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NOTES ON SOME RECENT METHODS IN BIFURCATION THEORY

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1. Introduction

In many areas of applied mathematics the problem of finding the equilibrium states of a dynamical system reduces to solving an equation of the form

$$G(x) = 0 \tag{1.1}$$

where G is a mapping $\boldsymbol{R}^n \to \boldsymbol{R}^p$ (so (1.1) consists of p equations in n variables) or, more generally, G: $X \to Y$ where X, Y are Banach spaces. Often $p = n$, as for example when $G = \operatorname{grad} f$ for some function f: $\boldsymbol{R}^n \to \boldsymbol{R}$.

If one of the variables (call it λ) is thought of as being under control, such as an externally applied force, while the other variables represent the unknown state of the system, then solving

$$G_\lambda(x) \equiv G(\lambda; x) = 0 \tag{1.2}$$

as an equation in x as λ varies is what we mean by a *bifurcation problem*. In these notes we shall discuss some current approaches to finding the structure of solutions to equations of form (1.2). The methods will apply in many contexts where x is an infinite-dimensional variable, which among other things allows the apparently "static" nature of the problem to have an interpretation for some dynamic bifurcations (periodic solutions, etc) as illustrated for example in 6(a) below. We suppose that all data are smooth, i.e., infinitely differentiable or at least sufficiently differentiable for our purposes.

2. The Implicit Function Theorem

Let G_μ be a mapping $\boldsymbol{R}^n \to \boldsymbol{R}^n$ depending on an r-dimensional parameter μ, and suppose $G_0(0) = 0$. If the origin is a *regular point* for G_0, i.e., the derivative $DG_0(0)$

is an *invertible* linear operator $\mathbf{R}^n \to \mathbf{R}^n$ (i.e., nonsingular matrix) then $G_\mu(x) = 0$ has a unique solution $x = x_\mu$ close to $x = 0$ for μ sufficiently small, and x_μ varies as smoothly with μ as G_μ does. This is the Implicit Function Theorem (IFT) (Dieudonné [12]), equally valid with a Banach space in place of $\mathbf{R}^n$. Under these circumstances there is no actual bifurcation taking place near $\mu = 0$.

If $DG_0(0)$ is not invertible (certainly the case when G_μ maps from $\mathbf{R}^n$ to $\mathbf{R}^p$ with $n \neq p$) the IFT still yields results. Choose coordinates so that the null space (kernel) of $DG_0(0)$ corresponds to the first $k \geqslant 0$ coordinates in $\mathbf{R}^n$, and the range of $DG_0(0)$ corresponds to the last $(n-k)$ coordinates in $\mathbf{R}^p$. Write $u = (x_1, \ldots, x_k)$ and $v = (x_{k+1}, \ldots, x_n)$; also write

$$G_\mu(x) \equiv G_\mu(u, v) \equiv (H_\mu(u, v), K_\mu(u, v)) \tag{2.1}$$

where K_μ: $\mathbf{R}^n \to \mathbf{R}^{n-k}$ and H_μ: $\mathbf{R}^n \to \mathbf{R}^c$ with $c = p - n + k$. Now apply the IFT to the problem

$$K_{\mu,u}(v) \equiv K_\mu(u, v) = 0. \tag{2.2}$$

By construction $DK_{0,0}(0)$ is invertible, so there exists a unique solution $v = v_{\mu,u}$ to equation (2.2). Thus finding solutions x_μ to $G_\mu(x) = 0$ is the same as finding solutions to

$$\bar{H}_\mu(u) \equiv H_\mu(u, v_{\mu,u}) = 0 \tag{2.3}$$

which is now a problem in u, parametrized by μ. In other words G_μ: $\mathbf{R}^n \to \mathbf{R}^p$ has been replaced by $\bar{H}_\mu$: $\mathbf{R}^k \to \mathbf{R}^c$. The IFT cannot be pushed any further here because $D_0 \bar{H}(0)$ is identically zero (no linear terms in u in $\bar{H}_0$) and new techniques are necessary. This use of the IFT is the *Ljaponov–Schmidt process* for reducing a bifurcation problem to its essential core. Commonly $r = 1$ and $\mu \equiv \lambda$.

The reduction process is especially valuable in infinite-dimensional settings. If the dimensions k (= dim ker $DG_0(0)$) and c (= codim range $DG_0(0)$) are *finite* (that is to say $DG_0(0)$ is a *Fredholm operator*, which is not unusual in physical problems), the original problem in infinite dimensions is by this method reduced to a finite dimensional one. In this setting the equation (2.3) represents c equations in k variables (varying with parameter μ) which are often called the *bifurcation equations* for the original problem.

3. Methods for finding solutions

The problem of finding what the set of zeros of p functions in n (or $n+1$) variables looks like is of course a very fundamental one in mathematics. When the functions are polynomials, this problem is what algebraic geometry is (or was) all about. Let us mention a few methods for attacking it.

(i) *Pretending the functions are polynomials (by forgetting high order terms), and doing algebraic geometry.*

This is fine, provided the algebraic geometry problem can be solved. There is then the worry about whether the higher order terms seriously affect the solution or not.

(ii) *Trying solutions in the form of infinite series with unknown coefficients, substituting appropriately into* (1.2), *and reading off the coefficients.*

Methods of this type are generally known as *perturbation methods.* The drawbacks here are, firstly, deciding which variables to expand as series in which other variables without a qualitative picture of the solutions to start with, and, secondly, verifying that the expansions are indeed convergent series on some neighbourhood of the origin. There is then the uncertainty about whether all solutions have in fact been found by this approach.

(iii) *Particular simplifications.*

Sometimes it is easy to spot from the form of the Taylor series for $G(\lambda; x)$ what the solution set must be like. For example, suppose $c = k = 1$ and the bifurcation equation is known to have a solution $x = p_\lambda$, differentiable in λ, for λ in a neighbourhood of 0. Then the bifurcation equation has the form

$$(x - p_\lambda) H(\lambda; x) = 0 \tag{3.1}$$

and so if we take $u = x - p_\lambda$ as a new coordinate we convert (3.1) into the form $uF(\lambda; u) = 0$. Now if we find the zeros of F near $(0; p_0)$ we shall have found those of G. For example, if $\dfrac{\partial F}{\partial \lambda}(0; p_0) \neq 0$ the IFT implies the zeros of F near $(0; p_0)$ have the form of a graph $\lambda = \lambda(u)$, so the zeros of G consist of the u-axis plus the graph cutting it transversely at $(0; p_0)$. This is the core of the theorem of Crandall and Rabinowitz [10] in the differentiable case. See also Hale ([22], § 3), and Example 4.2(ii) below.

(iv) *Re-scaling the variables.*

If $n = p = 1$ a classical technique allows the solution branches to (1.2) near the origin to be expressed as series expansions of one variable (say x) in terms of *fractional* powers of the other (say λ). (See e.g. Walker [40], p. 97. This is usually carried out for complex variables, but is readily interpreted in the real case; it is sometimes called *Puiseux expansion.*) The key to this method is the geometry of the *Newton diagram*, meaning the set of all points (r, s) in the plane $\boldsymbol{R}^2$ where r, s are non-negative integers such that $\lambda^r x^s$ appears in the Taylor series for $G(\lambda, x)$ at $(0, 0)$. For $n > 1$ and for $p > 1$ things become more complicated, but the Newton diagram can still be usefully exploited ([25]). Setting $x = \varepsilon^a \xi$, $\lambda = \varepsilon^b \nu$ where a, b are determined from the analogue of the Newton diagram, the equation $G(\lambda, x) = 0$ can be converted to the form $E \cdot \bar{G}(\varepsilon, \nu, \xi) = 0$ where E is a $p \times p$ diagonal matrix whose entries are powers of ε, and $\bar{G}$ has the form

$$\bar{G}(\varepsilon, \nu, \xi) \equiv \bar{G}_0(\nu, \xi) + O(\varepsilon). \tag{3.2}$$

Now if $\bar{G}_0 = 0$ has a solution (v_0, ξ_0), and *if* the IFT can be applied there to $\bar{G}_{\varepsilon,v}(\xi) \equiv \bar{G}(\varepsilon, v; \xi)$ (i.e., if $D\bar{G}_{0,v_0}(\xi_0)$ is invertible) there is a unique nearby solution $\xi = \xi_{\varepsilon,v}$ which can in turn be interpreted as a solution for x in terms of λ. On the other hand, if $D\bar{G}_{0,v_0}(\xi_0)$ is *not* invertible more work is needed. See Sather [30] for discussion (on the real case in particular) and further references. See also Sattinger [32].

Rescaling methods of a slightly different flavour are effectively used also in multi-parameter problems by Hale and others: see Hale [22]. §§ 9–15).

(v) *Topological methods.*

There is a wide literature on the methods of *topological degree* theory and their applications: see for example [24]. This can be very successful in detecting the existence of solutions, but is less effective in counting them geometrically. Other more sophisticated topological invariants have also recently been brought to bear on bifurcation problems. See Alexander [1], Fadell and Rabinowitz [13], and also Conley [9] with particular reference to dynamic problems.

The methods sketched above are all attempts to bridge the frustrating gap between (a) those relatively easily-soluble problems (usually polynomial ones) that may be written down as examples and illustrations of various types of bifurcation behaviour, and (b) those problems that actually arise in applied mathematics. It would be delightful if the latter could somehow be persuaded to become the former.

In fact there is a sense in which this does indeed frequently happen. The basis for this is the work of Golubitsky and Schaeffer ([18]) on singularity theory as a tool for bifurcation theory, which we next try to explain.

4. Equivalence of bifurcation problems

Let G, H: $\boldsymbol{R}^n \to \boldsymbol{R}^p$ be two mappings with $G(0) = H(0) = 0$. It is natural to regard $G(x) = 0$ and $H(x) = 0$ as being *qualitatively the same problem* if there are sufficiently regular invertible coordinate changes ϕ in $\boldsymbol{R}^n$ (with $\phi(0) = 0$) and ψ in $\boldsymbol{R}^p$ (with $\psi(0) = 0$) such that

$$H(\phi(x)) \equiv \psi(G(x)), \tag{4.1}$$

so that ϕ, ψ convert the graph of G precisely into the graph of H as subsets of $\boldsymbol{R}^n \times \boldsymbol{R}^p$.

Suppose G, H to be *smooth*, and stipulate that ϕ, ψ and their inverses be also smooth. Then if (4.1) holds for all x in some neighbourhood of the origin we say G, H are *A-equivalent* at 0 in $\boldsymbol{R}^n$. This is current terminology from singularity theory (cf. Gibson [15]).

If G, H are A-equivalent their main features are qualitatively identical.

For example, the zero set for G is taken by ϕ to the zero set of H, and likewise for the set of *singular points* (those where the derivative matrix DG fails to have maximal rank); also ψ takes the range of G to that of H, and takes *singular values* of G (i.e., points $G(x)$ where x is a singular point) to those of H. These facts can often be used to show easily that certain pairs G, H are *not* A-equivalent: see the examples below.

This form of equivalence is very strong, saying that in converting graph G to graph H the rescaling ψ of the "y-axis" $\boldsymbol{R}^p$ must be the *same* over every point in the "x-axis" $\boldsymbol{R}^n$. In fact some rigidity can be relaxed to allow ψ to vary with x without losing much essential information. In this case (4.1) becomes

$$H(\phi(x)) \equiv \psi_x(G(x)), \tag{4.2}$$

where we still assume $\psi_x(0) = 0$. In particular, the set of zeros for G will still be taken by ϕ to the set of zeros for H, and the "degree of contact" (so far, as that makes sense) of the graphs of G, H at 0 will not be affected. This weaker equivalence is called *contact equivalence* (see papers of Mather) or *K-equivalence*, and turns out to be very appropriate for classifying bifurcation phenomena.

4.1. LEMMA. *In* (4.2) *there is no loss of generality in taking ψ_x to be a* linear *transformation L_x of $\boldsymbol{R}^p$.*

Proof. As $\psi_x(0) = 0$ we can write $\psi_x(y) \equiv q_x(y) \cdot y$ where $q_x(y)$ is the $p \times p$ matrix $\int_0^1 \frac{d}{dy}(\psi_x(ty))\,dt$. Then replace $q_x(y)$ by $L_x \equiv q_x(G(x))$ and we have $L_x \cdot G(x) = q_x(G(x)) \cdot G(x) = \psi_x(G(x))$. ∎

4.2. EXAMPLES. (i) $G(x) \equiv x^3 + x^4$, $H(x) \equiv x^3$ where $n = p = 1$. Then $G(x) \equiv x^3(1+x) = (x')^3$ where $x' = x(1+x)^{1/3}$. Hence G, H are K-equivalent (in fact A-equivalent) at 0 with $\phi(x) \equiv x(1+x)^{1/3}$ and $\psi_x(y) = \psi(y) \equiv y$. (Note ϕ is only defined for $|x| < 1$. Obviously G, H are not K-equivalent over all of $\boldsymbol{R}$, as their zero sets away from $x = 0$ are qualitatively different.) The same argument applies to $G(x) \equiv ax^m + (\text{higher order terms})$, $H(x) \equiv \pm x^m$ depending on the sign of $a \neq 0$.

(ii) Take $p = 1$ and any n, and let $G(x) \equiv Q(x) + h(x)$ where Q is a nondegenerate quadratic form and h has vanishing first and second derivatives. The Morse Lemma (see [21] or [28] for a proof) asserts that G is A-equivalent (hence K-equivalent) to Q, with $\psi(y) \equiv y$. As a special case with $n = 2$ and $Q(x) \equiv Q(\lambda; u) = \lambda u + au^2$ we recover the Crandall–Rabinowitz result ([10], § 3 (iii)).

(iii) $n = 1$, $p = 2$. $G(x) \equiv (x^2, x^3)$, $H(x) \equiv (x^2, 0)$. Here $G(x) \equiv L_x H(x)$ where L_x is $\begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix}$, so G, H are K-equivalent with $\phi(x) \equiv x$.

(iv) $n = p = 2$. $G(x_1, x_2) \equiv (x_1^3, x_2)$, $H(x_1, x_2) \equiv (x_1^3 + x_1^2 x_2, x_2)$. This time $H(x) = L_x G(x)$ with $L_x = \begin{vmatrix} 1 & x_1^2 \\ 0 & 1 \end{vmatrix}$, so again G, H are K-equivalent ($\varphi(x) \equiv x$).

Observe that in neither of cases (iii), (iv) could G and H be A-equivalent. In (iii) the ranges of G, H could not be converted to each other by a single ψ, while in (iv) the singular points cannot correspond under any ϕ (nor the singular values under ψ): check this! Nevertheless the zero sets for G, H do correspond (simply the origin in $\boldsymbol{R}^n$) in each example.

All this so far is for a single mapping G. Replacing $G(x)$ by $G(\lambda; x)$ we get a similar story for bifurcation problems using $L_{\lambda;x}$ and $\phi(\lambda; x)$, if we make the common sense restriction that the coordinate change $\phi(\lambda; x)$ should have the form

$$\phi(\lambda; x) \equiv (\mu(\lambda); \chi(\lambda; x)) \tag{4.3}$$

to distinguish state variables x from parameters λ in order to remain consistent with physical interpretations.

So much for definitions. What use can be made of them? It would be satisfying to be able to prove a statement of the following kind:

4.3. *Assertion.* Any bifurcation problem is K-equivalent to a polynomial problem.

Unfortunately, this assertion is false. However, in a certain sense a modified version is true:

4.4. *Assertion.* Any practical bifurcation problem is K-equivalent to a polynomial problem.

By "practical" we mean "arising from a problem in the real world". To explain this possibly startling assertion we turn to the extra ingredient that singularity theory adds to bifurcation theory, namely *structural stability*.

The idea here is that if a piece of mathematics is to model a phenomenon in the real world, then the essential features of the model should persist under small enough changes in the numerical input *because this input can in any case only be given to within a nonzero margin of error*. This was first explored in the context of ordinary differential equations (under the name of *roughness*) by Andronov and Pontryagin [2], and more recently developed extensively into many areas of mathematics and of natural philosophy by Thom [38]. Its role in singularity theory emerges in the works of Mather [27] and Arnol'd ([3], [4]) in particular, or the collected papers in [5].

5. Structural stability

Any mathematical system may be called structurally stable if when it is perturbed by a sufficiently small amount *quantitatively* it remains *qualitatively* the same as it was before. This only takes on a rigorous sense, however, if precise meanings are assigned to the terms "perturb" and "*qualitatively the same*". In our contex we take "qualitatively the same" to mean "K-equivalent".

5.1. DEFINITION. The mapping G: $\boldsymbol{R}^n \to \boldsymbol{R}^p$ with $G(0) = 0$ is *structurally stable* (more precisely: *K-stable*) if, given any 1-parameter family of mappings H^t: $\boldsymbol{R}^n \to \boldsymbol{R}^p$ with $H^0 = G$ there are coordinate changes ϕ^t, ψ^t_x converting H^t to G as in (4.2), viz

$$H^t(\phi^t(x)) \equiv \psi^t_x(G(x)) \tag{5.1}$$

for sufficiently small t. Here we do *not* assume H^t, ϕ^t and ψ^t_x take 0 to 0 (in the relevant spaces), but do assume all data smooth in all variables including t, and defined on some neighbourhoods of the origin in $\boldsymbol{R}^n \times \boldsymbol{R}$, $\boldsymbol{R}^p \times \boldsymbol{R}$ as appropriate.

The same definition applies to bifurcation problems, using $G(\lambda; x)$ instead of $G(x)$. Remember that each $\phi^t(\lambda; x)$ now has to have the special form as in (4.3).

Since the majority of interesting bifurcation studies in the literature are for the case $n = p$ (so that solutions form curves with branches meeting at bifurcation points) we shall make this assumption in what follows.

Experience shows that with the above definition (or any other sensible one!) most bifurcation problems are unlikely to be stable. Indeed, this is what the whole theory of *imperfection-sensitivity analysis* is concerned with: analysing the effect on a bifurcation problem of arbitrarily small changes in the equations. This observation represents a theorem, a simple consequence of the analysis of Golubitsky and Schaeffer.

5.2. THEOREM. *If $G(\lambda; x)$ is structurally stable at $(0; 0)$ then it is not a true bifurcation problem: either $DG_0(0)$ is nonsingular (in which case the* IFT *gives a unique solution for all small λ) or G is K-equivalent at $(0; 0)$ to*

$$F(\lambda; x_1, x_2, \ldots, x_n) \equiv (x_1^2 - \lambda, x_2, \ldots, x_n) \tag{5.2}$$

i.e., *the origin is a* limit point.

The possibility of removing the effect of λ from the last $n-1$ coordinates is another way of interpreting Ljapunov–Schmidt reduction in this case.

Thus all true bifurcation problems are structurally unstable, and so according to the philosophy of structural stability we should not use them to

model reality. If fact they are of course used extensively in this way, but they carry with them the knowledge that in practice a system will not behave according to the actual model but will, in virtue of "imperfections", adopt a behaviour described by some small perturbation of the original problem. To understand this practically and theoretically a natural step is to insert extra imperfection parameters $\alpha = (\alpha_1, \ldots, \alpha_r)$ into the problem and look at the *perturbed bifurcation problem*

$$G(\alpha, \lambda; x) \equiv 0, \tag{5.3}$$

hoping to analyze how the behaviour as a problem in $(\lambda; x)$ changes as α varies near 0 in $\mathbb{R}^r$.

The ideas of K-equivalence and structural stability apply equally well to perturbed bifurcation problems as they do to unperturbed problems or to single mappings: we insist now that $\phi(\alpha, \lambda; x)$ has the form

$$\phi(\alpha, \lambda; x) \equiv (\beta(\alpha), \mu(\alpha, \lambda); \chi(\alpha, \lambda; x)) \tag{5.4}$$

to protect the special status of the imperfection parameters. Golubitsky and Schaeffer study stability of perturbed bifurcation problems, and classify those which are stable with small r. From this analysis we extract:

5.3. THEOREM. *If $r \leqslant 2$ and $G(\alpha, \lambda; x)$ is stable at $(0, 0; 0)$ then G is K-equivalent at $(0, 0; 0)$ to $F(\alpha, \lambda; x) \equiv (h(\alpha, \lambda; x_1), x_2, \ldots, x_n)$ where h is one or other of the following:*

$r = 1$:

$h_1(\alpha, \lambda; x_1) \equiv x_1$,	*regular point*;
$h_2(\alpha, \lambda; x_1) \equiv x_1^2 - \lambda$,	*limit point*;
$h_3(\alpha, \lambda; x_1) \equiv x_1^2 - \lambda x_1 + \alpha$,	*unsymmetric bifurcation (exchange of stabilities)*;
$h_4(\alpha, \lambda; x_1) \equiv x_1^2 + \lambda^2 + \alpha$,	*isola (unusual in solid mechanics)*;
$h_5(\alpha, \lambda; x_1) \equiv x_1^3 + \alpha x_1 - \lambda$,	*hysteresis.*

$r = 2$: *The above examples (read α_1 for α on the right hand side of h_3, h_4, h_5) together with:*

$h_6(\alpha, \lambda; x_1) \equiv x_1^3 + \alpha_2 x_1^2 - \lambda x_1 - \alpha_1$,	*symmetric bifurcation*;
$h_7(\alpha, \lambda; x_1) \equiv x_1^4 + \alpha_2 x_1^2 + \alpha_1 x_1 - \lambda$,	*quartic "limit point"*;
$h_8(\alpha, \lambda; x_1) \equiv x_1^2 - \lambda^3 - \alpha_2 \lambda + \alpha_1$,	*(unusual in solid mechanics).*

In fact if $G(\alpha, \lambda; x)$ is stable and $r \leqslant 7$ the results in [18] show that G must still be K-equivalent to $F(\alpha, \lambda; x) \equiv (h(\alpha, \lambda; x_1), x_2, \ldots, x_n)$ for some h, i.e., there is only one essential variable x_1 and we are dealing with bifurcation "at a single eigenvalue". This looks like a severe limitation of the theory, for bifurcation at multiple eingenvalues is obviously very important

in solid mechanics. But such bifurcation normally arises through the imposition of *symmetry* on a physical structure, and symmetry is the key to extending Golubitsky and Schaeffer's theory to a great variety of interesting problems.

6. Symmetry

The importance of symmetry and symmetry-breaking for the understanding of natural phenomena has been recognized and profitably exploited in theoretical physics for many years, while the mathematical tools have found slower use in solid and fluid mechanics. It is only quite recently that the general role of symmetry in abstract bifurcation problems has begun to be seriously researched. Here briefly are some results obtained by explorations in various directions.

(a) *Implicit function theorem with symmetries.*

Suppose Γ is a group of linear transformations acting on $\boldsymbol{R}^n$ and on $\boldsymbol{R}^p$ (in possibly unrelated ways). If γ is an element of Γ, write γx for the effect of γ on x. Suppose also that the mapping G_λ respects these symmetries for every λ, namely $G_\lambda \gamma = \gamma G_\lambda$ for all γ in G. Then the interaction of the IFT with the Γ-actions may yield more results than either on its own.

With the notation as above for the Ljapunov–Schmidt reduction, it is straightforward to show (see e.g. [11]) that the coordinates can be chosen so that the Γ-actions preserve the decomposition of $\boldsymbol{R}^n$ as $\boldsymbol{R}^k \times \boldsymbol{R}^{n-k}$ and of $\boldsymbol{R}^p$ as $\boldsymbol{R}^c \times \boldsymbol{R}^{n-k}$, and then the reduced problem $\tilde{H}_\lambda$ also respects these symmetries.

6.1. THEOREM (Dancer [11], Vanderbauwhede [39]). *Suppose the only point of $\boldsymbol{R}^c$ which remains fixed under the Γ-action is $\{0\}$, which is the same as saying that any point of $\boldsymbol{R}^p$ fixed under the Γ-action must lie in $\boldsymbol{R}^{n-k}$ = range $DG_0(0)$. Then $x = x_\lambda \equiv (0, v(\lambda, 0))$ is a solution to $G_\lambda(x) = 0$ for small λ. (In fact it is the* only *solution of the form $(0, v)$, and it is fixed under the action of Γ.)*

Proof. By construction $G_\lambda(0, v(\lambda, 0))$ lies in $\boldsymbol{R}^c$. Thus for any γ in Γ it is also true that $\gamma G_\lambda(0, v(\lambda, 0))$ lies in $\boldsymbol{R}^c$. But $\gamma G_\lambda(0, v(\lambda, 0)) = G_\lambda \gamma(0, v(\lambda, 0)) = G_\lambda(0, \gamma v(\lambda, 0))$ so $\gamma v(\lambda, 0) = v(\lambda, 0)$ by uniqueness of $v(\lambda, u)$. Therefore $G_\lambda(0, v(\lambda, 0))$ is fixed by γ, which from our hypothesis implies $G_\lambda(0, v(\lambda, 0)) = 0$. ∎

Application (Vanderbauwhede [39]). *Nonlinear oscillations.*

Consider looking for 2π-periodic solutions of the ordinary differential equation

$$x'' + \phi(x) = \lambda p(t) \tag{6.1}$$

where x is a real variable, p is a continuous 2π-periodic function of t and denotes $\frac{d}{dt}$. Here ϕ is some C^2 function $\boldsymbol{R} \to \boldsymbol{R}$. Let x_0 be a 2π-periodic

solution to

$$x'' + \phi(x) = 0 \tag{6.2}$$

and seek solutions to (6.1) near x_0. This problem can be expressed in the form (1.2) by taking X (resp. Y) to be the space of C^2 (resp. continuous) 2π-periodic functions $\boldsymbol{R} \to \boldsymbol{R}$ and setting

$$G_\lambda(x) \equiv (x_0'' + x'') + \phi(x_0 + x) - \lambda p. \tag{6.3}$$

With sensible norms chosen for X, Y the mapping G_λ: $X \to Y$ is C^1 and satisfies the conditions for Ljapunov–Schmidt reduction at 0 with

$$\text{range } DG_0(0) = \{\text{all } y \text{ in } Y \text{ with } \int_0^{2\pi} x_0(t)\, y(t)\, dt = 0\}. \tag{6.4}$$

Let $\Gamma \simeq Z_2$ be the group of two elements $\{e, \tau\}$ with e the identity and τ acting on X and Y by $(\tau x)(t) = x(-t)$. If p is fixed by τ (i.e., p is an *even* function) it follows that G_λ is Γ-equivariant. The hypotheses of Theorem (6.1) can then be verified to hold if x_0 is an even function. Likewise if p and also ϕ are odd functions the hypotheses hold if x_0 is an odd function. Thus we get

6.2. PROPOSITION. (i) *Suppose p is even. Then from every* even *solution x_0 to* (6.2) *there bifurcates a 1-parameter family of even solutions x_λ to* (6.1);

(ii) *Suppose p and ϕ are odd. Then from every* odd *solution x_0 to* (6.2) *there bifurcates a 1-parameter family of* odd *solutions x_λ to* (6.1).

(b) *Symmetry in the higher order terms.*

Suppose $G_\lambda(x)$ has no linear terms in x when $\lambda = 0$, so that the IFT can give no further simplification. The next step is to look at quadratic terms or, if there are none of those, at cubic terms, and so on. Since Γ acts linearly on $\boldsymbol{R}^n$, $\boldsymbol{R}^p$, the terms $G_\lambda^k(x)$ of homogeneous degree k in x will themselves define a mapping G_λ^k: $\boldsymbol{R}^n \to \boldsymbol{R}^p$ which is Γ-equivariant, for each $k = 2, 3, \ldots$ The mere fact that G_λ^k is Γ-equivariant obviously imposes strong constraints on the coefficients in G_λ^k, and these constraints can be systematically determined in very many cases by existing methods of *group representation theory*. These can then be used to give information about possible bifurcation geometry quite independently of the actual physical data. It is only after this analysis that the physical data are then used to determine the choice among limited possibilities. This is an approach which has been developed and exploited fruitfully in Sattinger [33], [34].

Application (Sattinger). *The Bénard convection problem.*

When a layer of fluid is heated from below, convection causes instabilities and in certain circumstances can give rise to very regular patterns of flow which have a "cellular" structure: see for example the picture of hexagonal cells in Sattinger [34], taken from Koschmieder [23]. Supposing

the problem suitably idealized (assuming, for instance, that the fluid covers an infinite plane) so that it can be set up in appropriate function spaces in the form (1.2) (see Fife [14]), and assuming *a priori* invariance with respect to a discrete lattice of translational symmetries in the plane, Sattinger shows that the bifurcation equations (2.3) are 6-dimensional and the first coordinate of $\bar{H}_\lambda$ is forced to have one or other of the following forms:

(6.5)
$$\text{hexagonal lattice: } \lambda u_1 + c u_2 u_6 + (a u_1^2 u_4 + b u_1 (u_2 u_5 + u_3 u_6)) + (\text{order} \geqslant 4),$$
$$\text{square or rhombic lattice: } \lambda u_1 + a u_1^2 u_3 + b u_1 u_2 u_4 + (\text{order} \geqslant 4).$$

The remaining five coordinates are found by cyclic permutation of u_1, $u_2, \ldots, u_6$. The constants a, b, c depend on the physical data of the problem but the analysis of solutions of (6.5) up to terms of order 3 is now a matter of relatively straightforward algebra. In this way the symmetry assumptions about the problem have reduced the nonlinear calculations to manageable proportions. See [33] for further details, as well as discussions of stability and general interpretation.

The question of which possible stable state of lower symmetry a physical system will actually adopt (without *a priori* assumptions) when bifurcating from a state of higher symmetry is a very fundamental and important one, especially in physics of elementary particles. See discussions in Sattinger [31], which address the general problem as well as the case of Bénard convection.

(c) *Contact equivalence and symmetry.*

The previous method of analysis by symmetry properties of homogeneous quadratic, cubic, etc. terms using group representation theory is very effective for some problems, but does not seem to deal satisfactorily with questions of the influence of higher order terms, structural stability and imperfection-sensitivity. Golubitsky and Schaeffer [19] are able to handle these matters by incorporating Γ-equivariance into the theory of K-equivalence as outlined in § 4. Most of the ideas there continue to make sense when we insist that all the coordinate-changes involved should respect the action of Γ in an appropriate way. For example, two Γ-equivariant mappings G_λ, H_λ: $\boldsymbol{R}^n \to \boldsymbol{R}^p$ (where we suppose Γ acts on $\boldsymbol{R}^n$, $\boldsymbol{R}^p$ but not on the λ-axis) are called *Γ-K-equivalent* if they are K-equivalent as defined before, using a linear transformation $L_{\lambda;x}$ and a coordinate change $\phi(\lambda; x)$ as in (4.3), with additional conditions that L and $\phi = (\mu; \chi)$ satisfy the symmetry constraints:

$$\chi(\lambda; \gamma x) \equiv \gamma \chi(\lambda; x), \qquad L_{\lambda;\gamma x} \cdot \gamma \equiv \gamma L_{\lambda;x} \tag{6.6}$$

for every γ in Γ. From this we derive a definition of *Γ-structural stability* by analogy with Definition (5.1) and the remarks following it.

Application (Golubitsky and Schaeffer [19]). *Buckling of a rectangular plate.*

A model for buckling of a rectangular plate is provided by the Von

Karman equations (see e.g. Chow, Hale and Mallet-Paret [8]), which lead via Ljapunov–Schmidt reduction or finite-element approximation (Bauer, Keller and Reiss [6]) to a two-dimensional bifurcation problem of the form

$$G_\lambda(x_1, x_2) \equiv (p(x_1, x_2), q(x_1, x_2)) - \lambda(x_1, x_2) \tag{6.7}$$

where p, q begin at degree 3. Symmetry of the physical problem leads (see [19]) to symmetry of G: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ under the group $\Gamma \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ of pairs $(\varepsilon_1, \varepsilon_2)$ where $\varepsilon_i = \pm 1$ and the group operation is coordinatewise multiplication; the action of Γ is defined on $\boldsymbol{R}^2$ (both copies) by

$$\begin{aligned} &(+1, -1) \text{ takes } (x_1, x_2) \text{ to } (x_1, -x_2), \\ &(-1, +1) \text{ takes } (x_1, x_2) \text{ to } (-x_1, -x_2). \end{aligned} \tag{6.8}$$

After factoring out harmless constants and ruling out certain degenerate cases the general such G_λ can be shown to have the form

$$G_\lambda^\pm(x_1, x_2) \equiv (x_1^3 + bx_1 x_2^2, cx_1^2 x_2 \pm x_2^3) - \lambda(x_1, x_2) \tag{6.9}$$

up to degree 3, where $c \neq 1$ and $b \neq \pm 1$. Now this is certainly not structurally stable, and indeed Golubitsky and Schaeffer show that it needs the inclusion of 14 extra imperfection parameters in appropriate places to make it so: a quite intractable state of affairs. On the other hand, by restricting attention to *symmetry-respecting* perturbations and coordinate changes they show ([19]) that only *one* extra imperfection-parameter is then needed in addition to the parameters b, c already appearing on $G_\lambda^\pm$. Thus

$$F_{\alpha,b,c,\lambda}^\pm(x_1, x_2) \equiv G_\lambda^\pm(x_1, x_2) + \alpha(0, x_2) = 0 \tag{6.10}$$

is structurally stable as a perturbed bifurcation problem, and in particular the higher order terms which were omitted in (6.9) may be honestly forgotten. The great simplification and yet completeness that this achieves sheds light on the analyses of Bauer et al. [6] and Chow et al. [8], helps to organize information in Magnus and Poston [26], and is used by Schaeffer and Golubitsky [36] to explain the curious phenomenon of "mode jumping" in rectangular plates.

7. Conclusion

Singularity theory is a natural tool to use in bifurcation theory. It combines geometrical insight with rigorous analysis, and at its present state of development the theory of K-equivalence of Γ-equivariant mappings $G(\lambda; x)$ appears to handle very satisfactorily many of the problems of perturbed bifurcation theory in the possible presence of symmetry. Here we have given only the barest sketch: for full details see Golubitsky and Schaeffer [18], [19], and for further applications see Golubitsky and Keyfitz [16], Golubitsky, Keyfitz and Schaeffer [17], Golubitsky and Schaeffer [20] and Sattinger [31]. For an entertaining general survery of these and other applications of singularity theory see Stewart [37]. A good introductory reference for the ideas of singularity theory is Callahan [7]; for an accessible account of K-

equivalence and such more technical matters Gibson [15] is recommended. Of course much has been written on applications of singularity theory in the particular case of elementary catastrophe theory: see Poston and Stewart [29] and Zeeman [41] and references therein.

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