D)^{\boldsymbol{C}}\right)$ is non-vanishing, then

$$|\utilde{(\Phi_{-t})_D}|\circ|\psi|\circ(\Phi_t)_D = |(\Phi_t)_D^*\psi|.$$

The following relation is straightforward:

(117)
$$\langle\!\langle(\widetilde{\Phi_{-t}}\otimes\utilde{\Phi_{-t}^{1/2}})\circ\sigma\circ\Phi_t,\ (\widetilde{\Phi_{-t}}\otimes\utilde{\Phi_{-t}^{1/2}})\circ\varrho\circ\Phi_t\rangle\!\rangle = |\utilde{(\Phi_{-t})_D}|\circ\langle\!\langle\sigma,\varrho\rangle\!\rangle\circ\Phi_t$$

for $\sigma,\varrho\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_F^{1/2})$ being D-horizontal.

From (117) it follows that if $\sigma\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_F^{1/2})\cap H_F$, then $(\widetilde{\Phi_{-t}}\otimes\utilde{\Phi_{-t}^{1/2}})\circ\sigma\circ\Phi_t\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_{-t}^{1/2})\cap H_F$ and

(118)
$$\|(\widetilde{\Phi_{-t}}\otimes\utilde{\Phi_{-t}^{1/2}})\circ\sigma\circ\Phi_t\|_{H}^{2} = \|\sigma\|_F^2.$$

Thus, for $t\in\boldsymbol{R}$ the exists a unitary operator

$$U_t^f\colon H_F\to H_F$$

such that for $\sigma\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_F^{1/2})\cap H_F$,

(119)
$$U_t^f\sigma := (\widetilde{\Phi_{-t}}\otimes\utilde{\Phi_{-t}^{1/2}})\circ\sigma\circ\Phi_t.$$

$(U_t^f)_{t\in\boldsymbol{R}}$ forms a unitary one-parameter group.

We also have, in virtue of (115),

(120)
$$\left.\frac{d}{dt}\right|_{t=0}(U_t^f\sigma)(m) = i(Q_f\sigma)(m)$$

for $m\in M$ and $\sigma\in D_{Q_f}$.

III.22. THEOREM. 1. $(U_t^f)_{t\in\boldsymbol{R}}$ *is a strongly continuous one-parameter unitary group in* H_F.

2. *Let* $U_t^f \underset{t}{\equiv} \exp(it\tilde{Q}_f)$, *where* $\tilde{Q}_f$ *is the self-adjoint generator of this group. Then* $\tilde{Q}_f\supset Q_f$.

3. $\tilde{Q}_f = \overline{Q_f}$.

Proof. 1. Let $\mathrm{pr}_2\colon \boldsymbol{R}\times M\to M$ be the projection onto the second factor. Let

$$\boldsymbol{T} := \mathrm{pr}_2^*(\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2})$$

be a pull-back of $\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2}$ by pr_2. By shifts of the first variable, $\boldsymbol{R}$ acts on $\boldsymbol{R}\times M$ as a group of diffeomorphisms Ψ_τ, and these diffeomorphisms can be lifted to automorphisms $\widetilde{\Psi}_\tau$ of $\boldsymbol{T}$, as $\boldsymbol{T}$ is trivial along pr_2-fibres.

For $\sigma\in\Gamma(\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2})$ and $h\in C_0^\infty(\boldsymbol{R})$, let $K_h\sigma\in\Gamma(\boldsymbol{T})$,

$$K_h\sigma(\tau, m) := h(\tau)\Big[(\widetilde{\Phi_{-\tau}}\otimes \underset{\smile}{\Phi_\tau^{1/2}})\big(\sigma(\Phi_\tau(m))\big)\Big]. \tag{121}$$

We have

$$\begin{aligned} V_t(K_h\sigma)\restriction_{(\tau,m)} &:= (\widetilde{\Psi}_{-t}\circ K_h\sigma\circ \Psi_t)(\tau, m)\\ &= h(t+\tau)\Big[(\widetilde{\Phi_{-t-\tau}}\otimes \underset{\smile}{\Phi_{-t-\tau}^{1/2}})\big(\sigma(\Phi_{t+\tau}(m))\big)\Big]\\ &= h(t+\tau)\Big[(\widetilde{\Phi_{-\tau}}\otimes \underset{\smile}{\Phi_{-\tau}^{1/2}})\Big(\big((\widetilde{\Phi_{-t}}\otimes \underset{\smile}{\Phi_{-t}^{1/2}})\circ\sigma\circ\Phi_t\big)(\Phi_\tau(m))\Big)\Big]\\ &= K_{h_t}\big((\widetilde{\Phi_{-t}}\otimes \underset{\smile}{\Phi_{-t}^{1/2}})\circ\sigma\circ\Phi_t\big)\big|_{(\tau,m)}, \end{aligned} \tag{122}$$

where $h_t(\tau) := h(t+\tau)$.

Let $\vartheta\in\Gamma(\boldsymbol{T})$. $\vartheta(\tau,\cdot)$ is a section of $\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2}$. Let ϑ be such that for $\tau\in\boldsymbol{R}$, $\vartheta(\tau,\cdot)$ is D-horizontal. Suppose also that for each $\tau\in\boldsymbol{R}$, $\vartheta(\tau,\cdot)$ is of finite norm (see Definition III.18), and that

$$(\vartheta,\vartheta)_{\boldsymbol{T}} := \int_{\boldsymbol{R}} d\tau\Big(\int_{M/D}\langle\!\langle\vartheta(\tau,\cdot),\vartheta(\tau,\cdot)\rangle\!\rangle\Big) < +\infty.$$

For two such sections ϑ and $\eta\in\Gamma(\boldsymbol{T})$, we shall put

$$\begin{aligned} (\vartheta,\eta)_{\boldsymbol{T}} &:= \int_{\boldsymbol{R}} d\tau\Big(\int_{M/D}\langle\!\langle\vartheta(\tau,\cdot),\eta(\tau,\cdot)\rangle\!\rangle\Big)\\ &= \int_{M/D}\Big(\int_{\boldsymbol{R}} d\tau\langle\!\langle\vartheta(\tau,\cdot),\eta(\tau,\cdot)\rangle\!\rangle\Big). \end{aligned} \tag{123}$$

(The last equality can be easily proved by means of the Fubini–Tonelli Theorem.)

In virtue of (117), for $\sigma,\varrho\in\Gamma(\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2})\cap H_F$,

$$\begin{aligned} (K_h\sigma, K_h\varrho)_{\boldsymbol{T}} &= \int_{\boldsymbol{R}}|h(\tau)|^2\Big(\int_{M/D}|\underset{\smile}{(\Phi_{-t})_D}|\circ\langle\!\langle\sigma,\varrho\rangle\!\rangle\circ\Phi_t\Big)d\tau\\ &= \Big(\int_{\boldsymbol{R}}|h(\tau)|^2 d\tau\Big)(\sigma,\varrho)_F. \end{aligned} \tag{124}$$

Choosing h such that $\int_{\boldsymbol{R}} |h(\tau)|^2 d\tau = 1$, for $\sigma \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2}) \cap H_F$ we obtain

$$\begin{aligned}(125)\qquad (\sigma, U_t^f\sigma)_F &= \big(K_h\sigma, K_h(U_t^f\sigma)\big)_T = \big(K_h\sigma, V_t(K_{h_{-t}}\sigma)\big)_T \\ &= \big(K_h\sigma, V_t(K_{h_{-t}}\sigma - K_h\sigma)\big)_T + \big(K_h\sigma, V_t(K_h\sigma)\big)_T \\ &= \int_{\boldsymbol{R}} d\tau \Big(\int_{M/D} \langle\!\langle K_h\sigma(\tau, \cdot), K_{h_{-t}}\sigma(t+\tau, \cdot) - \\ &\quad - K_h\sigma(t+\tau, \cdot)\rangle\!\rangle \Big) + \int_{\boldsymbol{R}} d\tau \Big(\int_{M/D} \langle\!\langle K_h\sigma(\tau, \cdot), K_h\sigma(t+\tau, \cdot)\rangle\!\rangle \Big) \\ &= \int_{\boldsymbol{R}} d\tau\, \overline{h(\tau)} \big(h(\tau) - h(t+\tau)\big) \Big(\int_{M/D} \langle\!\langle U_\tau^f\sigma, U_{t+\tau}^f\sigma\rangle\!\rangle \Big) + \\ &\qquad + \int_{M/D} \Big(\int_{\boldsymbol{R}} \langle\!\langle K_h\sigma(\tau, \cdot), K_h\sigma(t+\tau, \cdot)\rangle\!\rangle d\tau \Big).\end{aligned}$$

But

$$\begin{aligned}(126)\qquad &\Big| \int_{\boldsymbol{R}} d\tau\, \overline{h(\tau)} \big(h(\tau) - h(t+\tau)\big) \Big(\int_{M/D} \langle\!\langle U_t^f\sigma, U_{t+\tau}^f\sigma\rangle\!\rangle \Big| \Big) \\ &\qquad \leqslant \int_{\boldsymbol{R}} d\tau\, |h(\tau)|\, |h(\tau) - h(t+\tau)|\, \|\sigma\|_F^2 \xrightarrow[t\to 0]{} 0.\end{aligned}$$

Now for $\vartheta \in \Gamma(\boldsymbol{T})$ such that for each $m \in M$, $\vartheta(\cdot, m)$ is square integrable on $\boldsymbol{R}$, let $\hat{\vartheta}(E, m)$ denote $\int_{\boldsymbol{R}} e^{iEs}\vartheta(s, m)\,ds$. In virtue of the Bounded Convergence Theorem, we have

$$\begin{aligned}(127)\qquad &\int_{M/D} \Big(\int_{\boldsymbol{R}} \langle\!\langle K_h\sigma(\tau, \cdot), K_h\sigma(\tau+t, \cdot)\rangle\!\rangle d\tau \Big) \\ &\quad = \frac{1}{2\pi} \int_{M/D} \Big(\int_{\boldsymbol{R}} e^{-iEt} \langle\!\langle \widehat{K_h\sigma}(E, \cdot), \widehat{K_h\sigma}(E, \cdot)\rangle\!\rangle dE \Big) \\ &\quad \xrightarrow[t\to 0]{} \frac{1}{2\pi} \int_{M/D} \Big(\int_{\boldsymbol{R}} \langle\!\langle \widehat{K_h\sigma}(E, \cdot), \widehat{K_h\sigma}(E, \cdot)\rangle\!\rangle dE \Big) \\ &\quad = (K_h\sigma, K_h\sigma)_T = (\sigma, \sigma)_F.\end{aligned}$$

From (125), (126) and (127), we get

$$(128)\qquad (\sigma, U_t^f\sigma)_F \xrightarrow[t\to 0]{} (\sigma, \sigma)_F \quad \text{for } \sigma \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2}) \cap H_F,$$

which proves the weak continuity on a dense subset of H_F and, consequently, strong continuity of $\boldsymbol{R} \ni t \mapsto U_t^f$ (U_t^f are unitary).

2. It can be seen that, when proving 1, we have in fact reduced the case to a much simpler one; when we have a Hilbert space of maps on $\boldsymbol{R}$ taking values in another Hilbert space, and our one-parameter unitary group is the group of shifts of the $\boldsymbol{R}$-variable. We shall now avail ourselves of this reduction.

Let $\sigma, \varrho \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2}) \cap H_F$, $\varrho \in D_{Q_f}$. From (115) and (122), we get

$$\frac{d}{dt}\Big|_{t=0} V_t(K_h\varrho) = K_h\big([(\nabla_{X_f} + if) \otimes \mathscr{L}_{X_f}^{1/2}]\varrho\big) + K_{h'}\varrho = K_h(iQ_f\varrho) + K_{h'}\varrho.$$

Now, choosing h again so that $\int\limits_{\boldsymbol{R}} |h(\tau)|^2 d\tau = 1$, we have

$$(129)\quad \Big(\sigma, \frac{1}{t}(U_t^f\varrho - \varrho) - iQ_f\varrho\Big)_{H_F} = \Big(K_h\sigma, \frac{1}{t}\big(K_h(U_t^f\varrho) - K_h\varrho\big) - K_h(iQ_f\varrho)\Big)_T$$

$$= \Big(K_h\sigma, \frac{1}{t}\big(V_t(K_{h_{-t}}\varrho) - K_h\varrho\big) - \frac{d}{dt}\Big|_{t=0} V_t(K_h\varrho) + K_{h'}\varrho\Big)_T$$

$$= \Big(K_h\sigma, \frac{1}{t}\big(V_t(K_{h_{-t}}\varrho - K_h\varrho)\big) + V_t(K_{h'_{-t}}\varrho)\Big)_T + \big(K_h\sigma, K_{h'}\varrho - V_t(K_{h'_{-t}}\varrho)\big)_T +$$

$$+ \Big(K_h\sigma, \frac{1}{t}\big(V_t(K_h\varrho) - K_h\varrho\big) - \frac{d}{dt}\Big|_{t=0} V_t(K_h\varrho)\Big)_T$$

$$= \int\limits_{\boldsymbol{R}} d\tau\,\overline{h(\tau)}\Big[\frac{1}{t}\big(h(\tau) - h(t+\tau)\big) + h'(\tau)\Big] \cdot \Big(\int\limits_{M/D} \langle\!\langle U_\tau^f\sigma, U_{t+\tau}^f\varrho\rangle\!\rangle\Big) +$$

$$+ \Big(\int\limits_{\boldsymbol{R}} \overline{h(\tau)}h'(\tau)\,d\tau\Big)(\sigma, \varrho - U_t^f\varrho)_F + \int\limits_{\boldsymbol{R}} d\tau\Big(\int\limits_{M/D} \langle\!\langle K_h\sigma(\tau, \cdot), \frac{1}{t}\big(K_h\varrho(t+\tau, \cdot) -$$

$$- K_h\varrho(\tau, \cdot)\big) - \frac{d}{dt}\Big|_{t=0} K_h\varrho(t+\tau, \cdot)\rangle\!\rangle\Big).$$

Now

$$(130)\quad \Big|\int\limits_{\boldsymbol{R}} d\tau\,\overline{h(\tau)}\Big[\frac{1}{t}\big(h(\tau) - h(t+\tau)\big) + h'(\tau)\Big]\Big(\int\limits_{M/D} \langle\!\langle U_\tau^f\sigma, U_{t+\tau}^f\varrho\rangle\!\rangle\Big)\Big|$$

$$\leqslant \int\limits_{\boldsymbol{R}} d\tau\,|h(\tau)|\,\Big|\frac{1}{t}\big(h(\tau) - h(t+\tau)\big) + h'(\tau)\Big|\,\|\sigma\|_F\|\varrho\|_F \xrightarrow[t\to 0]{} 0$$

uniformly for σ, such that $\|\sigma\|_F \leqslant 1$;

$$(131)\quad \Big|\Big(\int\limits_{\boldsymbol{R}} \overline{h(\tau)}h'(\tau)\,d\tau\Big)(\sigma, \varrho - U_t^f\varrho)_F\Big|$$

$$\leqslant \Big|\int\limits_{\boldsymbol{R}} h(\tau)h'(\tau)\,d\tau\Big|\,\|\varrho - U_t^f\varrho\|_F\|\sigma\|_F \xrightarrow[t\to 0]{} 0$$

uniformly for σ, such that $\|\sigma\|_F \leqslant 1$;

$$
(132)\quad \left|\int_{\boldsymbol{R}} d\tau \left(\int_{M/D} \langle\!\langle K_h\sigma(\tau,\cdot), \frac{1}{t}\big(K_h\varrho(t+\tau,\cdot) - K_h\varrho(\tau,\cdot)\big) - \right.\right.
$$

$$
\left.\left. - \frac{d}{dt}\Big|_{t=0} K_h\varrho(t+\tau,\cdot)\rangle\!\rangle\right)\right|
$$

$$
\leqslant \frac{1}{2\pi}\left|\int_{M/D}\left(\int_{\boldsymbol{R}}\left(\frac{1}{t}(e^{-iEt}-1)+iE\right)\langle\!\langle \widehat{K_h}\sigma(E,\cdot), \widehat{K_h}\varrho(E,\cdot)\rangle\!\rangle\, dE\right)\right|
$$

$$
\leqslant \frac{1}{2\pi}\left(\int_{M/D}\left(\int_{\boldsymbol{R}}\left|\frac{\frac{1}{t}(e^{-iEt}-1)+iE}{iE}\right|^2 \langle\!\langle iE\widehat{K_h}\varrho(E,\cdot), iEK_h\varrho(E,\cdot)\rangle\!\rangle\, dE\right)\right)^{1/2} \times
$$

$$
\times \left(\int_{M/D}\left(\int_{\boldsymbol{R}} \langle\!\langle \widehat{K_h}\sigma(E,\cdot), \widehat{K_h}\sigma(E,\cdot)\rangle\!\rangle\, dE\right)\right)^{1/2}
$$

$$
= \frac{1}{2\pi^{1/2}}\left(\int_{M/D}\left(\int_{\boldsymbol{R}}\left|\frac{\frac{1}{t}(e^{-iEt}-1)+iE}{iE}\right|^2 \times\right.\right.
$$

$$
\left.\left. \times \langle\!\langle iE\widehat{K_h}\varrho(E,\cdot), iE\widehat{K_h}\varrho(E,\cdot)\rangle\!\rangle\, dE\right)\right)^{1/2} \|\sigma\|_F \xrightarrow[t\to 0]{} 0
$$

uniformly for σ, such that $\|\sigma\|_F \leqslant 1$, in virtue of the Bounded Convergence Theorem.

(129), (130), (131) and (132) yield

$$
\left(\sigma, \frac{1}{t}(U_t^f\varrho - \varrho) - iQ_f\varrho\right)_F \xrightarrow[t\to 0]{} 0
$$

uniformly for σ, such that $\|\sigma\|_F \leqslant 1$, which proves 2.

3. First we show that Q_f is densely defined. Indeed, let $\varrho \in \Gamma(\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2}) \cap \cap H_F$. Then $\frac{1}{t}\int_0^t U_\tau^f\varrho\, d\tau \in \Gamma(\boldsymbol{L}\otimes \boldsymbol{D}_F^{1/2}) \cap H_F$. But

$$
U_{t'}^f\left(\frac{1}{t}\int_0^t U_\tau^f d\tau\right) = \frac{1}{t}\int_0^t U_{t'+\tau}^f\varrho\, d\tau = \frac{1}{t}\int_{t'}^{t+t'} U_\tau^f\varrho\, d\tau.
$$

Hence

$$
\frac{d}{dt'}\Big|_{t'=0} U_{t'}^f\left(\frac{1}{t}\int_0^t U_\tau^f\varrho\, d\tau\right) = \frac{1}{t}(U_t^f\varrho - \varrho) \in H_F,
$$

where the derivation is performed point-wise. Thus, in virtue of (115), (119) and (101),

$$\left(\frac{1}{t}\int\limits_0^t U^f_\tau \varrho\, d\tau\right) \in D_{Q_f} \quad \text{and} \quad Q_f\left(\frac{1}{t}\int\limits_0^t U^f_\tau \varrho\, d\tau\right) = \frac{i}{t}(\varrho - U^f_t \varrho).$$

But

$$\boldsymbol{R} \ni \tau \mapsto U^f_\tau \varrho \in H_F$$

is continuous, and so

$$\frac{1}{t}\int\limits_0^t U^f_\tau \varrho\, d\tau \xrightarrow[t\to 0]{} \varrho \quad \text{in } H_F.$$

We also note that if $\varrho \in D_{Q_f}$, then $U^f_t \varrho \in D_{Q_f}$ and $Q_f U^f_t \varrho = U^f_t Q_f \varrho$ (see (120)). Now 3 follows immediately from the following:

LEMMA (Renouard — see [11]). *Let A be a densely defined operator in a Hilbert space, and $\tilde{A}$ its self-adjoint extension. If D_A is $e^{it\tilde{A}}$-invariant for $t \in \boldsymbol{R}$, then $\tilde{A} = \bar{A}$.*

Proof of lemma. Let $K_\pm$ be deficiency subspaces of A, $K_\pm = \{\sigma \in D_{A^*} : (A^* \pm i)\sigma = 0\}$. $K_\pm$ are closed and $e^{it\tilde{A}}$ invariant, as for $\sigma \in K_\pm$ and $\varrho \in D_A$,

$$\begin{aligned}\left((A \pm i)\varrho, e^{it\tilde{A}}\sigma\right) &= \left(e^{-it\tilde{A}}(A \pm i)\varrho, \sigma\right)\\ &= \left(e^{-it\tilde{A}}(\tilde{A} \pm i)\varrho, \sigma\right) = \left((A \pm i)e^{-it\tilde{A}}\varrho, \sigma\right)\\ &= \left(e^{-it\tilde{A}}\varrho, (A^* \mp i)\sigma\right) = 0,\end{aligned}$$

and hence $e^{it\tilde{A}}\sigma \in D_{A^*}$, and $(A^* \pm i)e^{it\tilde{A}}\sigma = 0$. Now $\tilde{A}\restriction K_\pm$ is an infinitesimal generator of $(e^{it\tilde{A}}\restriction K_\pm)_{t\in\boldsymbol{R}}$, and therefore $D_{\tilde{A}} \cap K_\pm$ is dense in $K_\pm$. But if $\sigma \in D_{\tilde{A}} \cap K_\pm$, then

$$(\tilde{A} \pm i)\sigma = (A^* \pm i)\sigma = 0,$$

as A^* is an extension of $\tilde{A}$. But $\tilde{A}$ is self-adjoint, hence $\sigma = 0$ and, consequently, $K_\pm = \{0\}$, which proves our lemma. □ ■

III.23. COROLLARY. *Let $f \in C^\infty_{FF}(M)$ be real, such that X_f is complete. Then Q_f is a densely defined essentially self-adjoint operator.*

III.24. PROPOSITION. *Let $f, g \in C^\infty_{FF}(M)$ and $\sigma \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}^{1/2}_F)$. Then*

$$(133)\quad [(\nabla_{X_f} + if) \otimes \mathscr{L}^{1/2}_{X_f}, (\nabla_{X_g} + ig) \otimes \mathscr{L}^{1/2}_{X_g}] = -\left((\nabla_{\{f,g\}} + i\{f,g\}) \otimes \mathscr{L}^{1/2}_{\{f,g\}}\right).$$

Proof. We shall prove (133) locally. Let, locally, $\sigma = s \otimes \varphi$, where $s \in \Gamma_{\mathrm{loc}}(\boldsymbol{L})$ and $\varphi \in \Gamma_{\mathrm{loc}}(\boldsymbol{D}^{1/2}_F)$. Thus, locally,

$$(134)\quad [(\nabla_{X_f} + if) \otimes \mathscr{L}^{1/2}_{X_f}, (\nabla_{X_g} + ig) \otimes \mathscr{L}^{1/2}_{X_g}]\sigma = ([\nabla_{X_f} + if, \nabla_{X_g} + ig]s) \otimes \varphi + \\ + s \otimes [\mathscr{L}^{1/2}_{X_f}, \mathscr{L}^{1/2}_{X_g}]\varphi = ([\nabla_{X_f}, \nabla_{X_g}]s) \otimes \varphi - 2i\{f,g\} s \otimes \varphi + s \otimes [\mathscr{L}^{1/2}_{X_f}, \mathscr{L}^{1/2}_{X_g}]\varrho.$$

But by (29),

$$(135)\qquad \begin{aligned}[\nabla_{X_f}, \nabla_{X_g}]s &= iX_f(\langle s_* X_g | a\rangle)s - iX_g(\langle s^* X_f | a\rangle)s \\ &= i\langle X_f, X_g | d(s^* a)\rangle s + i\langle [X_f, X_g] | s^* a\rangle s \\ &= i\langle X_f, X_g | \omega\rangle s + \nabla_{[X_f, X_g]}s = i\{f, g\} + \nabla_{[X_f, X_g]}s.\end{aligned}$$

$[\mathscr{L}^{1/2}_{X_f}, \mathscr{L}^{1/2}_{X_g}]$ is a $[X_f, X_g]$-derivation of $\Gamma_{\mathrm{loo}}(D^{1/2}_F)$ related to $[\mathscr{L}_{X_f}, \mathscr{L}_{X_g}] = \mathscr{L}_{[X_f, X_g]}$ by (4). Thus,

$$(136)\qquad [\mathscr{L}^{1/2}_{X_f}, \mathscr{L}^{1/2}_{X_g}] = [\mathscr{L}_{X_f}, \mathscr{L}_{X_g}]^{1/2} = \mathscr{L}^{1/2}_{[X_f, X_g]}.$$

Substitution of (136) and (135) to (134) yields

$$\begin{aligned}&[(\nabla_{X_f} + if)\otimes \mathscr{L}^{1/2}_{X_f}, (\nabla_{X_g} + ig)\otimes \mathscr{L}^{1/2}_{X_g}]\sigma \\ &= \big((\nabla_{[X_f, X_g]} - i\{f, g\})s\big)\otimes\varphi + s\otimes\mathscr{L}^{1/2}_{[X_f, X_g]}\varphi = -[(\nabla_{X_{\{f,g\}}} + i\{f, g\})\otimes\mathscr{L}^{1/2}_{X_{\{f,g\}}}]\sigma,\end{aligned}$$

as $[X_f, X_g] = -X_{\{f,g\}}$, which proves (134). ∎

III.25. Corollary. *If f, g are real elements of $C^\infty_{FF}(M)$, then*

$$(137)\qquad [Q_f, Q_g]\varrho = iQ_{\{f,g\}}\varrho$$

for such $\varrho \in H_F$ that both sides of (137) *make sense.*

Thus the Kostant quantization procedure realizes, partly, the Dirac program of representing classical physical values by self-adjoint operators in a Hilbert space of states, so that the Poisson bracket should pass into the commutator (up to the constant i).

III.26. Example (cotangent bundle). Let (M, ω), a quantum bundle and a polarization with a square root structure be as in Examples III.2, 7, 11 and 19. $C^\infty_{FF}(M)$ is composed of functions f whose canonical fields X_f project by $\varkappa_*$ to a vector field $\utilde{X}_f$ on $\mathscr{X}$. For $g \in C^\infty(M)$ constant along $\varkappa$-fibres, let $\utilde{g}$ denote such a function on $\mathscr{X}$ that $g = \utilde{g}\circ\varkappa$. Simple calculations yield (in the notation of Example III.19)

$$(138)\qquad \utilde{Q_f\varrho} = (f \utilde{- \langle X_f | \sigma\rangle})\utilde{\varrho} - i\mathscr{L}^{1/2}_{\utilde{X}_f}\utilde{\varrho}$$

for real $f \in C^\infty_{FF}(M)$ and $\varrho \in D_{Q_f}$.

For $f \in C^\infty_F(M)$,

$$(139)\qquad \utilde{Q_f\varrho} = f\utilde{\varrho}.$$

III.27. Example (complex projective plane). Let (M, ω), a quantum bundle and a polarization with a square root structure be as in Examples III.8, 12 and 20. We shall follow here the notation adopted in these examples.

$C^{\infty}_{FF}(M)$ is spanned by functions x, y, z and a constant function

$$[z_1, z_2] \xrightarrow{x} \frac{m+1}{2}\,\frac{z_1\bar{z}_2 + z_2\bar{z}_1}{|z_1|^2+|z_2|^2},$$

$$[z_1, z_2] \xrightarrow{y} \frac{i(m+1)}{2}\,\frac{z_1\bar{z}_2 - z_2\bar{z}_1}{|z_1|^2+|z_2|^2},$$

$$[z_1, z_2] \xrightarrow{z} \frac{m+1}{2}\,\frac{|z_1|^2 - |z_2|^2}{|z_1|^2+|z_2|^2}$$

(x, y, z are proportional to $\varkappa_i$, $i = 1, 2, 3$, in the notation of (32)). Functions x, y, z generate a Lie subalgebra of $C^{\infty}_{FF}(M)$, which is isomorphic to $su(2)$. Straightforward calculations yield:

$$Q_x(\{a_j\}) = \{b_j\},$$

where

(140) $$b_0 = \tfrac{1}{2}a_1; \quad b_j = \tfrac{1}{2}[(j+1)a_{j+1} + (m+1-j)a_{j-1}], \quad 1 \leqslant j \leqslant m-1;$$
$$b_m = \tfrac{1}{2}a_{m-1},$$

$$Q_y(\{a_j\}) = \{c_j\},$$

where

(141) $$c_0 = \frac{i}{2}a_1; \quad c_j = \frac{i}{2}[(j+1)a_{j+1} - (m+1-j)a_{j-1}], \quad 1 \leqslant j \leqslant m-1;$$
$$c_m = -\frac{i}{2}a_{m-1},$$

(142) $$Q_z(\{a_j\}) = \left\{\left(\frac{m}{2} - j\right)a_j\right\}.$$

This gives the standard Wigner representation of $su(2)$ for spin equal to $m/2$.

IV. KERNEL QUANTIZATION

Throughout Chapter IV we shall assume that M is connected and simply connected.

A. Geometric quantization of canonical diffeomorphisms

Let F be a polarization for (M, ω) and $(\boldsymbol{D}_F^{1/2}, \iota)$ a square root structure for F. We shall repeat some considerations from Chapter III Section E in a slightly different context.

Let V: $M \to M$ be a canonical diffeomorphism, i.e. let

$$V^* \omega = \omega. \tag{143}$$

Since M is connected and simply connected, there exists an automorphism $\tilde{V}$ of the bundle $\boldsymbol{L}$

$$\begin{tikzcd} L \arrow[r, "\tilde{V}"] \arrow[d] & L \arrow[d] \\ M \arrow[r, "V"] & M \end{tikzcd} \tag{144}$$

such that $\tilde{V}^* \alpha = \alpha$, $\tilde{V}$ preserves the hermitian structure on $\boldsymbol{L}$, and $\tilde{V}$ is unique up to the multiplication of L by a constant factor from S^1 (see [6]).

If V additionally preserves the polarization F, then it also defines a bundle automorphism $\underset{\sim}{V}$ of $\boldsymbol{D}_F$

$$\begin{tikzcd} D_F \arrow[r, "\underset{\sim}{V}"] \arrow[d] & D_F \arrow[d] \\ M \arrow[r, "V"] & M \end{tikzcd} \tag{145}$$

such that if $\chi \in \Gamma_{\mathrm{loc}}(\boldsymbol{D}_F)$, then

$$\underset{\sim}{V}^{-1} \circ \chi \circ V = V^* \chi. \tag{146}$$

IV.1. LEMMA. *There exists a bundle automorphism* $\underset{\sim}{V}^{1/2}$

(147)
$$\begin{tikzcd} D_F^{1/2} \arrow[r, "\underset{\sim}{V}^{1/2}"] \arrow[d] & D_F^{1/2} \arrow[d] \\ M \arrow[r, "V"] & M \end{tikzcd}$$

such that

(148)
$$(\underset{\sim}{V}^{1/2}x)\underline{\otimes}(\underset{\sim}{V}^{1/2}x) = \underset{\sim}{V}(x\underline{\otimes}x)$$

for $x \in D_F^{1/2}$. $\underset{\sim}{V}^{1/2}$ *is unique up to the sign.*

Proof. Let $(0_\alpha)_{\alpha \in A}$ be an open covering of M by simply connected sets, and for each $\alpha \in A$, let φ_α be a non-vanishing section of $D_F^{1/2}$ over 0_α. Suppose, additionally, that for each $\alpha \in A$, $D_F^{1/2} \restriction_{V^{-1}(0_\alpha)}$ is trivial. Then for each α there exists a section ψ'_α of $D_F^{1/2}$ over $V^{-1}(0_\alpha)$ such that

(149)
$$\psi'_\alpha \underline{\otimes} \psi'_\alpha = V^*(\varphi_\alpha \underline{\otimes} \varphi_\alpha),$$

which is easily verified as the problem can be reduced to the taking of a smooth square root of a (smooth) non-vanishing function defined on a simply connected set. For $(\alpha, \beta) \in A \times A$ such that $0_\alpha \cap 0_\beta \neq \emptyset$, let

$$\varphi_\beta = g_{\beta\alpha}\varphi_\alpha \quad \text{on } 0_\alpha \cap 0_\beta.$$

In virtue of (149), we have

$$\psi'_\beta = \sigma_{\beta\alpha}(g_{\beta\alpha} \circ V)\psi'_\alpha,$$

where $\sigma_{\beta\alpha}$ is locally constant on $V^{-1}(0_\alpha) \cap V^{-1}(0_\beta)$ and takes values in $Z_2 = \{-1, 1\}$. It is easily verified that $(\sigma_{\beta\alpha})$ is a Čech 1-cocycle for the covering $(V^{-1}(0_\alpha))_{\alpha \in A}$. Since $H^1(M, Z_2) = 0$ (in virtue of the Cohomological Theorem on Universal Coefficients), we have (passing if necessary to a finer covering)

$$\sigma_{\beta\alpha} = \gamma_\beta \gamma_\alpha^{-1},$$

where γ_α is a locally constant Z_2-valued function on $V^{-1}(0_\alpha)$.

Let

$$\psi_\alpha := \gamma_\alpha^{-1}\psi'_\alpha.$$

Then we can define $\underset{\sim}{V}^{1/2}$ by

(150)
$$\underset{\sim}{V}^{1/2} \circ \psi_\alpha = \varphi_\alpha \circ V.$$

It is easily verified that (150) uniquely defines $\underset{\sim}{V}^{1/2}$. Since the family (ψ_α) can be chosen up to the simultaneous change of sign, $\underset{\sim}{V}^{1/2}$ is unique up to the sign. ∎

Now let V_D be a diffeomorphism induced on M/D by V. We have a bundle automorphism $\underset{\sim}{|V_D|}$

$$\begin{tikzcd}
{|\Lambda^{2n-k}T^*(M/D)^C|} \arrow[r, "\underset{\sim}{|V_D|}"] \arrow[d] & {|\Lambda^{2n-k}T^*(M/D)^C|} \arrow[d] \\
M/D \arrow[r, "V_D"] & M/D
\end{tikzcd} \tag{151}$$

such that if $\sigma \in \Gamma(\Lambda^{2n-k}T^*(M/D)^C)$ is non-vanishing, then

$$\underset{\sim}{|V_D|}^{-1} \circ |\sigma| \circ V_D = |V_D^* \sigma|.$$

The automorphisms in (144) and (147) define a bundle automorphism $\tilde{V} \otimes \underset{\sim}{V}^{1/2}$

$$\begin{tikzcd}
L \otimes D_F^{1/2} \arrow[r, "\tilde{V} \otimes \underset{\sim}{V}^{1/2}"] \arrow[d] & L \otimes D_F^{1/2} \arrow[d] \\
M \arrow[r, "V"] & M
\end{tikzcd} \tag{152}$$

Verification of the following facts is straightforward.

IV.2. PROPOSITION.

1. *If* $\sigma \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2})$ *is* F*-horizontal, then* $(\tilde{V} \otimes \underset{\sim}{V}^{1/2})^{-1} \circ \sigma \circ V$ *is also* F*-horizontal.*

2. $\langle\!\langle (\tilde{V} \otimes \underset{\sim}{V}^{1/2})^{-1} \circ \sigma \circ V, (\tilde{V} \otimes \underset{\sim}{V}^{1/2})^{-1} \circ \varrho \circ V \rangle\!\rangle = \underset{\sim}{|V_D|}^{-1} \circ \langle\!\langle \sigma, \varrho \rangle\!\rangle \circ V_D$ *for* $\sigma, \varrho \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2})$ *being* D*-horizontal* (*compare* (117)).

3. *If* $\sigma \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2}) \cap \boldsymbol{H}_F$, *then* $(\tilde{V} \otimes \underset{\sim}{V}^{1/2}) \circ \sigma \circ V \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2}) \cap \boldsymbol{H}_F$ *and* $\|(\tilde{V} \otimes \underset{\sim}{V}^{1/2})^{-1} \circ \sigma \circ V\|_F^2 = \|\sigma\|_F^2$ (*compare* (118)).

Thus, there exists a unitary operator $U_{\tilde{V} \otimes \underset{\sim}{V}^{1/2}}\colon \boldsymbol{H}_F \to \boldsymbol{H}_F$, *such that for* $\sigma \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}_F^{1/2}) \cap \boldsymbol{H}_F$,

$$U_{\tilde{V} \otimes \underset{\sim}{V}^{1/2}} \sigma = (\tilde{V} \otimes \underset{\sim}{V}^{1/2})^{-1} \circ \sigma \circ V. \tag{153}$$

B. Distinguished kernels

Let F_1 and F_2 be two polarizations for (M, ω).

IV.3. DEFINITION. F_1 and F_2 are called ***transversal*** if for each $m \in M$, $(F_1)_m \cap (F_2)_m = \{0_m\}$.

Let $(\boldsymbol{D}_{F_1}^{1/2}, \iota_1)$ and $(\boldsymbol{D}_{F_2}^{1/2}, \iota_2)$ be square root structures for F_1 and F_2, respectively. Suppose that F_1 and F_2 are transversal. One can easily see that $\boldsymbol{D}_{F_1} \otimes \boldsymbol{D}_{F_2}$ is isomorphic to $\Lambda^{2n}\boldsymbol{T}^*(M)^C$, the isomorphism being defined by external multiplication on fibres. We have bundle isomorphisms

$$(154)\quad (\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2})\otimes(\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2})\simeq(\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_1}^{1/2})\otimes(\boldsymbol{D}_{F_2}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2})$$
$$\overset{\iota_1\otimes\iota_2}{\simeq}\boldsymbol{D}_{F_1}\otimes\boldsymbol{D}_{F_2}\simeq\Lambda^{2n}\boldsymbol{T}^*(M)^{C}.$$

Let us denote the composed isomorphism by χ. Thus, $\big((\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2}),\chi\big)$ is a square root of $\Lambda^{2n}\boldsymbol{T}^*(M)^{C}$.

IV.4. DEFINITION. We say that a section $(\overset{n}{\omega})^{1/2}\in\Gamma(\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2})$ is an *adjustment* of square root structures $(\boldsymbol{D}_{F_1}^{1/2},\iota_1)$ and $(\boldsymbol{D}_{F_2}^{1/2},\iota_2)$, or that $(\overset{n}{\omega})^{1/2}$ *adjusts* $(\boldsymbol{D}_{F_1}^{1/2},\iota_1)$ and $(\boldsymbol{D}_{F_2}^{1/2},\iota_2)$ if

$$(155)\qquad (\overset{n}{\omega})^{1/2}\underline{\otimes}(\overset{n}{\omega})^{1/2}=\chi\circ\big((\overset{n}{\omega})^{1/2}\otimes(\overset{n}{\omega})^{1/2}\big)=\underbrace{\omega\wedge\ldots\wedge\omega}_{n\text{ times}}=:\overset{n}{\omega}.$$

Let us note that if $Z:=\{\lambda\in D_{F_1}^{1/2}\otimes D_{F_2}^{1/2}\colon\ \chi(\lambda\otimes\lambda)\in\overset{n}{\omega}(M)\}$, then Z together with a mapping $Z\ni\lambda\mapsto\chi(\lambda\otimes\lambda)\in\overset{n}{\omega}(M)$ is a covering space of the connected simply connected space $\overset{n}{\omega}(M)\subset\Lambda^{2n}\boldsymbol{T}^*(M)^{C}$ (M is connected and simply connected). Thus connected components of Z determine two adjustments of $(\boldsymbol{D}_{F_1}^{1/2},\iota_1)$ and $(\boldsymbol{D}_{F_2}^{1/2},\iota_2)$, differing by sign.

We shall assume for the rest of the paper that (F_1,F_2) *is a pair of transversal polarizations for* (M,ω) *with square root structures* $(\boldsymbol{D}_{F_1}^{1/2},\iota_1)$ *and* $(\boldsymbol{D}_{F_2}^{1/2},\iota_2)$, *adjusted by* $(\overset{n}{\omega})^{1/2}\in\Gamma(\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2})$.

Let $\boldsymbol{L}^*$ be the dual bundle of $\boldsymbol{L}$. One can provide $\boldsymbol{L}^*$ with a dual connection, i.e. such one that

$$(156)\qquad \langle s\,|\,\nabla_X^*s^*\rangle=X(\langle s\,|\,s^*\rangle)-\langle\nabla_X s\,|\,s^*\rangle$$

if $s\in\Gamma(\boldsymbol{L})$ and $s^*\in\Gamma(\boldsymbol{L}^*)$, and X is a vector field on M (∇^* is a covariant derivative defined by the dual connection).

Let $\mathrm{pr}_i\colon M\times M\to M$, $i=1,2$, be the projection onto the i-th factor. Let

$$(157)\qquad \boldsymbol{P}:=\mathrm{pr}_1^*(\boldsymbol{L}\otimes\boldsymbol{D}_{F_1}^{1/2})\otimes\mathrm{pr}_2^*(\boldsymbol{L}^*\otimes\boldsymbol{D}_{F_2}^{1/2}).$$

$\boldsymbol{P}$ is a one-dimensional complex bundle over $M\times M$. Let $\Delta:=\{(m,m)\in M\times M\}$.

Let $K\in\Gamma(\boldsymbol{P})$.

By $\overbrace{K(m_1,\cdot)}$ we shall denote a section of $\boldsymbol{L}^*\otimes\boldsymbol{D}_{F_2}^{1/2}$ obtained from $K\restriction_{\{m_1\}\times M}$ by a natural identification of $\boldsymbol{P}\restriction_{\{m_1\}\times M}$ and $\boldsymbol{L}^*\otimes\boldsymbol{D}_{F_2}^{1/2}$ (determined up to a constant).

By $\overbrace{K(\cdot,m_2)}$ we shall denote a section of $\boldsymbol{L}\otimes\boldsymbol{D}_{F_1}^{1/2}$ obtained from $K\restriction_{M\times\{m_2\}}$ by a natural identification of $\boldsymbol{P}\restriction_{M\times\{m_2\}}$ and $\boldsymbol{L}\otimes\boldsymbol{D}_{F_1}^{1/2}$ (determined up to a constant).

By $\widetilde{K\restriction_\Delta}$ we shall denote a section of $\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2}$ obtained from $K\restriction_\Delta$ by the natural identification of $\boldsymbol{P}\restriction_\Delta$ and $\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2}$.

IV.5. DEFINITION. A non-zero section $K\in\Gamma(\boldsymbol{P})$ will be called a *distinguished kernel* for the pair of polarizations (F_1, F_2) with square root structures $(\boldsymbol{D}_{F_1}^{1/2}, \iota_1)$ and $(\boldsymbol{D}_{F_2}^{1/2}, \iota_2)$, adjusted by $(\overset{n}{\omega})^{1/2}$, or (with a slight abuse of language) a distinguished kernel for (F_1, F_2) if:

1. for each $m_1\in M$, $\widetilde{K(m_1, \cdot)}$ is F_2-horizontal (i.e. $(\nabla_X^*\otimes\mathscr{L}_X^{1/2})\big(\widetilde{K(m_1, \cdot)}\big) = 0$ for X taking values in F_2);

2. for each $m_2\in M$, $\widetilde{K(\cdot, m_2)}$ is F_1-horizontal;

3. $\widetilde{K\restriction_\Delta}$ is proportional to $(\overset{n}{\omega})^{1/2}$.

Let

$H_{F_i}^0 := \{\varrho\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_{F_i}^{1/2})\cap H_{F_i}\colon\ \varrho_{E_i}(\operatorname{suppt}\varrho)$ is a compact subset of $M/E_i\}$.

$H_{F_i}^0$ is dense in H_{F_i}, $i = 1, 2$, since, if $\varrho\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_{F_i}^{1/2})\cap H_{F_i}$ and $f\in C_0^\infty(M/E)$, then $f\varrho\in H_{F_i}^0$.

Let $\varrho_1\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_{F_1}^{1/2})\cap H_{F_1}$. For $m_2\in M$, let $K(\cdot, m_2) = \widetilde{K(\cdot, m_2)}\otimes\sigma_{m_2}$, where $\sigma_{m_2}\in(L^*\otimes D_{F_2}^{1/2})_{m_2}$. We shall put

$$\langle\!\langle\varrho_1(\cdot), K(\cdot, m_2)\rangle\!\rangle := \langle\!\langle\varrho_1(\cdot), \widetilde{K(\cdot, m_2)}\rangle\!\rangle\otimes\sigma_{m_2}. \tag{158}$$

Suppose that on M/D_1, $\langle\!\langle\varrho_1(\cdot), K(\cdot, m_2)\rangle\!\rangle$ defines a measure with bounded absolute variation. Then we shall put

$$(\varrho_1(\cdot), K(\cdot, m_2))_{F_1} := \Big(\int\limits_{M/D_1}\langle\!\langle\varrho_1(\cdot), K(\cdot, m_2)\rangle\!\rangle\Big)\otimes\sigma_{m_2}. \tag{159}$$

Now

$$m_2\mapsto(\varrho_1(\cdot), K(\cdot, m_2))_{F_1}$$

is a not necessarily smooth section of $\boldsymbol{L}^*\otimes\boldsymbol{D}_{F_2}^{1/2}$. Let $\varrho_2\in\Gamma(\boldsymbol{L}\otimes\boldsymbol{D}_{F_1}^{1/2})\cap H_{F_1}$. We shall identify

$$m_2\mapsto\varrho_2(m_2)\otimes(\varrho_1(\cdot), K(\cdot, m_2))_{F_1}$$

with a not necessarily smooth section of $\boldsymbol{D}_{F_1}^{1/2}\otimes\boldsymbol{D}_{F_2}^{1/2}$ in an obvious way, and we shall put

$$\{\varrho_1, \varrho_2\}_K(m_2) := \frac{\varrho_2(m_2)\otimes(\varrho_1(\cdot), K(\cdot, m_2))_{F_1}}{\left(\overset{n}{\omega}\right)^{1/2}(m_2)}. \tag{160}$$

IV.6. DEFINITION. Let K be a distinguished kernel for (F_1, F_2). We say that K is *regular* if:

1. for $\varrho_1\in H_{F_1}^0$ and $m_2\in M$, $\langle\!\langle\varrho_1(\cdot), K(\cdot, m_2)\rangle\!\rangle$ defines on M/D_1 a measure with bounded absolute variation;

2. $m_2 \mapsto (\varrho_1(\cdot), K(\cdot, m_2))_{F_1}$ is a smooth section of $L^* \otimes D^{1/2}_{F_2}$, which is F_2-horizontal;

3. for $\varrho_2 \in H^0_{F_1}$, $\{\varrho_1, \varrho_2\}_K \in \mathscr{L}^1(\overset{n}{\omega})$;

4. $\left|\int\limits_M \{\varrho_1, \varrho_2\}_K \overset{n}{\omega}\right| \leqslant \|\varrho_1\|_{F_1} \|\varrho_2\|_{F_1} \cdot$ const, and the left-hand side is not identically zero.

Let K be a regular distinguished kernel for (F_1, F_2). By $(\cdot, \cdot)^K_{F_1}$ we shall denote the $\frac{1}{2}$-linear continuous form on H_{F_1}, defined for $\varrho_1, \varrho_2 \in H^0_{F_1}$ by

$$(\varrho_1, \varrho_2)^K_{F_1} := \int\limits_M \{\varrho_1, \varrho_2\}_K \overset{n}{\omega}. \tag{161}$$

IV.7. DEFINITION. Let K be a distinguished kernel for (F_1, F_2). We say that K is *reproducing* if:

$$\begin{aligned} &1. \text{ it is regular;} \\ &2. \ (\cdot, \cdot)^K_{F_1} = (\cdot, \cdot)_{F_1}. \end{aligned} \tag{162}$$

We shall be interested in situations in which a distinguished kernel exists and is unique up to a constant factor. One such situation will be described in the proposition below, and another (involving kählerian polarizations) will be given later.

IV.8. PROPOSITION. *Suppose that* F_1 *and* F_2 *are real, and that*

$$\varrho_{D_1 D_2}\colon M \to M/D_1 \times M/D_2, \quad \textit{where } \varrho_{D_1 D_2}(m) := \big(\varrho_{D_1}(m), \varrho_{D_2}(m)\big),$$

is a diffeomorphism. Then there exists a distinguished kernel for (F_1, F_2), *which is unique up to a constant factor.*

Proof. One can easily see that, in virtue of our assumptions about M, F_1 and F_2, maximal integral submanifolds of D_i, $i = 1, 2$, are simply connected. Thus the connection in L defines a parallelism in L over integral submanifolds of D_i, and one can define a bundle L/D_i possessing M/D_i as the base space (see [11]). We have a canonical isomorphism

$$L \simeq \varrho^*_{D_i}(L/D_i). \tag{163}$$

Analogously, for L^* we define L^*/D_i and

$$L^* \simeq \varrho^*_{D_i}(L^*/D_i). \tag{164}$$

D_{F_i} is also parallel over integral submanifolds of D_i, where the parallelism is defined by an isomorphism

$$D_{F_i} \simeq \varrho^*_{D_i}\big(\Lambda^n T^*(M/D_i)^C\big). \tag{165}$$

Let $x \in M/D_i$, U be its open connected simply connected neighbourhood, and σ a non-vanishing n-form on U. Then

$$\{\lambda \in D^{1/2}_{F_i} : \lambda \underline{\otimes} \lambda \text{ is in the range of } \varrho^*_{D_i}\sigma\}$$

is a covering space for the simply connected range of $\varrho^*_{D_i}\sigma$ in $\Lambda^n T^*(M)^C/ /(F_i \cap \overline{F_i})$ and, hence, is composed of two disconnected components. Thus, there exists a (smooth) section $\varphi \in \Gamma_{\varrho^{-1}_{D_i}(U)}(\boldsymbol{D}^{1/2}_{F_i})$ such that $\varphi \underline{\otimes} \varphi = \varrho^*_{D_i}\sigma$. Such sections define a parallelism in $\boldsymbol{D}^{1/2}_{F_i}$ over integral submanifolds of D_i, and enable us to define a vector bundle $\boldsymbol{D}^{1/2}_{F_i}/D^i$. We also have a canonical isomorphism

$$\varrho^*_{D_i}(\boldsymbol{D}^{1/2}_{F_i}/D_i) \simeq \boldsymbol{D}^{1/2}_{F_i}. \tag{166}$$

Let us consider a bundle

$$\boldsymbol{S} := \mathrm{pr}_1^*(\boldsymbol{L}/D_1 \otimes \boldsymbol{D}^{1/2}_{F_1}/D_1) \otimes \mathrm{pr}_2^*(\boldsymbol{L}^*/D_2 \otimes \boldsymbol{D}^{1/2}_{F_2}/D_2) \tag{167}$$

over $M/D_1 \times M/D_2$, where pr_1 and pr_2 are projections of $M/D_1 \times M/D_2$ onto M/D_1 and M/D_2, respectively. Let $\varrho_{D_1} \times \varrho_{D_2}\colon M \times M \to M/D_1 \times M/D_2$ be the product of ϱ_{D_1} and ϱ_{D_2}. The isomorphisms (163), (164) and (166) produce isomorphisms

$$\boldsymbol{P} \simeq (\varrho_{D_1} \times \varrho_{D_2})^*(\boldsymbol{S}) \tag{168}$$

and

$$\boldsymbol{D}^{1/2}_{F_1} \otimes \boldsymbol{D}^{1/2}_{F_2} \simeq \boldsymbol{L} \otimes \boldsymbol{D}^{1/2}_{F_1} \otimes \boldsymbol{L}^* \otimes \boldsymbol{D}^{1/2}_{F_2} \simeq \varrho^*_{D_1D_2}(\boldsymbol{S}). \tag{169}$$

The isomorphisms (168) and (196) allow us in turn to compose a bundle morphism $\varkappa$

$$\begin{tikzcd}
P \arrow[r, "\varkappa'"] \arrow[rr, bend left, "\varkappa"] \arrow[d] & L \otimes D^{1/2}_{F_1} \otimes L^* \otimes D^{1/2}_{F_2} \arrow[r] \arrow[d] & D^{1/2}_{F_1} \otimes D^{1/2}_{F_2} \arrow[d] \\
M \times M \arrow[r, "\varrho^{-1}_{D_1D_2}\circ(\varrho_{D_1}\times\varrho_{D_2})"] & M \arrow[r, "\mathrm{id}_M"] & M
\end{tikzcd} \tag{170}$$

Now $(\overset{n}{\omega})^{1/2}$ is a section of $\boldsymbol{D}^{1/2}_{F_1} \otimes \boldsymbol{D}^{1/2}_{F_2}$. Let K be the unique section of $\boldsymbol{P}$, which makes the following diagram commutative:

$$\begin{tikzcd}
P \arrow[r, "\varkappa"] \arrow[d] & D^{1/2}_{F_1} \otimes D^{1/2}_{F_2} \arrow[d] \\
M \times M \arrow[r, "\varrho^{-1}_{D_1D_2}\circ(\varrho_{D_1}\times\varrho_{D_2})"] \arrow[u, bend left=60, "K"] & M \arrow[u, bend right=60, "(\overset{n}{\omega})^{1/2}"']
\end{tikzcd} \tag{171}$$

It follows immediately from this construction that K is a distinguished kernel. Its uniqueness (up to a constant factor) is also straightforward. ■

We shall examine more closely the situation considered in Proposition IV.8: F_1, F_2 are real and $\varrho_{D_1D_2}$ is a diffeomorphism of M onto $M/D_1 \times M/D_2$. Let ν be the bundle anti-isomorphism of L and L^*, defined by the hermitian structure of L

(172)
$$\begin{tikzcd} L \arrow[r, "\nu"] \arrow[d] & L^* \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$

The complex conjugation defines a bundle anti-isomorphism δ_i of D_{F_i}, $i = 1, 2$,

(173)
$$\begin{tikzcd} D_{F_i} \arrow[r, "\delta_i"] \arrow[d] & D_{F_i} \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$

As in Lemma IV.1, we can show that there exists a bundle anti-isomorphism $\delta_i^{1/2}$ of $D_{F_i}^{1/2}$

(174)
$$\begin{tikzcd} D_{F_i} \arrow[r, "\delta_i^{1/2}"] \arrow[d] & D_{F_i} \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$

such that for $\varphi, \psi \in (D_{F_i}^{1/2})_x$, $(\delta_i^{1/2}\varphi)\underline{\otimes}(\delta_i^{1/2}\psi) = \delta_i(\varphi \underline{\otimes} \psi)$, determined up to the sign.

$\nu \otimes \delta_i^{1/2}$ is a bundle anti-isomorphism of $L \otimes D_{F_i}^{1/2}$ and $L^* \otimes D_{F_i}^{1/2}$, which transforms F-horizontal sections to F-horizontal ones, $i = 1, 2$.

IV.9. Definition. We say that F_1 is *isometrically* related to F_2 if for $\varrho \in H_{F_1}^0$ and $\sigma \in H_{F_2}^0$,

$$\frac{\varrho \otimes [(\nu \otimes \delta_i^{1/2}) \circ \sigma]}{(\overset{n}{\omega})^{1/2}} \in \mathscr{L}^1(\overset{n}{\omega}),$$

and if there exists an isometric operator $V_{F_1F_2}\colon H_{F_1} \to H_{F_2}$ such that

(175)
$$\int_M \frac{\varrho \otimes [(\nu \otimes \delta^{1/2}) \cdot \sigma]}{(\overset{n}{\omega})^{1/2}} (\overset{n}{\omega}) = (\sigma, V_{F_1F_2}\varrho)_{F_2} \cdot \mathrm{const}$$

(as usually we consider $\varrho \otimes [(\nu \otimes \delta_2^{1/2}) \circ \sigma]$ as a section of $D_{F_1}^{1/2} \otimes D_{F_2}^{1/2}$).

If $V_{F_1F_2}$ is unitary, then we say that F_1 is *unitarily* related to F_2 (see [4]).

$V_{F_1F_2}$ is often called a *generalized Fourier transform* relating F_1 to F_2.

The following proposition shows a close connection between reproducing distinguished kernels and isometric operators relating two polarizations.

IV.10. PROPOSITION. *Let F_1, F_2 be real and $\varrho_{D_1D_2}$ be a diffeomorphism of M onto $M/D_1 \times M/D_2$. Let K be a distinguished kernel for (F_1, F_2), and suppose that K is reproducing for (F_1, F_2). Then F_1 is isometrically related to F_2, and for $\varrho \in H_{F_1}^0$ we can put*

$$(V_{F_1F_2}\varrho)(m_2) = (\nu^{-1} \otimes \delta_2^{1/2})\big[\big(\varrho(\cdot), K(\cdot, m_2)\big)_{F_1}\big] \cdot \mathrm{const}. \tag{176}$$

If there exists a reproducing distinguished kernel also for (F_2, F_1), then F_1 is unitarily related to F_2.

Proof. We shall start by proving the following:

LEMMA. *For $\varrho \in H_{F_1}^0$ and $\sigma \in H_{F_2}^0$,*

(177)

$$\int_M \frac{\varrho \otimes [(\nu \otimes \delta_2^{1/2}) \circ \sigma]}{(\overset{n}{\omega})^{1/2}} (\overset{n}{\omega}) = A \int_{M/D_2} \langle\!\langle \sigma(\cdot\cdot), [(\nu^{-1} \otimes \delta_2^{1/2}) \big(\varrho(\cdot), K(\cdot, \cdot\cdot)\big)_{F_1}] \rangle\!\rangle$$

where A does not depend on ϱ nor on σ.

Proof of lemma. Throughout the proof of lemma, we shall identify the points of M, with their images by $\varrho_{D_1D_2}$ in $M/D_1 \times M/D_2$. Let $m = (m', m'') \in M$. In a contractible neighbourhood U' of m' in M/D_1, there exist real 1-forms $\chi_1', \ldots, \chi_n'$, linearly independent at each point, and in a contractible neighbourhood U'' of m'' in M/D_2, real 1-forms $\chi_1'', \ldots, \chi_n''$, also linearly independent at each point. Let $\eta_j' := \varrho_{D_1}^* \chi_j'$ and $\eta_k'' := \varrho_{D_2}^* \chi_k''$, $1 \leqslant j, k \leqslant n$. In a neighbourhood $U := U' \times U''$ of m, there exist (smooth) sections of $D_{F_1}^{1/2}$ and $D_{F_2}^{1/2}$, respectively, $(\eta_1' \wedge \ldots \wedge \eta_n')^{1/2}$ and $(\eta_1'' \wedge \ldots \wedge \eta_n'')^{1/2}$, such that

$$(\eta_1' \wedge \ldots \wedge \eta_n')^{1/2} \underline{\otimes} (\eta_1' \wedge \ldots \wedge \eta_n')^{1/2} = \eta_1' \wedge \ldots \wedge \eta_n'$$

and

$$(\eta_1'' \wedge \ldots \wedge \eta_n'')^{1/2} \underline{\otimes} (\eta_1'' \wedge \ldots \wedge \eta_n'') = \eta_1'' \wedge \ldots \wedge \eta_n'',$$

and non-vanishing sections $s, t \in \Gamma(L)$, F_1-and F_2-horizontal, respectively.

Suppose that

$$\operatorname{suppt} \varrho \subset \varrho_{D_1}^{-1}(U') \quad \text{and} \quad \operatorname{suppt} \sigma \subset \varrho_{D_2}^{-1}(U'').$$

Then

$$\varrho\restriction_U = (f\circ\varrho_{D_1})s\otimes(\eta_1\wedge\ \dots\ \wedge\eta_n')^{1/2},$$
$$\sigma\restriction_U = (g\circ\varrho_{D_2})t\otimes(\eta_1''\wedge\ \dots\ \wedge\eta_n''),$$

where $f\in C_0^\infty(U')$ and $g\in C_0^\infty(U'')$.

We can suppose that

$$\overset{n}{\omega} = a^2(\eta_1'\wedge\ \dots\ \wedge\eta_n'\wedge\eta_1''\wedge\ \dots\ \wedge\eta_n'') \tag{178}$$

with a being a positive function,

$$(\overset{n}{\omega})^{1/2} = a(\eta_1'\wedge\ \dots\ \wedge\eta_n')^{1/2}\otimes(\eta_1''\wedge\ \dots\ \wedge\eta_n'')^{1/2}. \tag{179}$$

We check that for $m_1 = (m_1', m_1'')$ and $m_2 = (m_2', m_2'')$, $m_1, m_2\in U$,

$$K(m_1, m_2) = B\frac{a(m_3)}{\big(t(m_3), s(m_3)\big)}\times$$
$$\times s(m_1)\otimes(\eta_1'\wedge\dots\ \wedge\eta_n')^{1/2}(m_1)\otimes\nu\big(t(m_2)\big)\otimes(\eta_1''\wedge\ \dots\ \wedge\eta_n'')^{1/2}(m_2),$$

where $m_3 = (m_1', m_2'')$, and B does not depend on m_1 nor on m_2. By (44), (53), (56) and (158), we have

$$\langle\!\langle\varrho(\cdot), K(\cdot, m_2)\rangle\!\rangle(m_1') = \tag{180}$$

$$B\left(\frac{\overline{f(m_1')}a(m_3)|s(m_3)|^2}{\big(t(m_3), s(m_3)\big)}|\chi_1'\wedge\ \dots\ \wedge\chi_n'|(m_1')\right)\otimes\nu\big(t(m_2)\big)\otimes(\eta_1''\ \dots\ \wedge\eta_n'')^{1/2}(m_2).$$

By (159),

$$\big(\varrho(\cdot), K(\cdot, m_2)\big)_{F_1} = B\left(\int\limits_{M/D_1'}\frac{\overline{f(\cdot)}a(\cdot, m_2'')|s(\cdot, m_2'')|^2}{\big(t(\cdot, m_2''), s(\cdot, m_2'')\big)}|\chi_1'\wedge\ \dots\right. \tag{181}$$
$$\left.\dots\wedge\chi_n'|(\cdot)\right)\nu\big(t(m_2)\big)\otimes(\eta_1''\wedge\ \dots\ \wedge\eta_n'')^{1/2}(m_2).$$

Further,

$$\langle\!\langle\sigma(\cdot\cdot), \big[(\nu^{-1}\otimes\delta_2^{1/2})\big(\varrho(\cdot), K(\cdot, \cdot\cdot)\big)_{F_1}\big]\rangle\!\rangle\ (m_2'') \tag{182}$$
$$= \pm B\overline{g(m_2'')}\left(\int\limits_{M/D}\frac{\overline{f(\cdot)}a(\cdot, m_2'')|s(\cdot, m_2'')|^2}{\big(s(\cdot, m_2''), t(\cdot, m_2'')\big)}|\chi_1'\wedge\ \dots\right.$$
$$\left.\dots\wedge\chi_n'|(\cdot)\right)|t(m_2)|^2|\chi_1''\wedge\ \dots\ \wedge\chi_n''|(m_2''),$$

where the sign "$-$" appears if $\delta_2^{1/2}\circ(\eta_1''\wedge\ldots\wedge\eta_n'')^{1/2} = -(\eta_1''\wedge\ldots\wedge\eta_n'')^{1/2}$. Hence,

$$\begin{aligned}(183)\quad &\int\limits_{M/D_2}\langle\!\langle\sigma(\cdot\cdot),\, [(\nu^{-1}\otimes\delta_2^{1/2})\big(\varrho(\cdot), K(\cdot,\cdot\cdot)\big)_{F_1}]\rangle\!\rangle \\ &= \pm B\int\limits_{M/D_1\times M/D_2}\frac{\overline{g(m_2'')}f(m_1')a(m_1', m_2'')\,|s(m_1', m_2'')|^2\,|t(m_1', m_2'')|^2}{\big(s(m_1', m_2''), t(m_1', m_2'')\big)}\times \\ &\qquad\times|\eta_1'\wedge\ldots\wedge\eta_n'\wedge\eta_1''\wedge\ldots\wedge\eta_n''|(m_1', m_2'') \\ &= \pm B\int\limits_{M/D_1\times M/D_2}\overline{g(m_2'')}f(m_1')a(m_1', m_2'')\big(t(m_1', m_2''), s(m_1', m_2'')\big)\times \\ &\qquad\times|\eta_1'\wedge\ldots\wedge\eta_n'\wedge\eta_1''\wedge\ldots\wedge\eta_n''|(m_1', m_2''),\end{aligned}$$

as s and t are proportional at each point.

On the other hand,

$$\begin{aligned}(184)\quad &\int\limits_M\frac{\varrho\otimes[(\nu\otimes\delta_2^{1/2})\circ\sigma]}{(\overset{n}{\omega})^{1/2}}\overset{n}{\omega} = \pm\int\limits_M\overline{g(m_2'')}f(m_1')a(m_1', m_2')\times \\ &\qquad\times\big(t(m_1', m_2''), s(m_1', m_2'')\big)(\eta_1'\wedge\ldots\wedge\eta_n'\wedge\eta_1''\wedge\ldots\wedge\eta_2'')(m_1', m_2''),\end{aligned}$$

where the sign "$-$" appears if $\delta_2^{1/2}\cdot(\eta_1''\wedge\ldots\wedge\eta_n'')^{1/2} = -(\eta_1''\wedge\ldots\wedge\eta_n'')$. Thus

$$\begin{aligned}(185)\quad &\int\limits_M\frac{\varrho\otimes[(\nu\otimes\delta_2^{1/2})\circ\sigma]}{(\overset{n}{\omega})^{1/2}}\overset{n}{\omega} = \pm\int\limits_{M/D_1\times M/D_2}\overline{g(m_2'')}f(m_1')a(m_1', m_2'')\times \\ &\qquad\times\big(t(m_1', m_2''), s(m_1', m_2'')\big)|\eta_1'\wedge\ldots\wedge\eta_n'\wedge\eta_1''\wedge\ldots\wedge\eta_n''|(m_1', m_2'').\end{aligned}$$

Comparing (183) and (185), we get (177) with $A = 1/B$ for our special case.

The general case when $\varrho\in H^0_{F_1}$, $\sigma\in H^0_{F_2}$ are arbitrary can be reduced to the considered case by means of partitions of unity on M/D_1 and M/D_2. □

For $\varrho\in H^0_{F_1}$, let

$$(186)\qquad (V_{F_1F_2}\varrho)(m_2) := A^{1/2}(\nu^{-1}\otimes\delta_2^{1/2})\big(\big(\varrho(\cdot), K(\cdot, m_2)\big)_{F_1}\big).$$

We note that $V_{F_1F_2}\varrho\in\Gamma(L\otimes D_{F_2}^{1/2})$ and depends linearly on $\varrho\in H^0_{F_1}$. For $\sigma\in H^0_{F_2}$, (177) and (186) yield

$$(187)\qquad \int\limits_M\frac{\varrho\otimes[(\nu\otimes\delta_2^{1/2})\circ\sigma]}{(\overset{n}{\omega})^{1/2}}\overset{n}{\omega} = A^{1/2}\int\limits_{M/D_2}\langle\!\langle\sigma, V_{F_1F_2}\varrho\rangle\!\rangle.$$

Now let $\varphi_k \in C_0^\infty(M/D_2)$, $0 \leqslant \varphi_k \leqslant 1$ and $\varphi_k(r) \nearrow 1$ for each $r \in M/D_2$, and let $\psi_k := \varphi_k \circ \varrho_{D_2}$. $\psi_k(V_{F_1F_2}\varrho) \in H^0_{F_2}$, and from (187) we get

$$\overline{A^{1/2}} \int\limits_M \psi_k(m_2) \frac{\varrho(m_2) \otimes \big(\varrho(\cdot), K(\cdot, m_2)\big)_{F_1}}{(\overset{n}{\omega})^{1/2}(m_2)} \overset{n}{\omega}(m_2) \tag{188}$$

$$= \int\limits_M \psi_k \frac{\varrho \otimes [(\nu \otimes \delta_2^{1/2}) \circ V_{F_1F_2}\varrho]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega} = A^{1/2} \int\limits_{M/D_2} \psi_k \langle\!\langle V_{F_1F_2}\varrho, V_{F_1F_2}\varrho \rangle\!\rangle .$$

But K is regular. Therefore,

$$m_2 \mapsto \frac{\varrho(m_2) \otimes \big(\varrho(\cdot), K(\cdot, m_2)\big)_{F_1}}{(\overset{n}{\omega})^{1/2}(m_2)}$$

belongs to $\mathscr{L}^1(\overset{n}{\omega})$. Hence,

$$\lim_{k\to\infty} \int\limits_M \psi_k(m_2) \frac{\varrho(m_2) \otimes \big(\varrho(\cdot), K(\cdot, m_2)\big)_{H_{F_1}}}{(\overset{n}{\omega})^{1/2}(m_2)} \overset{n}{\omega}(m_2)$$

$$= \int\limits_M \frac{\varrho(m_2) \otimes \big(\varrho(\cdot), K(\cdot, m_2)\big)_{F_1}}{(\overset{n}{\omega})^{1/2}(m_2)} \overset{n}{\omega}(m_2) = \int\limits_M \{\varrho, \varrho\}_K \overset{n}{\omega} = (\varrho, \varrho)^K_{F_1} = (\varrho, \varrho)_{F_1}.$$

Thus,

$$\lim_{k\to\infty} \int\limits_{M/D_2} \psi_k \langle\!\langle V_{F_1F_2}\varrho, V_{F_1F_2}\varrho \rangle\!\rangle \xrightarrow[k\to\infty]{} \frac{\overline{A^{1/2}}}{A^{1/2}} \|\varrho\|^2_{F_1}. \tag{189}$$

So $\overline{A^{1/2}} = A^{1/2}$, and

$$\int\limits_{M/D_2} \langle\!\langle V_{F_1F_2}\varrho, V_{F_1F_2}\varrho \rangle\!\rangle = \|\varrho\|^2_{F_1}. \tag{190}$$

Hence

$$V_{F_1F_2}\varrho \in H_{F_2} \text{ and } V_{F_1F_2} \text{ maps } H^0_{F_1} \text{ into } H_{F_2}$$

isometrically and thus can be defined as an isometric operation from H_{F_1} into H_{F_2}.

If also for (F_2, F_1) there exists a reproducing distinguished kernel, we can show similarly that F_2 is isometrically related to F_1, where the relating isometry $V_{F_2F_1}$, as can easily be seen from (187), is equal to $\lambda V^*_{F_1F_2}$. Thus, if $\sigma \in H_{F_2}$ is orthogonal to the range of $V_{F_1F_2}$, then

$$(\sigma, V_{F_1F_2}\varrho)_{F_2} = (V^*_{F_1F_2}\sigma, \varrho)_{F_2} = \frac{1}{\lambda}(V_{F_2F_1}\sigma, \varrho)_{F_2}$$

and, consequently, $\sigma = 0$. Thus, $V_{F_1F_2}$ is unitary and, changing at most the phase of $V_{F_2F_1}$, we can write $V_{F_2F_1} = V^*_{F_1F_2}$. ∎

Coming back to the general case of not necessarily real polarizations, let $V\colon M \to M$ be a canonical diffeomorphism preserving F_1 and F_2. Let $\tilde{V}$, $\utilde{V}_1^{1/2}$, $\utilde{V}_2^{1/2}$

(191)
$$\begin{tikzcd}
L \arrow[r, "\tilde{V}"] \arrow[d] & L \arrow[d] & D_{F_1}^{1/2} \arrow[r, "\utilde{V}_1^{1/2}"] \arrow[d] & D_{F_1}^{1/2} \arrow[d] & D_{F_2}^{1/2} \arrow[r, "\utilde{V}_2^{1/2}"] \arrow[d] & D_{F_2}^{1/2} \arrow[d] \\
M \arrow[r, "V"] & M & M \arrow[r, "V"] & M & M \arrow[r, "V"] & M
\end{tikzcd}$$

be automorphisms as those given in (144) and (147). For L^*, we also have a bundle automorphism $(\tilde{V}^{-1})^*$

(192)
$$\begin{tikzcd}
L^* \arrow[r, "(\tilde{V}^{-1})^*"] \arrow[d] & L^* \arrow[d] \\
M \arrow[r, "V"] & M
\end{tikzcd}$$

which preserves the connection of L^*.

The bundle automorphisms from (191) and (192) allow us to define, in an evident way, a bundle automorphism $\widetilde{V \times V}$

(193)
$$\begin{tikzcd}
P \arrow[r, "\widetilde{V \times V}"] \arrow[d] & P \arrow[d] \\
M \times M \arrow[r, "V \times V"] & M \times M
\end{tikzcd}$$

IV.11. Proposition. *Let K be a distinguished kernel for (F_1, F_2). Then*

$$\widetilde{V \times V} \circ K \circ (V \times V)^{-1}$$

is also a distinguished kernel.

Proof. Our proposition follows at once from the fact that the bundle automorphisms $\tilde{V} \otimes \utilde{V}_1^{1/2}$ and $(\tilde{V}^{-1})^* \otimes \utilde{V}_2^{1/2}$

$$\begin{tikzcd}
L \otimes D_{F_1}^{1/2} \arrow[r, "\tilde{V} \otimes \utilde{V}_1^{1/2}"] \arrow[d] & L \otimes D_{F_2}^{1/2} \arrow[d] & L^* \otimes D_{F_2}^{1/2} \arrow[r, "(\tilde{V}^{-1})^* \otimes \utilde{V}_2^{1/2}"] \arrow[d] & L^* \otimes D_{F_2}^{1/2} \arrow[d] \\
M \arrow[r, "V"] & M & M \arrow[r, "V"] & M
\end{tikzcd}$$

transform F_1- and F_2-horizontal sections of $L \otimes D^{1/2}_{F_1}$ and $L^* \otimes D^{1/2}_{F_2}$ to F_1 and F_2-horizontal sections, respectively, and from the fact that the bundle automorphism $\utilde{V_1^{1/2}} \otimes \utilde{V_2^{1/2}}$

(194)
$$\begin{tikzcd}
D^{1/2}_{F_1} \otimes D^{1/2}_{F_2} \arrow[r, "\utilde{V_1^{1/2}} \otimes \utilde{V_2^{1/2}}"] \arrow[d] & D^{1/2}_{F_1} \otimes D^{1/2}_{F_2} \arrow[d] \\
M \arrow[r, "V"] & M
\end{tikzcd}$$

transforms $(\overset{n}{\omega})^{1/2}$ into $\pm(\overset{n}{\omega})^{1/2}$ in virtue of (143), (148) and (155). ∎

Let $Q_{F_1F_2}$ denote the group of canonical diffeomorphisms of M preserving F_1 and F_2. Let $\widetilde{Q_{F_1F_2}}$ denote the group of all bundle automorphisms $\tilde{V} \otimes \utilde{V_1^{1/2}}$ obtained for $V \in Q_{F_1F_2}$. We have a short exact sequence of groups

$$1 \to S^1 \to \widetilde{Q_{F_1F_2}} \to Q_{F_1F_2} \to 1. \tag{195}$$

Unitary operators $U_{\tilde{V} \otimes \utilde{V_1^{1/2}}}$, defined as in (153), provide a unitary representation U of $\widetilde{Q_{F_1F_2}}$ in H_{F_1},

$$\widetilde{Q_{F_1F_2}} \ni \tilde{V} \otimes \utilde{V_1^{1/2}} \xrightarrow{U} U^{-1}_{\tilde{V} \otimes \utilde{V^{1/2}}} \in \operatorname{Aut}(H_{F_1}).$$

IV.12. Proposition. *Suppose that K is the unique (up to a constant factor) distinguished kernel, and that K is regular. Suppose also that the representation*

$$U\colon \widetilde{Q_{F_1F_2}} \to \operatorname{Aut}(H_{F_1})$$

is irreducible. Then K, after a suitable normalization, is reproducing.

Proof. Let $V \times V$ and $\widetilde{V \times V}$ be as in (193). By Proposition IV.11, $\widetilde{V \times V} \cdot K \cdot (V \times V)^{-1}$ is proportional to K and a closer examination of the restrictions of both sections of P to Δ shows that

$$\widetilde{V \times V} \cdot K \cdot (V \times V)^{-1} = \pm K, \tag{196_1}$$

the sign depending on whether the bundle automorphism $\utilde{V_1^{1/2}} \otimes \utilde{V_2^{1/2}}$ in (194) preserves $(\overset{n}{\omega})^{1/2}$ or changes its sign. So for $\varrho_1, \varrho_2 \in H^0_{F_1}$, we have (see (160))

$$\{U_{\tilde{V} \otimes \utilde{V_1^{1/2}}} \varrho_1, U_{\tilde{V} \otimes \utilde{V_1^{1/2}}} \varrho_2\}_K = \pm \{\varrho_1, \varrho_2\}_{\widetilde{V \times V} \cdot K \cdot (V \times V)^{-1}} \circ V = \{\varrho_1, \varrho_2\}_K \circ V \tag{196_2}$$

as the sign "$-$" in the middle term of (196_2) appears if and only if it appears in (196_1). Hence,

$$(U_{\tilde{V}\otimes V_1^{1/2}}\varrho_1,\ U_{\tilde{V}\otimes V_2^{1/2}}\varrho_2)^K_{F_1} = (\varrho_1,\varrho_2)^K_{F_1}. \tag{197}$$

Let $A \in B(H_{F_1})$ be defined by (see (76) and (161), and 4. in Definition IV.6)

$$(\cdot,\cdot)^K_{F_1} = (A\cdot,\cdot)_{F_1}.$$

(197) shows that A commutes with $U_{\tilde{V}\otimes V_1^{1/2}}$, and the thesis of our proposition follows immediately from the irreducibility of U and the Schur Lemma. ■

We shall describe another situation in which a distinguished kernel exists.

IV.13. PROPOSITION. *Suppose that F_1 is kählerian, $H_{F_1} \neq \{0\}$, and $F_2 = \overline{F_1}$. Suppose that the group $Q_{F_1F_2}$ acts transitively on M (i.e. M is a Klein manifold). Then there exists a reproducing distinguished kernel, which is the unique (up to a constant factor) distinguished kernel for (F_1, F_2).*

Proof. We shall first show that L and $D^{1/2}_{F_1}$ carry, in our case, a natural structure of holomorphic bundles (see [12]).

Locally, $L \simeq (M\times C, \mathrm{pr}_1, M)$ and the connection form $a \simeq \frac{dz}{iz} + \mathrm{pr}_1^*\theta$, where θ is a 1-form on M. Since F_1 is involutive, it defines a complex structure on M, for which F_1 is the sub-bundle of anti-holomorphic vectors. We have $\theta = \theta_{10}+\theta_{01}$, where θ_{10} is of (1, 0)-type and θ_{01} of (0, 1)-type.

$$\omega = d\theta = \partial\theta_{10}+\bar\partial\theta_{10}+\partial\theta_{01}+\bar\partial\theta_{01}.$$

Since F_1 is ω-isotropic, $\bar\partial\theta_{01} = 0$. Thus, by the Grothendic–Dolbeault Lemma, there exists locally a function f, such that

$$\bar\partial f = -\theta_{01}.$$

It is easily verified that e^{if} defines a local F_1-horizontal section of L. Thus, locally, there exist F_1-horizontal non-vanishing sections of L. Since two such sections differ by a holomorphic factor, we can treat L as a holomorphic bundle over M. Similarly, there exist local non-vanishing F_1-horizontal sections φ of $D^{1/2}_{F_1}$ (such that $\varphi\otimes\varphi$ is a local holomorphic n-form on M), and also here two such sections differ by a holomorphic factor. Thus we can provide $D^{1/2}_{F_1}$ with a structure of a holomorphic bundle over M. Therefore $L\otimes D^{1/2}_{F_1}$ possesses a natural structure of a holomorphic bundle. The holomorphic and the F_1-horizontal sections of $L\otimes D^{1/2}_{F_1}$ coincide.

LEMMA 1 (see [2], Proposition 1.5.11). $H_{F_1}\cap\Gamma(L\otimes D_1^{1/2}) = H_{F_1}$. *Thus, no completion procedure is needed to define* H_{F_1}.

Proof of Lemma 1. We must show that any Cauchy sequence (ϱ_n) of holomorphic sections of $L\otimes D_{F_1}^{1/2}$ converges in H_{F_1} to a holomorphic section. Let $\sigma\in\Gamma_U(L\otimes D_{F_1}^{1/2})$ be holomorphic, non-vanishing, and U an open subset of M. Then

$$\varrho_n{\restriction}_U = f_n\sigma \tag{198}$$

and

$$\langle\!\langle \varrho_n-\varrho_m, \varrho_n-\varrho_m\rangle\!\rangle_{F_1} = |f_n-f_m|^2\langle\!\langle \sigma, \sigma\rangle\!\rangle_{F_1} \quad \text{on } U.$$

$\langle\!\langle \sigma, \sigma\rangle\!\rangle_{F_1}$ is a non-vanishing smooth absolute volume element on U. But for any differential operator $\mathscr{D}$ on U with smooth coefficients, any compact subset $\mathscr{K}\subset U$, and any positive measure μ on U with smooth non-vanishing density, there exists $C_{\mathscr{K},\mathscr{D},\mu}$ such that

$$\sup_{\mathscr{K}}|\mathscr{D}f| \leqslant C_{\mathscr{K},\mathscr{D},\mu}\Big(\int_U |f|^2 d\mu\Big)^{1/2} \tag{199}$$

for any holomorphic function f on U – see [14]. Thus,

$$\sup_{\mathscr{K}}|\mathscr{D}(f_n-f_m)| \xrightarrow[n,m\to\infty]{} 0$$

– (f_m) is a Cauchy sequence in the topology of compact convergence of all derivatives. Hence, (f_m) converge in this topology to a holomorphic function f on U. It can easily be checked that local sections $f\sigma$ (taken for different σ) define a holomorphic section ϱ of $L\otimes D_{F_1}^{1/2}$. For C being a measurable subset of U, we have

$$\int_C \langle\!\langle \varrho-\varrho_N, \varrho-\varrho_N\rangle\!\rangle_{F_1} \leqslant \sup_{m\geqslant N}\int_C \langle\!\langle \varrho_m-\varrho_N, \varrho_m-\varrho_N\rangle\!\rangle_{F_1},$$

as $f_n\to f$ in $L^2(U, \langle\!\langle \sigma, \sigma\rangle\!\rangle_{F_1})$, and, consequently,

$$\int_M \langle\!\langle \varrho-\varrho_N, \varrho-\varrho_N\rangle\!\rangle_{F_1} \leqslant \sup_{m\geqslant N}\int_M \langle\!\langle \varrho_m-\varrho_N, \varrho_m-\varrho\rangle\!\rangle_{F_1} \xrightarrow[N\to\infty]{} 0.$$

Thus, $\varrho\in H_{F_1}$ and $\varrho_N \xrightarrow[N\to\infty]{} \varrho$ in H_{F_1}. ∎

Let us bear in mind that the hermitian structure in L defines a bundle anti-isomorphism ν of L and L^*,

$$\begin{tikzcd} L \arrow[r, "\nu"] \arrow[d] & L^* \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd} \tag{172}$$

The complex conjugation defines a bundle anti-isomorphism δ

$$(200)\qquad \begin{tikzcd} D_{F_1} \arrow[r, "\delta"] \arrow[d] & D_{F_2} \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}.$$

Analogously, as in Lemma IV.1, we can show that there exists a bundle anti-isomorphism $\delta^{1/2}$

$$(201)\qquad \begin{tikzcd} D^{1/2}_{F_1} \arrow[r, "\delta^{1/2}"] \arrow[d] & D^{1/2}_{F_2} \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$

determined up to the sign.

Thus, we have a bundle anti-isomorphism $\nu\otimes\delta^{1/2}$

$$(202)\qquad \begin{tikzcd} L\otimes D^{1/2}_{F_1} \arrow[r, "\nu\otimes\delta^{1/2}"] \arrow[d] & L^*\otimes D^{1/2}_{F_2} \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$

LEMMA 2. *Let* $\varrho_1, \varrho_2 \in \Gamma(L\otimes D^{1/2}_{F_1})$. *The following relation holds*:

$$(203)\qquad \frac{\varrho_2\otimes[(\nu\otimes\delta^{1/2})\cdot\varrho_1]}{(\overset{n}{\omega})^{1/2}} = A\,\frac{\langle\!\langle \varrho_1, \varrho_2\rangle\!\rangle_{F_1}}{|\overset{n}{\omega}|},$$

where, as usual, we identify the sections of $L\otimes D^{1/2}_{F_1}\otimes L^*\otimes D^{1/2}_{F_2}$ *and of* $D^{1/2}_{F_1}\otimes D^{1/2}_{F_2}$. A *in* (203) *is a complex number and does not depend on the point nor on* ϱ_1 *and* ϱ_2, $|A| = n!$.

Proof of Lemma 2. We shall check (203) locally. Let $\eta_1, \ldots, \eta_n$ be local 1-forms on M, linearly independent at each point and vanishing on F_1-directions. Let $(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}$ be a local section of $D^{1/2}_{F_1}$ such that

$$(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}\underline{\otimes}(\eta_1\wedge\ldots\wedge\eta_n)^{1/2} = \eta_1\wedge\ldots\wedge\eta_n.$$

We have, locally,

$$(204)\qquad \varrho_i = s_i\otimes(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}, \quad \text{where } s_i\in\Gamma_{\mathrm{loc}}(L),\ i = 1, 2,$$

$$(205)\qquad \omega = \sum_{i,j} B^{ij}\eta_i\wedge\overline{\eta}_j,$$

$$(206)\qquad \overset{n}{\omega} = (-1)^{n(n-1)/2} n!\det(B^{ij})(\eta_1\wedge\ldots\wedge\eta_n)\wedge(\overline{\eta_1}\wedge\ldots\wedge\overline{\eta_n}),$$

$$(207)\qquad (\overset{n}{\omega})^{1/2} = \pm a_n(n!)^{1/2}\big(\det(B^{ij})\big)^{1/2}(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}\otimes\delta^{1/2}[(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}],$$

where

$$a_n = \begin{cases} 1 & \text{if } n = 4k \text{ or } n = 4k+1, \\ i & \text{if } n = 4k+2 \text{ or } n = 4k+3. \end{cases}$$

Now, locally,

$$\frac{\varrho_2 \otimes [(\nu \otimes \delta^{1/2}) \circ \varrho_1]}{(\overset{n}{\omega})^{1/2}} = \pm \frac{(s_1, s_2)}{a_n (n!)^{1/2} (\det(B^{ij}))^{1/2}}. \tag{208}$$

By (44) and (53),

$$\langle\!\langle \varrho_1, \varrho_2 \rangle\!\rangle_{F_1} = \frac{(s_1, s_2)}{|\hat{\omega}^n(\eta_1, \ldots, \eta_n, \overline{\eta_1}, \ldots, \overline{\eta_n})|^{1/2}} |\eta_1 \wedge \ldots \wedge \eta_n \wedge \overline{\eta_1} \wedge \ldots \wedge \overline{\eta_n}|.$$

Thus, locally,

$$\frac{\langle\!\langle \varrho_1, \varrho_2 \rangle\!\rangle_{F_1}}{|\overset{n}{\omega}|} = \frac{(s_1, s_2)}{n! \, |\det(B^{ij})| \, |\hat{\omega}^n(\eta_1, \ldots, \eta_n, \overline{\eta_1}, \ldots, \overline{\eta_n})|^{1/2}}. \tag{209}$$

We must still compute

$$|\hat{\omega}^n(\eta_1, \ldots, \eta_n, \overline{\eta_1}, \ldots, \overline{\eta_n})| = |\langle \eta_1^{\#}, \ldots, \eta_n^{\#}, \overline{\eta_1}^{\#}, \ldots, \overline{\eta_n}^{\#} | \overset{n}{\omega} \rangle|$$

$$= n! \, |\det(B^{ij})| \, \big| \langle \eta_1^{\#}, \ldots, \eta_n^{\#}, \overline{\eta_1}^{\#}, \ldots, \overline{\eta_n}^{\#} | \eta_1 \wedge \ldots \wedge \eta_n \wedge \overline{\eta_1} \wedge \ldots \wedge \overline{\eta_n} \rangle \big|.$$

Let

$$C_{ij} := \langle \eta_j^{\#}, \overline{\eta_i}^{\#} | \omega \rangle = \langle \overline{\eta_i}^{\#} | \eta_j \rangle = \overline{\langle \eta_i^{\#} | \overline{\eta_j} \rangle}.$$

We have

$$\overline{\eta_j} = \overline{\eta_j}^{\#} \lrcorner \, \omega = \overline{\eta_j}^{\#} \lrcorner \Big(\sum_{i,k} B^{ik} \eta_i \wedge \overline{\eta_k} \Big) = \sum_{i,k} C_{ji} B^{ik} \overline{\eta_k},$$

as $\langle \overline{\eta_j}^{\#} | \overline{\eta_i} \rangle = 0$.

Thus, the matrix (C_{ij}) is inverse to (B^{ij}).

Now

$$|\langle \eta_1^{\#}, \ldots, \eta_n^{\#}, \overline{\eta_1}^{\#}, \ldots, \overline{\eta_n}^{\#} | \eta_1 \wedge \ldots \wedge \eta_n \wedge \overline{\eta_1} \ldots \wedge \overline{\eta_n} \rangle|$$

$$= \det(C_{ij}) \overline{\det(C_{ij})} = \frac{1}{|\det(B^{ij})|^2}.$$

Hence

$$|\hat{\omega}^n(\eta_1, \ldots, \eta_n, \overline{\eta_1}, \ldots, \overline{\eta_n})| = \frac{n!}{|\det(B^{ij})|}. \tag{210}$$

(209) and (210) yield

$$\frac{\langle\!\langle \varrho_1, \varrho_2 \rangle\!\rangle_{F_1}}{|\overset{n}{\omega}|} = \frac{(s_1, s_2)}{(n!)^{3/2} |\det(B^{ij})|^{1/2}}. \tag{211}$$

From (205) we conclude that $\det(B^{ij})$ is non-vanishing, real or purely imaginary (ω is real). Thus, comparing (208) and (211), we get the thesis of Lemma 2. □

If ϱ is an F_1-horizontal section of $\boldsymbol{L} \otimes \boldsymbol{D}_{F_1}^{1/2}$, then $(\nu \otimes \delta^{1/2}) \circ \varrho$ is F_2-horizontal.

Let $\sigma \in \Gamma_U(\boldsymbol{L} \otimes \boldsymbol{D}_{F_1}^{1/2})$ be a non-vanishing F_1-horizontal section. Let $m_2 \in U \subset M$. For $\varrho \in H_{F_1}$, let $\tilde{\varrho}^{\sigma}$ be a holomorphic function defined by

$$\varrho \restriction_U = \tilde{\varrho}^{\sigma} \sigma. \tag{212}$$

Then, in virtue of (199),

$$H_{F_1} \ni \varrho \to \tilde{\varrho}^{\sigma}(m_2) \in \boldsymbol{C}$$

is a linear continuous form on H_{F_1}. Thus, there exists $\vartheta_{m_2}^{\sigma} \in H_{F_1}$ such that

$$\tilde{\varrho}^{\sigma}(m_2) = (\vartheta_{m_2}^{\sigma}, \varrho)_{F_1}. \tag{213}$$

Let $\tau \in \Gamma_W(\boldsymbol{L} \otimes \boldsymbol{D}_{F_1}^{1/2})$ be another non-vanishing F_1-horizontal section. Let

$$\vartheta_{m_2}^{\sigma} \restriction_W = \tilde{\vartheta}_{m_2}^{\sigma,\tau} \tau. \tag{214}$$

LEMMA 3.

$$W \times U \ni (m_1, m_2) \mapsto \tilde{\vartheta}_{m_2}^{\sigma,\tau}(m_1) \in \boldsymbol{C} \tag{215}$$

is smooth, holomorphic in m_1*-variables and anti-holomorphic in* m_2*-variables.*

Proof of Lemma 3. Without loss of generality, we can assume that U and W are open balls in $\boldsymbol{R}^{2n} = \boldsymbol{C}^n$.

Let

$$\mathscr{D}^{a} := \frac{\partial^{|a|}}{\partial x_1^{a_1} \ldots \partial x_{2n}^{a_{2n}}}, \qquad a := (a_1, \ldots, a_{2n}),\ |a| := a_1 + \ldots + a_{2n}. \tag{216}$$

We note that if $\mathscr{D}$ is a (complex) linear combination of $\mathscr{D}^{a}$, then

$$H_{F_1} \ni \varrho \mapsto \mathscr{D}\tilde{\varrho}^{\sigma}(m_2) \in \boldsymbol{C}$$

is also a continuous linear form on H_{F_1} (see (199)). Let $\vartheta_{\mathscr{D},m_2}^{\sigma} \in H_{F_1}$ be such that

$$\mathscr{D}\tilde{\varrho}^{\sigma}(m_2) = (\vartheta_{\mathscr{D},m_2}^{\sigma}, \varrho)_{F_1}, \tag{217}$$

and let

$$\vartheta_{\mathscr{D},m_2}^{\sigma} \restriction_W = \tilde{\vartheta}_{\mathscr{D},m_2}^{\sigma,\tau} \tau. \tag{218}$$

First we note that for each $m_2 \in M$,

$$W \ni m_1 \mapsto \tilde{\vartheta}_{m_2}^{\sigma,\tau}(m_1) \quad \text{is holomorphic.}$$

Now we shall show that

$$\mathscr{D}^a_{m_2}\tilde{\vartheta}^{\sigma,\tau}_{m_2}(m_1) = \tilde{\vartheta}^{\sigma,\tau}_{\mathscr{D}^a,m_2}(m_1) \quad \text{for } |a| = 1 \tag{219}$$

(the index m_2 at $\mathscr{D}^a$ shows that the latter acts on m_2-variables).

We have

$$\begin{aligned} \left\| \frac{1}{t}(\vartheta^\sigma_{(m_2+te} - \vartheta^\sigma_{m_2}) - \vartheta^\sigma_{\nabla_e,m_2} \right\|_{F_1} &= \sup_{\substack{\varrho\in H_{F_1}\\ \|\varrho\|_{F_1}<1}} \left| \left(\frac{1}{t}(\vartheta^\sigma_{(m_2+te} - \vartheta^\sigma_{m_2}) - \vartheta^\sigma_{\nabla_e,m_2}, \varrho \right)_{F_1} \right| \\ &= \sup_{\substack{\varrho\in H_{F_1}\\ \|\varrho\|_{F_1}<1}} \left| \frac{1}{t}\left(\tilde{\varrho}^\sigma(m_2+te) - \tilde{\varrho}^\sigma(m_2)\right) - \nabla_e\tilde{\varrho}^\sigma(m_2) \right| \\ &\leqslant C \sup_{\substack{\varrho\in H_{F_1}\\ \|\varrho\|_{F_1}<1}} \sup_{\substack{|\gamma|=2\\ m\in\mathscr{K}}} |\mathscr{D}^\gamma\tilde{\varrho}^\sigma(m)|\,\|e\|^2|t| \leqslant C_1|t| \end{aligned} \tag{220}$$

for sufficiently small t ($\mathscr{K}$ being a compact neighbourhood of m_2 in U) (we have again used (199)).

Thus,

$$\frac{1}{t}(\vartheta^\sigma_{(m_2+te)} - \vartheta^\sigma_{m_2}) \xrightarrow[t\to 0]{} \vartheta^\sigma_{\nabla_e,m_2} \quad \text{in } H_{F_1}, \tag{221}$$

and so, in virtue of (199),

$$\frac{1}{t}\left(\tilde{\vartheta}^{\sigma,\tau}_{(m_2+te)}(m_1) - \tilde{\vartheta}^{\sigma,\tau}_{m_2}(m_1)\right) \xrightarrow[t\to 0]{} \tilde{\vartheta}^{\sigma,\tau}_{\nabla_e,m_2}(m_1), \tag{222}$$

which proves (219).

Now if $\mathscr{D}$ is a first order derivation in the direction of F_2, then, by (222),

$$\mathscr{D}_{m_2}\tilde{\vartheta}^{\sigma,\tau}_{m_2}(m_1) = \tilde{\vartheta}^{\sigma,\tau}_{\overline{\mathscr{D}},m_2}(m_1)$$

($\tilde{\vartheta}^{\sigma,\tau}_{\mathscr{D},m_2}$ depends aniti-linearly on $\mathscr{D}$).

But for any $\varrho\in H_{F_1}$,

$$(\vartheta^\sigma_{\overline{\mathscr{D}},m_2}, \varrho)_{H_{F_1}} = \overline{\mathscr{D}}\tilde{\varrho}^\sigma(m_2) = 0.$$

Hence $\vartheta^\sigma_{\overline{\mathscr{D}},m_2} = 0$, and consequently, $\mathscr{D}_{m_2}\tilde{\vartheta}^{\sigma,\tau}_{m_2} = 0$.

Thus, (215) is holomorphic in m_1-variables and anti-holomorphic in m_2-variables, and also (by Hartog's Theorem) smooth, which is what we were to show. ■

Let us put

$$(223)\quad K(m_1, m_2) := \vartheta^{\sigma}_{m_2}(m_1) \otimes \left[(\nu \otimes \delta^{1/2})\big(\sigma(m_2)\big)\right] \in (\boldsymbol{L} \otimes \boldsymbol{D}^{1/2}_{F_1})_{m_1} \otimes (\boldsymbol{L}^* \otimes \boldsymbol{D}^{1/2}_{F_2})_{m_2}.$$

It is easily verified that $K(m_1, m_2)$ does not depend on the choice of the section σ, and thus, is well defined for all $(m_1, m_2) \in M \times M$. We can interprete $K(\cdot, \cdot)$ as a section of the bundle $\boldsymbol{P}$ (see (157)). Lemma 2 implies that K is smooth, $\overgroup{K(\cdot, m_2)}$ is F_1-horizontal, and $\overgroup{K(m_1, \cdot)}$ is F_2-horizontal. We shall show that $\overgroup{K\restriction_{\Delta}} = C(\overset{n}{\omega})^{1/2}$, $C \neq 0$, proving that K is a distinguished kernel.

Let $\tilde{V}$, $(\tilde{V}^{-1})^*$, $V_1^{1/2}$, $V_2^{1/2}$, $\overgroup{V \times V}$ be bundle automorphisms as in (191), (192) and (193). We note that

$$(224)\qquad \tilde{V}^* \cdot \nu = \nu \cdot \tilde{V}^{-1}.$$

Moreover,

$$(225)\qquad (\utilde{V_2})^{-1} \cdot \delta = \delta \cdot (\utilde{V_1})^{-1}.$$

Hence,

$$(226)\qquad (\utilde{V_2^{1/2}})^{-1} \cdot \delta^{1/2} = \pm\, \delta^{1/2} \cdot (\utilde{V_1^{1/2}})^{-1}.$$

For the section $K \in \Gamma(\boldsymbol{P})$, we have

$$\begin{aligned}&\overgroup{V \times V}^{-1}\Big(K\big(V(m_1), V(m_2)\big)\Big)\\ &\qquad = \overgroup{V \times V}^{-1}\Big[\vartheta^{\sigma'}_{V(m_2)}\big(V(m_1)\big) \otimes \big[(\nu \otimes \delta^{1/2})\big(\sigma'\big(V(m_2)\big)\big)\big]\Big],\end{aligned}$$

where by σ' we denote the transformed section σ:

$$\sigma' := (\tilde{V} \otimes \utilde{V_1^{1/2}}) \circ \sigma \circ V^{-1} \in \Gamma_{V(U)}(\boldsymbol{L} \otimes \boldsymbol{D}^{1/2}_{F_1}).$$

Thus,

$$\begin{aligned}(227)\quad &\overgroup{V \times V}^{-1}\Big(K\big(V(m_1), V(m_2)\big)\Big)\\ &= \Big[(\tilde{V} \otimes \utilde{V_1^{1/2}})^{-1}\Big(\vartheta^{\sigma'}_{V(m_2)}\big(V(m_1)\big)\Big)\Big] \otimes\\ &\qquad\qquad \otimes\Big[\big(\tilde{V}^* \otimes (\utilde{V_2^{1/2}})^{-1}\big) \cdot (\nu \otimes \delta^{1/2})\big(\sigma'\big(V(m_2)\big)\big)\Big]\\ &= \pm\,[(U_{\tilde{V} \otimes \utilde{V_1^{1/2}}}\,\vartheta^{\sigma'}_{V(m_2)})(m_1)] \otimes \Big[(\nu \otimes \delta^{1/2}) \circ (\tilde{V} \otimes \utilde{V_1^{1/2}})^{-1}\big(\sigma'\big(V(m_2)\big)\big)\Big]\\ &= \pm\,[(U_{\tilde{V} \otimes \utilde{V_1^{1/2}}}\,\vartheta^{\sigma'}_{V(m_2)})(m_1)] \otimes \big[(\nu \otimes \delta^{1/2})\big(\sigma(m_2)\big)\big].\end{aligned}$$

But for $\varrho \in H_{F_1}$,

$$(228)\quad (V_{\utilde{\tilde{V} \otimes} V_1^{1/2}}\,\vartheta^{\sigma'}_{V(m_2)}, \varrho)_{F_1} = (\vartheta^{\sigma'}_{V(m_2)}, U^{-1}_{\utilde{\tilde{V} \otimes} V_1^{1/2}}\,\varrho)_{F_1} = \overgroup{U^{-1}_{\utilde{\tilde{V} \otimes} V_1^{1/2}}\,\varrho^{\sigma'}}\big(V(m_2)\big).$$

Moreover, we have

$$(229)\quad (U^{-1}_{\tilde{V}\otimes \underset{\smile}{V_1^{1/2}}}\varrho)\big(V(m_2)\big) = (\tilde{V}\otimes \underset{\smile}{V_1^{1/2}})\big(\varrho(m_2)\big) = \tilde{\varrho}^{\sigma}(m_2)(\tilde{V}\otimes \underset{\smile}{V_1^{1/2}})\big(\sigma(m_2)\big)$$
$$= \tilde{\varrho}^{\sigma}(m_2)\sigma'\big(V(m_2)\big).$$

Hence (by (228) and (229)),

$$(U_{\tilde{V}\otimes \underset{\smile}{V_1^{1/2}}}\vartheta^{\sigma'}_{V(m_2)}, \varrho)_{F_1} = \tilde{\varrho}^{\sigma}(m_2) = (\vartheta^{\sigma}_{m_2}, \varrho)_{F_1},$$

and

$$U_{\tilde{V}\otimes \underset{\smile}{V_1^{1/2}}}\vartheta^{\sigma'}_{V(m_2)} = \vartheta^{\sigma}_{m_2}.$$

Thus, (227) reads

$$(230)\quad \overline{V\times V}^{-1}\Big(K\big(V(m_1), V(m_2)\big)\Big) = \pm\vartheta^{\sigma}_{m_2}(m_1)\otimes\big[(\nu\otimes\delta^{1/2})\big(\sigma(m_2)\big)\big]$$
$$= \pm K(m_1, m_2),$$

where the sign in the right-hand side does not depend on the point (m_1, m_2).

(230) implies

$$(231)\quad (\underset{\smile}{V_1^{1/2}}\otimes \underset{\smile}{V_1^{1/2}})^{-1}\Big(\overline{K{\restriction}_{\Delta}}\big(V(m)\big)\Big) = \pm\overline{K{\restriction}_{\Delta}}(m).$$

We also have

$$(232)\quad (\underset{\smile}{V_1^{1/2}}\otimes \underset{\smile}{V_2^{1/2}})^{-1}\Big((\overset{n}{\omega})^{1/2}\big(V(m)\big)\Big) = \pm(\overset{n}{\omega})^{1/2}(m).$$

Since, according to our assumptions, $Q_{F_1F_2}$ acts transitively on M, from (231) and (232) we obtain

$$(233)\quad \overline{K{\restriction}_{\Delta}}(m) = \pm C(\overset{n}{\omega})^{1/2}(m).$$

As both sides of (233) are smooth and $(\overset{n}{\omega})^{1/2}$ is non-vanishing, the sign in (233) does not depend on the point, and can be absorbed into C. Therefore,

$$(234)\quad \overline{K{\restriction}_{\Delta}} = C(\overset{n}{\omega})^{1/2},$$

and, since K is not identically zero ($H_{F_1} \neq \{0\}$), $C \neq 0$, which implies that $\overline{K{\restriction}_{\Delta}}$ is not identically zero in virtue of the following:

LEMMA 4. *Let* $(0, 0)\in W\times U$, *where* W *and* U *are open connected subsets of* C^n. *Let*

$$W\times U \ni (v^1, \dots, v^n;\ w^1, \dots, w^n) \mapsto \Phi(v^1, \dots, v^n;\ w^1, \dots, w^n) \in C$$

be a (smooth) function, holomorphic in the v*-variables and anti-holomorphic in the* w*-variables. Suppose that* $f(z; z) \equiv 0$ *if* $(z; z)\in W\times U$ $(z\in C^n)$. *Then* $\Phi \equiv 0$.

Proof of Lemma 4. Let $z^j := v^j + \overline{w^j}$, and $z^{n+j} := i(v^j - \overline{w^j})$, $1 \leqslant j \leqslant n$. Let

$$\tilde{\Phi}(z^1, \ldots, z^{2n}) := \Phi(v^1, \ldots, v^n; w^1, \ldots, w^n).$$

$\tilde{\Phi}$ is a holomorphic function on a connected neighbourhood θ of zero in C^{2n}, and $\tilde{\Phi} = 0$ on $\{(z^1, \ldots, z^{2n})\colon \operatorname{Im} z^1 = 0, \ldots, \operatorname{Im} z^{2n} = 0\}$. Hence, by the "Edge of the Wedge" Theorem and the Analitical Continuation Theorem, $\tilde{\Phi} = 0$. ■

Thus, K is a distinguished kernel. Lemma 4 implies also that K is the unique distinguished kernel (up to a constant factor). We must show that K is reproducing.

Let $\varrho_1 \in H^0_{F_1} = H_{F_1}$. For each $m_2 \in M$, $\overline{K(\cdot, m_2)} \in H_{F_1}$ and

$$(235)\qquad \bigl(\varrho_1(\cdot), K(\cdot, m_2)\bigr)_{F_1} = (\varrho_1, \vartheta^{\sigma}_{m_2})_{F_1}(\nu \otimes \delta^{\frac{1}{2}})\bigl(\sigma(m_2)\bigr)$$
$$= \overline{\tilde{\varrho}^{\sigma}_1(m_2)}(\nu \otimes \delta^{\frac{1}{2}})\bigl(\sigma(m_2)\bigr) = (\nu \otimes \delta^{\frac{1}{2}})\bigl(\varrho_1(m_2)\bigr)$$

— see (159). Now by (203) for $\varrho_2 \in H_{F_1}$

$$(236)\qquad \frac{\varrho_2(m_2) \otimes \bigl(\varrho_1, K(\cdot, m_2)\bigr)_{F_1}}{\bigl(\overset{n}{\omega}\bigr)^{\frac{1}{2}}(m_2)} = \frac{\varrho_2(m_2) \otimes \bigl[(\nu \otimes \delta^{\frac{1}{2}})\bigl(\varrho_1(m_2)\bigr)\bigr]}{\bigl(\overset{n}{\omega}\bigr)^{\frac{1}{2}}(m_2)}$$
$$= A\,\frac{\langle\!\langle \varrho_1, \varrho_2 \rangle\!\rangle_{F_1}(m_2)}{\bigl|\overset{n}{\omega}\bigr|(m_2)}.$$

Thus (see (160)),

$$(237)\qquad \{\varrho_1, \varrho_2\}_K = A\,\frac{\langle\!\langle \varrho_1, \varrho_2 \rangle\!\rangle_{F_1}}{\bigl|\overset{n}{\omega}\bigr|},$$

and (see (161))

$$(238)\qquad (\varrho_1, \varrho_2)^K_{F_1} = A(\varrho_1, \varrho_2)_{F_1}.$$

Hence (see Definitions IV.6 and 7), $\frac{1}{A}K$ is reproducing. ■

IV.14. **Remark.** The construction of the reproducing distinguished kernel given in the proof of Proposition IV.13 shows a close connection between the reproducing kernels considered here and the Bergman–Aronszajn kernels (see [8]).

IV.15. EXAMPLE (cotangent bundle). Let (M, ω) and a quantum bundle be as in Examples III.2 and 7. We shall assume, additionally, that $\mathscr{X} = \boldsymbol{R}^n$. Let $(x^1, \ldots, x^n)$ be the canonical coordinate chart for $\boldsymbol{R}^n$, $(x^1, \ldots, x^n; p_1, \ldots, p_n)$ be the induced chart for $T^*(\mathscr{X})$, and $(x^1, \ldots, x^n;$

$p_1, \ldots, p_n; z)$ that for L. Using these coordinates we can write

$$\sigma = \sum_{j=1}^{n} p_j dx^j, \tag{239}$$

$$\omega = -\sum_{j=1}^{n} dp_j \wedge dx^j, \tag{240}$$

$$\alpha = \frac{dz}{iz} - \sum_{j=1}^{i} p_j dx^j. \tag{241}$$

Let F_1 be spanned by vector fields $\partial/\partial p_j$ and F_2 by fields $\partial/\partial x_k$, $1 \leqslant j, k \leqslant n$. Let $(\boldsymbol{D}_{F_1}^{1/2}, \iota_1)$ and $(\boldsymbol{D}_{F_2}^{1/2}, \iota_2)$ be square root structures for F_1 and F_2, respectively. Let $(d^n x)^{1/2}$ and $(d^n p)^{1/2}$ be sections of $\boldsymbol{D}_{F_1}^{1/2}$ and $\boldsymbol{D}_{F_2}^{1/2}$, respectively, such that

$$(d^n x)^{1/2} \underline{\otimes} (d^n x)^{1/2} = dx^1 \wedge \ldots \wedge dx^n =: d^n x, \tag{242}$$

and

$$(d^n p)^{1/2} \underline{\otimes} (d^n p)^{1/2} = dp_1 \wedge \ldots \wedge dp_n =: d^n p. \tag{243}$$

These sections define trivializations of $\boldsymbol{D}_{F_1}^{1/2}$ and $\boldsymbol{D}_{F_2}^{1/2}$.

Thus, $\boldsymbol{P}$ is also trivial, and to an element $K \in \Gamma(\boldsymbol{P})$ we shall assign a function $\tilde{K}$ on $M \times M$, in an evident manner. Let

$$\left(\overset{n}{\omega}\right) = (n!)^{1/2} S_n (d^n x)^{1/2} \otimes (d^n p)^{1/2}, \tag{244}$$

where $S_n = 1$ if $n = 4k$ or $n = 4k+1$, and $S_n = i$ if $n = 4k+2$ or $n = 4k+3$.

It is easily verified that

$$\tilde{K}(y, q; x, p) = \frac{S_n}{(2\pi)^n (n!)^{1/2}} \exp\left(i \sum_{m=1}^{n} p_m (y^m - x^m)\right), \tag{245}$$

where

$$x := (x^1, \ldots, x^n), \qquad p := (p_1, \ldots, p_n),$$
$$y := (y^1, \ldots, y^n), \qquad q := (q_1, \ldots, q_n),$$

corresponds to a reproducing distinguished kernel K for (F_1, F_2). The situation presented here is an example of the general one described in Propositions IV.10 and 12.

Now let us take F_1 to be spanned by $\partial/\partial w_j$, where $w_j = x^j + ip_j$, $j = 1, \ldots, n$, and $F_2 = \overline{F_1}$ (F_1 is positive). Let $(d^n \bar{w})^{1/2}$ and $(d^n w)^{1/2}$ be sections of $\boldsymbol{D}_{F_1}^{1/2}$ and $\boldsymbol{D}_{F_2}^{1/2}$, respectively, such that

$$(d^n \bar{w})^{1/2} \underline{\otimes} (d^n \bar{w})^{1/2} = d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_n =: d^n \bar{w}, \tag{246}$$

and

$$(d^n w)^{1/2} \otimes (d^n w)^{1/2} = dw_1 \wedge \ldots \wedge dw_n := d^n w. \tag{247}$$

As before, these sections define trivializations of $D_{F_1}^{1/2}$ and $D_{F_2}^{1/2}$. Assigning a function $\tilde{\sigma}$ on M to a section $\sigma \epsilon \Gamma(L \otimes D_{F_1}^{1/2})$ in an obvious way, we can obtain

$$H_{F_1} = \Big\{\sigma \epsilon (L \otimes D_{F_1}^{1/2}) \colon \tilde{\sigma}(w) = \exp\Big(\tfrac{1}{8}\sum_{j=1}^{n} (\overline{w_j} - w_j)^2\Big) \overline{f_\sigma(w)},$$
$$\text{where } f_\sigma \text{ is holomorphic on } M\Big\},$$

and

$$\|\sigma\|_{F_1}^2 = \frac{1}{(2^n n!)^{1/2}} \int_M |\tilde{\sigma}(w)|^2 |d^n\overline{w} \wedge d^n w| < +\infty. \tag{248}$$

We shall also, as before, represent $K \epsilon \Gamma(P)$ by a function $\tilde{K} \epsilon C^\infty(M \times M)$ in an obvious way. Let

$$\binom{n}{\omega}^{1/2} = \left(\frac{n!}{2^n}\right)^{1/2} R_n (d^n\overline{w})^{1/2} \otimes (d^n w)^{1/2}, \tag{249}$$

where

$$R_n := \begin{cases} 1 & \text{if } n = 2k, \\ \dfrac{1-i}{(2)^{1/2}} & \text{if } n = 2k+1. \end{cases} \tag{250}$$

Then we can check that

$$\tilde{K}(w'; w) := \frac{1}{(2^n n!)^{1/2}(2\pi)^n} R_n \exp\Big(\sum_{j=1}^{n} \big(\tfrac{1}{8}(\overline{w'_j} - w'_j)^2 + \tfrac{1}{8}(\overline{w_j} - w_j)^2 -$$
$$- \tfrac{1}{4}(\overline{w'_j} - w_j)^2\big)\Big), \tag{251}$$

where $w' := (w'_1, \ldots, w'_n)$ and $w := (w_1, \ldots, w_n)$, corresponds to a reproducing distinguished kernel K for (F_1, F_2). The situation described above is an example of the general one presented in Proposition IV.13. The kernel (251) has, in fact, become well known in the Segal–Bargmann representation theory (see [3]).

In the 3-rd case we shall take F_1 to be spanned by $\partial/\partial p_j$ and F_2 to be spanned by $\partial/\partial w_k$, $1 \leqslant j, k \leqslant n$. Let, as before, $(d^n x)^{1/2}$ and $(d^n\overline{w})^{1/2}$ be sections of $D_{F_1}^{1/2}$ and $D_{F_2}^{1/2}$, respectively, such that (242) and (246) hold. As in the previously considered cases, for the polarizations under consideration we shall assign to a section K of the bundle P, a function $\tilde{K}$ on $M \times M$, in an obvious way. Let

$$\binom{n}{\omega}^{1/2} = (n!)^{1/2} \overline{R_n} (d^n x)^{1/2} \otimes (d^n\overline{w})^{1/2}. \tag{252}$$

We can check that

$$\tilde{K}(y, q; w) := \frac{\overline{R_n}}{(2\pi)^n (n!)^{1/2}} \exp\Big(\sum_{j=1}^{n} \big(\tfrac{1}{2}(\overline{w_j} - y_j)^2 - \tfrac{1}{8}(\overline{w_j} - w_j)^2\big)\Big) \tag{253}$$

corresponds to a reproducing distinguished kernel K for (F_1, F_2).

If, conversely, we take F_1 to be spanned by $\partial/\partial w_k$ and F_2 by $\partial/\partial p_j$, $1 \leqslant j, k \leqslant n$, then, proceeding as in the previous cases, we get

$$\tilde{K}(w'; x, p) := C \exp\Big(\sum_{j=1}^{n} \big(\tfrac{1}{8}(\overline{w_j'} - w_j')^2 - \tfrac{1}{2}(w_j' - x_j)^2\big)\Big) \tag{254}$$

as the function corresponding to a distinguished kernel K for (F_1, F_2). In this case, however, K is not regular.

IV.16. EXAMPLE (complex projective plane). Let (M, ω_s) and a quantum bundle over (M, ω_s) be as in Example III.8 (for $s = m/2$). Let $F_1 := F$ and $F_2 := \overline{F}$, where F has been defined in Example III.12. We shall also make use of the notation introduced in Example III.20. Let us set $(D_{F_1}^{1/2}, \iota_1) := (D_F^{1/2}, \iota)$, where $(D_F^{1/2}, \iota)$ is as in Example III.20. $(D_{F_2}^{1/2}, \iota_2)$ will be similarly defined. Namely, we shall put

$$D_{F_2}^{1/2} := (\{1\} \times 0_1 \times C) \cup (\{2\} \times 0_2 \times C)/\sim D_{F_2}^{1/2}, \tag{255}$$

where

$$(1, [1, \mathrm{w}], z) \sim D_{F_2}^{1/2} (2, [\mathrm{v}, 1], z') \Leftrightarrow \frac{1}{\mathrm{v}} = \mathrm{w} \quad \text{and} \quad z' = i\mathrm{w}z. \tag{256}$$

The bundle projection π will be defined on representing elements as the projection onto the second factor.

On $\pi^{-1}(0_i) \subset D_{F_2}^{1/2}$ one can define diffeomorphic maps H_i: $\pi^{-1}(0_i) \to C^2$:

$$H_1([1, [1, \mathrm{w}], z]) := (\mathrm{w}, z), \qquad H_2([2, [\mathrm{v}, 1], z]) := (\mathrm{v}, z).$$

Let us put

$$\iota_2\big(H_1^{-1}(\mathrm{w}, z_1) \otimes H_1^{-1}(\mathrm{w}, z_2)\big) := z_1 z_2 dw([1, \mathrm{w}]),$$
$$\iota_2\big(H_2^{-1}(\mathrm{v}, z_1) \otimes H_2^{-1}(\mathrm{v}, z_2)\big) := z_1 z_2 dv([\mathrm{v}, 1]).$$

We easily check that ι_2 is a bundle isomorphism of $D_{F_2}^{1/2} \otimes D_{F_2}^{1/2}$ onto $\Lambda^n T^*(M)^C/F_2 = D_{F_2}$, and thus, $(D_{F_2}^{1/2}, \iota_2)$ is a square root structure for F_2. Now

$$\begin{aligned} 0_1 \ni [1, \mathrm{w}] &\xrightarrow{(dw)^{1/2}} H_1^{-1}(\mathrm{w}, 1) \in D_{F_2}^{1/2}, \\ 0_2 \ni [\mathrm{v}, 1] &\xrightarrow{(dv)^{1/2}} H_2^{-1}(\mathrm{v}, 1) \in D_{F_2}^{1/2} \end{aligned} \tag{257}$$

are non-vanishing horizontal sections of $D_{F_2}^{1/2}$, defined on 0_1 and 0_2, respectively, and

$$(dw)^{1/2} \underline{\otimes} (dw)^{1/2} = dw, \qquad (dv)^{1/2} \underline{\otimes} (dv)^{1/2} = dv.$$

We shall choose an adjustment of $(\boldsymbol{D}^{1/2}_{F_1}, \iota_1)$ and $(\boldsymbol{D}^{1/2}_{F_2}, \iota_2)$, $\left(\overset{n}{\omega}\right)^{1/2} \in \Gamma(\boldsymbol{D}^{1/2}_{F_1} \otimes \boldsymbol{D}^{1/2}_{F_2})$ so that

$$\left(\overset{n}{\omega}\right)^{1/2} \restriction_{0_1} = \frac{1-i}{(2)^{1/2}} (m+1)^{1/2} \frac{(d\overline{w})^{1/2} \otimes (dw)^{1/2}}{1+|w|^2}, \tag{258}$$

$$\left(\overset{n}{\omega}\right)^{1/2} \restriction_{0_2} = \frac{1-i}{(2)^{1/2}} (m+1)^{1/2} \frac{(d\bar{v})^{1/2} \otimes (dv)^{1/2}}{1+|v|^2}. \tag{259}$$

Let $s^m_1 \in \Gamma_{0_1}(\boldsymbol{L}_{m/2})$ be an F_1-horizontal section defined by (86), and let $(s^m_1)^* \in \Gamma_{0_1}(\boldsymbol{L}^*_{m/2})$ be defined by

$$\langle t \,|\, (s^m_1)^* \rangle \underset{t \in \Gamma_{0_1}(\boldsymbol{L})}{=\!=\!=} (s^m_1, t). \tag{260}$$

$(s^m_1)^*$ is non-vanishing and F_2-horizontal.

We note that s^m_1, $(s^m_1)^*$, $(d\overline{w})^{1/2}$ and $(dw)^{1/2}$ define trivializations of $\boldsymbol{L}_{m/2}\restriction_{0_1}$, $\boldsymbol{L}^*_{m/2}\restriction_{0_1}$, $\boldsymbol{D}^{1/2}_{F_1}\restriction_{0_1}$ and $\boldsymbol{D}^{1/2}_{F_2}\restriction_{0_1}$, respectively, and thus, also a trivialization of $\boldsymbol{P}\restriction_{0_1\times 0_1}$. Hence, if $K \in \Gamma(\boldsymbol{P})$, then we can represent $K\restriction_{0_1\times 0_1}$ by a function $\tilde{K}$ on $0_1 \times 0_1$, and $\tilde{K}$ uniquely determines K. We can check that

$$\tilde{K}([1, \mathrm{w}']; [1, \mathrm{w}]) := \frac{1-i}{(2)^{1/2}} \frac{(m+1)^{1/2}}{2\pi} (1+\overline{\mathrm{w}'}\mathrm{w})^m \tag{261}$$

defines a reproducing distinguished kernel K for (F_1, F_2). The situation presented above provides an example of the general case described in Proposition IV.13.

C. Kernel representation of quantized operators

We shall start with the following:

IV.17. Lemma. *Let $\varrho \in \Gamma(\boldsymbol{L} \otimes \boldsymbol{D}^{1/2}_{F_1})$ be F_1-horizontal and $\sigma \in \Gamma(\boldsymbol{L}^* \otimes \boldsymbol{D}^{1/2}_{F_2})$ be F_2-horizontal. Suppose that X and Y are vector fields on M, preserving F_1 and F_2, respectively.*

Let $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, where X_1, Y_1 take values in F_1, and X_2, Y_2 in F_2. Then

$$\frac{[(\nabla_X \otimes \mathscr{L}^{1/2}_X)\varrho] \otimes \sigma}{\left(\overset{n}{\omega}\right)^{1/2}} = -\frac{1}{2} \frac{\mathscr{L}_{X_2}\overset{n}{\omega}}{\overset{n}{\omega}} \frac{\varrho \otimes \sigma}{\left(\overset{n}{\omega}\right)^{1/2}} + \frac{\mathscr{L}_{X_2}\left(\dfrac{\varrho \otimes \sigma}{\left(\overset{n}{\omega}\right)^{1/2}} \overset{n}{\omega}\right)}{\overset{n}{\omega}}, \tag{262}$$

$$\frac{\varrho \otimes [(\nabla^*_Y \otimes \mathscr{L}^{1/2}_Y)\sigma]}{\left(\overset{n}{\omega}\right)^{1/2}} = -\frac{1}{2} \frac{\mathscr{L}_{Y_1}\overset{n}{\omega}}{\overset{n}{\omega}} \frac{\varrho \otimes \sigma}{\left(\overset{n}{\omega}\right)^{1/2}} + \frac{\mathscr{L}_{Y_1}\left(\dfrac{\varrho \otimes \sigma}{\left(\overset{n}{\omega}\right)^{1/2}} \overset{n}{\omega}\right)}{\overset{n}{\omega}}, \tag{263}$$

where, as usual, we identify the sections of $(\boldsymbol{L}\otimes \boldsymbol{D}^{1/2}_{F_1})\otimes(\boldsymbol{L}^*\otimes \boldsymbol{D}^{1/2}_{F_2})$ *and of* $\boldsymbol{D}^{1/2}_{F_1}\otimes \boldsymbol{D}^{1/2}_{F_2}$.

Proof. We shall prove (262) only, as the proof of (263) proceeds in full analogy.

First, note that X_2 preserves both F_1 and F_2. In the following, we shall denote by $\mathscr{L}^{1/2}_{X_2}$ some different operations:

1. operation on $\Gamma_{\mathrm{loc}}(\boldsymbol{D}^{1/2}_{F_1})$;
2. operation on $\Gamma_{\mathrm{loc}}(\boldsymbol{D}^{1/2}_{F_2})$;
3. operation on $\Gamma_{\mathrm{loc}}(\boldsymbol{D}^{1/2}_{F_1}\otimes \boldsymbol{D}^{1/2}_{F_2})$ — a tensor product of the preceding ones (see Proposition II.4).

Since $\boldsymbol{D}^{1/2}_{F_1}\otimes \boldsymbol{D}^{1/2}_{F_2}$ together with a bundle isomorphism χ, given by (154), is a square root of $\Lambda^{2n}\boldsymbol{T}^*(\boldsymbol{M})^{\boldsymbol{C}}$, we can also define an operation $\mathscr{L}^{1/2}_{X_2}$ on $\Gamma_{\mathrm{loc}}(\boldsymbol{D}^{1/2}_{F_1}\otimes \boldsymbol{D}^{1/2}_{F_2})$ by using the procedure described in Proposition II.5. It is easily verified that this operation coincides with 3 above. This remark becomes useful when proving (262).

It sufficies to prove (262) locally. Thus, we can put

$$\varrho = s\otimes\varphi, \qquad \sigma = t^*\otimes\psi,$$

where $s\in\Gamma(\boldsymbol{L})$, $\varphi\in\Gamma(\boldsymbol{D}^{1/2}_{F_1})$, $t^*\in\Gamma(\boldsymbol{L}^*)$, $\psi\in\Gamma(\boldsymbol{D}^{1/2}_{F_2})$. We have

$$\begin{aligned}
(264)\qquad [(\nabla_X\otimes\mathscr{L}^{1/2}_X)\varrho]\otimes\sigma &= [(\nabla_X s)\otimes\varphi + s\otimes(\mathscr{L}^{1/2}_X\varphi)]\otimes(t^*\otimes\psi)\\
&= \langle\nabla_X s\,|\,t^*\rangle\varphi\otimes\psi + \langle s\,|\,t^*\rangle(\mathscr{L}^{1/2}_X\varphi)\otimes\psi\\
&= \langle\nabla_{X_1} s\,|\,t^*\rangle\varphi\otimes\psi + \langle s\,|\,t^*\rangle(\mathscr{L}^{1/2}_{X_1}\varphi)\otimes\psi +\\
&\quad + \langle\nabla_{X_2} s\,|\,t^*\rangle\varphi\otimes\psi + \langle s\,|\,t^*\rangle(\mathscr{L}^{1/2}_{X_2}\varphi)\otimes\psi\\
&= [(\nabla_{X_1}\otimes\mathscr{L}^{1/2}_{X_1})\varrho]\otimes\sigma + X_2(\langle s\,|\,t^*\rangle)\varphi\otimes\psi -\\
&\quad - \langle s\,|\,\nabla^*_{X_2}t^*\rangle\varphi\otimes\psi + \langle s\,|\,t^*\rangle\mathscr{L}^{1/2}_{X_2}(\varphi\otimes\psi) -\\
&\quad - \langle s\,|\,t^*\rangle\varphi\otimes(\mathscr{L}^{1/2}_{X_2}\psi) = \mathscr{L}^{1/2}_{X_2}(\langle s\,|\,t^*\rangle\varphi\otimes\psi) -\\
&\quad - \varrho\otimes[(\nabla^*_{X_2}\otimes\mathscr{L}^{1/2}_{X_2})\sigma] = \mathscr{L}^{1/2}_{X_2}(\varrho\otimes\sigma).
\end{aligned}$$

From (4) and (155), however, we get

$$\begin{aligned}
(265)\qquad \frac{\mathscr{L}^{1/2}_{X_2}(\varrho\otimes\sigma)}{(\overset{n}{\omega})^{1/2}} &= X_2\left(\frac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}\right) + \frac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}\,\frac{\mathscr{L}^{1/2}_{X_2}(\overset{n}{\omega})^{1/2}}{(\overset{n}{\omega})^{1/2}}\\
&= X_2\left(\frac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}\right) + \frac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}\,\frac{\mathscr{L}_{X_2}\overset{n}{\omega}}{\overset{n}{\omega}} - \frac{1}{2}\,\frac{\mathscr{L}_{X_2}\overset{n}{\omega}}{\overset{n}{\omega}}\,\frac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}\\
&= \frac{\mathscr{L}_{X_2}\left(\dfrac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}\overset{n}{\omega}\right)}{\overset{n}{\omega}} - \frac{1}{2}\,\frac{\mathscr{L}_{X_2}\overset{n}{\omega}}{\overset{n}{\omega}}\,\frac{\varrho\otimes\sigma}{(\overset{n}{\omega})^{1/2}}.
\end{aligned}$$

Comparing (264) and (265), we get (262). ∎

Let $f\in C^{\infty}(M)$ and X_f be the canonical vector field defined by f. Let $X_f=(X_f)_1+(X_f)_2$, where $(X_f)_i$ takes values in F_i, $i=1,2$.

Now we prove the following theorem giving a kernel representation for operators Q_f.

IV.18. THEOREM. *Suppose that K is a reproducing distinguished kernel for (F_1, F_2). Let $f\in C^{\infty}_{F_1F_1}(M)$ be real. Let $\varrho_1\in H^0_{F_1}$ and $\varrho_2\in D_{Q_f}\cap H^0_{F_1}$. Then*

$$(266)\qquad (\varrho_1, Q_f\varrho_2)_{F_1}=\int\limits_M\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\overset{n}{\omega}$$

if $\left(f+\dfrac{i}{2}\,\dfrac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\in\mathscr{L}^1(\overset{n}{\omega})$, *and*

$$(267)\qquad \int\limits_M\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})=0.$$

Proof. (160) yields

$$(268)\qquad \{\varrho_1, Q_f\varrho_2\}_K(m_2)=\frac{[Q_f\varrho_2(m_2)]\otimes\big(\varrho_1(\cdot), K(\cdot, m_2)\big)_{F_1}}{(\overset{n}{\omega})^{1/2}(m_2)}$$
$$=\frac{1}{i}\,\frac{[(\nabla_{X_f}\otimes\mathscr{L}^{1/2}_{X_f})\varrho_2](m_2)\otimes\big(\varrho_1(\cdot), K(\cdot, m_2)\big)_{F_1}}{(\overset{n}{\omega})^{1/2}(m_2)}+$$
$$+f(m_2)\{\varrho_1,\varrho_2\}_K(m_2).$$

From (268) and (262),

$$(269)\qquad \{\varrho_1, Q_f\varrho_2\}_K=\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K-i\,\frac{\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})}{\overset{n}{\omega}}.$$

Thus, by (161) and (162),

$$(270)\qquad (\varrho_1, Q_f\varrho_2)_{F_1}=\int\limits_M\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\overset{n}{\omega}-$$
$$-i\int\limits_M\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})$$
$$=\int\limits_M\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K$$

if

$$\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\in\mathscr{L}^1(\overset{n}{\omega}),$$

and

$$\int\limits_M\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})=0.\quad\blacksquare$$

IV.19. Definition. We shall say that f is *F_1F_2-proper* if $(X_f)_2$ has a support whose intersection with any maximal connected integral submanifold of E_2 is compact.

IV.20. Corollary. *Let K be a reproducing distinguished kernel for (F_1,F_2). Let $f\in C^\infty_{F_1F_1}(M)$ be real and F_1F_2-proper. Let $\varrho_1\in H^0_{F_1}$ and $\varrho_2\in D_{Q_f}\cap H^0_{F_1}$. Then* (266) *holds if*

$$\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\in\mathscr{L}^1(\overset{n}{\omega}).$$

Proof. We must only show that $J:=\int\limits_M\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})=0$. Let $g_k\in C_0^\infty(M/E_2)$, $0\leqslant g_k\leqslant 1$, and $g_k(r)\nearrow 1$ for each $r\in M/E_2$, and let $h_k:=g_k\circ\varrho_{E_2}$. By (269) and 3 of Definition IV.6,

$$\frac{\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})}{\overset{n}{\omega}}\in\mathscr{L}^1(\overset{n}{\omega})$$

when our assumptions hold. Hence,

$$\begin{aligned}J&=\lim_{k\to\infty}\int\limits_M h_k\mathscr{L}_{(X_f)_2}(\{\varrho_1,\varrho_2\}_K\overset{n}{\omega})=\lim_{k\to\infty}\int\limits_M\mathscr{L}_{(X_f)_2}[h_k\{\varrho_1,\varrho_2\}_K\overset{n}{\omega}]\\&=\lim_{k\to\infty}\int\limits_M d[h_k\{\varrho_1,\varrho_2\}_K((X_f)_2\,\lrcorner\,\overset{n}{\omega})]=0\end{aligned}$$

as $h_k\{\varrho_1,\varrho_2\}_K((X_f)_2\,\lrcorner\,\overset{n}{\omega})$ has a compact support.

IV.21. Proposition. *Let F_1, F_2 be real and suppose that $\varrho_{D_1D_2}$ is a diffeomorphism of M onto $M/D_1\times M/D_2$. Let K be a reproducing distinguished kernel for (F_1,F_2). Then for real $f\in C^\infty_{F_1F_1}(M)$, $\varrho_1\in H^0_{F_1}$ and $\varrho_2\in H^0_{F_1}\cap D_{Q_f}$* (266) *holds if*

$$\left(f+\frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\in\mathscr{L}^1(\overset{n}{\omega}).\tag{266}$$

Proof. Similarly, as in the proof of Corollary IV.20, let $g_k \in C_0^\infty(M/D_2)$, $0 \leqslant g_k \leqslant 1$, $g_k(r) \nearrow 1$ for each $r \in M/D_2$, and let $h_k := g_k \circ \varrho_{D_2}$. Then

$$\int_M \mathscr{L}_{(X_f)_2}(\{\varrho_1, \varrho_2\}_K \overset{n}{\omega}) = \lim_{k\to\infty} \int_M \mathscr{L}_{(X_f)_2}(h_k\{\varrho_1, \varrho_2\}_K \overset{n}{\omega}) = 0,$$

as, firstly,

$$\frac{\mathscr{L}_{(X_f)_2}(\{\varrho_1, \varrho_2\}_K \overset{n}{\omega})}{\overset{n}{\omega}} \in \mathscr{L}^1(\overset{n}{\omega})$$

and, secondly, $h_k\{\varrho_1, \varrho_2\}_K$ is of compact support. ■

IV.22. PROPOSITION. *Let F_1, F_2 be real, and suppose that $\varrho_{D_1D_2}$ is a diffeomorphism of M onto $M/D_1 \times M/D_2$. Let K be a reproducing distinguished kernel for (F_1, F_2). Let $V_{F_1F_2}\colon H_{F_1} \to H_{F_2}$ be an isometric operator relating F_1 and F_2* (see Definition IV.9 and Proposition IV.10). *Then for real $f \in C^\infty_{F_2F_2}(M)$, $\varrho_1 \in H^0_{F_1}$ and $\varrho_2 \in H^0_{F_1}$, such that $V_{F_1F_2}\varrho_2 \in D_{Q_f}$,*

$$(271)\qquad \big(V_{F_1F_2}\varrho_1, Q_f(V_{F_1F_2}\varrho_2)\big)_{F_2} = \int_M \left(f + \frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1, \varrho_2\}_K \overset{n}{\omega}$$

if

$$\left(f + \frac{i}{2}\,\frac{\mathscr{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1, \varrho_2\}_K \in \mathscr{L}^1(\overset{n}{\omega}).$$

(Q_f denotes here an operator in H_{F_2} obtained by Konstant quantization of f).

Proof. Throughout this proof we shall use the notation of Proposition IV.10 and its proof. We have (with $h_k = g_k \circ \varrho_{D_2}$ as in the proof of Proposition IV.21)

$$(272)\qquad \big(h_k V_{F_1F_2}\varrho_1, Q_f(V_{F_1F_2}\varrho_2)\big)_{F_2} = \big(Q_f(h_k V_{F_1F_2}\varrho_1), V_{F_1F_2}\varrho_2\big)_{F_2}$$

in virtue of the following:

LEMMA. *Let $\sigma_1, \sigma_2 \in D_{Q_f} \subset H_{F_2}$. Let $f \in C^\infty_{F_2F_2}(M)$ be real. Then*

$$(273)\qquad \langle\!\langle Q_f\sigma_1, \sigma_2\rangle\!\rangle = \langle\!\langle \sigma_1, Q_f\sigma_2\rangle\!\rangle + i\,|\mathscr{L}_{\underset{\sim}{X}_f}|\langle\!\langle \sigma_1, \sigma_2\rangle\!\rangle,$$

where $\underset{\sim}{X}_f$ is a vector field on M/D_2 obtained by projection of X_f by means of $(\varrho_{D_2})_*$ $(f \in C^\infty_{F_2F_2}(M))$.

Indeed, by the lemma, (272) holds if

$$\int_{M/D_2} |\mathscr{L}_{\underset{\sim}{X}_f}|(\langle\!\langle h_k V_{F_1F_2}\varrho_1, V_{F_1F_2}\varrho_2\rangle\!\rangle) = \int_{M/D_2} |\mathscr{L}_{\underset{\sim}{X}_f}|(g_k\langle\!\langle V_{F_1F_2}\varrho_1, V_{F_1F_2}\varrho_2\rangle\!\rangle) = 0,$$

which does as g_k is of compact support. Now, by (187), from (272) we get

$$(274)\qquad \big((h_k V_{F_1F_2}\varrho_1),\, Q_f(V_{F_1F_2}\varrho_2)\big)_{F_2}$$

$$= \frac{1}{A^{1/2}} \int\limits_M \frac{\varrho_2 \otimes [(\nu \otimes \delta_2^{1/2}) \circ Q_f(h_k V_{F_1F_2}\varrho_1)]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega}$$

$$= \frac{1}{A^{1/2}} \int\limits_M \frac{\varrho_2 \otimes \left[(\nu \otimes \delta_2^{1/2}) \left(\frac{1}{i} (\nabla_{X_f} \otimes \mathscr{L}^{1/2}_{X_f})(h_k V_{F_1F_2}\varrho_1)\right)\right]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega} +$$

$$+ \frac{1}{A^{1/2}} \int\limits_M h_k f \frac{\varrho_2 \otimes [(\nu \otimes \delta_2^{1/2})(V_{F_1F_2}\varrho_1)]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega}.$$

Thus, by (263),

$$(275)\qquad \big((h_k V_{F_1F_2}\varrho_1),\, Q_f(V_{F_1F_2}\varrho_2)\big)_{F_2}$$

$$= \frac{i}{A^{1/2}} \int\limits_M \frac{\varrho_2 \otimes \left[(\nabla^*_{X_f} \otimes \mathscr{L}^{1/2}_{X_f})\left((\nu \otimes \delta_2^{1/2})(h_k V_{F_1F_2}\varrho_1)\right)\right]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega} +$$

$$+ \frac{1}{A^{1/2}} \int\limits_M h_k f \frac{\varrho_2 \otimes [(\nu \otimes \delta_2^{1/2}) \circ (V_{F_1F_2}\varrho_1)]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega}$$

$$= \frac{1}{A^{1/2}} \int\limits_M h_k \left(f - \frac{i}{2} \frac{\mathscr{L}(X_f)_1 \overset{n}{\omega}}{\overset{n}{\omega}}\right) \frac{\varrho_2 \otimes [(\nu \otimes \delta_2^{1/2}) \circ (V_{F_1F_2}\varrho_1)]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega} +$$

$$+ \frac{i}{A^{1/2}} \int\limits_M \mathscr{L}_{(X_f)_1} \left(h_k \frac{\varrho_2 \otimes [(\nu \otimes \delta_2^{1/2})(V_{F_1F_2}\varrho_1)]}{(\overset{n}{\omega})^{1/2}} \overset{n}{\omega}\right).$$

From (275), (186) and (190) we get

$$(276)\qquad \big((h_k V_{F_1F_2}\varrho_1),\, Q_f(V_{F_1F_2}\varrho_2)\big)_{F_2}$$

$$= \int\limits_M h_k(m_2) \left(f - \frac{i}{2} \frac{\mathscr{L}_{(X_f)_1} \overset{n}{\omega}}{\overset{n}{\omega}}\right)(m_2) \frac{\varrho_2(m_2) \otimes \big(\varrho_1(\cdot),\, K(\cdot, m_2)\big)_{F_1}}{(\overset{n}{\omega})^{1/2}(m_2)} \cdot \overset{n}{\omega}(m_2)$$

$$= \int\limits_M h_k \left(f - \frac{i}{2} \frac{\mathscr{L}_{(X_f)_1} \overset{n}{\omega}}{\overset{n}{\omega}}\right) \{\varrho_1, \varrho_2\}_K \overset{n}{\omega}.$$

Passing with k to infinity on both sides of (276) and noting that $\mathscr{L}_{(X_f)_1}\overset{n}{\omega} = -\mathscr{L}_{(X_f)_2}\overset{n}{\omega}$, as $\mathscr{L}_{X_f}\overset{n}{\omega} = 0$, we obtain the required result.

Proof of the lemma. A little more generally, we shall show that if $f\in C^{\infty}_{F_1F_2}(M)$ is real, and if $\sigma_1,\sigma_2\in\Gamma(L\otimes D^{1/2}_{F_2})$ and are F_2-horizontal, then

$$|\mathscr{L}_{X_f}|(\langle\!\langle\sigma_1,\sigma_2\rangle\!\rangle) \sim = \langle\!\langle\big((\nabla_{X_f}+if)\otimes\mathscr{L}^{1/2}_{X_f}\big)(\sigma_1),\sigma_2\rangle\!\rangle+\langle\!\langle\sigma_1,\big((\nabla_{F_f}+if)\otimes\mathscr{L}^{1/2}_{X_f}\big)(\sigma_2)\rangle\!\rangle.$$

Let $\chi_1,\dots,\chi_n$ be local real 1-forms on M/D_2, linearly independent at each point. Let $\eta_j:=\varrho^*_{D_2}\chi_j$, $j=1,\dots,n$. Let $(\eta_1\wedge\dots\wedge\eta_n)^{1/2}$ be a local section of $D^{1/2}_{F_2}$ such that

$$(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\underline{\otimes}(\eta_1\wedge\dots\wedge\eta_n)^{1/2}=\eta_1\wedge\dots\wedge\eta_n.$$

Locally, we have

$$\sigma_1=s\otimes(\eta_1\wedge\dots\wedge\eta_n)^{1/2},\qquad \sigma_2=t\otimes(\eta_1\wedge\dots\wedge\eta_n)^{1/2},$$

with $s,t\in\Gamma_{\text{loc}}(L)$. Thus, locally (see (53) and (56)),

$$\begin{aligned}(277)\quad &|\varrho^*_{D_2}|\langle\!\langle\big((\nabla_{X_f}+if)\otimes\mathscr{L}^{1/2}_{X_f}\big)(\sigma_1),\sigma_2\rangle\!\rangle+|\varrho^*_{D_2}|\langle\!\langle\sigma_1,\big((\nabla_{X_f}+\\ &+if)\otimes\mathscr{L}^{1/2}_{X_f}\big)(\sigma_2)\rangle\!\rangle=\langle\!\langle\big((\nabla_{X_f}+if)s\big)\otimes(\eta_1\wedge\dots\wedge\eta_n)^{1/2}+\\ &+s\otimes\big(\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\big),t\otimes(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle\!\rangle_{F_2}+\\ &+\langle\!\langle s\otimes(\eta_1\wedge\dots\wedge\eta_n)^{1/2},\big((\nabla_{X_f}+if)t\big)\otimes(\eta_1\wedge\dots\wedge\eta_n)^{1/2}+\\ &+t\otimes\big(\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\big)\rangle\!\rangle_{F_2}\\ &=(\nabla_{X_f}s,t)\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &+(s,t)\langle\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &+(s,\nabla_{X_f}t)\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &+(s,t)\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}\\ &=X_f\big((s,t)\big)\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &+(s,t)[\langle\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &+\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}].\end{aligned}$$

But in the same neighbourhood

$$\begin{aligned}(278)\quad &|\varrho^*_{D_2}|\,|\underline{\mathscr{L}}_{X_f}|\langle\!\langle\sigma_1,\sigma_2\rangle\!\rangle=|\mathscr{L}_{X_f}|\langle\!\langle\sigma_1,\sigma_2\rangle\!\rangle\\ &=|\mathscr{L}_{X_f}|[(s,t)\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}]\\ &=X_f\big((s,t)\big)\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &\quad+(s,t)\,|\mathscr{L}_{X_f}|\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}.\end{aligned}$$

Thus, it suffices to show that

$$\begin{aligned}(279)\quad &\langle\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}+\\ &\quad+\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},\mathscr{L}^{1/2}_{X_f}(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F}\\ &=|\mathscr{L}_{X_f}|\langle(\eta_1\wedge\dots\wedge\eta_n)^{1/2},(\eta_1\wedge\dots\wedge\eta_n)^{1/2}\rangle_{F_2}.\end{aligned}$$

But by (4),

$$\mathcal{L}^{1/2}_{X_f}(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}=\frac{1}{2}\,\frac{\mathcal{L}_{X_f}(\eta_1\wedge\ldots\wedge\eta_n)}{\eta_1\wedge\ldots\wedge\eta_n}\,(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}.$$

Thus, the left-hand side of (279) is equal to (see (44))

$$\begin{aligned}\frac{\mathcal{L}_{X_f}(\eta_1\wedge\ldots\wedge\eta_n)}{\eta_1\wedge\ldots\wedge\eta_n}&\langle(\eta_1\wedge\ldots\wedge\eta_n)^{1/2},(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}\rangle_{F_2}\\&=\frac{\mathcal{L}_{X_f}(\eta_1\wedge\ldots\wedge\eta_n)}{\eta_1\wedge\ldots\wedge\eta_n}\,|\eta_1\wedge\ldots\wedge\eta_n|.\end{aligned}$$

This is the same expression as we have on the right-hand side of (279), as by (44) and (17),

$$\begin{aligned}|\mathcal{L}_{X_f}|\langle(\eta_1\wedge\ldots\wedge\eta_n)^{1/2},(\eta_1\wedge\ldots\wedge\eta_n)^{1/2}\rangle_{F_2}&=|\mathcal{L}_{X_f}|\,|\eta_1\wedge\ldots\wedge\eta_n|\\&=\frac{\mathcal{L}_{X_f}(\eta_1\wedge\ldots\wedge\eta_n)}{\eta_1\wedge\ldots\wedge\eta_n}\,|\eta_1\wedge\ldots\wedge\eta_n|.\end{aligned}$$

Thus, the lemma, and, consequently, Proposition IV.22, are proved. ■

Theorem IV.18, Corollary IV.20 and Propositions IV.21 and 22 suggest that we can treat (266) as a generalized prescription for quantization of real functions $f\in C^\infty_{F_1F_1}(M)+C^\infty_{F_2F_2}(M)=:C^\infty_{F_1F_2}(M)$. Namely, to real $f\in C^\infty_{F_1F_2}(M)$, we shall assign a $1\frac{1}{2}$-linear form $Q^K_f(\cdot,\cdot)$ defined by

$$Q^K_f(\varrho_1,\varrho_2):=\int\limits_M\left(f+\frac{i}{2}\,\frac{\mathcal{L}_{(X_f)_2}\overset{n}{\omega}}{\overset{n}{\omega}}\right)\{\varrho_1,\varrho_2\}_K\overset{n}{\omega}\tag{280}$$

for $\varrho_1,\varrho_2\in H^0_{F_1}$, such that the $2n$-form under the integral sign on the right-hand side of (280) defines a measure with finite absolute variation. This formal quantization prescription makes sense only if we are given a reproducing distinguished kernel K for (F_1,F_2). Propositions IV.21 and 22 show that under the special assumptions made there when we deal with two real polarizations isometrically related by an isometry $V_{F_1F_2}$, our prescription can be, roughly speaking, formulated as follows:

if $f=f_1+f_2$, *where* $f_1\in C^\infty_{F_1F_1}(M)$ *and* $f_2\in C^\infty_{F_2F_2}(M)$, *we take* $Q_{f_1}+$ $+V^*_{F_1F_2}Q_{f_2}V_{F_1F_2}$ *as the quantized operator.*

The problem arises whether the $1\frac{1}{2}$-linear form Q^K_f, given by (280), defines an operator, and if it does — whether the operator is symmetric or self adjoint. These are very deep problems both in their geometry and, as examples with trivial geometry show (see Example IV.23 below), also analytically. They will not be considered here (see [5]).

Using reproducing distinguished kernels, one can define other formal prescriptions for quantization of real $f \in C^\infty(M)$. For example, take the one given by

$$\overset{a-n}{Q}_f(\varrho_1, \varrho_2) := \int_M f\{\varrho_1, \varrho_2\}_K \overset{n}{\omega}, \tag{281}$$

which we shall call an anti-normal ordering prescription. (266) and (280) show its connections to Kostant's and our extended quantization procedures.

Now we shall show what (280) and (281) look like for the most simple and instructive cases.

IV.23. Example (cotangent bundle). Let (M, ω) and a quantum bundle be as in Example IV.15. Let F_1 be spanned by vector fields $\partial/\partial p_j$ and F_2 by fields $\partial/\partial x^k$, $1 \leqslant j, k \leqslant n$. Let $(d^n x)^{1/2}$ and $(d^n p)^{1/2}$ also be as in Example IV.15. Since $L = M \times C$ and $(d^n x)^{1/2}$ determines a trivialization of $D_{F_1}^{1/2}$, we shall represent $\varrho \in \Gamma(L \otimes D_{F_1}^{1/2})$ by a function $\tilde{\varrho}$ on M, in an obvious way. We note that ϱ is F_1-horizontal if and only if $\tilde{\varrho}$ is of the form $\tilde{\varrho} = \tilde{\tilde{\varrho}} \circ \varrho_{D_1}$, where $\tilde{\tilde{\varrho}} \in C^\infty(M/D_1)$. We also have

$$\varrho \in H^0_{F_1} \Leftrightarrow \tilde{\tilde{\varrho}} \in C_0^\infty(M/D_1).$$

Now

$$\{\varrho_1, \varrho_2\}_K(x, p) = \frac{1}{(2\pi)^n n!} \int_{\mathfrak{X}} \overline{\tilde{\tilde{\varrho}}_1(y)} \exp\left(i \sum_{j=1}^{n} p_j(y^j - x^j)\right) \cdot \tilde{\tilde{\varrho}}_2(x)\, |d^n y|.$$

Thus, if $f \in C^\infty_{F_1 F_2}(M)$, then

$$Q_f^K(\varrho_1, \varrho_2) = \frac{1}{(2\pi)^n} \int_M \left(\int_{\mathfrak{X}} \tilde{\tilde{\varrho}}_1(y) \exp\left(i \sum_{j=1}^{n} p_j(y^j - x^j)\right) \times \right.$$
$$\left. \times \left(f(x, p) - \frac{i}{2} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x^k \partial p_k} \right) \tilde{\tilde{\varrho}}_2(x)\, |d^n y| \right) |d^n x|\, |d^n p|. \tag{282}$$

If, for example, $n = 1$ and f is a polynomial, i.e. if $f(x, p) = Ax^a + Bx^b p + Cxp^c + Dp^d$, then for $\varrho_1, \varrho_2 \in H^0_{F_1}$,

$$Q_f^K(\varrho_1, \varrho_2) = \left(\varrho_1, (A\hat{x}^a + \tfrac{1}{2}B\hat{x}^b\hat{p} + \tfrac{1}{2}B\hat{p}\hat{x}^b + \tfrac{1}{2}C\hat{x}\hat{p}^c + \tfrac{1}{2}C\hat{p}^c\hat{x} + D\hat{p}^d)\varrho_2\right)_{F_1},$$

where $\widetilde{\widetilde{\hat{x}\varrho}}(x) = x\tilde{\tilde{\varrho}}(x)$, and $\widetilde{\widetilde{\hat{p}\varrho}}(x) = \frac{1}{i}\frac{d}{dx}\tilde{\tilde{\varrho}}(x)$ for $\varrho \in H^0_{F_1}$. Thus we have obtained in particular a standard quantization prescription for functions being sums of a part dependent on x and a part dependent on p only.

For anti-normal ordering procedure we get

$$\overset{a-n}{Q_F}(\varrho_1, \varrho_2) = \frac{1}{(2\pi)^n} \int\limits_M \Big(\int\limits_{\mathfrak{X}} \overline{\tilde{\tilde{\varrho}}_1(y)} \exp\Big(i \sum_{j=1}^{n} p_j(y^j - x^j)\Big) \times$$

$$\times f(x, p) \tilde{\tilde{\varrho}}_2(x) \, |d^n y| \Big) |d^n x| \, |d^n p|,$$

and in the case $n = 1$, for $f(x, p) = \sum\limits_{n,m} A_{nm} x^n p^m$ and $\varrho_1, \varrho_2 \in H^0_{F_1}$ we get

$$\overset{a-n}{Q_f}(\varrho_1, \varrho_2) = \Big(\varrho_1, \sum_{n,m} A_{nm} \hat{p}^m \hat{x}^n \varrho_2\Big)_{F_1}.$$

Let us now take F_1 to be spanned by $\partial/\partial w_j$ $(w_j = x^j + ip_j)$, and $F_2 = \overline{F_1}$. If $f \in C^\infty(M)$ and $\sigma_1, \sigma_2 \in H^0_{F_1} = H_{F_1}$, then (in the notation of (248))

$$\overset{a-n}{Q_f}(\sigma_1, \sigma_2) = \frac{1}{(2^n n!)(4\pi)^n} \int\limits_M \Big(\int\limits_M \exp\Big(\tfrac{1}{4} \sum_{j=1}^{n} \big((\overline{w'_j} - w'_j)^2 +$$

$$+ (\overline{w_j} - w_j)^2 - (\overline{w'_j} - w_j)^2\big) f^1_\sigma(w') f(w) \overline{f^2_\sigma(w)} \, |d^n \overline{w'} \wedge d^n w'| \Big) |d^n \overline{w} \wedge d^n w|.$$

If $n = 1$ and $f = \sum\limits_{n,m} A_{nm} w^n \overline{w}^m$, then

$$\overset{a-n}{Q_f}(\sigma_1, \sigma_2) = \sum_{n,m} (a^{+^n} \sigma_1, A_{nm} a^{+^m} \sigma_2)_{F_1},$$

where $\overline{a^+ \sigma}(w) = \overline{w} \tilde{\sigma}(w)$ (in the notation of (248)).

This shows why we have called $\overset{a-n}{Q_f}$ an *anti-normal ordering quantization*.

IV.24. EXAMPLE (complex projective plane). Let (M, ω_s), a quantum bundle, a pair of polarizations (F_1, F_2) with adjusted square root structures, and a reproducing distinguished kernel be as in Example IV.16 (for $s = m/2$). We shall also use the notation introduced in Examples III.8, 12 and 20.

Let $\varrho_i \in H^0_{F_1} = H_{F_1}$, $\varrho_i \restriction_{\mathfrak{O}_1} = \overline{\psi_1 \circ \bar{w}} \, s_1^m \otimes (d\overline{w})^{1/2}$, where $\psi_i(w) = \sum\limits_{j=0}^{m} \overline{a_i^j} w^j$, $i = 1, 2$. Then if f is a (real) linear combination of x, y, z (see Example

III.27), (280) reads

$$(283) \qquad Q_f^K(\varrho_1, \varrho_2) = \frac{(m+1)^{3/2}}{2\pi} \int\limits_{M \times M} \psi_1(w') \left[\frac{(1+\overline{w'}w)^m}{(1+|w'|^2)^{m+2}(1+|w|^2)^{m+2}} \right] \times$$

$$\times \left(\frac{m+2}{m+1} f(w) \right) \overline{\psi_2(w)} \, |dw' \wedge d\overline{w'}| \, |dw \wedge d\overline{w}| .$$

(283) yields a kernel representation of operators of the Wigner representation of su(2) for spin $m/2$ (compare [15]).

In the considered case, the anti-normal ordering quantizations of x, y, z differ only by the factor $\frac{m+2}{m+1}$ from those given by (280) and (283).

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DISSERTATIONES MATHEMATICAE
CXXVIII

ERRATA

Page, line	For:	Read:
$31^{5,7}$	X_f	$\underset{\sim}{X_f}$
31_6	$D_F^{1/1}$	$D_F^{1/2}$
33_{13}	$\Phi_t\|_H^2$	$\Phi_t\|_F^2$
48_3	$\delta^{1/2}$	$\delta_2^{1/2}$