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Nomographic functions

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INTRODUCTION

Let Ω_x , Ω_y and Ω_z be three sets of any kind. They may be sets of numbers, sets of vectors, they may also be sets of another kind. We assume only that they are not empty sets. Let Ω be the Cartesian product $\Omega_x \times \Omega_y \times \Omega_z$ ⁽¹⁾.

Let $F(x, y, z)$, where $x \in \Omega_x$, $y \in \Omega_y$, $z \in \Omega_z$, be a function defined on the set Ω , the values of $F(x, y, z)$ being real numbers.

In the present paper I shall give a method of determining whether the function $F(x, y, z)$ can be written in the form

$$(1) \quad F(x, y, z) \equiv \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix}$$

where

$$(2) \quad X_i \equiv X_i(x), \quad Y_i \equiv Y_i(y), \quad Z_i \equiv Z_i(z) \quad (i = 1, 2, 3)$$

(the values of the functions $X_i(x)$, $Y_i(y)$, $Z_i(z)$ being real numbers) and \equiv is the symbol of identity.

The determinant (1) is called a *Massau determinant*.

An equivalence of two Massau determinants will be defined. Then I shall solve the problem of the number of non-equivalent Massau determinants of a function $F(x, y, z)$ and I shall show how all these determinants can be constructed.

The problem whether the function $F(x, y, z)$ could be written in the form (1) is connected with the question when for an equation

$$(3) \quad F(x, y, z) = 0$$

a straight-line nomogram or an alignment chart could be designed. For, if any three straight lines of the nomogram for the equation (3), corresponding to three fixed values of the variables x, y, z respectively, have

⁽¹⁾ The Cartesian product $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ of the sets $\Omega_1, \Omega_2, \dots, \Omega_n$ is the set of all systems (x_1, x_2, \dots, x_n) , where $x_i \in \Omega_i$ ($i = 1, 2, \dots, n$). The symbol $x_i \in \Omega_i$ means that x_i is an element of the set Ω_i .

in the Cartesian co-ordinate system ξ, η the equations

$$(4) \quad \begin{aligned} X_1 \xi + X_2 \eta + X_3 &= 0, \\ Y_1 \xi + Y_2 \eta + Y_3 &= 0, \\ Z_1 \xi + Z_2 \eta + Z_3 &= 0, \end{aligned}$$

then they have a common point if and only if

$$(5) \quad \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix} = 0.$$

On the other hand, if the scale equations of an alignment chart for the equation (3), corresponding to the variables x, y, z respectively, are

$$(6) \quad \begin{aligned} \xi &= \frac{X_1}{X_3}, & \eta &= \frac{X_2}{X_3}, \\ \xi &= \frac{Y_1}{Y_3}, & \eta &= \frac{Y_2}{Y_3}, & X_3 Y_3 Z_3 &\neq 0, \\ \xi &= \frac{Z_1}{Z_3}, & \eta &= \frac{Z_2}{Z_3}, \end{aligned}$$

then any three points, corresponding to three fixed values of the variables x, y, z respectively, are co-linear if and only if

$$(7) \quad \begin{vmatrix} \frac{X_1}{X_3} & \frac{X_2}{X_3} & 1 \\ \frac{Y_1}{Y_3} & \frac{Y_2}{Y_3} & 1 \\ \frac{Z_1}{Z_3} & \frac{Z_2}{Z_3} & 1 \end{vmatrix} = 0.$$

The equality (7) with the condition $X_3 Y_3 Z_3 \neq 0$ is equivalent to (5).

The problem whether the function $F(x, y, z)$ could be written in the form (1) is also connected with the problem when for an equation

$$(8) \quad F(t_1, t_2, \dots, t_6) = 0$$

a nomogram consisting of three binary fields could be designed (fig. 1). For, if the binary-field equations in the Cartesian co-ordinate system ξ, η are

$$(9) \quad \begin{aligned} \xi &= T_1(t_1, t_2), & \eta &= T_2(t_1, t_2), \\ \xi &= T_3(t_3, t_4), & \eta &= T_4(t_3, t_4), \\ \xi &= T_5(t_5, t_6), & \eta &= T_6(t_5, t_6), \end{aligned}$$

and if

$$(10) \quad x \equiv (t_1, t_2), \quad y \equiv (t_3, t_4), \quad z \equiv (t_5, t_6)$$

and

$$(11) \quad \begin{aligned} T_1(t_1, t_2) &\equiv \frac{X_1}{X_3}, & T_2(t_1, t_2) &\equiv \frac{X_2}{X_3}, \\ T_3(t_3, t_4) &\equiv \frac{Y_1}{Y_3}, & T_4(t_3, t_4) &\equiv \frac{Y_2}{Y_3}, & X_3 Y_3 Z_3 &\neq 0, \\ T_5(t_5, t_6) &\equiv \frac{Z_1}{Z_3}, & T_6(t_5, t_6) &\equiv \frac{Z_2}{Z_3}, \end{aligned}$$

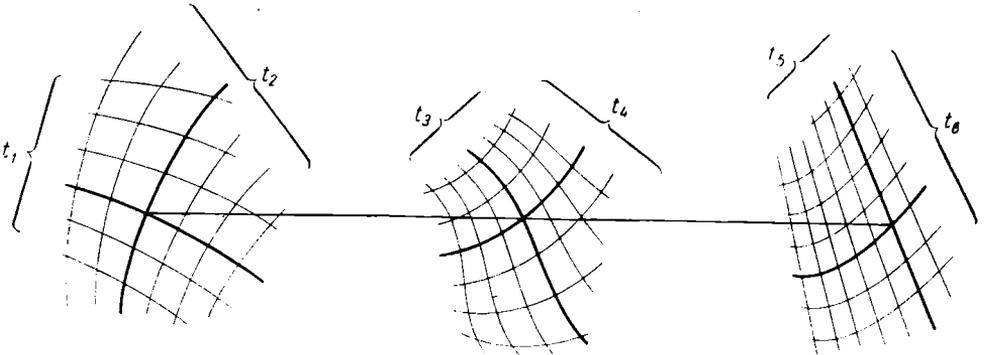


Fig. 1.

then any three points corresponding to three fixed values of the variables x, y, z respectively are co-linear if and only if the equality (7), equivalent to (5), holds.

The problem whether a function $F(x, y, z)$ can be written in the form (1) is a narrowing of the broader problem of determining when the dependence of z upon x and y given by the equation $g(x, y, z) = 0$ may be expressed by the vanishing of a function (1). Both problems are well known as the fundamental problems of nomography and they are nearly as old as nomography itself. As far as I know, only T. H. Gronwall has given a solution of the broader one [2]. He finds a necessary and sufficient condition in the existence of a common solution of two partial differential equations of the second order. Several solutions of the narrower problem have also been found. The oldest one of those which I know was given by E. Duporcq [1]. He finds a necessary and sufficient condition in the existence of 9 numbers for which a system of functional equations is solvable. In the literature on nomography there is a common

opinion that both above-mentioned solutions are generally of no use for the effective construction of Massau determinants.

In the year 1915 there appeared a paper by O. D. Kellogg [3] in which for the first time the idea of linear independence of functions was used and necessary and sufficient conditions of the existence of a form

$$F(x, y, z) \equiv F_1(y, z)X_1(x) + F_2(y, z)X_2(x) + F_3(y, z)X_3(x)$$

for a function $F(x, y, z)$ were given. But Kellogg's criteria for the existence of a form (1) for a function $F(x, y, z)$ were unnecessarily complicated and led to computations too long and troublesome for practical use.

In recent years several papers by I. A. Vilner [5], [6] have appeared. Unfortunately, as far as I know, the proofs of the theorems given in those papers are not yet published. As far as it can be deduced from those papers, the author has solved the problem of the existence of a form (1) for a function $F(x, y, z)$ in another way and more generally than it will be presented in this paper. His solution is effective. But it is incomplete, because the author has investigated only the regular cases. And yet it is the irregular cases that are of greatest use in nomography. Moreover, I. A. Vilner has not used the idea of equivalence of Massau determinants and has not solved the problem of the uniqueness of the representation (1). Furthermore, I believe that my method will be simpler in practice.

In the present paper written before I read those by Vilner I have chosen a less general way, which makes it possible to analyse all the possible cases and to solve in this way the problem of uniqueness.

Although the proofs of the two fundamental theorems which will be presented here are rather long, owing to the great number of cases considered, the theorems themselves will be convenient for practical use.

The paper is written in such a way that practising engineers will find no difficulty in understanding the material if they are familiar with the algebra of matrices and determinants.

In the whole paper, without further mention, numbers are to be regarded as real numbers.

I. LINEAR DEPENDENCE AND INDEPENDENCE OF FUNCTIONS

Let

$$(12) \quad T_i \equiv T_i(t) \quad (i = 1, 2, \dots, r)$$

be functions defined on a set θ and taking numerical values. Their values for $t = t_j$ will be denoted as follows:

$$(13) \quad T_i^j = T_i(t_j) \quad (i = 1, 2, \dots, r; j = 1, 2, \dots, s).$$

DEFINITION 1. The functions T_1, T_2, \dots, T_n are to be called *linearly dependent* if and only if there exist numbers q_1, q_2, \dots, q_n , at least one of them differing from zero, such that

$$(14) \quad q_1 T_1 + q_2 T_2 + \dots + q_n T_n \equiv 0,$$

that is,

$$(15) \quad q_1 T_1(t) + q_2 T_2(t) + \dots + q_n T_n(t) = 0$$

for every $t \in \theta$.

If the functions T_1, T_2, \dots, T_n are not linearly dependent, we shall call them *linearly independent*. Thus, if the functions T_1, T_2, \dots, T_n are linearly independent, then the identity (14) holds if and only if

$$q_1 = q_2 = \dots = q_n = 0.$$

LEMMA 1. *If the functions T_1, T_2, \dots, T_n are linearly dependent then the functions T_1, T_2, \dots, T_k , where $k > n$, are also linearly dependent. If the functions T_1, T_2, \dots, T_n are linearly independent, then the functions T_1, T_2, \dots, T_k , where $k < n$, are also linearly independent.*

LEMMA 2. *If the functions T_1, T_2, \dots, T_n are linearly independent, none of them is identically zero.*

The proofs of lemmas 1 and 2 follow immediately from the definition of linearly dependent functions.

THEOREM 1. *The functions T_1, T_2, \dots, T_n are linearly independent if and only if there exist such elements $t_j \in \theta$ ($j = 1, 2, \dots, n$) that*

$$(16) \quad \begin{array}{cccc} T_1^1 & T_2^1 & \dots & T_n^1 \\ T_1^2 & T_2^2 & \dots & T_n^2 \\ \dots & \dots & \dots & \dots \\ T_1^n & T_2^n & \dots & T_n^n \end{array} \neq 0.$$

Proof. Necessity. If the functions $T_1, T_2, \dots, T_k, T_{k+1}$ are linearly dependent, then the identity (14), where $n = k+1$, must be satisfied for q_1, q_2, \dots, q_n , at least one of them differing from zero. Hence the system of equations (19) must have a non-zero solution for every $t_{k+1} \in \theta$, which implies the condition (17).

Sufficiency. If the identity (17) holds, then it follows by (18) from the expansion of the determinant (17) by the elements of the last row that the functions $T_1, T_2, \dots, T_k, T_{k+1}$ are linearly dependent. Thus the theorem is proved.

Owing to Theorem 2 we may construct a method of determining whether given functions T_1, T_2, \dots, T_n are linearly dependent or not.

If $T_1 \equiv 0$, the functions T_1, T_2, \dots, T_n are, by Lemma 2, linearly dependent.

If $T_1 \neq 0$, we find an element $t_1 \in \theta$ such that $T_1^1 \neq 0$. If

$$(20) \quad \begin{vmatrix} T_1^1 & T_2^1 \\ T_1 & T_2 \end{vmatrix} \equiv 0.$$

then the functions T_1, T_2 are linearly dependent by Theorem 2 and consequently the functions T_1, T_2, \dots, T_n are also linearly dependent by Lemma 1.

If the condition (20) is not satisfied, then the functions T_1, T_2 are linearly independent. If moreover $n > 2$, we find such an element $t_2 \in \theta$ that

$$\begin{vmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{vmatrix} \neq 0.$$

If

$$(21) \quad \begin{vmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1 & T_2 & T_3 \end{vmatrix} \equiv 0,$$

then the functions T_1, T_2, T_3 are linearly dependent by Theorem 2 and consequently the functions T_1, T_2, \dots, T_n are also linearly dependent by Lemma 1.

On the other hand, if the condition (21) is not satisfied, then the functions T_1, T_2, T_3 are linearly independent. If, moreover, $n > 3$, we find such an element $t_3 \in \theta$ that

$$\begin{vmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{vmatrix} \neq 0.$$

Prolonging this procedure, we answer step-by-step whether the functions T_1, T_2, T_3, T_4 , the functions T_1, T_2, T_3, T_4, T_5 etc. are linearly dependent, until we obtain the answer for the functions T_1, T_2, \dots, T_n .

EXAMPLE 1. We shall show that the functions of a numerical variable x :

$$F_1(x) \equiv x, \quad F_2(x) \equiv e^x, \quad F_3(x) \equiv e^{-x}, \quad 0 \leq x \leq 1,$$

are linearly independent.

Since $F_1(x) \not\equiv 0$, we find that $F_1(1) = 1 \neq 0$.

Since

$$\begin{vmatrix} F_1(1) & F_2(1) \\ F_1(x) & F_2(x) \end{vmatrix} \equiv \begin{vmatrix} 1 & e \\ x & e^x \end{vmatrix} \equiv e^x - ex \neq 0,$$

we find that

$$\begin{vmatrix} F_1(1) & F_2(1) \\ F_1(0) & F_2(0) \end{vmatrix} = \begin{vmatrix} 1 & e \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Since

$$\begin{vmatrix} F_1(1) & F_2(1) & F_3(1) \\ F_1(0) & F_2(0) & F_3(0) \\ F_1(x) & F_2(x) & F_3(x) \end{vmatrix} \equiv \begin{vmatrix} 1 & e & \frac{1}{e} \\ 0 & 1 & 1 \\ x & e^x & e^{-x} \end{vmatrix} \equiv \left(e - \frac{1}{e}\right)x - e^x + e^{-x} \neq 0,$$

the functions $F_1(x), F_2(x), F_3(x)$ are linearly independent.

The procedure applied in Example 1 may sometimes be simplified by using Theorem 1 immediately.

EXAMPLE 2. We shall show that the functions of a pair of numerical variables (x, y) :

$$\begin{aligned} F_1(x, y) &\equiv 1, & F_2(x, y) &\equiv x, & \dots, & F_{m+1}(x, y) &\equiv x^m, \\ F_{m+2}(x, y) &\equiv y, & F_{m+3}(x, y) &\equiv xy, & \dots, & F_{2m+2}(x, y) &\equiv x^m y, \\ & \dots & & & & & \\ F_{mn+n+1}(x, y) &\equiv y^n, & F_{mn+n+2}(x, y) &\equiv xy^n, & \dots, & F_{(m+1)(n+1)}(x, y) &\equiv x^m y^n, \\ & & & & & & 0 \leq x \leq c, \quad 0 \leq y \leq d \end{aligned}$$

are linearly independent (c and d being arbitrary positive numbers).

Let a_1, a_2, \dots, a_r , where $r = (m+1)(n+1)$, be arbitrary but different numbers from the interval $[0, \min(c, \sqrt[m+1]{d})]$. Then

$$\begin{vmatrix} F_1(a_1, a_1^{m+1}) & F_2(a_1, a_1^{m+1}) & \dots & F_r(a_1, a_1^{m+1}) & 1 & a_1 & a_1^2 & \dots & a_1^{r-1} \\ F_1(a_2, a_2^{m+1}) & F_2(a_2, a_2^{m+1}) & \dots & F_r(a_2, a_2^{m+1}) & 1 & a_2 & a_2^2 & \dots & a_2^{r-1} \\ \dots & \dots \\ F_1(a_r, a_r^{m+1}) & F_2(a_r, a_r^{m+1}) & \dots & F_r(a_r, a_r^{m+1}) & 1 & a_r & a_r^2 & \dots & a_r^{r-1} \end{vmatrix} =$$

is a Vandermonde determinant (see e. g. [4], p. 130) and therefore differs from zero.

Hence, the functions $F_1(x, y), F_2(x, y), \dots, F_r(x, y)$ are linearly independent by Theorem 1, where θ is the rectangle $0 \leq x \leq c, 0 \leq y \leq d$, its elements are points (x, y) , elements t_j are the points (a_j, a_j^{m+1}) , and $T_i \equiv F_i(x, y)$ ($i, j = 1, 2, \dots, r$).

Now let $T \equiv T(t)$ be a function with numerical values, defined on the same set θ as the functions T_1, T_2, \dots, T_n . The values of that function for $t = t_j$ ($j = 1, 2, \dots, s$) will be denoted as follows:

$$T^j = T(t_j).$$

THEOREM 3. *If the functions T_1, T_2, \dots, T_n are linearly independent and the functions T, T_1, T_2, \dots, T_n are already linearly dependent, then there exists exactly one system of numbers p_1, p_2, \dots, p_n such that*

$$(22) \quad T \equiv p_1 T_1 + p_2 T_2 + \dots + p_n T_n.$$

Proof. If the functions T, T_1, T_2, \dots, T_n are linearly dependent, there exist, from the definition, numbers q, q_1, q_2, \dots, q_n , at least one of them differing from zero, such that

$$(23) \quad qT + q_1 T_1 + q_2 T_2 + \dots + q_n T_n \equiv 0.$$

If $q = 0$, then the functions T_1, T_2, \dots, T_n would be linearly dependent, contrary to the assumption made. Therefore $q \neq 0$ and (23) implies the identity (22), where $p_i = -q_i/q$ ($i = 1, 2, \dots, n$). If another system of numbers $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_n$ existed, such that

$$(24) \quad T \equiv \bar{p}_1 T_1 + \bar{p}_2 T_2 + \dots + \bar{p}_n T_n,$$

then the identities (22) and (24) would imply

$$(p_1 - \bar{p}_1) T_1 + (p_2 - \bar{p}_2) T_2 + \dots + (p_n - \bar{p}_n) T_n \equiv 0.$$

Since the functions T_1, T_2, \dots, T_n are linearly independent, we should have $p_i - \bar{p}_i = 0$, i. e., $\bar{p}_i = p_i$ ($i = 1, 2, \dots, n$). This completes the proof.

If the functions T, T_1, T_2, \dots, T_n are given, a method of evaluating the numbers p_1, p_2, \dots, p_n may easily be constructed. Namely, there

Since the system (30) must have a non-zero solution in q_1, q_2, \dots, q_n , the condition (29) must be satisfied.

Sufficiency. If the equality (29) holds, the rows of its determinant are linearly dependent, i. e., there exist numbers r_1, r_2, \dots, r_n , at least one of which differs from zero, such that

$$(31) \quad r_1 p_{1i} + r_2 p_{2i} + \dots + r_n p_{ni} = 0 \quad (i = 1, 2, \dots, n).$$

From (26) and (31) we obtain

$$r_1 S_1 + r_2 S_2 + \dots + r_n S_n \equiv 0.$$

Therefore the functions S_1, S_2, \dots, S_n are then linearly dependent.

Thus Theorem 5 is proved.

II. RANK OF A FUNCTION

Let Φ be the Cartesian product of two arbitrary non-empty sets Φ_u and Φ_v . Let

$$U_i \equiv U_i(u), \quad V_i \equiv V_i(v), \quad \bar{U}_i \equiv \bar{U}_i(u), \quad \bar{V}_i \equiv \bar{V}_i(v) \\ (i = 1, 2, \dots, r)$$

be functions defined on the sets Φ_u and Φ_v respectively and taking numerical values. Their values for $u = u_j$ and $v = v_j$ will be denoted as follows:

$$U_i^j = U_i(u_j), \quad V_i^j = V_i(v_j), \quad \bar{U}_i^j = \bar{U}_i(u_j), \quad \bar{V}_i^j = \bar{V}_i(v_j) \\ (i = 1, 2, \dots, r; \quad j = 1, 2, \dots, s).$$

Let

$$G \equiv G(u, v), \quad u \in \Phi_u, \quad v \in \Phi_v,$$

be a function defined on the set Φ and taking numerical values. We shall denote its values for $u = u_j$ and $v = v_k$ by

$$G^{jk} = G(u_j, v_k) \quad (j, k = 1, 2, \dots, s).$$

We introduce moreover the functions

$$G^{\cdot k} \equiv G(u, v_k), \quad G^j \equiv G(u_j, v) \quad (j, k = 1, 2, \dots, s).$$

DEFINITION 2. The function G is said to be of *rank* n ($n > 1$) if and only if there exist functions $U_1, U_2, \dots, U_n; V_1, V_2, \dots, V_n$ such that

$$(32) \quad G \equiv U_1 V_1 + U_2 V_2 + \dots + U_n V_n$$

and there are no functions $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{n-1}; \bar{V}_1, \bar{V}_2, \dots, \bar{V}_{n-1}$ such that

$$G \equiv \bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2 + \dots + \bar{U}_{n-1} \bar{V}_{n-1}.$$

The function G is said to be of *rank zero* if and only if it equals zero identically. It is said to be of *rank 1* if and only if there exist functions $U_1 \not\equiv 0$ and $V_1 \not\equiv 0$ such that $G \equiv U_1 V_1$.

The function G is said to be of *rank greater than* n if and only if there are no functions $U_1, U_2, \dots, U_n; V_1, V_2, \dots, V_n$ satisfying the identity

(32). I. e., the function G is said to be of rank greater than n if it is either of rank $m > n$ or of no finite rank.

THEOREM 6. *The function G is of rank greater than n if and only if there exist in Φ_u elements u_1, u_2, \dots, u_{n+1} and in Φ_v elements v_1, v_2, \dots, v_{n+1} , such that*

$$(33) \quad \begin{vmatrix} G^{11} & G^{12} & \dots & G^{1,n+1} \\ G^{21} & G^{22} & \dots & G^{2,n+1} \\ \dots & \dots & \dots & \dots \\ G^{n+1,1} & G^{n+1,2} & \dots & G^{n+1,n+1} \end{vmatrix} \neq 0.$$

Proof. Necessity. We carry out the proof by induction. If the function G is of rank greater than zero, then $G \neq 0$ and there exist elements $u_1 \in \Phi_u$ and $v_1 \in \Phi_v$ such that $G^{11} \neq 0$. Thus the necessity of our condition is obvious for $n = 0$. Let us now assume that for $n = k$, if the function G is of rank greater than n , there exist elements $u_1, u_2, \dots, u_{n+1}; v_1, v_2, \dots, v_{n+1}$ such that the condition (33) holds, but for $n = k+1$ such elements do not exist, i. e.,

$$(34) \quad \begin{vmatrix} G^{11} & G^{12} & \dots & G^{1,k+1} & G^1. \\ G^{21} & G^{22} & \dots & G^{2,k+1} & G^2. \\ \dots & \dots & \dots & \dots & \dots \\ G^{k+1,1} & G^{k+1,2} & \dots & G^{k+1,k+1} & G^{k+1,.} \\ G^{.,1} & G^{.,2} & \dots & G^{.,k+1} & G \end{vmatrix} \equiv 0,$$

where

$$(35) \quad a = \begin{vmatrix} G^{11} & G^{12} & \dots & G^{1,k+1} \\ G^{21} & G^{22} & \dots & G^{2,k+1} \\ \dots & \dots & \dots & \dots \\ G^{k+1,1} & G^{k+1,2} & \dots & G^{k+1,k+1} \end{vmatrix} \neq 0$$

and the function G is of rank greater than $k+1$.

Expanding the determinant (34) by the elements of the last row we should then obtain by the condition (35)

$$(36) \quad G \equiv -G^{.,1} \frac{A_1}{a} - G^{.,2} \frac{A_2}{a} - \dots - G^{.,k+1} \frac{A_{k+1}}{a},$$

where A_i ($i = 1, 2, \dots, k+1$) are the cofactors of $G^{.,i}$. Thus, the function G would be of rank not exceeding $k+1$, contrary to the assumption made. It follows that if our condition is necessary for $n = k$ it is also necessary for $n = k+1$.

Thus the necessity of our condition is established.

Sufficiency. If there exist in Φ_u and Φ_v respectively elements u_1, u_2, \dots, u_{n+1} and v_1, v_2, \dots, v_{n+1} such that the condition (33) is satisfied, then the function G is not of rank zero. If we had

$$G \equiv U_1 V_1 + U_2 V_2 + \dots + U_k V_k,$$

where $k \leq n$, then we should have

$$G^{ij} = U_1^i V_1^j + U_2^i V_2^j + \dots + U_k^i V_k^j$$

and after the substitution in (33)

$$\begin{vmatrix} U_1^1 V_1^1 + \dots + U_k^1 V_k^1 & U_1^1 V_1^2 + \dots + U_k^1 V_k^2 & \dots & U_1^1 V_1^{n+1} + \dots + U_k^1 V_k^{n+1} \\ U_1^2 V_1^1 + \dots + U_k^2 V_k^1 & U_1^2 V_1^2 + \dots + U_k^2 V_k^2 & \dots & U_1^2 V_1^{n+1} + \dots + U_k^2 V_k^{n+1} \\ \dots & \dots & \dots & \dots \\ U_1^{n+1} V_1^1 + \dots + U_k^{n+1} V_k^1 & U_1^{n+1} V_1^2 + \dots + U_k^{n+1} V_k^2 & \dots & U_1^{n+1} V_1^{n+1} + \dots + U_k^{n+1} V_k^{n+1} \end{vmatrix}$$

$$= \begin{vmatrix} U_1^1 & U_2^1 & \dots & U_k^1 & \overbrace{0 \dots 0}^{n+1-k \text{ columns}} \\ U_1^2 & U_2^2 & \dots & U_k^2 & 0 \dots 0 \\ \dots & \dots & \dots & \dots & \dots \\ U_1^{n+1} & U_2^{n+1} & \dots & U_k^{n+1} & 0 \dots 0 \end{vmatrix} \cdot \begin{vmatrix} V_1^1 & V_1^2 & \dots & V_1^{n+1} \\ V_2^1 & V_2^2 & \dots & V_2^{n+1} \\ \dots & \dots & \dots & \dots \\ V_k^1 & V_k^2 & \dots & V_k^{n+1} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{vmatrix} = 0,$$

$\left. \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \right\} \begin{matrix} n+1-k \\ \text{rows} \end{matrix}$

contrary to the condition (33). It follows that the function G must be of rank greater than n whenever the condition (33) is satisfied.

This completes the proof.

THEOREM 7. Let $u_i \in \Phi_u$ and $v_j \in \Phi_v$ ($i, j = 1, 2, \dots, k$) be such elements that the condition (35) with $k+1$ replaced by k is satisfied. Then the function G is of rank n ($n \geq k$) if and only if the function

$$(37) \quad H \equiv \begin{vmatrix} G^{11} & G^{12} & \dots & G^{1k} & G^{1.} \\ G^{21} & G^{22} & \dots & G^{2k} & G^{2.} \\ \dots & \dots & \dots & \dots & \dots \\ G^{k1} & G^{k2} & \dots & G^{kk} & G^{k.} \\ G^{.1} & G^{.2} & \dots & G^{.k} & G \end{vmatrix}$$

is of rank $n-k$.

Proof. Necessity. If the function G is of rank n , there exist functions $U_1, U_2, \dots, U_n; V_1, V_2, \dots, V_n$, such that the identity (32)

holds. The identity (37) may then be rewritten in the form

$H \equiv$

$$\begin{vmatrix} U_1^1 V_1^1 + \dots + U_n^1 V_n^1 & U_1^1 V_1^2 + \dots + U_n^1 V_n^2 & \dots & U_1^1 V_1^k + \dots + U_n^1 V_n^k & U_1^1 V_1 + \dots + U_n^1 V_n \\ U_1^2 V_1^1 + \dots + U_n^2 V_n^1 & U_1^2 V_1^2 + \dots + U_n^2 V_n^2 & \dots & U_1^2 V_1^k + \dots + U_n^2 V_n^k & U_1^2 V_1 + \dots + U_n^2 V_n \\ \dots & \dots & \dots & \dots & \dots \\ U_1^k V_1^1 + \dots + U_n^k V_n^1 & U_1^k V_1^2 + \dots + U_n^k V_n^2 & \dots & U_1^k V_1^k + \dots + U_n^k V_n^k & U_1^k V_1 + \dots + U_n^k V_n \\ U_1^1 V_1^1 + \dots + U_n^1 V_n^1 & U_1^1 V_1^2 + \dots + U_n^1 V_n^2 & \dots & U_1^1 V_1^k + \dots + U_n^1 V_n^k & U_1^1 V_1 + \dots + U_n^1 V_n \end{vmatrix}$$

$$\equiv \sum_{1 \leq i_1, i_2, \dots, i_{k+1} \leq n} \begin{vmatrix} U_{i_1}^1 V_{i_1}^1 & U_{i_2}^1 V_{i_2}^2 & \dots & U_{i_k}^1 V_{i_k}^k & U_{i_{k+1}}^1 V_{i_{k+1}}^1 \\ U_{i_1}^2 V_{i_1}^1 & U_{i_2}^2 V_{i_2}^2 & \dots & U_{i_k}^2 V_{i_k}^k & U_{i_{k+1}}^2 V_{i_{k+1}}^1 \\ \dots & \dots & \dots & \dots & \dots \\ U_{i_1}^k V_{i_1}^1 & U_{i_2}^k V_{i_2}^2 & \dots & U_{i_k}^k V_{i_k}^k & U_{i_{k+1}}^k V_{i_{k+1}}^1 \\ U_{i_1}^1 V_{i_1}^1 & U_{i_2}^1 V_{i_2}^2 & \dots & U_{i_k}^1 V_{i_k}^k & U_{i_{k+1}}^1 V_{i_{k+1}}^1 \end{vmatrix}$$

$$\equiv \sum_{1 \leq i_1, i_2, \dots, i_{k+1} \leq n} \begin{vmatrix} U_{i_1}^1 & U_{i_2}^1 & \dots & U_{i_k}^1 & U_{i_{k+1}}^1 \\ U_{i_1}^2 & U_{i_2}^2 & \dots & U_{i_k}^2 & U_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ U_{i_1}^k & U_{i_2}^k & \dots & U_{i_k}^k & U_{i_{k+1}}^k \\ U_{i_1}^1 & U_{i_2}^1 & \dots & U_{i_k}^1 & U_{i_{k+1}}^1 \end{vmatrix} \cdot V_{i_1}^1 V_{i_2}^2 \dots V_{i_k}^k V_{i_{k+1}}^1$$

Hence we have

$$(38) \quad H \equiv \sum_{1 \leq i_1 < i_2 < \dots < i_{k+1} \leq n} \begin{vmatrix} U_{i_1}^1 & U_{i_2}^1 & \dots & U_{i_k}^1 & U_{i_{k+1}}^1 \\ U_{i_1}^2 & U_{i_2}^2 & \dots & U_{i_k}^2 & U_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ U_{i_1}^k & U_{i_2}^k & \dots & U_{i_k}^k & U_{i_{k+1}}^k \\ U_{i_1}^1 & U_{i_2}^1 & \dots & U_{i_k}^1 & U_{i_{k+1}}^1 \end{vmatrix} \begin{vmatrix} V_{i_1}^1 & V_{i_2}^1 & \dots & V_{i_k}^1 & V_{i_{k+1}}^1 \\ V_{i_1}^2 & V_{i_2}^2 & \dots & V_{i_k}^2 & V_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ V_{i_1}^k & V_{i_2}^k & \dots & V_{i_k}^k & V_{i_{k+1}}^k \\ V_{i_1}^1 & V_{i_2}^1 & \dots & V_{i_k}^1 & V_{i_{k+1}}^1 \end{vmatrix}$$

If all the determinants

$$(39) \quad \begin{vmatrix} U_{i_1}^1 & U_{i_2}^1 & \dots & U_{i_k}^1 \\ U_{i_1}^2 & U_{i_2}^2 & \dots & U_{i_k}^2 \\ \dots & \dots & \dots & \dots \\ U_{i_1}^k & U_{i_2}^k & \dots & U_{i_k}^k \end{vmatrix} \quad (1 \leq i_1 < i_2 < \dots < i_k \leq n)$$

equalled zero, there would then be $H \equiv 0$, by (38). Moreover, by (37), the function G would be of rank not exceeding k , which could be verified by expanding the determinant (37) by the elements of the last row. But the function G is of rank $n \geq k$ by assumption. Thus $n = k$ and the function $H \equiv 0$ would obviously be of rank $n - k$ as desired.

If at least one of the determinants (39) differs from zero, we assume that the indices in (32) are arranged in such a manner that

$$(40) \quad c = \begin{vmatrix} U_1^1 & U_2^1 & \dots & U_k^1 \\ U_1^2 & U_2^2 & \dots & U_k^2 \\ \dots & \dots & \dots & \dots \\ U_1^k & U_2^k & \dots & U_k^k \end{vmatrix} \neq 0.$$

Then we introduce the following notations:

$$(41) \quad U_{i_1, i_2, \dots, i_{k+1}} \equiv \begin{vmatrix} U_{i_1}^1 & U_{i_2}^1 & \dots & U_{i_k}^1 & U_{i_{k+1}}^1 \\ U_{i_1}^2 & U_{i_2}^2 & \dots & U_{i_k}^2 & U_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ U_{i_1}^k & U_{i_2}^k & \dots & U_{i_k}^k & U_{i_{k+1}}^k \\ U_{i_1} & U_{i_2} & \dots & U_{i_k} & U_{i_{k+1}} \end{vmatrix},$$

$$V_{i_1, i_2, \dots, i_{k+1}} \equiv \begin{vmatrix} V_{i_1}^1 & V_{i_2}^1 & \dots & V_{i_k}^1 & V_{i_{k+1}}^1 \\ V_{i_1}^2 & V_{i_2}^2 & \dots & V_{i_k}^2 & V_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ V_{i_1}^k & V_{i_2}^k & \dots & V_{i_k}^k & V_{i_{k+1}}^k \\ V_{i_1} & V_{i_2} & \dots & V_{i_k} & V_{i_{k+1}} \end{vmatrix}.$$

Now we are in a position to rewrite the identity (38) in the form

$$(42) \quad H \equiv \sum_{1 \leq i_1 < i_2 < \dots < i_{k+1} \leq n} U_{i_1, i_2, \dots, i_{k+1}} V_{i_1, i_2, \dots, i_{k+1}}.$$

We next introduce the functions

$$(43) \quad P_m \equiv P_m(u) \equiv \begin{vmatrix} U_1^1 & U_2^1 & \dots & U_k^1 & U_m^1 \\ U_1^2 & U_2^2 & \dots & U_k^2 & U_m^2 \\ \dots & \dots & \dots & \dots & \dots \\ U_1^k & U_2^k & \dots & U_k^k & U_m^k \\ U_1 & U_2 & \dots & U_k & U_m \end{vmatrix} \quad (m = 1, 2, \dots, n).$$

Then we have by (40)

$$(44) \quad P_m \equiv c U_m + L_m,$$

where

$$(45) \quad L_m \equiv p_{m1} U_1 + p_{m2} U_2 + \dots + p_{mk} U_k$$

and p_{mj} ($j = 1, 2, \dots, k$) are the cofactors of U_j in the determinant (43).

We introduce also the notations

$$P_m^j = P_m(u_j), \quad L_m^j = L_m(u_j) \quad (j = 1, 2, \dots, k).$$

By (43) we have

$$P_m^j = 0$$

and therefore by (44) and (40)

$$(46) \quad U_m^j = -\frac{1}{c} L_m^j \quad (j = 1, 2, \dots, k).$$

It follows from (41), (44) and (46) that

$$(47) \quad U_{i_1, i_2, \dots, i_{k+1}} \equiv \frac{(-1)^{k+1}}{c^{k+1}} \begin{vmatrix} L_{i_1}^1 & L_{i_2}^1 & \dots & L_{i_k}^1 & L_{i_{k+1}}^1 \\ L_{i_1}^2 & L_{i_2}^2 & \dots & L_{i_k}^2 & L_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ L_{i_1}^k & L_{i_2}^k & \dots & L_{i_k}^k & L_{i_{k+1}}^k \\ L_{i_1} & L_{i_2} & \dots & L_{i_k} & L_{i_{k+1}} \end{vmatrix} + \frac{(-1)^k}{c^{k+1}} \begin{vmatrix} L_{i_1}^1 & L_{i_2}^1 & \dots & L_{i_k}^1 & L_{i_{k+1}}^1 \\ L_{i_1}^2 & L_{i_2}^2 & \dots & L_{i_k}^2 & L_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ L_{i_1}^k & L_{i_2}^k & \dots & L_{i_k}^k & L_{i_{k+1}}^k \\ P_{i_1} & P_{i_2} & \dots & P_{i_k} & P_{i_{k+1}} \end{vmatrix}.$$

But by (45)

$$\begin{vmatrix} L_{i_1}^1 & L_{i_2}^1 & \dots & L_{i_k}^1 & L_{i_{k+1}}^1 \\ L_{i_1}^2 & L_{i_2}^2 & \dots & L_{i_k}^2 & L_{i_{k+1}}^2 \\ \dots & \dots & \dots & \dots & \dots \\ L_{i_1}^k & L_{i_2}^k & \dots & L_{i_k}^k & L_{i_{k+1}}^k \\ L_{i_1} & L_{i_2} & \dots & L_{i_k} & L_{i_{k+1}} \end{vmatrix} \equiv \begin{vmatrix} U_1^1 & U_2^1 & \dots & U_k^1 & 0 \\ U_1^2 & U_2^2 & \dots & U_k^2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ U_1^k & U_2^k & \dots & U_k^k & 0 \\ U_1 & U_2 & \dots & U_k & 0 \end{vmatrix} \begin{vmatrix} p_{i_1,1} & p_{i_2,1} & \dots & p_{i_{k+1},1} \\ p_{i_1,2} & p_{i_2,2} & \dots & p_{i_{k+1},2} \\ \dots & \dots & \dots & \dots \\ p_{i_1,k} & p_{i_2,k} & \dots & p_{i_{k+1},k} \\ 0 & 0 & \dots & 0 \end{vmatrix} \equiv 0.$$

Hence the identity (47) may be rewritten in the form

$$(48) \quad U_{i_1, i_2, \dots, i_{k+1}} \equiv r_{i_1, i_2, \dots, i_{k+1}}^{i_1} P_{i_1} + r_{i_1, i_2, \dots, i_{k+1}}^{i_2} P_{i_2} + \dots + r_{i_1, i_2, \dots, i_{k+1}}^{i_{k+1}} P_{i_{k+1}},$$

where

$$r_{i_1, i_2, \dots, i_{k+1}}^{i_j} = \frac{(-1)^{k+1}}{c^{k+1}} l_{i_1, i_2, \dots, i_{k+1}}^{i_j},$$

$l_{i_1, i_2, \dots, i_{k+1}}^{i_j}$ being the cofactor of P_{i_j} in the second determinant of (47).

It follows from (42) and (48) that

$$(49) \quad H \equiv \sum_{m=1}^n P_m R_m,$$

where R_m are functions of the variable v only. Since by (43)

$$P_m \equiv 0 \quad \text{for} \quad m = 1, 2, \dots, k,$$

the identity (49) becomes

$$H \equiv \sum_{m=k+1}^n P_m R_m.$$

Hence the function H is of rank not exceeding $n-k$. Now, let us prove that it is exactly of rank $n-k$ by showing that its rank is not less than $n-k$.

By (35) and (37) we have

$$(50) \quad G \equiv \frac{1}{a} H - G^{.1} \frac{A_1}{a} - G^{.2} \frac{A_2}{a} - \dots - G^{.k} \frac{A_k}{a},$$

where A_i ($i = 1, 2, \dots, k$) are the cofactors of $G^{.i}$ in the determinant (37) and therefore functions of the variable v only. If the function H were of rank less than $n-k$, then, by (50), the function G would be of rank less than n , contrary to the assumption made. Hence the function H is exactly of rank $n-k$ as desired.

Thus the necessity is established.

Sufficiency. If the function H is of rank $n-k$, then, by (50), the function G is of rank not exceeding n . If the function G was of rank $m < n$, then, by the proof already given, the function H would be of rank $m-k < n-k$, contrary to the assumption made. Therefore the function G is exactly of rank n as desired.

Thus Theorem 7 is proved.

It follows from Theorem 7 in particular that for $n = k$ in (37) we have $H \equiv 0$. It also follows from the same theorem that if the function G is of rank n ($n > 0$), the function

$$(51) \quad G_1 \equiv G_1(u, v) \equiv \frac{1}{G^{11}} \begin{vmatrix} G^{11} & G^{1.} \\ G^{.1} & G \end{vmatrix} \quad (G^{11} \neq 0)$$

is of rank $n-1$. From (51) we obtain

$$(52) \quad G \equiv \frac{G^{.1} G^{1.}}{G^{11}} + G_1.$$

Furthermore we introduce the functions

$$(53) \quad G_i \equiv G_i(u, v) \equiv \frac{1}{G_{i-1}^{ii}} \left| \begin{array}{cc} G_{i-1}^{ii} & G_{i-1}^i \\ G_{i-1}^i & G_{i-1} \end{array} \right| \quad (G_{i-1}^{ii} \neq 0, \quad i = 2, 3, \dots, n),$$

where

$$G_{i-1}^{ii} = G_{i-1}(u_i, v_i), \quad G_{i-1}^i = G_{i-1}(u, v_i), \quad G_{i-1} = G_{i-1}(u_i, v),$$

u_i and v_i being arbitrary elements of the sets Φ_u and Φ_v respectively, such that $G_{i-1}^{ii} \neq 0$. In order to make (53) valid for $i = 1$ too, we set $G_0 \equiv G$.

By Theorem 7 the function G_i is of rank $n - i$. Therefore, for $i = 1, 2, \dots, n$ it is $G_{i-1} \neq 0$ and there always exist such elements $u_i \in \Phi_u$ and $v_i \in \Phi_v$ that $G_{i-1}^{ii} \neq 0$. But we have

$$(54) \quad G_n \equiv 0.$$

Since it follows from (53) that

$$(55) \quad G_{i-1} \equiv \frac{G_{i-1}^i G_{i-1}^i}{G_{i-1}^{ii}} + G_i \quad (i = 1, 2, \dots, n),$$

then, under the condition that the function G is of rank n , we have by (52) and (55)

$$(56) \quad G \equiv \frac{G^1 G^1}{G^{11}} + \frac{G_1^2 G_1^2}{G_1^{22}} + \dots + \frac{G_{n-1}^n G_{n-1}^n}{G_{n-1}^{nn}}.$$

Thus we have a simple method of finding the rank of a given function G and the functions $U_1, U_2, \dots, U_n; V_1, V_2, \dots, V_n$ suitable for the identity (32). Namely, we construct first the sequence of the functions G_1, G_2, \dots by evaluating them successively according to (53). If in this sequence we meet a function $G_n \equiv 0$, it is the last term of the sequence and the function G is of rank n . Then we can choose

$$(57) \quad U_i \equiv \frac{G_{i-1}^i}{G_{i-1}^{ii}}, \quad V_i \equiv G_{i-1}^i,$$

where $G_{i-1}^i, G_{i-1}^i, G_{i-1}^{ii}$ being already evaluated when the sequence has been constructed. By (56) the functions (57) satisfy the identity (32).

If in the sequence we meet a function $G_n \neq 0$, then the function G is of rank greater than n and we can find the next term of the sequence. If this sequence is infinite, the function G is of no finite rank.

EXAMPLE 4. We shall determine whether the function of a pair of numerical variables (x, y)

$$(58) \quad F \equiv F(x, y) \\ \equiv 2x^4y^4 - 2x^3y^3 + x^3y^2 - 2x^4 + x^3y + x^2y^2 + y^4 + x^2y + xy^2 + 2y^3 + xy + y^2 + y - 1,$$

defined for $x \geq 0, y \geq 0$, is of rank 3.

We set $u \equiv x, v \equiv y, F \equiv G$ and find successively the functions (53) as follows.

Since $F \neq 0$ we find that $F(0, 0) = -1 \neq 0$. Then

$$G_1 \equiv G_1(x, y) \equiv \frac{1}{-1} \begin{vmatrix} -1 & y^4 + 2y^3 + y^2 + y - 1 \\ -2x^4 - 1 & 2x^4y^4 - 2x^3y^3 + x^3y^2 - 2x^4 + x^3y + x^2y^2 + \\ & + y^4 + x^2y + xy^2 + 2y^3 + xy + y^2 + y - 1 \end{vmatrix} \\ \equiv -4x^4y^3 - 2x^4y^2 - 2x^3y^3 - 2x^4y + x^3y^2 + x^3y + x^2y^2 + x^2y + xy^2 + xy \neq 0.$$

Since $G_1(1, 1) = -4 \neq 0$, we have by (53)

$$G_2 \equiv G_2(x, y)$$

$$\equiv \frac{1}{-4} \begin{vmatrix} -4 & -6y^3 + y^2 + y \\ -8x^4 + 2x^2 + 2x & -4x^4y^3 - 2x^4y^2 - 2x^3y^3 - 2x^4y + x^3y^2 + \\ & + x^3y + x^2y^2 + x^2y + xy^2 + xy \end{vmatrix} \\ \equiv 8x^4y^3 - 4x^4y^2 - 2x^3y^3 - 4x^4y + x^3y^2 - 3x^2y^3 + x^3y + \frac{3}{2}x^2y^2 - 3xy^3 + \frac{3}{2}x^2y + \\ + \frac{3}{2}xy^2 + \frac{3}{2}xy \neq 0.$$

Since $G_2(2, 2) = 470 \neq 0$, we have by (53)

$$G_3 \equiv G_3(x, y)$$

$$\equiv \frac{1}{470} \begin{vmatrix} 470 & 94y^3 - 47y^2 - 47y \\ 40x^4 - 10x^3 - 15x^2 - 15x & 8x^4y^3 - 4x^4y^2 - 2x^3y^3 - 4x^4y + \\ & + x^3y^2 - 3x^2y^3 + x^3y + \frac{3}{2}x^2y^2 - \\ & - 3xy^3 + \frac{3}{2}x^2y + \frac{3}{2}xy^2 + \frac{3}{2}xy \end{vmatrix} \equiv 0.$$

Hence the function (58) is of rank 3. We have by (56)

$$F \equiv (2x^4 + 1)(y^4 + 2y^3 + y^2 + y - 1) + (2x^4 - \frac{1}{2}x^2 - \frac{1}{2}x)(-6y^3 + y^2 + y) + \\ + (4x^4 - x^3 - \frac{3}{2}x^2 - \frac{3}{2}x)(2y^3 - y^2 - y),$$

which could be verified directly.

THEOREM 8. *Let the function G be given in the form*

$$(59) \quad G \equiv U_1V_1 + U_2V_2 + \dots + U_nV_n.$$

Then it is of rank n if and only if the functions U_1, U_2, \dots, U_n and the functions V_1, V_2, \dots, V_n — separately treated — are linearly independent.

Proof. Necessity. If the function G is of rank n , there exist by Theorem 6 elements $u_i \in \Phi_u$ and $v_i \in \Phi_v$ ($i = 1, 2, \dots, n$) such that — retaining the previous notations — we have

$$(60) \quad d = \begin{vmatrix} G^{11} & G^{12} & \dots & G^{1n} \\ G^{21} & G^{22} & \dots & G^{2n} \\ \dots & \dots & \dots & \dots \\ G^{n1} & G^{n2} & \dots & G^{nn} \end{vmatrix} \neq 0.$$

But we have now by (59)

$$(61) \quad G^{ij} = U_1^i V_1^j + U_2^i V_2^j + \dots + U_n^i V_n^j \quad (i, j = 1, 2, \dots, n)$$

and after substituting in (60)

$$(62) \quad d = \begin{vmatrix} U_1^1 V_1^1 + \dots + U_n^1 V_n^1 & U_1^1 V_1^2 + \dots + U_n^1 V_n^2 & \dots & U_1^1 V_1^n + \dots + U_n^1 V_n^n \\ U_1^2 V_1^1 + \dots + U_n^2 V_n^1 & U_1^2 V_1^2 + \dots + U_n^2 V_n^2 & \dots & U_1^2 V_1^n + \dots + U_n^2 V_n^n \\ \dots & \dots & \dots & \dots \\ U_1^n V_1^1 + \dots + U_n^n V_n^1 & U_1^n V_1^2 + \dots + U_n^n V_n^2 & \dots & U_1^n V_1^n + \dots + U_n^n V_n^n \end{vmatrix} \\ = \begin{vmatrix} U_1^1 & U_2^1 & \dots & U_n^1 \\ U_1^2 & U_2^2 & \dots & U_n^2 \\ \dots & \dots & \dots & \dots \\ U_1^n & U_2^n & \dots & U_n^n \end{vmatrix} \begin{vmatrix} V_1^1 & V_1^2 & \dots & V_1^n \\ V_2^1 & V_2^2 & \dots & V_2^n \\ \dots & \dots & \dots & \dots \\ V_n^1 & V_n^2 & \dots & V_n^n \end{vmatrix} \neq 0.$$

It follows that

$$(63) \quad \begin{vmatrix} U_1^1 & U_2^1 & \dots & U_n^1 \\ U_1^2 & U_2^2 & \dots & U_n^2 \\ \dots & \dots & \dots & \dots \\ U_1^n & U_2^n & \dots & U_n^n \end{vmatrix} \neq 0, \quad \begin{vmatrix} V_1^1 & V_2^1 & \dots & V_n^1 \\ V_1^2 & V_2^2 & \dots & V_n^2 \\ \dots & \dots & \dots & \dots \\ V_1^n & V_2^n & \dots & V_n^n \end{vmatrix} \neq 0$$

and by Theorem 1 the functions U_1, U_2, \dots, U_n and the functions V_1, V_2, \dots, V_n — separately treated — are linearly independent.

Sufficiency. If the functions U_1, U_2, \dots, U_n and the functions V_1, V_2, \dots, V_n — separately treated — are linearly independent, then there exist by Theorem 1 elements $u_i \in \Phi_u$ and $v_i \in \Phi_v$ ($i = 1, 2, \dots, n$) such that the conditions (63) are satisfied. Furthermore, by (61) and (62) the condition (60) is then also satisfied. It follows by Theorem 6 that the function G is of rank greater than $n-1$. But by (59) it is of rank not exceeding n . Therefore it is exactly of rank n .

Thus Theorem 8 is proved.

Theorem 8 enables us sometimes to simplify the procedure of determining the rank of a given function G .

EXAMPLE 5. Let us determine whether the function of a pair of numerical variables (x, y)

$$F(x, y) \equiv x^3 y^2 + x^2 y^2 - x^3 y - x^2 - xy - 2x,$$

defined for all x and y , is of rank 3.

In order to do it we group the terms according to the powers of x and thus we obtain

$$F(x, y) \equiv x^3(y^2 - y) + x^2(y^2 - 1) - x(y + 2).$$

By Example 2 the functions x^3, x^2, x are linearly independent. Now we shall show that the functions

$$H_1(y) \equiv y^2 - y, \quad H_2(y) \equiv y^2 - 1, \quad H_3(y) \equiv -y - 2,$$

are also linearly independent. Indeed, substituting successively $y = 0, 1$ and -1 we obtain

$$\begin{vmatrix} H_1(0) & H_2(0) & H_3(0) \\ H_1(1) & H_2(1) & H_3(1) \\ H_1(-1) & H_2(-1) & H_3(-1) \end{vmatrix} = \begin{vmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 2 & 0 & -1 \end{vmatrix} = 6 \neq 0$$

and therefore by Theorem 1 the functions $H_1(y), H_2(y), H_3(y)$ are linearly independent. Hence by Theorem 8 the function $F(x, y)$ is of rank 3.

THEOREM 9. *A necessary and sufficient condition for the identity*

$$(64) \quad U_1 V_1 + U_2 V_2 + \dots + U_n V_n \equiv \bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2 + \dots + \bar{U}_n \bar{V}_n,$$

where the functions U_1, U_2, \dots, U_n and the functions V_1, V_2, \dots, V_n — separately treated — are linearly independent, is the existence of such a matrix of numerical coefficients

$$(65) \quad \mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

with the determinant

$$(66) \quad a = |\mathfrak{A}| \neq 0$$

that

$$(67) \quad \bar{U}_i \equiv a_{i1} U_1 + a_{i2} U_2 + \dots + a_{in} U_n \quad (i = 1, 2, \dots, n),$$

$$(68) \quad \bar{V}_i \equiv b_{i1} V_1 + b_{i2} V_2 + \dots + b_{in} V_n$$

and the identities (67) in the form

$$(74) \quad \bar{\mathfrak{U}} \equiv \mathfrak{U}\mathfrak{U}.$$

Substituting (74) in (73) we obtain

$$(75) \quad \bar{\mathfrak{Q}}\mathfrak{U}\mathfrak{U} \equiv \mathfrak{Q}\mathfrak{U}.$$

Since the functions U_1, U_2, \dots, U_n are by our assumption linearly independent, the identity (75) implies

$$(76) \quad \bar{\mathfrak{Q}}\mathfrak{U} \equiv \mathfrak{Q}.$$

Since the functions $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n$ are linearly independent, it follows from (74) by Theorem 5 that the matrix (65) with a_{ij} defined by (71) is non-singular. Then we obtain from (76)

$$(77) \quad \mathfrak{Q} \equiv \mathfrak{Q}\mathfrak{U}^{-1},$$

\mathfrak{U}^{-1} being the inverse of the matrix \mathfrak{U} , and (77) implies the identities (68). Thus the necessity is established.

Sufficiency. If there exists such a matrix (65) with the determinant (66) that the identities (67) and (68), i. e., (74) and (77) hold, then multiplying both sides of (77) on the right by \mathfrak{U} we obtain the identity (76). Multiplying both sides of (76) on the right by \mathfrak{U} we obtain the identity (75). Finally, replacing $\mathfrak{U}\mathfrak{U}$ by $\bar{\mathfrak{U}}$ we obtain the identity (73), i. e. (64), as desired.

This completes the proof of Theorem 9.

Theorem 9 enables us to find all the possible forms of the type

$$G \equiv U_1 V_1 + U_2 V_2 + \dots + U_n V_n$$

for a function G of rank n if one of them is already given. Thus, Theorem 9 is a complement of Theorem 7, which has enabled us to compute one of such forms.

COROLLARY 9.1. *The function $G \equiv U_1 V_1$ of rank 1 can be written in the form $G \equiv \bar{U}_1 \bar{V}_1$ if and only if there exists such a number $a \neq 0$ that*

$$\bar{U}_1 \equiv aU_1, \quad \bar{V}_1 \equiv \frac{1}{a} V_1.$$

COROLLARY 9.2. *The function*

$$G \equiv U_1 V_1 + U_2 V_2$$

of rank 2 can be written in the form

$$G \equiv \bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2$$

if and only if there exist numbers $a_{11}, a_{12}, a_{21}, a_{22}$ satisfying the condition

$$a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

such that

$$\bar{U}_1 \equiv a_{11} U_1 + a_{12} U_2, \quad \bar{V}_1 \equiv \frac{a_{22}}{a} V_1 - \frac{a_{21}}{a} V_2,$$

$$\bar{U}_2 \equiv a_{21} U_1 + a_{22} U_2, \quad \bar{V}_2 \equiv -\frac{a_{12}}{a} V_1 + \frac{a_{11}}{a} V_2.$$

Corollaries 9.1 and 9.2 are particular cases of Theorem 9.

COROLLARY 9.3. *A necessary and sufficient condition for the identity (64), where the functions U_1, U_2, \dots, U_n and the functions V_1, V_2, \dots, V_n — separately treated — are linearly independent, is the existence of a matrix (65) with the determinant (66), such that*

$$(78) \quad \begin{aligned} \bar{U}_i &\equiv a_{i1} U_1 + a_{i2} U_2 + \dots + a_{in} U_n \\ \bar{V}_i &\equiv a_{1i} \bar{V}_1 + a_{2i} \bar{V}_2 + \dots + a_{ni} \bar{V}_n \end{aligned} \quad (i = 1, 2, \dots, n).$$

This corollary follows from the identities (74) and (76).

EXAMPLE 6. In Example 4 we have obtained for the function (58) the form

$$\begin{aligned} F &\equiv (2x^4 + 1)(y^4 + 2y^3 + y^2 + y - 1) + (2x^4 - \frac{1}{2}x^2 - \frac{1}{2}x)(-6y^3 + y^2 + y) + \\ &\quad + (4x^4 - x^3 - \frac{3}{2}x^2 - \frac{3}{2}x)(2y^3 - y^2 - y). \end{aligned}$$

Since

$$\begin{aligned} y^4 + 2y^3 + y^2 + y - 1 &\equiv 1 \cdot (y^4 - 1) + 2 \cdot y^3 + 1 \cdot (y^2 + y), \\ -6y^3 + y^2 + y &\equiv 0 \cdot (y^4 - 1) - 6 \cdot y^3 + 1 \cdot (y^2 + y), \\ 2y^3 - y^2 - y &\equiv 0 \cdot (y^4 - 1) + 2 \cdot y^3 - 1 \cdot (y^2 + y), \end{aligned}$$

we have by Corollary 9.3

$$\begin{aligned} F &\equiv [1 \cdot (2x^4 + 1) + 0 \cdot (2x^4 - \frac{1}{2}x^2 - \frac{1}{2}x) + 0 \cdot (4x^4 - x^3 - \frac{3}{2}x^2 - \frac{3}{2}x)] \cdot (y^4 - 1) + \\ &\quad + [2 \cdot (2x^4 + 1) - 6 \cdot (2x^4 - \frac{1}{2}x^2 - \frac{1}{2}x) + 2 \cdot (4x^4 - x^3 - \frac{3}{2}x^2 - \frac{3}{2}x)] \cdot y^3 + \\ &\quad + [1 \cdot (2x^4 + 1) + 1 \cdot (2x^4 - \frac{1}{2}x^2 - \frac{1}{2}x) - 1 \cdot (4x^4 - x^3 - \frac{3}{2}x^2 - \frac{3}{2}x)] \cdot (y^2 + y) \end{aligned}$$

and after reduction

$$F \equiv (2x^4 + 1)(y^4 - 1) + (-2x^3 + 2)y^3 + (x^3 + x^2 + x + 1)(y^2 + y).$$

We see that Corollary 9.3 enables us sometimes to simplify the form of a given function.

THEOREM 10. *A necessary and sufficient condition for the identity*

$$(79) \quad U_1 V_1 + U_2 V_2 + \dots + U_n V_n \equiv \bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2 + \dots + \bar{U}_n \bar{V}_n + \bar{U}_{n+1} \bar{V}_{n+1},$$

where the functions U_1, U_2, \dots, U_n and the functions V_1, V_2, \dots, V_n — separately treated — are linearly independent, is the existence of such an index k and such a matrix of numerical coefficients

$$(80) \quad \mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,k-1} & 0 & a_{1k} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,k-1} & 0 & a_{2k} & \dots & a_{2n} \\ \dots & \dots \\ a_{k-1,1} & a_{k-1,2} & \dots & a_{k-1,k-1} & 0 & a_{k-1,k} & \dots & a_{k-1,n} \\ a_{k1} & a_{k2} & \dots & a_{k,k-1} & 1 & a_{kk} & \dots & a_{kn} \\ a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,k-1} & 0 & a_{k+1,k} & \dots & a_{k+1,n} \\ \dots & \dots \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,k-1} & 0 & a_{n+1,k} & \dots & a_{n+1,n} \end{bmatrix}$$

with the determinant

$$(81) \quad a = |\mathfrak{A}| \neq 0,$$

that

$$(82) \quad \bar{U}_i \equiv a_{i1} U_1 + a_{i2} U_2 + \dots + a_{in} U_n \quad (i = 1, 2, \dots, n+1),$$

$$(83) \quad \bar{V}_i \equiv b_{i1} V_1 + b_{i2} V_2 + \dots + b_{i,k-1} V_{k-1} + b_{ik} \bar{V}_k + b_{i,k+1} V_k + \dots + b_{i,n+1} V_n \\ (i = 1, 2, \dots, k-1, k+1, \dots, n+1),$$

or, conversely,

$$(84) \quad \bar{V}_i \equiv a_{i1} V_1 + a_{i2} V_2 + \dots + a_{in} V_n \quad (i = 1, 2, \dots, n+1),$$

$$(85) \quad \bar{U}_i \equiv b_{i1} U_1 + b_{i2} U_2 + \dots + b_{i,k-1} U_{k-1} + b_{ik} \bar{U}_k + b_{i,k+1} U_k + \dots + \\ + b_{i,n+1} U_n \quad (i = 1, 2, \dots, k-1, k+1, \dots, n+1),$$

where

$$(86) \quad b_{ij} = \frac{a_{ij}^*}{a} \quad (i, j = 1, 2, \dots, n+1),$$

a_{ij}^* being the cofactor of the element lying in the i -th row and the j -th column of the determinant (81), and \bar{V}_k in (83) or \bar{U}_k in (85) is an arbitrary function.

Proof. Necessity. By Theorem 8 the function $U_1 V_1 + U_2 V_2 + \dots + U_n V_n$ is of rank n . Then by the same theorem and (79) the functions $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{n+1}$ or the functions $\bar{V}_1, \bar{V}_2, \dots, \bar{V}_{n+1}$ are linearly dependent. Owing to the symmetry of those two cases it is sufficient to prove the theorem when the functions $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{n+1}$ are linearly dependent.

In this case there exist such an index k ($1 \leq k \leq n+1$) and such numbers c_1, c_2, \dots, c_n that

$$(87) \quad \bar{U}_k \equiv c_1 \bar{U}_1 + c_2 \bar{U}_2 + \dots + c_{k-1} \bar{U}_{k-1} + c_k \bar{U}_{k+1} + \dots + c_n \bar{U}_{n+1}.$$

If the identity (79) holds, we substitute (87) in it and after a rearrangement we have

$$(88) \quad U_1 V_1 + U_2 V_2 + \dots + U_n V_n \equiv \bar{U}_1 (\bar{V}_1 + c_1 \bar{V}_k) + \bar{U}_2 (\bar{V}_2 + c_2 \bar{V}_k) + \dots + \bar{U}_{k-1} (\bar{V}_{k-1} + c_{k-1} \bar{V}_k) + \bar{U}_{k+1} (\bar{V}_{k+1} + c_k \bar{V}_k) + \dots + \bar{U}_{n+1} (\bar{V}_{n+1} + c_n \bar{V}_k).$$

It follows by Theorem 9 that there exists a matrix of numerical coefficients

$$(89) \quad \mathcal{A}_0 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{k-1,1} & a_{k-1,2} & \dots & a_{k-1,n} \\ a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,n} \\ \dots & \dots & \dots & \dots \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,n} \end{bmatrix}$$

with the determinant

$$(90) \quad a_0 = |\mathcal{A}_0| \neq 0$$

such that the identities (82) hold for $i = 1, 2, \dots, k-1, k+1, \dots, n+1$. Substituting them in (87) we obtain moreover the identity (82) for $i = k$, where

$$a_{kj} = a_{1j} c_1 + a_{2j} c_2 + \dots + a_{k-1,j} c_{k-1} + a_{k+1,j} c_k + \dots + a_{n+1,j} c_n \quad (j = 1, 2, \dots, n).$$

Introducing the notations

$$\mathfrak{U} \equiv \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}, \quad \bar{\mathfrak{U}} \equiv \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \vdots \\ \bar{U}_n \\ \bar{U}_{n+1} \end{bmatrix}, \quad \mathfrak{V} \equiv [V_1 \ V_2 \ \dots \ V_n],$$

$$\bar{\mathfrak{V}} \equiv [\bar{V}_1 \ \bar{V}_2 \ \dots \ \bar{V}_n \ \bar{V}_{n+1}],$$

$$\mathfrak{V}_1 \equiv [V_1 \ V_2 \ \dots \ V_{k-1} \ \bar{V}_k \ V_k \ \dots \ V_n],$$

(91)

$$\mathcal{A}_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,n} \end{bmatrix}$$

we are now in a position to rewrite the identity (79) in the form

$$(92) \quad \mathfrak{B}\mathfrak{U} \equiv \mathfrak{B}\bar{\mathfrak{U}}$$

and the identities (82) in the form

$$(93) \quad \bar{\mathfrak{U}} \equiv \mathfrak{U}_1\mathfrak{U}.$$

Substituting (93) in (92) we obtain

$$(94) \quad \mathfrak{B}\mathfrak{U}_1\mathfrak{U} \equiv \mathfrak{B}\mathfrak{U}.$$

Since the functions U_1, U_2, \dots, U_n are — by our assumption — linearly independent, then (94) implies

$$(95) \quad \mathfrak{B}\mathfrak{U}_1 \equiv \mathfrak{B}.$$

Hence

$$(96) \quad \bar{\mathfrak{B}}\mathfrak{U} \equiv \mathfrak{B}_1,$$

where \mathfrak{U} is the matrix (80) obtained by augmenting \mathfrak{U}_1 with the column consisting of zeros except the k -th element, which is 1. Then by (90) follows (81) and we obtain from (96)

$$(97) \quad \bar{\mathfrak{B}} \equiv \mathfrak{B}_1\mathfrak{U}^{-1},$$

\mathfrak{U}^{-1} being the inverse of the matrix \mathfrak{U} . The identity (97) implies the identities (83).

Thus the necessity of our condition is established.

Sufficiency. If there exist an index k and a matrix (80) with the determinant (81) such that the identities (82) and (83) hold, we complete them with the identity

$$\bar{V}_k \equiv b_{k1}V_1 + b_{k2}V_2 + \dots + b_{k,k-1}V_{k-1} + b_{kk}\bar{V}_k + b_{k,k+1}V_{k+1} + \dots + b_{k+1,n}V_n,$$

where $b_{km} = 0$ for $k \neq m$ and $b_{kk} = 1$. Now we are in a position to write them in the forms (93) and (97) respectively. The identity (97) implies (96), next (95), and (94). Substituting (93) in (94) we obtain the identity (92), i. e. the identity (79).

This completes the proof of our theorem.

COROLLARY 10.1. *The function $G \equiv U_1V_1$ of rank 1 can be written in the form*

$$(98) \quad G \equiv \bar{U}_1\bar{V}_1 + \bar{U}_2\bar{V}_2$$

if and only if there exist such two numbers a_1, a_2 , at least one of them differing from zero, that

$$(99) \quad \bar{U}_1 \equiv a_1U_1, \quad \bar{U}_2 \equiv a_2U_1, \quad a_1\bar{V}_1 + a_2\bar{V}_2 \equiv V_1,$$

or

$$(100) \quad \bar{V}_1 \equiv a_1 V_1, \quad \bar{V}_2 \equiv a_2 V_1, \quad a_1 \bar{U}_1 + a_2 \bar{U}_2 \equiv U_1.$$

This follows from (93) and (95) by Theorem 10 in the case $n = 1$.

We shall write the condition that at least one of the numbers a_1, a_2 differs from zero in the form

$$a_1^2 + a_2^2 > 0.$$

COROLLARY 10.2. *The function*

$$(101) \quad G \equiv U_1 V_1 + U_2 V_2$$

of rank 2 can be written in the form

$$(102) \quad G \equiv \bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2 + \bar{U}_3 \bar{V}_3,$$

if and only if there exist such numbers $a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}$ satisfying the condition

$$(103) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

that one of the six following cases occurs:

$$(104) \quad \begin{aligned} \bar{U}_p &\equiv a_{11} U_1 + a_{12} U_2, & \bar{U}_q &\equiv a_{21} U_1 + a_{22} U_2, & \bar{U}_r &\equiv a_{31} U_1 + a_{32} U_2, \\ \bar{V}_p &\equiv \frac{a_{22}}{a} V_1 - \frac{a_{21}}{a} V_2 - \frac{a_{22} a_{31} - a_{21} a_{32}}{a} \bar{V}_r, \\ \bar{V}_q &\equiv -\frac{a_{12}}{a} V_1 + \frac{a_{11}}{a} V_2 + \frac{a_{12} a_{31} - a_{11} a_{32}}{a} \bar{V}_r \end{aligned}$$

(the function \bar{V}_r being arbitrary), or

$$(105) \quad \begin{aligned} \bar{V}_p &\equiv a_{11} V_1 + a_{12} V_2, & \bar{V}_q &\equiv a_{21} V_1 + a_{22} V_2, & \bar{V}_r &\equiv a_{31} V_1 + a_{32} V_2, \\ \bar{U}_p &\equiv \frac{a_{22}}{a} U_1 - \frac{a_{21}}{a} U_2 - \frac{a_{22} a_{31} - a_{21} a_{32}}{a} \bar{U}_r, \\ \bar{U}_q &\equiv -\frac{a_{12}}{a} U_1 + \frac{a_{11}}{a} U_2 + \frac{a_{12} a_{31} - a_{11} a_{32}}{a} \bar{U}_r \end{aligned}$$

(the function \bar{U}_r being arbitrary), where

$$p = 1, q = 2, r = 3, \quad \text{or} \quad p = 2, q = 3, r = 1,$$

$$\text{or} \quad p = 3, q = 1, r = 2.$$

This follows from Theorem 10, because in the case of $n = 2$ and $k = 3$ the matrix (80) has the form

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}, \quad a = |\mathfrak{A}| \neq 0,$$

and hence by (86)

$$b_{11} = \frac{a_{22}}{a}, \quad b_{12} = -\frac{a_{21}}{a}, \quad b_{13} = \frac{a_{21}a_{32} - a_{22}a_{31}}{a},$$

$$b_{21} = -\frac{a_{12}}{a}, \quad b_{22} = \frac{a_{11}}{a}, \quad b_{23} = \frac{a_{12}a_{31} - a_{11}a_{32}}{a}.$$

III. NOMOGRAPHIC FUNCTIONS

Let Ω be the Cartesian product of three arbitrary non-empty sets Ω_x , Ω_y and Ω_z . Let $F(x, y, z)$, where $x \in \Omega_x$, $y \in \Omega_y$, $z \in \Omega_z$, be a function defined on the set Ω and taking numerical values.

DEFINITION 3. The function $F(x, y, z)$ is said to be of rank n with respect to x if and only if, when considered as a function of two variables x and (y, z) , it is of rank n . Similarly, the function $F(x, y, z)$ is said to be of rank n with respect to y (or z) if and only if, when considered as a function of two variables y and (x, z) (or z and (x, y)), it is of rank n .

We define analogously the function of rank greater than n with respect to x , y or z .

DEFINITION 4. The function $F \equiv F(x, y, z)$ is to be called *nomographic* if and only if

1° there exist such functions

$$X_i \equiv X_i(x), \quad Y_i \equiv Y_i(y), \quad Z_i \equiv Z_i(z) \\ (i = 1, 2, 3; x \in \Omega_x, y \in \Omega_y, z \in \Omega_z)$$

with numerical values that

$$(106) \quad F \equiv \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix},$$

2° the function F is of rank greater than 1 with respect to each of the variables x, y, z .

In order to explain the second of those conditions let us suppose that the function F is of rank 1 with respect to x . There exist then such functions $X(x)$ and $K(y, z)$ that $F \equiv X(x)K(y, z)$. But then the equation $F = 0$, for which a nomogram would be required, might be split into two trivial equations

$$X(x) = 0 \quad \text{and} \quad K(y, z) = 0$$

for which there is no need of nomograms⁽¹⁾.

⁽¹⁾ For the completeness of our considerations some remarks concerning the case when condition 2° fails will be given in Chapter VII.

Every determinant of the type (106) will be called a **Massau form** of the function F .

DEFINITION 5. Two Massau forms of a function F

$$F \equiv \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix} \equiv \begin{vmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{vmatrix}$$

will be called *equivalent* if and only if there exists a matrix of numbers

$$\mathcal{U} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with the determinant

$$a = |\mathcal{U}| \neq 0$$

and two numbers d_1, d_2 satisfying the condition

$$ad_1d_2 = 1$$

such that

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \equiv \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\ d_1\bar{Y}_1 & d_1\bar{Y}_2 & d_1\bar{Y}_3 \\ d_2\bar{Z}_1 & d_2\bar{Z}_2 & d_2\bar{Z}_3 \end{bmatrix} \cdot \mathcal{U},$$

that is,

$$\begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & X_3 \\ \frac{1}{d_1}Y_1 & \frac{1}{d_1}Y_2 & \frac{1}{d_1}Y_3 \\ \frac{1}{d_2}Z_1 & \frac{1}{d_2}Z_2 & \frac{1}{d_2}Z_3 \end{bmatrix} \cdot \mathcal{U}^{-1}.$$

where \mathcal{U}^{-1} is the inverse of the matrix \mathcal{U} .

If each of two Massau forms of a given function F is equivalent to a third, then they are equivalent one to another. We see that the equivalence of Massau forms is a true equivalence relation.

If all the Massau forms of a nomographic function are equivalent in pairs, we shall call it a *uniquely nomographic function*. If a nomographic function has exactly two non-equivalent Massau forms, we shall call it *doubly nomographic*. If a nomographic function has exactly $k > 2$ non-equivalent Massau forms, we shall call it *k-nomographic*.

The equivalence of two Massau forms of a function $F(x, y, z)$ has a simple geometrical interpretation. It means, namely, that each of the two nomograms for the equation $F(x, y, z) = 0$ which correspond to those Massau forms of $F(x, y, z)$ can be obtained from the other by a projective transformation. It does not occur if the corresponding Massau forms are non-equivalent.

LEMMA 3. *If in the form (106) two columns are interchanged and the signs in one of the rows are replaced by the opposite ones, then the resulting Massau form is equivalent to the form (106).*

Proof. Let us interchange, for example, the second and the third column in (106) and let us change the signs in the second row. Then

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ -Y_1 & -Y_3 & -Y_2 \\ Z_1 & Z_3 & Z_2 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & X_3 \\ -1 \cdot Y_1 & -1 \cdot Y_2 & -1 \cdot Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We see that the resulting Massau form is equivalent to the former one. There are nine possible cases and for each of them the proof is analogous.

LEMMA 4. *If to one of the columns of the form (106) a linear combination of the remaining ones is added, then the resulting Massau form is equivalent to the form (106).*

Proof. Let us add to the first column of the form (106) the second one multiplied by a and the third one multiplied by b . We then obtain

$$\begin{bmatrix} X_1 + aX_2 + bX_3 & X_2 & X_3 \\ Y_1 + aY_2 + bY_3 & Y_2 & Y_3 \\ Z_1 + aZ_2 + bZ_3 & Z_2 & Z_3 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}.$$

We see that the resulting Massau form is equivalent to the former one. There are three possible cases and in each of them the proof is analogous.

LEMMA 5. *If one of the rows in the form (106) is multiplied by $a \neq 0$ and one of the columns multiplied by $1/a$, then the resulting Massau form is equivalent to the form (106).*

Proof. Let us multiply the first row of the determinant (106) by $a \neq 0$ and the second column by $1/a$. We then obtain

$$\begin{bmatrix} aX_1 & X_2 & aX_3 \\ Y_1 & \frac{1}{a}Y_2 & Y_3 \\ Z_1 & \frac{1}{a}Z_2 & Z_3 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & X_3 \\ \frac{1}{a}Y_1 & \frac{1}{a}Y_2 & \frac{1}{a}Y_3 \\ \frac{1}{a}Z_1 & \frac{1}{a}Z_2 & \frac{1}{a}Z_3 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}.$$

We see that the resulting Massau form is equivalent to the former one. There are nine possible cases and in each of them the proof is analogous.

LEMMA 6. *If one of the rows (or columns) in the form (106) is multiplied by $a \neq 0$ and another by $1/a$, then the resulting Massau form is equivalent to the form (106).*

Proof. Let us multiply the first row of the determinant (106) by $a \neq 0$ and the second by $1/a$. We then obtain

$$\begin{bmatrix} aX_1 & aX_2 & aX_3 \\ \frac{1}{a} Y_1 & \frac{1}{a} Y_2 & \frac{1}{a} Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & X_3 \\ \frac{1}{a^2} Y_1 & \frac{1}{a^2} Y_2 & \frac{1}{a^2} Y_3 \\ \frac{1}{a} Z_1 & \frac{1}{a} Z_2 & \frac{1}{a} Z_3 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}.$$

We see that the resulting Massau form is equivalent to the former one. There are six possible cases and in each of them the proof is analogous.

THEOREM 11. *If the function of rank 2 with respect to x*

$$(107) \quad F \equiv X_1 G_1 + X_2 G_2,$$

where the functions

$$X_i \equiv X_i(x), \quad G_i \equiv G_i(y, z) \quad (i = 1, 2; x \in \Omega_x, y \in \Omega_y, z \in \Omega_z)$$

take numerical values, is nomographic, then there exists for each of its Massau forms an equivalent form

$$(108) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{vmatrix},$$

in which the functions X_1 and X_2 are the same as in (107) and the functions

$$\bar{Y}_i \equiv \bar{Y}_i(y), \quad \bar{Z}_i \equiv \bar{Z}_i(z) \quad (i = 1, 2, 3; y \in \Omega_y, z \in \Omega_z),$$

taking numerical values, satisfy the identities

$$(109) \quad G_1 \equiv \bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2, \quad G_2 \equiv \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3.$$

Proof. If the function F is nomographic, then there exist such functions

$$\begin{aligned} \bar{\bar{X}}_i &\equiv \bar{\bar{X}}_i(x), & \bar{\bar{Y}}_i &\equiv \bar{\bar{Y}}_i(y), & \bar{\bar{Z}}_i &\equiv \bar{\bar{Z}}_i(z) \\ & & & & & (i = 1, 2, 3; x \in \Omega_x, y \in \Omega_y, z \in \Omega_z), \end{aligned}$$

taking numerical values, that

$$(110) \quad F \equiv \begin{bmatrix} \bar{\bar{X}}_1 & \bar{\bar{X}}_2 & \bar{\bar{X}}_3 \\ \bar{\bar{Y}}_1 & \bar{\bar{Y}}_2 & \bar{\bar{Y}}_3 \\ \bar{\bar{Z}}_1 & \bar{\bar{Z}}_2 & \bar{\bar{Z}}_3 \end{bmatrix},$$

and

$$(111) \quad F \equiv \bar{\bar{X}}_1 \bar{\bar{G}}_1 + \bar{\bar{X}}_2 \bar{\bar{G}}_2 + \bar{\bar{X}}_3 \bar{\bar{G}}_3,$$

where

$$(112) \quad \bar{\bar{G}}_1 \equiv \bar{\bar{Y}}_2 \bar{\bar{Z}}_3 - \bar{\bar{Y}}_3 \bar{\bar{Z}}_2, \quad \bar{\bar{G}}_2 \equiv \bar{\bar{Y}}_3 \bar{\bar{Z}}_1 - \bar{\bar{Y}}_1 \bar{\bar{Z}}_3, \quad \bar{\bar{G}}_3 \equiv \bar{\bar{Y}}_1 \bar{\bar{Z}}_2 - \bar{\bar{Y}}_2 \bar{\bar{Z}}_1.$$

It follows by Corollary 10.2 that there exists such a non-singular matrix

$$\mathfrak{U} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}, \quad a = |\mathfrak{U}| \neq 0,$$

that

$$(113) \quad \begin{bmatrix} \bar{\bar{X}}_p \\ \bar{\bar{X}}_q \\ \bar{\bar{X}}_r \end{bmatrix} \equiv \mathfrak{U} \cdot \begin{bmatrix} X_1 \\ X_2 \\ 0 \end{bmatrix},$$

or

$$(114) \quad \begin{bmatrix} \bar{\bar{G}}_p \\ \bar{\bar{G}}_q \\ \bar{\bar{G}}_r \end{bmatrix} \equiv \mathfrak{U} \cdot \begin{bmatrix} G_1 \\ G_2 \\ 0 \end{bmatrix},$$

where

$$p = 1, q = 2, r = 3, \quad \text{or} \quad p = 2, q = 3, r = 1, \\ \text{or} \quad p = 3, q = 1, r = 2.$$

If the identity (113) holds, we introduce the matrix

$$(115) \quad \begin{bmatrix} X_1 & X_2 & 0 \\ \bar{\bar{Y}}_1 & \bar{\bar{Y}}_2 & \bar{\bar{Y}}_3 \\ \bar{\bar{Z}}_1 & \bar{\bar{Z}}_2 & \bar{\bar{Z}}_3 \end{bmatrix} \equiv \begin{bmatrix} \bar{\bar{X}}_p & \bar{\bar{X}}_q & \bar{\bar{X}}_r \\ a \bar{\bar{Y}}_p & a \bar{\bar{Y}}_q & a \bar{\bar{Y}}_r \\ \bar{\bar{Z}}_p & \bar{\bar{Z}}_q & \bar{\bar{Z}}_r \end{bmatrix} (\mathfrak{U}^{-1})^*,$$

$(\mathfrak{U}^{-1})^*$ being the transpose of the inverse of the matrix \mathfrak{U} . Its determinant is the desired Massau form equivalent to (110), since by Lemma 3 the Massau forms

$$\begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{X}_p & \bar{X}_q & \bar{X}_r \\ \bar{Y}_p & \bar{Y}_q & \bar{Y}_r \\ \bar{Z}_p & \bar{Z}_q & \bar{Z}_r \end{bmatrix}$$

are equivalent and by (115) the forms

$$\begin{bmatrix} X_1 & X_2 & 0 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{X}_p & \bar{X}_q & \bar{X}_r \\ \bar{Y}_p & \bar{Y}_q & \bar{Y}_r \\ \bar{Z}_p & \bar{Z}_q & \bar{Z}_r \end{bmatrix}$$

are also equivalent. Furthermore, by (107) and Theorem 8 the functions X_1 and X_2 are linearly independent and, therefore, the identities (107) and (108) imply the identities (109).

If the identity (114) held we should introduce the matrix

$$(116) \quad \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \tilde{X}_3 \\ \tilde{Y}_1 & \tilde{Y}_2 & \tilde{Y}_3 \\ \tilde{Z}_1 & \tilde{Z}_2 & \tilde{Z}_3 \end{bmatrix} \equiv \begin{bmatrix} \bar{X}_p & \bar{X}_q & \bar{X}_r \\ \frac{1}{a} \bar{Y}_p & \frac{1}{a} \bar{Y}_q & \frac{1}{a} \bar{Y}_r \\ \bar{Z}_p & \bar{Z}_q & \bar{Z}_r \end{bmatrix} \cdot \mathfrak{A}$$

and, therefore, by (110) we should have

$$(117) \quad F' \equiv \begin{vmatrix} \tilde{X}_1 & \tilde{X}_2 & \tilde{X}_3 \\ \tilde{Y}_1 & \tilde{Y}_2 & \tilde{Y}_3 \\ \tilde{Z}_1 & \tilde{Z}_2 & \tilde{Z}_3 \end{vmatrix},$$

the functions

$$\tilde{X}_i \equiv \tilde{X}_i(x), \quad \tilde{Y}_i \equiv \tilde{Y}_i(y), \quad \tilde{Z}_i \equiv \tilde{Z}_i(z) \\ (i = 1, 2, 3; x \in \Omega_x, y \in \Omega_y, z \in \Omega_z)$$

taking numerical values. Furthermore, we should obtain from (116)

$$\begin{bmatrix} \tilde{Y}_2 \tilde{Z}_3 - \tilde{Y}_3 \tilde{Z}_2 \\ \tilde{Y}_3 \tilde{Z}_1 - \tilde{Y}_1 \tilde{Z}_3 \\ \tilde{Y}_1 \tilde{Z}_2 - \tilde{Y}_2 \tilde{Z}_1 \end{bmatrix} \equiv \mathfrak{A}^{-1} \begin{bmatrix} \bar{G}_p \\ \bar{G}_q \\ \bar{G}_r \end{bmatrix}$$

and then by (114)

$$\begin{bmatrix} \tilde{Y}_2 \tilde{Z}_3 - \tilde{Y}_3 \tilde{Z}_2 \\ \tilde{Y}_3 \tilde{Z}_1 - \tilde{Y}_1 \tilde{Z}_3 \\ \tilde{Y}_1 \tilde{Z}_2 - \tilde{Y}_2 \tilde{Z}_1 \end{bmatrix} \equiv \begin{bmatrix} G_1 \\ G_2 \\ 0 \end{bmatrix}.$$

Thus we should have

$$(118) \quad \tilde{Y}_1 \tilde{Z}_2 - \tilde{Y}_2 \tilde{Z}_1 \equiv 0.$$

If $\tilde{Y}_1 \equiv 0$, then we should have $\tilde{Y}_2 \equiv 0$ or $\tilde{Z}_1 \equiv 0$. If $\tilde{Y}_1 \equiv \tilde{Y}_2 \equiv 0$, then by (117) the function F would be of rank 0 or 1 with respect to y , which would contradict the assumption that F is a nomographic function. Analogously, we could not have $\tilde{Y}_1 \equiv \tilde{Z}_1 \equiv 0$. Therefore, we should have $\tilde{Y}_1 \neq 0$ and, similarly, $\tilde{Y}_2 \neq 0$, $\tilde{Z}_1 \neq 0$, and $\tilde{Z}_2 \neq 0$. It would follow by (118) that the functions \tilde{Y}_1 , \tilde{Y}_2 and the functions \tilde{Z}_1 , \tilde{Z}_2 — separately treated — are linearly dependent and there would exist such a number m that $\tilde{Y}_2 \equiv m\tilde{Y}_1$, $\tilde{Z}_2 \equiv m\tilde{Z}_1$. But then we should have

$$F \equiv (m\tilde{X}_1 - \tilde{X}_2) \begin{vmatrix} \tilde{Y}_1 & \tilde{Y}_3 \\ \tilde{Z}_1 & \tilde{Z}_3 \end{vmatrix}$$

and the function F would be of rank 0 or 1 with respect to x , which would contradict the assumption that F is a nomographic function.

Thus, if F is a nomographic function, the identity (114) cannot occur. This completes the proof of Theorem 11.

THEOREM 12. *If the function of rank 3 with respect to x*

$$(119) \quad F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3,$$

where the functions

$$X_i \equiv X_i(x), \quad G_i \equiv G_i(y, z) \quad (i = 1, 2, 3; x \in \Omega_x, y \in \Omega_y, z \in \Omega_z)$$

take numerical values, is nomographic, then there exists for each of its Massau forms an equivalent form

$$(120) \quad F \equiv \begin{vmatrix} X_1 & X_2 & X_3 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{vmatrix},$$

in which the functions X_1 , X_2 , and X_3 are the same as in (119) and the functions

$$\bar{Y}_i \equiv \bar{Y}_i(y), \quad \bar{Z}_i \equiv \bar{Z}_i(z) \quad (i = 1, 2, 3; y \in \Omega_y, z \in \Omega_z),$$

taking numerical values, satisfy the identities

$$(121) \quad G_1 \equiv \bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2, \quad G_2 \equiv \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3, \quad G_3 \equiv \bar{Y}_1 \bar{Z}_2 - \bar{Y}_2 \bar{Z}_1.$$

Proof. If the function F is nomographic and an identity (110) holds then we have (111) with (112). It follows by Theorem 9 that there exists

such a non-singular matrix \mathfrak{U} with the determinant

$$a = |\mathfrak{U}| \neq 0$$

that

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \equiv \mathfrak{U} \cdot \begin{bmatrix} \bar{\bar{X}}_1 \\ \bar{\bar{X}}_2 \\ \bar{\bar{X}}_3 \end{bmatrix}.$$

We then introduce the matrix

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \\ \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 \end{bmatrix} \equiv \begin{bmatrix} \bar{\bar{X}}_1 & \bar{\bar{X}}_2 & \bar{\bar{X}}_3 \\ \frac{1}{a} \bar{\bar{Y}}_1 & \frac{1}{a} \bar{\bar{Y}}_2 & \frac{1}{a} \bar{\bar{Y}}_3 \\ \bar{\bar{Z}}_1 & \bar{\bar{Z}}_2 & \bar{\bar{Z}}_3 \end{bmatrix} \cdot \mathfrak{U}^*,$$

\mathfrak{U}^* being the transpose of the matrix \mathfrak{U} . Its determinant is the desired Massau form equivalent to (110). By (119) and Theorem 8 the functions X_1, X_2, X_3 are linearly independent and, therefore, the identities (119) and (120) imply the identities (121).

Thus, our theorem is proved.

THEOREM 13. *If the function F is nomographic, it is of rank 2 or 3 with respect to each of the variables x, y, z . Furthermore, if*

$$(122) \quad F \equiv X_1 G_1 + X_2 G_2$$

when it is of rank 2 with respect to x , and

$$(123) \quad F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3$$

when it is of rank 3 with respect to x , (the functions $X_i \equiv X_i(x), G_i \equiv G_i(y, z)$ ($i = 1, 2, 3; x \in \Omega_x, y \in \Omega_y, z \in \Omega_z$) taking numerical values), then each of the functions G_1, G_2, G_3 is of rank 1 or 2.

Proof. If the function F is nomographic, it has a Massau form (106) and, therefore, it can be written in the form

$$F \equiv X_1(Y_2 Z_3 - Y_3 Z_2) + X_2(Y_3 Z_1 - Y_1 Z_3) + X_3(Y_1 Z_2 - Y_2 Z_1).$$

Thus, F is a function of rank not greater than 3 with respect to x . Since by definition it is of rank greater than 1 with respect to x , it can be only of rank 2 or 3 with respect to x . We prove analogously that F is a function of rank 2 or 3 with respect to each of the variables y and z .

If the function F is given in the form (122), there exist by Theorem 11 such functions

$$(124) \quad \bar{Y}_i \equiv \bar{Y}_i(y), \quad \bar{Z}_i \equiv \bar{Z}_i(z) \quad (i = 1, 2, 3; y \in \Omega_y, z \in \Omega_z)$$

with numerical values, that the identities (109) hold. Similarly, if the function F is given in the form (123), there exist by Theorem 12 such functions (124) that the identities (121) hold. It follows from (109) and (121) that the functions G_1, G_2, G_3 are of rank not greater than 2. Since they cannot equal zero identically (because in the case (122) the function F is of rank 2 and in the case (123) of rank 3 with respect to x), they can only be of rank 1 or 2.

Thus our theorem is proved.

In order to reduce considerably the number of all cases which are to be considered, we now make the following three preliminary assumptions.

The first preliminary assumption. The rank r_x of the function $F \equiv F(x, y, z)$ with respect to x is not greater than the rank r_y with respect to y and the rank r_y is not greater than the rank r_z with respect to z , i. e.,

$$(125) \quad r_x \leq r_y \leq r_z.$$

The first preliminary assumption will not restrict the generality of our considerations, because for every function $F(x, y, z)$ we shall be able to change the variables in such a way that the inequality (125) will be satisfied.

The second preliminary assumption. If the function F is given in the form (122) or (123), the rank r_1 of the function G_1 is not less than the rank r_2 of the function G_2 and r_2 is not less than the rank r_3 of the function G_3 , i. e.,

$$(126) \quad r_3 \leq r_2 \leq r_1.$$

Also the second preliminary assumption will not restrict the generality, since a suitable rearrangement of the indices in the identities (122) or (123) will always be possible. (See Lemma 3.)

By Theorem 13 and the first preliminary assumption, for a nomographic function F only 4 cases, excluding one another, are possible:

- I. The function F is of rank 2 with respect to each of the variables x, y , and z .
- II. The function F is of rank 2 with respect to each of the variables x and y , but of rank 3 with respect to z .
- (127) III. The function F is of rank 2 with respect to x and of rank 3 with respect to each of the variables y and z .
- IV. The function F is of rank 3 with respect to each of the variables x, y and z .

By Theorem 13 and the second preliminary assumption, for a nomographic function F only 7 cases, excluding one another, are possible:

(128)

- | | | |
|------------------------------------|--------------------------|---------------------------------|
| 1. $F = X_1G_1 + X_2G_2,$ | $G_1 = Y_1Z_1,$ | $G_2 = Y_3Z_3,$ |
| 2. ,, | $G_1 = Y_1Z_1 + Y_2Z_2,$ | ,, |
| 3. ,, | ,, | $G_2 = Y_3Z_3 + Y_4Z_4,$ |
| 4. $F = X_1G_1 + X_2G_2 + X_3G_3.$ | $G_1 = Y_1Z_1,$ | $G_2 = Y_3Z_3,$ $G_3 = Y_5Z_5.$ |
| 5. ,, | $G_1 = Y_1Z_1 + Y_2Z_2,$ | ,, ,, |
| 6. ,, | ,, | $G_2 = Y_3Z_3 + Y_4Z_4,$,, |
| 7. ,, | ,, | ,, $G_3 = Y_5Z_5 + Y_6Z_6.$ |

where the functions

$$(129) \quad Y_i = Y_i(y) \neq 0, \quad Z_i = Z_i(z) \neq 0$$

$$(i = 1, 2, \dots, 6; y \in \Omega_y, z \in \Omega_z)$$

take numerical values and, by Theorem 8, the functions $Y_1, Y_2,$ the functions $Y_3, Y_4,$ the functions $Y_5, Y_6,$ the functions $Z_1, Z_2,$ the functions Z_3, Z_4 and the functions Z_5, Z_6 — separately treated — are linearly independent; moreover, in the first three cases (128) the functions X_1, X_2 and the functions $G_1, G_2,$ and in the last four cases (128) the functions X_1, X_2, X_3 and the functions G_1, G_2, G_3 — separately treated — are linearly independent.

The third preliminary assumption.

In the second case (128):

The functions Y_1, Y_3 may be linearly dependent only if the functions Y_2, Y_3 are also linearly dependent.

In the third case (128):

1° The functions Y_1, Y_2, Y_4 may be linearly independent only if the functions Y_1, Y_2, Y_3 are also linearly independent.

2° If the functions Y_1, Y_2, Y_3 are linearly independent, then the functions Y_2, Y_3, Y_4 may be linearly dependent only if the functions Y_1, Y_3, Y_4 are also linearly dependent.

3° If the functions Y_1, Y_2, Y_3 and the functions Y_1, Y_2, Y_4 — separately treated — are linearly dependent, then the functions Z_1, Z_2, Z_3 may be linearly independent only if the functions Z_1, Z_2, Z_4 are also linearly independent.

4° If the functions Y_1, Y_2, Y_3 and the functions Y_1, Y_2, Y_4 — separately treated — are linearly dependent and the functions Z_1, Z_2, Z_4 linearly independent, then the functions Z_1, Z_3, Z_4 may be linearly dependent only if the functions Z_2, Z_3, Z_4 are also linearly dependent.

In the seventh case (128):

1° The functions Y_1, Y_2, Y_m ($m = 4, 5, \text{ or } 6$) may be linearly independent only if the functions Y_1, Y_2, Y_3 are also linearly independent.

2° If the functions Y_1, Y_2, Y_3 are linearly independent, then the functions Y_2, Y_3, Y_4 may be linearly dependent only if the functions Y_1, Y_3, Y_4 are also linearly dependent.

3° If the functions Y_1, Y_2, Y_3 are linearly independent, then the functions Y_1, Y_2, Y_6 may be linearly dependent only if the functions Y_1, Y_2, Y_5 are also linearly dependent.

The third preliminary assumption, like the first two preliminary assumptions will not restrict the generality of our considerations, because a suitable rearrangement of indices in the identities (128) will always be possible.

IV. TWO FUNDAMENTAL THEOREMS

We shall retain the notations used in Theorem 11, Theorem 12, and the three preliminary assumptions, and we shall prove two theorems which will be called *fundamental*.

We shall first distinguish the following seven cases, excluding one another, which will be called *principal*:

Principal cases.

I. The function F is of rank 2 with respect to each of the variables x, y and z .

$$F \equiv X_1 G_1 + X_2 G_2.$$

The functions G_1 and G_2 are both of rank 1.

II. The function F is of rank 2 with respect to each of the variables x, y and z .

$$F \equiv X_1 G_1 + X_2 G_2.$$

The function G_1 is of rank 2 and the function G_2 of rank 1.

III. The function F is of rank 2 with respect to each of the variables x, y and z .

$$F \equiv X_1 G_1 + X_2 G_2.$$

The functions G_1 and G_2 are both of rank 2.

$$G_1 \equiv Y_1 Z_1 + Y_2 Z_2, \quad G_2 \equiv Y_3 Z_3 + Y_4 Z_4.$$

$$(130) \quad Y_3 \equiv m_{31} Y_1 + m_{32} Y_2, \quad Z_3 \equiv n_{31} Z_1 + n_{32} Z_2,$$

$$Y_4 \equiv m_{41} Y_1 + m_{42} Y_2, \quad Z_4 \equiv n_{41} Z_1 + n_{42} Z_2^{(1)}.$$

$$(131) \quad (m_{31} n_{31} - m_{32} n_{32} + m_{41} n_{41} - m_{42} n_{42})^2 +$$

$$+ 4(m_{32} n_{31} + m_{42} n_{41})(m_{31} n_{32} + m_{41} n_{42}) \geq 0.$$

IV. The function F is of rank 2 with respect to each of the variables x and y , but of rank 3 with respect to z .

$$F \equiv X_1 G_1 + X_2 G_2.$$

The function G_1 is of rank 2 and the function G_2 is of rank 1.

(¹) Here and further on the coefficients m_{ij} and n_{ij} ($i = 1, 2, \dots, 6; j = 1, 2, 3, 4$) are numbers.

V. The function F is of rank 2 with respect to each of the variables x and y , but of rank 3 with respect to z .

$$F \equiv X_1 G_1 + X_2 G_2.$$

The functions G_1 and G_2 are both of rank 2.

$$G_1 \equiv Y_1 Z_1 + Y_2 Z_2, \quad G_2 \equiv Y_3 Z_3 + Y_4 Z_4.$$

The functions Z_1, Z_2, Z_4 are linearly independent.

$$(132) \quad \begin{aligned} Y_3 &\equiv m_{31} Y_1 + m_{32} Y_2, & Z_3 &\equiv n_{31} Z_1 + n_{32} Z_2 + n_{34} Z_4, \\ Y_4 &\equiv m_{41} Y_1 + m_{42} Y_2, \end{aligned}$$

$$(133) \quad n_{31} = -\frac{m_{32} n_{34} + m_{42}}{m_{31} n_{34} + m_{41}} n_{32} \quad (m_{31} n_{34} + m_{41} \neq 0).$$

VI. The function F is of rank 2 with respect to x and of rank 3 with respect to each of the variables y and z .

$$F \equiv X_1 G_1 + X_2 G_2.$$

The functions G_1 and G_2 are both of rank 2.

$$G_1 \equiv Y_1 Z_1 + Y_2 Z_2, \quad G_2 \equiv Y_3 Z_3 + Y_4 Z_4.$$

The functions Y_1, Y_2, Y_3 and the functions Z_1, Z_2, Z_4 — separately treated — are linearly independent.

$$(134) \quad Y_4 \equiv m_{41} Y_1 + m_{42} Y_2 + m_{43} Y_3, \quad Z_3 \equiv n_{31} Z_1 + n_{32} Z_2 + n_{34} Z_4.$$

$$(135) \quad n_{31} = -\frac{m_{42}}{m_{41}} n_{32} \quad (m_{41} \neq 0),$$

$$n_{34} = -m_{43}.$$

VII. The function F is of rank 3 with respect to each of the variables x, y and z .

$$F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3.$$

Each of the functions G_1, G_2, G_3 is of rank 2.

$$G_1 \equiv Y_1 Z_1 + Y_2 Z_2, \quad G_2 \equiv Y_3 Z_3 + Y_4 Z_4, \quad G_3 \equiv Y_5 Z_5 + Y_6 Z_6.$$

The functions Y_1, Y_2, Y_3 and the functions Z_1, Z_2, Z_4 — separately treated — are linearly independent.

$$(136) \quad \begin{aligned} Y_4 &\equiv m_{41} Y_1 + m_{42} Y_2 + m_{43} Y_3, & Z_3 &\equiv n_{31} Z_1 + n_{32} Z_2 + n_{34} Z_4, \\ Y_5 &\equiv m_{51} Y_1 + m_{52} Y_2 + m_{53} Y_3, & Z_5 &\equiv n_{51} Z_1 + n_{52} Z_2 + n_{54} Z_4, \\ Y_6 &\equiv m_{61} Y_1 + m_{62} Y_2 + m_{63} Y_3, & Z_6 &\equiv n_{61} Z_1 + n_{62} Z_2 + n_{64} Z_4. \end{aligned}$$

$$\begin{aligned}
 n_{31} &= -\frac{m_{42}}{m_{41}} n_{32} \quad (m_{41} \neq 0), \\
 n_{34} &= -m_{43}, \\
 n_{51} &= \frac{n_{32} n_{54}}{m_{41} m_{63}} m_{62} \quad (m_{63} \neq 0), \\
 n_{52} &= -\frac{n_{32} n_{54}}{m_{41} m_{63}} m_{61}, \\
 n_{61} &= -\frac{n_{32} n_{54}}{m_{41} m_{63}} m_{52}, \\
 n_{62} &= \frac{n_{32} n_{54}}{m_{41} m_{63}} m_{51}, \\
 n_{64} &= -\frac{m_{53}}{m_{63}} n_{54}.
 \end{aligned}
 \tag{137}$$

We shall also distinguish the following Massau forms, which we shall call principal:

Principal Massau forms:
 For the first principal case:

$$\begin{aligned}
 (138) \quad & \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & 0 & Z_1 \end{array} \right| \quad \text{and} \quad \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ Y_3 & 0 & Y_1 \\ 0 & -Z_1 & -Z_3 \end{array} \right|.
 \end{aligned}$$

For the second principal case ($Y_3 = m_{31} Y_1 + m_{32} Y_2, Z_3 = n_{31} Z_1 + n_{32} Z_2$):

a) If $m_{31} n_{31} + m_{32} n_{32} \neq 0, n_{32} \neq 0$:

$$\begin{aligned}
 (139) \quad & \left\{ \begin{array}{l} \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & \frac{m_{31}}{m_{32}} Y_1 + Y_2 \\ m_{32} n_{31} Z_1 + m_{32} n_{32} Z_2 & -Z_2 & Z_1 - \frac{m_{31}}{m_{32}} Z_2 \end{array} \right| \\ \text{and} \\ \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ m_{31} n_{32} Y_1 + m_{32} n_{32} Y_2 & -Y_2 & Y_1 - \frac{n_{31}}{n_{32}} Y_2 \\ 0 & -Z_1 & -\frac{n_{31}}{n_{32}} Z_1 - Z_2 \end{array} \right| \end{array} \right. \quad (m_{32} \neq 0).
 \end{aligned}$$

b) If $m_{31}n_{31} + m_{32}n_{32} \neq 0$, $n_{32} = 0$:

$$(140) \quad \left\{ \begin{array}{l} \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & \frac{m_{31}}{m_{32}} Y_1 + Y_2 \\ m_{32}n_{31}Z_1 + m_{32}n_{32}Z_2 & -Z_2 & Z_1 - \frac{m_{31}}{m_{32}} Z_2 \end{array} \right| \\ \text{and} \\ \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ m_{31}n_{31}Y_1 + m_{32}n_{31}Y_2 & -Y_1 & -Y_2 \\ 0 & Z_2 & -Z_1 \end{array} \right| \quad (m_{32} \neq 0). \end{array} \right.$$

c) If $m_{31}n_{31} + m_{32}n_{32} = 0$:

$$(141) \quad \left\{ \begin{array}{l} \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & \frac{m_{31}}{m_{32}} Y_1 + Y_2 \\ m_{32}n_{31}Z_1 + m_{32}n_{32}Z_2 & -Z_2 & Z_1 - \frac{m_{31}}{m_{32}} Z_2 \end{array} \right| \quad (m_{32} \neq 0). \end{array} \right.$$

For the third principal case:

a) If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} \neq 0$, where

$$(142) \quad \begin{aligned} r_{31} &= m_{31}n_{31} + m_{41}n_{41}, & r_{32} &= m_{32}n_{31} + m_{42}n_{41}, \\ r_{41} &= m_{31}n_{32} + m_{41}n_{42}, & r_{42} &= m_{32}n_{32} + m_{42}n_{42}, \end{aligned}$$

and if $r_{32} \neq 0$:

$$(143) \quad \left\{ \begin{array}{l} \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ r_{31}Y_1 + r_{32}Y_2 & -Y_1 & \frac{p_1 - r_{42}}{r_{32}} Y_1 + Y_2 \\ p_1Z_2 & -Z_2 & \frac{p_1 - r_{42}}{r_{32}} Z_2 - Z_1 \end{array} \right| \\ \text{and} \\ \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ r_{31}Y_1 + r_{32}Y_2 & -Y_1 & \frac{p_2 - r_{42}}{r_{32}} Y_1 + Y_2 \\ p_2Z_2 & -Z_2 & \frac{p_2 - r_{42}}{r_{32}} Z_2 - Z_1 \end{array} \right|, \end{array} \right.$$

where p_1 and p_2 are the two different solutions of the equation in p :

$$(144) \quad p^2 - (r_{31} + r_{42})p + (r_{31}r_{42} - r_{32}r_{41}) = 0.$$

b) If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} \neq 0$ and $r_{32} \neq 0$:

$$(145) \quad \left\{ \begin{array}{l} \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ r_{41}Y_1 + r_{42}Y_2 & -Y_2 & r_{31}Y_1 \\ Z_1 & -\frac{1}{r_{31}}Z_1 & -Z_2 \end{array} \right| \\ \text{and} \\ \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ r_{31}Y_1 - Y_1 & \frac{r_{41}}{r_{42} - r_{31}}Y_1 + Y_2 \\ r_{42}Z_2 - Z_2 & \frac{r_{41}}{r_{42} - r_{31}}Z_2 - Z_1 \end{array} \right| \end{array} \right. \quad (r_{31}(r_{42} - r_{31}) \neq 0).$$

c) If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} = 0$ and $r_{32} \neq 0$:

$$(146) \quad \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ r_{31}Y_1 + r_{32}Y_2 & -Y_1 & \frac{r_{31} - r_{42}}{2r_{32}}Y_1 + Y_2 \\ \frac{r_{31} + r_{42}}{2}Z_2 - Z_2 & -Z_2 & \frac{r_{31} - r_{42}}{2r_{32}}Z_2 - Z_1 \end{array} \right|.$$

d) If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} = 0$ and $r_{32} = 0$:

$$(147) \quad \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ r_{41}Y_1 + r_{42}Y_2 & -Y_2 & r_{31}Y_1 \\ Z_1 & -\frac{1}{r_{31}}Z_1 & -Z_2 \end{array} \right| \quad (r_{31} \neq 0).$$

For the fourth principal case ($Y_3 = m_{31}Y_1 + m_{32}Y_2$):

$$(148) \quad \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & \frac{m_{31}}{m_{32}}Y_1 + Y_2 \\ m_{32}Z_3 - Z_2 & -Z_2 & Z_1 - \frac{m_{31}}{m_{32}}Z_2 \end{array} \right| \quad (m_{32} \neq 0).$$

For the fifth principal case:

$$(149) \quad \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ m_{31}n_{32}Y_1 + m_{32}n_{32}Y_2 & -Y_2 & Y_1 + \frac{m_{32}n_{34} + m_{42}}{m_{31}n_{34} + m_{41}}Y_2 \\ (m_{31}n_{34} + m_{41})Z_4 & -Z_1 & \frac{m_{32}n_{34} + m_{42}}{m_{31}n_{34} + m_{41}}Z_1 - Z_2 \end{array} \right| \quad (m_{31}n_{34} + m_{41} \neq 0).$$

For the sixth principal case:

$$(150) \quad \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ n_{32}Y_3 & -Y_2 & Y_1 + \frac{m_{42}}{m_{41}}Y_2 \\ m_{41}Z_4 & -Z_1 & \frac{m_{42}}{m_{41}}Z_1 - Z_2 \end{array} \right| \quad (m_{41} \neq 0).$$

For the seventh principal case:

$$(151) \quad \left| \begin{array}{ccc} X_1 + (m_{51}m_{62} - & X_2 + (m_{51}m_{63} - & \frac{n_{54}^2}{m_{41}^2 m_{63}} X_3 \\ -m_{52}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} X_3 & -m_{53}m_{61}) \frac{n_{54}}{m_{41}m_{63}} X_3 & \\ n_{32}Y_3 & -Y_2 & Y_1 + \frac{m_{42}}{m_{41}}Y_2 \\ m_{41}Z_4 & -Z_1 & \frac{m_{42}}{m_{41}}Z_1 - Z_2 \end{array} \right| \quad (m_{41}m_{63} \neq 0),$$

where

$$(152) \quad r = (m_{41}m_{52} - m_{42}m_{51})m_{63} - (m_{41}m_{62} - m_{42}m_{61})m_{53}.$$

We are now in a position to formulate both fundamental theorems.

FIRST FUNDAMENTAL THEOREM. *The function F is nomographic if and only if — under the three preliminary assumptions — one of the principal cases occurs.*

SECOND FUNDAMENTAL THEOREM. *If the function F is nomographic — under the three preliminary assumptions — then:*

a) *it is doubly nomographic whenever there occurs the first principal case, or the second principal case with $m_{31}n_{31} + m_{32}n_{32} \neq 0$, or the third principal case with $(r_{31} - r_{42})^2 + 4r_{32}r_{41} \neq 0$; then every Massau form of the*

function F is equivalent to one of the two corresponding principal Massau forms, each of them being non-equivalent to the other,

b) it is uniquely nomographic in the remaining cases (also in the second principal case with $m_{31}n_{31} + m_{32}n_{32} = 0$ and in the third principal case with $(r_{31} - r_{42})^2 + 4r_{32}r_{41} = 0$); then every Massau form of the function F is equivalent to the corresponding principal Massau form.

Proof of the First Fundamental Theorem

Necessity. If the function F is nomographic, then by Theorem 13 and the three preliminary assumptions one of the cases (127) and one of the cases (128) must occur. Moreover, the first three cases (127) can occur only together with the first three cases (128) and the last case (127), together with the last four cases (128). Therefore, we have only the following cases:

- I.1, i. e., the first case (127) and the first case (128),
- I.2, " " " " " " " " second " "
- I.3, " " " " " " " " third " "
- II.1, i. e., the second case (127) and the first case (128),
- II.2, " " " " " " " " second " "
- II.3, " " " " " " " " third " "
- III.1, i. e., the third case (127) and the first case (128),
- III.2, " " " " " " " " second " "
- III.3, " " " " " " " " third " "
- IV.4, i. e., the fourth case (127) and the fourth case (128),
- IV.5, " " " " " " " " fifth " "
- IV.6, " " " " " " " " sixth " "
- IV.7, " " " " " " " " seventh " "

Since the cases I.1, I.2, and II.2 are obviously identical with the first, the second, and the fourth principal case respectively, we must consider only the remaining cases.

Case I.3. In this case we have

$$(153) \quad F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(Y_3Z_3 + Y_4Z_4),$$

i. e.,

$$F \equiv Y_1 \cdot X_1Z_1 + Y_2 \cdot X_1Z_2 + Y_3 \cdot X_2Z_3 + Y_4 \cdot X_2Z_4.$$

If the functions Y_1, Y_2, Y_3, Y_4 were linearly independent, then the functions $X_1Z_1, X_1Z_2, X_2Z_3, X_2Z_4$ would by Theorem 8 be linearly dependent, since the function F is in the present case of rank 2 with respect to y . Then, there would exist numbers c_1, c_2, c_3, c_4 , at least one of them differing from zero, such that

$$c_1X_1Z_1 + c_2X_1Z_2 + c_3X_2Z_3 + c_4X_2Z_4 \equiv X_1(c_1Z_1 + c_2Z_2) + X_2(c_3Z_3 + c_4Z_4) \equiv 0.$$

It would follow by the linear independence of the functions X_1, X_2 that

$$c_1 Z_1 + c_2 Z_2 \equiv 0, \quad c_3 Z_3 + c_4 Z_4 \equiv 0,$$

and next, by the linear independence of the functions Z_1, Z_2 and of the functions Z_3, Z_4 — separately treated —

$$c_1 = c_2 = 0 \quad \text{and} \quad c_3 = c_4 = 0,$$

which is a contradiction. Therefore the functions Y_1, Y_2, Y_3, Y_4 must be linearly dependent.

If the functions Y_1, Y_2, Y_3 were linearly independent, there would exist by Theorem 3 such numbers p_1, p_2, p_3 that

$$Y_4 \equiv p_1 Y_1 + p_2 Y_2 + p_3 Y_3.$$

Then we should have by (153)

$$F \equiv (X_1 Z_1 + p_1 X_2 Z_4) Y_1 + (X_1 Z_2 + p_2 X_2 Z_4) Y_2 + (X_2 Z_3 + p_3 X_2 Z_4) Y_3.$$

Since the function F is of rank 2 with respect to y , there would exist numbers q_1, q_2, q_3 , at least one of them differing from zero, such that

$$\begin{aligned} & q_1 (X_1 Z_1 + p_1 X_2 Z_4) + q_2 (X_1 Z_2 + p_2 X_2 Z_4) + q_3 (X_2 Z_3 + p_3 X_2 Z_4) \\ & \equiv X_1 (q_1 Z_1 + q_2 Z_2) + X_2 (p_1 q_1 Z_4 + p_2 q_2 Z_4 + q_3 Z_3 + p_3 q_3 Z_4) \equiv 0. \end{aligned}$$

It would follow by the linear independence of the functions X_1, X_2 that

$$q_1 Z_1 + q_2 Z_2 \equiv 0,$$

$$p_1 q_1 Z_4 + p_2 q_2 Z_4 + q_3 Z_3 + p_3 q_3 Z_4 \equiv q_3 Z_3 + (p_1 q_1 + p_2 q_2 + p_3 q_3) Z_4 \equiv 0,$$

and next, by the linear independence of the functions Z_1, Z_2 and of the functions Z_3, Z_4

$$q_1 = q_2 = q_3 = 0,$$

which is a contradiction. Therefore the functions Y_1, Y_2, Y_3 are linearly dependent and, by symmetry, the functions Y_1, Y_2, Y_4 are also linearly dependent. Since the functions Y_1 and Y_2 are linearly independent, there exist by Theorem 3 such numbers $m_{31}, m_{32}, m_{41}, m_{42}$ that the two identities in the first column (130) hold.

We similarly prove that the remaining identities (130) also hold.

Since the functions Y_1, Y_2 , the functions Y_3, Y_4 , the functions Z_1, Z_2 , and the functions Z_3, Z_4 — separately treated — are linearly independent, we have by Theorem 5

$$(154) \quad m_{31} m_{42} - m_{32} m_{41} \neq 0, \quad n_{31} n_{42} - n_{32} n_{41} \neq 0.$$

If the function F is nomographic, then, by Theorem 11, there exists for each of its Massau forms an equivalent form (108) such that the identities (109) hold. Then we have

$$(155) \quad \bar{Y}_2\bar{Z}_3 - \bar{Y}_3\bar{Z}_2 \equiv Y_1Z_1 + Y_2Z_2, \quad \bar{Y}_3\bar{Z}_1 - \bar{Y}_1\bar{Z}_3 \equiv Y_3Z_3 + Y_4Z_4.$$

Let us introduce the functions

$$(156) \quad \tilde{Y}_3 \equiv n_{31}Y_3 + n_{41}Y_4, \quad \tilde{Y}_4 \equiv n_{32}Y_3 + n_{42}Y_4.$$

We have then by (130)

$$(157) \quad Y_3Z_3 + Y_4Z_4 \equiv Y_3(n_{31}Z_1 + n_{32}Z_2) + Y_4(n_{41}Z_1 + n_{42}Z_2) \\ \equiv \tilde{Y}_3Z_1 + \tilde{Y}_4Z_2$$

and we can replace (155) by

$$(158) \quad \bar{Y}_2\bar{Z}_3 - \bar{Y}_3\bar{Z}_2 \equiv Y_1Z_1 + Y_2Z_2, \quad \bar{Y}_3\bar{Z}_1 - \bar{Y}_1\bar{Z}_3 \equiv \tilde{Y}_3Z_1 + \tilde{Y}_4Z_2.$$

Now, there exist by Corollary 9.2 such numbers $a_{ij}, b_{ij} (i, j = 1, 2)$ satisfying the conditions

$$(159) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0,$$

that

$$(160) \quad \begin{aligned} \bar{Y}_2 &\equiv a_{11}Y_1 + a_{12}Y_2, & \bar{Z}_3 &\equiv \frac{a_{22}}{a}Z_1 - \frac{a_{21}}{a}Z_2, \\ \bar{Y}_3 &\equiv a_{21}Y_1 + a_{22}Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a}Z_1 - \frac{a_{11}}{a}Z_2, \\ \bar{Y}_3 &\equiv b_{11}\tilde{Y}_3 + b_{12}\tilde{Y}_4, & \bar{Z}_1 &\equiv \frac{b_{22}}{b}Z_1 - \frac{b_{21}}{b}Z_2, \\ \bar{Y}_1 &\equiv b_{21}\tilde{Y}_3 + b_{22}\tilde{Y}_4, & \bar{Z}_3 &\equiv \frac{b_{12}}{b}Z_1 - \frac{b_{11}}{b}Z_2. \end{aligned}$$

From (160) we obtain

$$(161) \quad \begin{aligned} a_{21}Y_1 + a_{22}Y_2 &\equiv b_{11}\tilde{Y}_3 + b_{12}\tilde{Y}_4, \\ \frac{a_{22}}{a}Z_1 - \frac{a_{21}}{a}Z_2 &\equiv \frac{b_{12}}{b}Z_1 - \frac{b_{11}}{b}Z_2. \end{aligned}$$

But it follows from (156) and (130) that

$$(162) \quad \tilde{Y}_3 \equiv r_{31}Y_1 + r_{32}Y_2, \quad \tilde{Y}_4 \equiv r_{41}Y_1 + r_{42}Y_2,$$

where

$$(163) \quad \begin{aligned} r_{31} &= m_{31}n_{31} + m_{41}n_{41}, & r_{32} &= m_{32}n_{31} + m_{42}n_{41}, \\ r_{41} &= m_{31}n_{32} + m_{41}n_{42}, & r_{42} &= m_{32}n_{32} + m_{42}n_{42}, \end{aligned}$$

and by (154)

$$(164) \quad \begin{vmatrix} r_{31} & r_{32} \\ r_{41} & r_{42} \end{vmatrix} = \begin{vmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{vmatrix} \begin{vmatrix} n_{31} & n_{32} \\ n_{41} & n_{42} \end{vmatrix} \neq 0.$$

Therefore, by Theorem 4, we obtain from (161) and (162)

$$(165) \quad \begin{aligned} a_{21} &= r_{31}b_{11} + r_{41}b_{12}, & a_{22} &= r_{32}b_{11} + r_{42}b_{12}, \\ a_{22}b &= b_{12}a, & a_{21}b &= b_{11}a. \end{aligned}$$

Substituting the first two equalities (165) in the remaining two ones we obtain after a rearrangement

$$(166) \quad \begin{aligned} r_{32}b \cdot b_{11} + (r_{42}b - a) \cdot b_{12} &= 0, \\ (r_{31}b - a) \cdot b_{11} + r_{41}b \cdot b_{12} &= 0. \end{aligned}$$

Since by (159) at least one of the numbers b_{11} , b_{12} differs from zero, it follows that

$$\begin{vmatrix} r_{32}b & r_{42}b - a \\ r_{31}b - a & r_{41}b \end{vmatrix} = 0,$$

i. e.,

$$(167) \quad p^2 - (r_{31} + r_{42})p + (r_{31}r_{42} - r_{32}r_{41}) = 0,$$

where

$$(168) \quad p = \frac{a}{b}.$$

Since the equation (167) must have a solution in p , then

$$(169) \quad (r_{31} + r_{42})^2 - 4(r_{31}r_{42} - r_{32}r_{41}) = (r_{31} - r_{42})^2 + 4r_{32}r_{41} \geq 0,$$

i. e., by (163) the condition (131).

We see that for a nomographic function F case I.3 implies the third principal case.

Case II.1. In this case we should have

$$F \equiv X_1 Y_1 Z_1 + X_2 Y_3 Z_3.$$

Then the rank of the function F with respect to z would not exceed 2, which is a contradiction. We see that the case II.1 cannot occur.

Case II.3. In this case we also have the identity (153). Therefore we prove, in the same way as in Case I.3, that there exist numbers $m_{31}, m_{32}, m_{41}, m_{42}$ satisfying the identities in the first column (132) and that the functions Z_1, Z_2, Z_3, Z_4 are linearly dependent.

If the functions Z_1, Z_2, Z_3 and the functions Z_1, Z_2, Z_4 — separately treated — were linearly dependent, then we should prove — as in case I.3 — the existence of such numbers $n_{31}, n_{32}, n_{41}, n_{42}$ that the identities in the second column (130) would hold. Substituting them in (153) we should obtain

$$F \equiv (X_1 Y_1 + n_{31} X_2 Y_3 + n_{41} X_2 Y_4)Z_1 + (X_1 Y_2 + n_{32} X_2 Y_3 + n_{42} X_2 Y_4)Z_2,$$

which would contradict the assumption that the function F is of rank 3 with respect to z . It follows that either the functions Z_1, Z_2, Z_3 or the functions Z_1, Z_2, Z_4 are linearly independent. Then by the third preliminary assumption the functions Z_1, Z_2, Z_4 are linearly independent and by Theorem 3 there exist such numbers n_{31}, n_{32}, n_{34} that the last identity (132) holds.

If $n_{32} = 0$, then the functions Z_1, Z_3, Z_4 would be linearly dependent and by the third preliminary assumption the functions Z_2, Z_3, Z_4 would also be linearly dependent. Since by Theorem 3 the numbers n_{31}, n_{32}, n_{34} are unique, we should also have $n_{31} = 0$ and $Z_3 \equiv n_{34} Z_4$, which contradicts the linear independence of the functions Z_3, Z_4 . Therefore

$$(170) \quad n_{32} \neq 0.$$

If the function F is nomographic, then by Theorem 11 there exists for each of its Massau forms an equivalent form (108) such that the identities (109) hold. Then we have the identities (155) and — by Corollary 9.2 — there exist such numbers a_{ij}, b_{ij} ($i, j = 1, 2$) satisfying the conditions (159) that

$$(171) \quad \begin{aligned} \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_3 &\equiv \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2, \\ \bar{Y}_3 &\equiv a_{21} Y_1 + a_{22} Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2, \\ \bar{Y}_3 &\equiv b_{11} Y_3 + b_{12} Y_4, & \bar{Z}_1 &\equiv \frac{b_{22}}{b} Z_3 - \frac{b_{21}}{b} Z_4, \\ \bar{Y}_1 &\equiv b_{21} Y_3 + b_{22} Y_4, & \bar{Z}_3 &\equiv \frac{b_{12}}{b} Z_3 - \frac{b_{11}}{b} Z_4. \end{aligned}$$

From (171) we obtain

$$(172) \quad a_{21} Y_1 + a_{22} Y_2 \equiv b_{11} Y_3 + b_{12} Y_4, \quad \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2 \equiv \frac{b_{12}}{b} Z_3 - \frac{b_{11}}{b} Z_4.$$

Hence by (132) and Theorem 4

$$(173) \quad \begin{aligned} a_{21} &= m_{31}b_{11} + m_{41}b_{12}, & a_{22} &= m_{32}b_{11} + m_{42}b_{12}, \\ \frac{a_{22}}{a} &= n_{31}\frac{b_{12}}{b}, & \frac{a_{21}}{a} &= -n_{32}\frac{b_{12}}{b}, & b_{12}n_{34} &= b_{11}. \end{aligned}$$

It follows from (159) and the last equation (173) that

$$(174) \quad b_{12} \neq 0.$$

Substituting the last equation (173) in the first two we obtain after a rearrangement

$$(175) \quad a_{21} = (m_{31}n_{34} + m_{41})b_{12}, \quad a_{22} = (m_{32}n_{34} + m_{42})b_{12},$$

and substituting (175) in the third and the fourth equation (173) we obtain (after a rearrangement)

$$(176) \quad \begin{aligned} n_{31}a - (m_{32}n_{34} + m_{42})b &= 0, \\ n_{32}a + (m_{31}n_{34} + m_{41})b &= 0. \end{aligned}$$

Since by (159) the system of equations (176) must have a non-zero solution in a and b , we obtain

$$(177) \quad (m_{31}n_{34} + m_{41})n_{31} + (m_{32}n_{34} + m_{42})n_{32} = 0.$$

If $m_{31}n_{34} + m_{41} = 0$, then by (170) we should obtain $m_{32}n_{34} + m_{42} = 0$ and hence $m_{31}m_{42} - m_{32}m_{41} = 0$. Then by (132) and Theorem 5 the functions Y_3, Y_4 would be linearly dependent and by Theorem 8 the function G_2 would not be of rank 2, which is a contradiction. It follows that $m_{31}n_{34} + m_{41} \neq 0$ and we obtain from (177) the condition (133).

We see that for a nomographic function F case II.3 implies the fifth principal case.

Case III.1. We show, as in case II.1, that case III.1 cannot occur.

Case III.2. If this case occurred we should have

$$\begin{aligned} F &\equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2Y_3Z_3 \equiv Y_1 \cdot X_1Z_1 + Y_2 \cdot X_1Z_2 + Y_3 \cdot X_2Z_3 \\ &\equiv Z_1 \cdot X_1Y_1 + Z_2 \cdot X_1Y_2 + Z_3 \cdot X_2Y_3. \end{aligned}$$

Hence by Theorem 8 the functions Y_1, Y_2, Y_3 and the functions Z_1, Z_2, Z_3 — separately treated — would be linearly independent.

If the function F were nomographic, then by Theorem 11 there would exist for each of its Massau forms an equivalent form (108) such that the identities (109) would hold. Then we should have

$$\bar{Y}_2\bar{Z}_3 - \bar{Y}_3\bar{Z}_2 \equiv Y_1Z_1 + Y_2Z_2, \quad \bar{Y}_3\bar{Z}_1 - \bar{Y}_1\bar{Z}_3 \equiv Y_3Z_3,$$

and by Corollaries 9.2 and 10.1 there would exist such numbers a_{11} , a_{12} , a_{21} , a_{22} , b_1 , b_2 satisfying the conditions

$$(178) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad b_1^2 + b_2^2 > 0,$$

that

$$\bar{Y}_2 = a_{11} Y_1 + a_{12} Y_2, \quad \bar{Z}_3 = \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2,$$

$$\bar{Y}_3 = a_{21} Y_1 + a_{22} Y_2, \quad \bar{Z}_2 = \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2,$$

and

$$\bar{Y}_3 = b_1 Y_3, \quad \bar{Y}_1 = b_2 Y_3, \quad b_1 \bar{Z}_1 - b_2 \bar{Z}_3 = Z_3,$$

or

$$\bar{Z}_1 = b_1 Z_3, \quad \bar{Z}_3 = b_2 Z_3, \quad b_1 \bar{Y}_3 - b_2 \bar{Y}_1 = Y_3.$$

It would follow that

$$b_1 Y_3 = a_{21} Y_1 + a_{22} Y_2 \quad \text{or} \quad b_2 Z_3 = \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2.$$

Since — as we have shown — the functions Y_1 , Y_2 , Y_3 and the functions Z_1 , Z_2 , Z_3 — separately treated — would be linearly independent, each of the last two identities would imply $a_{21} = a_{22} = 0$, which is a contradiction of (178).

It follows that case III.2 cannot occur.

Case III.3. In this case we have once more the identity (153) and we show, in the same way as in case I.3, that the functions Y_1 , Y_2 , Y_3 , Y_4 and the functions Z_1 , Z_2 , Z_3 , Z_4 — separately treated — are linearly dependent.

If the functions Y_1 , Y_2 , Y_3 and also the functions Y_1 , Y_2 , Y_4 — separately treated — were linearly dependent, then — by linear independence of the functions Y_1 , Y_2 and by Theorem 3 — there would exist such numbers m_{31} , m_{32} , m_{41} , m_{42} that

$$Y_3 = m_{31} Y_1 + m_{32} Y_2, \quad Y_4 = m_{41} Y_1 + m_{42} Y_2.$$

Substituting these identities in (153) we should have after a rearrangement

$$F = Y_1(X_1 Z_1 + m_{31} X_2 Z_3 + m_{41} X_2 Z_4) + Y_2(X_1 Z_2 + m_{32} X_2 Z_3 + m_{42} X_2 Z_4),$$

i. e., a contradiction of the assumption that the function F is of rank 3 with respect to y . Therefore, either the functions Y_1 , Y_2 , Y_3 or the functions Y_1 , Y_2 , Y_4 are linearly independent. By the third preliminary assumption the functions Y_1 , Y_2 , Y_3 are linearly independent. Then

there exist by Theorem 3 such numbers m_{41}, m_{42}, m_{43} that the first identity (134) holds. We prove in a similar way that either the functions Z_1, Z_2, Z_3 or the functions Z_1, Z_2, Z_4 are linearly independent.

Let us suppose that the functions Z_1, Z_2, Z_3 are linearly independent. There exist then by Theorem 3 such numbers n_{41}, n_{42}, n_{43} that

$$(179) \quad Z_4 \equiv n_{41}Z_1 + n_{42}Z_2 + n_{43}Z_3.$$

If the function F is nomographic, then — by Theorem 11 — there exists for each of its Massau forms an equivalent form (108) such that the identities (109) hold. Then we have the identities (155) and — by Corollary 9.2 — there exist such numbers a_{ij}, b_{ij} ($i, j = 1, 2$) satisfying the conditions (159) that the identities (171) hold. From (171) we obtain (172) and then — by Theorem 4, the first identity (134) and (179)

$$(180) \quad \begin{aligned} a_{21} &= m_{41}b_{12}, & a_{22} &= m_{42}b_{12}, & b_{11} &= -m_{43}b_{12}, \\ \frac{a_{22}}{a} &= -n_{41}\frac{b_{11}}{b}, & \frac{a_{21}}{a} &= n_{42}\frac{b_{11}}{b}, & b_{12} &= n_{43}b_{11}. \end{aligned}$$

It follows by (159) from the third and the last equation (180) that $n_{43} \neq 0$. Therefore, we obtain from (179)

$$(181) \quad Z_3 \equiv -\frac{n_{41}}{n_{43}}Z_1 - \frac{n_{42}}{n_{43}}Z_2 + \frac{1}{n_{43}}Z_4.$$

If the functions Z_1, Z_2, Z_4 were linearly dependent, then — by the linear independence of the functions Z_1, Z_2 — there would exist such numbers p_1, p_2 that

$$Z_4 \equiv p_1Z_1 + p_2Z_2$$

and, next, by (181)

$$Z_3 \equiv \left(-\frac{n_{41}}{n_{43}} + \frac{p_1}{n_{43}}\right)Z_1 + \left(-\frac{n_{42}}{n_{43}} + \frac{p_2}{n_{43}}\right)Z_2,$$

which would contradict the assumption that the functions Z_1, Z_2, Z_3 are linearly independent. Therefore, if the functions Z_1, Z_2, Z_3 are linearly independent, then the functions Z_1, Z_2, Z_4 are also linearly independent.

Thus we have shown that the functions Z_1, Z_2, Z_4 are linearly independent. There exist then by Theorem 3 such numbers n_{31}, n_{32}, n_{34} that the second identity (134) holds.

We have already shown that there exist in the present case such numbers a_{ij}, b_{ij} ($i, j = 1, 2$) satisfying the conditions (159) that the

identities (172) hold. By Theorem 4 the identities (172) and (134) imply

$$(182) \quad \begin{aligned} a_{21} &= m_{41} b_{12}, & a_{22} &= m_{42} b_{12}, & b_{11} &= -m_{43} b_{12}, \\ \frac{a_{22}}{a} &= n_{31} \frac{b_{12}}{b}, & \frac{a_{21}}{a} &= -n_{32} \frac{b_{12}}{b}, & n_{34} b_{12} &= b_{11}. \end{aligned}$$

From the third of those equations we obtain by (159)

$$(183) \quad b_{12} \neq 0.$$

Substituting the first two equations (182) in the fourth and the fifth one we obtain by (183)

$$(184) \quad n_{31} a - m_{42} b = 0, \quad n_{32} a + m_{41} b = 0.$$

If $m_{41} = 0$, then by (134) the functions Y_2, Y_3, Y_4 would be linearly dependent and by the third preliminary assumption the functions Y_1, Y_3, Y_4 would also be linearly dependent. Then by (134) we should have $m_{42} = 0$ and, next, $Y_4 \equiv m_{43} Y_3$. Thus the functions Y_3, Y_4 would be linearly dependent, which is a contradiction. It follows that $m_{41} \neq 0$ and from the second equation (184) we obtain

$$b = -\frac{n_{32}}{m_{41}} a.$$

Substituting this in the first equation (184) we obtain by (159) the first condition (135). The third and the last equation (182) imply by (183) the second condition (135).

We see that for a nomographic function F Case III.3 implies the sixth principal case.

Case IV.4. In this case we should have

$$(185) \quad F \equiv X_1 Y_1 Z_1 + X_2 Y_3 Z_3 + X_3 Y_5 Z_5.$$

By Theorem 8 the functions Y_1, Y_3, Y_5 and the functions Z_1, Z_3, Z_5 — separately treated — would be linearly independent.

If the function F were nomographic, then by Theorem 12 there would exist for each of its Massau forms an equivalent form (120) such that the identities (121) would hold. Then we should have

$$(186) \quad \begin{aligned} \bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2 &\equiv Y_1 Z_1, & \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3 &\equiv Y_3 Z_3, \\ \bar{Y}_1 \bar{Z}_2 - \bar{Y}_2 \bar{Z}_1 &\equiv Y_5 Z_5, \end{aligned}$$

and — by Corollary 10.1 — there would exist such numbers a_1, a_2, b_1, b_2 ,

c_1, c_2 satisfying the conditions

$$(187) \quad a_1^2 + a_2^2 > 0, \quad b_1^2 + b_2^2 > 0, \quad c_1^2 + c_2^2 > 0,$$

that

$$(188) \quad \bar{Y}_2 \equiv a_1 Y_1, \quad \bar{Y}_3 \equiv a_2 Y_1, \quad a_1 \bar{Z}_3 - a_2 \bar{Z}_2 \equiv Z_1,$$

or

$$(189) \quad \bar{Z}_3 \equiv a_1 Z_1, \quad \bar{Z}_2 \equiv a_2 Z_1, \quad a_1 \bar{Y}_2 - a_2 \bar{Y}_3 \equiv Y_1,$$

$$(190) \quad \bar{Y}_3 \equiv b_1 Y_3, \quad \bar{Y}_1 \equiv b_2 Y_3, \quad b_1 \bar{Z}_1 - b_2 \bar{Z}_3 \equiv Z_3,$$

or

$$(191) \quad \bar{Z}_1 \equiv b_1 Z_3, \quad \bar{Z}_3 \equiv b_2 Z_3, \quad b_1 \bar{Y}_3 - b_2 \bar{Y}_1 \equiv Y_3,$$

$$(192) \quad \bar{Y}_1 \equiv c_1 Y_5, \quad \bar{Y}_2 \equiv c_2 Y_5, \quad c_1 \bar{Z}_2 - c_2 \bar{Z}_1 \equiv Z_5,$$

or

$$(193) \quad \bar{Z}_2 \equiv c_1 Z_5, \quad \bar{Z}_1 \equiv c_2 Z_5, \quad c_1 \bar{Y}_1 - c_2 \bar{Y}_2 \equiv Y_5.$$

If the identities (188) and (190) held, then we should have $a_2 Y_1 \equiv b_1 Y_3$ and — by the linear independence of the functions $Y_1, Y_3 - a_2 = b_1 = 0$. Then $a_1 \bar{Z}_3 \equiv Z_1, -b_2 \bar{Z}_3 \equiv Z_3$. Since by (187) we should have $a_1 \neq 0, b_2 \neq 0$, we should obtain $Z_1 \equiv -a_1 Z_3 / b_2$, which contradicts the linear independence of the functions Z_1, Z_3, Z_5 . Thus the identities (188) and (190) cannot be simultaneous.

In a similar way we show that the identities (188) and (192) cannot be simultaneous and — analogously — the identities (191) and (193). It follows that the identities (188) cannot occur at all.

We show in a similar way that also the identities (189) cannot occur. It follows that the whole case IV.4 cannot occur.

Case IV.5. In this case we should have

$$(194) \quad F \equiv X_1(Y_1 Z_1 + Y_2 Z_2) + X_2 Y_3 Z_3 + X_3 Y_5 Z_5.$$

If the function F were nomographic, then by Theorem 12 there would exist for each of its Massau forms an equivalent form (120) such that the identities (121) would hold. Then we should have

$$\bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2 \equiv Y_1 Z_1 + Y_2 Z_2, \quad \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3 \equiv Y_3 Z_3,$$

$$\bar{Y}_1 \bar{Z}_2 - \bar{Y}_2 \bar{Z}_1 \equiv Y_5 Z_5,$$

and — by Corollaries 9.2 and 10.1 — there would exist such numbers $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1, c_2$ satisfying the conditions

$$(195) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad b_1^2 + b_2^2 > 0, \quad c_1^2 + c_2^2 > 0,$$

that

$$(196) \quad \bar{Y}_2 \equiv a_{11} Y_1 + a_{12} Y_2, \quad \bar{Z}_3 \equiv \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2,$$

$$(197) \quad \bar{Y}_3 \equiv a_{21} Y_1 + a_{22} Y_2, \quad \bar{Z}_2 \equiv \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2,$$

$$(197) \quad \bar{Y}_3 \equiv b_1 Y_3, \quad \bar{Y}_1 \equiv b_2 Y_3, \quad b_1 \bar{Z}_1 - b_2 \bar{Z}_3 \equiv Z_3,$$

or

$$(198) \quad \bar{Z}_1 \equiv b_1 Z_3, \quad \bar{Z}_3 \equiv b_2 Z_3, \quad b_1 \bar{Y}_3 - b_2 \bar{Y}_1 \equiv Y_3,$$

$$(199) \quad \bar{Y}_1 \equiv c_1 Y_5, \quad \bar{Y}_2 \equiv c_2 Y_5, \quad c_1 \bar{Z}_2 - c_2 \bar{Z}_1 \equiv Z_5,$$

or

$$(200) \quad \bar{Z}_2 \equiv c_1 Z_5, \quad \bar{Z}_1 \equiv c_2 Z_5, \quad c_1 \bar{Y}_1 - c_2 \bar{Y}_2 \equiv Y_5.$$

If the identities (196), (197), and (199) held, then we should have

$$b_1 Y_3 \equiv a_{21} Y_1 + a_{22} Y_2, \quad c_2 Y_5 \equiv a_{11} Y_1 + a_{12} Y_2.$$

Since by (195) at least one of the numbers a_{21} , a_{22} and at least one of the numbers a_{11} , a_{12} would differ from zero, then the functions Y_1 , Y_2 , Y_3 and the functions Y_1 , Y_2 , Y_5 would be linearly dependent. Since the functions Y_1 , Y_2 are linearly independent, we should have

$$b_1 \neq 0, \quad c_2 \neq 0,$$

and hence

$$Y_3 \equiv \frac{a_{21}}{b_1} Y_1 + \frac{a_{22}}{b_1} Y_2, \quad Y_5 \equiv \frac{a_{11}}{c_2} Y_1 + \frac{a_{12}}{c_2} Y_2.$$

Substituting these identities in (194) we should have after a rearrangement

$$F \equiv Y_1 \left(X_1 Z_1 + \frac{a_{21}}{b_1} X_2 Z_3 + \frac{a_{11}}{c_2} X_3 Z_5 \right) + Y_2 \left(X_1 Z_2 + \frac{a_{22}}{b_1} X_2 Z_3 + \frac{a_{12}}{c_2} X_3 Z_5 \right),$$

which would contradict the assumption that the function F is of rank 3 with respect to y .

Thus the identities (196), (197), and (199) cannot be simultaneous.

If the identities (196), (197), and (200) held, then we should have

$$b_1 Y_3 \equiv a_{21} Y_1 + a_{22} Y_2, \quad b_2 c_1 Y_3 - a_{11} c_2 Y_1 - a_{12} c_2 Y_2 \equiv Y_5,$$

where — as before — $b_1 \neq 0$. It would then follow that

$$Y_3 \equiv \frac{a_{21}}{b_1} Y_1 + \frac{a_{22}}{b_1} Y_2, \quad Y_5 \equiv \left(\frac{a_{21} b_2 c_1}{b_1} - a_{11} c_2 \right) Y_1 + \left(\frac{a_{22} b_2 c_1}{b_1} - a_{12} c_2 \right) Y_2.$$

Substituting these identities in (194) we should obtain once more a contradiction of the assumption that the function F is of rank 3 with respect to y .

Thus, neither the identities (196), (197), and (199) nor the identities (196), (197), and (200) can be simultaneous. It follows that the identities (197) cannot occur at all. We show analogously that the identities (198) cannot occur. It follows that the whole case IV.5 cannot occur.

Case IV.6. In this case we should have

$$(201) \quad F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(Y_3Z_3 + Y_4Z_4) + X_3Y_5Z_5.$$

If the function F were nomographic, then by Theorem 12 there would exist for each of its Massau forms an equivalent form (120) such that the identities (121) would hold. Then we should have

$$\begin{aligned} \bar{Y}_2\bar{Z}_3 - \bar{Y}_3\bar{Z}_2 &\equiv Y_1Z_1 + Y_2Z_2, & \bar{Y}_3\bar{Z}_1 - \bar{Y}_1\bar{Z}_3 &\equiv Y_3Z_3 + Y_4Z_4, \\ \bar{Y}_1\bar{Z}_2 - \bar{Y}_2\bar{Z}_1 &\equiv Y_5Z_5, \end{aligned}$$

and — by Corollaries 9.2 and 10.1 — there would exist such numbers a_{ij} , b_{ij} , c_i ($i, j = 1, 2$) satisfying the conditions

$$(202) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0, \quad c_1^2 + c_2^2 > 0,$$

that

$$(203) \quad \begin{aligned} \bar{Y}_2 &\equiv a_{11}Y_1 + a_{12}Y_2, & \bar{Z}_3 &\equiv \frac{a_{22}}{a}Z_1 - \frac{a_{21}}{a}Z_2, \\ \bar{Y}_3 &\equiv a_{21}Y_1 + a_{22}Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a}Z_1 - \frac{a_{11}}{a}Z_2, \\ \bar{Y}_3 &\equiv b_{11}Y_3 + b_{12}Y_4, & \bar{Z}_1 &\equiv \frac{b_{22}}{b}Z_3 - \frac{b_{21}}{b}Z_4, \\ \bar{Y}_1 &\equiv b_{21}Y_3 + b_{22}Y_4, & \bar{Z}_3 &\equiv \frac{b_{12}}{b}Z_3 - \frac{b_{11}}{b}Z_4, \end{aligned}$$

and, moreover,

$$(204) \quad \bar{Y}_1 \equiv c_1Y_5, \quad \bar{Y}_2 \equiv c_2Y_5, \quad c_1\bar{Z}_2 - c_2\bar{Z}_1 \equiv Z_5,$$

or

$$(205) \quad \bar{Z}_2 \equiv c_1Z_5, \quad \bar{Z}_1 \equiv c_2Z_5, \quad c_1\bar{Y}_1 - c_2\bar{Y}_2 \equiv Y_5.$$

If the identities (203) and (204) held, we should have

$$(206) \quad \begin{aligned} c_1Y_5 &\equiv b_{21}Y_3 + b_{22}Y_4, & c_2Y_5 &\equiv a_{11}Y_1 + a_{12}Y_2, \\ a_{21}Y_1 + a_{22}Y_2 &\equiv b_{11}Y_3 + b_{12}Y_4, \end{aligned}$$

whence by (202)

$$Y_3 \equiv \frac{a_{21}b_{22}}{b}Y_1 + \frac{a_{22}b_{22}}{b}Y_2 - \frac{b_{12}c_1}{b}Y_5,$$

$$Y_4 \equiv \frac{-a_{21}b_{21}}{b}Y_1 - \frac{a_{22}b_{21}}{b}Y_2 + \frac{b_{11}c_1}{b}Y_5.$$

But — by (202), (206), and the linear independence of the functions $Y_1, Y_2 - c_2 \neq 0$ and from the second identity (206) we should have

$$Y_5 \equiv \frac{a_{11}}{c_2}Y_1 + \frac{a_{12}}{c_2}Y_2.$$

Hence we should obtain

$$(207) \quad Y_3 \equiv m_{31}Y_1 + m_{32}Y_2, \quad Y_4 \equiv m_{41}Y_1 + m_{42}Y_2,$$

$$Y_5 \equiv m_{51}Y_1 + m_{52}Y_2,$$

where

$$m_{31} = \frac{a_{21}b_{22}}{b} - \frac{a_{11}b_{12}c_1}{bc_2}, \quad m_{32} = \frac{a_{22}b_{22}}{b} - \frac{a_{12}b_{12}c_1}{bc_2},$$

$$m_{41} = -\frac{a_{21}b_{21}}{b} + \frac{a_{11}b_{11}c_1}{bc_2}, \quad m_{42} = -\frac{a_{22}b_{21}}{b} + \frac{a_{12}b_{11}c_1}{bc_2},$$

$$m_{51} = \frac{a_{11}}{c_2}, \quad m_{52} = \frac{a_{12}}{c_2}.$$

Substituting (207) in (201) we should obtain after a rearrangement

$$F \equiv Y_1(X_1Z_1 + m_{31}X_2Z_3 + m_{41}X_2Z_4 + m_{51}X_3Z_5) +$$

$$+ Y_2(X_1Z_2 + m_{32}X_2Z_3 + m_{42}X_2Z_4 + m_{52}X_3Z_5),$$

which would contradict the assumption that the function F is of rank 3 with respect to y .

Thus the identities (203) and (204) cannot be simultaneous.

We show analogously that the identities (203) and (205) also cannot be simultaneous. It follows that Case IV.6 cannot occur at all.

Case IV.7. In this case we have

$$(208) \quad F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(Y_3Z_3 + Y_4Z_4) + X_3(Y_5Z_5 + Y_6Z_6).$$

If the function F is nomographic, there exists — by Theorem 12 — for each of its Massau forms an equivalent form (120) such that the identities (121) hold. Then we have

$$\bar{Y}_2\bar{Z}_3 - \bar{Y}_3\bar{Z}_2 \equiv Y_1Z_1 + Y_2Z_2, \quad \bar{Y}_3\bar{Z}_1 - \bar{Y}_1\bar{Z}_3 \equiv Y_3Z_3 + Y_4Z_4,$$

$$\bar{Y}_1\bar{Z}_2 - \bar{Y}_2\bar{Z}_1 \equiv Y_5Z_5 + Y_6Z_6,$$

and there exist by Corollary 9.2 such numbers a_{ij}, b_{ij}, c_{ij} ($i, j = 1, 2$) satisfying the conditions

$$(209) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0, \quad c = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0,$$

that

$$(210) \quad \begin{aligned} \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_3 &\equiv \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2, \\ \bar{Y}_3 &\equiv a_{21} Y_1 + a_{22} Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2, \\ \bar{Y}_3 &\equiv b_{11} Y_3 + b_{12} Y_4, & \bar{Z}_1 &\equiv \frac{b_{22}}{b} Z_3 - \frac{b_{21}}{b} Z_4, \\ \bar{Y}_1 &\equiv b_{21} Y_3 + b_{22} Y_4, & \bar{Z}_3 &\equiv \frac{b_{12}}{b} Z_3 - \frac{b_{11}}{b} Z_4, \\ \bar{Y}_1 &\equiv c_{11} Y_5 + c_{12} Y_6, & \bar{Z}_2 &\equiv \frac{c_{22}}{c} Z_5 - \frac{c_{21}}{c} Z_6, \\ \bar{Y}_2 &\equiv c_{21} Y_5 + c_{22} Y_6, & \bar{Z}_1 &\equiv \frac{c_{12}}{c} Z_5 - \frac{c_{11}}{c} Z_6. \end{aligned}$$

Hence we have

$$(211) \quad \begin{aligned} b_{21} Y_3 + b_{22} Y_4 &\equiv c_{11} Y_5 + c_{12} Y_6, & \frac{b_{22}}{b} Z_3 - \frac{b_{21}}{b} Z_4 &\equiv \frac{c_{12}}{c} Z_5 - \frac{c_{11}}{c} Z_6, \\ a_{11} Y_1 + a_{12} Y_2 &\equiv c_{21} Y_5 + c_{22} Y_6, & \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2 &\equiv \frac{c_{22}}{c} Z_5 - \frac{c_{21}}{c} Z_6, \\ a_{21} Y_1 + a_{22} Y_2 &\equiv b_{11} Y_3 + b_{12} Y_4, & \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2 &\equiv \frac{b_{12}}{b} Z_3 - \frac{b_{11}}{b} Z_4. \end{aligned}$$

If the functions Y_1, Y_2, Y_3 were linearly dependent, then by the third preliminary assumption the functions Y_1, Y_2, Y_m ($m = 4, 5, 6$) would also be linearly dependent. Since the functions Y_1, Y_2 are linearly independent, there would exist by Theorem 3 such numbers m_{i1}, m_{i2} ($i = 3, 4, 5, 6$) that

$$Y_i \equiv m_{i1} Y_1 + m_{i2} Y_2 \quad (i = 3, 4, 5, 6).$$

Substituting these identities in (208) we should obtain

$$\begin{aligned} F &\equiv Y_1(X_1 Z_1 + m_{31} X_2 Z_3 + m_{41} X_2 Z_4 + m_{51} X_3 Z_5 + m_{61} X_3 Z_6) + \\ &\quad + Y_2(X_1 Z_2 + m_{32} X_2 Z_3 + m_{42} X_2 Z_4 + m_{52} X_3 Z_5 + m_{62} X_3 Z_6), \end{aligned}$$

which would contradict the assumption that the function F is of rank 3 with respect to y .

Thus, the functions Y_1, Y_2, Y_3 are linearly independent. Since by (209) at least one of the numbers a_{21}, a_{22} differs from zero, then by the last identity in the first column (211) the functions Y_1, Y_2, Y_3, Y_4 are linearly dependent. There exist then by Theorem 3 such numbers m_{41}, m_{42}, m_{43} that the first identity (136) holds.

Since — by the first two identities in the first column (211) and by the third condition (209) — there exist such numbers p_{ij} ($i = 5, 6$; $j = 1, 2, 3, 4$) that

$$Y_5 \equiv p_{51} Y_1 + p_{52} Y_2 + p_{53} Y_3 + p_{54} Y_4,$$

$$Y_6 \equiv p_{61} Y_1 + p_{62} Y_2 + p_{63} Y_3 + p_{64} Y_4,$$

then there exist, by the first identity (136), such numbers m_{ij} ($i = 5, 6$; $j = 1, 2, 3$) that the last two identities in the first column (136) hold.

If the functions Z_1, Z_2, Z_4 were linearly dependent, there would exist — by the linear independence of the functions Z_1, Z_2 and Theorem 3 — such numbers n_{41}, n_{42} that

$$(212) \quad Z_4 \equiv n_{41} Z_1 + n_{42} Z_2.$$

Substituting this identity in the last identity (211) we should obtain

$$\frac{b_{12}}{b} Z_3 \equiv \left(\frac{a_{22}}{a} + n_{41} \frac{b_{11}}{b} \right) Z_1 + \left(-\frac{a_{21}}{a} + n_{42} \frac{b_{11}}{b} \right) Z_2.$$

Since by the third identity in the first column (211)

$$(213) \quad b_{12} \neq 0,$$

(the functions Y_1, Y_2, Y_3 being linearly independent and by (209) at least one of the numbers a_{21}, a_{22} differing from zero), we should obtain

$$(214) \quad Z_3 \equiv n_{31} Z_1 + n_{32} Z_2,$$

where

$$n_{31} = \frac{a_{22} b}{b_{12} a} + n_{41} \frac{b_{11}}{b_{12}}, \quad n_{32} = -\frac{a_{21} b}{b_{12} a} + n_{42} \frac{b_{11}}{b_{12}}.$$

Moreover, by the first two identities in the second column (211) there exist such numbers r_{ij} ($i = 5, 6$; $j = 1, 2, 3, 4$) that

$$(215) \quad Z_5 \equiv r_{51} Z_1 + r_{52} Z_2 + r_{53} Z_3 + r_{54} Z_4, \quad Z_6 \equiv r_{61} Z_1 + r_{62} Z_2 + r_{63} Z_3 + r_{64} Z_4.$$

Then by (212) and (214) there would exist such numbers $n_{51}, n_{52}, n_{61}, n_{62}$ that

$$Z_5 \equiv n_{51} Z_1 + n_{52} Z_2, \quad Z_6 \equiv n_{61} Z_1 + n_{62} Z_2.$$

$G_2 \equiv Y_3 Z_3 + Y_4 Z_4$ would not be of rank 2, which is a contradiction. Thus we have

$$(218) \quad m_{41} \neq 0.$$

If $m_{63} = 0$, then by (136) the functions Y_1, Y_2, Y_6 would be linearly dependent and by the third preliminary assumption the functions Y_1, Y_2, Y_6 would also be linearly dependent. It would follow by (136) that also $m_{53} = 0$ and by the third equation (216)

$$b_{21} + m_{43} b_{22} = 0.$$

But this equation and the ninth equation (216) would then imply $b_{11} b_{22} - b_{12} b_{21} = 0$, which is a contradiction of (209). Therefore, we have

$$(219) \quad m_{63} \neq 0.$$

By (219) we obtain from the sixth equation (216)

$$(220) \quad c_{22} = -\frac{m_{53}}{m_{63}} c_{21},$$

where by (209)

$$(221) \quad c_{21} \neq 0.$$

Substituting (220) in the fifteenth equation (216) we obtain by (221)

$$(222) \quad n_{64} = -\frac{m_{53}}{m_{63}} n_{64}.$$

The ninth and the last equation (216) imply by (217)

$$(223) \quad n_{34} = -m_{43}.$$

Substituting (222) and (223) in the twelfth equation (216) we obtain

$$-\frac{1}{b} (b_{21} + m_{43} b_{22}) = \frac{n_{54}}{m_{63} c} (m_{53} c_{11} + m_{63} c_{12}),$$

i. e., by the third equation (216)

$$(224) \quad -\frac{1}{b} (m_{53} c_{11} + m_{63} c_{12}) = -\frac{n_{54}}{m_{63} c} (m_{53} c_{11} + m_{63} c_{12}).$$

If $m_{53} c_{11} + m_{63} c_{12} = 0$, then we could write the third equation (216) in the form

$$b_{21} + m_{43} b_{22} = 0.$$

This equation and the ninth equation (216) would imply $b_{11} b_{22} - b_{12} b_{21} = 0$, which is a contradiction of (209). It follows that

$$(225) \quad m_{53} c_{11} + m_{63} c_{12} \neq 0$$

and we obtain from (224)

$$(226) \quad c = -\frac{n_{54}}{m_{63}} b.$$

Substituting the seventh equation (216) in the seventeenth one we obtain by (217) and (218)

$$(227) \quad b = -\frac{n_{32}}{m_{41}} a.$$

The equations (226) and (227) imply

$$(228) \quad c = \frac{n_{32} n_{54}}{m_{41} m_{63}} a.$$

The ninth equation (216) can be written in the form

$$(229) \quad b_{11} = -m_{43} b_{12}.$$

Therefore, by the third equation (216)

$$(230) \quad b = b_{11} b_{22} - b_{12} b_{21} = -(b_{21} + m_{43} b_{22}) b_{12} = -(m_{53} c_{11} + m_{63} c_{12}) b_{12}.$$

On the other hand, we have by (220)

$$(231) \quad c = c_{11} c_{22} - c_{12} c_{21} = -(m_{53} c_{11} + m_{63} c_{12}) \frac{c_{21}}{m_{63}}.$$

Substituting (230) and (231) in (226) we obtain

$$-(m_{53} c_{11} + m_{63} c_{12}) \frac{c_{21}}{m_{63}} = \frac{n_{54}}{m_{63}} (m_{53} c_{11} + m_{63} c_{12}) b_{12},$$

whence by (225)

$$(232) \quad c_{21} = -n_{54} b_{12}.$$

The equations (220) and (232) imply

$$(233) \quad c_{22} = \frac{m_{53} n_{54}}{m_{63}} b_{12}.$$

Substituting (232) and (233) in the fourth and the fifth equation (216), we obtain

$$(234) \quad a_{11} = -(m_{51} m_{63} - m_{53} m_{61}) \frac{n_{54}}{m_{63}} b_{12},$$

$$a_{12} = -(m_{52} m_{63} - m_{53} m_{62}) \frac{n_{54}}{m_{63}} b_{12}.$$

The seventh and the eighth equation (216) are

$$(235) \quad a_{21} = m_{41} b_{12}, \quad a_{22} = m_{42} b_{12}.$$

The equations (234) and (235) imply

$$\begin{aligned} a &= a_{11} a_{22} - a_{12} a_{21} \\ &= \left[-(m_{51} m_{63} - m_{53} m_{61}) \frac{n_{54}}{m_{63}} m_{42} + (m_{52} m_{63} - m_{53} m_{62}) \frac{n_{54}}{m_{63}} m_{41} \right] b_{12}^2, \end{aligned}$$

i. e.,

$$(236) \quad a = \frac{n_{54}}{m_{63}} r b_{12}^2,$$

where by (209)

$$(237) \quad r = (m_{41} m_{52} - m_{42} m_{51}) m_{63} - (m_{41} m_{62} - m_{42} m_{61}) m_{53} \neq 0.$$

The equations (227) and (236) imply

$$(238) \quad b = -\frac{n_{32} n_{54}}{m_{41} m_{63}} r b_{12}^2,$$

the equations (228) and (236) imply

$$(239) \quad c = \frac{n_{32} n_{54}^2}{m_{41} m_{63}^2} r b_{12}^2.$$

Substituting (232) and (239) in (231) we obtain

$$(240) \quad \frac{n_{32} n_{54}^2}{m_{41} m_{63}^2} r b_{12}^2 = (m_{53} c_{11} + m_{63} c_{12}) \frac{n_{54}}{m_{63}} b_{12}.$$

Since (209) and (238) imply

$$(241) \quad n_{32} \neq 0, \quad n_{54} \neq 0,$$

we obtain from (240) by (217)

$$(242) \quad c_{12} = -\frac{m_{53}}{m_{63}} c_{11} + \frac{n_{32} n_{54}}{m_{41} m_{63}^2} r b_{12}.$$

On the other hand, the first two equations (216) imply

$$(m_{51} c_{11} + m_{61} c_{12}) m_{42} = (m_{52} c_{11} + m_{62} c_{12}) m_{41},$$

i. e.,

$$(243) \quad (m_{41} m_{52} - m_{42} m_{51}) c_{11} + (m_{41} m_{62} - m_{42} m_{61}) c_{12} = 0.$$

Substituting (242) in (243) we obtain after a rearrangement

$$\begin{aligned} [(m_{41}m_{52} - m_{42}m_{61})m_{63} - (m_{41}m_{62} - m_{42}m_{61})m_{53}] \frac{c_{11}}{m_{63}} + \\ + (m_{41}m_{62} - m_{42}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}^2} r b_{12} = 0, \end{aligned}$$

whence by (237)

$$(244) \quad c_{11} = -(m_{41}m_{62} - m_{42}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12}.$$

The equations (242) and (244) imply

$$c_{12} = [(m_{41}m_{62} - m_{42}m_{61})m_{53} + r] \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12},$$

i.e., by (237)

$$(245) \quad c_{12} = (m_{41}m_{52} - m_{42}m_{51}) \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12}.$$

Substituting (244) and (245) in the first equation (216) we obtain by (218)

$$b_{22} = [-(m_{41}m_{62} - m_{42}m_{61})m_{51} + (m_{41}m_{52} - m_{42}m_{51})m_{61}] \frac{n_{32}n_{54}}{m_{41}^2m_{63}} b_{12},$$

i.e., after a rearrangement,

$$(246) \quad b_{22} = -(m_{51}m_{62} - m_{52}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12}.$$

Substituting (244), (245), and (246) in the third equation (216), we obtain

$$\begin{aligned} b_{21} = [(m_{41}m_{52} - m_{42}m_{51})m_{63} - (m_{41}m_{62} - m_{42}m_{61})m_{53} + \\ + (m_{51}m_{62} - m_{52}m_{61})m_{43}] \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12}, \end{aligned}$$

i. e., by (237)

$$(247) \quad b_{21} = [r + (m_{51}m_{62} - m_{52}m_{61})m_{43}] \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12}.$$

Substituting (235) and (227) in the sixteenth equation (216) we obtain by (209) and (217) the first condition (137).

The second condition (137) is identical to (223).

In order to prove the third condition (137) let us subtract the tenth equation (216) multiplied by c_{21} from the thirteenth one multiplied by c_{11} . Thus we obtain

$$n_{51} = \frac{a_{12}c_{11}}{a} - n_{31} \frac{b_{22}c_{21}}{b}.$$

Substituting here (234), (244), (236), (246), (232), (238), and the first equation (137), we have

$$\begin{aligned}
 n_{51} &= \frac{(m_{52}m_{63} - m_{53}m_{62}) \frac{n_{54}}{m_{63}} b_{12} (m_{41}m_{62} - m_{42}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12}}{\frac{n_{54}}{m_{63}} r b_{12}^2} + \\
 &\quad + \frac{m_{42}}{m_{41}} n_{32} \frac{(m_{51}m_{62} - m_{52}m_{61}) \frac{n_{32}n_{54}^2}{m_{41}m_{63}} b_{12}^2}{\frac{n_{32}n_{54}}{m_{41}m_{63}} r b_{12}^2} \\
 &= [(m_{52}m_{63} - m_{53}m_{62})(m_{41}m_{62} - m_{42}m_{61}) - m_{42}m_{63}(m_{51}m_{62} - m_{52}m_{61})] \times \\
 &\quad \times \frac{n_{32}n_{54}}{m_{41}m_{63}r} \\
 &= [(m_{41}m_{52} - m_{42}m_{51})m_{63} - (m_{41}m_{62} - m_{42}m_{61})m_{53}] \frac{n_{32}n_{54}}{m_{41}m_{63}r} m_{62},
 \end{aligned}$$

i. e., by (237) the third condition (137).

Subtracting the eleventh equation (216) multiplied by c_{21} from the fourteenth one multiplied by c_{11} , we obtain analogously the fourth condition (137).

Subtracting the tenth equation (216) multiplied by c_{22} from the thirteenth one multiplied by c_{12} , we obtain analogously the fifth condition (137).

Subtracting the eleventh equation (216) multiplied by c_{22} from the fourteenth one multiplied by c_{12} , we obtain analogously the sixth condition (137).

The last condition (137) is identical to (222).

Thus, we see that for a nomographic function F Case IV.7 implies the seventh principal case.

This completes the proof of necessity of the conditions formulated in the First Fundamental Theorem.

Sufficiency. We now have to show that in each of the seven principal cases the function F is nomographic.

Principal case I. Since the functions G_1 and G_2 are both of rank 1, there exist such functions $Y_i \equiv Y_i(y)$, $Z_i \equiv Z_i(z)$ ($i = 1, 3$), that

$$G_1 \equiv Y_1 Z_1, \quad G_2 \equiv Y_3 Z_3.$$

Hence we have

$$F \equiv X_1 Y_1 Z_1 + X_2 Y_2 Z_2 \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_2 \\ Z_1 & 0 & Z_2 \end{vmatrix}$$

and the function F is obviously nomographic.

Principal case II. Now there exist functions $Y_i \equiv Y_i(y) \neq 0$, $Z_i \equiv Z_i(z) \neq 0$ ($i = 1, 2, 3$), such that

$$G_1 \equiv Y_1 Z_1 + Y_2 Z_2, \quad G_2 \equiv Y_3 Z_3,$$

the functions Y_1, Y_2 and the functions Z_1, Z_2 being by Theorem 8 linearly independent. Hence we have

$$(248) \quad F \equiv X_1(Y_1 Z_1 + Y_2 Z_2) + X_2 Y_3 Z_3 \equiv Y_1 \cdot X_1 Z_1 + Y_2 \cdot X_1 Z_2 + Y_3 \cdot X_2 Z_3.$$

Suppose the functions Y_1, Y_2, Y_3 were linearly independent. Since the function F is of rank 2 with respect to y , then by Theorem 8 the functions $X_1 Z_1, X_1 Z_2, X_2 Z_3$ would be linearly dependent, i. e., there exist such numbers q_1, q_2, q_3 , at least one of them differing from zero, that

$$q_1 X_1 Z_1 + q_2 X_1 Z_2 + q_3 X_2 Z_3 \equiv X_1(q_1 Z_1 + q_2 Z_2) + X_2 \cdot q_3 Z_3 \equiv 0.$$

Since the functions X_1, X_2 are linearly independent, it would follow that

$$q_1 Z_1 + q_2 Z_2 \equiv 0, \quad q_3 Z_3 \equiv 0.$$

Moreover, since the functions Z_1, Z_2 are linearly independent and $Z_3 \neq 0$ it follows that $q_1 = q_2 = q_3 = 0$, which is a contradiction. Therefore, the functions Y_1, Y_2, Y_3 are linearly dependent. We show analogously that the functions Z_1, Z_2, Z_3 are also linearly dependent. Since the functions Y_1, Y_2 and the functions Z_1, Z_2 — separately treated — are linearly independent, there exist by Theorem 3 such numbers $m_{31}, m_{32}, n_{31}, n_{32}$ that

$$(249) \quad Y_3 \equiv m_{31} Y_1 + m_{32} Y_2, \quad Z_3 \equiv n_{31} Z_1 + n_{32} Z_2.$$

If $m_{32} = 0$, then the functions Y_1, Y_3 would be linearly dependent and by the third preliminary assumption the functions Y_2, Y_3 would also be linearly dependent. It would follow that also $m_{31} = 0$ and $Y_3 \equiv 0$, which is a contradiction. Therefore,

$$(250) \quad m_{32} \neq 0.$$

By (249) and (250) we are now in a position to write the function F in the form

$$(251) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & -\frac{1}{m_{32}}Z_2 & Z_1 - \frac{m_{31}}{m_{32}}Z_2 \end{vmatrix}.$$

Indeed, the identity (251) is equivalent to

$$F \equiv X_1 \left(Y_1 Z_1 - \frac{m_{31}}{m_{32}} Y_1 Z_2 + \frac{1}{m_{32}} Y_3 Z_2 \right) + X_2 Y_3 Z_3$$

and, by (249), equivalent to (248) as desired.

Thus the function F is nomographic.

Principal case III. Let $p = p_1$ be a solution of the equation (144). It exists by the condition (131). If $r_{32} \neq 0$, we can write

$$(252) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ r_{31} Y_1 + r_{32} Y_2 & -Y_1 & \frac{p_1 - r_{42}}{r_{32}} Y_1 + Y_2 \\ p_1 Z_2 & -Z_2 & \frac{p_1 - r_{42}}{r_{32}} Z_2 - Z_1 \end{vmatrix},$$

because the identity (252) is equivalent to

$$F \equiv X_1 (Y_1 Z_1 + Y_2 Z_2) + X_2 \left(\frac{p_1^2 - p_1 r_{42}}{r_{32}} Y_1 Z_2 + p_1 Y_2 Z_2 - r_{31} \frac{p_1 - r_{42}}{r_{32}} Y_1 Z_2 - (p_1 - r_{42}) Y_2 Z_2 + r_{31} Y_1 Z_1 + r_{32} Y_2 Z_1 \right),$$

i. e., by (144), to

$$F \equiv X_1 (Y_1 Z_1 + Y_2 Z_2) + X_2 (r_{31} Y_1 Z_1 + r_{32} Y_2 Z_1 + r_{41} Y_1 Z_2 + r_{42} Y_2 Z_2)$$

and next, by (130) and (142), to

$$(253) \quad F \equiv X_1 (Y_1 Z_1 + Y_2 Z_2) + X_2 (Y_3 Z_3 + Y_4 Z_4),$$

as desired.

If $r_{32} = 0$, then it follows from (164) that $r_{31} \neq 0$ and we can write

$$(254) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ r_{41} Y_1 + r_{42} Y_2 & -Y_2 & r_{31} Y_1 \\ Z_1 & -\frac{1}{r_{31}} Z_1 & -Z_2 \end{vmatrix},$$

because the identity (254) is equivalent to

$$F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(r_{31}Y_1Z_1 + r_{41}Y_1Z_2 + r_{42}Y_2Z_2),$$

i. e., by (130) and (142), to the identity (253), as desired.

Either by (252) or by (254) the function F is nomographic.

Principal case IV. Since the function G_1 is of rank 2 and the function G_2 of rank 1, there exist functions $Y_i \equiv Y_i(y)$, $Z_i \equiv Z_i(z)$ ($i = 1, 2, 3$) such that the identity (248) holds and the functions Y_1, Y_2 and the functions Z_1, Z_2 — separately treated — are, by Theorem 8, linearly independent. We show, as in the second principal case, that the functions Y_1, Y_2, Y_3 are linearly dependent, i. e., there exist by Theorem 3 such numbers m_{31} and m_{32} that

$$(255) \quad Y_3 \equiv m_{31}Y_1 + m_{32}Y_2.$$

We show, as in the second principal case, that (250) holds. Thus, we can write

$$(256) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & -\frac{1}{m_{32}}Z_2 & Z_1 - \frac{m_{31}}{m_{32}}Z_2 \end{vmatrix},$$

since the identity (256) is equivalent to

$$F \equiv X_1 \left(Y_1Z_1 - \frac{m_{31}}{m_{32}}Y_1Z_2 + \frac{1}{m_{32}}Y_3Z_2 \right) + X_2Y_3Z_3,$$

i. e., by (255) equivalent to the identity (248), as desired.

By (256) the function F is nomographic.

Principal case V. The proof of (170) is also valid here.

In the present case we have

$$F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(Y_3Z_3 + Y_4Z_4),$$

i. e., by (132), (133) and (170)

$$\begin{aligned} F &\equiv X_1 \left(Y_1Z_1 + Y_2Z_2 + \frac{n_{31}}{n_{32}}Y_2Z_1 - \frac{n_{31}}{n_{32}}Y_2Z_1 \right) + \\ &\quad + X_2(Y_3Z_3 + m_{41}Y_1Z_4 + m_{42}Y_2Z_4 + n_{34}Y_3Z_4 - n_{34}Y_3Z_4) \\ &\equiv X_1 \left[-\frac{1}{n_{32}}Y_2(-n_{31}Z_1 - n_{32}Z_2) + \left(Y_1 - \frac{n_{31}}{n_{32}}Y_2 \right) Z_1 \right] + \\ &\quad + X_2(Y_3Z_3 + m_{41}Y_1Z_4 + m_{42}Y_2Z_4 + m_{31}n_{34}Y_1Z_4 + m_{32}n_{34}Y_2Z_4 - n_{34}Y_3Z_4) \end{aligned}$$

$$\begin{aligned}
 &\equiv X_1 \left[-\frac{1}{n_{32}} Y_2 (n_{34} Z_4 - Z_3) + \left(Y_1 - \frac{n_{31}}{n_{32}} Y_2 \right) Z_1 \right] + \\
 &\quad + X_2 [Y_1 (m_{31} n_{34} + m_{41}) Z_4 + Y_2 (m_{32} n_{34} + m_{42}) Z_4 - Y_3 (n_{34} Z_4 - Z_3)] \\
 &\equiv X_1 \left[-\frac{1}{n_{32}} Y_2 (n_{34} Z_4 - Z_3) + \left(Y_1 - \frac{n_{31}}{n_{32}} Y_2 \right) Z_1 \right] + \\
 &\quad + X_2 \left[\left(Y_1 - \frac{n_{31}}{n_{32}} Y_2 \right) (m_{31} n_{34} + m_{41}) Z_4 - Y_3 (n_{34} Z_4 - Z_3) \right] \\
 &\equiv \left[\begin{array}{ccc} X_1 & X_2 & 0 \\ Y_3 & -\frac{1}{n_{32}} Y_2 & Y_1 - \frac{n_{31}}{n_{32}} Y_2 \\ (m_{31} n_{34} + m_{41}) Z_4 & -Z_1 & n_{34} Z_4 - Z_3 \end{array} \right] .
 \end{aligned}$$

Thus the function F is nomographic.

Principal case VI. In the present case we have

$$(257) \quad m_{41} \neq 0$$

and $F \equiv X_1 (Y_1 Z_1 + Y_2 Z_2) + X_2 (Y_3 Z_3 + Y_4 Z_4)$, i. e., by (134), (135)

and (257)

$$\begin{aligned}
 F &\equiv X_1 \left(Y_1 Z_1 + Y_2 Z_2 + \frac{m_{42}}{m_{41}} Y_2 Z_1 - \frac{m_{42}}{m_{41}} Y_2 Z_1 \right) + \\
 &\quad + X_2 (n_{31} Y_3 Z_1 + n_{32} Y_3 Z_2 + n_{34} Y_3 Z_4 + m_{41} Y_1 Z_4 + m_{42} Y_2 Z_4 + m_{43} Y_3 Z_4) \\
 &\equiv X_1 \left[-Y_2 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) + \left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) Z_1 \right] + \\
 &\quad + X_2 \left(-\frac{m_{42}}{m_{41}} n_{32} Y_3 Z_1 + n_{32} Y_3 Z_2 + m_{41} Y_1 Z_4 + m_{42} Y_2 Z_4 \right) \\
 &\equiv X_1 \left[-Y_2 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) + \left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) Z_1 \right] + \\
 &\quad + X_2 \left[\left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) m_{41} Z_4 - n_{32} Y_3 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) \right] \\
 &\equiv \left[\begin{array}{ccc} X_1 & X_2 & 0 \\ n_{32} Y_3 & -Y_2 & Y_1 + \frac{m_{42}}{m_{41}} Y_2 \\ m_{41} Z_4 & -Z_1 & \frac{m_{42}}{m_{41}} Z_1 - Z_2 \end{array} \right] .
 \end{aligned}$$

Thus the function F is nomographic.

Principal case VII. In the present case we have

$$(258) \quad m_{41} \neq 0, \quad m_{63} \neq 0.$$

We shall prove that the function F can be written in the form (151), i. e., in the form

$$(259) \quad F \equiv X_1 \left[-Y_2 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) + \left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) Z_1 \right] + \\ + X_2 \left[\left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) m_{41} Z_4 - n_{32} Y_3 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) \right] + \\ + X_3 \left\{ (m_{51} m_{62} - m_{52} m_{61}) \frac{n_{32} n_{54}}{m_{41} m_{63}} \left[-Y_2 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) + \right. \right. \\ \left. \left. + \left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) Z_1 \right] + (m_{51} m_{63} - m_{53} m_{61}) \frac{n_{54}}{m_{41} m_{63}} \times \right. \\ \left. \times \left[\left(Y_1 + \frac{m_{42}}{m_{41}} Y_2 \right) m_{41} Z_4 - n_{32} Y_3 \left(\frac{m_{42}}{m_{41}} Z_1 - Z_2 \right) \right] + \right. \\ \left. + \frac{n_{54} r}{m_{41}^2 m_{63}} [-n_{32} Y_3 Z_1 + m_{41} Y_2 Z_4] \right\} \\ \equiv X_1 (Y_1 Z_1 + Y_2 Z_2) + X_2 \left[Y_3 \left(-\frac{m_{42}}{m_{41}} n_{32} Z_1 + n_{32} Z_2 - m_{43} Z_4 \right) + \right. \\ \left. + (m_{41} Y_1 + m_{42} Y_2 + m_{43} Y_3) Z_4 \right] + \\ + X_3 \left\{ (m_{51} m_{62} - m_{52} m_{61}) \frac{n_{32} n_{54}}{m_{41} m_{63}} (Y_1 Z_1 + Y_2 Z_2) + \right. \\ \left. + (m_{51} m_{63} - m_{53} m_{61}) \frac{n_{54}}{m_{63}} Y_1 Z_4 + \right. \\ \left. + \left[(m_{51} m_{63} - m_{53} m_{61}) \frac{m_{42} n_{54}}{m_{41} m_{63}} + \frac{n_{54} r}{m_{41} m_{63}} \right] Y_2 Z_4 + \right. \\ \left. + \left[-(m_{51} m_{63} - m_{53} m_{61}) \frac{m_{42} n_{32} n_{54}}{m_{41}^2 m_{63}} - \frac{n_{32} n_{54}}{m_{41}^2 m_{63}} r \right] Y_3 Z_1 + \right. \\ \left. + (m_{51} m_{63} - m_{53} m_{61}) \frac{n_{32} n_{54}}{m_{41} m_{63}} Y_3 Z_2 \right\}.$$

But in the present case we have

$$F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(Y_3Z_3 + Y_4Z_4) + X_3(Y_5Z_5 + Y_6Z_6),$$

i. e., by (136)

$$(260) \quad F \equiv X_1(Y_1Z_1 + Y_2Z_2) + \\ + X_2(n_{31}Y_3Z_1 + n_{32}Y_3Z_2 + n_{34}Y_3Z_4 + m_{41}Y_1Z_4 + \\ + m_{42}Y_2Z_4 + m_{43}Y_3Z_4) + X_3[(m_{51}n_{51} + m_{61}n_{61})Y_1Z_1 + \\ + (m_{52}n_{51} + m_{62}n_{61})Y_2Z_1 + (m_{53}n_{51} + m_{63}n_{61})Y_3Z_1 + \\ + (m_{51}n_{52} + m_{61}n_{62})Y_1Z_2 + (m_{52}n_{52} + m_{62}n_{62})Y_2Z_2 + \\ + (m_{53}n_{52} + m_{63}n_{62})Y_3Z_2 + (m_{51}n_{54} + m_{61}n_{64})Y_1Z_4 + \\ + (m_{52}n_{54} + m_{62}n_{64})Y_2Z_4 + (m_{53}n_{54} + m_{63}n_{64})Y_3Z_4].$$

Thus it is sufficient to show that the identities (259) and (260) are equivalent, i. e.,

$$(261) \quad \left\{ \begin{array}{l} n_{31} = -\frac{m_{42}}{m_{41}}n_{32}, \\ n_{34} = -m_{43}, \\ m_{51}n_{51} + m_{61}n_{61} = (m_{51}m_{62} - m_{52}m_{61})\frac{n_{32}n_{54}}{m_{41}m_{63}}, \\ m_{52}n_{51} + m_{62}n_{61} = 0, \\ m_{53}n_{51} + m_{63}n_{61} = -(m_{51}m_{63} - m_{53}m_{61})\frac{m_{42}n_{32}n_{54}}{m_{41}^2m_{63}} - \frac{n_{32}n_{54}}{m_{41}^2m_{63}}r, \\ m_{51}n_{52} + m_{61}n_{62} = 0, \\ m_{52}n_{52} + m_{62}n_{62} = (m_{51}m_{62} - m_{52}m_{61})\frac{n_{32}n_{54}}{m_{41}m_{63}}, \\ m_{53}n_{52} + m_{63}n_{62} = (m_{51}m_{63} - m_{53}m_{61})\frac{n_{32}n_{54}}{m_{41}m_{63}}, \\ m_{51}n_{54} + m_{61}n_{64} = (m_{51}m_{63} - m_{53}m_{61})\frac{n_{54}}{m_{63}}, \\ m_{52}n_{54} + m_{62}n_{64} = (m_{51}m_{63} - m_{53}m_{61})\frac{m_{42}n_{54}}{m_{41}m_{63}} + \frac{n_{54}}{m_{41}m_{63}}r, \\ m_{53}n_{54} + m_{63}n_{64} = 0. \end{array} \right.$$

The first two equations (261) are identical to the first two conditions (137) and hence they are satisfied.

By the third and the fifth condition (137) we have

$$m_{51}n_{51} + m_{61}n_{61} = m_{51}m_{62} \frac{n_{32}n_{54}}{m_{41}m_{63}} - m_{52}m_{61} \frac{n_{32}n_{54}}{m_{41}m_{63}}.$$

It follows that the third equation (261) is also satisfied.

The third and the fifth condition (137) imply also the fourth equation (261). Moreover, we have

$$(262) \quad m_{53}n_{51} + m_{63}n_{61} = (m_{53}m_{62} - m_{52}m_{63}) \frac{n_{32}n_{54}}{m_{41}m_{63}}.$$

On the other hand, we have by (152)

$$(263) \quad - (m_{51}m_{63} - m_{53}m_{61}) \frac{m_{42}n_{32}n_{54}}{m_{41}^2m_{63}} - \frac{n_{32}n_{54}}{m_{41}^2m_{63}} r \\ = \frac{n_{32}n_{54}}{m_{41}^2m_{63}} [- (m_{51}m_{63} - m_{53}m_{61})m_{42} - (m_{41}m_{52} - m_{42}m_{51})m_{63} + \\ + (m_{41}m_{62} - m_{42}m_{61})m_{53}] \\ = - \frac{n_{32}n_{54}}{m_{41}m_{63}} (m_{53}m_{62} - m_{52}m_{63}).$$

The equations (262) and (263) imply the fifth equation (261).

The fourth and the sixth condition (137) imply the sixth, the seventh, and the eighth equation (261).

The seventh condition (137) implies the ninth equation (261) and

$$(264) \quad m_{52}n_{54} + m_{62}n_{64} = (m_{52}m_{63} - m_{53}m_{62}) \frac{n_{54}}{m_{63}}.$$

On the other hand, we have by (152)

$$(265) \quad (m_{51}m_{63} - m_{53}m_{61}) \frac{m_{42}n_{54}}{m_{41}m_{63}} + \frac{n_{54}}{m_{41}m_{63}} r \\ = \frac{n_{54}}{m_{41}m_{63}} [(m_{51}m_{63} - m_{53}m_{61})m_{42} + (m_{41}m_{52} - m_{42}m_{51})m_{63} - \\ - (m_{41}m_{62} - m_{42}m_{61})m_{53}] \\ = \frac{n_{54}}{m_{63}} (m_{52}m_{63} - m_{53}m_{62}).$$

The equations (264) and (265) imply the tenth equation (261).

The last equation (261) is equivalent to the seventh condition (137).

Since all the equations (261) are satisfied, the function F can be written in the form (151). Thus it is nomographic.

This completes the proof of the First Fundamental Theorem.

Proof of the Second Fundamental Theorem

We shall prove our theorem for each principal case separately.

Principal case I. In this case we have

$$(266) \quad F \equiv X_1 Y_1 Z_1 + X_2 Y_3 Z_3,$$

where

$$Y_1 \equiv Y_1(y) \neq 0, \quad Y_3 \equiv Y_3(y) \neq 0, \quad Z_1 \equiv Z_1(z) \neq 0, \quad Z_3 \equiv Z_3(z) \neq 0 \\ (y \in \Omega_y, z \in \Omega_z).$$

If the function F is nomographic then for each of its Massau forms there exists — by Theorem 11 — an equivalent form (108) such that the identities (109) hold. Thus we have

$$\bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2 \equiv Y_1 Z_1, \quad \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3 \equiv Y_3 Z_3,$$

and there exist by Corollary 10.1 such numbers a_1, a_2, b_1, b_2 satisfying the conditions

$$(267) \quad a_1^2 + a_2^2 > 0, \quad b_1^2 + b_2^2 > 0,$$

that

$$(268) \quad \bar{Y}_2 \equiv a_1 Y_1, \quad \bar{Y}_3 \equiv a_2 Y_1, \quad a_1 \bar{Z}_3 - a_2 \bar{Z}_2 \equiv Z_1,$$

or

$$(269) \quad \bar{Z}_3 \equiv a_1 Z_1, \quad \bar{Z}_2 \equiv a_2 Z_1, \quad a_1 \bar{Y}_2 - a_2 \bar{Y}_3 \equiv Y_1,$$

and

$$(270) \quad \bar{Y}_3 \equiv b_1 Y_3, \quad \bar{Y}_1 \equiv b_2 Y_3, \quad b_1 \bar{Z}_1 - b_2 \bar{Z}_3 \equiv Z_3,$$

or

$$(271) \quad \bar{Z}_1 \equiv b_1 Z_3, \quad \bar{Z}_3 \equiv b_2 Z_3, \quad b_1 \bar{Y}_3 - b_2 \bar{Y}_1 \equiv Y_3.$$

We shall first show that the identities (268) and (270) cannot be simultaneous. Indeed, if $a_2 \neq 0$, we should have from (268) $Y_1 \equiv \bar{Y}_3/a_2$ and then

$$(272) \quad \bar{Y}_2 \equiv \frac{a_1}{a_2} \bar{Y}_3.$$

If $\bar{Y}_3 \equiv 0$, then we should also have $\bar{Y}_2 \equiv 0$, but this, by (109), would imply $G_1 \equiv 0$, which contradicts the assumption that F is a function of rank 2 with respect to x . Therefore, $\bar{Y}_3 \neq 0$ and by (270) $b_1 \neq 0$. Next we should have $Y_3 \equiv \bar{Y}_3/b_1$ and

$$(273) \quad \bar{Y}_1 \equiv \frac{b_2}{b_1} \bar{Y}_3.$$

It would follow from (108), (272) and (273) that the function F is of rank 0 or 1 with respect to y , which is a contradiction.

On the other hand, if $a_2 = 0$, then by (268) $\bar{Y}_3 \equiv 0$ and by (108) the function F would be of rank 0 or 1 with respect to z , which is also a contradiction. It follows that the identities (268) and (270) cannot be simultaneous.

We show analogously that the identities (269) and (271) cannot be simultaneous. Hence we have to consider only the following two cases:

Case A: the identities (268) and (271) occur simultaneously.

Case B: " (269) " (270) " "

We shall now show that in Case A

$$(274) \quad a_2 \neq 0, \quad b_2 \neq 0.$$

If $a_2 = 0$, then by (268) $\bar{Y}_3 \equiv 0$ and by (108) the function F would be of rank 0 or 1 with respect to z , which is a contradiction. Hence $a_2 \neq 0$. We show analogously that also $b_2 \neq 0$. Then we obtain from (268), (271):

$$\begin{aligned} \bar{Y}_1 &\equiv \frac{a_2 b_1}{b_2} Y_1 - \frac{1}{b_2} Y_3, & \bar{Z}_1 &\equiv b_1 Z_3, \\ \bar{Y}_2 &\equiv a_1 Y_1, & \bar{Z}_2 &\equiv -\frac{1}{a_2} Z_1 + \frac{a_1 b_2}{a_2} Z_3, \\ \bar{Y}_3 &\equiv a_2 Y_1, & \bar{Z}_3 &\equiv b_2 Z_3. \end{aligned}$$

Thus we can rewrite (108) in the form

$$(275) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ \frac{a_2 b_1}{b_2} Y_1 - \frac{1}{b_2} Y_3 & a_1 Y_1 & a_2 Y_1 \\ b_1 Z_3 & -\frac{1}{a_2} Z_1 + \frac{a_1 b_2}{a_2} Z_3 & b_2 Z_3 \end{vmatrix}.$$

Since

$$\begin{aligned} &\begin{bmatrix} X_1 & X_2 & 0 \\ \frac{a_2 b_1}{b_2} Y_1 - \frac{1}{b_2} Y_3 & a_1 Y_1 & a_2 Y_1 \\ b_1 Z_3 & -\frac{1}{a_2} Z_1 + \frac{a_1 b_2}{a_2} Z_3 & b_2 Z_3 \end{bmatrix} \\ &\begin{bmatrix} X_1 & X_2 & 0 \\ -\frac{1}{b_2} Y_2 & 0 & -\frac{1}{b_2} Y_1 \\ 0 & \frac{1}{a_2} (-Z_1) & \frac{1}{a_2} (-Z_3) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_2 b_1 & -a_1 b_2 & -a_2 b_2 \end{bmatrix}, \end{aligned}$$

then the Massau form (275) is equivalent to the second form (138). Thus in Case A each of the Massau forms of the function F is equivalent to the second form (138).

We analogously show that in Case B each of the Massau forms of the function F is equivalent to the first form (138).

Both forms (138) are possible, since their expansions give $X_1 Y_1 Z_1 + X_2 Y_3 Z_3$, as desired. If they were equivalent, there would exist such numbers c_{ij} ($i, j = 1, 2, 3$) and d_1, d_2 that

$$\begin{bmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & 0 & Z_1 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & 0 \\ d_1 Y_3 & 0 & d_1 Y_1 \\ 0 & -d_2 Z_1 & -d_2 Z_3 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

and

$$(276) \quad d_1 d_2 \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 1.$$

But it would then follow that

$$\begin{aligned} c_{13} X_1 + c_{23} X_2 &\equiv 0, \\ c_{13} d_1 Y_3 + c_{33} d_1 Y_1 &\equiv Y_3, \\ -c_{23} d_2 Z_1 - c_{33} d_2 Z_3 &\equiv Z_1. \end{aligned}$$

Since the functions X_1, X_2 are linearly independent, then $c_{13} = c_{23} = 0$ and we should have

$$Y_3 \equiv c_{33} d_1 Y_1, \quad Z_3 \equiv -\frac{1}{c_{33} d_2} Z_1 \quad (\text{by (276) there is } c_{33} d_2 \neq 0).$$

It would follow that

$$F \equiv X_1 Y_1 Z_1 + X_2 Y_3 Z_3 = \left(X_1 - \frac{d_1}{d_2} X_2 \right) Y_1 Z_1,$$

which contradicts the assumption that the function F is of rank 2 with respect to each of the variables x, y and z .

Therefore, the Massau forms (138) are not equivalent, the function F is in the first principal case doubly nomographic and each of its Massau forms is equivalent to one of the forms (138).

Principal case II. In this case we have (248) with (249) and (250).

If the function F is nomographic, then for each of its Massau forms there exists — by Theorem 11 — an equivalent form (108) such that

the identities (109) hold. Thus we have

$$\bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2 \equiv Y_1 Z_1 + Y_2 Z_2, \quad \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3 \equiv Y_3 Z_3,$$

and there exist, by Corollaries 9.2 and 10.1, such numbers a_{11} , a_{12} , a_{21} , a_{22} , b_1 , b_2 satisfying the conditions

$$(277) \quad a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad b_1^2 + b_2^2 > 0,$$

that

$$(278) \quad \begin{aligned} \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_3 &\equiv \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2, \\ \bar{Y}_3 &\equiv a_{21} Y_1 + a_{22} Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2, \end{aligned}$$

and

$$(279) \quad \bar{Y}_3 \equiv b_1 Y_3, \quad \bar{Y}_1 \equiv b_2 Y_3, \quad b_1 \bar{Z}_1 - b_2 \bar{Z}_3 \equiv Z_3,$$

or

$$(280) \quad \bar{Z}_1 \equiv b_1 Z_3, \quad \bar{Z}_3 \equiv b_2 Z_3, \quad b_1 \bar{Y}_3 - b_2 \bar{Y}_1 \equiv Y_3.$$

Thus we have to consider the following two cases:

Case A: the identities (278) and (279) occur simultaneously.

Case B: ,, (278) ,, (280) ,, ,,

In Case A we have

$$b_1 Y_3 \equiv a_{21} Y_1 + a_{22} Y_2.$$

Since the functions Y_1 , Y_2 are linearly independent and by (277) at least one of the numbers a_{21} , a_{22} differs from zero, we must have $b_1 \neq 0$ and

$$(281) \quad Y_3 \equiv \frac{a_{21}}{b_1} Y_1 + \frac{a_{22}}{b_1} Y_2.$$

The identities (249) and (281) imply

$$(282) \quad a_{21} = b_1 m_{31}, \quad a_{22} = b_1 m_{32}.$$

We then have by (278), (279), and (282)

$$\begin{aligned} \bar{Y}_1 &\equiv b_2 Y_3, & \bar{Z}_1 &\equiv \frac{b_2 m_{32}}{a} Z_1 - \frac{b_2 m_{31}}{a} Z_2 + \frac{1}{b_1} Z_3, \\ \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2, \\ \bar{Y}_3 &\equiv b_1 Y_3, & \bar{Z}_3 &\equiv \frac{b_1 m_{32}}{a} Z_1 - \frac{b_1 m_{31}}{a} Z_2. \end{aligned}$$

Therefore, we can rewrite (108) in the form

(283)

$$F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ b_2 Y_3 & a_{11} Y_1 + a_{12} Y_2 & b_1 Y_3 \\ \frac{b_2 m_{32}}{a} Z_1 - \frac{b_2 m_{31}}{a} Z_2 + \frac{1}{b_1} Z_3 & \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2 & \frac{b_1 m_{32}}{a} Z_1 - \frac{b_1 m_{31}}{a} Z_2 \end{vmatrix}.$$

Since by (249) and (250)

$$\begin{aligned} & \begin{bmatrix} X_1 & X_2 & 0 \\ b_2 Y_3 & a_{11} Y_1 + a_{12} Y_2 & b_1 Y_3 \\ \frac{b_2 m_{32}}{a} Z_1 - \frac{b_2 m_{31}}{a} Z_2 + \frac{1}{b_1} Z_3 & \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2 & \frac{b_1 m_{32}}{a} Z_1 - \frac{b_1 m_{31}}{a} Z_2 \end{bmatrix} \\ & \equiv \begin{bmatrix} X_1 & X_2 & 0 \\ 0 & \frac{a}{b_1 m_{32}} Y_1 & \frac{a}{b_1 m_{32}} \left(\frac{m_{31}}{m_{32}} Y_1 + Y_2 \right) \\ \frac{1}{b_1 m_{32}} (m_{32} n_{31} Z_1 + m_{32} n_{32} Z_2) & \frac{1}{b_1 m_{32}} (-Z_2) & \frac{1}{b_1 m_{32}} \left(Z_1 - \frac{m_{31}}{m_{32}} Z_2 \right) \end{bmatrix} \times \\ & \quad \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{b_1 b_2 m_{32}^2}{a} & \frac{a_{12} b_1 m_{32}}{a} & \frac{b_1^2 m_{32}^2}{a} \end{bmatrix}, \end{aligned}$$

where by (282)

$$a = a_{11} a_{22} - a_{12} a_{21} = (m_{32} a_{11} - m_{31} a_{12}) b_1,$$

then the Massau form (283) is equivalent to the first form (139), i. e., the first form (140), i. e., the form (141). It follows that in Case A each of the Massau forms of the function F is equivalent to the first form (139), i. e., the first form (140), i. e., the form (141).

Analogously, we show that in Case B each of the Massau forms of the function F is equivalent to the second form (139) if $n_{32} \neq 0$, and to the second form (140) if $n_{32} = 0$.

The two forms (139) are equivalent if and only if there exist such numbers c_{ij} ($i, j = 1, 2, 3$) and d_1, d_2 that

$$\begin{bmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & \frac{m_{31}}{m_{32}} Y_1 + Y_2 \\ m_{32}n_{31}Z_1 + m_{32}n_{32}Z_2 & -Z_2 & Z_1 - \frac{m_{31}}{m_{32}} Z_2 \end{bmatrix} \\ \equiv \begin{bmatrix} X_1 & X_2 & 0 \\ d_1(m_{31}n_{32}Y_1 + m_{32}n_{32}Y_2) & -d_1Y_2 & d_1\left(Y_1 - \frac{n_{31}}{n_{32}}Y_2\right) \\ 0 & -d_2Z_1 & -d_2\left(\frac{n_{31}}{n_{32}}Z_1 + Z_2\right) \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

and

$$(284) \quad d_1 d_2 \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 1,$$

i. e., if and only if the following simultaneous identities are satisfied:

$$c_{11}X_1 + c_{21}X_2 \equiv X_1, \quad c_{12}X_1 + c_{22}X_2 \equiv X_2, \quad c_{13}X_1 + c_{23}X_2 \equiv 0,$$

$$c_{11}d_1n_{32}(m_{31}Y_1 + m_{32}Y_2) - c_{21}d_1Y_2 + c_{31}d_1\left(Y_1 - \frac{n_{31}}{n_{32}}Y_2\right) \equiv 0,$$

$$c_{12}d_1n_{32}(m_{31}Y_1 + m_{32}Y_2) - c_{22}d_1Y_2 + c_{32}d_1\left(Y_1 - \frac{n_{31}}{n_{32}}Y_2\right) \equiv Y_1,$$

$$c_{13}d_1n_{32}(m_{31}Y_1 + m_{32}Y_2) - c_{23}d_1Y_2 + c_{33}d_1\left(Y_1 - \frac{n_{31}}{n_{32}}Y_2\right) \equiv \frac{m_{31}}{m_{32}}Y_1 + Y_2,$$

$$-c_{21}d_2Z_1 - c_{31}d_2\left(\frac{n_{31}}{n_{32}}Z_1 + Z_2\right) \equiv m_{32}n_{31}Z_1 + m_{32}n_{32}Z_2,$$

$$-c_{22}d_2Z_1 - c_{32}d_2\left(\frac{n_{31}}{n_{32}}Z_1 + Z_2\right) \equiv -Z_2,$$

$$-c_{23}d_2Z_1 - c_{33}d_2\left(\frac{n_{31}}{n_{32}}Z_1 + Z_2\right) \equiv Z_1 - \frac{m_{31}}{m_{32}}Z_2,$$

with the equation (284).

Since the functions X_1, X_2 , the functions Y_1, Y_2 and the functions Z_1, Z_2 — separately treated — are linearly independent, this system of

identities is by (250) equivalent to

$$\begin{aligned}
 c_{11} &= 1, & c_{21} &= 0, & c_{12} &= 0, & c_{22} &= 1, & c_{13} &= 0, & c_{23} &= 0, \\
 c_{11}d_1m_{31}n_{32} + c_{31}d_1 &= 0, & c_{11}d_1m_{32}n_{32} - c_{21}d_1 - c_{31}d_1\frac{n_{31}}{n_{32}} &= 0, \\
 c_{12}d_1m_{31}n_{32} + c_{32}d_1 &= 1, & c_{12}d_1m_{32}n_{32} - c_{22}d_1 - c_{32}d_1\frac{n_{31}}{n_{32}} &= 0, \\
 c_{13}d_1m_{31}n_{32} + c_{33}d_1 &= \frac{m_{31}}{m_{32}}, & c_{13}d_1m_{32}n_{32} - c_{23}d_1 - c_{33}d_1\frac{n_{31}}{n_{32}} &= 1, \\
 -c_{21}d_2 - c_{31}d_2\frac{n_{31}}{n_{32}} &= m_{32}n_{31}, & -c_{31}d_2 &= m_{32}n_{31}, \\
 -c_{22}d_2 - c_{32}d_2\frac{n_{31}}{n_{32}} &= 0, & -c_{32}d_2 &= -1, \\
 -c_{23}d_2 - c_{33}d_2\frac{n_{31}}{n_{32}} &= 1, & -c_{33}d_2 &= -\frac{m_{31}}{m_{32}},
 \end{aligned}$$

with the equation (284), i. e. (d_1 and d_2 differing from zero by (284)), to the system of equations:

$$\begin{aligned}
 c_{11} &= c_{22} = 1, & c_{12} &= c_{13} = c_{21} = c_{23} = 0, \\
 d_1m_{31}n_{32} + c_{31}d_1 &= 0, & d_1m_{32}n_{32} - c_{31}d_1\frac{n_{31}}{n_{32}} &= 0, \\
 c_{32}d_1 &= 1, & -d_1 - c_{32}d_1\frac{n_{31}}{n_{32}} &= 0, \\
 c_{33}d_1 &= \frac{m_{31}}{m_{32}}, & -c_{33}d_1\frac{n_{31}}{n_{32}} &= 1, \\
 -c_{31}d_2\frac{n_{31}}{n_{32}} &= m_{32}n_{31}, & -c_{31}d_2 &= m_{32}n_{31}, \\
 -d_2 - c_{32}d_2\frac{n_{31}}{n_{32}} &= 0, & -c_{32}d_2 &= -1, \\
 -c_{33}d_2\frac{n_{31}}{n_{32}} &= 1, & -c_{33}d_2 &= -\frac{m_{31}}{m_{32}}, \\
 d_1d_2c_{33} &= 1.
 \end{aligned}$$

This system has a unique solution:

$$\begin{aligned}
 c_{11} &= c_{22} = 1, & c_{12} &= c_{21} = c_{13} = c_{23} = 0, \\
 c_{31} &= -m_{31}n_{32}, & c_{32} &= \frac{m_{31}}{m_{32}}, & c_{33} &= \frac{m_{31}^2}{m_{32}^2}, & d_1 &= -\frac{n_{31}}{n_{32}}, & d_2 &= -\frac{n_{31}}{n_{32}},
 \end{aligned}$$

if and only if

$$(285) \quad m_{31}n_{31} + m_{32}n_{32} = 0.$$

Thus, the two forms (139) are equivalent if and only if the condition (285) is satisfied. Both forms (139) are possible, since their expansions give by (249) the identity (248), as desired.

We show in a similar way that the Massau forms (140) are equivalent if and only if the condition (285) is satisfied and that both those forms are possible.

Therefore: If $m_{31}n_{31} + m_{32}n_{32} \neq 0$, the function F is doubly nomographic and each of its Massau forms is equivalent to one of the two non-equivalent forms (139) when $n_{32} \neq 0$, and to one of the two non-equivalent forms (140) when $n_{32} = 0$. If $m_{31}n_{31} + m_{32}n_{32} = 0$ the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (141). This exactly was to be proved in the second principal case.

Principal case III. In this case we have the identity (153) and the conditions (154). If the function F is nomographic, then — by Theorem 11 — for each of its Massau forms there exists an equivalent form (108) such that the identities (155) hold. We introduce the functions (156) and we have the identities (157) and (158). Now, there exist such numbers a_{ij}, b_{ij} ($i, j = 1, 2$) satisfying the conditions (159) that the identities (160) hold. From (160) we obtain (161). We also have the identities (162) with (163) and the condition (164). It follows next the system of equations (165), which implies (166) and (167) with (168) and (169).

By (168) the system of equations (166) is equivalent to

$$(286) \quad r_{32}b_{11} + (r_{42} - p)b_{12} = 0, \quad (r_{31} - p)b_{11} + r_{41}b_{12} = 0.$$

We shall consider the following two cases: case A: $r_{32} \neq 0$, case B: $r_{32} = 0$.

Case A. We obtain from (286)

$$(287) \quad b_{11} = \frac{p - r_{42}}{r_{32}} b_{12}$$

and making use of (165)

$$(288) \quad a_{21} = \frac{p - r_{42}}{r_{32}} pb_{12}, \quad a_{22} = pb_{12}.$$

It follows from (287) by (159) that

$$(289) \quad b_{12} \neq 0.$$

By (160), (287) and (288) we have

$$\begin{aligned} \bar{Y}_1 &\equiv b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4, & \bar{Z}_1 &\equiv \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2, \\ \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{pb} Z_1 - \frac{a_{11}}{pb} Z_2, \\ \bar{Y}_3 &\equiv \left(\frac{p-r_{42}}{r_{32}} \tilde{Y}_3 + \tilde{Y}_4 \right) b_{12}, & \bar{Z}_3 &\equiv \left(Z_1 - \frac{p-r_{42}}{r_{32}} Z_2 \right) \frac{b_{12}}{b}, \end{aligned}$$

and the Massau form (108) can be rewritten as follows:

$$(290) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4 & a_{11} Y_1 + a_{12} Y_2 & \left(\frac{p-r_{42}}{r_{32}} \tilde{Y}_3 + \tilde{Y}_4 \right) b_{12} \\ \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2 & \frac{a_{12}}{pb} Z_1 - \frac{a_{11}}{pb} Z_2 & \left(Z_1 - \frac{p-r_{42}}{r_{32}} Z_2 \right) \frac{b_{12}}{b} \end{vmatrix}.$$

Now we shall show that

$$(291) \quad \begin{vmatrix} X_1 & X_2 & 0 \\ b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4 & a_{11} Y_1 + a_{12} Y_2 & \left(\frac{p-r_{42}}{r_{32}} \tilde{Y}_3 + \tilde{Y}_4 \right) b_{12} \\ \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2 & \frac{a_{12}}{pb} Z_1 - \frac{a_{11}}{pb} Z_2 & \left(Z_1 - \frac{p-r_{42}}{r_{32}} Z_2 \right) \frac{b_{12}}{b} \end{vmatrix} \\ \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ -\frac{b}{b_{12}} (r_{31} Y_1 + r_{32} Y_2) & -\frac{b}{b_{12}} (-Y_1) & -\frac{b}{b_{12}} \left(\frac{p-r_{42}}{r_{32}} Y_1 + Y_2 \right) \\ \frac{1}{pb_{12}} pZ_2 & \frac{1}{pb_{12}} (-Z_2) & \frac{1}{pb_{12}} \left(\frac{p-r_{42}}{r_{32}} Z_2 - Z_1 \right) \end{vmatrix} \times \\ \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{pb_{12} b_{22}}{b} & -\frac{a_{12} b_{12}}{b} & -\frac{pb_{12}^2}{b} \end{vmatrix}.$$

Multiplying the matrices on the right side of (291) we obtain the following identities:

$$(292) \quad \left\{ \begin{array}{l} X_1 \equiv X_1, \quad X_2 \equiv X_2, \quad 0 \equiv 0, \\ b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4 \equiv -\frac{b}{b_{12}} (r_{31} Y_1 + r_{32} Y_2) + pb_{22} \left(\frac{p-r_{42}}{r_{32}} Y_1 + Y_2 \right), \\ a_{11} Y_1 + a_{12} Y_2 \equiv \frac{b}{b_{12}} Y_1 + a_{12} \left(\frac{p-r_{42}}{r_{32}} Y_1 + Y_2 \right), \\ \left(\frac{p-r_{42}}{r_{32}} \tilde{Y}_3 + \tilde{Y}_4 \right) b_{12} \equiv pb_{12} \left(\frac{p-r_{42}}{r_{32}} Y_1 + Y_2 \right), \\ \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2 \equiv \frac{1}{b_{12}} Z_2 - \frac{b_{22}}{b} \left(\frac{p-r_{42}}{r_{32}} Z_2 - Z_1 \right), \\ \frac{a_{12}}{pb} Z_1 - \frac{a_{11}}{pb} Z_2 \equiv -\frac{1}{pb_{12}} Z_2 - \frac{a_{12}}{pb} \left(\frac{p-r_{42}}{r_{32}} Z_2 - Z_1 \right), \\ \left(Z_1 - \frac{p-r_{42}}{r_{32}} Z_2 \right) \frac{b_{12}}{b} \equiv -\frac{b_{12}}{b} \left(\frac{p-r_{42}}{r_{32}} Z_2 - Z_1 \right). \end{array} \right.$$

The first three identities (292) and the last one are obviously true. The remaining identities — by Theorem 4 — hold if and only if

$$(293) \quad \left\{ \begin{array}{l} b_{21} r_{31} + b_{22} r_{41} = -\frac{b}{b_{12}} r_{31} + pb_{22} \frac{p-r_{42}}{r_{32}}, \\ b_{21} r_{32} + b_{22} r_{42} = -\frac{b}{b_{12}} r_{32} + pb_{22}, \quad a_{11} = \frac{b}{b_{12}} + a_{12} \frac{p-r_{42}}{r_{32}}, \\ \frac{p-r_{42}}{r_{32}} r_{31} + r_{41} = p \frac{p-r_{42}}{r_{32}}, \quad p-r_{42} + r_{42} = p, \\ -\frac{b_{21}}{b} = \frac{1}{b_{12}} - \frac{b_{22}}{b} \frac{p-r_{42}}{r_{32}}, \quad -\frac{a_{11}}{pb} = -\frac{1}{pb_{12}} - \frac{a_{12}}{pb} \frac{p-r_{42}}{r_{32}}. \end{array} \right.$$

The first equation (293) can be rewritten in the form

$$b_{21} r_{31} + b_{22} r_{41} = -\frac{b_{11} b_{22} - b_{12} b_{21}}{b_{12}} r_{31} + b_{22} \frac{p^2 - pr_{42}}{r_{32}}$$

and it is equivalent to

$$b_{22} \left(\frac{p^2 - pr_{42} - r_{32} r_{41}}{r_{32}} - \frac{b_{11}}{b_{12}} r_{31} \right) = 0,$$

i. e., by (287)

$$b_{22} (p^2 - pr_{42} - r_{32} r_{41} - pr_{31} + r_{31} r_{42}) = 0$$

and this equation is satisfied by (167). Thus the first equation (293) is true.

We show analogously that the second equation (293) is also true.

In order to prove the third equation (293) we notice that by (168) and (288)

$$a = a_{11}a_{22} - a_{12}a_{21} = \left(a_{11} - \frac{p - r_{42}}{r_{32}} a_{12} \right) pb_{12} = pb,$$

whence

$$a_{11} - \frac{p - r_{42}}{r_{32}} a_{12} = \frac{b}{b_{12}},$$

i. e., the third equality (293).

The fourth equation (293) is true, since it is equivalent to (167). The fifth one is trivial. The sixth equation (293) is equivalent to the second one and the seventh equation (293) is equivalent to the third one. Therefore, all the equations (293) are true. It follows by (291) that the Massau form (290) is equivalent to

$$(294) \quad F = \begin{vmatrix} X_1 & X_2 & 0 \\ r_{31} Y_1 + r_{32} Y_2 & -Y_1 & \frac{p - r_{42}}{r_{32}} Y_1 + Y_2 \\ pZ_2 & -Z_2 & \frac{p - r_{42}}{r_{32}} Z_2 - Z_1 \end{vmatrix}$$

and we show, as for the identity (252), that the identity (294) is true. Hence in Case A each of the Massau forms of the function F is equivalent to (294), where, however, p must be a solution of the equation (144). Thus in Case A each of the Massau forms of the function F is equivalent to one of the forms (143).

The Massau forms (143) are equivalent if and only if there exist such numbers c_{ij} ($i, j = 1, 2, 3$) and d_1, d_2 that

$$(295) \quad \begin{bmatrix} X_1 & X_2 & 0 \\ r_{31} Y_1 + r_{32} Y_2 & -Y_1 & \frac{p_1 - r_{42}}{r_{32}} Y_1 + Y_2 \\ p_1 Z_2 & -Z_2 & \frac{p_1 - r_{42}}{r_{32}} Z_2 - Z_1 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & 0 \\ d_1(r_{31} Y_1 + r_{32} Y_2) & -d_1 Y_1 & d_1 \left(\frac{p_2 - r_{42}}{r_{32}} Y_1 + Y_2 \right) \\ d_2 p_2 Z_2 & -d_2 Z_2 & d_2 \left(\frac{p_2 - r_{42}}{r_{32}} Z_2 - Z_1 \right) \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

and

$$(296) \quad d_1 d_2 \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 1.$$

Multiplying on the right side of (295) the first row of the first matrix by the third column of the second one, we obtain

$$c_{13} X_1 + c_{23} X_2 \equiv 0,$$

whence — by the linear independence of the functions X_1, X_2 —

$$c_{13} = c_{23} = 0.$$

Now multiplying the second row of the first matrix by the third column of the second one, we obtain

$$\frac{p_1 - r_{42}}{r_{32}} Y_1 + Y_2 \equiv \left(\frac{p_2 - r_{42}}{r_{32}} Y_1 + Y_2 \right) d_1 c_{33},$$

whence — by the linear independence of the functions Y_1, Y_2 —

$$\frac{p_1 - r_{42}}{r_{32}} = \frac{p_2 - r_{42}}{r_{32}} d_1 c_{33}, \quad 1 = d_1 c_{33},$$

and next

$$p_1 = p_2.$$

Thus the two forms (143) are equivalent if and only if the equation (144) has a unique solution, i. e., if and only if

$$(r_{31} + r_{42})^2 - 4(r_{31} r_{42} - r_{32} r_{41}) = (r_{31} - r_{42})^2 + 4r_{32} r_{41} = 0.$$

Then $p = (r_{31} + r_{42})/2$ and each of the forms (143) can be rewritten in the form (146). Thus we have obtained the following result for Case A:

If $(r_{31} - r_{42})^2 + 4r_{32} r_{41} \neq 0$, the function F is doubly nomographic and each of its Massau forms is equivalent to one of the non-equivalent forms (143). If $(r_{31} - r_{42})^2 + 4r_{32} r_{41} = 0$, the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (146).

Case B. Now we have $r_{32} = 0$ and by (164)

$$(297) \quad r_{31} \neq 0, \quad r_{42} \neq 0.$$

It follows from (167) that

$$p = r_{31} \quad \text{OR} \quad p = r_{42}.$$

Therefore, we shall consider the following three sub-cases:

B-1: $p = r_{31} \neq r_{42}$,

B-2: $p = r_{42} \neq r_{31}$,

B-3: $p = r_{31} = r_{42}$.

Case B-1. Now we have by (286)

$$(298) \quad b_{12} = 0$$

and then by (159)

$$(299) \quad b_{11} \neq 0.$$

Moreover, it follows from (159), (165) and (298) that

$$(300) \quad a_{21} = r_{31}b_{11}, \quad a_{22} = 0,$$

and

$$(301) \quad a = a_{11}a_{22} - a_{12}a_{21} = -a_{12}a_{21} = -r_{31}a_{12}b_{11} \neq 0,$$

$$b = b_{11}b_{22} - b_{12}b_{21} = b_{11}b_{22} \neq 0,$$

and from (168)

$$(302) \quad a = r_{31}b.$$

The equalities (301) and (302) imply by (297) and (299)

$$(303) \quad a_{12} = -b_{22}, \quad a = r_{31}b_{11}b_{22}.$$

It follows from (160), (298), (300), (301), and (303) that

$$\bar{Y}_1 \equiv b_{21}\tilde{Y}_3 + b_{22}\tilde{Y}_4, \quad \bar{Z}_1 \equiv \frac{1}{b_{11}}Z_1 - \frac{b_{21}}{b_{11}b_{22}}Z_2,$$

$$\bar{Y}_2 \equiv a_{11}Y_1 - b_{22}Y_2, \quad \bar{Z}_2 \equiv -\frac{1}{r_{31}b_{11}}Z_1 - \frac{a_{11}}{r_{31}b_{11}b_{22}}Z_2,$$

$$\bar{Y}_3 \equiv b_{11}\tilde{Y}_3, \quad \bar{Z}_3 \equiv -\frac{1}{b_{22}}Z_2.$$

Therefore, we can rewrite (108) in the form

$$(304) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ b_{21}\tilde{Y}_3 + b_{22}\tilde{Y}_4 & a_{11}Y_1 - b_{22}Y_2 & b_{11}\tilde{Y}_3 \\ \frac{1}{b_{11}}Z_1 - \frac{b_{21}}{b_{11}b_{22}}Z_2 & -\frac{1}{r_{31}b_{11}}Z_1 - \frac{a_{11}}{r_{31}b_{11}b_{22}}Z_2 & -\frac{1}{b_{22}}Z_2 \end{vmatrix}.$$

We simply verify by (162) that

$$\begin{bmatrix} X_1 & X_2 & 0 \\ b_{21}\tilde{Y}_3 + b_{22}\tilde{Y}_4 & a_{11}Y_1 - b_{22}Y_2 & b_{11}\tilde{Y}_3 \\ \frac{1}{b_{11}}Z_1 - \frac{b_{21}}{b_{11}b_{22}}Z_2 & -\frac{1}{r_{31}b_{11}}Z_1 - \frac{a_{11}}{r_{31}b_{11}b_{22}}Z_2 & -\frac{1}{b_{22}}Z_2 \end{bmatrix} \\ = \begin{bmatrix} X_1 & X_2 & 0 \\ b_{22}(r_{41}Y_1 + r_{42}Y_2) & b_{22}(-Y_2) & b_{22}r_{31}Y_1 \\ \frac{1}{b_{11}}Z_1 & \frac{1}{b_{11}}\left(-\frac{1}{r_{31}}Z_1\right) & \frac{1}{b_{11}}(-Z_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{b_{21}}{b_{22}} & \frac{a_{11}}{r_{31}b_{22}} & \frac{b_{11}}{b_{22}} \end{bmatrix}.$$

Thus the Massau form (304) and also the form (108) are equivalent in Case B-1 to the first form (145).

Case B-2. Now we have by (286) and (159)

$$(305) \quad b_{12} \neq 0$$

and

$$(306) \quad b_{11} = \frac{r_{41}}{r_{42} - r_{31}} b_{12}.$$

Moreover, it follows from (165) and (306) that

$$(307) \quad a_{21} = \frac{r_{41}r_{42}}{r_{42} - r_{31}} b_{12}, \quad a_{22} = r_{42} b_{12},$$

and

$$(308) \quad a = a_{11}a_{22} - a_{12}a_{21} = \left(a_{11} - \frac{r_{41}}{r_{42} - r_{31}} a_{12} \right) r_{42} b_{12}, \\ b = b_{11}b_{22} - b_{12}b_{21} = \left(\frac{r_{41}}{r_{42} - r_{31}} b_{22} - b_{21} \right) b_{12}.$$

Since by (168)

$$(309) \quad a = r_{42}b,$$

we obtain from (308) by (297) and (305)

$$a_{11} - \frac{r_{41}}{r_{42} - r_{31}} a_{12} = \frac{r_{41}}{r_{42} - r_{31}} b_{22} - b_{21}.$$

Hence we have

$$(310) \quad a_{11} = \frac{r_{41}}{r_{42} - r_{31}} (a_{12} + b_{22}) - b_{21}$$

and

$$(311) \quad a = \left(\frac{r_{41}}{r_{42} - r_{31}} b_{22} - b_{21} \right) r_{42} b_{12}.$$

It follows from (160), (306), (307), (308), (309), and (310) that

$$\bar{Y}_1 \equiv b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4,$$

$$\bar{Y}_2 \equiv \left[\frac{r_{41}}{r_{42} - r_{31}} (a_{12} + b_{22}) - b_{21} \right] Y_1 + a_{12} Y_2,$$

$$\bar{Y}_3 \equiv \frac{r_{41} r_{42}}{r_{42} - r_{31}} b_{12} Y_1 + r_{42} b_{12} Y_2,$$

$$\bar{Z}_1 \equiv \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2,$$

$$\bar{Z}_2 \equiv \frac{a_{12}}{r_{42} b} Z_1 - \left[\frac{r_{41}}{r_{42} - r_{31}} (a_{12} + b_{22}) - b_{21} \right] \frac{1}{r_{42} b} Z_2,$$

$$\bar{Z}_3 \equiv \left(Z_1 - \frac{r_{41}}{r_{42} - r_{31}} Z_2 \right) \frac{b_{12}}{b}.$$

Therefore, we can rewrite (108) in the form

$$(312) \quad F \equiv$$

$$\left| \begin{array}{ccc} X_1 & X_2 & 0 \\ b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4 & \left[\frac{r_{41}}{r_{42} - r_{31}} (a_{12} + b_{22}) - b_{21} \right] Y_1 + a_{12} Y_2 & \frac{r_{41} r_{42}}{r_{42} - r_{31}} b_{12} Y_1 + r_{42} b_{12} Y_2 \\ \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2 & \frac{a_{12}}{r_{42} b} Z_1 - \left[\frac{r_{41}}{r_{42} - r_{31}} (a_{12} + b_{22}) - b_{21} \right] \frac{1}{r_{42} b} Z_2 & \left(Z_1 - \frac{r_{41}}{r_{42} - r_{31}} Z_2 \right) \frac{b_{12}}{b} \end{array} \right|.$$

We simply verify by (162) and (308) that

$$\begin{bmatrix} X_1 & X_2 & 0 \\ b_{21} \tilde{Y}_3 + b_{22} \tilde{Y}_4 & \left[\frac{r_{41}}{r_{42} - r_{31}} (a_{12} + b_{22}) - \right. & \left. \frac{r_{41} r_{42}}{r_{42} - r_{31}} b_{12} Y_1 + r_{42} b_{12} Y_2 \right. \\ & \left. - b_{21} \right] Y_1 + a_{12} Y_2 & \\ \frac{b_{22}}{b} Z_1 - \frac{b_{21}}{b} Z_2 & \frac{a_{12}}{r_{42} b} Z_1 - \left[\frac{r_{41}}{r_{42} - r_{31}} (a_{12} + \right. & \left. \left(Z_1 - \frac{r_{41}}{r_{42} - r_{31}} Z_2 \right) \frac{b_{12}}{b} \right. \\ & \left. + b_{22}) - b_{21} \right] \frac{1}{r_{42} b} Z_2 & \end{bmatrix}$$

$$\equiv \begin{bmatrix} X_1 & X_2 & 0 \\ -\frac{b}{b_{12}} r_{31} Y_1 & -\frac{b}{b_{12}} (-Y_1) & -\frac{b}{b_{12}} \left(\frac{r_{41}}{r_{42} - r_{31}} Y_1 + Y_2 \right) \\ \frac{1}{r_{42} b_{12}} r_{42} Z_2 & \frac{1}{r_{42} b_{12}} (-Z_2) & \frac{1}{r_{42} b_{12}} \left(\frac{r_{41}}{r_{42} - r_{31}} Z_2 - Z_1 \right) \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{r_{42} b_{12} b_{22}}{-b} & \frac{a_{12} b_{12}}{-b} & \frac{r_{42} b_{12}^2}{-b} \end{bmatrix}$$

Thus, the Massau form (312) and also the form (108) are equivalent in Case B-2 to the second form (145).

Case B-3. Now we have by (286)

$$(313) \quad r_{41} b_{12} = 0.$$

If $r_{41} = 0$, then we should have by (162)

$$\tilde{Y}_3 \equiv r_{31} Y_1, \quad \tilde{Y}_4 \equiv r_{42} Y_2 \equiv r_{31} Y_2,$$

and by (157)

$$G_2 \equiv Y_3 Z_3 + Y_4 Z_4 \equiv \tilde{Y}_3 Z_1 + \tilde{Y}_4 Z_2 \equiv r_{31} (Y_1 Z_1 + Y_2 Z_2) \equiv r_{31} G_1.$$

Thus the functions G_1, G_2 would be linearly dependent and by Theorem 8 the function F would not be of rank 2 with respect to x , which is a contradiction. It follows that

$$(314) \quad r_{41} \neq 0$$

and by (313)

$$(315) \quad b_{12} = 0.$$

Then we have by (159) the condition (299) and the equations (300), (301), (302), (303). It follows that in Case B-3 we can also rewrite (108) in the form (304) equivalent to the first form (145), i. e., to the form (147).

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} \neq 0$ and $r_{32} = 0$, then $r_{31} - r_{42} \neq 0$, both cases B-1 and B-2 are possible and each of the Massau forms of the function F is equivalent to one of the forms (145). Both forms (145) are possible, since by expanding them we obtain after simple computations

$$F \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(\tilde{Y}_3Z_1 + \tilde{Y}_4Z_2) \equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2(Y_3Z_3 + Y_4Z_4)$$

as desired.

If the forms (145) were equivalent, there would exist such numbers c_{ij} ($i, j = 1, 2, 3$) and d_1, d_2 that

$$(316) \quad \begin{bmatrix} X_1 & X_2 & 0 \\ r_{41}Y_1 + r_{42}Y_2 & -Y_2 & r_{31}Y_1 \\ Z_1 & -\frac{1}{r_{31}}Z_1 & -Z_2 \end{bmatrix} \equiv \begin{bmatrix} X_1 & X_2 & 0 \\ d_1r_{31}Y_1 & -d_1Y_1 & d_1\left(\frac{r_{41}}{r_{42}-r_{31}}Y_1 + Y_2\right) \\ d_2r_{42}Z_2 & -d_2Z_2 & d_2\left(\frac{r_{41}}{r_{42}-r_{31}}Z_2 - Z_1\right) \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

and

$$(317) \quad d_1d_2 \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 1.$$

Multiplying on the right side of (316) the first row of the first matrix by the third column of the second one, we should obtain by (316)

$$c_{13}X_1 + c_{23}X_2 \equiv 0,$$

whence by the linear independence of the functions X_1, X_2

$$(318) \quad c_{13} = c_{23} = 0.$$

Multiplying the second row of the first matrix by the third column of the second one we should obtain by (316) and (318)

$$r_{31} Y_1 \equiv d_1 c_{33} \left(\frac{r_{41}}{r_{42} - r_{31}} Y_1 + Y_2 \right),$$

whence by the linear independence of the functions Y_1, Y_2

$$d_1 c_{33} = 0,$$

which is a contradiction of (317). It follows that the forms (145) are not equivalent.

Thus, if $(r_{31} - r_{42})^2 + 4r_{32}r_{41} \neq 0$ and $r_{32} = 0$, then the function F is doubly nomographic and each of its Massau forms is equivalent to one of the non-equivalent forms (145).

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} = 0$ and $r_{32} = 0$, then $r_{31} = r_{42}$ and we have the case B-3. Then the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (147).

This completes the proof in the third principal case.

Principal case IV. Now we have

$$F \equiv X_1 G_1 + X_2 G_2,$$

where

$$G_1 \equiv Y_1 Z_1 + Y_2 Z_2, \quad G_2 \equiv Y_3 Z_3,$$

the functions $Y_1 \equiv Y_1(y)$, $Y_2 \equiv Y_2(y)$ and the functions $Z_1 \equiv Z_1(z)$, $Z_2 \equiv Z_2(z)$ — separately treated — being linearly independent. Moreover, we have the identity (255).

If the function F is nomographic, then for each of its Massau forms there exists — by Theorem 11 — an equivalent form (108) such that the identities (109) hold. Thus we have

$$\bar{Y}_2 \bar{Z}_3 - \bar{Y}_3 \bar{Z}_2 \equiv Y_1 Z_1 + Y_2 Z_2, \quad \bar{Y}_3 \bar{Z}_1 - \bar{Y}_1 \bar{Z}_3 \equiv Y_3 Z_3,$$

and there exist by Corollaries 9.2 and 10.1 such numbers $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ satisfying the conditions (277) that the identities (278) and (279) or (280) hold.

If the identities (278) and (280) were simultaneous, we should have

$$b_2 Z_3 \equiv \frac{a_{22}}{a} Z_1 - \frac{a_{21}}{a} Z_2.$$

Since the functions Z_1, Z_2 are linearly independent and by (277) at least one of the numbers a_{21}, a_{22} differs from zero, we should have $b_2 \neq 0$ and

$$Z_3 \equiv \frac{a_{22}}{ab_2} Z_1 - \frac{a_{21}}{ab_2} Z_2.$$

It would follow that

$$\begin{aligned}
 F &\equiv X_1(Y_1Z_1 + Y_2Z_2) + X_2Y_3Z_3 \\
 &\equiv \left(X_1Y_1 + \frac{a_{22}}{ab_2} X_2Y_3 \right) Z_1 + \left(X_1Y_2 - \frac{a_{21}}{ab_2} X_2Y_3 \right) Z_2,
 \end{aligned}$$

which contradicts the assumption that the function F is of rank 3 with respect to z . It follows that in the fourth principal case the identities (280) cannot occur and the functions Z_1, Z_2, Z_3 must be linearly independent.

The identities (278) and (279) imply (281), whence we have (282) and we can write (108) in the form (283) (by (277) and (282) it is $b_1 \neq 0$).

Since the proof of (250) is valid also in the present case and by (255)

$$\begin{aligned}
 &\left[\begin{array}{ccc} X_1 & X_2 & 0 \\ b_2 Y_3 & a_{11} Y_1 + a_{12} Y_2 & b_1 Y_3 \\ \frac{b_2 m_{32}}{a} Z_1 - \frac{b_2 m_{31}}{a} Z_2 + \frac{1}{b_1} Z_3 & \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2 & \frac{b_1 m_{32}}{a} Z_1 - \frac{b_1 m_{31}}{a} Z_2 \end{array} \right] \\
 &\equiv \left[\begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & \frac{a}{b_1 m_{32}} Y_1 & \frac{a}{b_1 m_{32}} \left(\frac{m_{31}}{m_{32}} Y_1 + Y_2 \right) \\ \frac{1}{b_1 m_{32}} m_{32} Z_3 & \frac{1}{b_1 m_{32}} (-Z_2) & \frac{1}{b_1 m_{32}} \left(Z_1 - \frac{m_{31}}{m_{32}} Z_2 \right) \end{array} \right] \times \\
 &\qquad \times \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{b_1 b_2 m_{32}^2}{a} & \frac{a_{12} b_1 m_{32}}{a} & \frac{b_1^2 m_{32}^2}{a} \end{array} \right]
 \end{aligned}$$

where by (282)

$$a = a_{11} a_{22} - a_{12} a_{21} = (m_{32} a_{11} - m_{31} a_{12}) b_1,$$

the Massau form (283) and thus the form (108) are equivalent to the form (148). It follows by Theorem 11 that the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (148).

Principal case V. The proof of (170) is valid also here.

If the function F is nomographic, then for each of its Massau forms there exists by Theorem 11 an equivalent form (108) such that the identities (109) hold. Then we have the identities (155) and — by Corollary 9.2 — there exist such numbers a_{ij}, b_{ij} ($i, j = 1, 2$) satisfying the condi-

tions (159) that the identities (171) and (172) hold. Hence we obtain (173), (174), (175), (176), and from (176) by (170)

$$(319) \quad a = - \frac{m_{31}n_{34} + m_{41}}{n_{32}} b.$$

By (171), (175), (319), and the last equation (173) we have

$$\begin{aligned} \bar{Y}_1 &\equiv b_{21} Y_3 + b_{23} Y_4, & \bar{Z}_1 &\equiv \frac{b_{22}}{b} Z_3 - \frac{b_{21}}{b} Z_4, \\ \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_2 &\equiv - \frac{n_{32}}{(m_{31}n_{34} + m_{41})b} (a_{12}Z_1 - a_{11}Z_2), \\ \bar{Y}_3 &\equiv [(m_{31}n_{34} + m_{41}) Y_1 + (m_{32}n_{34} + m_{42}) Y_2] b_{12}, \\ & & \bar{Z}_3 &\equiv - \frac{(m_{32}n_{34} + m_{42})n_{32}b_{12}}{(m_{31}n_{34} + m_{41})b} Z_1 + n_{32} \frac{b_{12}}{b} Z_2, \end{aligned}$$

and by (132) with (133) we can rewrite (108) in the form

$$(320) \quad F \equiv \begin{array}{|ccc} X_1 & X_2 & 0 \\ \hline (m_{31}b_{21} + m_{41}b_{22}) Y_1 + & a_{11} Y_1 + a_{12} Y_2 & [(m_{31}n_{34} + m_{41}) Y_1 + \\ + (m_{32}b_{21} + m_{42}b_{22}) Y_2 & & + (m_{32}n_{34} + m_{42}) Y_2] b_{12} \\ \hline - \frac{m_{32}n_{34} + m_{42}}{m_{31}n_{34} + m_{41}} \frac{b_{22}}{n_{32}} \frac{b}{b} Z_1 + & \frac{n_{32}(a_{12}Z_1 - a_{11}Z_2)}{(m_{31}n_{34} + m_{41})b} & - \frac{(m_{32}n_{34} + m_{42})}{(m_{31}n_{34} + m_{41})} \times \\ \hline + n_{32} \frac{b_{22}}{b} Z_2 + \left(n_{34} \frac{b_{22}}{b} - \frac{b_{21}}{b} \right) Z_4 & & \times \frac{n_{32}b_{12}}{b} Z_1 + n_{32} \frac{b_{12}}{b} Z_2 \end{array},$$

where by the last equation (173)

$$(321) \quad b = b_{11}b_{22} - b_{12}b_{21} = (n_{34}b_{22} - b_{21})b_{12}.$$

Subtracting in (320) the third column multiplied by b_{22}/b_{12} from the first one, we obtain by (321)

$$(322) \quad F \equiv \begin{array}{|ccc} X_1 & X_2 & 0 \\ \hline - (m_{31} Y_1 + & a_{11} Y_1 + a_{12} Y_2 & [(m_{31}n_{34} + m_{41}) Y_1 + \\ + m_{32} Y_2] \frac{b}{b_{12}} & & + (m_{32}n_{34} + m_{42}) Y_2] b_{12} \\ \hline \frac{1}{b_{12}} Z_4 & - \frac{n_{32}}{(m_{31}n_{34} + m_{41})b} (a_{12}Z_1 - a_{11}Z_2) & - \frac{(m_{32}n_{34} + m_{42})n_{32}b_{12}}{(m_{31}n_{34} + m_{41})b} Z_1 + \\ & & + \frac{n_{32}b_{12}}{b} Z_2 \end{array}$$

By Lemma 4 the forms (108) and (322) are equivalent.

Multiplying in (322) the third row by $m_{31}n_{34} + m_{41}$ and the third column by $\frac{1}{m_{31}n_{34} + m_{41}}$, we obtain

$$(323) \quad F \equiv \begin{array}{|ccc|} \hline X_1 & X_2 & 0 \\ \hline -(m_{31} Y_1 + & a_{11} Y_1 + a_{12} Y_2 & \left(Y_1 + \frac{m_{32} n_{34} + m_{42}}{m_{31} n_{34} + m_{41}} Y_2 \right) b_{12} \\ + m_{32} Y_2) \frac{b}{b_{12}} & & \\ \hline (m_{31} n_{34} + & - \frac{n_{32}}{b} (a_{12} Z_1 - a_{11} Z_2) & - \frac{(m_{32} n_{34} + m_{42}) n_{32} b_{12}}{(m_{31} n_{34} + m_{41}) b} Z_1 + \frac{n_{32} b_{12}}{b} Z_2 \\ + m_{41}) \frac{1}{b_{12}} Z_4 & & \\ \hline \end{array}$$

By Lemma 5 the forms (322) and (323) and hence the forms (108) and (323) are equivalent.

Subtracting in (323) the third column multiplied by a_{11}/b_{12} from the second one, we obtain

$$(324) \quad F' \equiv \begin{array}{|ccc|} \hline X_1 & X_2 & 0 \\ \hline -(m_{31} Y_1 + & \left(a_{12} - \frac{m_{32} n_{34} + m_{42}}{m_{31} n_{34} + m_{41}} a_{11} \right) Y_2 & \left(Y_1 + \frac{m_{32} n_{34} + m_{42}}{m_{31} n_{34} + m_{41}} Y_2 \right) b_{12} \\ + m_{32} Y_2) \frac{b}{b_{12}} & & \\ \hline (m_{31} n_{34} + & \left(\frac{(m_{32} n_{34} + m_{42}) n_{32} a_{11}}{(m_{31} n_{34} + m_{41}) b} - \frac{(m_{32} n_{34} + m_{42})}{(m_{31} n_{34} + m_{41})} \times \right. \\ + m_{41}) \frac{1}{b_{12}} Z_4 & \left. - \frac{n_{32} a_{12}}{b} \right) Z_1 & \times \frac{n_{32} b_{12}}{b} Z_1 + \frac{n_{32} b_{12}}{b} Z_2 \\ \hline \end{array}$$

Since by (175) and (319)

$$\begin{aligned} a &= a_{11} a_{22} - a_{12} a_{21} = [(m_{32} n_{34} + m_{42}) a_{11} - (m_{31} n_{34} + m_{41}) a_{12}] b_{12} = \\ &= - \frac{m_{31} n_{34} + m_{41}}{n_{32}} b, \end{aligned}$$

the identity (324) can be rewritten in the form

$$(325) \quad F \equiv \begin{array}{|ccc|} \hline X_1 & X_2 & 0 \\ \hline -(m_{31} Y_1 + m_{32} Y_2) \frac{b}{b_{12}} & \frac{b}{n_{32} b_{12}} Y_2 & \left(Y_1 + \frac{m_{32} n_{34} + m_{42}}{m_{31} n_{34} + m_{41}} Y_2 \right) b_{12} \\ \hline (m_{31} n_{34} + m_{41}) \frac{1}{b_{12}} Z_4 & - \frac{1}{b_{12}} Z_1 & - \frac{(m_{32} n_{34} + m_{42}) n_{32} b_{12}}{(m_{31} n_{34} + m_{41}) b} Z_1 + \frac{n_{32} b_{12}}{b} Z_2 \\ \hline \end{array}$$

By Lemma 4 the forms (323) and (325) and hence the forms (108) and (325) are equivalent.

Multiplying in (325) the second row by $-n_{32}b_{12}/b$ and the third column by $-b/n_{32}b_{12}$, and then the third row by b_{12} and the third column by $1/b_{12}$, we obtain the form (149). By Lemma 5 the forms (325) and (149) and hence the forms (108) and (149) are equivalent. Thus the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (149).

Principal case VI. If the function F is nomographic, then — by Theorem 11 — for each of its Massau forms there exists an equivalent form (108) such that the identities (109) hold. Then we have the identities (155) and — by Corollary 9.2 — there exist such numbers a_{ij}, b_{ij} ($i, j = 1, 2$) satisfying the conditions (159) that the identities (171) and (172) hold. Then — by Theorem 4 and (134) — we obtain (182), (183), (184). By (182) we have

$$(326) \quad a = a_{11}a_{22} - a_{12}a_{21} = (m_{42}a_{11} - m_{41}a_{12})b_{12},$$

then by (184)

$$(327) \quad b = b_{11}b_{22} - b_{12}b_{21} = -(m_{43}b_{22} + b_{21})b_{12} = -\frac{n_{32}}{m_{41}}a,$$

and hence by (183)

$$(328) \quad n_{32}(m_{42}a_{11} - m_{41}a_{12}) = m_{41}(m_{43}b_{22} + b_{21}).$$

By (171), (182), (134) and (135) we have

$$\bar{Y}_1 \equiv m_{41}b_{22}Y_1 + m_{42}b_{22}Y_2 + (m_{43}b_{22} + b_{21})Y_3,$$

$$\bar{Z}_1 \equiv -\frac{m_{42}n_{32}}{m_{41}} \cdot \frac{b_{22}}{b}Z_1 + n_{32} \frac{b_{22}}{b}Z_2 - (m_{43}b_{22} + b_{21}) \frac{1}{b}Z_4,$$

$$\bar{Y}_2 \equiv a_{11}Y_1 + a_{12}Y_2, \quad \bar{Z}_2 \equiv \frac{a_{12}}{a}Z_1 - \frac{a_{11}}{a}Z_2,$$

$$\bar{Y}_3 \equiv (m_{41}Y_1 + m_{42}Y_2)b_{12}, \quad \bar{Z}_3 \equiv (m_{42}Z_1 - m_{41}Z_2) \frac{b_{12}}{a},$$

i. e., by (327)

$$\bar{Y}_1 \equiv m_{41}b_{22}Y_1 + m_{42}b_{22}Y_2 + \frac{n_{32}a}{m_{41}b_{12}}Y_3,$$

$$\bar{Z}_1 \equiv \frac{m_{42}b_{22}}{a}Z_1 - \frac{m_{41}b_{22}}{a}Z_2 + \frac{1}{b_{12}}Z_4,$$

$$\begin{aligned} \bar{Y}_2 &\equiv a_{11} Y_1 + a_{12} Y_2, & \bar{Z}_2 &\equiv \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2, \\ \bar{Y}_3 &\equiv (m_{41} Y_1 + m_{42} Y_2) b_{12}, & \bar{Z}_3 &\equiv (m_{42} Z_1 - m_{41} Z_2) \frac{b_{12}}{a}, \end{aligned}$$

and we can rewrite (108) in the form

$$(329) \quad F \equiv \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ m_{41} b_{22} Y_1 + m_{42} b_{22} Y_2 + \frac{n_{32} a}{m_{41} b_{12}} Y_3 & a_{11} Y_1 + a_{12} Y_2 & (m_{41} Y_1 + m_{42} Y_2) b_{12} \\ \frac{m_{42} b_{22}}{a} Z_1 - \frac{m_{41} b_{22}}{a} Z_2 + \frac{1}{b_{12}} Z_4 & \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2 & (m_{42} Z_1 - m_{41} Z_2) \frac{b_{12}}{a} \end{array} \right|.$$

Subtracting in (329) the third column multiplied by b_{22}/b_{12} from the first one, we obtain

$$(330) \quad F \equiv \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ \frac{n_{32} a}{m_{41} b_{12}} Y_3 & a_{11} Y_1 + a_{12} Y_2 & (m_{41} Y_1 + m_{42} Y_2) b_{12} \\ \frac{1}{b_{12}} Z_4 & \frac{a_{12}}{a} Z_1 - \frac{a_{11}}{a} Z_2 & (m_{42} Z_1 - m_{41} Z_2) \frac{b_{12}}{a} \end{array} \right|.$$

By Lemma 4 the forms (108) and (330) are equivalent.

Subtracting in (330) the third column multiplied by $a_{11}/m_{41} b_{12}$ from the second one, we obtain by (326)

$$(331) \quad F \equiv \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ \frac{n_{32} a}{m_{41} b_{12}} Y_3 & -\frac{a}{m_{41} b_{12}} Y_2 & (m_{41} Y_1 + m_{42} Y_2) b_{12} \\ \frac{1}{b_{12}} Z_4 & -\frac{1}{m_{41} b_{12}} Z_1 & (m_{42} Z_1 - m_{41} Z_2) \frac{b_{12}}{a} \end{array} \right|.$$

By Lemma 4 the forms (330) and (331), and hence the forms (108) and (331) are equivalent.

Multiplying in (331) the second row by $m_{41} b_{12}/a$ and the third column by $a/m_{41} b_{12}$ and then the third row by $m_{41} b_{12}$ and the third column by $1/m_{41} b_{12}$, we obtain the form (150). By Lemma 5 the forms (331) and (150), and hence the forms (108) and (150), are equivalent.

It follows by Theorem 11 that the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (150).

Principal case VII. If the function F is nomographic, then for each of its Massau forms there exists — by Theorem 12 — an equivalent form (120) such that the identities (121) hold. Then we have

$$\begin{aligned}\bar{Y}_2\bar{Z}_3 - \bar{Y}_3\bar{Z}_2 &\equiv Y_1Z_1 + Y_2Z_2, & \bar{Y}_3\bar{Z}_1 - \bar{Y}_1\bar{Z}_3 &\equiv Y_3Z_3 + Y_4Z_4, \\ \bar{Y}_1\bar{Z}_2 - \bar{Y}_2\bar{Z}_1 &\equiv Y_5Z_5 + Y_6Z_6,\end{aligned}$$

and there exist by Corollary 9.2 such numbers a_{ij}, b_{ij}, c_{ij} ($i, j = 1, 2$) satisfying the conditions (209) that the identities (210) and (211) hold. By (136) and Theorem 4 we obtain from (211) the equations (216) and then (217), (229), (232), (233), (234), (235), (236), (237), (238), (239), (241), (244), (245), (246), and (247). Thus we have by (136), (137) and (210):

$$\begin{aligned}\bar{Y}_1 &\equiv b_{21}Y_3 + b_{22}Y_4 \equiv m_{41}b_{22}Y_1 + m_{42}b_{22}Y_2 + (b_{21} + m_{43}b_{22})Y_3 \\ &\equiv [-(m_{51}m_{62} - m_{52}m_{61})(m_{41}Y_1 + m_{42}Y_2) + rY_3] \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12},\end{aligned}$$

$$\begin{aligned}\bar{Y}_2 &\equiv a_{11}Y_1 + a_{12}Y_2 \\ &\equiv -[(m_{51}m_{63} - m_{53}m_{61})Y_1 + (m_{52}m_{63} - m_{53}m_{62})Y_2] \frac{n_{64}}{m_{63}} b_{12},\end{aligned}$$

$$\bar{Y}_3 \equiv a_{21}Y_1 + a_{22}Y_2 \equiv (m_{41}Y_1 + m_{42}Y_2)b_{12},$$

$$\begin{aligned}\bar{Z}_1 &\equiv \frac{b_{22}}{b}Z_3 - \frac{b_{21}}{b}Z_4 \equiv n_{31}\frac{b_{22}}{b}Z_1 + n_{32}\frac{b_{22}}{b}Z_2 + \frac{n_{34}b_{22} - b_{21}}{b}Z_4 \\ &\equiv (m_{41}Z_2 - m_{42}Z_1)\frac{n_{32}b_{22}}{m_{41}b} - \frac{b_{21} + m_{43}b_{22}}{b}Z_4 \\ &\equiv (m_{41}Z_2 - m_{42}Z_1)\frac{(m_{51}m_{62} - m_{52}m_{61})n_{32}}{m_{41}rb_{12}} + \frac{1}{b_{12}}Z_4,\end{aligned}$$

$$\bar{Z}_2 \equiv \frac{a_{12}}{a}Z_1 - \frac{a_{11}}{a}Z_2 \equiv [(m_{51}m_{63} - m_{53}m_{61})Z_2 - (m_{52}m_{63} - m_{53}m_{62})Z_1] \frac{1}{rb_{12}},$$

$$\bar{Z}_3 \equiv \frac{a_{22}}{a}Z_1 - \frac{a_{21}}{a}Z_2 \equiv (m_{42}Z_1 - m_{41}Z_2)\frac{m_{63}}{n_{54}rb_{12}},$$

and we can rewrite (120) in the form

$$(332) \quad F \equiv \begin{array}{|c|c|c|} \hline & X_1 & X_2 & X_3 \\ \hline [- (m_{51}m_{62} - m_{52}m_{61})(m_{41}Y_1 + & - [(m_{51}m_{63} - m_{53}m_{61})Y_1 + & (m_{41}Y_1 + \\ + m_{42}Y_2) + rY_3] \frac{n_{32}n_{54}}{m_{41}m_{63}} b_{12} & + (m_{52}m_{63} - m_{53}m_{62})Y_2] \frac{n_{54}}{m_{63}} b_{12} & + m_{42}Y_2) b_{12} \\ (m_{41}Z_2 - m_{42}Z_1) \times & [(m_{51}m_{63} - m_{53}m_{61})Z_2 - & (m_{42}Z_1 - \\ \times \frac{(m_{51}m_{62} - m_{52}m_{61})n_{32}}{m_{41}rb_{12}} + \frac{1}{b_{12}} Z_4 & - (m_{52}m_{63} - m_{53}m_{62})Z_1] \frac{1}{rb_{12}} & - m_{41}Z_2) \frac{m_{63}}{n_{54}rb_{12}} \\ \hline \end{array}$$

Adding in (332) the third column multiplied by $(m_{51}m_{62} - m_{52}m_{61}) \times \times n_{32}n_{54}/m_{41}m_{63}$ to the first one and then the third column multiplied by $(m_{51}m_{63} - m_{53}m_{61})n_{54}/m_{41}m_{63}$ to the second one, we obtain

$$(333) \quad F \equiv \begin{array}{|c|c|c|} \hline X_1 + (m_{51}m_{62} - & X_2 + (m_{51}m_{63} - m_{53}m_{61}) \frac{n_{54}}{m_{41}m_{63}} X_3 & X_3 \\ - m_{52}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} X_3 & & \\ \frac{n_{32}n_{54}}{m_{41}m_{63}} rb_{12} Y_3 & [(m_{51}m_{63} - m_{53}m_{61})m_{42} - & (m_{41}Y_1 + \\ - (m_{52}m_{63} - m_{53}m_{62})m_{41}] \frac{n_{54}}{m_{41}m_{63}} b_{12} Y_2 & + m_{42}Y_2) b_{12} \\ \frac{1}{b_{12}} Z_4 & [(m_{51}m_{63} - m_{53}m_{61})m_{42} - & (m_{42}Z_1 - \\ - (m_{52}m_{63} - m_{53}m_{62})m_{41}] \frac{1}{m_{41}rb_{12}} Z_1 & - m_{41}Z_2) \frac{m_{63}}{n_{54}rb_{12}} \\ \hline \end{array}$$

Since by (237)

$$\begin{aligned} & (m_{51}m_{63} - m_{53}m_{61})m_{42} - (m_{52}m_{63} - m_{53}m_{62})m_{41} \\ & = -[(m_{41}m_{52} - m_{42}m_{51})m_{63} - (m_{41}m_{62} - m_{42}m_{61})m_{53}] = -r, \end{aligned}$$

the identity (333) can be rewritten in the form

$$(334) \quad F \equiv \begin{array}{|c|c|c|} \hline X_1 + (m_{51}m_{62} - & X_2 + (m_{51}m_{63} - & X_3 \\ - m_{52}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} X_3 & - m_{53}m_{61}) \frac{n_{54}}{m_{41}m_{63}} X_3 & \\ \frac{n_{32}n_{54}}{m_{41}m_{63}} rb_{12} Y_3 & - \frac{n_{54}}{m_{41}m_{63}} rb_{12} Y_2 & (m_{41}Y_1 + m_{42}Y_2) b_{12} \\ \frac{1}{b_{12}} Z_4 & - \frac{1}{m_{41}b_{12}} Z_1 & (m_{42}Z_1 - m_{41}Z_2) \frac{m_{63}}{n_{54}rb_{12}} \\ \hline \end{array}$$

By Lemma 4 the forms (120) and (334) are equivalent.

Multiplying the second row of the determinant (334) by $m_{41}m_{63}/n_{54}rb_{12}$ and the third column by $n_{54}rb_{12}/m_{41}m_{63}$, and then the third row by $m_{41}b_{12}$ and the third column by $1/m_{41}b_{12}$, we obtain the form (151). By Lemma 5 the forms (334) and (151), and hence the forms (120) and (151), are equivalent.

It follows by Theorem 12 that in the seventh principal case the function F is uniquely nomographic and each of its Massau forms is equivalent to the form (151).

This completes the proof of the Second Fundamental Theorem.

V. SCHEME OF COMPUTATIONS

We are now in a position to construct a scheme of computations which will enable us to determine whether a given function is nomographic and, moreover, to find all possible Massau forms of a nomographic function. We retain the previous notations.

Scheme I

1. If $F \equiv F(x, y, z) \equiv 0$, the function F is not nomographic. If $F \not\equiv 0$, we find such elements $x_1 \in \Omega_x, y_1 \in \Omega_y, z_1 \in \Omega_z$ that

$$(335) \quad F(x_1, y_1, z_1) \neq 0.$$

2. If

$$(336) \quad F_{x_1} \equiv F_{x_1}(x, y, z) \equiv \frac{1}{F(x_1, y_1, z_1)} \left| \begin{array}{cc} F(x_1, y_1, z_1) & F(x_1, y, z) \\ F(x, y_1, z_1) & F(x, y, z) \end{array} \right| \equiv 0$$

or

$$(337) \quad F_{y_1} \equiv F_{y_1}(x, y, z) \equiv \frac{1}{F(x_1, y_1, z_1)} \left| \begin{array}{cc} F(x_1, y_1, z_1) & F(x, y_1, z) \\ F(x_1, y, z_1) & F(x, y, z) \end{array} \right| \equiv 0$$

or

$$(338) \quad F_{z_1} \equiv F_{z_1}(x, y, z) \equiv \frac{1}{F(x_1, y_1, z_1)} \left| \begin{array}{cc} F(x_1, y_1, z_1) & F(x, y, z_1) \\ F(x_1, y_1, z) & F(x, y, z) \end{array} \right| \equiv 0,$$

the function F is not nomographic. If none of the identities (336), (337), (338) is satisfied, we find such elements $x_i \in \Omega_x, y_i \in \Omega_y, z_i \in \Omega_z$ ($i = 2, 3, 4$) that

$$(339) \quad F_{x_1}(x_2, y_2, z_2) \neq 0, \quad F_{y_1}(x_3, y_3, z_3) \neq 0, \quad F_{z_1}(x_4, y_4, z_4) \neq 0.$$

3. If

$$(340) \quad F_{x_2} \equiv F_{x_2}(x, y, z) \equiv \frac{1}{F_{x_1}(x_2, y_2, z_2)} \left| \begin{array}{cc} F_{x_1}(x_2, y_2, z_2) & F_{x_1}(x_2, y, z) \\ F_{x_1}(x, y_2, z_2) & F_{x_1}(x, y, z) \end{array} \right| \equiv 0$$

we introduce the functions

$$(341) \quad \begin{aligned} X_1 &\equiv X_1(x) \equiv F(x, y_1, z_1), & X_2 &\equiv X_2(x) \equiv F_{x_1}(x, y_2, z_2), \\ G_1 &\equiv G_1(y, z) \equiv \frac{F(x_1, y, z)}{F(x_1, y_1, z_1)}, & G_2 &\equiv G_2(y, z) \equiv \frac{F_{x_1}(x_2, y, z)}{F_{x_1}(x_2, y_2, z_2)}. \end{aligned}$$

We shall call this case X2, since the function F is then of rank 2 with respect to x .

If $F_{x_2} \neq 0$, we find such elements $x_5 \in \Omega_x$, $y_5 \in \Omega_y$, $z_5 \in \Omega_z$ that

$$(342) \quad F_{x_2}(x_5, y_5, z_5) \neq 0.$$

Then, if

$$(343) \quad \left| \begin{array}{cc} F_{x_2}(x_5, y_5, z_5) & F_{x_2}(x_5, y, z) \\ F_{x_2}(x, y_5, z_5) & F_{x_2}(x, y, z) \end{array} \right| \equiv 0,$$

we have a case which will be called X3, since the function F is then of rank 3 with respect to x .

If the identity (343) fails, the function F is not nomographic.

4. If

$$(344) \quad F_{y_2} \equiv F_{y_2}(x, y, z) \equiv \frac{1}{F_{y_1}(x_3, y_3, z_3)} \left| \begin{array}{cc} F_{y_1}(x_3, y_3, z_3) & F_{y_1}(x, y_3, z) \\ F_{y_1}(x_3, y, z_3) & F_{y_1}(x, y, z) \end{array} \right| \equiv 0,$$

we have a case which will be called Y2, since the function F is then of rank 2 with respect to y .

If $F_{y_2} \neq 0$, we find such elements $x_6 \in \Omega_x$, $y_6 \in \Omega_y$, $z_6 \in \Omega_z$ that

$$(345) \quad F_{y_2}(x_6, y_6, z_6) \neq 0.$$

Then, if

$$(346) \quad \left| \begin{array}{cc} F_{y_2}(x_6, y_6, z_6) & F_{y_2}(x, y_6, z) \\ F_{y_2}(x_6, y, z_6) & F_{y_2}(x, y, z) \end{array} \right| \equiv 0,$$

we have a case which will be called Y3, since the function F is then of rank 3 with respect to y .

If the condition (346) fails, the function F is not nomographic.

5. If

$$(347) \quad F_{z_2} \equiv F_{z_2}(x, y, z) \equiv \frac{1}{F_{z_1}(x_4, y_4, z_4)} \left| \begin{array}{cc} F_{z_1}(x_4, y_4, z_4) & F_{z_1}(x, y, z_4) \\ F_{z_1}(x_4, y_4, z) & F_{z_1}(x, y, z) \end{array} \right| \equiv 0,$$

we have a case which will be called Z2, since the function F is then of rank 2 with respect to z .

If $F_{z_2} \neq 0$, we find such elements $x_7 \in \Omega_x$, $y_7 \in \Omega_y$, $z_7 \in \Omega_z$ that

$$(348) \quad F_{z_2}(x_7, y_7, z_7) \neq 0.$$

Then, if

$$(349) \quad \begin{vmatrix} F_{z_2}(x_7, y_7, z_7) & F_{z_2}(x, y, z_7) \\ F_{z_2}(x_7, y_7, z) & F_{z_2}(x, y, z) \end{vmatrix} \equiv 0,$$

we have a case which will be called Z3, since the function F is then of rank 3 with respect to z .

If the condition (349) fails, the function F is not nomographic.

6. If the cases X2, Y2, Z2 or Z3 occur simultaneously, we pass to scheme II.

If the cases X2, Y3, Z2 occur simultaneously, we replace the function F by the function

$$(350) \quad F^*(x, y, z) \equiv F(x, z, y), \quad x \in \Omega_x, y \in \Omega_z, z \in \Omega_y,$$

which we obtain from the function F by interchanging the symbols y and z . Then we keep the functions X_1, X_2 from (341) unaltered, but we replace the functions G_1 and G_2 from (341) by

$$(351) \quad G_1 \equiv G_1(y, z) \equiv \frac{F(x_1, z, y)}{F(x_1, y_1, z_1)}, \quad G_2 \equiv G_2(y, z) \equiv \frac{F_{z_1}(x_2, z, y)}{F_{z_1}(x_2, y_2, z_2)}$$

$(y \in \Omega_z, z \in \Omega_y)$

(i. e., we interchange in G_1 and G_2 the symbols y and z). Thus we have for the function F^* the cases X2, Y2, Z3 and we pass to scheme II.

If the cases X2, Y3, Z3 occur simultaneously, we pass to scheme II.

If the cases X3, Y2, Z2 occur simultaneously, we replace the function F by the function

$$(352) \quad F^{**} \equiv F^{**}(x, y, z) \equiv F(z, y, x), \quad x \in \Omega_z, y \in \Omega_y, z \in \Omega_x,$$

which we obtain from the function F by interchanging the symbols x and z . Then we introduce the functions

$$(353) \quad \begin{aligned} X_1 &\equiv X_1(x) \equiv F(x_1, y_1, x), & X_2 &\equiv X_2(x) \equiv F_{z_1}(x_4, y_4, x), \\ G_1 &\equiv G_1(y, z) \equiv \frac{F(z, y, z_1)}{F(x_1, y_1, z_1)}, & G_2 &\equiv G_2(y, z) \equiv \frac{F_{z_1}(z, y, z_4)}{F_{z_1}(x_4, y_4, z_4)}, \end{aligned}$$

$x \in \Omega_z, y \in \Omega_y, z \in \Omega_x,$

where the functions $F(x_1, y_1, x), F_{z_1}(x_4, y_4, x), F(z, y, z_1), F_{z_1}(z, y, z_4)$ may be obtained by interchanging the symbols x and z in the functions $F(x_1, y_1, z), F_{z_1}(x_4, y_4, z), F(x, y, z_1), F_{z_1}(x, y, z_4)$ calculated already in (338) and (347). Thus we obtain for the function F^{**} the cases X2, Y2, Z3. Then we pass to scheme II.

If the cases X3, Y2, Z3 occur simultaneously, we replace the function F by the function

$$(354) \quad F^{***} \equiv F^{***}(x, y, z) \equiv F(y, x, z), \quad x \in \Omega_y, y \in \Omega_x, z \in \Omega_z,$$

which we obtain from the function F by interchanging the symbols x and y . Then we introduce the functions

$$(355) \quad \begin{aligned} X_1 &\equiv X_1(x) \equiv F(x_1, x, z_1), & X_2 &\equiv X_2(x) \equiv F_{y_1}(x_3, x, z_3), \\ G_1 &\equiv G_1(y, z) \equiv \frac{F(y, y_1, z)}{F(x_1, y_1, z_1)}, & G_2 &\equiv G_2(y, z) \equiv \frac{F_{y_1}(y, y_3, z)}{F_{y_1}(x_3, y_3, z_3)}, \end{aligned}$$

where the functions $F(x_1, x, z_1)$, $F(y, y_1, z)$, $F_{y_1}(x_3, x, z_3)$, $F_{y_1}(y, y_3, z)$ may be obtained by interchanging the symbols x and y in the functions $F(x_1, y, z_1)$, $F(x, y_1, z)$, $F_{y_1}(x_3, y, z_3)$, $F_{y_1}(x, y_3, z)$, calculated already in (337) and (344). Thus we obtain for the function F^{***} the cases X2, Y3, Z3. Then we pass to scheme II.

If the cases X3, Y3, Z2 occur simultaneously, we replace the function F by the function (352) and we introduce the functions (353). Thus we obtain for the function F^{**} the cases X2, Y3, Z3. Then we pass to scheme II.

Finally if the cases X3, Y3, Z3 occur simultaneously, we introduce the functions

$$(356) \quad \begin{aligned} X_1 &\equiv X_1(x) \equiv F(x, y_1, z_1), & X_2 &\equiv X_2(x) \equiv F_{x_1}(x, y_2, z_2), \\ & & X_3 &\equiv X_3(x) \equiv F_{x_2}(x, y_5, z_5), \\ G_1 &\equiv G_1(y, z) \equiv \frac{F(x_1, y, z)}{F(x_1, y_1, z_1)}, & G_2 &\equiv G_2(y, z) \equiv \frac{F_{x_1}(x_2, y, z)}{F_{x_1}(x_2, y_2, z_2)}, \\ & & G_3 &\equiv G_3(y, z) \equiv \frac{F_{x_2}(x_5, y, z)}{F_{x_2}(x_5, y_5, z_5)}, \end{aligned}$$

and we pass to scheme II.

Remark. Scheme I may be simplified if we are able to write the function F immediately

1° in the form

$$(357) \quad F \equiv X_1(x)G_1(y, z) + X_2(x)G_2(y, z)$$

or

$$(358) \quad F \equiv X_1(x)G_1(y, z) + X_2(x)G_2(y, z) + X_3(x)G_3(y, z),$$

the functions X_1, X_2 and the functions G_1, G_2 in the case (357), the functions X_1, X_2, X_3 and the functions G_1, G_2, G_3 in the case (358) — separately treated — being linearly independent;

2° in the form

$$(359) \quad F \equiv Y_1(y)H_1(x, z) + Y_2(y)H_2(x, z)$$

or

$$(360) \quad F \equiv Y_1(y)H_1(x, z) + Y_2(y)H_2(x, z) + Y_3(y)H_3(x, z),$$

the functions $Y_1(y)$, $Y_2(y)$ and the functions $H_1(x, z)$, $H_2(x, z)$ in the case (359), the functions $Y_1(y)$, $Y_2(y)$, $Y_3(y)$ and the functions $H_1(x, z)$, $H_2(x, z)$, $H_3(x, z)$ in the case (360) — separately treated — being linearly independent, and

3° in the form

$$(361) \quad F \equiv Z_1(z)K_1(x, y) + Z_2(z)K_2(x, y)$$

or

$$(362) \quad F \equiv Z_1(z)K_1(x, y) + Z_2(z)K_2(x, y) + Z_3(z)K_3(x, y),$$

the functions $Z_1(z)$, $Z_2(z)$ and the functions $K_1(x, y)$, $K_2(x, y)$ in the case (361), the functions $Z_1(z)$, $Z_2(z)$, $Z_3(z)$ and the functions $K_1(x, y)$, $K_2(x, y)$, $K_3(x, y)$ in the case (362) — separately treated — being linearly independent. Then (357) gives the case X2, (358) the case X3, (359) the case Y2, (360) the case Y3, (361) the case Z2, and (362) the case Z3.

Scheme II

1. We find such elements y_8 and z_8 that

$$(363) \quad G_1(y_8, z_8) \neq 0.$$

If

$$(364) \quad G_{11} \equiv G_{11}(y, z) \equiv \frac{1}{G_1(y_8, z_8)} \begin{vmatrix} G_1(y_8, z_8) & G_1(y_8, z) \\ G_1(y, z_8) & G_1(y, z) \end{vmatrix} \equiv 0,$$

we introduce the functions

$$(365) \quad Y_1 \equiv Y_1(y) \equiv G_1(y, z_8), \quad Z_1 \equiv Z_1(z) \equiv \frac{G_1(y_8, z)}{G_1(y_8, z_8)}.$$

This case will be called G_11 .

If $G_{11} \neq 0$, we find such elements y_9 , z_9 that

$$(366) \quad G_{11}(y_9, z_9) \neq 0.$$

Then, if

$$(367) \quad \begin{vmatrix} G_{11}(y_9, z_9) & G_{11}(y_9, z) \\ G_{11}(y, z_9) & G_{11}(y, z) \end{vmatrix} \equiv 0,$$

we introduce the functions

$$(368) \quad \begin{aligned} Y_1 &\equiv Y_1(y) \equiv G_1(y, z_8), & Y_2 &\equiv Y_2(y) \equiv G_{11}(y, z_9), \\ Z_1 &\equiv Z_1(z) \equiv \frac{G_1(y_8, z)}{G_1(y_8, z_8)}, & Z_2 &\equiv Z_2(z) \equiv \frac{G_{11}(y_9, z)}{G_{11}(y_9, z_9)}. \end{aligned}$$

This case will be called $G_1 2$.

If the condition (367) fails, the function F is not nomographic.

2. We find such elements y_{10}, z_{10} that

$$(369) \quad G_2(y_{10}, z_{10}) \neq 0.$$

If

$$(370) \quad G_{21} \equiv G_{21}(y, z) \equiv \frac{1}{G_2(y_{10}, z_{10})} \begin{vmatrix} G_2(y_{10}, z_{10}) & G_2(y_{10}, z) \\ G_2(y, z_{10}) & G_2(y, z) \end{vmatrix} \equiv 0,$$

we introduce the functions

$$(371) \quad Y_3 \equiv Y_3(y) \equiv G_2(y, z_{10}), \quad Z_3 \equiv Z_3(z) \equiv \frac{G_2(y_{10}, z)}{G_2(y_{10}, z_{10})}.$$

This case will be called $G_2 1$.

If $G_{21} \neq 0$, we find such elements y_{11}, z_{11} that

$$(372) \quad G_{21}(y_{11}, z_{11}) \neq 0.$$

Then, if

$$(373) \quad \begin{vmatrix} G_{21}(y_{11}, z_{11}) & G_{21}(y_{11}, z) \\ G_{21}(y, z_{11}) & G_{21}(y, z) \end{vmatrix} \equiv 0,$$

we introduce the functions

$$(374) \quad \begin{aligned} Y_3 &\equiv Y_3(y) \equiv G_2(y, z_{10}), & Y_4 &\equiv Y_4(y) \equiv G_{21}(y, z_{11}), \\ Z_3 &\equiv Z_3(z) \equiv \frac{G_2(y_{10}, z)}{G_2(y_{10}, z_{10})}, & Z_4 &\equiv Z_4(z) \equiv \frac{G_{21}(y_{11}, z)}{G_{21}(y_{11}, z_{11})}. \end{aligned}$$

This case will be called $G_2 2$.

If the condition (373) fails, the function F is not nomographic.

3. If we have obtained in scheme I the cases X3, Y3, Z3, we find, moreover, such elements y_{12}, z_{12} that

$$(375) \quad G_3(y_{12}, z_{12}) \neq 0.$$

If

$$(376) \quad G_{31} \equiv G_{31}(y, z) \equiv \frac{1}{G_3(y_{12}, z_{12})} \begin{vmatrix} G_3(y_{12}, z_{12}) & G_3(y_{12}, z) \\ G_3(y, z_{12}) & G_3(y, z) \end{vmatrix} \equiv 0,$$

we introduce the functions

$$(377) \quad Y_5 \equiv Y_5(y) \equiv G_3(y, z_{12}), \quad Z_5 \equiv Z_5(z) \equiv \frac{G_3(y_{12}, z)}{G_3(y_{12}, z_{12})}.$$

This case will be called $G_3 1$.

If $G_{31} \neq 0$, we find such elements y_{13}, z_{13} that

$$(378) \quad G_{31}(y_{13}, z_{13}) \neq 0.$$

Then, if

$$(379) \quad \begin{vmatrix} G_{31}(y_{13}, z_{13}) & G_{31}(y_{13}, z) \\ G_{31}(y, z_{13}) & G_{31}(y, z) \end{vmatrix} = 0,$$

we introduce the functions

$$(380) \quad \begin{aligned} Y_5 &\equiv Y_5(y) \equiv G_3(y, z_{12}), & Y_6 &\equiv Y_6(y) \equiv G_{31}(y, z_{13}), \\ Z_5 &\equiv Z_5(z) \equiv \frac{G_3(y_{12}, z)}{G_3(y_{12}, z_{12})}, & Z_6 &\equiv Z_6(z) \equiv \frac{G_{31}(y_{13}, z)}{G_{31}(y_{13}, z_{13})}. \end{aligned}$$

This case will be called $G_3 2$.

If the condition (379) fails, the function F is not nomographic.

Remark. The above-mentioned calculations in scheme II may be omitted if we are able to write immediately the following identities:

$$(381) \quad G_1 \equiv Y_1 Z_1$$

or

$$(382) \quad G_1 \equiv Y_1 Z_1 + Y_2 Z_2,$$

the functions Y_1, Y_2 and the functions Z_1, Z_2 — separately treated — being linearly independent,

$$(383) \quad G_2 \equiv Y_3 Z_3$$

or

$$(384) \quad G_2 \equiv Y_3 Z_3 + Y_4 Z_4,$$

the functions Y_3, Y_4 and the functions Z_3, Z_4 — separately treated — being linearly independent,

$$(385) \quad G_3 \equiv Y_5 Z_5$$

or

$$(386) \quad G_3 \equiv Y_5 Z_5 + Y_6 Z_6,$$

the functions Y_5, Y_6 and the functions Z_5, Z_6 — separately treated — being linearly independent.

Then, (381) gives the case G_{11} , (382) the case G_{12} , (383) the case G_{21} , (384) the case G_{22} , (385) the case G_{31} , and (386) the case G_{32} .

Moreover, the forms (357), (358) in scheme I and the forms (382), (384), (386) in scheme II may sometimes be simplified by use of Corollary 9.3, as has been shown in Example 6.

4. Owing to the introduction of the functions F^* , F^{**} , F^{***} we obtain from scheme I only the cases X2, Y2, Z2, or the cases X2, Y2, Z3, or the cases X2, Y3, Z3, or the cases X3, Y3, Z3.

If we have the cases X2, Y2, Z2, $G_{1,1}$, $G_{2,1}$, we now pass directly to scheme III.

If we have the cases X2, Y2, Z2, $G_{1,2}$, $G_{2,1}$, we now pass directly to scheme IV.

If we have the cases X2, Y2, Z2, $G_{1,1}$, $G_{2,2}$, we interchange the indices as follows:

$$(387) \quad \frac{\text{before the change } |X_1|X_2|G_1|G_2|Y_1|Y_3|Y_4|Z_1|Z_3|Z_4|}{\text{after the change } |X_2|X_1|G_2|G_1|Y_3|Y_1|Y_2|Z_3|Z_1|Z_2|}$$

Thus we obtain the cases X2, Y2, Z2, $G_{1,2}$, $G_{2,1}$ and we now pass directly to scheme IV.

If we have the cases X2, Y2, Z2, $G_{1,2}$, $G_{2,2}$, we now pass directly to scheme V.

If we have the cases X2, Y2, Z3, $G_{1,1}$, $G_{2,1}$, the function F is not nomographic.

If we have the cases X2, Y2, Z3, $G_{1,2}$, $G_{2,1}$, we now pass directly to the scheme VI.

If we have the cases X2, Y2, Z3, $G_{1,1}$, $G_{2,2}$, we interchange the indices as in (387). Thus we obtain the cases X2, Y2, Z3, $G_{1,2}$, $G_{2,1}$ and we now pass directly to scheme VI.

If we have the cases X2, Y2, Z3, $G_{1,2}$, $G_{2,2}$, we now pass directly to scheme VII.

If we have the cases X2, Y3, Z3 and the case $G_{1,1}$ or $G_{2,1}$, the function F is not nomographic.

If we have the cases X2, Y3, Z3, $G_{1,2}$, $G_{2,2}$, we now pass directly to scheme VIII.

If we have the cases X3, Y3, Z3 and the case $G_{1,1}$ or $G_{2,1}$ or $G_{3,1}$, the function F is not nomographic.

If we have the cases X3, Y3, Z3, $G_{1,2}$, $G_{2,2}$, $G_{3,2}$, we now pass directly to scheme IX.

Scheme III

The function F is here doubly nomographic and each of its Massau forms is equivalent to one of the following two non-equivalent forms:

$$(388) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & 0 & Z_1 \end{vmatrix} \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ Y_3 & 0 & Y_1 \\ 0 & -Z_1 & -Z_3 \end{vmatrix}.$$

Scheme IV

We find such elements $y_{14}, y_{15}, z_{14}, z_{15}$ that

$$(389) \quad \begin{vmatrix} Y_1(y_{14}) & Y_2(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} Z_1(z_{14}) & Z_2(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) \end{vmatrix} \neq 0$$

and we solve in $m_{31}, m_{32}, n_{31}, n_{32}$ the simultaneous equations

$$(390) \quad \begin{aligned} Y_1(y_{14})m_{31} + Y_2(y_{14})m_{32} &= Y_3(y_{14}), \\ Y_1(y_{15})m_{31} + Y_2(y_{15})m_{32} &= Y_3(y_{15}), \end{aligned}$$

and

$$(391) \quad \begin{aligned} Z_1(z_{14})n_{31} + Z_2(z_{14})n_{32} &= Z_3(z_{14}), \\ Z_1(z_{15})n_{31} + Z_2(z_{15})n_{32} &= Z_3(z_{15}). \end{aligned}$$

It is worth while to verify that

$$(392) \quad Y_3 \equiv m_{31} Y_1 + m_{32} Y_2, \quad Z_3 \equiv n_{31} Z_1 + n_{32} Z_2^{(1)}.$$

If $m_{32} = 0$, we interchange the indices as follows:

$$(393) \quad \frac{\text{before the change } \begin{vmatrix} Y_1 & Y_2 & Z_1 & Z_2 & m_{31} & m_{32} & n_{31} & n_{32} \end{vmatrix}}{\text{after the change } \begin{vmatrix} Y_2 & Y_1 & Z_2 & Z_1 & m_{32} & m_{31} & n_{32} & n_{31} \end{vmatrix}}.$$

Thus we always obtain

$$(394) \quad m_{32} \neq 0.$$

Now if $m_{31}n_{31} + m_{32}n_{32} \neq 0$ and $n_{32} \neq 0$, the function F is doubly nomographic and each of its Massau forms is equivalent to one of the following two non-equivalent forms

$$(395) \quad F \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & -\frac{1}{m_{31}}Z_2 & Z_1 - \frac{m_{31}}{m_{32}}Z_2 \end{vmatrix} \equiv \begin{vmatrix} X_1 & X_2 & 0 \\ Y_3 & -\frac{1}{n_{32}}Y_2 & Y_1 - \frac{n_{31}}{n_{32}}Y_2 \\ 0 & -Z_1 & -Z_3 \end{vmatrix}.$$

If $m_{31}n_{31} + m_{32}n_{32} \neq 0$ and $n_{32} = 0$, the function F is doubly nomographic and each of its Massau forms is equivalent to one of the following

(¹) The computations of (389), (390), and (391) may be omitted if we are able to write the identities (392) immediately. These identities must be satisfied, since they are identical to the identities (249).

two non-equivalent forms

$$(396) \quad F \equiv \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & -\frac{1}{m_{32}}Z_2 & Z_1 - \frac{m_{31}}{m_{32}}Z_2 \end{array} \right| \equiv \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ n_{31}Y_3 & -Y_1 & -Y_2 \\ 0 & Z_2 & -Z_1 \end{array} \right|.$$

If $m_{31}n_{31} + m_{32}n_{32} = 0$, the function F is uniquely nomographic and each of its Massau forms is equivalent to the form

$$(397) \quad F \equiv \left| \begin{array}{ccc} X_1 & X_2 & 0 \\ 0 & Y_1 & Y_3 \\ Z_3 & -\frac{1}{m_{32}}Z_2 & Z_1 - \frac{m_{31}}{m_{32}}Z_2 \end{array} \right|.$$

Scheme V

We find such elements $y_{14}, y_{15}, z_{14}, z_{15}$ that

$$(398) \quad \left| \begin{array}{cc} Y_1(y_{14}) & Y_2(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) \end{array} \right| \neq 0 \quad \text{and} \quad \left| \begin{array}{cc} Z_1(z_{14}) & Z_2(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) \end{array} \right| \neq 0$$

and we solve in m_{ij}, n_{ij} ($i = 3, 4; j = 1, 2$) the simultaneous equations

$$(399) \quad \begin{aligned} Y_1(y_{14})m_{i1} + Y_2(y_{14})m_{i2} &= Y_i(y_{14}) \\ Y_1(y_{15})m_{i1} + Y_2(y_{15})m_{i2} &= Y_i(y_{15}) \end{aligned} \quad (i = 3, 4),$$

and

$$(400) \quad \begin{aligned} Z_1(z_{14})n_{i1} + Z_2(z_{14})n_{i2} &= Z_i(z_{14}) \\ Z_1(z_{15})n_{i1} + Z_2(z_{15})n_{i2} &= Z_i(z_{15}) \end{aligned} \quad (i = 3, 4).$$

It is worth while to verify that

$$(401) \quad \begin{aligned} Y_3 &\equiv m_{31}Y_1 + m_{32}Y_2, & Z_3 &\equiv n_{31}Z_1 + n_{32}Z_2, \\ Y_4 &\equiv m_{41}Y_1 + m_{42}Y_2, & Z_4 &\equiv n_{41}Z_1 + n_{42}Z_2, \end{aligned} \quad (1)$$

We now compute the numbers

$$(402) \quad \begin{aligned} r_{31} &= m_{31}n_{31} + m_{41}n_{41}, & r_{32} &= m_{32}n_{31} + m_{42}n_{41}, \\ r_{31} &= m_{31}n_{32} + m_{41}n_{42}, & r_{42} &= m_{32}n_{32} + m_{42}n_{42}. \end{aligned}$$

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} < 0$, the function F is not nomographic.

(¹) The computations of (398), (399), and (400) may be omitted if we are able to write the identities (401) immediately. They must be satisfied by the proof given for the identity (153).

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} > 0$ and $r_{32} \neq 0$, we find the solutions $p = p_1$ and $p = p_2$ of the equation

$$(403) \quad p^2 - (r_{31} + r_{42})p + (r_{31}r_{42} - r_{32}r_{41}) = 0.$$

The function F is then doubly nomographic and each of its Massau forms is equivalent to one of the following two non-equivalent forms

$$(404) \quad F \equiv \left| \begin{array}{ccc|c} X_1 & X_2 & 0 & \\ r_{31}Y_1 + r_{32}Y_2 & -Y_1 & \frac{p_i - r_{42}}{r_{32}}Y_1 + Y_2 & \\ p_iZ_2 & -Z_2 & \frac{p_i - r_{42}}{r_{32}}Z_2 - Z_1 & \end{array} \right| \quad (i = 1, 2).$$

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} > 0$ and $r_{32} = 0$, the function F is also doubly nomographic and each of its Massau forms is equivalent to one of the following two non-equivalent forms

$$(405) \quad F \equiv \left| \begin{array}{ccc|c} X_1 & X_2 & 0 & \\ r_{41}Y_1 + r_{42}Y_2 & -Y_2 & r_{31}Y_1 & \\ Z_1 & -\frac{1}{r_{31}}Z_1 & -Z_2 & \end{array} \right| \equiv \left| \begin{array}{ccc|c} X_1 & X_2 & 0 & \\ r_{31}Y_1 & -Y_1 & \frac{r_{41}}{r_{42} - r_{31}}Y_1 + Y_2 & \\ r_{42}Z_2 & -Z_2 & \frac{r_{41}}{r_{42} - r_{31}}Z_2 - Z_1 & \end{array} \right|.$$

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} = 0$ and $r_{32} \neq 0$, the function F is uniquely nomographic and each of its Massau forms is equivalent to the form

$$(406) \quad F \equiv \left| \begin{array}{ccc|c} X_1 & X_2 & 0 & \\ r_{31}Y_1 + r_{32}Y_2 & -Y_1 & \frac{r_{31} - r_{42}}{2r_{32}}Y_1 + Y_2 & \\ \frac{r_{31} + r_{42}}{2}Z_2 & -Z_2 & \frac{r_{31} - r_{42}}{2r_{32}}Z_2 - Z_1 & \end{array} \right|.$$

If $(r_{31} - r_{42})^2 + 4r_{32}r_{41} = 0$ and $r_{32} = 0$, the function F is uniquely nomographic and each of its Massau forms is equivalent to the form

$$(407) \quad F \equiv \left| \begin{array}{ccc|c} X_1 & X_2 & 0 & \\ r_{41}Y_1 + r_{42}Y_2 & -Y_2 & r_{31}Y_1 & \\ Z_1 & -\frac{1}{r_{31}}Z_1 & -Z_2 & \end{array} \right|.$$

Scheme VI

We find such elements $y_{14}, y_{15}, z_{14}, z_{15}$ that

$$(408) \quad \begin{vmatrix} Y_1(y_{14}) & Y_2(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) \end{vmatrix} \neq 0, \quad \begin{vmatrix} Z_1(z_{14}) & Z_2(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) \end{vmatrix} \neq 0,$$

and we solve in $m_{31}, m_{32}, n_{31}, n_{32}$ the simultaneous equations

$$(409) \quad \begin{aligned} Y_1(y_{14})m_{31} + Y_2(y_{14})m_{32} &= Y_3(y_{14}), \\ Y_1(y_{15})m_{31} + Y_2(y_{15})m_{32} &= Y_3(y_{15}), \end{aligned}$$

and

$$(410) \quad \begin{aligned} Z_1(z_{14})n_{31} + Z_2(z_{14})n_{32} &= Z_3(z_{14}), \\ Z_1(z_{15})n_{31} + Z_2(z_{15})n_{32} &= Z_3(z_{15}). \end{aligned}$$

It is worth while to verify that

$$(411) \quad Y_3 \equiv m_{31}Y_1 + m_{32}Y_2, \quad Z_3 \equiv n_{31}Z_1 + n_{32}Z_2^{(1)}.$$

If $m_{32} = 0$, we interchange the indices as follows:

$$(412) \quad \frac{\text{before the change}}{\text{after the change}} \quad \begin{vmatrix} Y_1 & Y_2 & Z_1 & Z_2 & m_{31} & m_{32} \\ Y_2 & Y_1 & Z_2 & Z_1 & m_{32} & m_{31} \end{vmatrix}.$$

Thus, we always have

$$(413) \quad m_{32} \neq 0.$$

Then the function F is uniquely nomographic.

If in scheme I we have immediately obtained the cases X2, Y2, Z3 for the function F , then each of its Massau forms is equivalent to the form

$$(414) \quad F \equiv \begin{vmatrix} X_1(x) & X_2(x) & 0 \\ 0 & Y_1(y) & Y_3(y) \\ Z_3(z) & -\frac{1}{m_{32}}Z_2(z) & Z_1(z) - \frac{m_{31}}{m_{32}}Z_2(z) \end{vmatrix}.$$

If in scheme I we have obtained the cases X2, Y2, Z3 for the function F^* , then each of the Massau forms of the function F is equivalent to the form

(¹) The computations of (408), (409), and (410) may be omitted if we are able to write the first condition (411) and to prove the linear independence of the functions Z_1, Z_2, Z_3 immediately. The conditions (411) must be satisfied, since the first one of them is identical to (255) and it is easy to show that $Z_3 \equiv n_{31}Z_1 + n_{32}Z_2$ implies the case Z2 instead of Z3.

$$(415) \quad F \equiv \begin{vmatrix} X_1(x) & X_2(x) & 0 \\ -Z_3(y) & \frac{1}{m_{32}} Z_2(y) & \frac{m_{31}}{m_{32}} Z_2(y) - Z_1(y) \\ 0 & Y_1(z) & Y_3(z) \end{vmatrix}.$$

If in scheme I we have obtained the cases X2, Y2, Z3 for the function F^{**} , then each of the Massau forms of the function F is equivalent to the form

$$(416) \quad F \equiv \begin{vmatrix} -Z_3(x) & \frac{1}{m_{32}} Z_2(x) & \frac{m_{31}}{m_{32}} Z_2(x) - Z_1(x) \\ 0 & Y_1(y) & Y_3(y) \\ X_1(z) & X_2(z) & 0 \end{vmatrix}.$$

Scheme VII

We find such elements $y_{14}, y_{15}, z_{14}, z_{15}$ that

$$(417) \quad \begin{vmatrix} Y_1(y_{14}) & Y_2(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} Z_1(z_{14}) & Z_2(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) \end{vmatrix} \neq 0,$$

and we solve in m_{ij}, q_{ij} ($i = 3, 4; j = 1, 2$) the simultaneous equations

$$(418) \quad \begin{aligned} Y_1(y_{14})m_{i1} + Y_2(y_{14})m_{i2} &= Y_i(y_{14}) \\ Y_1(y_{15})m_{i1} + Y_2(y_{15})m_{i2} &= Y_i(y_{15}) \end{aligned} \quad (i = 3, 4),$$

and

$$(419) \quad \begin{aligned} Z_1(z_{14})q_{i1} + Z_2(z_{14})q_{i2} &= Z_i(z_{14}) \\ Z_1(z_{15})q_{i1} + Z_2(z_{15})q_{i2} &= Z_i(z_{15}) \end{aligned} \quad (i = 3, 4).$$

It is worth while to verify that

$$(420) \quad Y_3 \equiv m_{31} Y_1 + m_{32} Y_2, \quad Y_4 \equiv m_{41} Y_1 + m_{42} Y_2,$$

and at least one of the identities

$$(421) \quad Z_3 \equiv q_{31} Z_1 + q_{32} Z_2, \quad Z_4 \equiv q_{41} Z_1 + q_{42} Z_2$$

fails⁽¹⁾. If the second identity (421) holds, we interchange the indices as follows:

⁽¹⁾ The computations of (417), (418), and (419) may be omitted if we are able to write the identities (420) and to prove the linear independence of the functions Z_1, Z_2, Z_3 or Z_1, Z_2, Z_4 immediately.

$$(422) \quad \frac{\text{before the change}}{\text{after the change}} \quad \left| \begin{array}{c|c|c|c|c|c} Y_3 & Y_4 & Z_3 & Z_4 & m_{3i} & m_{4i} \\ \hline Y_4 & Y_3 & Z_4 & Z_3 & m_{4i} & m_{3i} \end{array} \right| \quad (i = 1, 2).$$

Thus the second condition (421) always fails, i. e., the functions Z_1, Z_2, Z_4 are linearly independent.

Now we find such an element z_{16} that

$$(423) \quad \left| \begin{array}{c|c|c} Z_1(z_{14}) & Z_2(z_{14}) & Z_4(z_{14}) \\ \hline Z_1(z_{15}) & Z_2(z_{15}) & Z_4(z_{15}) \\ \hline Z_1(z_{16}) & Z_2(z_{16}) & Z_4(z_{16}) \end{array} \right| \neq 0$$

and we solve in n_{31}, n_{32}, n_{34} the simultaneous equations

$$(424) \quad \begin{aligned} Z_1(z_{14})n_{31} + Z_2(z_{14})n_{32} + Z_4(z_{14})n_{34} &= Z_3(z_{14}), \\ Z_1(z_{15})n_{31} + Z_2(z_{15})n_{32} + Z_4(z_{15})n_{34} &= Z_3(z_{15}), \\ Z_1(z_{16})n_{31} + Z_2(z_{16})n_{32} + Z_4(z_{16})n_{34} &= Z_3(z_{16}). \end{aligned}$$

It is worth while to verify that

$$(425) \quad Z_3 \equiv n_{31}Z_1 + n_{32}Z_2 + n_{34}Z_4^{(1)}.$$

If $n_{32} = 0$, we interchange the indices as follows:

$$(426) \quad \frac{\text{before the change}}{\text{after the change}} \quad \left| \begin{array}{c|c|c|c|c|c|c|c} Y_1 & Y_2 & Z_1 & Z_2 & m_{i1} & m_{i2} & n_{i1} & n_{i2} \\ \hline Y_2 & Y_1 & Z_2 & Z_1 & m_{i2} & m_{i1} & n_{i2} & n_{i1} \end{array} \right| \quad (i = 3, 4).$$

Thus we always have

$$(427) \quad n_{32} \neq 0.$$

Now if the condition

$$(428) \quad n_{31} = -\frac{m_{32}n_{34} + m_{42}}{m_{31}n_{34} + m_{41}} n_{32} \quad (m_{31}n_{34} + m_{41} \neq 0)$$

fails, the function F is not nomographic.

If the condition (428) holds, the function F is uniquely nomographic. Then, if in scheme I we have immediately obtained the cases X2, Y2, Z3 for the function F , each of its Massau forms is equivalent to the form

⁽¹⁾ The computations of (423) and (424) may be omitted if we are able to write the identity (425) immediately. The identities (420) and (425) must be satisfied by the proof given for the case II.3 on the page 55.

$$(429) \quad F \equiv \begin{vmatrix} X_1(x) & X_2(x) & 0 \\ n_{32} Y_3(y) & -Y_2(y) & Y_1(y) - \frac{n_{31}}{n_{32}} Y_2(y) \\ (m_{31}n_{34} + m_{41})Z_4(z) & -Z_1(z) & -\frac{n_{31}}{n_{32}} Z_1(z) - Z_2(z) \end{vmatrix}.$$

If we have obtained in scheme I the cases X2, Y2, Z3 for the function F^* , then each of the Massau forms of the function F is equivalent to the form

$$(430) \quad F \equiv \begin{vmatrix} X_1(x) & X_2(x) & 0 \\ -(m_{31}n_{34} + m_{41})Z_4(y) & Z_1(y) & \frac{n_{31}}{n_{32}} Z_1(y) + Z_2(y) \\ n_{32} Y_3(z) & -Y_2(z) & Y_1(z) - \frac{n_{31}}{n_{32}} Y_2(z) \end{vmatrix}.$$

If we have obtained in scheme I the cases X2, Y2, Z3 for the function F^{**} , then each of the Massau forms of the function F is equivalent to the form

$$(431) \quad F \equiv \begin{vmatrix} -(m_{31}n_{34} + m_{41})Z_4(x) & Z_1(x) & \frac{n_{31}}{n_{32}} Z_1(x) + Z_2(x) \\ n_{32} Y_3(y) & -Y_2(y) & Y_1(y) - \frac{n_{31}}{n_{32}} Y_2(y) \\ X_1(z) & X_2(z) & 0 \end{vmatrix}.$$

Scheme VIII

We find such elements y_{14}, y_{15} that

$$(432) \quad \begin{vmatrix} Y_1(y_{14}) & Y_2(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) \end{vmatrix} \neq 0$$

and we solve in $p_{31}, p_{32}, p_{41}, p_{42}$ the simultaneous equations

$$(433) \quad \begin{aligned} Y_1(y_{14})p_{i1} + Y_2(y_{14})p_{i2} &= Y_i(y_{14}) \\ Y_1(y_{15})p_{i1} + Y_2(y_{15})p_{i2} &= Y_i(y_{15}) \end{aligned} \quad (i = 3, 4).$$

We verify that at least one of the conditions

$$(434) \quad Y_3 \equiv p_{31} Y_1 + p_{32} Y_2, \quad Y_4 \equiv p_{41} Y_1 + p_{42} Y_2$$

fails⁽¹⁾.

⁽¹⁾ The computations (432) and (433) may be omitted if we are able to determine immediately whether the functions Y_1, Y_2, Y_i ($i = 3, 4$) are linearly dependent.

If the first identity (434) holds, we interchange the indices as follows:

$$(435) \quad \begin{array}{l} \text{before the change} \\ \text{after the change} \end{array} \quad \left| \begin{array}{cccc|cc} Y_3 & Y_4 & Z_3 & Z_4 & p_{3j} & p_{4j} \\ Y_4 & Y_3 & Z_4 & Z_3 & p_{4j} & p_{3j} \end{array} \right| \quad (j = 1, 2).$$

Thus the first identity (434) always fails, i. e., the functions Y_1, Y_2, Y_3 are linearly independent.

We now find such elements z_{14}, z_{15} that

$$(436) \quad \left| \begin{array}{cc} Z_1(z_{14}) & Z_2(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) \end{array} \right| \neq 0.$$

If

$$(437) \quad \left| \begin{array}{ccc} Z_1(z_{14}) & Z_2(z_{14}) & Z_4(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) & Z_4(z_{15}) \\ Z_1(z) & Z_2(z) & Z_4(z) \end{array} \right| \equiv 0,$$

the function F is not nomographic. If the condition (437) fails, we find such an element z_{16} that

$$(438) \quad \left| \begin{array}{ccc} Z_1(z_{14}) & Z_2(z_{14}) & Z_4(z_{14}) \\ Z_1(z_{15}) & Z_2(z_{15}) & Z_4(z_{15}) \\ Z_1(z_{16}) & Z_2(z_{16}) & Z_4(z_{16}) \end{array} \right| \neq 0$$

and we solve in n_{31}, n_{32}, n_{34} the simultaneous equations

$$(439) \quad \begin{aligned} Z_1(z_{14})n_{31} + Z_2(z_{14})n_{32} + Z_4(z_{14})n_{34} &= Z_3(z_{14}), \\ Z_1(z_{15})n_{31} + Z_2(z_{15})n_{32} + Z_4(z_{15})n_{34} &= Z_3(z_{15}), \\ Z_1(z_{16})n_{31} + Z_2(z_{16})n_{32} + Z_4(z_{16})n_{34} &= Z_3(z_{16}). \end{aligned}$$

We also find such an element y_{16} that

$$(440) \quad \left| \begin{array}{ccc} Y_1(y_{14}) & Y_2(y_{14}) & Y_3(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) & Y_3(y_{15}) \\ Y_1(y_{16}) & Y_2(y_{16}) & Y_3(y_{16}) \end{array} \right| \neq 0$$

and we solve in m_{41}, m_{42}, m_{43} the simultaneous equations

$$(441) \quad \begin{aligned} Y_1(y_{14})m_{41} + Y_2(y_{14})m_{42} + Y_3(y_{14})m_{43} &= Y_4(y_{14}), \\ Y_1(y_{15})m_{41} + Y_2(y_{15})m_{42} + Y_3(y_{15})m_{43} &= Y_4(y_{15}), \\ Y_1(y_{16})m_{41} + Y_2(y_{16})m_{42} + Y_3(y_{16})m_{43} &= Y_4(y_{16}). \end{aligned}$$

It is worth while to verify that

$$(442) \quad Y_4 \equiv m_{41}Y_1 + m_{42}Y_2 + m_{43}Y_3, \quad Z_3 \equiv n_{31}Z_1 + n_{32}Z_2 + n_{34}Z_4^{(1)}.$$

(1) The computations (432)-(441) may be omitted if we are able to write immediately the identities (442).

If $m_{41} = 0$, we interchange the indices as follows:

$$(443) \quad \begin{array}{c} \text{before the change} \\ \hline \text{after the change} \end{array} \left| \begin{array}{cccc|cccc} Y_1 & Y_2 & Z_1 & Z_2 & m_{41} & m_{42} & n_{31} & n_{32} \\ \hline Y_2 & Y_1 & Z_2 & Z_1 & m_{42} & m_{41} & n_{32} & n_{31} \end{array} \right|.$$

Thus we always have

$$(444) \quad m_{41} \neq 0.$$

If at least one of the conditions

$$(445) \quad n_{31} = -\frac{m_{42}}{m_{41}} n_{32}, \quad n_{34} = -m_{43}$$

fails, the function F is not nomographic.

If both conditions (445) are satisfied, the function F is uniquely nomographic. Then, if we have immediately obtained from scheme I the cases X2, Y3, Z3 for the function F , each of its Massau forms is equivalent to the form

$$(446) \quad F \equiv \left| \begin{array}{ccc} X_1(x) & X_2(x) & 0 \\ n_{32} Y_3(y) & -Y_2(y) & Y_1(y) + \frac{m_{42}}{m_{41}} Y_2(y) \\ m_{41} Z_4(z) & -Z_1(z) & \frac{m_{42}}{m_{41}} Z_1(z) - Z_2(z) \end{array} \right|.$$

If the cases X2, Y3, Z3 have been obtained for the function F^{**} , then each of the Massau forms of the function F is equivalent to the form

$$(447) \quad F \equiv \left| \begin{array}{ccc} -m_{41} Z_4(x) & Z_1(x) & Z_2(x) - \frac{m_{42}}{m_{41}} Z_1(x) \\ n_{32} Y_3(y) & -Y_2(y) & Y_1(y) + \frac{m_{42}}{m_{41}} Y_2(y) \\ X_1(z) & X_2(z) & 0 \end{array} \right|.$$

Finally, if the cases X2, Y3, Z3 have been obtained for the function F^{***} , then each of the Massau forms of the function F is equivalent to the form

$$(448) \quad F \equiv \left| \begin{array}{ccc} n_{32} Y_3(x) & -Y_2(x) & Y_1(x) + \frac{m_{42}}{m_{41}} Y_2(x) \\ -X_1(y) & -X_2(y) & 0 \\ m_{41} Z_4(z) & -Z_1(z) & \frac{m_{42}}{m_{41}} Z_1(z) - Z_2(z) \end{array} \right|.$$

Scheme IX

We find such elements y_{14}, y_{15} that

$$(449) \quad \begin{vmatrix} Y_1(y_{14}) & Y_2(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) \end{vmatrix} \neq 0$$

and we solve in p_{i1}, p_{i2} ($i = 3, 4, 5, 6$) the simultaneous equations

$$(450) \quad \begin{aligned} Y_1(y_{14})p_{i1} + Y_2(y_{14})p_{i2} &= Y_i(y_{14}) \\ Y_1(y_{15})p_{i1} + Y_2(y_{15})p_{i2} &= Y_i(y_{15}) \end{aligned} \quad (i = 3, 4, 5, 6).$$

We verify that at least one of the identities

$$(451) \quad Y_i \equiv p_{i1}Y_1 + p_{i2}Y_2 \quad (i = 3, 4, 5, 6)$$

fails.

If the identities (451) hold for $i = 3$ and $i = 4$, we interchange the indices as follows:

$$(452) \quad \frac{\text{before the change } |X_2|X_3|G_2|G_3|Y_3|Y_4|Y_5|Y_6|Z_3|Z_4|Z_5|Z_6|}{\text{after the change } |X_3|X_2|G_3|G_2|Y_5|Y_6|Y_3|Y_4|Z_5|Z_6|Z_3|Z_4|} \\ \frac{\text{before the change } |p_{3j}|p_{4j}|p_{5j}|p_{6j}|}{\text{after the change } |p_{5j}|p_{6j}|p_{3j}|p_{4j}|} \quad (j = 1, 2).$$

Thus at least one of the identities

$$(453) \quad Y_3 \equiv p_{31}Y_1 + p_{32}Y_2, \quad Y_4 \equiv p_{41}Y_1 + p_{42}Y_2$$

fails. If the first identity (453) holds, we interchange the indices as follows:

$$(454) \quad \frac{\text{before the change } |Y_3|Y_4|Z_3|Z_4|p_{3j}|p_{4j}|}{\text{after the change } |Y_4|Y_3|Z_4|Z_3|p_{4j}|p_{3j}|} \quad (j = 1, 2).$$

Thus the first identity (453) always fails, i. e., the functions Y_1, Y_2, Y_3 are linearly independent.

Now we find such an element y_{16} that

$$(455) \quad \begin{vmatrix} Y_1(y_{14}) & Y_2(y_{14}) & Y_3(y_{14}) \\ Y_1(y_{15}) & Y_2(y_{15}) & Y_3(y_{15}) \\ Y_1(y_{16}) & Y_2(y_{16}) & Y_3(y_{16}) \end{vmatrix} \neq 0$$

and we solve in m_{i1}, m_{i2}, m_{i3} the simultaneous equations

$$\begin{aligned}
 & Y_1(y_{14})m_{i1} + Y_2(y_{14})m_{i2} + Y_3(y_{14})m_{i3} = Y_i(y_{14}) \\
 (456) \quad & Y_1(y_{15})m_{i1} + Y_2(y_{15})m_{i2} + Y_3(y_{15})m_{i3} = Y_i(y_{15}) \quad (i = 4, 5, 6). \\
 & Y_1(y_{16})m_{i1} + Y_2(y_{16})m_{i2} + Y_3(y_{16})m_{i3} = Y_i(y_{16})
 \end{aligned}$$

It is worth while to verify that

$$(457) \quad Y_i \equiv m_{i1} Y_1 + m_{i2} Y_2 + m_{i3} Y_3 \quad (i = 4, 5, 6).$$

If $m_{41} = 0$, we interchange the indices as follows:

$$(458) \quad \frac{\text{before the change} \quad \left| \begin{array}{c|c|c|c|c|c} Y_1 & Y_2 & Z_1 & Z_2 & m_{i1} & m_{i2} \end{array} \right|}{\text{after the change} \quad \left| \begin{array}{c|c|c|c|c|c} Y_2 & Y_1 & Z_2 & Z_1 & m_{i2} & m_{i1} \end{array} \right|} \quad (i = 4, 5, 6).$$

Thus we always have

$$(459) \quad m_{41} \neq 0.$$

If $m_{53} = m_{63} = 0$, the function F is not nomographic. If

$$(460) \quad m_{53}^2 + m_{63}^2 > 0$$

and $m_{63} = 0$, we interchange the indices as follows:

$$(461) \quad \frac{\text{before the change} \quad \left| \begin{array}{c|c|c|c|c|c} Y_5 & Y_6 & Z_5 & Z_6 & m_{5j} & m_{6j} \end{array} \right|}{\text{after the change} \quad \left| \begin{array}{c|c|c|c|c|c} Y_6 & Y_5 & Z_6 & Z_5 & m_{6j} & m_{5j} \end{array} \right|} \quad (j = 1, 2, 3).$$

Thus we always have

$$(462) \quad m_{63} \neq 0.$$

Now we find such elements z_{14}, z_{15} that

$$(463) \quad \left| \begin{array}{c|c} Z_1(z_{14}) & Z_2(z_{14}) \\ \hline Z_1(z_{15}) & Z_2(z_{15}) \end{array} \right| \neq 0.$$

If

$$(464) \quad Z^*(z) \equiv \left| \begin{array}{c|c|c} Z_1(z_{14}) & Z_2(z_{14}) & Z_4(z_{14}) \\ \hline Z_1(z_{15}) & Z_2(z_{15}) & Z_4(z_{15}) \\ \hline Z_1(z) & Z_2(z) & Z_4(z) \end{array} \right| \equiv 0,$$

the function F is not nomographic. If $Z^*(z) \not\equiv 0$, we find such an element z_{16} that

$$(465) \quad Z^*(z_{16}) = \left| \begin{array}{c|c|c} Z_1(z_{14}) & Z_2(z_{14}) & Z_4(z_{14}) \\ \hline Z_1(z_{15}) & Z_2(z_{15}) & Z_4(z_{15}) \\ \hline Z_1(z_{16}) & Z_2(z_{16}) & Z_4(z_{16}) \end{array} \right| \neq 0$$

and we solve in n_{i1}, n_{i2}, n_{i4} the simultaneous equations

$$\begin{aligned}
 & Z_1(z_{14})n_{i1} + Z_2(z_{14})n_{i2} + Z_4(z_{14})n_{i4} = Z_i(z_{14}) \\
 (466) \quad & Z_1(z_{15})n_{i1} + Z_2(z_{15})n_{i2} + Z_4(z_{15})n_{i4} = Z_i(z_{15}) \quad (i = 3, 5, 6). \\
 & Z_1(z_{16})n_{i1} + Z_2(z_{16})n_{i2} + Z_4(z_{16})n_{i4} = Z_i(z_{16})
 \end{aligned}$$

It is worth while to verify that

$$(467) \quad Z_i \equiv n_{i1}Z_1 + n_{i2}Z_2 + n_{i4}Z_4 \quad (i = 3, 5, 6).$$

Now if at least one of the conditions

$$\begin{aligned}
 (468) \quad & n_{31} = -\frac{m_{42}}{m_{41}}n_{32}, & n_{61} &= -\frac{n_{32}n_{54}}{m_{41}m_{63}}m_{52}, \\
 & n_{34} = -m_{43}, & n_{62} &= \frac{n_{32}n_{54}}{m_{41}m_{63}}m_{51}, \\
 & n_{51} = \frac{n_{32}n_{54}}{m_{41}m_{63}}m_{62}, & n_{64} &= -\frac{m_{53}}{m_{63}}n_{54} \\
 & n_{52} = -\frac{n_{32}n_{54}}{m_{41}m_{63}}m_{61},
 \end{aligned}$$

fails, the function F is not nomographic.

If all the conditions (468) hold, the function F is uniquely nomographic and each of its Massau forms is equivalent to the form

$$(469) \quad F \equiv \left| \begin{array}{ccc}
 X_1 + (m_{51}m_{62} - & X_2 + (m_{51}m_{63} - & \frac{n_{54}^2}{m_{41}^2 m_{63}} X_3 \\
 - m_{52}m_{61}) \frac{n_{32}n_{54}}{m_{41}m_{63}} X_3 & - m_{53}m_{61}) \frac{n_{54}}{m_{41}m_{63}} X_3 & \\
 n_{32} Y_3 & - Y_2 & Y_1 + \frac{m_{42}}{m_{41}} Y_2 \\
 m_{41} Z_4 & - Z_1 & \frac{m_{42}}{m_{41}} Z_1 - Z_2
 \end{array} \right|,$$

where

$$(470) \quad r = (m_{41}m_{52} - m_{42}m_{51})m_{63} - (m_{41}m_{62} - m_{42}m_{61})m_{53}.$$

Remark. The computations of (449)-(466) may be omitted if we are able to write immediately the identities (457) and (467) with the conditions (459) and (462).

VI. EXAMPLES

We retain the previous notations.

I. Let us determine whether the function

$$(471) \quad F \equiv F(x, y, z) \equiv X + Y + Z,$$

where

$$X \equiv X(x) \neq \text{const}, \quad Y \equiv Y(y) \neq \text{const}, \quad Z \equiv Z(z) \neq \text{const},$$

and the variables x, y, z take numerical values, is nomographic.

We have

$$(472) \quad F \equiv X \cdot 1 + 1 \cdot (Y + Z).$$

We find such numbers x_1 and x_2 that

$$X(x_1) \neq X(x_2).$$

(It is always possible, since $X \neq \text{const.}$) We see that then

$$\begin{vmatrix} X(x_1) & 1 \\ X(x_2) & 1 \end{vmatrix} \neq 0.$$

It follows that the functions

$$(473) \quad X_1 \equiv X_1(x) \equiv X, \quad X_2 \equiv X_2(x) \equiv 1,$$

are linearly independent. Analogously, the functions

$$(474) \quad G_1 \equiv G_1(y, z) \equiv 1, \quad G_2 \equiv G_2(y, z) \equiv Y + Z,$$

are also linearly independent.

Since by (473), (474) and (472) it is

$$F \equiv X_1 G_1 + X_2 G_2,$$

the function F is of rank 2 with respect to x , i. e., we have the case X2 from scheme I.

By the symmetry of the equation (471) the function F is of rank 2 also with respect to each of the variables y and z . Thus we have the cases X2, Y2, Z2 from scheme I.

Moreover, we have

$$G_1 \equiv Y_1 Z_1 \quad \text{and} \quad G_2 \equiv Y_3 Z_3 + Y_4 Z_4,$$

where

$$Y_1 \equiv 1, \quad Z_1 \equiv 1, \quad Y_3 \equiv Y, \quad Z_3 \equiv 1, \quad Y_4 \equiv 1, \quad Z_4 \equiv Z.$$

Since the functions Y_3, Y_4 and the functions Z_3, Z_4 — separately treated — are linearly independent like the functions (473), we have the cases $G_1 1$ and $G_2 2$ from scheme II. Therefore, we interchange the indices according to (387) and thus we have

$$\begin{aligned} X_1 &\equiv 1, & X_2 &\equiv X, & Y_1 &\equiv Y, & Y_2 &\equiv 1, & Y_3 &\equiv 1, \\ G_1 &\equiv Y + Z, & G_2 &\equiv 1, & Z_1 &\equiv 1, & Z_2 &\equiv Z, & Z_3 &\equiv 1, \\ G_1 &\equiv Y_1 Z_1 + Y_2 Z_2, & G_2 &\equiv Y_3 Z_3, \end{aligned}$$

and we pass to scheme IV.

We see that the identities (392) may be written immediately. We have, namely,

$$Y_3 \equiv 0 \cdot Y_1 + 1 \cdot Y_2 \quad \text{and} \quad Z_3 \equiv 1 \cdot Z_1 + 0 \cdot Z_2.$$

Thus we have

$$m_{31} = 0, \quad m_{32} = 1, \quad n_{31} = 1, \quad n_{32} = 0,$$

and the condition (394) is satisfied.

Since $m_{31} n_{31} + m_{32} n_{32} = 0$, the function (471) is uniquely nomographic and, by (397), each of its Massau forms is equivalent to the form

$$F \equiv \begin{vmatrix} 1 & X & 0 \\ 0 & Y & 1 \\ 1 & -Z & 1 \end{vmatrix}.$$

II. Let us consider the function

$$(475) \quad F \equiv F(x, y, z) \equiv XY - Z,$$

where

$$X \equiv X(x) \neq \text{const}, \quad Y \equiv Y(y) \neq \text{const}, \quad Z \equiv Z(z) \neq \text{const},$$

the values of the variables x, y, z being numbers.

We have

$$F \equiv X \cdot Y - 1 \cdot Z \equiv Y \cdot X - 1 \cdot Z \equiv 1 \cdot XY - Z \cdot 1,$$

where the functions:

X	and	1	as functions of one variable x ,
Y	,,	$-Z$,, ,, two variables y and z ,
Y	,,	1	,, ,, one variable y ,
X	,,	$-Z$,, ,, two variables x and z ,
1	,,	$-Z$,, ,, one variable z ,
XY	,,	1	,, ,, two variables x and y ,

are obviously linearly independent. Thus we have the cases X_2, Y_2, Z_2 from scheme I and we may set

$$\begin{aligned} X_1 &\equiv X_1(x) \equiv X, & X_2 &\equiv X_2(x) \equiv 1, \\ G_1 &\equiv G_1(y, z) \equiv Y_1 Z_1, & G_2 &\equiv G_2(y, z) \equiv Y_3 Z_3, \end{aligned}$$

where

$$\begin{aligned} Y_1 &\equiv Y_1(y) \equiv Y, & Y_3 &\equiv Y_3(y) \equiv 1, \\ Z_1 &\equiv Z_1(z) \equiv 1, & Z_3 &\equiv Z_3(z) \equiv -Z. \end{aligned}$$

Thus we have the cases $G_1, 1$ and $G_2, 1$ from scheme II and, therefore, we now pass to scheme III. We see that the function (475) is doubly nomographic and each of its Massau forms is — by (388) — equivalent to one of the following two non-equivalent forms

$$F \equiv \begin{vmatrix} X & 1 & 0 \\ 0 & Y & 1 \\ -Z & 0 & 1 \end{vmatrix} \equiv \begin{vmatrix} X & 1 & 0 \\ 1 & 0 & Y \\ 0 & -1 & Z \end{vmatrix}.$$

III. Let us consider the function

$$(476) \quad F \equiv F(x, y, z) \equiv XZ' + YZ'' + 1,$$

where

$$\begin{aligned} X &\equiv X(x) \not\equiv \text{const}, & Y &\equiv Y(y) \not\equiv \text{const}, \\ Z' &\equiv Z'(z) \not\equiv \text{const}, & Z'' &\equiv Z''(z) \not\equiv \text{const} \end{aligned}$$

the values of the variables x, y, z being numbers.

Since

$$F \equiv X \cdot Z' + 1 \cdot (YZ'' + 1) \equiv Y \cdot Z'' + 1 \cdot (XZ' + 1)$$

and the functions

X	and	1	as functions of one variable x ,
Z'	,,	$YZ'' + 1$,, ,, two variables y and z ,
Y	,,	1	,, ,, one variable y ,
Z''	,,	$XZ' + 1$,, ,, two variables x and z ,

are obviously, linearly independent, then the function F is of rank 2 with respect to each of the variables x and y , i. e., we have the cases X2 and Y2 from scheme I. The rank of the function F with respect to z depends upon the linear dependence or independence of the functions Z' , Z'' and $Z''' \equiv Z'''(z) \equiv 1$. The following three cases are possible:

A. The functions Z' , Z'' , Z''' are linearly independent.

B. The functions Z' , Z'' are linearly independent but the functions Z' , Z'' , Z''' are linearly dependent.

C. The functions Z' , Z'' are linearly dependent.

In the case A the function F is of rank 3 with respect to z , since

$$F \equiv Z' \cdot X + Z'' \cdot Y + Z''' \cdot 1$$

and the functions X , Y and 1 as functions of two variables x and y are obviously linearly independent. Thus we have the cases X2, Y2, Z3 from scheme I and

$$\begin{aligned} X_1 &\equiv X_1(x) \equiv 1, & X_2 &\equiv X_2(x) \equiv X, \\ G_1 &\equiv G_1(y, z) \equiv Y_1 Z_1 + Y_2 Z_2, & G_2 &\equiv G_2(y, z) \equiv Y_3 Z_3, \end{aligned}$$

where

$$Y_1 \equiv Y, \quad Y_2 \equiv 1, \quad Y_3 \equiv 1, \quad Z_1 \equiv Z'', \quad Z_2 \equiv 1, \quad Z_3 \equiv Z'.$$

Since the functions Y_1 , Y_2 and the functions Z_1 , Z_2 — separately treated — are linearly independent, we have the cases $G_1 2$ and $G_2 1$ from scheme II, and therefore we pass to scheme VI.

We already know that the functions Z_1 , Z_2 , Z_3 are linearly independent. Moreover,

$$Y_3 \equiv 0 \cdot Y_1 + 1 \cdot Y_2.$$

Thus we immediately have the conditions (411), where

$$m_{31} = 0 \quad \text{and} \quad m_{32} = 1.$$

The condition (413) is satisfied. The function F is uniquely nomographic and each of its Massau forms is — by (414) — equivalent to the form

$$F \equiv \begin{vmatrix} 1 & X & 0 \\ 0 & Y & 1 \\ Z' & -1 & Z'' \end{vmatrix}.$$

In the case B there exist such numbers p_1 , p_2 that

$$(477) \quad 1 \equiv p_1 Z' + p_2 Z'',$$

and hence

$$F \equiv (X + p_1) Z' + (Y + p_2) Z''.$$

Since the functions $X + p_1$ and $Y + p_2$, as functions of two variables x, y , are obviously linearly independent, the function F is of rank 2 with respect to z . We have the cases X2, Y2, Z2 from scheme I and

$$(478) \quad F \equiv X_1 G_1 + X_2 G_2,$$

where

$$(479) \quad \begin{aligned} X_1 &\equiv X_1(x) \equiv 1, & X_2 &\equiv X_2(x) \equiv X, \\ G_1 &\equiv G_1(y, z) \equiv Y_1 Z_1 + Y_2 Z_2, & G_2 &\equiv G_2(y, z) \equiv Y_3 Z_3, \end{aligned}$$

and

$$(480) \quad Y_1 \equiv Y, \quad Y_2 \equiv 1, \quad Y_3 \equiv 1, \quad Z_1 \equiv Z'', \quad Z_2 \equiv 1, \quad Z_3 \equiv Z'.$$

The functions Y_1, Y_2 and the functions Z_1, Z_2 — separately treated — are linearly independent. Hence we have the cases $G_1 2, G_2 1$ from scheme II and we pass to scheme IV.

We see that the identities (392) may be written immediately. We have, namely,

$$Y_3 \equiv 0 \cdot Y_1 + 1 \cdot Y_2 \quad \text{and} \quad Z_3 \equiv -\frac{p_2}{p_1} Z_1 + \frac{1}{p_1} Z_2$$

(the identity (477) with the condition $Z'' \neq \text{const}$ implies $p_1 \neq 0$), and hence

$$m_{31} = 0, \quad m_{32} = 1, \quad n_{31} = -\frac{p_2}{p_1}, \quad n_{32} = \frac{1}{p_1}.$$

The condition (394) is satisfied. Since $m_{31} n_{31} + m_{32} n_{32} = 1/p_1 \neq 0$ and $n_{32} \neq 0$, the function F is then doubly nomographic and each of its Massau forms is equivalent to one of the following two non-equivalent forms:

$$F \equiv \begin{vmatrix} 1 & X & 0 \\ 0 & Y & 1 \\ Z' & -1 & Z'' \end{vmatrix} \equiv \begin{vmatrix} 1 & X & 0 \\ 1 & -p_1 & Y + p_2 \\ 0 & -Z'' & -Z' \end{vmatrix}.$$

In the case C there exists such a number p that $Z' \equiv pZ''$ and therefore,

$$F \equiv Z''(pX + Y) + 1 \cdot 1.$$

Since the functions Z'' and 1 as functions of one variable z and the functions $pX + Y$ and 1 as functions of two variables x and y are obviously linearly independent, then the function F is of rank 2 with respect to z . Thus we have the cases X2, Y2, Z2 from scheme I. Since the identities (478), (479), and (480) hold also in the present case, we have the cases

G_12, G_21 from scheme II and we pass to scheme IV. We see that the identities (392) may be written immediately. We have, namely,

$$Y_3 \equiv 0 \cdot Y_1 + 1 \cdot Y_2 \quad \text{and} \quad Z_3 \equiv pZ_1 + 0 \cdot Z_2.$$

Hence

$$m_{31} = 0, \quad m_{32} = 1, \quad n_{31} = p, \quad n_{32} = 0.$$

The condition (394) is satisfied.

Since $m_{31}n_{31} + m_{32}n_{32} = 0$, the function F is uniquely nomographic and each of its Massau forms is equivalent to the form

$$F \equiv \begin{vmatrix} 1 & X & 0 \\ 0 & Y & 1 \\ Z' & -1 & Z'' \end{vmatrix}.$$

IV. Let us consider the function

$$(481) \quad F \equiv F(x, y, z) \equiv xy + yz - xz + 1,$$

where the variables x, y, z take numerical values. Since

$$F \equiv x(y - z) + 1 \cdot (yz + 1) \equiv y(x + z) + 1 \cdot (1 - xz) \equiv z(y - x) + 1 \cdot (xy + 1)$$

and the functions

x	and	1	as	the functions	of	one	variable	x ,		
$y - z$,,	$yz + 1$,,	,,	,,	,,	,,	two	variables	y and z ,
y	,,	1	,,	,,	,,	,,	,,	one	variable	y ,
$x + z$,,	$1 - xz$,,	,,	,,	,,	,,	two	variables	x and z ,
z	,,	1	,,	,,	,,	,,	,,	one	variable	z ,
$y - x$,,	$xy + 1$,,	,,	,,	,,	,,	two	variables	x and y

are obviously linearly independent, then we have the cases X2, Y2, Z2 from scheme I and

$$F \equiv X_1G_1 + X_2G_2,$$

where

$$\begin{aligned} X_1 &\equiv X_1(x) \equiv x, & X_2 &\equiv X_2(x) \equiv 1, \\ G_1 &\equiv G_1(y, z) \equiv y - z, & G_2 &\equiv G_2(y, z) \equiv yz + 1. \end{aligned}$$

We see that

$$G_1 \equiv Y_1Z_1 + Y_2Z_2, \quad G_2 \equiv Y_3Z_3 + Y_4Z_4,$$

where

$$\begin{aligned} Y_1 &\equiv y, & Y_2 &\equiv 1, & Y_3 &\equiv y, & Y_4 &\equiv 1, \\ Z_1 &\equiv 1, & Z_2 &\equiv -z, & Z_3 &\equiv z, & Z_4 &\equiv 1. \end{aligned}$$

Since the functions Y_1, Y_2 , the functions Z_1, Z_2 , the functions Y_3, Y_4 and the functions Z_3, Z_4 — separately treated — are linearly independent, we have the cases G_12, G_22 from scheme II and, therefore, we pass to scheme V.

We see that the identities (401) may be written immediately. We have, namely,

$$\begin{aligned} Y_3 &\equiv 1 \cdot Y_1 + 0 \cdot Y_2, & Z_3 &\equiv 0 \cdot Z_1 - 1 \cdot Z_2, \\ Y_4 &\equiv 0 \cdot Y_1 + 1 \cdot Y_2, & Z_4 &\equiv 1 \cdot Z_1 + 0 \cdot Z_2. \end{aligned}$$

Thus we have obtained

$$\begin{aligned} m_{31} &= 1, & m_{32} &= 0, & m_{41} &= 0, & m_{42} &= 1, \\ n_{31} &= 0, & n_{32} &= -1, & n_{41} &= 1, & n_{42} &= 0. \end{aligned}$$

Hence by (402)

$$\begin{aligned} r_{31} &= 0, & r_{32} &= 1, & r_{41} &= -1, & r_{42} &= 0, \\ (r_{31} - r_{42})^2 + 4r_{32}r_{41} &= -4 < 0. \end{aligned}$$

It follows that the function (481) is not nomographic.

V. Let us consider the function

$$(482) \quad F \equiv F(x, y, z) \equiv \frac{x}{y+z} + 1,$$

where the variables x, y, z take numerical values with the condition $y+z \neq 0$.

In order to determine whether the function (482) is nomographic we use scheme I. We see that for $x_1 = 0, y_1 = 0, z_1 = 1$ the condition (335) is satisfied. Moreover, by (336), (337) and (338)

$$\begin{aligned} F_{x_1} &\equiv F_{x_1}(x, y, z) \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & 1 \\ x+1 & \frac{x}{y+z} + 1 \end{array} \right| \equiv \frac{x}{y+z} - x \neq 0, \\ F_{y_1} &\equiv F_{y_1}(x, y, z) \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & \frac{x}{z} + 1 \\ 1 & \frac{x}{y+z} + 1 \end{array} \right| \equiv \frac{x}{y+z} - \frac{x}{z} \neq 0, \\ F_{z_1} &\equiv F_{z_1}(x, y, z) \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & \frac{x}{y+1} + 1 \\ 1 & \frac{x}{y+z} + 1 \end{array} \right| \equiv \frac{x}{y+z} - \frac{x}{y+1} \neq 0. \end{aligned}$$

We see that for

$$\begin{aligned} x_2 = 1, & \quad y_2 = 1, & \quad z_2 = 1, \\ x_3 = 1, & \quad y_3 = 1, & \quad z_3 = 1, \\ x_4 = 1, & \quad y_4 = 1, & \quad z_4 = 0, \end{aligned}$$

the conditions (339) are satisfied. Moreover, we have by (340)

$$F_{x_2} \equiv F_{x_2}(x, y, z) \equiv -2 \left| \begin{array}{ccc} -\frac{1}{2} & \frac{1}{y+z} & -1 \\ \frac{x}{2} & \frac{x}{y+z} & -x \end{array} \right| \equiv 0$$

and by (341)

$$\begin{aligned} X_1 \equiv X_1(x) &\equiv x+1, & X_2 \equiv X_2(x) &\equiv -\frac{x}{2}, \\ G_1 \equiv G_1(y, z) &\equiv 1, & G_2 \equiv G_2(y, z) &\equiv \frac{-2}{y+z} + 2. \end{aligned}$$

We next have by (344)

$$\begin{aligned} F_{y_2} \equiv F_{y_2}(x, y, z) &\equiv -2 \left| \begin{array}{ccc} -\frac{1}{2} & \frac{x}{z+1} & -\frac{x}{z} \\ \frac{1}{y+1} & -1 & \frac{x}{y+z} - \frac{x}{z} \end{array} \right| \\ &\equiv \frac{x}{y+z} - \frac{x}{z} - \frac{2xy}{(y+1)(z+1)} + \frac{2xy}{(y+1)z} \neq 0. \end{aligned}$$

We see that for $x_6 = 1, y_6 = 2, z_6 = 2$ the condition (345) is satisfied. Moreover, we have by (346)

$$\begin{aligned} &\left| \begin{array}{cccc} -\frac{1}{36} & & \frac{x}{z+2} - \frac{x}{z} - \frac{4x}{3(z+1)} + \frac{4x}{3z} & \\ \frac{1}{y+2} - \frac{1}{2} - \frac{2y}{3(y+1)} + \frac{2y}{2(y+1)} & \frac{x}{y+z} - \frac{x}{z} - \frac{2xy}{(y+1)(z+1)} + \frac{2xy}{(y+1)z} & & \end{array} \right| \\ &\equiv -\frac{x}{(y+2)(z+2)} + \frac{4x}{3(y+2)(z+1)} - \frac{x}{3(y+2)z} - \frac{xy}{3(y+1)(z+2)} + \\ &\quad + \frac{xy}{2(y+1)(z+1)} - \frac{xy}{6(y+1)z} - \frac{x}{36(y+z)} + \frac{x}{2(z+2)} - \frac{2x}{3(z+1)} + \\ &\quad + \frac{7x}{36z} \neq 0. \end{aligned}$$

Thus the function (482) is not nomographic.

Let us notice that by Example I the function

$$F^* \equiv x + y + z$$

is nomographic. We see that although the equations $F = 0$ and $F^* = 0$ with the condition $y + z \neq 0$ are equivalent, one of the functions F, F^* is nomographic, but not the other one. Let us suppose that the equation $F \equiv F(x, y, z) = 0$ is called nomographic if and only if the function F is nomographic. Then we might sometimes obtain by transformations of a non-nomographic equation a nomographic one and vice versa.

VI. Let us consider the function

$$(483) \quad F \equiv F(x, y, z) \equiv -2 - x^2 - y + z + x^2 y^2 + e^x z - e^x y^2 z - x^2 \sqrt{z} + e^x \sqrt{z}(y + 2),$$

where the variables x, y, z take numerical values with the condition $z \geq 0$. In order to determine whether the function (483) is nomographic we use the scheme.

Since

$$F(0, 0, 0) = -2 \neq 0,$$

the condition (335) is satisfied with $x_1 = 0, y_1 = 0, z_1 = 0$. Moreover, by (336), (337) and (338) we have

$$\begin{aligned}
 F_{x_1} &\equiv -\frac{1}{2} \begin{vmatrix} -2 & -2 - y + 2z - y^2 z + y\sqrt{z} + 2\sqrt{z} \\ -2 - x^2 & -2 - x^2 - y + z + x^2 y^2 + e^x z - e^x y^2 z - x^2 \sqrt{z} + e^x y\sqrt{z} + 2e^x \sqrt{z} \end{vmatrix} \\
 &\equiv -\frac{1}{2}(-x^2 y + 2z - 2x^2 y^2 - 2y^2 z + 2x^2 z - x^2 y^2 z + 4x^2 \sqrt{z} + 4\sqrt{z} + 2y\sqrt{z} + x^2 y\sqrt{z} - 2e^x z + 2e^x y^2 z - 4e^x \sqrt{z} - 2e^x y\sqrt{z}) \neq 0, \\
 F_{y_1} &\equiv -\frac{1}{2} \begin{vmatrix} -2 & -2 - x^2 + z + e^x z - x^2 \sqrt{z} + 2e^x \sqrt{z} \\ -2 - y & -2 - x^2 - y + z + x^2 y^2 + e^x z - e^x y^2 z - x^2 \sqrt{z} + e^x y\sqrt{z} + 2e^x \sqrt{z} \end{vmatrix} \\
 &\equiv -\frac{1}{2}(yz - x^2 y - 2x^2 y^2 - x^2 y\sqrt{z} + e^x yz + 2e^x y^2 z) \neq 0, \\
 F_{z_1} &\equiv -\frac{1}{2} \begin{vmatrix} -2 & -2 - x^2 - y + x^2 y^2 \\ -2 + 2z + 2\sqrt{z} & -2 - x^2 - y + z + x^2 y^2 + e^x z - e^x y^2 z - x^2 \sqrt{z} + e^x y\sqrt{z} + 2e^x \sqrt{z} \end{vmatrix} \\
 &\equiv -z - yz - x^2 z + x^2 y^2 z - 2\sqrt{z} - y\sqrt{z} - 2x^2 \sqrt{z} + x^2 y^2 \sqrt{z} + e^x z - e^x y^2 z + 2e^x \sqrt{z} + e^x y\sqrt{z} \neq 0.
 \end{aligned}$$

Since

$$F_{x_1}(1, 1, 1) = \frac{1}{2}(6e - 9) \neq 0,$$

$$F_{y_1}(1, 1, 1) = -\frac{3}{2}(e - 1) \neq 0, \quad F_{z_1}(1, 1, 1) = 3e - 6 \neq 0,$$

we have the conditions (339) with

$$x_2 = x_3 = x_4 = y_2 = y_3 = y_4 = z_2 = z_3 = z_4 = 1.$$

Now we have by (340)

$$F_{x_2} \equiv \frac{2}{6e - 9} \begin{vmatrix} \frac{1}{2}(6e - 9) & -\frac{1}{2}(-y + 4z - 2ez - 2y^2 - 3y^2z + 2ey^2z + \\ & + 8\sqrt{z} - 4e\sqrt{z} + 3y\sqrt{z} - 2ey\sqrt{z}) \\ -\frac{1}{2}(6 + 3x^2 - 6e^x) & -\frac{1}{2}(-x^2y + 2z - 2x^2y^2 - 2y^2z + 2x^2z - \\ & - x^2y^2z + 4x^2\sqrt{z} + 4\sqrt{z} + 2y\sqrt{z} + x^2y\sqrt{z} - \\ & - 2e^x z + 2e^x y^2 z - 4e^x \sqrt{z} - 2e^x y \sqrt{z}) \end{vmatrix} \\ \equiv \frac{1}{2e - 3} (y - z - x^2y + ex^2y + 2y^2 + x^2z - ex^2z - 2x^2y^2 + 2ex^2y^2 - 2\sqrt{z} + \\ + 2x^2\sqrt{z} - 2ex^2\sqrt{z} - e^x y + e^x z - 2e^x y^2 + 2e^x \sqrt{z}) \neq 0.$$

Since

$$F_{x_2}(-1, 1, 0) = \frac{3(e^2 - 1)}{e(2e - 3)} \neq 0,$$

we have the condition (342) with $x_5 = -1$, $y_5 = 1$, $z_5 = 0$. Then we have by (343)

$$\begin{vmatrix} \frac{3(e^2 - 1)}{e(2e - 3)} & \frac{1}{2e - 3} \left(ey - ez + 2ey^2 - 2e\sqrt{z} - \frac{1}{e}y + \frac{1}{e}z - \right. \\ & \left. - \frac{2}{e}y^2 + \frac{2}{e}\sqrt{z} \right) \\ \frac{3 - 3x^2 + 3ex^2 - 3e^x}{2e - 3} & \frac{1}{2e - 3} (y - z - x^2y + ex^2y + 2y^2 + x^2z - \\ & - ex^2z - 2x^2y^2 + 2ex^2y^2 - 2\sqrt{z} + 2x^2\sqrt{z} - \\ & - 2ex^2\sqrt{z} - e^x y + e^x z - 2e^x y^2 + 2e^x \sqrt{z}) \end{vmatrix} \equiv 0.$$

Thus we have the case X3.

Furthermore, we have by (344)

$$F_{y_2} \equiv \frac{-1}{6e - 6} \begin{vmatrix} 3e - 3 & z - 3x^2 - x^2\sqrt{z} + 3e^x z \\ -y + ey - 2y^2 + 2ey^2 & yz - x^2y - 2x^2y^2 - x^2y\sqrt{z} + e^x yz + 2e^x y^2 z \end{vmatrix} \\ \equiv -\frac{1}{3} (yz - y^2z - x^2y\sqrt{z} + x^2y^2\sqrt{z}) \neq 0.$$

Since

$$F_{y_2}(-1, -1, 4) = \frac{4}{3} \neq 0,$$

we have the condition (345) with $x_6 = y_6 = -1$, $z_6 = 4$. Then we have by (346)

$$\begin{vmatrix} \frac{4}{3} & -\frac{1}{3}(-2z + 2x^2\sqrt{z}) \\ -\frac{1}{3}(2y - 2y^2) & -\frac{1}{3}(yz - y^2z - x^2y\sqrt{z} + x^2y^2\sqrt{z}) \end{vmatrix} \equiv 0.$$

Thus we have the case Y3.

Moreover, we have by (347)

$$\begin{aligned} F_{z_2} &\equiv \frac{1}{3e-6} \begin{vmatrix} 3e-6 & -3-2y-3x^2+2x^2y^2+3e^x+e^xy-e^xy^2 \\ -2z-4\sqrt{z}+3e\sqrt{z} & -z-yz-x^2z+x^2y^2z-2\sqrt{z}-y\sqrt{z}- \\ & -2x^2\sqrt{z}+x^2y^2\sqrt{z}+e^xz-e^xy^2z+ \\ & +2e^x\sqrt{z}+e^xy\sqrt{z} \end{vmatrix} \\ &\equiv \frac{1}{3e-6} (-3ez+2yz-3eyz-3ex^2z-2x^2y^2z+3ex^2y^2z+3e\sqrt{z}-2y\sqrt{z}+ \\ & +3ey\sqrt{z}+3ex^2\sqrt{z}+2x^2y^2\sqrt{z}-3ex^2y^2\sqrt{z}+3e^{x+1}z+4e^xy^2z+ \\ & +2e^xyz-3e^{x+1}y^2z-3e^{x+1}\sqrt{z}-2e^xy\sqrt{z}-4e^xy^2\sqrt{z}+ \\ & +3e^{x+1}y^2\sqrt{z}) \neq 0. \end{aligned}$$

Since

$$F_{z_2}(-1, -1, 4) = \frac{1}{3e-6} \left(\frac{4}{e} - 8 \right) \neq 0,$$

we have the condition (348) with $x_7 = y_7 = -1$, $z_7 = 4$. Then we have by (349)

$$\begin{vmatrix} \frac{1}{3e-6} \left(\frac{4}{e} - 8 \right) & \frac{1}{3e-6} (-6e+4y-6ey-6ex^2-4x^2y^2+ \\ & +6ex^2y^2+6e^{x+1}+8e^xy^2+4e^xy- \\ & -6e^{x+1}y^2) \\ \frac{1}{3e-6} \left(-4z+\frac{2}{e}z+ \right. & \frac{1}{3e-6} (-3ez+2yz-3eyz-3ex^2z-2x^2y^2z+ \\ & +3ex^2y^2z+3e\sqrt{z}-2y\sqrt{z}+3ey\sqrt{z}+ \\ & +3ex^2\sqrt{z}+2x^2y^2\sqrt{z}-3ex^2y^2\sqrt{z}+ \\ & +3e^{x+1}z+4e^xy^2z+2e^xyz-3e^{x+1}y^2z- \\ & -3e^{x+1}\sqrt{z}-2e^xy\sqrt{z}-4e^xy^2\sqrt{z}+ \\ & \left. +3e^{x+1}y^2\sqrt{z}) \right) \end{vmatrix} \equiv 0.$$

Thus we have the cases X3, Y3, Z3 and by (356)

$$X_1 \equiv -2 - x^2, \quad X_2 \equiv -\frac{1}{2}(6 + 3x^2 - 6e^x), \quad X_3 \equiv \frac{3 - 3x^2 + 3ex^2 - 3e^x}{2e - 3},$$

$$G_1 \equiv -\frac{1}{2}(-2 - y + 2z - y^2z + y\sqrt{z} + 2\sqrt{z}),$$

$$G_2 \equiv \frac{-1}{6e - 9}(-y + 4z - 2ez - 2y^2 - 3y^2z + 2ey^2z + 8\sqrt{z} - 4e\sqrt{z} + 3y\sqrt{z} - 2ey\sqrt{z}),$$

$$G_3 \equiv \frac{e}{3(e^2 - 1)} \left(ey - ez + 2ey^2 - 2e\sqrt{z} - \frac{1}{e}y + \frac{1}{e}z - \frac{2}{e}y^2 + \frac{2}{e}\sqrt{z} \right) \\ \equiv \frac{1}{3}(y - z + 2y^2 - 2\sqrt{z}).$$

Now we pass to scheme II. Since

$$G_1(0, 0) = 1 \neq 0,$$

we have the condition (363) with $y_0 = z_0 = 0$. Then we have by (364)

$$G_{11} \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & 1 - z - \sqrt{z} \\ 1 + \frac{1}{2}y & 1 + \frac{1}{2}y - z + \frac{1}{2}y^2z - \frac{1}{2}y\sqrt{z} - \sqrt{z} \end{array} \right| \\ \equiv \frac{1}{2}y^2z + \frac{1}{2}yz \neq 0.$$

Since

$$G_{11}(1, 1) = 1 \neq 0,$$

we have the condition (366) with $y_0 = z_0 = 1$. Then we have by (367)

$$\left| \begin{array}{cc} 1 & z \\ \frac{1}{2}y^2 + \frac{1}{2}y & \frac{1}{2}y^2z + \frac{1}{2}yz \end{array} \right| \equiv 0,$$

and we introduce by (368) the functions

$$Y_1 \equiv 1 + \frac{1}{2}y, \quad Y_2 \equiv \frac{1}{2}y^2 + \frac{1}{2}y,$$

$$Z_1 \equiv 1 - z - \sqrt{z}, \quad Z_2 \equiv z.$$

We have the case G_12 .

Since

$$G_2(-1, 0) = \frac{1}{6e - 9} \neq 0,$$

we have the condition (369) with $y_{10} = -1$, $z_{10} = 0$. We then have by (370)

$$G_{21} \equiv (6e-9) \left[\begin{array}{cc} \frac{1}{6e-9} & -\frac{1}{6e-9}(-1+z+5\sqrt{z}-2e\sqrt{z}) \\ \frac{1}{6e-9}(y+2y^2) & -\frac{1}{6e-9}(-y+4z-2ez-2y^2-3y^2z+ \\ & +2ey^2z+8\sqrt{z}-4e\sqrt{z}+3y\sqrt{z}-2ey\sqrt{z}) \end{array} \right] \\ \equiv \frac{1}{6e-9}(-4z+2ez-8\sqrt{z}+4e\sqrt{z}+yz+5y^2z-2ey^2z+2y\sqrt{z}+ \\ +10y^2\sqrt{z}-4ey^2\sqrt{z}) \neq 0.$$

Since

$$G_{21}(0, 1) = \frac{6e-12}{6e-9} \neq 0,$$

we have the condition (372) with $y_{11} = 0$, $z_{11} = 1$. Then we have by (373)

$$\left[\begin{array}{cc} \frac{6e-12}{6e-9} & \frac{1}{6e-9}(-4z+2ez-8\sqrt{z}+4e\sqrt{z}) \\ \frac{1}{6e-9}(6e-12+ \\ +3y+15y^2-6ey^2) & \frac{1}{6e-9}(-4z+2ez-8\sqrt{z}+4e\sqrt{z}+yz+ \\ +5y^2z-2ey^2z+2y\sqrt{z}+10y^2\sqrt{z}- \\ -4ey^2\sqrt{z}) \end{array} \right] \equiv 0,$$

and we introduce by (374) the functions

$$Y_3 \equiv \frac{1}{6e-9}(y+2y^2), \quad Y_4 \equiv \frac{1}{6e-9}(6e-12+3y+15y^2-6ey^2),$$

$$Z_3 \equiv 1-z-5\sqrt{z}+2e\sqrt{z}, \quad Z_4 \equiv \frac{1}{6e-12}(-4z+2ez-8\sqrt{z}+4e\sqrt{z}) \\ \equiv \frac{1}{3}(z+2\sqrt{z}).$$

We have the case $G_2 2$.

Since

$$G_3(-1, 0) = \frac{1}{3} \neq 0,$$

we have the condition (375) with $y_{12} = -1$, $z_{12} = 0$. Then we have by (376)

$$G_{31} \equiv 3 \left[\begin{array}{cc} \frac{1}{3} & \frac{1}{3}(1-z-2\sqrt{z}) \\ \frac{1}{3}(y+2y^2) & \frac{1}{3}(y-z+2y^2-2\sqrt{z}) \end{array} \right] \\ \equiv \frac{1}{3}(-z-2\sqrt{z}+yz+2y^2z+2y\sqrt{z}+4y^2\sqrt{z}) \neq 0.$$

Since

$$G_{31}(0, 1) = -1 \neq 0,$$

we have the condition (378) with $y_{13} = 0$, $z_{13} = 1$. Then we have by (379)

$$\begin{vmatrix} -1 & \frac{1}{3}(-z-2\sqrt{z}) \\ -1+y+2y^2 & \frac{1}{3}(-z-2\sqrt{z}+yz+2y^2z+2y\sqrt{z}+4y^2\sqrt{z}) \end{vmatrix} \equiv 0$$

and we introduce by (380) the functions

$$\begin{aligned} Y_5 &\equiv \frac{1}{3}(y+2y^2), & Y_6 &\equiv -1+y+2y^2, \\ Z_5 &\equiv 1-z-2\sqrt{z}, & Z_6 &\equiv \frac{1}{3}(z+2\sqrt{z}). \end{aligned}$$

Thus we have the cases G_12 , G_22 , G_32 from scheme II and we pass to scheme IX.

Since

$$\begin{vmatrix} Y_1(0) & Y_2(0) \\ Y_1(1) & Y_2(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{vmatrix} = 1 \neq 0,$$

we have the condition (449) with $y_{14} = 0$, $y_{15} = 1$. Now we may write the equations (450) for $i = 3$:

$$p_{31} = 0, \quad \frac{3}{2}p_{31} + p_{32} = \frac{1}{2e-3},$$

whence

$$p_{31} = 0, \quad p_{32} = \frac{1}{2e-3}.$$

We verify that

$$Y_3 \neq p_{31}Y_1 + p_{32}Y_2 \equiv \frac{1}{2e-3}Y_2.$$

Thus the functions Y_1 , Y_2 , Y_3 are linearly independent and we may pass directly to the condition (455). We see that for $y_{16} = -1$ this condition may be written in the form

$$\begin{vmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & \frac{1}{2e-3} \\ \frac{1}{2} & 0 & \frac{1}{6e-9} \end{vmatrix} = \frac{1}{6e-9} \neq 0$$

and the equations (456) in the form

$$\begin{aligned} m_{i1} &= Y_i(0) \\ \frac{3}{2} m_{i1} + m_{i2} + \frac{1}{2e-3} m_{i3} &= Y_i(1) \quad (i = 4, 5, 6). \\ \frac{1}{2} m_{i1} + \frac{1}{6e-9} m_{i3} &= Y_i(-1) \end{aligned}$$

Hence we have

$$\begin{aligned} m_{i1} &= Y_i(0), \quad m_{i2} = Y_i(1) - 3Y_i(-1), \\ m_{i3} &= (6e-9)[Y_i(-1) - \frac{1}{2}Y_i(0)] \end{aligned} \quad (i = 4, 5, 6)$$

and we find that

$$\begin{aligned} m_{41} &= \frac{6e-12}{6e-9} = \frac{2e-4}{2e-3}, & m_{42} &= \frac{2}{2e-3}, & m_{43} &= -3e+6, \\ m_{51} &= 0, & m_{52} &= 0, & m_{53} &= 2e-3, \\ m_{61} &= -1, & m_{62} &= 2, & m_{63} &= \frac{6e-9}{2}. \end{aligned}$$

We verify that the identities (457) hold. The conditions (459) and (462) are satisfied.

Since

$$\begin{vmatrix} Z_1(0) & Z_2(0) \\ Z_1(1) & Z_2(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \neq 0,$$

we have the condition (463) with $z_{14} = 0$, $z_{15} = 1$. Then we have by (464)

$$Z^*(z) \equiv \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1-z-\sqrt{z} & z & \frac{1}{3}(z+2\sqrt{z}) \end{vmatrix} \equiv \frac{2}{3}\sqrt{z} - \frac{2}{3}z \neq 0$$

and

$$Z^*(4) = -\frac{4}{3} \neq 0.$$

Thus we have the condition (465) with $z_{16} = 4$. Now we write the equations (466):

$$\begin{aligned} n_{i1} &= Z_i(0) \\ -n_{i1} + n_{i2} + n_{i4} &= Z_i(1) \quad (i = 3, 5, 6), \\ -5n_{i1} + 4n_{i2} + \frac{8}{3}n_{i4} &= Z_i(4) \end{aligned}$$

whence

$$n_{i1} = Z_i(0), \quad n_{i2} = \frac{7}{4}n_{i1} + \frac{3}{4}Z_i(4) - 2Z_i(1), \quad n_{i4} = n_{i1} - n_{i2} + Z_i(1)$$

($i = 3, 5, 6$).

Hence we compute

$$\begin{aligned} n_{31} &= 1, & n_{32} &= -e + 2, & n_{34} &= 3e - 6, \\ n_{51} &= 1, & n_{52} &= \frac{1}{2}, & n_{54} &= -\frac{3}{2}, \\ n_{61} &= 0, & n_{62} &= 0, & n_{64} &= 1, \end{aligned}$$

and we verify that the identities (467) and the conditions (468) are satisfied. It follows that the function (483) is uniquely nomographic and each of its Massau forms is by (469) equivalent to the form

$$(484) \quad F \equiv \begin{vmatrix} -2 - x^2 & \frac{3}{2e-4}(-2e + 3 + 3x^2 - 2ex^2 - 3e^x + 2e^x + 1) & \frac{6(2e-3)}{(2e-4)^2}(1 - x + ex^2 - e^x) \\ -\frac{e-2}{6e-9}(y + 2y^2) & -\frac{1}{2}y^2 - \frac{1}{2}y & 1 + \frac{y}{2} + \frac{1}{2e-4}(y + y^2) \\ \frac{2e-4}{6e-9}(z + 2\sqrt{z}) & -1 + z + \sqrt{z} & \frac{1}{e-2}(1 - z - \sqrt{z}) - z \end{vmatrix}.$$

The form (484) may be replaced by a simpler one. For example, we add the third column of the determinant (484) multiplied by $e - 2$ to the second one. Thus we obtain

$$(485) \quad F \equiv \begin{vmatrix} -2 - x^2 & \frac{3}{2}(2e-3)x^2 & \frac{6(2e-3)}{(2e-4)^2}(1 - x^2 + ex^2 - e^x) \\ -\frac{e-2}{6e-9}(y + 2y^2) & (e-2)\left(1 + \frac{y}{2}\right) & 1 + \frac{y}{2} + \frac{1}{2e-4}(y + y^2) \\ \frac{2e-4}{6e-9}(z + 2\sqrt{z}) & -(e-2)z & \frac{1}{e-2}(1 - z - \sqrt{z}) - z \end{vmatrix}.$$

By Lemma 4 the forms (484) and (485) are equivalent.

Now we multiply the first row of (485) by $(2e - 4)/(6e - 9)$ and the first column by $(6e - 9)/(2e - 4)$. Thus we obtain

$$(486) \quad F \equiv \begin{vmatrix} -2 - x^2 & (e-2)x^2 & \frac{1}{e-2}(1 - x^2 + ex^2 - e^x) \\ -\frac{1}{2}y - y^2 & (e-2)\left(1 + \frac{y}{2}\right) & 1 + \frac{y}{2} + \frac{1}{2e-4}(y + y^2) \\ z + 2\sqrt{z} & -(e-2)z & \frac{1}{e-2}(1 - z - \sqrt{z}) - z \end{vmatrix}.$$

By Lemma 5 the forms (485) and (486), and hence also the forms (484) and (486), are equivalent.

We next multiply the second column in (486) by $1/(e-2)$ and the third one by $e-2$. We obtain

$$(487) \quad F \equiv \begin{vmatrix} -2-x^2 & x^2 & 1-x^2+ex^2-e^x \\ -\frac{1}{2}y-y^2 & 1+\frac{1}{2}y & e-2+\frac{1}{2}ey-\frac{1}{2}y+\frac{y^2}{2} \\ z+2\sqrt{z} & -z & 1+z-ez-\sqrt{z} \end{vmatrix}.$$

By Lemma 6 the forms (486) and (487), and hence the forms (484) and (487), are equivalent.

Now we add the second column of (487) to the first one, and then to the third one, and we obtain

$$(488) \quad F \equiv \begin{vmatrix} -2 & x^2 & 1+ex^2-e^x \\ 1-y^2 & 1+\frac{1}{2}y & e-1+\frac{1}{2}ey+\frac{1}{2}y^2 \\ 2\sqrt{z} & -z & 1-ez-\sqrt{z} \end{vmatrix}.$$

By Lemma 4 the forms (487) and (488), and hence the forms (484) and (488), are equivalent.

We next add the first column of (488) multiplied by $\frac{1}{2}$ and the second column multiplied by $-e$ to the third one and we obtain

$$(489) \quad F \equiv \begin{vmatrix} -2 & x^2 & -e^x \\ 1-y^2 & 1+\frac{1}{2}y & -\frac{1}{2} \\ 2\sqrt{z} & -z & 1 \end{vmatrix}.$$

By Lemma 4 the forms (488) and (489), and hence the forms (484) and (489), are equivalent.

Finally we multiply the second row of (489) by 2 and the first column by $\frac{1}{2}$, and then the third row by -1 and the third column by -1 . Thus we obtain

$$(490) \quad F \equiv \begin{vmatrix} -1 & x^2 & e^x \\ 1-y^2 & 2+y & 1 \\ -\sqrt{z} & z & 1 \end{vmatrix}.$$

By Lemma 5 the forms (489) and (490), and hence the forms (484) and (490), are equivalent. It follows that each of the Massau forms of the function (483) is equivalent to the form (490).

Remark. We might obtain the form (490) in a considerably simpler way by grouping the terms in the identity (483), as in the previous example. Here we have used the general method described in the previous chapter, for we wanted to illustrate it. It is however, a well-known fact that general methods are not the simplest ones.

VII. We shall consider the function

$$(491) \quad T(t_1, t_2, t_3, t_4, t_5) \equiv t_3 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + \\ + t_2 t_4 t_5 + t_1 t_2 t_3 t_5$$

as a function of the following three variables:

$$(492) \quad x \equiv (t_1, t_2), \quad y \equiv (t_3, t_4), \quad z \equiv t_5,$$

where the variables t_1, t_2, t_3, t_4, t_5 take numerical values.

We set

$$F \equiv F(x, y, z) \equiv T(t_1, t_2, t_3, t_4, t_5).$$

Since

$$T(0, 0, 1, 0, 0) = 1 \neq 0,$$

we have the condition (335), where

$$x_1 = (0, 0), \quad y_1 = (1, 0), \quad z_1 = 0,$$

and by (336), (337) and (338)

$$F_{x_1} \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & t_3 - t_4 \\ 1 & t_3 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{array} \right| \\ \equiv -t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \neq 0,$$

$$F_{y_1} \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & 1 - t_1 t_5 - t_2 t_5 + t_1 t_2 t_5 \\ t_3 - t_4 & t_3 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{array} \right| \\ \equiv -t_1 t_2 t_4 + t_1 t_2 t_4 t_5 \neq 0,$$

$$F_{z_1} \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & t_3 - t_4 - t_1 t_2 t_4 \\ 1 & t_3 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{array} \right| \\ \equiv -t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \neq 0.$$

We see that for

$$x_2 = x_3 = (1, 1), \quad y_2 = y_3 = (0, 1), \quad z_2 = z_3 = 0, \\ x_4 = (0, 1), \quad y_4 = (0, 1), \quad z_4 = 1,$$

the conditions (339) are satisfied. Then we have by (340)

$$F_{x_2} \equiv \frac{1}{-1} \left| \begin{array}{cc} -1 & -t_4 - t_3 t_5 + 2t_4 t_5 \\ -t_1 t_2 & -t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{array} \right| \\ \equiv -t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + 2t_1 t_2 t_3 t_5 - 2t_1 t_2 t_4 t_5 \neq 0.$$

Since for

$$x_5 = (0, 1), \quad y_5 = (0, 1), \quad z_5 = 1$$

the condition (342) is satisfied, we obtain by (343)

$$\left| \begin{array}{cc} 1 & -t_3 t_5 + t_4 t_5 \\ t_1 + t_2 - 2t_1 t_2 & -t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + 2t_1 t_2 t_3 t_5 - 2t_1 t_2 t_4 t_5 \end{array} \right| \equiv 0.$$

Thus we have the case X3.

Now we have by (344)

$$F_{y_2} \equiv \frac{1}{-1} \left| \begin{array}{cc} -1 & -t_1 t_2 + t_1 t_2 t_5 \\ -t_4 & -t_1 t_2 t_4 + t_1 t_2 t_4 t_5 \end{array} \right| \equiv 0$$

and by (347)

$$F_{z_2} \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & -t_1 t_3 + t_1 t_4 - t_2 t_3 + t_2 t_4 + t_1 t_2 t_3 \\ t_5 & -t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{array} \right| \equiv 0.$$

Thus we have the cases X3, Y2, Z2 and, according to (352), we replace the identities (492) by

$$x \equiv t_5, \quad y \equiv (t_3, t_4), \quad z \equiv (t_1, t_2),$$

and we introduce by (353) the functions

$$X_1 \equiv X_1(x) \equiv 1, \quad X_2 \equiv X_2(x) \equiv t_5,$$

$$G_1 \equiv G_1(y, z) \equiv t_3 - t_4 - t_1 t_2 t_4,$$

$$G_2 \equiv G_2(y, z) \equiv -t_1 t_3 + t_1 t_4 - t_2 t_3 + t_2 t_4 + t_1 t_2 t_3.$$

Now we pass to scheme II. We see that for

$$y_8 = (1, 0), \quad z_8 = (0, 0)$$

the condition (363) is satisfied. We have by (364)

$$G_{11} \equiv \frac{1}{1} \left| \begin{array}{cc} 1 & 1 \\ t_3 - t_4 & t_3 - t_4 - t_1 t_2 t_4 \end{array} \right| \equiv -t_1 t_2 t_4 \neq 0$$

and for

$$y_9 = (0, 1), \quad z_9 = (1, 1),$$

the condition (366) is satisfied. Then we have by (367)

$$\begin{vmatrix} -1 & -t_1 t_2 \\ -t_4 & -t_1 t_2 t_4 \end{vmatrix} \equiv 0.$$

Thus we have the case $G_1 2$ and we introduce by (368) the functions

$$Y_1 \equiv t_3 - t_4, \quad Y_2 \equiv -t_4, \quad Z_1 \equiv 1, \quad Z_2 \equiv t_1 t_2.$$

Now we see that for

$$y_{10} = (0, 1), \quad z_{10} = (1, 0),$$

the condition (369) is satisfied and we have by (370)

$$G_{21} \equiv \frac{1}{1} \begin{vmatrix} 1 & t_1 + t_2 \\ -t_3 + t_4 & -t_1 t_3 + t_1 t_4 - t_2 t_3 + t_2 t_4 + t_1 t_2 t_3 \end{vmatrix} \equiv t_1 t_2 t_3 \neq 0.$$

The condition (372) is satisfied for

$$y_{11} = (1, 0), \quad z_{11} = (1, 1).$$

Then we have by (373)

$$\begin{vmatrix} 1 & t_1 t_2 \\ t_3 & t_1 t_2 t_3 \end{vmatrix} \equiv 0$$

and we introduce by (374) the functions

$$Y_3 \equiv -t_3 + t_4, \quad Y_4 \equiv t_3, \quad Z_3 \equiv t_1 + t_2, \quad Z_4 \equiv t_1 t_2.$$

Thus we have the cases $X2, Y2, Z3, G_1 2, G_2 2$, and therefore we pass to scheme VII.

We see that the conditions (417) are satisfied for

$$y_{14} = (1, 0), \quad y_{15} = (-1, -1), \quad z_{14} = (0, 0), \quad z_{15} = (1, 1).$$

Therefore the equations (418) may be written in the form

$$m_{31} = -1, \quad m_{32} = 0, \quad m_{41} = 1, \quad m_{42} = -1,$$

and the equations (419) in the form

$$q_{31} = 0, \quad q_{32} = 2, \quad q_{41} = 0, \quad q_{42} = 1.$$

We verify that the identities (420) hold, but the first identity (421) fails. Since the second identity (421) holds, we interchange the indices according to (422) and obtain

$$Y_3 \equiv t_3, \quad Y_4 \equiv -t_3 + t_4, \quad Z_3 \equiv t_1 t_2, \quad Z_4 \equiv t_1 + t_2, \\ m_{31} = 1, \quad m_{32} = -1, \quad m_{41} = -1, \quad m_{42} = 0.$$

We see that the identity (425) may be written immediately. We have, namely,

$$Z_3 \equiv 0 \cdot Z_1 + 1 \cdot Z_2 + 0 \cdot Z_4.$$

Hence

$$n_{31} = n_{34} = 0, \quad n_{32} = 1,$$

and the conditions (427) and (428) are satisfied.

It follows that the function (491) as a function of the variables (492) is uniquely nomographic and each of its Massau forms is — by (431) — equivalent to the form

$$T(t_1, t_2, t_3, t_4, t_5) \equiv \begin{vmatrix} t_1 + t_2 & 1 & t_1 t_2 \\ t_3 & t_4 & t_3 - t_4 \\ 1 & t_5 & 0 \end{vmatrix}.$$

VIII. We shall show that the same function (491) as a function of the following three variables:

$$(493) \quad x \equiv t_1, \quad y \equiv (t_2, t_3), \quad z \equiv (t_4, t_5),$$

is not nomographic.

We see that for

$$x_1 = 0, \quad y_1 = (0, 1), \quad z_1 = (0, 0),$$

the condition (335) is satisfied and the identity (338) may be written in the form

$$F_{z1} \equiv \frac{1}{1} \begin{vmatrix} 1 & t_3 \\ 1 - t_4 & t_3 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{vmatrix} \\ \equiv t_3 t_4 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \neq 0.$$

Furthermore, we see that for

$$x_4 = 0, \quad y_4 = (0, 0), \quad z_4 = (1, 0),$$

the last of the conditions (339) is satisfied and the identity (347) may be written in the form

$$F_{z2} \equiv \frac{1}{-1} \begin{vmatrix} -1 & t_3 - 1 - t_1 t_2 \\ -t_4 & t_3 t_4 - t_4 - t_1 t_2 t_4 - t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{vmatrix} \\ \equiv -t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \neq 0.$$

Finally we see that for

$$x_7 = 0, \quad y_7 = (1, 0), \quad z_7 = (1, 1),$$

the condition (348) is satisfied and we have by (349)

$$\begin{vmatrix} 1 & -t_1 t_3 + t_1 - t_2 t_3 + t_2 + t_1 t_2 t_3 \\ t_4 t_5 & -t_1 t_3 t_5 + t_1 t_4 t_5 - t_2 t_3 t_5 + t_2 t_4 t_5 + t_1 t_2 t_3 t_5 \end{vmatrix}$$

$$= -t_1 t_3 t_5 - t_2 t_3 t_5 + t_1 t_2 t_3 t_5 + t_1 t_3 t_4 t_5 + t_2 t_3 t_4 t_5 - t_1 t_2 t_3 t_4 t_5 \neq 0.$$

Since the condition (349) is not satisfied, the function (491) as a function of the three variables (493) is not nomographic.

We see from the examples VII and VIII that the same function (491) is nomographic when regarded as a function of the three variables (492), and is not nomographic when regarded as a function of the three variables (493).

VII. SUPPLEMENTARY REMARKS

- I. A function $F \equiv F(x, y, z)$ is called *nomographic* if and only if
- (494) 1° it has at least one Massau form,
 2° it is of rank greater than 1 with respect to each of the variables x, y and z .

We may ask when a function which does not satisfy the second of these conditions has a Massau form and, moreover, how many non-equivalent forms it may then have.

A function that satisfies the first condition (494) but not the second one is to be called *quasi-nomographic*. If all its Massau forms are equivalent in pairs, it will be called *uniquely quasi-nomographic*. If it has exactly two non-equivalent Massau forms, it will be called *doubly quasi-nomographic*. If it has exactly $k > 2$ non-equivalent Massau forms, it will be called *k-quasi-nomographic*.

The following two theorems may be proved⁽¹⁾:

THEOREM A. *A function $F \equiv F(x, y, z)$ with numerical values and $x \in \Omega_x, y \in \Omega_y, z \in \Omega_z$ is quasi-nomographic if and only if it identically equals zero or is of rank 1 with respect to one of the variables x, y, z and of rank 2 or 1 with respect to the remaining ones.*

THEOREM B. *If a function F is quasi-nomographic, then:*

1° *it is 15-quasi-nomographic if $F \equiv 0$ and each of the sets $\Omega_x, \Omega_y, \Omega_z$ consists of a single element,*

2° *it is 4-quasi-nomographic if the function F is of rank 1 with respect to each of the variables x, y, z and one of the sets $\Omega_x, \Omega_y, \Omega_z$ consists of exactly two elements, while each of the remaining two consists of a single element,*

3° *it is uniquely quasi-nomographic if the function F is of rank 1 with respect to each of the variables x, y, z and each of the sets $\Omega_x, \Omega_y, \Omega_z$ consists of a single element,*

4° *it is ∞ -quasi-nomographic in each of the remaining cases.*

⁽¹⁾ We omit their proofs, because quasi-nomographic functions are of no use in nomography.

II. Owing to the three preliminary assumptions we have introduced only seven principal cases. If we got rid of some of those assumptions, then the number of principal cases would increase.

The second and the third preliminary assumption are not troublesome, because a suitable rearrangement of indices is easily performable. The first preliminary assumption is the most troublesome, because we must compute the rank of the given function with respect to each of the variables in order to know whether this assumption is satisfied or not.

We may get rid of the first preliminary assumption by the introduction of six new principal cases symmetrical in x, y, z with respect to the fourth, the fifth, and the sixth principal cases. There would then be 13 preliminary cases. The author has chosen the scheme with the first preliminary assumption and with only 7 principal cases, because it seems to be simpler.

III. In nomography some forms of equations are well known and called *canonical*. Our theory enables us to construct all possible canonical forms for nomographic functions. For example, the first principal case yields

$$F \equiv X_1 Y_1 Z_1 + X_2 Y_3 Z_3,$$

where the functions X_1, X_2 , the functions Y_1, Y_3 and the functions Z_1, Z_3 — separately treated — are linearly independent. If

$$Z_1 \equiv 1, \quad X_2 \equiv 1, \quad Y_3 \equiv -1,$$

we obtain

$$F \equiv X_1 Y_1 - Z_3,$$

where

$$X_1 \neq \text{const}, \quad Y_1 \neq \text{const}, \quad Z_3 \neq \text{const}.$$

Similarly, the fourth principal case yields

$$F \equiv X_1(Y_1 Z_1 + Y_2 Z_2) + X_2(m_{31} Y_1 + m_{32} Y_2) Z_3,$$

where the functions X_1, X_2 , the functions Y_1, Y_2 and the functions Z_1, Z_2, Z_3 — separately treated — are linearly independent and $m_{32} \neq 0$. Setting

$$m_{32} X_2 \equiv X, \quad \frac{m_{31}}{m_{32}} = k,$$

we obtain

$$F \equiv X_1(Y_1 Z_1 + Y_2 Z_2) + X(k Y_1 + Y_2) Z_3.$$

If

$$X_1 \equiv 1, \quad Y_1 \equiv 1, \quad Z_3 \equiv 1,$$

we have the form

$$F \equiv kX + Z_1 + Y_2(X + Z_2),$$

where

$$X \neq \text{const}, \quad Y_2 \neq \text{const}, \quad Z_1 \neq \text{const}, \quad Z_2 \neq \text{const},$$

and k is an arbitrary numerical coefficient.

In a similar way we may construct many other canonical forms for nomographic functions. But canonical forms are useful only in the simplest cases. In the more difficult cases they are of no use, because it is too difficult to perceive that a given function may be written in a given canonical form. Supplementary difficulties arise from the fact that the number of all canonical forms may be very great.

The general method presented in this paper is always applicable.

IV. There are several methods of constructing Massau forms for a given function in nomography. They require, however, much skill and experience on the part of the computer and, therefore, they are useless in many cases. Moreover, they do not permit us to determine whether a given function is uniquely or doubly nomographic. Nevertheless, such methods may often require less computation than the general method presented in this paper. On the other hand, a trained computer will simplify the general method in particular cases.

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