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An algebraic and Kripke-style approach to a certain extension of intuitionistic logic

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Introduction

As is well known, Gödel first observed the existence of infinitely many logics between classical logic and intuitionistic logic (IL). Umezawa was the first to study these logics systematically and he called them intermediate logics. One of the examples of the intermediate logics is a logic which is obtained by adding formula DIS as a new axiom to the schema of axioms of IL. This formula is of the form

$$\forall x(\varphi(x) \cup \psi) \Rightarrow (\forall x \varphi(x) \cup \psi),$$

where x does not appear in ψ. The new logic is called DI-*logic*.

It turns out that DIL is a fragment of a certain extension of IL which is not an intermediate logic. More precisely, let L be a first order language of IL. We extend this language by adding two new propositional connectives to L, namely the so called Brouwerian negation, denoted by $\ulcorner$, and Brouwerian implication, denoted by $\dot{-}$. We call this logic Heyting–Brouwer logic (briefly H–B logic). It is obvious that H–B logic is not an intermediate one. DIL is a fragment of H–B logic in the sense that if a formula a does not contain $\dot{-}$ and $\ulcorner$ then a is a predicate tautology of H–B logic iff a is a predicate tautology of DIL.

However, the study of DIL has not been the immediate reason for the introduction of H–B logic. The propositional calculus of H–B logic, and the connection between this logic and semi-Boolean algebras (which play an analogous role for H–B logic to that played by Boolean algebras for classical logic), have been investigated in [16]. From those investigations it appeared that an intuitionistic logic with two negations and two implications, dual to itself, would have a more elegant algebraic and model-theoretic theory than an ordinary intuitionistic logic.

The purpose of this paper is to develop that theory. Two model-theoretic approaches are particularly important in the semantic study of non-classical calculi: algebraic and relational models. For first order calculi, algebraic modelling was introduced by Mostowski [14] and further developed by Henkin, Rasiowa and others. Relational models were first mentioned by Beth [1], then extended by Dyson and Kreisler [3] and further developed by Kripke [12] and independently by Beth and De Jongh [11] and Grzegorczyk [9], [10].

The purpose of the present paper is to use algebraic and relational models to investigate H–B logic.

The paper consists of three chapters.

Chapter I is devoted to the semi-Boolean algebras with infinite joins and meets. These algebras can be used for an algebraic interpretation of H–B logic. We also consider algebras called *D-pseudo-Boolean algebras* (DPBA), which were examined in another way by Görnemann [8]. The main results of this chapter are the constructions of Q-filters and Q-ideals and the special kind of the representation theorem for these algebras.

This theorem states:

For every semi-Boolean algebra (or for DPBA) $\mathfrak{A}$ *there exist an order topology* $\mathfrak{O}_{\leqslant}$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{O}_{\leqslant}$ *preserving at most enumerable joins and meets in* $\mathfrak{A}$.

In the proof of this theorem the Rasiowa–Sikorski lemma is used. The main difficulty of the proof of the Rasiowa–Sikorski lemma is to find enough Q-filters in the sense that the function assigning to every $a \in A$ the set of all Q-filters containing a is injective and preserves the semi-Boolean operations $\Rightarrow$, $\dot{-}$. This is reduced to the following conditions:

(a) if $c, d \in A$, $\nabla \in \mathfrak{O}_{\leqslant}$ and $c \Rightarrow d \notin \nabla$, then there exists a $\nabla' \in \mathfrak{O}_{\leqslant}$ such that $\nabla \subset \nabla'$, $c \in \nabla'$ and $d \notin \nabla'$, where $\mathfrak{O}_{\leqslant}$ is the set of all Q-filters;

(b) if $c, d \in A$, $\nabla \in \mathfrak{O}_{\leqslant}$ and $c \dot{-} d \notin \nabla$ then, for every $\nabla' \in \mathfrak{O}_{\leqslant}$ such that $\nabla' \subset \nabla$, if $c \in \nabla'$ then $d \in \nabla'$, where $\mathfrak{O}_{\leqslant}$ is the set of all Q-filters.

The results of this chapter are used in the treatment of algebraic and semantic (Kripke) models for H–B logic, which is done in Chapter II (and for DI logic, which is done in § 3 of Chapter III). In this chapter, H–B logic is described. First we introduce some basic notions dealing with algebraic models and prove the basic theorems, such as the deduction theorem, the reduction theorem for consistency, completeness (first and second form). Then we examine saturated H–B theories. We prove that every H–B theory can be extended — in an inessential way — to a saturated H–B theory. This fact is used to prove the Craig interpolation lemma and the Robinson consistency theorem. The last part of this section is devoted to Kripke models for H–B logic. It is seen that models of this type for H–B logic are with constant domains. This fact allows us to show an equivalence between algebraic models and Kripke models for this logic. This result is not true for IL.

The third chapter is devoted to model theory for H–B logic. A large number of basic theorems for classical logic are proved for this logic, eg. Łoś's theorem, an analogue of a result of Frayne and Scott on elementary equivalence and ultrafilter, Keisler's model extension theorem, etc. To prove this theorems we use methods which are connected with Kripke style models [2].

When this paper was in print I found some gaps in the proof of the theorems in §9 II. Professor H. Ono had informed me that these gaps could be removed easly using similar methods to that used to prove the Rasiowa–Sikorski lemma for semi-Boolean algebras.

We will use the following abbreviations:

$\bigvee a$ – there exists an a,

$\bigwedge a$ – for every a,

& – and,

iff – if and only if,

$\vdash a$ – a is provable in $\mathscr{T}$ and $\vdash_{\mathscr{T}} a$ – a is a theorem of $\mathscr{T}$,

■ – end of the proof.

I am deeply obliged to Professor H. Rasiowa for her guidance. I would also like to express my gratitude to Professor S. K. Thomason and Professor T. Traczyk for their suggestions to improve the clarity of the received results and for pointing out several ambiguities.

Chapter I

Semi-Boolean algebras

1. Semi-Boolean algebras.

DEFINITION. We shall say that an abstract algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ is a *semi-Boolean algebra* provided that

(i) $(A, \cup, \cap, \Rightarrow)$ is a relatively pseudo-complemented lattice,

(ii) $\dot{-}$ is a binary operation which satisfies the following condition:

$$a \dot{-} b \leqslant x \quad \text{if and only if} \quad a \leqslant b \cup x \text{ for any } a, b, x \in A.$$

The operation $\dot{-}$ will be called the *pseudo-difference.* This operation is dual to the relative pseudo-complement $\Rightarrow$.

We observe that in every semi-Boolean algebra there exist the greatest element $\bigvee$ and the least element $\bigwedge$, namely

$$\bigvee = a \Rightarrow a$$

and

$$\bigwedge = a \dot{-} a.$$

Thus every element $a \in A$ has the $\cap$-complement, namely

$$\neg a = a \Rightarrow \bigwedge$$

and every element $a \in A$ has the $\cup$-complement, namely

$$\ulcorner a = \bigvee \dot{-} a.$$

In [16] it was proved that the definition of a semi-Boolean algebra given above is equivalent to the following one: An abstract algebra $(A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$ will be called a *semi-Boolean algebra* provided that $(A, \cup, \cap, \Rightarrow, \neg)$ is a pseudo-Boolean algebra [15] and $(A, \cup, \cap, \dot{-}, \ulcorner)$ is a *Brouwerian algebra* [13]. A more detailed exposition of the properties of semi-Boolean algebras is given in [16].

Some examples of semi-Boolean algebras have been given in [16]. Now we shall construct a new example of a semi-Boolean algebra, which will be very useful in our consideration.

DEFINITION. By a *quasi-ordered* set we mean an ordered pair $\mathcal{G} = \langle G, \leqslant \rangle$, where G is a non-empty set and $\leqslant$ is a transitive and reflexive relation in G. Now, let $B \subset G$. We call B *open* if whenever $x \in B$ and $x \leqslant y$, then $y \in B$.

For $\mathcal{O}(\mathcal{G})$ we take the collection of all open subsets of $\mathcal{G}$ and for the ordering relation $\leqslant$ we take set inclusion. We note that the algebra $(\mathcal{O}(\mathcal{G}), \cup, \cap)$, where the operations $\cup$ and $\cap$ are just the ordinary union and intersection, respectively, is a distributive lattice with the unit element G and the zero element $\emptyset$ ($\emptyset$ — the empty set). Now, let $\Rightarrow$, $\dot{-}$ be two new operations in $\mathcal{O}(\mathcal{G})$ defined by the formulas: for every $B, C \in \mathcal{O}(\mathcal{G})$

(1) $B \Rightarrow C = \{x \in G \colon \bigwedge y \in G \text{ if } x \leqslant y \,\&\, y \in B \text{ then } y \in C\}$,

(2) $B \dot{-} C = \{x \in G \colon \bigvee y \in G, y \leqslant x \,\&\, y \in B \,\&\, y \notin C\}$.

By an easy verification we can prove the following:

1.1. *The algebra* $\mathfrak{O}(\mathcal{G}) = (\mathcal{O}(\mathcal{G}), \cup, \cap, \Rightarrow, \dot{-})$, *where* $\mathcal{O}(\mathcal{G})$ *is the family of all open sets of a quasi-ordered set* $\mathcal{G}$, *the operations* $\cup$ *and* $\cap$ *are the set-theoretical union and intersection, respectively, the operations* $\Rightarrow$ *and* $\dot{-}$ *are defined by* (1) *and* (2), *respectively, is a complete semi-Boolean algebra.* ∎

1.1 yields an important example of a semi-Boolean algebra. In the sequel, this algebra will be called an *order topology*. This example of a semi-Boolean algebra is typical because we have the following representation theorem:

1.2. *For every semi-Boolean algebra* $\mathfrak{A}$ *there exists an order topology* $\mathfrak{O}(\mathcal{G})$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{O}(\mathcal{G})$.

Proof. Let us denote by G the set of all prime filters of a semi-Boolean algebra $\mathfrak{A}$ and let h be defined as usual, namely

$$h(a) = \{\nabla \in G \colon a \in \nabla\}, \quad \text{for every } a \in A.$$

It is obvious that the system $\langle G, \leqslant\rangle$ — where $\leqslant$ is set-theoretical inclusion — is a quasi-ordered set, and $h(a)$ is open for every $a \in A$. We denote by $\mathcal{O}(\mathcal{G})$ the class of all open sets of $\langle G, \leqslant\rangle$. It is well known that h preserves the join $\cup$ and the meet $\cap$ and for every $\nabla \in G$, $a, b \in A$,

(3) $a \Rightarrow b \in \nabla$ iff for every $\nabla_1 \in G$ such that if $\nabla \subset \nabla_1$ and $a \in \nabla_1$ then $b \in \nabla_1$.

The proof of this condition may be found in [4] and it shows that

(4) $h(a \Rightarrow b) = h(a) \Rightarrow h(b)$,

where the sign $\Rightarrow$ on the right side of (4) is defined by (1). To show that h is the required monomorphism it is sufficient to prove that h preserves $\dot{-}$.

For this purpose we denote by $\tilde{G}$ the class of all prime ideals of the semi-Boolean algebra $\mathfrak{A}$. It is obvious that

(5) $\nabla \in G$ iff $\Delta \in \tilde{G}$, where $\Delta = A - \nabla$.

We can prove by using the same method [22] as in proving (3) that for every $\Delta \in \tilde{G}$,

(6) $a \dot{-} b \in \Delta$ iff for every $\Delta_1 \in \tilde{G}$ if $\Delta \subset \Delta_1$ and $b \in \Delta_1$ then $a \in \Delta_1$ for any $a, b \in A$.

On account of (6) and (5) we infer that for any $\nabla \in G$ and for every $a, b \in A$,

(7) $a \dot{-} b \in \nabla$ iff there exists a $\nabla_1 \in G$ such that $\nabla_1 \subset \nabla$, $a \in \nabla_1$ and $b \notin \nabla_1$.

Condition (7) proves that

(8) $h(a \dot{-} b) = h(a) \dot{-} h(b)$,

where the sign $\dot{-}$ on the right side of (8) is defined by (2). This completes the proof of 1.1. ∎

The following statement follows from 1.1 and [4].

1.3. *For every pseudo-Boolean algebra $\mathfrak{A}$ there exist a complete semi-Boolean algebra $\mathfrak{A}'$ and a monomorphism g from $\mathfrak{A}$ to $\mathfrak{A}'$.* ∎

1.4. *For every Brouwerian algebra $\mathfrak{A}$ there exist a complete semi-Boolean algebra $\mathfrak{A}'$ and a monomorphism h from $\mathfrak{A}$ to $\mathfrak{A}'$.*

The proof of 1.4 is similar to the proof of 1.3. In this case the notion of prime filters is replaced by the notion of prime ideals. ∎

It has been proved in [16] that

1.5. *Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra. If the infinite join $\bigcup_{t \in T} a_t$ exists in $\mathfrak{A}$, then for every $a, b \in A$ the joins $\bigcup_{t \in T} (a_t \cap a)$, $\bigcup_{t \in T} (a_t \dot{-} a)$ and $\bigcup_{t \in T} \big((b \cap a_t) \dot{-} a\big)$ exist and the meets $\bigcap_{t \in T} (a_t \Rightarrow a)$ and $\bigcap_{t \in T} \big((b \cap a_t) \Rightarrow a\big)$ also exist and*

(9) $a \cap \bigcup_{t \in T} a_t = \bigcup_{t \in T} (a_t \cap a)$,

(10) $\big(\bigcup_{t \in T} a_t\big) \Rightarrow a = \bigcap_{t \in T} (a_t \Rightarrow a)$,

(11) $\big(\bigcup_{t \in T} a_t\big) \dot{-} a = \bigcup_{t \in T} (a_t \div a)$,

(12) $\big(b \cap \bigcup_{t \in T} a_t\big) \Rightarrow a = \bigcap_{t \in T} \big((b \cap a_t) \dot{-} a\big)$,

(13) $\big(b \cap \bigcup_{t \in T} a_t\big) \dot{-} a = \bigcup_{t \in T} \big((b \cap a_t) \dot{-} a\big)$.

If the infinite meet $\bigcap_{t \in T} b_t$ exists in $\mathfrak{A}$, then for every $a, b \in A$ the joins $\bigcup_{t \in T} (b \dot{-} a_t)$ and $\bigcup_{t \in T} \big(b \dot{-} (a_t \cup a)\big)$ exist and the meets $\bigcap_{t \in T} (a \cup b_t)$, $\bigcap_{t \in T} (a \Rightarrow b_t)$

and $\bigcap_{t\in T}(a\Rightarrow(b_t\cup b))$ also exist and

(14) $a\cup\bigcap_{t\in T} b_t = \bigcap_{t\in T}(a\cup b_t)$,

(15) $a\Rightarrow\bigcap_{t\in T} b_t = \bigcap_{t\in T}(a\Rightarrow b_t)$,

(16) $a\dot{-}\bigcap_{t\in T} b_t = \bigcup_{t\in T}(b\dot{-}a_t)$,

(17) $a\Rightarrow\left(\bigcap_{t\in T} b_t\cup b\right) = \bigcap_{t\in T}\left(a\Rightarrow(b_t\cup b)\right)$,

(18) $a\dot{-}\left(\bigcap_{t\in T} b_t\cup b\right) = \bigcup_{t\in T}\left(a\dot{-}(b_t\cup b)\right)$. ■

DEFINITION. We say that a pseudo-Boolean algebras is a *D-pseudo-Boolean algebra* (briefly DPBA) if condition (14) is satisfied and by DBA we denote a *Brouwerian algebra* such that condition (9) is fulfilled.

By 1.5 we have

1.6. *Every semi-Boolean algebra is a DPBA.* ■

1.7. *Every semi-Boolean algebra is a DBA.* ■

2. Q-filters in semi-Boolean algebras. Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \sqcap)$ be a semi-Boolean algebra and let (Q) be a set of infinite joins and meets in $\mathfrak{A}$:

(Q) $$a_s = \bigcup_{a\in A_s} a, \quad s\in S, \quad b_s = \bigcap_{b\in B_s} b, \quad s\in S'.$$

DEFINITION. A prime filter ∇ is said to be a $\bigcap$*-filter* provided that

(f_1) for every $s\in S'$ if $B_s\subset\nabla$ then $b_s\in\nabla$.

A prime filter ∇ is said to be a $\bigcup$*-filter* provided that

(f_2) for every $s\in S$ if $a_s\in\nabla$ then $A_s\cap\nabla\neq\emptyset$.

A prime filter ∇ is called a *Q-filter* provided that (f_1) and (f_2) are satisfied.

Sometimes we say that ∇ preserves joins and meets in (Q) if ∇ is a Q-filter.

DEFINITION. A prime ideal Δ is called a $\bigcup$*-ideal* provided that

(i_1) for every $s\in S$ if $A_s\subset\Delta$ then $a_s\in\Delta$.

A prime Δ ideal is said to be a $\bigcap$*-ideal* provided that

(i_2) for every $s\in S$ if $b_s\in\Delta$ then $B_s\cap\Delta\neq\emptyset$.

A prime Δ ideal is said to be a *Q-ideal* provided that (i_1) and (i_2) are satisfied.

Sometimes we say that Δ preserves joins and meets in (Q) if Δ is a Q-ideal.

Now, we denote by $(x]$ the class of all elements $y \in A$ such that $y \leqslant x$, where A is the set of all elements of a semi-Boolean algebra $\mathfrak{A}$.

DEFINITION. A non-void subset A_0 of A is said to be *dense in* $(A, \leqslant)$, the relation $\leqslant$ being the lattice ordering in A, if for every $p \in A$ there exists an $a \in A_0$ such that $a \leqslant p$.

This definition can be formulated as follows:

A_0 is dense in A if for every $p \in A$

$$A_0 \cap (p] \neq \emptyset.$$

Let $\mathfrak{M}$ be a class of dense subsets of A.

DEFINITION. A filter ∇ is said to be $\mathfrak{M}$-*generic* if it has the following property:

if D is dense in A and $D \in \mathfrak{M}$ then $D \cap \nabla \neq \emptyset$.

2.1. *Let* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \div, \neg, \Gamma)$ *be a semi-Boolean algebra and let the set* S *in the definition of* (Q) *be at most enumerable. Then for every* $x \in A - \{\wedge\}$ *there exist a class* $\mathfrak{M}$ *of dense subsets of* A *and a* $\bigcup$*-filter* ∇ *such that* ∇ *is* $\mathfrak{M}$*-generic and* $x \in \nabla$.

Proof. Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \div, \neg, \Gamma)$ be a semi-Boolean algebra and, for all $n \in \omega$, $A_n \subset A$, let $a_n = \bigcup^{\mathfrak{A}}_{a \in A_n} a$ exist. Denote by S_n the set

$$(\neg a_n] \cup \bigcup_{a \in A_n} (a].$$

We prove that, for every $n \in \omega$, S_n is dense in $(A - \{\wedge\}, \leqslant)$. For this purpose it is sufficient to show that for any $n \in \omega$ and $p \in A - \{\wedge\}$,

$$S_n \cap (p] \neq \emptyset.$$

Let $p \in A - \{\wedge\}$: then either $p \in (\neg a_n]$ or $p \notin (\neg a_n]$. In the first case $p \in S_n$ and $S_n \cap (p] \neq \emptyset$. Suppose that $p \notin (\neg a_n]$, i.e. the relation $p \leqslant \neg a_n$ does not hold. Thus $p \cap a_n \neq \wedge$ and we infer that $\bigcup_{a \in A_n} (p \cap a) \neq \wedge$. The last inequality implies that there exists an $a_0 \in A_n$ such that $p \cap a_0 \neq \wedge$ and this proves that $(p] \cap (a_0] \neq \emptyset$. Thus there exists a $p_0 \neq \wedge$ such that $p_0 \in (p] \cap (a_0]$. So $p_0 \in \bigcup_{a \in A_n} (a]$ and $S_n \cap (p] \neq \emptyset$. This proves that, for all $n \in \omega$, S_n is dense in $(A - \{\wedge\}, \leqslant)$.

Now, denote by M the class of all S_n. Let $(\beta_n)_{n \in \omega}$ be a sequence of elements of $A - \{\wedge\}$ constructed in the following way:

$$\beta_0 = x,$$
$$\beta_1 \in S_1 \cap (\beta_0],$$
$$\beta_2 \in S_2 \cap (\beta_1],$$

$$\beta_n \in S_n \cap (\beta_{n-1}].$$

Let ∇ be a filter generated by $(\beta_n)_{n\in\omega}$. It is obvious that ∇ is $\mathfrak{M}$-generic and $x \in \nabla$. We can assume that ∇ is a maximal filter. If not, then we can extend it to a maximal ∇' which is also $\mathfrak{M}$-generic and $x \in \nabla'$. Now we prove that ∇ satisfies the condition (f_2). Let n be an arbitrary but fixed. By our assumption ∇ is $\mathfrak{M}$-generic. Thus $\nabla \cap S_n \neq \emptyset$; i.e. there exists $p \in \nabla$ such that $p \in S_n$. This implies that

either $p \in (\neg a_n]$ or $p \in \bigcup_{a\in A_n} (a]$; hence
either $p \leqslant \neg a_n$ or there exists an $a_0 \in A_n$ such that $p \leqslant a_0$; hence
either $\neg a_n \in \nabla$ or there exists an $a_0 \in A_n$ such that $a_0 \in \nabla$; hence
either $a_n \notin \nabla$ or there exists an $a_0 \in A_n$ such that $a_0 \in \nabla$; hence

$$\text{if } a_n \in \nabla \text{ then } A_n \cap \nabla \neq \emptyset,$$

which proves that ∇ is the required $\bigcup$-filter. ∎

The next lemma we can prove in a similar way to 2.1.

2.2. *Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \Gamma)$ be a semi-Boolean algebra and let the set S be at most enumerable. Then for every $x \in A - \{\vee\}$ there exists a class $\mathfrak{M}$ of dense subsets of $\mathfrak{A}$ and a $\bigcap$-ideal Δ such that Δ is $\mathfrak{M}$-generic and $x \in \Delta$.* ∎

Now we prove theorem which is analogous to the Rasiowa–Sikorski lemma.

2.3. *Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra and let the set* (Q) *be a set of joins and meets of the following form*

$$a_{2n} = \bigcup_{a\in A_{2n}} a, \qquad n \in \omega;$$
$$b_{2n+1} = \bigcap_{b\in B_{2n+1}} b, \qquad n \in \omega.$$

Let x, y be the elements of A such that the relation $x \leqslant y$ does not hold. Then there exists a Q-filter (a Q-ideal) such that $x \in \nabla$ and $y \notin \nabla$ ($x \notin \Delta$ and $y \in \Delta$).

Proof. The proof of this theorem is preceded by the following remark:

(R) *In every semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$, for any $a, b \in A$ $a \leqslant b$ iff there exists a $c \in A$ such that $a \dot{-} b \in (c]$ and $a \Rightarrow b \in [c)$, where $[c) = \{x \in A: c \leqslant x\}$.*

The proof of (R) is by an easy verification.

Now, we define two sequences, $(a_n)_{n\in\omega}$ and $(\beta_n)_{n\in\omega}$, of the elements of A such that

(i) $a_0 = y$, $\beta_0 = x$;

(ii) $a_{n-1} \leqslant a_n$ and $\beta_{n-1} \geqslant \beta_n$ for $n > 0$;

(iii) either $\beta_{2n+1} \leqslant b_{2n+1}$ or there exists a $b \in B_{2n+1}$ such that $b \leqslant a_{2n+1}$ for every $n \in \omega$, and either there exists an $a \in A_{2n}$ such that $\beta_{2n} \leqslant a$ or $a_{2n} \leqslant a_{2n}$ for every $n \in \omega$;

(iv) the relation $\beta_n \leqslant a_n$ does not hold for any $n \in \omega$.

Suppose that, for $k \in \omega$, $a_0, \ldots, a_{2k}$ and $\beta_0, \ldots, \beta_{2k}$ are constructed such that (i)–(iv) are fulfilled. On account of (iv) the relation $\beta_{2k} \leqslant a_{2k}$ does not hold. By (R) we infer that for every $c \in A$

$$\text{either } \beta_{2k} \dot{-} a_{2k} \notin (c] \text{ or } \beta_{2k} \Rightarrow a_{2k} \notin [c).$$

Putting $c = b_{2k+1}$, we obtain that

$$\text{either } \beta_{2k} \dot{-} a_{2k} \notin (b_{2k+1}] \text{ or } \beta_{2k} \Rightarrow a_{2k} \notin [b_{2k+1}).$$

Suppose that $\beta_{2k} \dot{-} a_{2k} \notin (b_{2k+1}]$. The condition $\beta_{2k} \dot{-} a_{2k} \in (b_{2k+1}]$ is equivalent to the following one: for every $b \in B_{2k+1}$, $\beta_{2k} \dot{-} a_{2k} \in (b]$. Thus, by our assumption there exists a $b \in B_{2k+1}$ such that $\beta_{2k} \dot{-} a_{2k} \notin (b]$. In this case we put $\beta_{2k+1} = \beta_{2k}$ and $a_{2k+1} = a_{2k} \cup b$. It is not difficult to check that β_{2k+1} and a_{2k+1} defined in this way satisfy (ii)–(iv).

Now, suppose that $\beta_{2k} \Rightarrow a_{2k} \notin [b_{2k+1})$. In this case we put $\beta_{2k+1} = b_{2k+1} \cap \beta_{2k}$ and $a_{2k+1} = a_{2k}$. Then (ii)–(iv) clearly hold for $n = 2k+1$.

We construct β_{2k+2} and a_{2k+2} in a similar way. By condition (iv) the relation $\beta_{2k+1} \leqslant a_{2k+1}$ does not hold. Using (R), we can assume that for every $c \in A$

$$\text{either } \beta_{2k+1} \dot{-} a_{2k+1} \notin (c] \text{ or } \beta_{2k+1} \Rightarrow a_{2k+1} \notin [c).$$

Assume $c = a_{2k+2}$ and let the first case be true, i.e. let $\beta_{2k+1} \dot{-} a_{2k+1} \notin (a_{2k+2}]$. Then $\beta_{2k+2} = \beta_{2k+1}$ and $a_{2k+2} = a_{2k+1} \cup a_{2k+2}$ satisfy (ii)–(iv).

Now, we observe that the condition $\beta_{2k+1} \Rightarrow a_{2k+1} \in [a_{2k+2})$ is equivalent to the following one: for every $a \in A_{2k+2}$, $\beta_{2k+1} \Rightarrow a_{2k+1} \in [a)$. Thus the condition $\beta_{2k+1} \Rightarrow a_{2k+1} \notin [a_{2k+2})$ yields that there exists an $a \in A_{2k+2}$ such that $\beta_{2k+1} \Rightarrow a_{2k+1} \notin [a)$. Putting $\beta_{2k+2} = a \cap \beta_{2k+1}$ and $a_{2k+2} = a_{2k+1}$, we find that (ii)–(iv) hold for $n = 2k+2$. Thus a_n and β_n are defined for all $n \in \omega$.

Let I be the ideal generated by the sequence $(a_n)_{n\in\omega}$ and let F be the filter generated by $(\beta_n)_{n\in\omega}$. Then by (iv), I and F are disjoint and

(v) either $b_{2n+1} \in F$ or there exists a $b \in B_{2n+1}$ such that $b \in I$, for $n \in \omega$,

(vi) either there exists an $a \in A_{2n}$ such that $a \in F$ or $a_{2n} \in I$, for any $n \in \omega$.

It is well known that in a distributive lattice every filter can be separated from an ideal which is disjoint from it by a prime filter, and every ideal can be separated from a filter disjoint from it by a prime ideal. Let ∇ be a prime filter containing F and such that ∇ is disjoint from I. Similarly, let Δ be a prime ideal such that Δ contains I and Δ is disjoint from ∇. It is obvious that $x \in \nabla$, $y \notin \nabla$ as well as $x \notin \Delta$ and $y \in \Delta$. By (v) and (vi) ∇ is the required Q-filter and Δ is the required Q-ideal, which completes the proof of 2.3. ■

3. Extensions of semi-Boolean algebras. In the sequel we assume that the sets S and S' in the definition of (Q) will be at most enumerable.

Now, we observe that by the definition of (Q) if, for every $n \in \omega$, $a_{2n} = \bigcup_{a \in A_{2n}} a$ and $b_{2n+1} = \bigcap_{b \in B_{2n+1}} b$ exist in $\mathfrak{A}$, then they belong to (Q). Thus and from Lemma 1.5 we infer that the joins

$$(*) \qquad \begin{array}{ll} \bigcup_{a \in A_{2n}} (a \dot{-} c), & \bigcup_{a \in A_{2n}} \big((d \cap a) \dot{-} c\big), \\ \bigcup_{b \in B_{2n+1}} (c \dot{-} b), & \bigcup_{b \in B_{2n+1}} \big(c \dot{-} (b \cup d)\big), \end{array}$$

exist for every $n \in \omega$ and $c, d \in A$ but they need not be in (Q). Similarly, the meets

$$(**) \qquad \begin{array}{ll} \bigcap_{a \in A_{2n}} (a \Rightarrow c), & \bigcap_{a \in A_{2n}} \big((a \cap d) \Rightarrow c\big), \\ \bigcap_{b \in B_{2n+1}} (c \Rightarrow b), & \bigcap_{b \in B_{2n+1}} \big(c \Rightarrow (b \cup d)\big), \end{array}$$

exist for every $n \in \omega$ and $c, d \in A$ but they need not be in (Q).

We will impose some properties on the sets A_{2n}, B_{2n+1} such that if a prime filter ∇ (a prime ideal Δ) preserves a_{2n}, b_{2n+1} then, for any $c, d \in A$, it preserves the infinite joins and meets given in $(*)$ $((**))$.

First we note that the following two statements follow from [18] and 1.2.

3.1. *Let* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \neg)$ *be a DPBA. Suppose that for every* $n \in \omega$, $A_{2n}, B_{2n+1} \subset A$ *and let*

(i) $a_{2n} = \bigcup_{a \in A_{2n}} a$ *and* $b_{2n+1} = \bigcap_{b \in B_{2n+1}} b$ *exist;*

(ii) *for any* $c \in A$ *and an arbitrary but fixed* n

$$\{a \Rightarrow c\colon\ a \in A_{2n}\} \in \{B_{2k+1}\colon\ k \in \omega\},$$
$$\{c \Rightarrow b\colon\ b \in B_{2n+1}\} \in \{B_{2k+1}\colon\ k \in \omega\};$$

(iii) *for any* $c, d \in A$ *and an arbitrary but fixed* n

$$\{(a \cap d) \Rightarrow c\colon\ a \in A_{2n}\} \in \{B_{2k+1}\colon\ k \in \omega\},$$
$$\{c \Rightarrow (b \cup d)\colon\ b \in B_{2n+1}\} \in \{B_{2k+1}\colon\ k \in \omega\}.$$

Then there exists a complete semi-Boolean algebra $\mathfrak{A}'$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{A}'$ *preserving all infinite joins* a_{2n} *and meets* b_{2n+1}, *for* $n \in \omega$. ∎

3.2. *Let* $\mathfrak{A} = (A, \cup, \cap, \dot{-}, \Gamma)$ *a DBA, and suppose that, for every* $n \in \omega$, $A_{2n}, B_{2n+1} \subset A$ *and*

(i) $a_{2n} = \bigcup_{a \in A_{2n}} a$ *and* $b_{2n+1} = \bigcap_{b \in B_{2n+1}} b$ *exist,*

(iv) *for any* $c \in A$ *and an arbitrary but fixed* n,

$$\{a \dot{-} c:\ a \in A_{2n}\} \in \{A_{2k}:\ k \in \omega\},$$
$$\{c \dot{-} b:\ b \in B_{2n+1}\} \in \{A_{2k}:\ k \in \omega\},$$

(v) *for any* $c, d \in A$ *and an arbitrary but fixed* n

$$\{(d \cap a) \dot{-} c:\ a \in A_{2n}\} \in \{A_{2k}:\ k \in \omega\},$$
$$\{c \dot{-} (b \cup d):\ b \in B_{2n+1}\} \in \{A_{2k}:\ k \in \omega\}.$$

Then there exists a complete semi-Boolean algebra $\mathfrak{A}'$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{A}'$ *preserving all infinite joins and meets in* (Q). ■

3.3. *Let* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ *be a semi-Boolean algebra. For every* $n \in \omega$ *suppose that* $A_{2n}, B_{2n+1} \subset A$ *are such that conditions* (i)–(v) *of* 3.1 *and* 3.2 *are satisfied.*

Then for every Q-filter ∇ *in* $\mathfrak{A}$ *such that* $a \Rightarrow b \notin \nabla$ *there exists a Q-filter* ∇' *such that* $a \in \nabla'$, $b \notin \nabla'$ *and* $\nabla \subset \nabla'$.

Proof. Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra and let ∇ be a Q-filter such that $a \Rightarrow b \notin \nabla$. We take the quotient algebra $\mathfrak{A}/\nabla$. On account of 2.7 [16] the algebra $\mathfrak{A}/\nabla$ need not be a semi-Boolean algebra but it is a DPBA [18]. By 3.1 this algebra can be extended to a semi-Boolean algebra $\mathfrak{A}'$. More precisely: there exists a semi-Boolean algebra $\mathfrak{A}'$ and a monomorphism h from $\mathfrak{A}/\nabla$ to $\mathfrak{A}'$ preserving all at most enumerable joins and meets in $\mathfrak{A}/\nabla$.

Now, we note that the relation $h(|a|) \leqslant h(|b|)$ does not hold. This follows from the fact that $a \Rightarrow b \notin \nabla$. By theorem 2.3 there exists a Q-filter $\tilde{\nabla}$ such that $h(|a|) \in \tilde{\nabla}$ and $h(|b|) \notin \tilde{\nabla}$. Let us set

$$\nabla' = \{x \in A:\ h(|x|) \in \tilde{\nabla}\}.$$

It is obvious that ∇' is a filter. Moreover, ∇' is a Q-filter as $\tilde{\nabla}$ is a Q-filter and h preserves all infinite joins and meets in $\mathfrak{A}/\nabla$. We observe that $a \in \nabla'$ and $b \notin \nabla'$. Now, let $x \in \nabla$; then $|x| = \bigvee_{\mathfrak{A}/\nabla}$ and $h(|x|) = \bigvee_{\mathfrak{A}'}$. Thus $h(|x|) \in \tilde{\nabla}$ and this gives $x \in \nabla'$, which proves that $\nabla \subset \nabla'$, i.e. ∇' is the required Q-filter. ■

In the sequel we assume that A_{2n}, B_{2n+1} always satisfy (i)–(v) of 3.1 and 3.2.

We denote by G the set of all Q-filters of a semi-Boolean algebra $\mathfrak{A}$. We take $\nabla \in G$.

3.4. $a \Rightarrow b \in \nabla$ *iff for every* $\nabla' \in G$ *such that* $\nabla \subset \nabla'$ *if* $a \in \nabla'$ *then* $b \in \nabla'$.

Proof. If $a \Rightarrow b \in \nabla$ then the lemma is obvious. On the other hand, suppose $a \Rightarrow b \notin \nabla$. On account of 3.3 we can construct a Q-filter ∇' such that $a \in \nabla'$ and $b \notin \nabla'$ and $\nabla \subset \nabla'$. ■

Using analogous methods to the proof of 3.4, we can prove the following lemma:

3.5. *Let* $\mathfrak{A}$ *be a semi-Boolean algebra and let* Δ *be a Q-ideal such that* $a \dot{-} b \notin \Delta$. *Then there exists a Q-ideal* Δ' *such that* $a \notin \Delta$, $b \in \Delta$ *and* $\Delta \subset \Delta'$.

The proof of 3.5 is analogous to the proof of 3.3 but, in this case the quotient algebra $\mathfrak{A}/\Delta$ is a DBA. On account of 3.2 the algebra $\mathfrak{A}/\Delta$ can be extended to a semi-Boolean algebra $\mathfrak{A}'$. We take for the required Q-ideal Δ' the set

$$\{x \in A : h(|x|) \in \tilde{\Delta}\},$$

where $\tilde{\Delta}$ is a Q-ideal such that $h(|a|) \notin \tilde{\Delta}$ and $h(|b|) \in \Delta$ and h is a function embedding $\mathfrak{A}/\Delta$ to $\mathfrak{A}'$ preserving all at most enumerable joins and meets in $\mathfrak{A}/\Delta$. ■

Let us denote by $\tilde{G}$ the set of all Q-ideals of a semi-Boolean algebra $\mathfrak{A}$ and let $\Delta \in \tilde{G}$. It follows from 3.5 that

3.6. *$a \dot{-} b \in \Delta$ iff for every $\Delta_1 \in \tilde{G}$ such that $\Delta \subset \Delta_1$ if $b \in \Delta_1$ then $a \in \Delta_1$.* ■

We observe that if ∇ is a Q-filter in a semi-Boolean algebra $\mathfrak{A}$ then $A - \nabla$ is a Q-ideal and

(1) $$\nabla \in G \quad \text{iff} \quad A - \nabla = \Delta \in \tilde{G}.$$

Let $\nabla \in G$.

3.7. *$a \dot{-} b \in \nabla$ iff there exists a $\nabla_1 \in G$ such that $\nabla_1 \subset \nabla$, $a \in \nabla_1$ and $b \notin \nabla_1$.*

Proof. Indeed, suppose that $a \dot{-} b \in \nabla$. By (1), $a \dot{-} b \notin \Delta$. Using 3.5, we have that there exists a $\Delta_1 \in \tilde{G}$ such that $\Delta \subset \Delta_1$, $b \in \Delta_1$ and $a \notin \Delta_1$. On account of (1) we infer that there exists $\nabla_1 = A - \Delta_1$ such that $\nabla_1 \subset \nabla = A - \Delta$ and $b \notin \nabla_1$ and $a \in \nabla_1$. On the other hand, the proof is obvious. ■

3.8. *Let $\mathfrak{A}$ be a semi-Boolean algebra and let the set* (Q) *be defined as usual and assume that A_{2n} and B_{2n+1} satisfy* (i)–(v) *of* 3.1 *and* 3.2. *Then there exists a monomorphism h from $\mathfrak{A}$ to an order topology preserving all infinite joins and meets in* (Q).

This theorem follows from 1.1, 2.3, 3.4 and 3.7. ■

Chapter II

Algebraic and semantic models for Heyting-Brouwer logic

1. Preliminaries. In this section we shall deal with a fixed formalized language $\mathscr{L}$ of the Heyting–Brouwer predicate calculus (briefly the H–B predicate calculus or simply H–B PC).

DEFINITION. A *language* $\mathscr{L}$ *of H–B PC* has as symbols the following:

(L_1) the free variables $x, y, z, x_1, y_1, z_1, \ldots$;

(L_2) the bounded variables $\xi, \eta, \xi_1, \xi_2, \ldots$;

(L_3) the symbols $\neg, \ulcorner, \cup, \cap, \Rightarrow, \dot{-}, \forall, \exists$;

(L_4) for each n, the n-ary function symbols $f, f_1, \ldots$ and the n-ary predicate symbols $p, p_1, \ldots$;

(L_5) auxiliary sings (and).

We shall always assume that the language $\mathscr{L}$ is enumerable.

We also assume that the binary predicate symbols must include the equality symbol $=$. A 0-ary function symbol is called a *constant* and we denote the set of constants by c.

The classes of all formulas F and terms T we define in usual way.

Let A_1 be the set containing all formulas of the forms:

(A_1) $\big((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma))\big)$,

(A_2) $(\alpha \Rightarrow (\alpha \cup \beta))$,

(A_3) $(\beta \Rightarrow (\alpha \cup \beta))$,

(A_4) $\big((\alpha \Rightarrow \gamma) \Rightarrow \big((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \cup \beta) \Rightarrow \gamma)\big)\big)$,

(A_5) $((\alpha \cap \beta) \Rightarrow \alpha)$,

(A_6) $((\alpha \cap \beta) \Rightarrow \beta)$,

(A_7) $\big((\gamma \Rightarrow \alpha) \Rightarrow \big((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \cap \beta))\big)\big)$,

(A_8) $\big((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \cap \beta) \Rightarrow \gamma)\big)$,

(A_9) $\big(((\alpha \cap \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma))\big)$,

(A_{10}) $((\alpha \Rightarrow \beta) \Rightarrow (\neg\beta \Rightarrow \neg\alpha))$,

(A_{11}) $\big(\alpha \Rightarrow (\beta \cup (\alpha \dot{-} \beta))\big)$,

(A_{12}) $((\alpha \dot{-} \beta) \Rightarrow \ulcorner(\alpha \Rightarrow \beta))$,

(A_{13}) $\big(((\alpha \dot{-} \beta) \dot{-} \gamma) \Rightarrow (\alpha \dot{-} (\beta \cup \gamma))\big)$,

(A_{14}) $(\neg(\alpha \dot{-} \beta) \Rightarrow (\alpha \Rightarrow \beta))$,

(A_{15}) $\big((\alpha \Rightarrow (\gamma \dot{-} \gamma)) \Rightarrow \neg\alpha\big)$,

(A_{16}) $\big(\neg\alpha \Rightarrow (\alpha \Rightarrow (\gamma \dot{-} \gamma))\big)$,

(A_{17}) $\big(((\gamma \Rightarrow \gamma) \dot{-} \alpha) \Rightarrow \ulcorner\alpha\big)$,

(A_{18}) $\big(\ulcorner\alpha \Rightarrow ((\gamma \Rightarrow \gamma) \dot{-} \alpha)\big)$,

where α, β, γ are any formulas in $\mathscr{L}$.

The formulas from A_1 are called *axioms* of H–B PC.

By a *consequence operation* in $\mathscr{L}$ we shall understand a mapping C of 2^F to 2^F such that, for every $F_1 \subset F$, $C(F_1)$ is the smallest set containing A_1, F_1 and closed under the rules of inference of the intuitionistic predicate calculus [15] strengthened by the schema (r)

$$\text{(r)} \qquad \frac{\alpha}{\neg\,\Gamma\alpha}$$

where α is any formula in $\mathscr{L}$.

We shall examine a deductive system

$$\mathscr{S} = (\mathscr{L}, C),$$

called the *H–B predicate calculus* based on the language $\mathscr{L}$, and a *formalized H–B predicate theory* based on $\mathscr{L}$

$$\mathscr{T} = (\mathscr{L}, C, A),$$

where A is any set of formulas. Formulas in $C(A)$ are called *theorems* of the theory $\mathscr{T}$. Theorems in $(\mathscr{L}, C, \emptyset)$ are called *derivable formulas*. If a formula α is derivable, then we shall write $\vdash\alpha$. The theory $\mathscr{T}$ is *consistent* if there exists a formula which is not a theorem of $\mathscr{T}$.

Let $(F, \cup, \cap, \Rightarrow, \dot{-}, \neg, \Gamma)$ be the algebra of formulas and let $\approx$ be a relation defined as follows:

(1) $\quad \alpha \approx \beta$ iff the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathscr{T}$.

It is well known that the relation $\approx$ is the congruence in the algebra of formulas. More precisely, denote by $\|\alpha\|$ the equivalence class containing α. Then we can prove in the usual way the following theorem:

1.1. *The algebra* $\mathfrak{A}(\mathscr{T}) = (F/\approx, \cup, \cap, \Rightarrow, \dot{-}, \neg, \Gamma)$ *of an H–B theory* $\mathscr{T}$ *is a semi-Boolean algebra. For any formulas* α, β

$$\text{(2)} \qquad \begin{aligned} \|\alpha\| \cup \|\beta\| &= \|(\alpha \cup \beta)\|, & \|\alpha\| \cap \|\beta\| &= \|(\alpha \cap \beta)\|, \\ \|\alpha\| \Rightarrow \|\beta\| &= \|(\alpha \Rightarrow \beta)\|, & \|\alpha\| \dot{-} \|\beta\| &= \|(\alpha \dot{-} \beta)\|, \\ \neg\|\alpha\| &= \|\neg\alpha\|, & \Gamma\|\alpha\| &= \|\Gamma\alpha\|. \end{aligned}$$

Moreover,

(3) $\quad \|\alpha\| \leqslant \|\beta\|$ *if and only if* $(\alpha \Rightarrow \beta)$ *is a theorem in* $\mathscr{T}$.

For every formula α,

(4) $\quad \|\alpha\| = \vee$ *if and only if* α *is a theorem in* $\mathscr{T}$.

Thus the semi-Boolean algebra $\mathfrak{A}(\mathscr{T})$ *is non-degenerate if and only if the theory* $\mathscr{T}$ *is consistent.*

For every formula $\beta(x)$,

$$\text{(Q)}\qquad \begin{aligned} \|\exists\xi\,\beta(\xi)\| &= \bigcup_{\tau\in T}\|\beta(\tau)\|,\\ \|\forall\xi\,\beta(\xi)\| &= \bigcap_{\tau\in T}\|\beta(\tau)\|. \end{aligned} \quad \blacksquare$$

It is obvious that every intuitionistic theorem is a theorem of H–B PC.

The next theorem shows the form of some theorems of H–B PC. We have chosen only those which are useful in the further parts of this paper.

1.2. *In every H–B predicate theory $\mathcal{T}$ the following formulas are theorems:*

(5) $(\neg\ulcorner a \Rightarrow a)$,

(6) $(\neg a \Rightarrow \ulcorner a)$,

(7) $(\ulcorner\ulcorner a \Rightarrow a)$,

(8) $(\ulcorner(a\cup\beta) \Rightarrow (\ulcorner a\cap\ulcorner\beta))$,

(9) $(\ulcorner(a\cap\beta) \Rightarrow (\ulcorner a\cup\ulcorner\beta))$,

(10) $((\ulcorner a\cup\ulcorner\beta) \Rightarrow \ulcorner(a\cap\beta))$,

(11) $(a\cup\ulcorner a)$,

(12) $(\forall\xi\,(\varphi(\xi)\cup\psi) \Rightarrow (\forall\xi\,\varphi(\xi)\cup\psi)$, ($\xi$ *does not appear in* ψ),

(13) $(\ulcorner\exists\xi\,\varphi(\xi) \Rightarrow \forall\xi\ulcorner\varphi(\xi))$,

(14) $(\ulcorner\forall\xi\,\varphi(\xi) \Rightarrow \exists\xi\,\ulcorner\varphi(\xi))$,

(15) $(\exists\xi\,\ulcorner\varphi(\xi) \Rightarrow \ulcorner\forall\xi\,\varphi(\xi))$,

(16) $(\neg\forall\xi\,\varphi(\xi) \Rightarrow \exists\xi\,\ulcorner\varphi(\xi))$,

(17) $(\forall\xi\,\neg\varphi(\xi) \Rightarrow \ulcorner\exists\xi\,\varphi(\xi))$.

The proof follows from 1.1 and the appropriate properties of semi-Boolean algebras. ∎

1.3. *An H–B theory $\mathcal{T}$ is consistent iff for any formula a, a and $\ulcorner a$ are not both theorems.*

Proof. Suppose that a and $\ulcorner a$ are theorems of $\mathcal{T}$. Then by (r) $\neg\ulcorner a$ and $\ulcorner a$ are also theorems. The formula $((\neg\ulcorner a\cap\ulcorner a) \Rightarrow \beta)$ is a theorem because it is of the form $((\neg\gamma\cap\gamma) \Rightarrow \eta)$ and every intuitionistic theorem is a theorem of H–B PC.

Thus every formula is a theorem of $\mathcal{T}$ and $\mathcal{T}$ is inconsistent. In the other direction the proof is obvious. ∎

On account of this lemma we infer that $\mathcal{T}$ is consistent if for no formula a, both formulas a and $\neg a$ are theorems in $\mathcal{T}$.

2. Algebraic structures. Let R be a realization of the language $\mathscr{L}$ described in § 1 in a non-void set D and in a complete semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$. It is well known that for a given realization R of $\mathscr{L}$ in a set $D \neq \emptyset$ and in a complete semi-Boolean algebra $\mathfrak{A}$ every formula a can be considered as a mapping

$$a_R\colon D^{V_a} \to A,$$

where V_a is the finite set of all free individual variables appearing in a. For this purpose it suffices

(a) to interpret all free or bound individual variables in a as variables ranging over the set D;

(b) to interpret every functor f in a as the mapping f_R: $D \times \ldots \times D \to D$;

(c) to interpret every predicate p in a as the mapping p_R: $D \times \times D \to A$;

(d) to interpret the logical connectives $\cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner$ as the signs of corresponding operations $\cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner$ in $\mathfrak{A}$;

(e) to interpret the quantifiers $\exists\xi$, $\forall\xi$ as the signs of the corresponding infinite operations $\bigcup_{\xi\in D}$, $\bigcap_{\xi\in D}$ in $\mathfrak{A}$.(¹)

We observe that if a is a closed formula, i.e. if V_a is empty, then a_R is a constant element in A.

Now if $v = \{v(u)\}_{u\in V}$ is any valuation in D, then a_R can be formulated as follows:

(a_0) $\bigcup_{c\in D}^{\mathfrak{A}} \beta_R(v_c^u)$ and $\bigcap_{c\in D}^{\mathfrak{A}} \beta_R(v_c^u)$ exist, where the valuation v_c^u is defined as follows:

$$v_c^u(u') = \begin{cases} v(u) & \text{if} \quad u \neq u', \\ c & \text{if} \quad u = u', \end{cases}$$

(a_1) $p(\tau_1, \ldots, \tau_k)_R(v) = p_R(\tau_{1R}(v), \ldots, \tau_{kR}(v))$, τ is a term,

(a_2) if, for every i, $1 \leqslant i \leqslant n$ and $v_1(u_i) = v_2(u_i)$ then $a_R(v_1) = a_R(v_2)$, where $v_1, v_2 \in D^V$, and $u_1, \ldots, u_n$ are all free variables in a,

(a_3) $(\beta \cup \gamma)_R(v) = \beta_R(v) \cup \gamma_R(v)$,

(a_4) $(\beta \cap \gamma)_R(v) = \beta_R(v) \cap \gamma_R(v)$,

(a_5) $(\beta \Rightarrow \gamma)_R(v) = \beta_R(v) \Rightarrow \gamma_R(v)$,

(a_6) $(\beta \dot{-} \gamma)_R(v) = \beta_R(v) \dot{-} \gamma_R(v)$,

(a_7) $(\neg\beta)_R(v) = \neg\beta_R(v)$,

(¹) According to condition (e) we need not to assume that $\mathfrak{A}$ is a complete semi-Boolean algebra. We must only require the infinite joins and meets mentioned in (e) to exist in $\mathfrak{A}$.

(a$_8$) $(\ulcorner\beta)_R(v) = \ulcorner\beta_R(v)$,

(a$_9$) $\exists\xi\,\beta(u/\xi)_R(v) = \bigcup_{c\in D}^{\mathfrak{A}}\beta_R(v_c^u)$,

(a$_{10}$) $\forall\xi\,\beta(u/\xi)_R(v) = \bigcap_{c\in D}^{\mathfrak{A}}\beta_R(v_c^u)$.

DEFINITION. By *an algebraic structure* $\mathfrak{M}_a$ (briefly *a*-structure) we shall understand a triple $\langle\mathfrak{A}, D, R\rangle$ where $\mathfrak{A}, D, R$ satisfy the above conditions and D is a countable set.

DEFINITION. We shall say that $\mathfrak{M}_a$ is *an algebraic model* (briefly *a-model*) *for a formula* $a\in F$ if $a_R(v) = \bigvee$ for $v\in D^V$ Sometimes we say that a is *valid* in $\mathfrak{M}_a$. $\mathfrak{M}_a$ is *an algebraic model for a set A* of formulas in $\mathscr{L}$ provided $\mathfrak{M}_a$ is an algebraic model for every formula a in A. In this case we say also that $\mathfrak{M}_a$ is *an algebraic model for the H–B theory* $\mathscr{T} = (\mathscr{L}, C, A)$. A formula a is called *algebraically valid* (a-valid) if for every realization R in any non-void set D and in any semi-Boolean algebra $\mathfrak{A}$ the a-structure $\langle\mathfrak{A}, D, R\rangle$ is an a-model for a.

We observe that no restriction was made on the set D, in which a realization R of $\mathscr{L}$ is given, except that D is countable and non-void set.

In particular, we can take as D the set of all terms T in $\mathscr{L}$. It is obvious that T is never empty since T contains the infinite set V of all free variables.

DEFINITION. A realization R of $\mathscr{L}$ is said to be a *canonical realization of terms* provided R is a realization in the set T of all terms (and in any semi-Boolean algebra) and if for every m-argument functor f in $\mathscr{L}$

(1) $$f_R(\tau_1, \dots, \tau_m) = f(\tau_1 \dots \tau_m),$$

where $\tau_1, \dots, \tau_m$ are any terms (on the left side of this equality f is an m-argument functor in $\mathscr{L}$ and on the right side f is an m-argument operation in T).

DEFINITION. By the *identity valuation* we shall understand the identity mapping from V into T, $i = \{u\}_{u\in V}$.

Let R^0 be the mapping defined on the set of all functors in $\mathscr{L}$ and on the set of all predicates in $\mathscr{L}$ such that

(i) R^0 restricted to the set of all functors in $\mathscr{L}$ is the canonical realization of terms in $\mathscr{L}$, i.e.

$$f_{R^0}\colon T^m \to T$$

is defined by (1) where $R = R^0$;

(ii) for every m-argument predicate p in $\mathscr{L}$ and for arbitrary terms $\tau_1, \dots, \tau_m$ in $\mathscr{L}$

(2) $$p_{R^0}(\tau_1, \dots, \tau_m) = h(\|p(\tau_1 \dots \tau_m)\|),$$

where h is a homomorphism preserving all the infinite joins and meets in (Q) (cf. p. 17 — we call it a *Q-homomorphism*) from $\mathfrak{A}(\mathcal{T})$ into a non-degenerate semi-Boolean algebra $\mathfrak{A}$.

2.1. *If R^0 is the canonical realization determined by a Q-homomorphism h from $\mathfrak{A}(\mathcal{T})$ into a semi-Boolean algebra, i.e. if conditions* (1) *and* (2) *are satisfied, then for every valuation v: $V \to T$ and for every formula a in $\mathcal{L}$*

$$a_{R^0}(v) = h(\|v'a\|),$$

where $v'a$ denotes the result of substitution v in a.

In particular, for the identity valuation i we have

$$a_{R^0}(i) = h(\|a\|).$$

This lemma follows immediately from the above definitions and 1.1. ∎

From this lemma and 1.1 we obtain

2.2. *If $\mathcal{T} = (\mathcal{L}, C, A)$ is a consistent H–B theory, then $\mathfrak{M}_a^0 = \langle \mathfrak{A}, T, R^0 \rangle$ is an a-model for $\mathcal{T}$, where R^0 is any canonical realization determined by a Q-homomorphism h from $\mathfrak{A}(\mathcal{T})$ into a non-degenerate semi-Boolean algebra $\mathfrak{A}$.*

If h is a Q-isomorphism, then for any formula a in $\mathcal{L}$

$$a \text{ is a theorem if and only if } a_{R^0}(i) = \bigvee,$$

where i is the identity valuation. ∎

3. Completeness theorems.

Completeness Theorem (second form). *An H–B theory is consistent if and only if it has an a-model.*

This theorem is part of the following theorem:

3.1. *For any H–B theory $\mathcal{T} = (\mathcal{L}, C, A)$ the following conditions are equivalent:*

(i) *$\mathcal{T}$ is consistent,*

(ii) *there exists an a-model for $\mathcal{T}$,*

(iii) *there exists an a-model $\mathfrak{M}_a = \langle \mathfrak{A}, D, R \rangle$ for $\mathcal{T}$ where $\mathfrak{A}$ is a complete semi-Boolean algebra,*

(iv) *there exists an a-model $\mathfrak{M}_a = \langle \mathfrak{A}, D, R \rangle$ for $\mathcal{T}$ where $\mathfrak{A}$ is an order topology.*

Proof. (i)$\Rightarrow$(ii): Suppose that $\mathcal{T}$ is consistent. Then the algebraic structure $\langle \mathfrak{A}, T, R^0 \rangle$ defined in 2.2 is an a-model for $\mathcal{T}$.

(ii)$\Rightarrow$(i): Suppose that $\mathcal{T}$ is inconsistent and $\mathfrak{M}_a = \langle \mathfrak{A}, D, R \rangle$ is an a-model for $\mathcal{T}$. Then both a and $\neg a$ are theorems of $\mathcal{T}$, i.e. $a_R(v) = \bigvee$

and $(\neg a)_R(v) = \mathsf{V}$ where $v \in D^V$ and V is the greatest element of $\mathfrak{A}$. By condtition (a_7) we infer that $\mathsf{V} = (\neg a)_R(v) = \neg(a_R(v)) = \neg\mathsf{V} = \Lambda$, a contradiction.

The remaining parts of this theorem follow from I, 1.3, 1.2 and 3.8. ∎

The following theorem is an immediate consequence of I, 1.2, 1.3, 3.8 and 1.1, and 2.2.

3.2. *For any consistent H–B theory* $\mathscr{T} = (\mathscr{L}, C, A)$ *and formula* $a \in \mathscr{L}$ *the following conditions are equivalent*:

(i) a *is a theorem of* $\mathscr{T}$,

(ii) a *is valid in every a-model for* $\mathscr{T}$,

(iii) a *is valid in every a-model* $\mathfrak{M}_a = (\mathfrak{A}, D, R)$ *for* $\mathscr{T}$ *where* $\mathfrak{A}$ *is a complete semi-Boolean algebra,*

(iv) a *is valid in every a-model* $\mathfrak{M}_a = (\mathfrak{A}, D, R)$ *for* $\mathscr{T}$ *where* $\mathfrak{A}$ *is an order topology.*

(v) a *is valid in* $\mathfrak{M}_a^0 = \langle\mathfrak{A}, T, R^0\rangle$, *where* $\mathfrak{M}_a^0$ *is defined as in* 2.2. ∎

If we assume in 3.2 the set A to be empty, then we obtain

COMPLETENESS THEOREM (first form, Gödel). *For any H–B predicate calculus* $\mathscr{S} = (\mathscr{L}, C)$ *and formula* $a \in \mathscr{L}$ *the following conditions are equivalent:*

(i) a *is a derivable formula,*

(ii) a *is a-valid in every a-model for* $\mathscr{S}$,

(iii) *for every realization* R *in any non-void set* D *and any complete semi-Boolean algebra the structure* $\langle\mathfrak{A}, D, R\rangle$ *is an a-model for* a,

(iv) *for every realization* R *in any non-void set* D *and any order topology* $\mathfrak{A}$ *the structure* $\langle\mathfrak{A}, D, R\rangle$ *is an a-model for* a,

(v) $a_{R^0}(i) = \mathsf{V}$ *for the canonical realization* R^0 *in* $\mathfrak{A}(\mathscr{S})$ *and the identity valuation* i. ∎

4. Deduction theorems.

DEDUCTION THEOREM. *Let* a *be a closed formula in* $\mathscr{L}$. *A formula* β *is a theorem in the H–B theory* $\mathscr{T}' = (\mathscr{L}, C, A \cup [a])$ *iff for some* n *the formula* $(Na \Rightarrow \beta)$ *is a theorem of* $\mathscr{T} = (\mathscr{L}, C, A)$, *where* N *means* n *times* $\neg\Gamma$.

Proof. If for some n $(Na \Rightarrow \beta) \in C(A)$ then $(Na \Rightarrow \beta)$ is a theorem of $\mathscr{T}'$ and by modus ponens and (r) β is a theorem of $\mathscr{T}'$.

Now let for some n $(Na \Rightarrow \beta) \notin C(A)$. Then $(a \Rightarrow \beta) \notin C(A)$ and by the second part of 2.2 we have that $(a \Rightarrow \beta)_{R^0}(i) \neq \mathsf{V}$, i.e. by 3.2(v), $(a \Rightarrow \beta)$ is not valid in $\mathfrak{M}_a^0 = \langle\mathfrak{A}, T, R^0\rangle$. We recall that R^0 is a canonical realization determined by a Q-homomorphism g from $\mathfrak{A}(\mathscr{T})$ to $\mathfrak{A}$. Hence

$$a_{R^0}(i) \Rightarrow \beta_{R^0}(i) \neq \mathsf{V}$$

and, since a is a closed formula, a_{R^0} is a constant function. By 2.1 we have

$$a_{R^0}(i) = g(\|a\|) \quad \text{and} \quad \beta_{R^0}(i) = g(\|\beta\|).$$

As the relation $g(\|a\|) \leqslant g(\|\beta\|)$ does not hold in $\mathfrak{A}$, hence by I,2.3 there exists a Q-filter ∇ such that

$$g(\|a\|) \in \nabla \quad \text{and} \quad g(\|\beta\|) \notin \nabla.$$

We consider the quotient algebra $\mathfrak{A}/\nabla$ and a natural homomorphism h from $\mathfrak{A}$ to $\mathfrak{A}/\nabla$, i.e.

$$h(a) = \|a\|.$$

Let $\gamma \in A$. Then $\gamma_{R^0}(i) = g(\|\gamma\|) = \mathsf{V}$. Thus $h(\gamma_{R^0}(i)) = h(g(\|\gamma\|)) = \mathsf{V}$, i.e. $\langle \mathfrak{A}, T, hR^0 \rangle$ is an a-model for $\mathscr{T}$. Moreover, it is an a-model for a since a_{hR^0} is identically equal to V (the unit element in $\mathfrak{A}/\nabla$). Thus $\langle \mathfrak{A}, T, hR^0 \rangle$ is an a-model for $\mathscr{T}'$. We observe that the formula β is not valid in this model since $\beta_{hR^0}(i) = h(\beta_{R^0}(i)) = hg(\|\beta\|) \neq \mathsf{V}$. So β is not a theorem in $\mathscr{T}'$. ∎

Let $\mathscr{T} = (\mathscr{L}, C, A)$ be a H–B theory and let F_0 be a set of formulas of $\mathscr{L}$. By $\mathscr{T}[F_0]$ we denote an H–B theory of the form $(\mathscr{L}, C, A \cup F_0)$.

REDUCTION THEOREM. *Let F_0 be a non-empty set of formulas of $\mathscr{L}$. A formula a is a theorem of $\mathscr{T}[F_0]$ iff there exist n and a theorem of $\mathscr{T}$ of the form $(N(a_1 \cap \ldots \cap a_n) \Rightarrow a)$ where each a_i is the closure of a formula in F_0 and N means n times $\neg\sqcap$.*

Proof. Suppose that $a \in C(A \cup F_0)$. We may assume that $a \notin C(A)$. Let $a_1', \ldots, a_n'$ be formulas from F_0 such that their closures $a_1, \ldots, a_n$ are used in the proof of a. Thus a is a theorem of $\mathscr{T}[\{a_1, \ldots, a_n\}]$, and by the deduction theorem we infer that the formula $(N(a_1 \cap \ldots \cap a_n) \Rightarrow a)$ is a theorem of $\mathscr{T}$. On the other hand, the proof is obvious. ∎

From this theorem we obtain

4.1. *A formula β is a theorem in an H–B theory $\mathscr{T} = (\mathscr{L}, C, A)$ with a non-empty set A of axioms iff there exists a conjunction a of closures of a finite numbers of axioms in A such that the implication $(Na \Rightarrow \beta)$ is derivable.* ∎

REDUCTION THEOREM FOR CONSISTENCY. *Let F_0 be a non-empty set of formulas of L. The theory $\mathscr{T}[F_0]$ is inconsistent iff there is a theorem of $\mathscr{T}$ which is a disjunction $\sqcap N$-negation of closures of distinct formulas in F_0.*

Proof. Suppose that such a theorem exists, i.e., for some $a_1', \ldots, a_n' \in F_0$, if $a_1, \ldots, a_n$ are their closures then

$$\vdash_{\mathscr{T}} (\sqcap N a_1 \cup \ldots \cup \sqcap N a_n).$$

Of course this formula is a theorem of $\mathscr{T}[F_0]$. Thus

$$\vdash_{\mathscr{T}[F_0]}(\ulcorner N\alpha_1 \cup \ldots \cup \ulcorner N\alpha_n) \quad \text{and} \quad \vdash_{\mathscr{T}[F_0]}(\alpha_1 \cap \ldots \cap \alpha_n)$$

which proves that every formula is a theorem of $\mathscr{T}[F_0]$.

Now suppose that $\mathscr{T}[F_0]$ is inconsistent. Then there exists a formula β such that β and $\neg\beta$ have the proofs in $\mathscr{T}[F_0]$. By the reduction theorem there are $\alpha_1', \ldots, \alpha_n'$ in F_0 such that, if $\alpha_1, \ldots, \alpha_n$ are their closures then

$$\vdash_{\mathscr{T}}(N(\alpha_1 \cap \ldots \cap \alpha_n) \Rightarrow (\beta \cap \neg\beta)).$$

Thus and by (A_{15}) we have

$$\vdash_{\mathscr{T}} \neg N(\alpha_1 \cap \ldots \cap \alpha_n)$$

which proves that

$$\vdash_{\mathscr{T}} \ulcorner N(\alpha_1 \cap \ldots \cap \alpha_n),$$

i.e.

$$\vdash_{\mathscr{T}}(\ulcorner N\alpha_1 \cup \ldots \cup \ulcorner N\alpha_n). \blacksquare$$

Sometimes we shall say that a set A is consistent (inconsistent) instead of saying that $\mathscr{T} = (\mathscr{L}, C, A)$ is consistent (inconsistent).

For any set A of formulas in $\mathscr{L}$ let the symbol V_{0A} denote the set of all $\|\alpha\| \in \mathfrak{A}(\mathscr{S})$ where α is in A and let

$$\|\alpha\| \in V_A \quad \text{iff} \quad \alpha \in C(A).$$

The next lemma follows immediately from the above definitions.

4.3. *The set* V_A *is a* $\bigcap$*-filter in* $A(\mathscr{S})$. *The set* V_{0A} $\bigcap$*-generates the* $\bigcap$*-filter* V_A. $\blacksquare$

4.4. (i) *An H–B theory* $\mathscr{T} = (\mathscr{L}, C, A)$ *is consistent if and only if the filter* $\bar{V}_A$ *is proper, where* $\bar{V}_A$ *denotes the set of all* $\|\alpha\|$ *such that* α *is a closed formula which is a theorem in* $\mathscr{T}$.

(ii) *A closed formula* α *is irrefutable in* $\mathscr{T}$, *i.e.* $\ulcorner\alpha$ *is not a theorem of* $\mathscr{T}$, *iff the theory H–B* $(\mathscr{L}, C, A \cup [\alpha])$ *is consistent.*

Proof. The first part follows from the definition $\bar{V}_A$. The second follows from (i) and from the fact that the filter generated by $\bar{V}_A$ and $\|\alpha\|$ is proper iff $\ulcorner\|\alpha\| \notin \bar{V}_A$. $\blacksquare$

5. Saturated H–B theories. Let A be a set of closed formulas of $\mathscr{L}$.

Definition. An H–B theory $\mathscr{T} = (\mathscr{L}, C, A)$ is said to be *saturated* provided it has the following properties:

(i) $\mathscr{T}$ is consistent,

(ii) for all $\alpha \in F$, if $\alpha \in C(A)$, then $\alpha \in A$,

(iii) for all $\alpha, \beta \in F$, if $(\alpha \cup \beta) \in A$, then $\alpha \in A$ or $\beta \in A$,

(iv) for all $x \in V$ and $\beta(x) \in F$, if $\exists \xi\, \beta(\xi) \in A$, then there exists a $\tau \in T$ such that $\beta(\tau) \in A$.

Sometimes we shall also say that A is a saturated set or a saturated set of the first kind (s.f.k.) instead of saying that $\mathscr{T}$ is saturated.

We observe that a "consistent theory $\mathscr{T}$ is saturated" means that it is prime (iii), rich (iv) and $C(A) = A$ (ii).

Let $\mathscr{L}$ and $\mathscr{L}'$ be two languages of the first order.

DEFINITION. Language $\mathscr{L}'$ is said to be an *extension* of $\mathscr{L}$ if and only if the set of free variables in $\mathscr{L}$, the set of all bound variables in $\mathscr{L}$, the sets of all m-argument functors in $\mathscr{L}$ ($m = 0, 1, \ldots$), and the sets of all m-argument predicates in $\mathscr{L}$ are all subsets of the corresponding sets of signs in $\mathscr{L}'$.

Let $\mathscr{L}'$ be an extension of $\mathscr{L}$.

DEFINITION. An H–B theory $\mathscr{T}' = (\mathscr{L}', C', A')$ is said to be an *extension* of an H–B theory $\mathscr{T} = (\mathscr{L}, C, A)$ provided that, for every formula a in $\mathscr{T}$, if a is a theorem of $\mathscr{T}$, then a is a theorem of $\mathscr{T}'$.

DEFINITION. An H–B theory $\mathscr{T}' = (\mathscr{L}', C', A')$ is said to be an *inessential extension* of an H–B theory $\mathscr{T} = (\mathscr{L}, C, A)$ provided that, for every formula a in $\mathscr{L}$, a is a theorem in $\mathscr{T}$ if and only if a is a theorem in $\mathscr{T}'$.

The aim of the first part of this section is to prove that every consistent H–B theory $\mathscr{T} = (\mathscr{L}, C, A)$ has a saturated inessential extension $\mathscr{T}' = (\mathscr{L}', C', A')$.

5.1. *Let A be a set of closed formulas and let $\mathscr{T} = (\mathscr{L}, C, A)$ be a consistent H–B theory. If $a \notin C(A)$, then there exists a saturated H–B theory $\mathscr{T}' = (\mathscr{L}', C', A')$ such that $A \subset A'$ and $a \notin A'$.*

Proof. Let the assumptions of 5.1 be satisfied. First we shall construct the language $\mathscr{L}'$. For this purpose we define by induction a sequence $\{L_n\}_{n \in N}$ of formalized languages $\mathscr{L}_n = \{A_n, T_n, F_n\}$, where A_n is the set of all signs in $\mathscr{L}_n$, T_n and F_n are the sets of all terms and formulas in $\mathscr{L}_n$, respectively, and we define a sequence of functions ψ^n ($n = 0, 1, \ldots$), such that

(1) $\mathscr{L}_0$ is identical with $\mathscr{L}$,

(2) $\mathscr{L}_{n+1}$ is an extension of $\mathscr{L}_n$ obtained by adding a set Ψ_n of functors, where Ψ_n is disjoint with the set A_n,

(3) Ψ^n is a one-to-one mapping from the set E_n onto Ψ_n, where E_n is the set of all existential formulas in $\mathscr{L}_n$ which do not occur in $\mathscr{L}_{n-1}$, such that if a is an existential formula in E_n with m free individual variables, then the image Ψ^n_a of a is an m-argument functor in Ψ_n.

Now, we take $\mathscr{L}' = \bigcup_n \mathscr{L}_n$. More exactly, the language $\mathscr{L}' = \{A', T', F'\}$ is obtained by adding the union Ψ of the sets

$$\Psi_0, \Psi_1, \Psi_2, \dots$$

to the set Φ of all functors in $\mathscr{L}$.

Denote by E' the set of all existential formulas in $\mathscr{L}'$, i.e. let E' be the union of disjoint sets E_n. Now, by (3) we have that all mappings Ψ^n determine together a one-to-one mapping ψ from E' onto Ψ such that

(4) ψ maps E_n onto Ψ_n,

(5) if a is an existential formula in $\mathscr{L}'$ with m free individual variables, i.e. if a is of the form

$$\exists \xi\, \beta(\xi, u_1, \dots, u_m),$$

then the image ψ_a of a is an m-argument functor in Ψ which does not occur in a.

Thus the language $\mathscr{L}'$ described above is an extension of $\mathscr{L}$.

Now, we must find the required set A'. Since $a \notin C(A)$ then by 1.1 $\|a\| \neq \mathsf{V}$. On account of I, 2.3 there exists a Q-filter ∇ in $\mathfrak{A}(\mathscr{T})$ such that $\|a\| \notin \nabla$. Let

$$A' = \{\beta \in L' \colon \|\beta\| \in \nabla \text{ and } \beta \text{ is a closed formula}\}.$$

It is obvious that $a \notin A'$ and $A \subset A'$.

We shall show that the theory $\mathscr{T}' = (\mathscr{L}', C', A')$ is the required saturated H–B theory.

(i) $\mathscr{T}'$ is consistent. Indeed, we observe that $a \notin C'(A')$. If $a \in C'(A')$ then by 4.1 there exists an η $(= \eta_1 \cap \dots \cap \eta_k,\ \eta_i \in A')$ such that $\vdash (N\eta \Rightarrow a)$. Since $\eta \in A'$, we have $N\|\eta\| \in \nabla$, which proves that $\|a\| \in \nabla$, a contradiction.

(ii) If $\gamma \in C'(A')$, then $\neg\ulcorner\gamma \in C'(A')$. By 4.1 there exist η, η', n such that $N\|\eta\| \leqslant \|\gamma\|$ and $(N+1)\|\eta'\| \leqslant \neg\ulcorner\|\gamma\|$. Because of this we have that $\|\gamma\| \in \nabla$ and $\neg\ulcorner\|\gamma\| \in \nabla$, which proves that $\gamma \in A'$.

As ∇ is a $\bigcup$-filter, we infer that $\mathscr{T}'$ is prime and rich, which completes the proof that $\mathscr{T}'$ is saturated. ∎

We observe that

5.2. *If $\mathscr{T} = (\mathscr{L}, C, A)$ is a saturated H–B theory, then ∇_A is a Q-filter.* ∎

DEFINITION. An H–B predicate theory $\mathscr{T} = (\mathscr{L}, C, A)$ is said to be *complete* if, for every closed formula a, exactly one of the formulas a and $\ulcorner a$ is a theorem.

5.3. *Let $\mathscr{T} = (\mathscr{L}, C, A)$ be a saturated H–B theory. Then $\mathscr{T}$ is complete.*

Proof. It follows by appropriate definitions. ■

DEFINITION. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be H–B theories. The *union* of $\mathscr{T}_1$ and $\mathscr{T}_2$, designated by $\mathscr{T}_1 \cup \mathscr{T}_2$, is a theory whose nonlogical symbols are the nonlogical symbols of $\mathscr{T}_1$ and the nonlogical symbols of $\mathscr{T}_2$, and whose nonlogical axioms are the nonlogical axioms of $\mathscr{T}_1$ and nonlogical axioms of $\mathscr{T}_2$.

5.4. *Let $\mathscr{T}_1 = (\mathscr{L}, C, A_1)$ and $\mathscr{T}_2 = (\mathscr{L}, C, A_2)$ be two consistent H–B theories and let $\mathscr{T} = (\mathscr{L}, C, A)$ be a complete H–B theory such that $A \subset A_1$ and $A \subset A_2$. Then the theory $\mathscr{T}_1 \cup \mathscr{T}_2$ is consistent.*

Proof. Suppose that $\mathscr{T}_1 \cup \mathscr{T}_2$ is not consistent. Then we infer by 4.1 that there must exist two formulas α' and β' such that $\alpha' \in A_1$ and $\beta' \in A_2$ and

$$\vdash \neg N(\alpha \cap \beta),$$

where α is the closure of α' and β is the closure of β'.

Thus, by the appropriate axioms and 1.2 (6) we obtain

$$\vdash (N\alpha \Rightarrow \neg N\beta).$$

On account of 1.2 (11) we infer that

$$\vdash (\ulcorner N\alpha \cup N\alpha).$$

Now, we observe that the formula $\ulcorner N\alpha$ cannot be a theorem of $\mathscr{T}$ because $\mathscr{T}_1$ is consistent. By our assumptions $\mathscr{T}$ is complete, and so we conclude that $N\alpha$ is a theorem of $\mathscr{T}$. Hence $N\beta$ is a theorem of $\mathscr{T}_2$ and $\mathscr{T}_2$ is not consistent, a contradiction. ■

6. Craig interpolation lemma. In this section, using the lemmas from § 5, we prove a modification of the Craig interpolation lemma. We show that this lemma is equivalent to the so-called Robinson consistency theorem.

First we formulate both theorems.

CRAIG INTERPOLATION LEMMA (CIL). *Let (L, C) be a consistent H–B predicate calculus and let α, β be two sentences such that $\vdash (\alpha \Rightarrow \beta)$. Then there exist a sentence γ and n such that*

$$\vdash (N\alpha \Rightarrow \gamma), \qquad \vdash (N\gamma \Rightarrow \neg (N-1) \neg \beta)$$

and every relation and function or constant symbol which occurs in γ occurs both in α and in β. (The identity symbol is allowed to occur in γ.)

ROBINSON CONSISTENCY THEOREM. *Let $\mathscr{T}_1 = (\mathscr{L}_1, C_1, A_1)$ and $\mathscr{T}_2 = (\mathscr{L}_2, C_2, A_2)$ be two consistent H–B theories. Let $\mathscr{T} = (\mathscr{L}, C, A)$ be a complete H–B theory such that $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$ and $A \subset A_1$ and $A \subset A_2$. Then the theory $\mathscr{T}_1 \cup \mathscr{T}_2$ is consistent.*

Using the Robinson consistency theorem, we prove CIL.

Proof of CIL. Let $\mathscr{L}_\alpha$ be a language containing only those relation, constant or function symbols of $\mathscr{L}$ which occur in α. Define $\mathscr{L}_\beta$ similarly. We assume that $\mathscr{L}_\alpha \cap \mathscr{L}_\beta \neq \emptyset$ (otherwise the proof of CIL is obvious) and $\beta \notin \mathscr{L}_\alpha$, $\alpha \notin \mathscr{L}_\beta$ (otherwise the proof of CIL is obvious). Let us put

$$(*) \qquad A_0 = \{\gamma \in \mathscr{L}_\alpha \cap \mathscr{L}_\beta \colon \gamma \text{ is a sentence and } \vdash (N\alpha \Rightarrow \gamma)\}.$$

If there are a $\gamma \in A_0$ and n such that $\vdash \big(N\gamma \Rightarrow \urcorner (N-1)\urcorner\beta\big)$, then this theorem is proved. Thus, suppose for every $\gamma \in A_0$ and for every n

$$\text{non} \vdash \big(N\gamma \Rightarrow \urcorner (N-1)\urcorner\beta\big).$$

Denote $\mathscr{L}_\alpha \cap \mathscr{L}_\beta$ by $\mathscr{L}_{\alpha\cap\beta}$ and observe that $\mathscr{T}_{\ulcorner\beta} = \big(\mathscr{L}_{\alpha\cap\beta}, C, A_0 \cup \{\ulcorner\beta\}\big)$ is a consistent H–B theory. Otherwise, there exists a $\gamma \in A_0$ such that $\vdash \urcorner(N\gamma \cap N\ulcorner\beta)$. Hence we have $\vdash (N\gamma \Rightarrow \urcorner N\ulcorner\beta)$ and by 1.2 (7) $\vdash \big(N\gamma \Rightarrow \urcorner (N-1)\urcorner\beta\big)$, a contradiction. On account of 5.1 we can extend $\mathscr{T}_{\ulcorner\beta}$ to a saturated H–B theory $\mathscr{T}' = (\mathscr{L}', C', A')$, where $\mathscr{L}_{\alpha\cap\beta} \subset \mathscr{L}'$ and $A_0 \cup \{\ulcorner\beta\} \subset A'$. We take the theory $\mathscr{T}'' = (\mathscr{L}_{\alpha\cap\beta}, C'', A'')$, where $A'' = A'|_{\mathscr{L}_{\alpha\cap\beta}}$. We observe that $\mathscr{T}''$ is complete (as $\mathscr{T}'$ is saturated). Now, we consider the theory $\mathscr{T}''_\alpha = (\mathscr{L}_\alpha, C, A'' \cup \{\alpha\})$. This theory is consistent. Otherwise, there is a $\gamma \in A''$ such that $\vdash (N\alpha \Rightarrow \urcorner N\gamma)$. Hence $\urcorner N\gamma \in A_0 \subset A''$ and $\mathscr{T}''$ is not consistent, a contradiction.

In this way we have two theories $\mathscr{T}'$ and $\mathscr{T}''_\alpha$ both consistent such that $\mathscr{T}' \cap \mathscr{T}''_\alpha$ is complete. Using the Robinson consistency theorem, we find that $\mathscr{T}' \cup \mathscr{T}''_\alpha$ is consistent. We observe that β and $\ulcorner\beta$ are theorems of $\mathscr{T}' \cup \mathscr{T}''_\alpha$ which proves that this theory is not consistent, a contradiction. Thus there must exist a $\gamma \in A_0$ and n such that $\vdash \big(N\gamma \Rightarrow \urcorner(N-1)\urcorner\beta\big)$.

Now, we assume the CIL and prove the Robinson consistency theorem.

Proof of the Robinson consistency theorem. Suppose that $\mathscr{T}_1 \cup \mathscr{T}_2$ is not consistent. Then there are α, β and n such that α is a conjunction of closures of finite numbers of formulas $\alpha_1, \ldots, \alpha_n \in A_1$ and $\beta = \beta_1 \cap \ldots \cap \beta_m$, where $\beta_i' \in A_2$ and β_i is closure of β_i', and

$$\vdash \urcorner(N\alpha \cap N\beta).$$

Thus we infer that $\vdash (N\alpha \Rightarrow \urcorner N\beta)$. By the Craig interpolation lemma, there are a sentence γ and n' such that

$$\vdash (N'N\alpha \Rightarrow \gamma) \quad \text{and} \quad \vdash \big(N'\gamma \Rightarrow \urcorner(N'-1)\urcorner N\beta\big)$$

and every relation, function or constant symbol occurring in γ occurs both α and β. Consequently, γ is a sentence of $\mathscr{L}_1 \cap \mathscr{L}_2$. Because of this we infer that either $\gamma \in C(A)$ or $\ulcorner\gamma \in C(A)$. If $\gamma \in C(A)$ then $\urcorner(N'-1)\urcorner N\beta \in C(A)$ and $\urcorner(N'-1)\urcorner N\beta \in C(A_2)$ and $\mathscr{T}_2$ is not consistent. Now, suppose that $\ulcorner\gamma \in C(A)$ then because the condition $\vdash (N'N\alpha \Rightarrow \gamma)$ implies $\vdash (\ulcorner\gamma \Rightarrow \ulcorner N'N\alpha)$

we infer that $\ulcorner N'N\alpha \in C(A)$. So $\mathscr{T}_1$ is not consistent. Hence we obtain that $\mathscr{T}_1 \cup \mathscr{T}_2$ must be consistent. ■

As a corollary we obtain

6.1. *The Robinson consistency theorem is equivalent to the Craig interpolation lemma.* ■

7. Theory of falsity. In this part we want to present another line of approach to the H–B logic.

By the completeness theorem for H–B PC we have

7.1. *A formula α is derivable if and only if α is a-valid.* ■

Definition. A formula α is said to be *a-false* if for every realization R in any non-void set D and any semi-Boolean algebra $\mathfrak{A}$, $\alpha_R(v) = \wedge$, where $v \in D^V$

Important Observation (I.O.). If for any $\alpha, \beta \in F$ the formula $(\alpha \Rightarrow \beta)$ is derivable, then for every valuation v and any a-structure $\mathfrak{M}_a = \langle \mathfrak{A}, D, R \rangle$, $\alpha_R(v) \leqslant \beta_R(v)$. The last inequality says that the element $\alpha_R(v) \dot{-} \beta_R(v)$ is equal to the least element in every semi-Boolean algebra, i.e. $\alpha_R(v) \dot{-} \beta_R(v) = \wedge$. This means that the formula $(\alpha \div \beta)$ is a-false. Thus we have

7.2. *A formula $(\alpha \Rightarrow \beta)$ is a-valid iff the formula $(\alpha \dot{-} \beta)$ is a-false.* ■

7.3. *The formulas of the following forms are a-false*

(1) $\big(((\beta \div \gamma) \dot{-} (\alpha \dot{-} \gamma)) \dot{-} ((\beta \dot{-} \alpha) \dot{-} \gamma)\big)$,

(2) $\big(((\gamma \div \alpha) \div (\gamma \dot{-} \beta)) \div (\beta \dot{-} \alpha)\big)$,

(3) $((\alpha \cap \beta) \dot{-} \alpha)$,

(4) $((\alpha \cap \beta) \dot{-} \beta)$,

(5) $\Big(\big((\gamma \div (\alpha \cap \beta)) \dot{-} (\gamma \dot{-} \beta)\big) \dot{-} (\gamma \dot{-} \alpha)\Big)$,

(6) $(\alpha \dot{-} (\alpha \cup \beta))$,

(7) $(\beta \dot{-} (\alpha \cup \beta))$,

(8) $\Big(\big(((\alpha \cup \beta) \dot{-} \gamma) \dot{-} (\beta \dot{-} \gamma)\big) \dot{-} (\alpha \dot{-} \gamma)\Big)$,

(9) $\big((\gamma \dot{-} (\alpha \cup \beta)) \dot{-} ((\gamma \dot{-} \beta) \dot{-} \alpha)\big)$,

(10) $\big(((\gamma \dot{-} \beta) \dot{-} \alpha) \dot{-} (\gamma \dot{-} (\gamma \cup \beta))\big)$,

(11) $\big(((\beta \Rightarrow \alpha) \cap \beta) \dot{-} \alpha\big)$,

(12) $((\ulcorner \alpha \dot{-} \ulcorner \beta) \dot{-} (\beta \dot{-} \alpha))$,

(13) $\big(((\beta \cap \gamma) \Rightarrow \alpha) \dot{-} (\gamma \Rightarrow (\beta \Rightarrow \gamma))\big)$

(14) $(\neg(\beta \dot{-} \alpha) \dot{-} (\beta \Rightarrow \alpha))$,

(15) $((\alpha \dot{-} \beta) \dot{-} \ulcorner(\beta \Rightarrow \alpha))$,

(16) $(\ulcorner\alpha \dot{-} ((\gamma \Rightarrow \gamma) \dot{-} \alpha))$,

(17) $(((\gamma \Rightarrow \gamma) \dot{-} \alpha) \dot{-} \ulcorner\alpha)$,

(18) $(\neg\alpha \dot{-} (\alpha \Rightarrow (\gamma \dot{-} \alpha)))$,

(19) $((\alpha \Rightarrow (\gamma \dot{-} \gamma)) \dot{-} \neg\alpha)$.

where $\alpha, \beta, \gamma \in \mathscr{L}$.

We show how to prove that a formula of form (1) is a-false. The proofs in the case of the remaining formulas are similar. By definition, to prove that the formula of form (1) is false it suffices to show that, for every valuation v and every a-structure $\mathfrak{M}_a = \langle \mathfrak{A}, D, R\rangle$,

$$(\beta_R(v) \dot{-} \gamma_R(v)) \dot{-} (\alpha_R(v) \dot{-} \gamma_R(v)) \leqslant (\beta_R(v) \dot{-} \alpha_R(v)) \dot{-} \gamma_R(v).$$

Let $\mathfrak{A}$ be a semi-Boolean algebra and v a valuation. We denote $\alpha_R(v) = a$, $\beta_R(v) = b$, $\gamma_R(v) = c$. On account of the appropriate properties of a semi-Boolean algebra [16], we find that

$$b \leqslant a \cup b \cup c \quad \text{implies} \quad b \leqslant a \cup (b \dot{-} a) \cup c.$$

The last inequality implies that

$$b \leqslant c \cup (a \dot{-} c) \cup ((b \dot{-} a) \dot{-} c).$$

But this inequality is equivalent to

$$(b \dot{-} c) \dot{-} (a \dot{-} c) \leqslant (b \dot{-} a) \dot{-} c,$$

which proves that the formula of form (1) is a-false. ∎

It is not difficult to see that if the formulas $(\alpha \dot{-} \beta)$ and β are a-false, then so is the formula α, and if the formula $\ulcorner \neg\alpha$ is a-false then so is the formula α. We can write this assertion in the form of the following rules:

$$\mathrm{r}_1^*: \frac{(\alpha \dot{-} \beta), \beta}{\alpha}; \qquad \mathrm{r}_2^*: \frac{\ulcorner \neg\alpha}{\alpha}.$$

These rules we understand in the following way: if the top formulas are a-false, so is the bottom one.

Now we extend the set of rules by the special rules for the quantifiers symbols, namely:

$$\mathrm{r}_3^*: \frac{\alpha(x) \dot{-} \beta}{\exists \xi\, \alpha(\xi) \dot{-} \beta}; \qquad \mathrm{r}_4^*: \frac{\beta \dot{-} \alpha(x)}{\beta \dot{-} \forall \xi\, \alpha(\xi)},$$

where β contains no occurrence of x and $\alpha(x)$ contains no quantifier binding ξ

$$r_5^*: \frac{\exists \xi\, \alpha(\xi) \dot{-} \beta}{\alpha(x) \dot{-} \beta}; \quad r_6^*: \frac{\alpha \dot{-} \forall \xi\, \beta(\xi)}{\alpha \dot{-} \beta(x)}.$$

Let A be a set of formulas. We shall treat the formulas of A as false formulas.

DEFINITION. We say that a formula α is *formally rejected* from a set A of formulas if there exists a finite sequence $\alpha_1, \ldots, \alpha_n$ of formulas such that the following conditions are satisfied:

(i) $\alpha_n = \alpha$,

(ii) α_1 is one of the formulas of the form (1)–(19) from 7.3, or it is a formula in A,

(iii) for every $1 < j \leqslant n$ either α_j is one of the formulas of form (1)–(19), or α_j is in A, or α_j is obtained from α_i and from $\alpha_k = (\alpha_j \dot{-} \alpha_i)$, where $1 \leqslant i < j$ and $1 \leqslant k < j$ by the application of rules r_1^*–r_6^*.

If a formula α is formally rejected from a set A, we shall write $A \dashv \alpha$. If $A = \emptyset$, then we shall write $\dashv \alpha$ and read: "α is formally rejected". Sometimes in such a case we say that α is an *antitheorem.*

Let $(F, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$ be the algebra of formulas of the language $\mathscr{L}$. It is not difficult to check that the relation $\sim$ defined as follows:

$\alpha \sim \beta$ if and only if the formulas $(\alpha \dot{-} \beta)$ and $(\beta \dot{-} \alpha)$ are both formally rejected

is a congruence relation in the algebra $(F, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$.

Now we consider the quotient algebra $(F/\sim, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$. Elements in this algebra will be denoted by $|\alpha|$ for $\alpha \in F$. The relation $\leqslant$ defined as follows: for any fomulas $\alpha, \beta \in F$ $|\alpha| \leqslant |\beta|$ if and only if the formula $(\alpha \dot{-} \beta)$ is formally rejected is an ordering relation in $F/\sim$.

7.4. *The algebra $(F/\sim, \cup, \cap, \Rightarrow, \dot{-}, \ulcorner, \neg)$ is a semi-Boolean algebra. Moreover, for any formulas $\alpha, \beta \in F$,*

$$|\alpha| \cup |\beta| = |(\alpha \cup \beta)|, \quad |\alpha| \cap |\beta| = |\alpha \cap \beta|,$$

$$|\alpha| \Rightarrow |\beta| = |(\alpha \Rightarrow \beta)|, \quad |\alpha| \dot{-} |\beta| = |(\alpha \dot{-} \beta)|,$$

$$\neg|\alpha| = |\neg \alpha|, \quad \ulcorner|\alpha| = |\ulcorner \alpha|.$$

For every formula α,

$$|\alpha| = \Lambda \quad \textit{if and only if} \quad \alpha \textit{ is an antitheorem.}$$

The semi-Boolean algebra $(F/\sim, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$ is non-degenerate if and only if there exists a formula α such that α is not an antitheorem.

For every formula $\beta(x)$,

$$\bigcup_{t\in T} |\beta(\tau)| = |\exists \xi\, \beta(\xi)|,$$

$$\bigcap_{t\in T} |\beta(\tau)| = |\forall \xi\, \beta(\xi)|.$$

The proof of this theorem is dual to the proof of 2.1 and we leave it out. ■

By 7.4 and the appropriate definitions we have

7.5. *For every formula* a, a *is a-false if and only if it is an antitheorem.* ■

On account of 7.5 we can write the above mentioned rules in the following forms:

$$r_1 : \frac{\dashv (a \dot{-} \beta), \dashv \beta}{\dashv a}; \quad r_2 : \frac{\dashv \ulcorner\ \urcorner a}{\dashv a}; \ldots \text{etc.}$$

We can formulate some theorems which are parallel to the deduction theorems for H–B PC and 5.1. The proofs of these theorems are analogous to those mentioned above and will be omitted.

7.6. *A closed formula* β *is formally rejected from the set* $A \cup \{a\}$, *i.e.* $A \cup \{a\} \dashv \beta$, *if and only if the formula* $(\beta \dot{-} a)$ *is formally rejected from* A, *i.e.* $A \dashv (\beta \dot{-} a)$. ■

7.7. *Let* A *be a nonempty set of formulas.* $A \dashv \beta$ *if and only if there exists a formula* a *of the form* $\exists c_1 \ldots a_1 \cup \ldots \cup \exists c_n \ldots a_n$, *where* $a_i \in A$, *such that the formula* $(\beta \dot{-} a)$ *is formally rejected, i.e.* $\dashv (\beta \dot{-} a)$. ■

DEFINITION. Let A be a set of closed formulas of $\mathscr{L}$. We say that A is a *saturated set of the second kind* (s.s.k) if

(s_1) for some $a \in F$, $A \not\vdash a$,

(s_2) for all $a \in F$, if $A \dashv a$, then $a \in A$,

(s_3) for all $a, \beta \in F$, if $(a \cap \beta) \in A$ then $a \in A$ or $\beta \in A$,

(s_4) for all $\beta(x) \in F$, if $\forall \xi\, \beta(\xi) \in A$ then there exists a constant c such that $\beta(c) \in A$.

7.8. *Let* A *be a nonempty set of closed formulas of* $\mathscr{L}$ *and let* $A' \not\dashv a$. *Then there exist a language* $\mathscr{L}'$ *and an s.s.k. set* A' *of formulas of* $\mathscr{L}'$ *such that* $A \subset A'$ *and* $a \notin A'$. ■

8. Kripke-style models.

DEFINITION. A *frame*(2) $\mathscr{A}$ for $\mathscr{L}$ consists of:

(c_1) a non-empty set $|\mathscr{A}|$, called the *universe* of $\mathscr{A}$.

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(2) frames in the sense admitted in this paper are often called interpretations or classical structures by other authors.

The elements of $|\mathscr{A}|$ are called the *individuals* of $\mathscr{A}$

(c_2) for each n-ary function symbol f of $\mathscr{L}$, an n-ary function $f_{\mathscr{A}}$ from $|\mathscr{A}|^n$ to $|\mathscr{A}|$. In particular, for each constant c of $\mathscr{L}$, $c_{\mathscr{A}}$ is an individual of $\mathscr{A}$.

(c_3) for each n-ary predicate symbol p of $\mathscr{L}$ other than $=$, an n-ary predicate $p_{\mathscr{A}}$ in $|\mathscr{A}|$.

Sometimes, we shall denote by $F_{\mathscr{A}}$, $V_{\mathscr{A}}$, ... etc. the sets of all formulas and variables ..., respectively, of the language $\mathscr{L}$.

DEFINITION. By the *Kripke structure* for $\mathscr{L}$ (k-structure) we understand a triple

$$\mathfrak{M}_k = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R} \rangle$$

in which

(k_1) $\mathbf{R}$ is a reflexive and a transitive relation on $\mathbf{T}$,

(k_2) for every $t \in \mathbf{T}$, $\mathscr{A}_t$ is a frame such that if $t\,\mathbf{R}\,s$, for $t, s \in \mathbf{T}$, then $|\mathscr{A}_t| \subseteq |\mathscr{A}_s|$. For every $t \in \mathbf{T}$, $|\mathscr{A}_t|$ is said to be a *domain*. In $\mathscr{A}_t$, equality is interpreted by a congruence relation $\equiv_t$. If p and f are n-ary predicate and function symbols, respectively, $a_1, \ldots$ $\ldots, a_n \in \mathscr{A}$ and $t\,\mathbf{R}\,s$, then

$$p_{\mathscr{A}_t}(a_1, \ldots, a_n) \quad \text{implies} \quad p_{\mathscr{A}_s}(a_1, \ldots, a_n)$$

and

$$f_{\mathscr{A}_t}(a_1, \ldots, a_n) = f_{\mathscr{A}_s}(a_1, \ldots, a_n).$$

DEFINITION. By a *valuation* (*assigment*) v in a structure $\mathfrak{M}_k$ we understand a map from the set of all individual variables to $\bigcup_{t \in \mathbf{T}} \mathscr{A}_t$. Let $t \in \mathbf{T}$, τ be a term and let v be a valuation in $\mathfrak{M}_k$. We define $\tau[v]_t$, by recursion on the length as follows

$$x_i[v]_t = v(x_i),$$

$$f(\tau_1, \ldots, \tau_n)[v]_t = \begin{cases} f_{\mathscr{A}_t}(\tau_1[v]_t, \ldots, \tau_n[v]_t) & \text{if } \tau_i[v]_t \in \mathscr{A}_t \\ & \quad \text{for } i = 1, \ldots, n; \\ \tau_j[v]_t & \text{where } j \text{ is the least } i \text{ such that } \tau_i[v]_t \in \mathscr{A}_t \\ & \quad \text{otherwise.} \end{cases}$$

DEFINITION. The *truth value of a formula* α in $\mathfrak{M}_k$ at a point t under v is defined below by induction on the length of α. It is denoted by $[\alpha(v)]_t$. According to former denotations we ought to write $[\alpha[v]_t]_t$. However, to facilitate our denotations we will write $[\alpha(v)]_t$ (even sometimes we will omit v) instead of writing that $[\alpha[v]_t]_t$.

(1) $[p(x_1, \ldots, x_n)(v)]_t = \top$ iff $p(a_1, \ldots, a_n) \in |\mathscr{A}_t|^n$, where $a_i = v(x_i) \in |\mathscr{A}_t|$;

(2) $[(\alpha \cup \beta)]_t = \top$ if $[\alpha]_t = \top$ or $[\beta]_t = \top$,
$[(\alpha \cup \beta)]_t = \bot$ if $[\alpha]_t = \bot$ and $[\beta]_t = \bot$;

(3) $[(\alpha \cap \beta)]_t = \top$ if $[\alpha]_t = \top$ and $[\beta]_t = \top$,
$[(\alpha \cap \beta)]_t = \bot$ if $[\alpha]_t = \bot$ or $[\beta]_t = \bot$;

(4) $[(\alpha \Rightarrow \beta)]_t = \top$ if for all s such that $t \mathbf{R} s$, $[\beta]_s = \top$ if $[\alpha]_s = \top$,
$[(\alpha \Rightarrow \beta)]_t = \bot$ if there exists an s such that $t \mathbf{R} s$, $[\alpha]_s = \top$ and $[\beta]_s = \bot$;

(5) $[(\alpha \dot{-} \beta)]_t = \top$ if there exists an s such that $s \mathbf{R} t$, $[\alpha]_s = \top$ and $[\beta]_s = \bot$,
$[(\alpha \dot{-} \beta)]_t = \bot$ if for all s such that $s \mathbf{R} t$, $[\alpha]_s = \bot$ if $[\beta]_s = \bot$;

(6) $[\urcorner \alpha]_t = \top$ if for all s such that $t \mathbf{R} s$, $[\alpha]_s = \bot$,
$[\urcorner \alpha]_t = \bot$ if there exists an s such that $t \mathbf{R} s$ and $[\alpha]_s = \top$;

(7) $[\ulcorner \alpha]_t = \top$ if there exists an s such that $s \mathbf{R} t$ and $[\alpha]_s = \bot$,
$[\ulcorner \alpha]_t = \bot$ if for all s such that $s \mathbf{R} t$, $[\alpha]_s = \top$;

(8) $[\exists \xi\ \beta(\xi)]_t = \top$ if there exists a $c \in |\mathscr{A}_t|$ such that $[\beta(c)]_t = \top$,
$[\exists \xi\ \beta(\xi)]_t = \bot$ otherwise;

(9) $[\forall \xi \beta(\xi)]_t = \top$ if for all $c \in |\mathscr{A}_t|$, $[\beta(c)]_t = \top$,
$[\forall \xi \beta(\xi)]_t = \bot$ otherwise.

Sometimes we shall write $[\alpha(v)]_t^{\mathfrak{A}} = \top$ if we want to underline that this equality is satisfied in a k-structure $\mathfrak{A}$ at the valuation v.

The intuitive motivation of those conditions is the following. Let $\mathbf{T}$ be interpreted as a collection of states of our knowledge. Thus $t \in \mathbf{T}$ may be considered as a collection of physical facts known at a particular time. If we have enough information to prove a formula α at the point of time t, we say that "t forces α" and write $[\alpha]_t = \top$. If we lack such information, we say: "t rejects α" and write $[\alpha]_t = \bot$. If $[\alpha]_t = \top$, we can say that α has been verified at the point t; if $[\alpha]_t = \bot$ then α has not been verified at t, but it might be verified later.

So, the first part of condition (5) says: to assert $(\alpha \dot{-} \beta)$ at a point of time t we need only to know that there exists an earlier time s such that our information or state of knowledge at that time is sufficient to verify α and is not sufficient to verify β.

In the same way we interpret the first part of (7). To assert $\ulcorner \alpha$ at a point of time t we need to know that there exists an earlier point of time s such that our information about α is not sufficient to verify α at the point s.

Similarly, if $(\alpha \dot{-} \beta)$ was not verified at the point t, then $(\alpha \dot{-} \beta)$ could be verified at no point earlier than t. Thus to reject $(\alpha \dot{-} \beta)$ at a point t we need to know that at any earlier time s if the state of our knowledge is not sufficient to verify β then the state of our knowledge is not sufficient to verify α.

DEFINITION. We shall say that $\mathfrak{M}_k$ is a *Kripke-style model* (briefly k-model) *for a formula* α iff $[\alpha]_t^{\mathfrak{M}_k} = \top$, for every $t \in \mathbf{T}$. Sometimes we say that α is k-valid in $\mathfrak{M}_k$. A structure $\mathfrak{M}_k$ is a *Kripke-style model for a set* A of formulas in $\mathscr{L}$ provided $\mathfrak{M}_k$ is a k-model for every formula $\alpha \in A$. In this case we say also that $\mathfrak{M}_k$ is a *k-model for an H–B theory* $\mathscr{T} = (\mathscr{L}, C, A)$. A formula α is said to be *k-valid* provided α is valid in every k-structure.

DEFINITION. We shall say that a k-structure has *constant domains* if all frames $\mathscr{A}_t$, $t \in \mathbf{T}$ have the same universe.

We observe that if $\mathfrak{M}_k$ is a k-structure with constant domains then for every formula

$$(\mathrm{R}_1) \quad \begin{array}{l} \text{If } [\alpha]_t^{\mathfrak{M}_k} = \top \text{ and } t\,\mathbf{R}\,s, \text{ then } [\alpha]_s^{\mathfrak{M}_k} = \top, \\ \text{If } [\alpha]_t^{\mathfrak{M}_k} = \bot \text{ and } s\,\mathbf{R}\,t, \text{ then } [\alpha]_s^{\mathfrak{M}_k} = \bot. \end{array}$$

The simple proof of this fact is omitted.

Now, we give an example of such a structure. Let $\mathbf{T}$ be the set of sequence of formulas of the form $(e, \varphi_1, \ldots, \varphi_n)$, where $e = (\alpha \Rightarrow \alpha)$ and $\vdash(\varphi_{i+1} \Rightarrow \varphi_i)$, $1 \leqslant i \leqslant n$. We recall that $\vdash$ means here being derivable in H–B PC. We assume that $(e) \in \mathbf{T}$. We define quasi-ordering relation $\leqslant$ on $\mathbf{T}$ as follows: $(e, \varphi_1, \ldots, \varphi_k) \leqslant (e, \varphi_1, \ldots, \varphi_n)$ holds either if $k < n$, then $(e, \varphi_1, \ldots, \varphi_k)$ is an initial segment of $(e, \varphi_1, \ldots, \varphi_n)$ or if $k = n$, then $(e, \varphi_1, \ldots, \varphi_{k-1})$ is an initial segment of $(e, \varphi_1, \ldots, \varphi_n)$ and $\vdash(\varphi_n \Rightarrow \varphi_k)$. For every $(e, \ldots, \varphi) \in \mathbf{T}$ we put $\mathscr{A}_{(e,\ldots,\varphi)}$ = set of all terms of the language $\mathscr{L}$. It may easily verify that this construction yields a well defined k-structure with base point (e).

Now, let us set

$$[p(a_1, \ldots, a_n)]_{(e,\ldots,\varphi)} = \top \quad \text{if and only if} \quad \vdash\big(\varphi \Rightarrow p(a_1, \ldots, a_n)\big)$$

and

$$[p(a_1, \ldots, a_n)]_{(e,\ldots,\varphi)} = \bot \quad \text{if and only if} \quad \dashv\big(p(a_1, \ldots, a_n) \dot{-} \varphi\big),$$

for every predicate symbol p of the language $\mathscr{L}$, where $\dashv$ means "formally rejected".

It has been proved in [6] that

8.1. *For any formula* γ *without* $\cup$, $\exists$, $\dot{-}$, $\ulcorner$ *we have*

$$[\gamma]_{(e,\ldots,\varphi)} = \top \quad \textit{if and only if} \quad \vdash(\varphi \Rightarrow \gamma). \ \blacksquare$$

Now we prove that

8.2. *For any* γ *without* $\cap, \forall, \Rightarrow, \urcorner$ *we have*

$$[\gamma]_{(e,\ldots,\varphi)} = \bot \quad \textit{if and only if} \quad \dashv(\gamma \dot{-} \varphi).$$

Proof. The proof is by induction on the length of γ. For atomic γ this holds by the definition. Suppose that this theorem is true for α and β and we shall prove it for $\gamma = (\alpha \cup \beta)$, $\gamma = (\alpha \dot{-} \beta)$, $\gamma = \ulcorner\alpha$, $\gamma = \exists\xi\, \alpha(\xi)$.

Case 1. Suppose that $\dashv((\alpha\cup\beta)\dot{-}\varphi)$. By 7.5, I.O., and the completeness theorem we have $\vdash((\alpha\cup\beta)\Rightarrow\varphi)$, which is equal to the following conditions: $\vdash(\alpha\Rightarrow\varphi)$ and $\vdash(\beta\Rightarrow\varphi)$. Thus these conditions are equivalent to $\dashv(\alpha\dot{-}\varphi)$ and $\dashv(\beta\dot{-}\varphi)$, which gives $[\alpha]_{(e,\ldots,\varphi)} = \bot$ and $[\beta]_{(e,\ldots,\varphi)} = \bot$, i.e. by (2) $[(\alpha\cup\beta)]_{(e,\ldots,\varphi)} = \bot$.

Case 2. We want to prove that

$$[(\alpha\dot{-}\beta)]_{(e,\ldots,\varphi)} = \bot \quad \text{if and only if} \quad \dashv((\alpha\dot{-}\beta)\dot{-}\varphi).$$

Suppose that $[(\alpha\dot{-}\beta)]_{(e,\ldots,\varphi)} = \bot$ and this lemma is true for α and β. By the definition of the truth value of $(\alpha\dot{-}\beta)$ we have that, for every sequence $(e,\ldots,\varphi')$ such that $(e,\ldots,\varphi') \leqslant (e,\ldots,\varphi)$, if $\dashv(\beta\dot{-}\varphi)$ then $\dashv(\alpha\dot{-}\varphi)$. We observe that $(e,\ldots,(\varphi\cup\beta)) \leqslant (e,\ldots,\varphi)$. Indeed, the formula $(\varphi\Rightarrow(\varphi\cup\beta))$ is provable in H–B PC. Thus and by (2), r_1^* § 7 we conclude that $\dashv(\varphi\dot{-}(\varphi\cup\beta))$ and $\dashv(\beta\dot{-}(\varphi\cup\beta))$, which proves that $\dashv(\alpha\dot{-}(\varphi\cup\beta))$. On account of 7.3 and r_1^* from § 7 we infer that $\dashv((\alpha\dot{-}\beta)\dot{-}\varphi)$.

Now let $\dashv((\alpha\dot{-}\beta)\dot{-}\varphi)$ and, for every sequence $(e,\ldots,\varphi')$ such that $(e,\ldots,\varphi') \leqslant (e,\ldots,\varphi)$, let $\dashv(\beta\dot{-}\varphi')$. Then, we have $\dashv(\varphi\dot{-}\varphi')$. Thus, by 7.3 (2) and r_1^* from § 7 we infer that $\dashv((\alpha\dot{-}\beta)\dot{-}\varphi')$ and by condition (1) of the same lemma and r_1^* we conclude that $\dashv(\alpha\dot{-}\varphi')$.

Case 3. The proof for $\gamma = \ulcorner\alpha$ is similar.

Case 4. Let $\gamma = \exists\xi\,\alpha(\xi)$. Suppose $[\exists\xi\,\alpha(\xi)]_{(e,\ldots,\varphi)} = \bot$. By definition, for every c, $[\alpha(c)]_{(e,\ldots,\varphi)} = \bot$. This implies, by the induction hypothesis, that $\dashv(\alpha(c)\dot{-}\varphi)$. Let $\dashv(\exists\xi\,\alpha(\xi)\dot{-}\varphi)$. Since $\dashv(\alpha(c)\dot{-}\exists\xi\,\alpha(\xi))$ (for every c), we infer that, for every c, $\dashv(\alpha(c)\dot{-}\varphi)$, which proves case 4. ∎

8.3. *Let* φ *be a formula from* $\mathscr{L}$. *For every formula* γ *which is not equivalent to* φ *such that either* $\vdash(\varphi\Rightarrow\gamma)$ *or* $\vdash(\gamma\Rightarrow\varphi)$,

$$[\gamma]_{(e,\ldots,\varphi)} = \top \quad \textit{if and only if} \quad \vdash(\varphi\Rightarrow\gamma).$$

Proof. The proof by induction on the length of γ. On account of 8.1, to prove this lemma it suffices to check only the cases where γ is $(\alpha\cup\beta)$, $(\alpha\dot{-}\beta)$, $\ulcorner\alpha$ and $\exists\xi\,\alpha(\xi)$.

For the proof, we may suppose that if $\vdash(\varphi\Rightarrow\gamma)$ then $\not\vdash(\gamma\Rightarrow\varphi)$. Indeed, suppose that $\vdash(\varphi\Rightarrow\gamma)$ and $\vdash(\gamma\Rightarrow\varphi)$ then γ is equivalent to φ, contrary to our assumption.

Now let γ be one of the formulas of the above mentioned forms.

Case 1. $\gamma=(\alpha\cup\beta)$. If $[(\alpha\cup\beta)]_{(e,\ldots,\varphi)}=\top$ then it is obvious that $\vdash(\varphi\Rightarrow(\alpha\cup\beta))$. Suppose that $\vdash(\varphi\Rightarrow(\alpha\cup\beta))$. Then we infer $\not\vdash((\alpha\cup\beta)\Rightarrow\varphi)$. On account of I.O. we have $\not\dashv((\alpha\cup\beta)\dot{-}\varphi)$, but this implies by 8.2 that $[\alpha]_{(e,\ldots,\varphi)}\neq\bot$ or $[\beta]_{(e,\ldots,\varphi)}\neq\bot$, i.e. $[\alpha]_{(e,\ldots,\varphi)}=\top$ or $[\beta]_{(e,\ldots,\varphi)}=\top$.

Case 2. $\gamma=(\alpha\dot{-}\beta)$. Let $\vdash(\varphi\Rightarrow(\alpha\dot{-}\beta))$. Then we have $\not\vdash((\alpha\dot{-}\beta)\Rightarrow\varphi)$ and $\not\dashv((\alpha\dot{-}\beta)\dot{-}\varphi)$. Using 8.2, we obtain that there exists a sequence $(e,\ldots,\varphi')$ such that $(e,\ldots,\varphi')\leqslant(e,\ldots,\varphi)$ and $[\alpha]_{(e,\ldots,\varphi')}\neq\bot$ and $[\beta]_{(e,\ldots,\varphi')}=\bot$, i.e. there exists a sequence $(e,\ldots,\varphi')$ such that $(e,\ldots,\varphi')\leqslant(e,\ldots,\varphi)$ and $[\alpha]_{(e,\ldots,\varphi')}=\top$ and $[\beta]_{(e,\ldots,\varphi')}=\bot$, which proves that $[(\alpha\dot{-}\beta)]_{(e,\ldots,\varphi)}=\top$.

Suppose that $[(\alpha\dot{-}\beta)]_{(e,\ldots,\varphi)}=\top$, i.e. there exists a sequence $(e,\ldots,\varphi')$ such that $(e,\ldots,\varphi')\leqslant(e,\ldots,\varphi)$ and $[\alpha]_{(e,\ldots,\varphi')}=\top$ and $[\beta]_{(e,\ldots,\varphi')}=\bot$. The condition $[\alpha]_{(e,\ldots,\varphi')}=\top$ implies $\dashv(\varphi'\dot{-}(\alpha\cup\beta))$ and this gives $\dashv(\varphi'\dot{-}(\beta\cup(\alpha\dot{-}\beta)))$. On account of case 1 we have $\dashv(\varphi'\dot{-}(\alpha\dot{-}\beta))$ or $\dashv(\varphi'\dot{-}\beta)$. We observe that the truth of the second condition means that $[\beta]_{(e,\ldots,\varphi')}=\top$ but this is a contradiction of our assumption. Thus we have $\dashv(\varphi'\dot{-}(\alpha\dot{-}\beta))$ and moreover $\dashv(\varphi\dot{-}\varphi')$. By 7.3 (2) and r_1^* from § 7 we obtain $\dashv(\varphi\dot{-}(\alpha\dot{-}\beta))$, i.e. $\vdash(\varphi\Rightarrow(\alpha\dot{-}\beta))$, which completes the proof of case 2.

The proofs of the remaining cases are similar. ■

8.4. *Let α be a nonderivable sentence of H–B PC. Then there exists a k-structure with constant domains in which α is not k-valid.*

Suppose that $\not\vdash\alpha$ but α is k-valid in every k-structure with constant domains. Thus, α is k-valid in the k-structure defined above. By the definitions and 8.3 we infer that

$$[\alpha]_{(e)}=\top \quad \text{if and only if} \quad \vdash(e\Rightarrow\alpha),$$

which proves that $\vdash\alpha$, a contradiction. ■

The converse theorem 8.4 is also true and it will be proved in § 10.

9. Canonical structures. Now we give another example of a k-structure with constant domains which shows a certain kind of duality of the H–B logic.

Let $\mathscr{A}$ be a frame for $\mathscr{L}$. Let $F_{\mathscr{A}}$ be a set of all formulas of $\mathscr{L}$.

DEFINITION. By an *$\mathscr{A}$-saturated set of the first kind* (*$\mathscr{A}$-s.f.k*) we mean any subset A of $F_{\mathscr{A}}$ which is s.f.k. (cf. the definition from § 5).

DEFINITION. By an *$\mathscr{A}$-saturated set of the second kind* (*$\mathscr{A}$-s.s.k.*) we mean any subset A of $F_{\mathscr{A}}$ which is s.s.k. (cf. the definition from § 7).

We denote by T the class of all sets which are $\mathscr{A}$-s.f.k. and by $\tilde{T}$ the class of all sets which are $\mathscr{A}$-s.s.k.

9.1. *For any* $A \subset F_{\mathscr{A}}$, $A \in T$ *if and only if* $F_{\mathscr{A}} - A \in \tilde{T}$.

Proof. Let A be an $\mathscr{A}$-s.f.k. and denote $F_{\mathscr{A}} - A$ by $\tilde{A}$. We shall prove that $\tilde{A}$ is $\mathscr{A}$-s.s.k. Suppose that for every $\eta \in F_{\mathscr{A}}$, $\tilde{A} \dashv \eta$. Let $\eta = (\gamma \Rightarrow \gamma)$. By 7.7 we have $\dashv ((\gamma \Rightarrow \gamma) \dot{-} \beta)$ where $\beta = \beta_1 \cup \ldots \cup \beta_n$, $\beta_i \in \tilde{A}$. On account of 7.4 we infer that $|(\gamma \Rightarrow \gamma)| \leqslant |\beta|$, i.e. $|((\gamma \Rightarrow \gamma) \Rightarrow \beta)| = \mathsf{V}$ and this proves $\vdash \beta$. So by our assumption that A is $\mathscr{A}$-s.f.k. we obtain $\beta_i \in A$ for i, a contradiction, which gives (s_1). On the other direction we proceed in the same way. The remaining cases we prove by an easy verification. ■

Let $\mathscr{A}_0$ be a frame for the language $\mathscr{L}_0$. Let $C_1, C_2, \ldots$ be a sequence of pairwise disjoint countable sets of constants. Define $\mathscr{L}_{i+1} = \mathscr{L}_i + C_{i+1}$, where $\mathscr{L} + C$ is the language obtained from $\mathscr{L}$ by adding all the constants of C. Let $\mathscr{A}_{i+1}$ be a frame for $\mathscr{L}_{i+1}$. Now let us denote by $\tilde{T}$ the class of all B such that, for some i, B is $\mathscr{A}_i$-s.s.k.

9.2. *For all* $B \in \tilde{T}$, $(\alpha \dot{-} \beta) \in B$ *if and only if for all* $B' \in \tilde{T}$ *such that* $B \subset B'$ *if* $\beta \in B'$ *then* $\alpha \in B'$.

Proof. Suppose that B is $\mathscr{A}_i$-s.s.k. and $(\alpha \dot{-} \beta) \notin B$. Then by 7.6 we have $B \cup \{\beta\} \not\dashv \alpha$. On account of 7.8 there exists a B' which is $\mathscr{A}_{i+1}$-s.s.k. and such that $\beta \in B'$ and $B \subset B'$ and $B' \not\dashv \alpha$. On the other hand the proof is obvious. ■

Now, we consider $\mathbf{T}$ as the class of all B which are $\mathscr{A}_i$-s.f.k. for some i.

9.3. $\mathbf{T}$ *has the following properties*:

(1) *for all* $B \in \mathbf{T}$, $(\alpha \cup \beta) \in B$ *if and only if* $\alpha \in B$ *or* $\beta \in B$,

(2) *for all* $B \in \mathbf{T}$, $(\alpha \cap \beta) \in B$ *if and only if* $\alpha \in B$ *and* $\beta \in B$,

(3) *for all* $B \in \mathbf{T}$, $(\alpha \Rightarrow \beta) \in B$ *if and only if, for all* $B' \in \mathbf{T}$ *such that* $B \subset B'$, *if* $\alpha \in B'$ *then* $\beta \in B'$,

(4) *for all* $B \in \mathbf{T}$, $\neg \alpha \in B$ *if and only if, for all* $B' \in \mathbf{T}$ *such that* $B \subset B'$, $\alpha \notin B'$,

(5) *for all* $B \in \mathbf{T}$, $(\alpha \dot{-} \beta) \in B$ *if and only if there exists a* $B' \in \mathbf{T}$ *such that* $B' \subset B$ *and* $\alpha \in B'$ *and* $\beta \notin B'$,

(6) *for all* $B \in \mathbf{T}$, $\ulcorner \alpha \in B$ *if and only if there exists a* $B' \in \mathbf{T}$ *such that* $B' \subset B$ *and* $\alpha \notin B'$,

(7) *for all* $B \in \mathbf{T}$, $\forall \xi\, \beta(\xi) \in B$ *if and only if, for all* $c \in C_{\mathscr{A}_i}$, $\beta(c) \in B$ (B *is* $\mathscr{A}_i$*-s.f.k.*),

(8) *for all* $B \in \mathbf{T}$, $\exists \xi\, \beta(\xi) \in B$ *if and only if there exists a* $c \in C_{\mathscr{A}_i}$ *such that* $\beta(c) \in B$ (B *is* $\mathscr{A}_i$*-s.f.k.*).

Proof. Conditions (1) and (2) are obvious. The proofs of (3) and (4) are given in [7]. (5) and (6) follow from 9.2 and 9.1. We prove (7). If $\forall \xi\, \beta(\xi) \in B$ then for all $c \in C_{\mathscr{A}_i}$, $\beta(c) \in B$. Suppose that $\forall \xi\, \beta(\xi) \notin B$.

Then $\forall\xi\beta(\xi)\in\tilde{B}$, where $\tilde{B} = F_{\mathscr{A}_i} - B$ and $\tilde{B}$ is an $\mathscr{A}_i$-s.s.k. Thus there exists a $c \in C_{\mathscr{A}_i}$ such that $\beta(c) \in \tilde{B}$. By 9.1 we infer that for some $c \in C_{\mathscr{A}_i}$, $\beta(c) \notin \tilde{B}$. The proof of (8) is similar. ∎

Now let $\mathbf{R}$ be the set-theoretical inclusion on $\mathbf{T}$. Put $\mathscr{A}_B = \bigcup_i (V_{\mathscr{A}_i} \cup C_{\mathscr{A}_i})$, for every $B \in \mathbf{T}$. It is clear that $\mathfrak{M}_k = \langle \mathscr{A}_B, \mathbf{T}, \mathbf{R}\rangle$, thus defined, is a k-structure.

DEFINITION. By a *canonical structure* we understand the structure $\langle \mathscr{A}_B, \mathbf{T}, \mathbf{R}\rangle$ defined above.

Now we define:

for every 0-ary predicate symbol p,

$$[p]_B = \top \quad \text{if} \quad p \in B,$$
$$[p]_B = \bot \quad \text{if} \quad p \notin B;$$

for every n-ary predicate symbol p $(n \geqslant 1)$,

$$[p(x_1, \ldots, x_n)(v)]_B = \top \quad \text{if} \quad p(a_1, \ldots, a_n) \in B,$$

where $a_i = v(x_i)$, $a_i \in \mathscr{A}_B$ for every $i = 1, \ldots, n$.

9.4. *Let $\mathscr{T} = (\mathscr{L}, C, A)$ be a saturated H–B theory. Then there exists a k-structure $\mathfrak{M}_k = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R}\rangle$ and a truth value of $a \in \mathscr{L}$ in $\mathfrak{M}_k$ under v such that for a certain $t_0 \in \mathbf{T}$*

$$a \in A \quad \textit{iff} \quad [a(v)]_{t_0} = \top$$

Proof. Let $\mathfrak{M}_k = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R}\rangle$ be a canonical structure. We recall that $\mathbf{T}$ is the set of all saturated sets. So $A \in \mathbf{T}$. We prove that

$$a \in A \quad \text{iff} \quad [a]_A = \top$$

We shall present only the case where a has one of the forms $\neg\beta$, $(\beta \dot{-} \gamma)$ and $\forall\xi\ a(\xi)$.

Case 1. a is $\neg\beta$ and 9.4 holds for β. By 9.3 (4) $\neg\beta \in A$ if and only if for all $A' \in \mathbf{T}$ such that $A \subset A'$, $\beta \notin A'$. By the induction hypothesis this takes place if and only if $[\beta]_{A'} = \bot$, for all $A \in \mathbf{T}$ such that $A \subset A'$. On account of the definition we have $[\neg\beta]_A = \top$.

Case 2. a is $(\beta \dot{-} \gamma)$ and 9.4 holds for β and γ. By 9.3 (5) $(\beta \dot{-} \gamma) \in A$ if and only if there exists an $A' \in \mathbf{T}$ such that $A' \subset A$, $\beta \in A'$ and $\gamma \notin A'$. By the hypothesis of induction this holds if and only if $[(\beta \dot{-} \gamma)]_A = \top$.

Case 3. a is $\forall\xi\,\beta(\xi)$. By 9.3 (7) $\forall\xi\,\beta(\xi) \in A$ if and only if for all $c \in \mathscr{A}_A$, $\beta(c) \in A$, i.e. $[\forall\xi\,\beta(\xi)]_A = \top$. ∎

9.5. *Let $\mathscr{T} = (\mathscr{L}, C, A)$ be a consistent H–B theory such that A is a closed set of formulas. Then there exists a k-structure of $\mathscr{L}$ and a truth* *value of* $a \in \mathscr{L}$ *in* $\mathfrak{M}_k$ *under* v *such that for a certain* $t_0 \in T$

$$[a]_{t_0} = \top \quad \textit{for every } a \in C(A).$$

Proof. By 5.1 there exists a saturated H–B theory $\mathscr{T}' = (\mathscr{L}', C, A')$ such that $A \subset A'$ and $C(A) \subset C(A')$. Using 9.4 we obtain that the required k-structure is the canonical structure and the required t_0 is A'. ■

10. Connections between *a*-models and *k*-models.

10.1. *Let* $\mathscr{T} = (\mathscr{L}, C, A)$ *be a consistent H–B theory. Let* $\mathfrak{M}_k = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R} \rangle$ *be a k-structure for* $\mathscr{L}$ *with constant, enumerable domains. Let a truth value of a formula* a *in* $\mathfrak{M}_k$ *be defined. Then the structure* $\langle \mathfrak{O}(\mathbf{T}), D, R \rangle$ *where* $\mathfrak{O}(\mathbf{T})$ *is the semi-Boolean algebra described in* I,1.1, $D = \mathscr{A}_t$ *and* R *is defined as follows: for any formula* a, $v \in D^V$

$$a_R(v) = \{t \in \mathbf{T}\colon [a(v)]_t = \top\}$$

is an a-model for $\mathscr{T}$ *if and only if* $\mathfrak{M}_k$ *is a k-model for* $\mathscr{T}$.

Proof. First we must show that $\mathfrak{M}_a = \langle \mathfrak{O}(\mathbf{T}), D, R \rangle$ is an a-structure. For the purpose we must check that the conditions (a_0)–(a_{10}) of § 2 are fulfilled. The conditions (a_0) follows from the fact that $\mathfrak{O}(\mathbf{T})$ is a complete semi-Boolean algebra.

We recall that in $\mathfrak{O}(\mathbf{T})$ the operations $\cup$, $\cap$ are the set theoretical union and intersection and the operations $\Rightarrow$, $\dot{-}$, $\urcorner$, $\ulcorner$ are given by the formulas:

$$\begin{aligned}
A \Rightarrow B &= \{t \in \mathbf{T}\colon \text{for every } u \in \mathbf{T} \text{ such that } t\,\mathbf{R}\,u \text{ if } u \in A, \text{ then } u \in B\},\\
A \dot{-} B &= \{t \in \mathbf{T}\colon \text{there exists an } s \in \mathbf{T} \text{ such that } s\,\mathbf{R}\,t,\ s \in A \text{ and } s \notin B\},\\
\urcorner A &= \{t \in \mathbf{T}\colon \text{for every } u \in \mathbf{T} \text{ such that } t\,\mathbf{R}\,u,\ u \notin A\},\\
\ulcorner A &= \{t \in \mathbf{T}\colon \text{there exists an } s \in \mathbf{T} \text{ such that } s\,\mathbf{R}\,t,\ s \notin A\},
\end{aligned}$$

where A, B are open subsets of $\mathbf{T}$.

Now, we demonstrate only case (a_5). In the remaining cases we proceed in a similar way.

We must show that

$$(a \dot{-} \beta)_R(v) = a_R(v) \dot{-} \beta_R(v).$$

By the appropriate definitions we have

$$\begin{aligned}
(a \dot{-} \beta)_R(v) &= \{t \in \mathbf{T}\colon [(a \dot{-} \beta)(v)]_t^{\mathfrak{M}_k} = \top\}\\
&= \{t \in \mathbf{T}\colon \text{there exists an } s \in \mathbf{T} \text{ such that } s\,\mathbf{R}\,t,\ [a(v)]_s^{\mathfrak{M}_k} = \top\\
&\quad \text{and } [\beta(v)]_s^{\mathfrak{M}_k} = \bot\} = \{t \in \mathbf{T}\colon \text{there exists an } s \in \mathbf{T} \text{ such}\\
&\quad \text{that } s\,\mathbf{R}\,t,\ s \in a_R(v) \text{ and } s \notin \beta_R(v)\} = a_R(v) \dot{-} \beta_R(v).
\end{aligned}$$

We observe that if $a \in A$ and $\mathfrak{M}_k$ is a k-model for a, then for every $t \in \mathbf{T}$, $v \in \mathscr{A}_t^V$, $[a(v)]_t^{\mathfrak{M}_k} = \top$. This is equivalent to $a_R(v) = \mathbf{T}$ — the greatest element of the semi-Boolean algebra $\mathfrak{O}(\mathbf{T})$. ■

10.2. *Let $\mathcal{T} = (\mathcal{L}, C, A)$ be a consistent H–B theory. Let $\mathfrak{M}_a = \langle \mathfrak{A}, D, R\rangle$ be an a-structure for $\mathcal{L}$. $\mathfrak{M}_a$ is an a-model for $\mathcal{T}$ if and only if the $\mathfrak{M}_k = \langle D, \mathbf{T}, \mathbf{R}\rangle$, where $\mathbf{T}$ is the set of all Q-filters in $\mathfrak{A}$ and R is the set inclusion relation $\mathbf{T}$ is a k-model for $\mathcal{T}$.*

Proof. By our assumption we have that the nodes of the tree $\mathbf{T}$ are Q-filters. So, for every valuation $v \in D^V$ and a formula a we define a truth value of a in $\mathfrak{M}_k$ as follows

$$[a(v)]_V = \top \quad \text{if} \quad a_R(v) \in V,$$
$$[a(v)]_V = \bot \quad \text{if} \quad a_R(v) \notin V.$$

Now, we must check that conditions (1)–(9) of § 8 are satisfied. We only show the cases (5) and (9). In the remaining cases we proceed in a similar way.

Case (5). Les us suppose that for a and β 10.2 is proved. By the definition we have $[(a \dot{-} \beta)(v)]_V^{\mathfrak{M}_k} = \top$ is equivalent to $(a \dot{-} \beta)_R(v) \in V$. On account of ($a_5$) § 2 and I,3.7, we have that $a_R(v) \dot{-} \beta_R(v) \in V$ if and only if there exists a $V' \in \mathbf{T}$ such that $V' \subset V$, $a_R(v) \in V'$ and $\beta_R(v) \notin V'$, i.e. there exists a $V' \in \mathbf{T}$ such that $V' \subset V$, $[a(v)]_V^{\mathfrak{M}_k} = \top$ and $[\beta(v)]_V^{\mathfrak{M}_k} = \bot$.

Case (9). Let us suppose that

$$[(\forall \xi\, a(\xi))(v)]_V^{\mathfrak{M}_k} = \top, \quad \text{i.e.} \quad \bigcap_{c \in D}^{\mathfrak{A}} a(v_c^u).$$

We remember that V is a Q-filter. So the last condition is equivalent to the one: for every $c \in D$, $a_R(v_c^u) \in V$, i.e. for every $c \in D$, $[a_R(v_c^u)]_V^{\mathfrak{M}_k} = \top$.

Now, let $a \in A$ and let $\mathfrak{M}_a$ be an a-model for a. Then for every $v \in D^V$, $a_R(v) = \bigvee$. Hence $a_R(v)$ belongs to every Q-filter, which proves that for every $V \in \mathbf{T}$, $[a(v)]_V^{\mathfrak{M}_k} = \top$ if and only if $a_R(v) = \bigvee$. ∎

10.3. *Let $\mathcal{T} = (\mathcal{L}, C, A)$ be a consistent H–B theory. For any formula a the following conditions are equivalent:*

(i) *a is a theorem of $\mathcal{T}$,*
(ii) *a is k-valid in every k-model for $\mathcal{T}$ with constant domains.*

Proof. (i)⇒(ii): If a is a theorem of $\mathcal{T}$, then by 3.2 a is a-valid in every a-model $\mathfrak{M}_a = \langle \mathfrak{A}, D, R\rangle$ for $\mathcal{T}$, where $\mathfrak{A}$ is a complete semi-Boolean algebra. Suppose that there exists a k-structure $\mathfrak{M}_k = \langle \mathcal{A}_t, \mathbf{T}, \mathbf{R}\rangle$ with constant domains such that for a certain $t_0 \in \mathbf{T}$ and a valuation $v \in D^V$, $D = \mathcal{A}_t$ for every $t \in \mathbf{T}$,

$$[a(v)]_{t_0} = \bot,$$

i.e. $\mathfrak{M}_k$ is not k-model for a.

By 10.1 there exists an a-structure $\mathfrak{M}_a = \langle \mathfrak{O}(\mathbf{T}), D, R\rangle$ such that $\mathfrak{O}(\mathbf{T})$ is a complete semi-Boolean algebra and for $\beta \in F$, $v \in D^V$

$$\beta_R(v) = \{t \in \mathbf{T}\colon\ [\beta(v)]_t = \top\}.$$

Moreover, we have that

$$a_R(v) \neq \mathsf{V}_{\mathfrak{O}(\mathbf{T})},$$

i.e. $\mathfrak{M}_a$ is not a-model for a. A contradiction which proves that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Suppose that a is not a theorem of $\mathcal{T}$ Because of 3.2 we infer that there exists an a-model $\mathfrak{M}_a = \langle \mathfrak{A}, D, R\rangle$ for $\mathbf{T}$ such that for $v \in D^V$

$$a_R(v) \neq \mathsf{V}_{\mathfrak{A}}.$$

By the definition of $\mathfrak{M}_a$ we have that D is a countable set and the infinite joins and meets

$$\text{(Q)} \qquad \bigcup_{c\in D}^{\mathfrak{A}} \beta_R(v_c^u) \quad \text{and} \quad \bigcap_{c\in D}^{\mathfrak{A}} \beta_R(v_c^u)$$

exist. Using I, 2.3, and 10.2 we have that for $\mathfrak{M}_k = \langle D, \mathbf{T}, \mathbf{R}\rangle$, where $\mathbf{T}$ is the set of all Q-filters in $\mathfrak{A}$ and for a certain Q-filter $V_0 \in \mathbf{T}$

$$[a(v)]_{V_0} = \bot$$

which proves that $\mathfrak{M}_k$ is not k-model for a. ■

If we assume in 10.3 the set A to be empty then we obtain

10.4 (strong completeness). *For any formula a the following two conditions are equivalent*:

(i) *a is a derivable formula,*

(ii) *every k-structure with constant domains is a k-model for a.* ■

Let $\mathcal{T} = (\mathcal{L}, C, A)$ be an H–B theory and $a \notin A$.

DEFINITION. We shall say that A *implies* a and write $A \models a$ if every k-model for $\mathcal{T}$ is a k-model for a.

10.5 (compactness). *Let $\mathcal{T} = (\mathcal{L}, C, A)$ be a consistent H–B theory and let $a \in A$. If $A \models a$, then for some finite subset A' of A, $A' \models a$.*

Proof. If $A \models a$, then by 10.3, $a \in C(A)$. Thus there exists a finite subset A' of A such that $a \in C(A')$, so $A' \models a$. ■

In a standard way we obtain

10.6 (Löwenheim–Skolem Theorem). *For a consistent H–B theory $\mathcal{T} = (\mathcal{L}, C, A)$, $A \models a$ iff every enumerable k-model for $\mathcal{T}$ is a k-model for a.* ■

Chapter III

Model theory for Heyting-Brouwer logic

1. Ultraproducts. In the sequel we shall denote k-structures by $\mathfrak{A}$, $\mathfrak{B}$ instead of $\mathfrak{M}_k$. We assume that $\mathbf{T}$ has the *base point* $\mathbf{O}$, i.e. for every $t \in \mathbf{T}$, $\mathbf{O}\,\mathbf{R}\,t$. This assumption is made only because of technical reason.

Let I be a nonempty set, and let D be an ultrafilter on I. Let $\{\mathfrak{A}_i\}_{i\in I}$ be a family of k-structures for $\mathscr{L}$, where $\mathfrak{A}_i = \langle \mathscr{A}_t^i, \mathbf{T}^i, \mathbf{R}^i, \mathbf{O}^i\rangle$.

DEFINITION. We call the structure $\langle \mathscr{B}_l, \mathbf{L}, \mathbf{S}, \mathbf{P}\rangle$ an *ultraproduct* of $\mathfrak{A}_i$ and denote it by $\prod_{i\in I} \mathfrak{A}_i/D$, if

(1) $\mathbf{L} = \prod_{i\in I} \mathbf{T}^i/D$, i.e. $\mathbf{L} = \{f_D\colon f \in \prod_{i\in I} \mathbf{T}^i\}$, where f_D is an equivalence class of the relation $\equiv_D$ on $\prod_{i\in I} \mathbf{T}^i$ which is defined in a normal way;

(2) $\mathbf{S} = \prod_{i\in I} \mathbf{R}^i/D$, i.e. $f_D\,\mathbf{S}\,g_D$ iff $f(i)\,\mathbf{R}^i\,g(i)$, $i \in I$;

(3) $\mathbf{P} = \langle \mathbf{O}^i\colon i \in I\rangle_D = \{g \in \prod_{i\in I} \mathbf{T}^i\colon \{i \in I\colon g(i) = \mathbf{O}^i\} \in D\}$;

(4) $\mathscr{B}_{f_D} = \prod_{i\in I} \mathscr{A}^i_{f(i)}/D$, for $f_D \in \mathbf{L}$.

The ultraproduct $\prod_{i\in I} \mathfrak{A}_i/D$ is a k-structure for $\mathscr{L}$, namely

(5) let p be an n-ary predicate symbol of $\mathscr{L}$. The interpretation of p in $\prod_{i\in I} \mathfrak{A}_i/D$ is a predicate P such that

$$P(f_D^1, \ldots, f_D^n) \text{ iff } \{i \in I\colon p_{\mathscr{A}^i_{f(i)}}(f^1(i), \ldots, f^n(i))\} \in D;$$

(6) let h be an n-ary function symbol of $\mathscr{L}$. Then h is interpreted in $\prod_{i\in I} \mathfrak{A}_i/D$ by the function H given by

$$H(f_D^1, \ldots, f_D^n) = \langle h_{\mathscr{A}^i_{f(i)}}(f^1(i), \ldots, f^n(i));\ i \in I\rangle_D;$$

(7) let c be a constant of $\mathscr{L}$. Then c is interpreted by the element $b \in \prod_{i\in I} \mathfrak{A}_i/D$, where $b = \langle a_i;\ i \in I\rangle_D$ and a_i is an interpretation of c in $\mathfrak{A}_i$.

If, for every $i \in I$, $\mathfrak{A}^i = \mathfrak{A}$ then $\prod_{i\in I} \mathfrak{A}^i/D$ is said to be the *ultrapower* and it is designated by $\mathfrak{A}_D^I$.

ŁOŚ'S THEOREM. *Let $I \neq \emptyset$ and let D be an ultrafilter on I. Let us denote a family of k-structures for $\mathscr{L}$ by $\{\mathfrak{A}_i\}_{i\in I}$ and the ultraproduct $\prod_{i\in I} \mathfrak{A}_i/D$ by $\mathfrak{B}$. Let v map V into $\prod_{i\in I} \mathscr{A}^i_{f(i)}$. Then*

(i) *for any term* $\tau(x_1, \ldots, x_n)$ *of* $\mathscr{L}$ *and elements* $v(x_1)_D, \ldots, v(x_n)_D \in B$

$$\tau_{\mathfrak{B}}(v(x_1)_D, \ldots, v(x_n)_D) = \langle \tau_{\mathfrak{A}_i}(v(x_1)(i), \ldots, v(x_n)(i)), i \in I \rangle_D;$$

(ii) *for a given formula* $\gamma(x_1, \ldots, x_n)$ *of* $\mathscr{L}$ *and elements* $v(x_1)_D, \ldots, v(x_n)_D \in \mathfrak{B}$ *we have*

$$[\gamma(v(x_1)_D, \ldots, v(x_n)_D)]^{\mathfrak{B}}_{f_D} = \top \quad \textit{iff} \quad \{i \in I\colon [\gamma(v(x_1)(i), \ldots, v(x_n)(i))]^{\mathfrak{A}_i}_{f(i)} = \top\} \in D;$$

(iii) *for any sentence* α *of* $\mathscr{L}$

$$[\alpha]^{\mathfrak{B}}_{f_D} = \top \quad \textit{iff} \quad \{i \in I\colon [\alpha]^{\mathfrak{A}_i}_{f(i)} = \top\} \in D.$$

Proof. Condition (iii) is an immediate consequence of (i) and (ii). The proofs of (i) and (ii) are by induction on the terms and formulas, respectively.

We prove (ii) only for the case where $\gamma(x_1, \ldots, x_n)$ is one of the forms $\ulcorner\alpha(x_1, \ldots, x_n)$, $(\alpha(x_1, \ldots, x_n) \dot{-} \beta(x_1, \ldots, x_n))$. To simplify our notation we shall leave out the free variables occurring in γ.

Case 1. $\gamma = \ulcorner\alpha$. Suppose that (ii) holds for α. Then the following four statements are equivalent:

$[\ulcorner\alpha(v_D)]^{\mathfrak{B}}_{f_D} = \top$, where $v_D = (v(x_1)_D, \ldots, v(x_n)_D)$;

there exists a g_D such that $g_D \mathbf{S} f_D$ and $[\alpha(v_D)]^{\mathfrak{B}}_{g_D} = \bot$;

there exists a $g \in \prod_{i \in I} \mathbf{T}^i$ such that $\{i \in I\colon g(i)\mathbf{R}^i f(i)\} \in D$ and $\{i \in I\colon [\alpha(v(i))]^{\mathfrak{A}_i}_{g(i)} = \bot\} \in D$;

$\{i \in I\colon [\ulcorner\alpha(v(i))]^{\mathfrak{A}_i}_{f(i)} = \top\} \in D$.

Case 2. $\gamma = (\alpha \dot{-} \beta)$. Suppose that (ii) holds for α and β. Then the following four statements are equivalent:

$[(\alpha \dot{-} \beta)(v_D)]^{\mathfrak{B}}_{f_D} = \top$;

there exists a g_D such that $g_D \mathbf{S} f_D$ and $[\alpha(v_D)]^{\mathfrak{B}}_{g_D} = \top$ and $[\beta(v_D)]^{\mathfrak{B}}_{g_D} = \bot$;

there exists a $g \in \prod_{i \in I} \mathbf{T}^i$ such that $\{i \in I\colon g(i)\mathbf{R}^i f(i)\} \in D$ and $\{i \in I\colon [\alpha(v(i))]^{\mathfrak{A}_i}_{g(i)} = \top\} \in D$ and $\{i \in I\colon [\beta(v(i))]^{\mathfrak{A}_i}_{g(i)} = \bot\} \in D$;

$\{i \in I\colon [(\alpha \dot{-} \beta)(v(i))]^{\mathfrak{A}_i}_{f(i)} = \top\} \in D$. ∎

DEFINITION. Let $\mathfrak{A} = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R}, \mathbf{O}\rangle$ and $\mathfrak{B} = \langle \mathscr{B}_l, \mathbf{L}, \mathbf{S}, \mathbf{P}\rangle$ be k-structures for $\mathscr{L}$. A map $m\colon \mathscr{A}_t \cup \mathbf{T} \to \mathscr{B}_l \cup \mathbf{L}$ is said to be a *monomorphism* if the following conditions hold:

(m_1) for $t \in \mathbf{T}$, $m(t) \in \mathbf{L}$, $m(\mathbf{O}) = \mathbf{P}$,

(m_2) for every $t, s \in \mathbf{T}$, if $t\mathbf{R}s$ then $m(t)\mathbf{S}m(s)$,

(m_3) $a \in |\mathscr{A}_t|$ iff $m(a) \in |\mathscr{B}_{m(t)}|$ and $a \equiv_t a'$ iff $m(a) \equiv_{m(t)} m(a')$,

(m_4) $m(f_{\mathscr{A}_t}(\tau_1, \ldots, \tau_n)) = f_{\mathscr{B}_{m(t)}}(m(\tau_1), \ldots, m(\tau_n))$,

(m_5) $p_{\mathscr{A}_t}(\tau_1, \ldots, \tau_n)$ iff $p_{\mathscr{B}_{m(t)}}(m(\tau_1), \ldots, m(\tau_n))$.

DEFINITION. A k-structure $\mathfrak{B}$ is said to be an *elementary extension* of an $\mathfrak{A}$, or a structure $\mathfrak{A}$ is *elementarily embedded in* $\mathfrak{B}$, iff

(e_1) there exists a monomorphism $m\colon \mathfrak{A}\to\mathfrak{B}$,

(e_2) for every $t\in\mathbf{T}$ and all valuations v in $\mathfrak{A}$

$$[a(v)]_t^{\mathfrak{A}} = \top \quad \text{iff} \quad [a(m\circ v)]_{m(t)}^{\mathfrak{B}} = \top .$$

In this case we shall use the notion $m\colon \mathfrak{A} \overset{\equiv}{\to} \mathfrak{B}$ and the map m will be called an *elementary extension.*

1.1. *For every k-structure $\mathfrak{A}$ and for every ultrafilter D on I, $I\neq\emptyset$, there exists a monomorphism m from $\mathfrak{A}$ to $\mathfrak{A}_D^I$ such that m is an elementary extension.*

Proof. We define m in the usual way, namely m is given by the formula $m(a) = \langle a_i\colon i\in I\rangle_D$ for $a\in|\mathscr{A}_t|$ and $m(t) = \langle t_i, i\in I\rangle_D$ for $t\in\mathbf{T}$. ∎

The monomorphism m defined above will be called a *canonical embedding.*

2. Model extension theorem. Let $\mathfrak{A}$ be a k-structure for $\mathscr{L}$. We expand the language $\mathscr{L}$ to a new language $\mathscr{L}'$ by adding a new constant symbol c_a, $a\in|\mathscr{A}_0|$. Now, we may expand $\mathfrak{A}$ to the k-structure

$$\mathfrak{A} = (A, a)_{a\in|\mathscr{A}_0|} = \langle \mathscr{A}_t', \mathbf{T}, \mathbf{R}, \mathbf{O}\rangle$$

for $\mathscr{L}$ by interpreting each new constant c_a by the element a. $\mathscr{A}_t'$ is the classical expansion of $\mathscr{A}_t$ to the language $\mathscr{L}'$.

THEOREM ON CONSTANTS. *Let $\mathscr{T}$ be an H–B theory in a language $\mathscr{L}$ and let $\mathfrak{A}$ be a k-structure for $\mathscr{L}$. If $\mathscr{T}'$ is obtained from $\mathscr{T}$ by adding new constants $c_{\mathscr{A}_0}$ (but no new nonlogical axioms), then for every formula a of $\mathscr{T}$ and every sequence $c_{a_1}, \ldots, c_{a_n}$ of new constants*

$$\vdash_{\mathscr{T}} a \quad \textit{iff} \quad \vdash_{\mathscr{T}'} a_{x_1,\ldots,x_n}(c_{a_1}, \ldots, c_{a_n}),$$

where $x_1, \ldots, x_n$ are all free variables in a and $a_{x_1,\ldots,x_n}(c_{a_1}, \ldots, c_{a_n})$ means that x_i is replaced by c_{a_i} in a.

The simple proof of this theorem is omitted. ∎

Let h be a function such that $\operatorname{dom}(h)\subseteq\mathbf{T}$, $\mathbf{O}\in\operatorname{dom}(h)$ and, for every $t\in\operatorname{dom}(h)$, let $h(t) = \langle h_1(t), h_2(t)\rangle$ where

(1) $h_1(t)\in|\mathscr{A}|^q$, $q<\omega$;

(2) $h_2(t)$ is a finite set of sentences of the form $a_{x_1,\ldots,x_n}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ where $\mathbf{a}_i$ is the name of $a_i = h_1(t)(i)$ for $i = 1, \ldots, n$, $n\leqslant q$.

DEFINITION. If a is a formula of $\mathscr{L}$, then by an *$\mathfrak{A}$-instance* of a we shall understand every formula a' belonging to $h_2(t)$.

2.1. *For every formula a, a is k-valid in $\mathfrak{A}$ iff $[a']_t^{\mathfrak{A}} = \top$ for every $\mathfrak{A}$-instance a' of a and every $t \in \mathbf{T}$.*

The proof, by induction on the length of a, is omitted.

Let Γ be a set of formulas of $\mathscr{L}$. Denote by $\Gamma(\mathfrak{A})$ the set of all $\mathfrak{A}$-instances of the formulas from Γ.

DEFINITION. The Γ *diagram* of $\mathfrak{A}$, denoted by $D_\Gamma(\mathfrak{A})$, is the set of all formulas $a \in \Gamma(\mathfrak{A})$ such that a is k-valid in $\mathfrak{A}$.

DEFINITION. We say that a k-structure $\mathfrak{A} = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R}, \mathbf{O}\rangle$ is *associated* with a k-structure $\mathfrak{B} = \langle \mathscr{B}_l, \mathbf{L}, \mathbf{S}, \mathbf{P}\rangle$ if there exists a pair of functions $\langle f, g\rangle$ such that:

(a_1) $\operatorname{dom}(f) \subset \mathbf{T}$, $\operatorname{rng}(f) \subset \mathbf{L}$, $\mathbf{O} \in \operatorname{dom}(f)$ and $f(\mathbf{O}) = \mathbf{P}$,

(a_2) if $t\,\mathbf{R}\,s$ then $f(t)\,\mathbf{S}\,f(s)$,

(a_3) $\operatorname{dom}(g) = \operatorname{dom}(f)$, if $t \in \operatorname{dom}(g)$ then $g(t)$ maps a subset of $|\mathscr{A}_t|$ onto a subset of $|\mathscr{B}_{f(t)}|$.

Sometimes we say that such a pair associates the k-structures $\mathfrak{A}$ and $\mathfrak{B}$.

DIAGRAM THEOREM. *Let $\mathfrak{A} = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R}, \mathbf{O}\rangle$ and $\mathfrak{B} = \langle \mathscr{B}_l, \mathbf{L}, \mathbf{S}, \mathbf{P}\rangle$ be k-structures for $\mathscr{L}$, and let $|\mathscr{A}_\mathbf{O}| \subset |\mathscr{B}_\mathbf{P}|$ and Γ be a set of formulas in $\mathscr{L}$ such that if a is an atomic formula or the negation of an atomic formula then $a \in \Gamma$. If $\mathfrak{B}^* = (\mathfrak{B}, b)_{b \in |\mathscr{B}_\mathbf{P}|}$ is a k-model of $D_\Gamma(\mathfrak{A})$, then there exist a set $I \neq \emptyset$, an ultrafilter D on I and a monomorphism h such that $h\colon \mathfrak{A} \to \mathfrak{B}_D^I$.*

Let I be a set of all pairs associating $\mathfrak{A}$ and $\mathfrak{B}$ such that f and $g(t)$ are finite for each $t \in \operatorname{dom}(g)$. Let $a \in D_\Gamma(\mathfrak{A})$. We recall that a is determined by the function $h = \langle h_1, h_2\rangle$.

Now let $T_a \subset I$ be the set of all pairs $\langle f, g\rangle \in I$ such that

(i) $\operatorname{dom}(h) \subset \operatorname{dom}(f) = \operatorname{dom}(g)$,

(ii) if $t \in \operatorname{dom}(h)$ then $\operatorname{rng}\big(h_1(t)\big) \subseteq \operatorname{dom}\big(g(t)\big)$,

(iii) if $t \in \operatorname{dom}(h)$, $a_{x_1,\ldots,x_n}(a_1, \ldots, a_n) \in h_2(t)$ and if v is valuation such that $v(x_i) = g(t)(a_i)$ for $i = 1, \ldots, n$, then $[a(v)]_{f(t)}^{\mathfrak{B}} = \top$.

We first show that T_a is a non-empty set. We define $\langle f, g\rangle$ as follows. As $\mathfrak{B}^*$ is a k-model of $D_\Gamma(\mathfrak{A})$ we infer that for $a \in D_\Gamma(\mathfrak{A})$

$$[a_{x_1,\ldots,x_n}(a_1, \ldots, a_n)]_\mathbf{P}^{\mathfrak{B}^*} = \top .$$

Put $\operatorname{dom}(f) = \operatorname{dom}(h)$, $f(\mathbf{O}) = \mathbf{P}$ and for every $t \in \operatorname{dom}(h)$

$$\big[\exists\, y_1, \ldots, \exists\, y_n\, a'(y_1, \ldots, y_n)\big]_{(t)}^{\mathfrak{B}^*} = \top ,$$

where $a'(y_1, \ldots, y_n)$ is the result of replacing all distinct names $\mathbf{a}_i$ in $a_{(x_1,\ldots,x_n)}(a_1, \ldots, a_n)$ by distinct free variables which did not occur previously in $a_{(x_1,\ldots,x_n)}(a_i, \ldots, a_n)$. Thus, we have defined the function f.

Now let us set $\operatorname{dom}(g) = \operatorname{dom}(f)$ and for $t \in \operatorname{dom}(f)$ let $\operatorname{dom}\big(g(t)\big) = \operatorname{rng}\big(h_1(t)\big)$. Suppose that $\operatorname{rng}\big(h_1(t)\big) = \{a_1, \ldots, a_q\}$, $n \leqslant q < \omega$. Then there exist $b_1, \ldots, b_q \in |\mathscr{B}_{f(t)}|$ such that if v is a valuation and $v(y_i) = b_i$

then

$$[a'(b_1, \ldots, b_n)]^{\mathfrak{B}}_{f(t)} = \top.$$

Putting $g(t)(a_i) = b_i$ for $i = 1, \ldots, n$, we have that the pair $\langle f, g\rangle$, where f and g are the functions defined above, belongs to T_a.

Now let

$$N = \{T_a\colon\ a \in \Gamma(\mathfrak{A})\}.$$

We shall show that N has the finite intersection property. Suppose that $a_1, \ldots, a_n \in \Gamma(\mathfrak{A})$. Then we have functions $h_1, \ldots, h_n$ such that $h_i = \langle h_{i,1}, h_{i,2}\rangle$ where $h_{i,1}, h_{i,2}$ satisfy (1) and (2). We want to find T_a such that $T_a \neq \emptyset$ and

$$T_a \subseteq T_{a_1} \cap \ldots \cap T_{a_n}.$$

Let $\operatorname{dom}(h) = \bigcup_{i=1}^{n} \operatorname{dom}(h_i)$ and let $E_t = \bigcup_{i=1}^{n} \operatorname{rng}\big(h_{i,1}(t)\big)$. For $a_1, \ldots, a_l \in E_t$ we define $h_1(t)(j) = a_j$. Let $h_2(t) = \bigcup_{i=1}^{n} h_{i,2}(t)$. We take $a \in h_2(t)$ and form T_a as before. It is not difficult to check that, for every $k = 1, \ldots, n$, $T_a \subseteq T_{a_k}$. Thus, N has the finite intersection property and it can be extended to an ultrafilter D on I.

Now, let $\mathfrak{B}^I_D$ be an ultrapower of $\mathfrak{B}$. We define a map m in the following way:

for every $t \in \mathbf{T}$, $m(t) = \overline{m}(t)_D$ where $\overline{m}\colon \mathbf{T} \to \mathbf{L}^I$ is given by the formula

$$\overline{m}(t)\,\langle f, g\rangle = \begin{cases} f(t) & \text{if } t \in \operatorname{dom}(f), \\ \mathbf{P} & \text{otherwise,} \end{cases}$$

and for every $a \in |\mathscr{A}_t|$, $m(a) = \overline{m}(a)_D$ where $\overline{m}\colon |\mathscr{A}_t| \to |\mathscr{B}^I_{m(t)}|$ is given by the formula

$$\overline{m}(a)(\langle f, g\rangle) = \begin{cases} g(t)(a) & \text{if } a \in \operatorname{dom}\big(g(t)\big), \\ b & \text{otherwise,} \end{cases}$$

where b is an arbitrary but fixed element of $|\mathscr{B}_{\mathbf{P}}|$.

To complete the proof we must show that m satisfies conditions (m_1)–(m_5). We prove the case (m_3) when $a, a' \in |\mathscr{A}_t|$ and $a \not\equiv_t a'$. The proofs of the remaining cases are similar. By our assumption, a formula α of the form $x \neq y$ belongs to Γ. Thus, let $h = \langle h_1, h_2\rangle$ be a function such that for $t \in \operatorname{dom}(h)$, $h_1(t)(1) = a$, $h_1(t)(2) = a'$ and $a \neq a' \in h_2(t)$. Put

$$Z = \big\{\langle f, g\rangle \in I\colon\ t \in \operatorname{dom}(f)\ \&\ \{a, a'\} \subset \operatorname{dom}\big(g(t)\big)\ \&\ [g(t)(a) \neq g(t)(a')]^{\mathfrak{B}}_{f(t)} = \top\big\}.$$

We observe that $T_a \subseteq Z$ and thus $Z \in D$. Since

$$Z \subseteq \{\langle f, g\rangle \in I\colon\ m(a)(\langle f, g\rangle) \not\equiv_t m(a')(\langle f, g\rangle)\} \in D,$$

we infer by the definition of $\equiv$ in $\mathfrak{B}_D^I$ that $m(a) \not\equiv_{m(t)} m(a')$.

Now we prove the last part of the diagram theorem. For this purpose let $a \in \Gamma$, $a_1, \ldots, a_n \in |\mathscr{A}_t|$, and suppose that

$$[a_{x_1,\ldots,x_n}(a_1, \ldots, a_n)]_t^{\mathfrak{A}} = \top .$$

Thus $a \in D_\Gamma(\mathfrak{A})$. We recall that a is determined by the function $h = \langle h_1, h_2\rangle$. Let T_a be defined as before and let

$$Z = \{(f, g) \in I\colon\ t \in \operatorname{dom}(f) \;\&\; a_1, \ldots, a_n \in \operatorname{dom}(g(t)) \;\&\; [(g(t)(a_1), \ldots \\ \ldots, g(t)(a_n))]_{f(t)}^{\mathfrak{B}} = \top\}.$$

Then $T_a \subseteq Z$ and so $Z \in D$. By the Łoś theorem we infer that

$$[a(m(a_1), \ldots, m(a_n))]_{m(t)}^{\mathfrak{B}_D^I} = \top \quad \text{iff} \quad Z \in D.$$

Thus

$$[a(v)]_t^{\mathfrak{A}} = \top \quad \text{iff} \quad [a(m\circ v)]_{m(t)}^{\mathfrak{B}_D^I} = \top$$

which completes the proof of the diagram theorem. ∎

DEFINITION. Let $\mathfrak{A} = \langle \mathscr{A}_t, \mathbf{T}, \mathbf{R}, \mathbf{O}\rangle$ and $\mathfrak{B} = \langle \mathscr{B}_l, \mathbf{L}, \mathbf{S}, \mathbf{P}\rangle$ be two k-structures for $\mathscr{L}$. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* provided that for every sentence a, $[a]_{\mathbf{O}}^{\mathfrak{A}} = \top$ iff $[a]_{\mathbf{P}}^{\mathfrak{B}} = \top$. We express this relationship between k-structures by $\equiv$.

2.2. *For every two k-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ iff there exist $I \neq \emptyset$ an ultrafilter D on I and a map m such that $m\colon \mathfrak{A} \overset{\equiv}{\rightarrow} \mathfrak{B}_D^I$.*

Proof. The proof is similar to the proof of the diagram theorem. ∎

DEFINITION. A class $\mathbf{K}$ of k-structures is said to be an *EC_Δ class* iff $\mathbf{K}$ is closed with respect to ultraproducts nad elementary equivalence.

In the classical case the EC_Δ class is equivalent to the class of all models of a certain theory $\mathscr{T}$.

In the case of intuitionistic logic this theorem is not true, namely we can prove that there exists a class $\mathbf{K} \in EC_\Delta$ such that $\mathbf{K}$ is not the class of all models of any intuitionistic theory [7].

Now we prove

2.3. *Let $\mathbf{K}$ be a class of k-structures for $\mathscr{L}$. $\mathbf{K} \in EC_\Delta$ iff $\mathbf{K}$ is the class of all k-models of a certain H–B theory $\mathscr{T} = (\mathscr{L}, C, A)$.*

Proof. It is obvious that if $\mathbf{K}$ is the class of all k-models of a certain H–B theory then $\mathbf{K} \in EC_\Delta$.

Let $\mathbf{K}$ be the class of all k-structures and let $\mathbf{K} \in EC_\Delta$. Denote by Γ the set of all sentences γ which are k-valid in every k-structure belonging to **K**, i.e.

$$\Gamma = \{\gamma\colon\ \forall \mathfrak{A} \in \mathbf{K}\,[\gamma]_{\mathbf{0}}^{\mathfrak{A}} = \top\}.$$

Let $\mathfrak{M}_k$ be a k-model for Γ. We want to show that $\mathfrak{M}_k \in \mathbf{K}$.

Let $\Gamma_0 = \{a_1, \ldots, a_n\}$ be a finite set of sentences such that $\mathfrak{M}_k$ is a k-model of Γ_0. We first show that there exists a k-model $\mathfrak{B}$ of Γ_0 such that $\mathfrak{B} \in \mathbf{K}$.

We observe that the sentence

$$a = (\ulcorner N\, a_1 \cup \ldots \cup \ulcorner Na_n)$$

where $a_1, \ldots, a_n$ are all elements of Γ_0 is false in $\mathfrak{M}_k$. So a is false in every k-structure belonging to **K**. Let $\mathfrak{B} \in \mathbf{K}$ be a structure such that

$$[a]_{\mathbf{P}}^{\mathfrak{B}} = \perp.$$

Thus for every i we have $[a_i]_{\mathbf{P}}^{\mathfrak{B}} = \top$. Hence we found a k-model for Γ_0 such that this model is in **K**.

Now let I be a set of all finite Γ_i such that $\mathfrak{M}_k$ is a k-model for Γ_i, i.e.

$$I = \{\Gamma_i\colon\ \Gamma_i \text{ is finite and } \mathfrak{M}_k \text{ is a } k\text{-model for } \Gamma_i\}.$$

We have $\Gamma_0 \in I$, and so I is a non-empty set. Moreover, for every $\Gamma_i \in I$ we can construct a k-model belonging to **K**. So let $\mathfrak{B}_{\Gamma_i}$ be a k-model for $\Gamma_i \in I$ such that $\mathfrak{B}_{\Gamma_i} \in \mathbf{K}$.

We put

$$J_{\Gamma_i} = \{\Gamma'\colon\ \mathfrak{B}_{\Gamma_i} \text{ is a } k\text{-model for } \Gamma'\}$$

and

$$N = \{J_{\Gamma_i}\colon\ \Gamma_i \in I\}.$$

We observe that I is closed under union and $J_{\Gamma_i} \cap J_{\Gamma_j} = J_{\Gamma_i \cup \Gamma_j}$. Thus N has the finite intersection property, and so it can be extended to an ultrafilter D. By an easy computation we have

$$\mathfrak{M}_k \equiv \prod_{\Gamma_i \in I} \mathfrak{B}_{\Gamma_i/D}$$

and by the definition of **K** we infer that $\mathfrak{M}_k \in \mathbf{K}$, which proves 2.3. ∎

DEFINITION. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two k-structures for $\mathscr{L}$, let Γ be a set of formulas of $\mathscr{L}$ and let m be a monomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. We say that $\mathfrak{B}$ is a *Γ-extension* of $\mathfrak{A}$ (and m is a *Γ-embedding* of $\mathfrak{A}$ in $\mathfrak{B}$) if, for all $a \in \Gamma$ and all assigments v in $\mathfrak{A}$, $[a(v)]_{\mathbf{0}}^{\mathfrak{A}} = \top$ implies $[a(m \circ v)]_{\mathbf{P}}^{\mathfrak{B}} = \top$.

DEFINITION. A set Γ of formulas is said to be *regular* if every formula of the form $x = y$ or $x \neq y$ is in Γ, and if for every formula a in Γ every formula of the form $a(x_1, \ldots, x_n)$ is in Γ.

Model Extension Theorem. *Let $\mathfrak{A}$ be a k-structure for $\mathscr{L}$, $\mathscr{T}$ a theory with language $\mathscr{L}$, and Γ a regular set of formulas in $\mathscr{L}$. Then $\mathfrak{A}$ possesses a Γ-extension which is a k-model of $\mathscr{T}$ iff every theorem of $\mathscr{T}$ which is a disjunction of $\ulcorner N$-negations of formulas in Γ is valid in $\mathfrak{A}$.*

Suppose that such a Γ-extension $\mathfrak{B}$ exists. We must show that if

$$\vdash_{\mathscr{T}}(\ulcorner Na_1 \cup \ldots \cup \ulcorner Na_n),$$

where each a_i is in $\Gamma(\mathfrak{A})$, then $a = (\ulcorner Na_1 \cup \ldots \cup \ulcorner Na_n)$ is k-valid in $\mathfrak{A}$, i.e., for every valuation v in $\mathfrak{A}$, $[a(v)]_0^{\mathfrak{A}} = \top$. Otherwise, there is a valuation v' of a such that $[(\ulcorner Na_1(v') \cup \ldots \cup \ulcorner Na_n(v'))]_0^{\mathfrak{A}} = \bot$. Thus, for every $i = 1, \ldots, n$ we have $[a_i(v')]_0^{\mathfrak{A}} = \top$. By our assumption we infer that $[a_i(m\circ v')]_0^{\mathfrak{A}} = \top$ for every $i = 1, \ldots, n$, i.e. there exists a valuation $m\circ v'$ such that $[a(m\circ v')]_0^{\mathfrak{B}} = \bot$. This is impossible since a is a theorem of $\mathscr{T}$ and $\mathfrak{B}$ is a k-model for $\mathscr{T}$.

Now, suppose that, for all n and all $a_1, \ldots, a_n \in \Gamma(\mathfrak{A})$, if $\vdash_{\mathscr{T}}(\ulcorner Na_1 \cup \cup \ulcorner Na_n)$ then $(\ulcorner Na_1 \cup \cup \ulcorner Na_n)$ is k-valid in $\mathfrak{A}$. We expand the language $\mathscr{L}$ to $\mathscr{L}' = \mathscr{L} + \{a\colon a \in |\mathscr{A}_0|\}$ and let $\mathscr{T}'$ be a theory with the language $\mathscr{L}'$. Let $\mathscr{T}''$ be a theory which is obtained from $\mathscr{T}'$ by adding to the set of specific axioms of $\mathscr{T}'$ the set of all $a \in \Gamma(\mathfrak{A})$ such that $[a]_0^{\mathfrak{A}'} = \top$ where $\mathfrak{A}' = (\mathfrak{A}, a)_{a\in|\mathscr{A}_0|}$. Now, we shall show that $\mathscr{T}''$ is a consistent theory. If not, then by the reduction theorem for consistency there would be $a_1'', \ldots, a_n'' \in \Gamma(\mathfrak{A})$ such that the formula $(\ulcorner Na_1' \cup \cup \ldots \cup \ulcorner Na_n')$ is a theorem of $\mathscr{T}'$. Since every a_i' is of the form $a'_{i(x_1,\ldots,x_m)}(\mathbf{a}_1^{(i)}, \ldots, \mathbf{a}_m^{(i)})$, $(\mathbf{a}_j^{(i)} \in |\mathscr{A}_0|)$, we infer by the theorem on constants that formula $(\ulcorner Na_1 \cup \ldots \cup \ulcorner Na_n)$ is a theorem of $\mathscr{T}$, where, for every $i = 1, \ldots, n$, a_i is obtained from a_i' by replacing all names $\mathbf{a}_j^{(i)}$, $j = 1, \ldots, n$, by new variables, e.g. $y_1, \ldots, y_m$. Thus by the regularity of Γ each a_i is in Γ. Moreover, by our assumption $(\ulcorner Na_1 \cup \cup \ldots \cup \ulcorner Na_n)$ is k-valid in $\mathfrak{A}$, i.e. for every valuation v

$$[(\ulcorner Na_1(v) \cup \ldots \cup \ulcorner Na_n(v))]_0^{\mathfrak{A}} = \top.$$

This proves that, for $i = 1, \ldots, n$ and every valuation v, $[a_i(v)]_0^{\mathfrak{A}} = \bot$. But for a valuation v' such that $v'(y_j) = a_j, j = 1, \ldots, m$, we have $[a_i(v')]_0^{\mathfrak{A}} = \top$, a contradiction. Hence $\mathscr{T}''$ is consistent and by the completeness theorem $\mathscr{T}''$ has a k-model $\mathfrak{C} = \langle \mathscr{C}_k, \mathbf{M}, \mathbf{W}, \mathbf{Q}\rangle$. We may assume that $a_{c_Q} = a$ for $a \in |\mathscr{A}_0|$. On account of the diagram theorem, there are an ultrapower $\mathfrak{C}_D^I$ and a monomorphism $m\colon \mathfrak{A} \to \mathfrak{C}_D^I$. By lemma 1.1 the canonical embedding $d\colon \mathfrak{C} \to \mathfrak{C}_D^I$ is an elementary embedding and $\mathscr{T} \subset \mathscr{T}''$. Thus, $\mathfrak{C}_D^I$ is a k-model for $\mathscr{T}$. Let $\mathfrak{C}' = \mathfrak{C}_D^I \upharpoonright \mathscr{L}$; then $\mathfrak{C}$ is a model for $\mathscr{T}''$.

Now we shall show that $\mathfrak{C}'$ is the required Γ-extension of $\mathfrak{A}$.

Let $a = a(x_1, \ldots, x_n) \in \Gamma$, where $x_1, \ldots, x_n$ are all the free variables of a and let v be a valuation in $\mathfrak{A}$ such that $v(x_i) = a_i$, $a_i \in |\mathscr{A}_0|$, $i = 1, \ldots, n$. Suppose that $[a(v)]_0^{\mathfrak{A}} = \top$ and $a' = a_{(x_1,\ldots,x_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, i.e. a' is an $\mathfrak{A}$-instance of a. By our assumption a' is an axiom of $\mathscr{T}''$. As $\mathfrak{C}$ is a model of $\mathscr{T}''$ we infer that $[a]_Q^{\mathfrak{C}} = \top$ and $[a']_{\mathbf{P}}^{\mathfrak{C}_D^I} = \top$, where $\mathbf{P}$ is the base point of the tree of $\mathfrak{C}_D^I$. It follows from the proof of the diagram theorem that a map m such that $m(a) = f_D$, where $f \in \mathfrak{C}^I$ and $f(i) = a$, is a monomorphism from $\mathfrak{A}$ to $\mathfrak{C}'$ and $[a(m \circ v)]_{\mathbf{P}}^{\mathfrak{C}} = \top$, which completes the proof of this theorem. ■

COROLLARY. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be k-structures for $\mathscr{L}$. Let Γ be a regular set of formulas in $\mathscr{L}$ and let Δ be a set of formulas containing every formula of the form $\forall \zeta_1, \ldots, \forall \zeta_n a$, where a is a disjunction of $\ulcorner N$-negations of formulas in Γ. If $\mathfrak{B}$ is a Δ-extension of $\mathfrak{A}$ then there exists a Γ-extension of $\mathfrak{B}$ which is an elementary extension of $\mathfrak{A}$.*

Proof. Suppose that all the assumptions are satisfied. Let Γ' be the set of all sentences of $\mathscr{L}$ which are k-valid in $\mathfrak{A}$. We show that there exists a Γ-extension of $\mathfrak{B}$ which is a model of Γ. By the model extension theorem, we only need to show that if $a_1, \ldots, a_n$ are in Γ' and $(\ulcorner N a_1 \cup \ldots \cup \ulcorner N a_n)$ is a theorem of Γ' then $(\ulcorner N a_1 \cup \ldots \cup \ulcorner N a_n)$ is k-valid in $\mathfrak{B}$.

Let $a_1, \ldots, a_n \in \Gamma'$ and let $\vdash_{\Gamma'} (\ulcorner N a_1 \cup \ldots \cup \ulcorner N a_n)$. The universal closure β of $(\ulcorner N a_1 \cup \ldots \cup \ulcorner N a_n)$ is a theorem of Γ' and so $[\beta]_0^{\mathfrak{A}} = \top$. By our assumption β is in Δ and $[\beta]_{\mathbf{P}}^{\mathfrak{B}} = \top$. Thus the formula $(\ulcorner N a_1 \cup \ldots \cup \ulcorner N a_n)$ is k-valid in $\mathfrak{B}$. By the model extension theorem, there exist a k-structure $\mathfrak{D}$ and a monomorphism m_1 from $\mathfrak{B}$ to $\mathfrak{D}$ such that m_1 is a Γ-embedding of $\mathfrak{B}$ in $\mathfrak{D}$ and $\mathfrak{D}$ is a k-model of Γ'. We observe that $\mathfrak{A} \equiv \mathfrak{D}$ and so by 2.2 there exist a non-empty set I, an ultrafilter D on I and a monomorphism h from $\mathfrak{A}$ to $\mathfrak{D}_D^I$ which is an elementary embedding. Now we want to show that there exists a Γ-embedding of $\mathfrak{B}$ in $\mathfrak{D}_D^I$. Let m be a canonical embedding of $\mathfrak{D}$ in $\mathfrak{D}_D^I$. Then the map $h_1 = m \circ m_1$ is the required Γ-embedding of $\mathfrak{B}$ in $\mathfrak{D}_D^I$. ■

Let $\langle C, \leqslant \rangle$ be a directed set and, for $c \in C$, let $\mathfrak{A}_c = \langle \mathscr{A}_i^c, \mathbf{T}^c, \mathbf{R}^c, \mathbf{O}^c \rangle$ be a k-structure for $\mathscr{L}$.

DEFINITION. A sequence $(\mathfrak{A}_c)_{c \in C}$ of k-structures is said to be a *chain* if, for every $c, d \in C$ such that $c \leqslant d$, $\mathfrak{A}_d$ is an extension of $\mathfrak{A}_c$.

DEFINITION. Let us assume that, for a chain $(\mathfrak{A}_c)_{c \in C}$ of k-structures, f_c is a function such that

(i) $f_c^c = \mathrm{id}$,

(ii) for $c \leqslant d$, $f_c^d \colon \mathfrak{A}_c \to \mathfrak{A}_d$ and f_c^d is a monomorphism,

(iii) for $c \leqslant d \leqslant e$, $f_d^e \circ f_c^d = f_c^e$.

DEFINITION. The *union of the chain* is the k-structure

$$\mathfrak{A}^\infty = \langle \mathscr{A}_t^\infty, \mathbf{T}^\infty, \mathbf{R}^\infty, \mathbf{O}^\infty \rangle$$

defined as follows:

Let $S = \bigcup \{\mathbf{T}^c \times \{c\} : c \in C\}$. Putting

$$\langle t, c\rangle \sim \langle s, d\rangle \quad \text{iff} \quad \bigvee e\big(c, d \leqslant e \,\&\, f_c^e(t) = f_d^e(s)\big)$$

we define the equivalence relation $\sim$ on S. Denote $\langle t, c\rangle/\!\sim$ by $[t, c]$ and set $\mathbf{T}^\infty = \{[t, c] : \langle t, c\rangle \in S\}$, $\mathbf{O}^\infty = [0^c, c]$ for an arbitrary $c \in C$. Now we define $\mathbf{R}^\infty$ on $\mathbf{T}^\infty$ as follows:

$$[t, c]\mathbf{R}^\infty[s, c] \quad \text{iff} \quad \bigvee e\big(c, d \leqslant e \,\&\, f_c^e(t)\mathbf{R}^c f_d^e(s)\big).$$

It is obvious that $\mathbf{R}^\infty$ is a quasi-ordering relation on $\mathbf{T}^\infty$, $\mathbf{O}^\infty \in \mathbf{T}^\infty$ and $\mathbf{O}^\infty$ is the base point of $\langle \mathbf{T}^\infty, \mathbf{R}^\infty\rangle$.

Put $Z = \bigcup\big\{|\mathscr{A}_t^c| \times \{t\} \times \{c\} : t \in \mathbf{T}^c, c \in C\big\}$ and let $\approx$ be the relation on Z defined by the formula

$$\langle a, t, c\rangle \approx \langle b, s, d\rangle \quad \text{iff} \quad \bigvee e\big(e, d \leqslant e \,\&\, f_c^e(a) = f_d^e(b) \,\&\, f_c^e(t) = f_d^e(s)\big).$$

This relation is an equivalence relation on Z.

Let us assume $[a, t, c] = \langle a, t, c\rangle/\!\approx$ and set

$$|\mathscr{A}_{[t,c]}^\infty| = \{[b, s, d] : b \in |\mathscr{A}_s^d| \,\&\, [s, d] = [t, c]\}.$$

Let p and f be the n-ary predicate and the function symbol, respectively, and let $q_i = [a_i, t_i, c_i] \in |\mathscr{A}_{[t,c]}^\infty|$ for $i = 1, \ldots, n$.

We define

$$P_{\mathscr{A}_{[t,c]}^\infty}(q_1, \ldots, q_n) \quad \text{iff} \quad \bigvee e\big(c, c_1, \ldots, c_n \leqslant e \,\&\, P_{\mathscr{A}_l^e}\big(f_{c_1}^e(a_1), \ldots, f_{c_n}^e(a_n)\big)\big)$$

where $l = f_c^e(t)$ and

$$f_{\mathscr{A}_{[t,c]}}(q_1, \ldots, q_n) = f_{\mathscr{A}_l^e}\big(f_{c_1}^e(a_1), \ldots, f_{c_n}^e(a_n)\big) \quad \text{where } e = f_c^e(t).$$

It is not difficult to check that $\mathfrak{A}^\infty = \langle \mathscr{A}_{[t,c]}^\infty, \mathbf{T}^\infty, \mathbf{R}^\infty, \mathbf{O}^\infty\rangle$, where $\mathscr{A}_{[t,c]}^\infty$, $\mathbf{T}^\infty, \mathbf{R}^\infty, \mathbf{O}^\infty$ are defined above, is a k-structure for $\mathscr{L}$.

Finally, we define $f_c^\infty : \mathfrak{A}_c \to \mathfrak{A}^\infty$ as follows: for $t \in T^c$ $f_c^\infty(t) = [t, c]$ and if $a \in |\mathscr{A}_t^c|$ then $f_c^\infty(a) = [a, t, c]$.

We observe that

$$(*) \qquad \text{for all } c \leqslant d, \quad f_d^\infty \circ f_c^d = f_c^\infty.$$

DEFINITION. An *elementary chain* is a chain $(\mathfrak{A}_c)_{c \in C}$ such that, for each c, d, $c \leqslant d$, $\mathfrak{A}_d$ is an elementary extension of $\mathfrak{A}_c$.

We may assume that the function f_c^d fixes this extension, i.e. conditions (e_1), (e_2) from page 47 are fulfilled by f_c^d.

TARSKI LEMMA. *Let $\langle C, \leqslant\rangle$ be a directed set and let $(\mathfrak{A}_c)_{c\in C}$ be an elementary chain, i.e. f_c^d: $\mathfrak{A}_c \to \mathfrak{A}_d$ for every c, d such that $c \leqslant d$. Then, for each $c \in C$, f_c^∞: $\mathfrak{A}_c \overset{\equiv}{\to} \mathfrak{A}^\infty$.*

Proof. Let $\mathfrak{A}^\infty = \langle \mathscr{A}_l^\infty, \mathbf{T}^\infty, \mathbf{R}^\infty, \mathbf{O}^\infty\rangle$ be constructed as above. It can easily be checked that the function f_c^∞ satisfies conditions (m_1)–(m_5). Now we prove the following assertion by induction on the length of formula γ:

$$[\gamma(v)]_s^{\mathfrak{A}_c} = \top \quad \text{iff} \quad [\gamma(f_c^\infty \circ v)]_t^{\mathfrak{A}^\infty} = \top \text{ where } t = f_c^\infty(s).$$

We limit our proof to the case where γ is of the form $(\alpha \dot{-} \beta)$ and the above assertion is true for α and β.

Let us suppose that for any valuation v in $\mathfrak{A}_c$ $[(\alpha \dot{-} \beta)(v)]_s^{\mathfrak{A}^\infty} = \top$. By the definition of the satisfaction and the induction hypothesis we infer that there exists a u, $u\,\mathbf{R}^c\,s$,

$$[\alpha(v)]_u^{\mathfrak{A}_c} = \top \quad \text{and} \quad [\beta(v)]_u^{\mathfrak{A}_c} = \bot.$$

Then, by the appropriate definitions, we have

$$f_c^\infty(u)\,\mathbf{R}^\infty f_c^\infty(s), \quad [\alpha(f_c^\infty \circ v)]_{t'}^{\mathfrak{A}^\infty} = \top \quad \text{and} \quad [\beta(f_c^\infty \circ v)]_{t'}^{\mathfrak{A}^\infty} = \bot,$$

where $t' = f_c^\infty(u)$.

Thus, on account of the satisfaction we obtain

$$[(\alpha \dot{-} \beta)(f_c^\infty \circ v)]_t^{\mathfrak{A}^\infty} = \top, \quad \text{where} \quad t = f_c^\infty(s).$$

Conversely suppose $[(\alpha \dot{-} \beta)(f_c^\infty \circ v)]_t^{\mathfrak{A}^\infty} = \top$, $t = f_c^\infty(s)$ for some $s \in \mathbf{T}^c$. Then there exists a $t' \in \mathbf{T}^\infty$ such that $t'\,\mathbf{R}^\infty\,t$ and

$$[\alpha(f_c^\infty \circ v)]_{t'}^{\mathfrak{A}^\infty} = \top \quad \text{and} \quad [\beta(f_c^\infty \circ v)]_{t'}^{\mathfrak{A}^\infty} = \bot.$$

We can assume that $t' = f_{c'}^\infty(u)$ for $c' \in C$ and $u \in \mathbf{T}^c$. Since $t'\,\mathbf{R}^\infty\,t$, we have $[u, c']\,\mathbf{R}^\infty[s, c]$. Thus, there exists a $d \in C$ such that $c', c \leqslant d$ and $f_c^d(u)\,\mathbf{R}^d f_c^d(s)$. We observe that by (*)

$$f_c^\infty \circ v = f_d^\infty \circ f_c^\infty \circ v.$$

Thus,

$$[\alpha(f_d^\infty \circ f_c^d \circ v)]_{t'}^{\mathfrak{A}^\infty} = \top \quad \text{and} \quad [\beta(f_d^\infty \circ f_c^d \circ v)]_{t'}^{\mathfrak{A}^\infty} = \bot$$

where $t' = f_d^\infty(f_c^d(u))$. Hence, by the induction hypothesis we infer

$$[\alpha(f_c^d \circ v)]_{\omega'}^{\mathfrak{A}_d} = \top \quad \text{and} \quad [\beta(f_c^d \circ v)]_{\omega'}^{\mathfrak{A}_d} = \bot,$$

where $\omega' = f_c^d(u)$. Now, if $\omega = f_c^d(s)$ then $\omega'\,\mathbf{R}^d\,\omega$, and so $[(\alpha \dot{-} \beta)(f_c^d \circ v)]_\omega^{\mathfrak{A}_d} = \top$. Now, if we use the fact that $\mathfrak{A}_d$ is an elementary extension of $\mathfrak{A}_c$, and moreover $f_c^d: \mathfrak{A}_c \overset{\equiv}{\to} \mathfrak{A}_d$, then $[(\alpha \dot{-} \beta)(v)]_s^{\mathfrak{A}_c} = \top$. ∎

3. Connections between H–B logic and DI logic. This section deals with an investigation of some logic stronger than intuitionism.

The first-order language $\mathscr{L}_D$ of this logic is the same as the first-order language of intuitionistic logic (IL). We observe that $\mathscr{L}_D$ is obtained from the language $\mathscr{L}$ which was described in II, § 1 by cancelling $\dot{-}$ and $\ulcorner$.

As the axioms of DI logic we assume all schemas of the axioms of IL and

(Dis) $$\forall \xi \left(\varphi(\xi) \cup \psi\right) \Rightarrow \left(\forall \xi\, \varphi(\xi) \cup \psi\right)$$

where ξ does not appear in ψ.

The rules of inference are the same as for IL. It is well known that the formula (Dis) is not a theorem of IL but this formula is a theorem of H–B logic.

We recall that a D-pseudo-Boolean algebra (DPBA) (I, § 1) is a pseudo-Boolean algebra such that

$$a \cup \bigcap_{t \in T} b_t = \bigcap_{t \in T} (a \cup b_t)$$

provided all the infinite meets in question exist.

We observe that the order topology described in I, § 1 is an example of DPBA. Moreover

3.1. *The algebra* $\mathfrak{O}_{\leqslant} = (\mathcal{O}_{\leqslant}, \cup, \cap, \Rightarrow, \neg)$ — *where* $\mathcal{O}_{\leqslant}$ *is the family of all open sets of a quasi-ordered set* $(C, \leqslant)$, *the operations* $\cup$ *and* $\cap$ *are the set-theoretical union and intersection, respectively, and the operations* $\Rightarrow$ *and* $\neg$ *are defined by*

$$B \Rightarrow C = \{x \in C\colon \forall y \in C \text{ if } x \leqslant y \;\&\; y \in B \text{ then } y \in C\},$$

$$\neg B = \{x \in C\colon \forall y \in C \text{ if } x \leqslant y \text{ then } y \notin B\},$$

respectively — is a complete DPBA. ∎

3.2. *For every DPBA* $\mathfrak{A}$ *there exist an order topology* $\mathfrak{O}_{\leqslant}$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{O}_{\leqslant}$.

The proofs of these lemmas is by an easy verification. ∎

Using the same methods as in I, § 2, we can prove

3.3. *Let* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \neg)$ *be a DPBA and let the set* (Q) *be defined as in* I, § 2. *Let* x, y *be elements of* A *such that the relation* $x \leqslant y$ *does not hold. Then there exists a Q-filter* V *such that* $x \in V$ *and* $y \notin V$. ∎

Finally we can prove

3.4. *Let* $\mathfrak{A}$ *be a DPBA and let* A_{2n}, B_{2n+1} *satisfy* (i)–(iii) *from* I, 3.1. *Then there exist an order topology* $\mathfrak{O}_{\leqslant}$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{O}_{\leqslant}$ *preserving all infinite joins* a_{2n} *and meets* b_{2n+1} *for every* $n \in \omega$. ∎

D-pseudo-Boolean algebras are characteristic for DI logic in the sense that we have the following lemma.

Let $(F, \cup, \cap, \Rightarrow, \urcorner)$ be the algebra of formulas of $\mathscr{L}_D$ and let $\approx$ be defined as usual.

3.5. *Let $\mathscr{T} = (\mathscr{L}, C, A)$ be a theory based on $\mathscr{L}_D$. The Lindenbaum algebra $\mathfrak{A}(\mathscr{T}) = (F/\approx, \cup, \cap, \Rightarrow, \urcorner)$ of the theory $\mathscr{T}$ is a DPBA. For any formula a, β*

$$\|(a \cup \beta)\| = \|a\| \cup \|\beta\|, \qquad \|(a \cap \beta)\| = \|a\| \cap \|\beta\|,$$

$$\|(a \Rightarrow \beta)\| = \|a\| \Rightarrow \|\beta\|, \qquad \|\urcorner a\| = \urcorner \|a\|,$$

$$\|a\| \leqslant \|\beta\| \quad \textit{iff} \quad (a \Rightarrow \beta) \in C(A).$$

For every formula a

$$\|a\| = \vee \quad \textit{iff} \quad a \in C(A).$$

For every formula $\beta(x)$

$$\|\exists \xi\, \beta(\xi)\| = \bigcup_{\tau \in T} \|\beta(\tau)\|,$$
$$\|\forall \xi\, \beta(\xi)\| = \bigcap_{\tau \in T} \|\beta(\tau)\|.$$

$\mathscr{T}$ is consistent iff $\mathfrak{A}(\mathscr{T})$ is non-degenerate. ■

The notions of an algebraic structure, a-validity, and a-models are introduced in the same way as in II, § 2.

In a standard way we can prove

COMPLETENESS THEOREM. *For any D-predicate calculus $(\mathscr{L}_D, C_D)$ and any formula $a \in \mathscr{L}_D$ the following conditions are equivalent:*

(i) *a is a derivable formula, i.e. $a \in C_D(\emptyset)$,*

(ii) *a is a-valid,*

(iii) *a is valid in every algebraic model $\mathfrak{M}_a = (\mathfrak{A}, D, R)$ for $(\mathscr{L}_D, C_D)$ where $\mathfrak{A}$ is a complete DPBA,*

(iv) *a is valid in every algebraic model $\mathfrak{M}_a = (\mathfrak{A}, D, R)$ for $(\mathscr{L}_D, C_D)$ where $\mathfrak{A}$ is an order topology.* ■

The next lemma shows the connection between DI logic and H–B logic.

3.6. *Let a be a formula from $\mathscr{L}$ (language of H–B logic). If a does not contain $\dot{-}$ and $\ulcorner$, then $a \in C(\emptyset)$ iff $a \in C_D(\emptyset)$.*

Proof. It is obvious that if a is in $C_D(\emptyset)$ then $a \in C(\emptyset)$.

Suppose that a is not in $C_D(\emptyset)$. By the completeness theorem for DI logic and appropriate definitions, there exists a D-pseudo-Boolean algebra $\mathfrak{A}$ such that $a_R(v) \neq \vee$ where $\vee$ is the greatest element of $\mathfrak{A}$. Let h be a natural monomorphism from $\mathfrak{A}$ to $\mathfrak{A}(\mathscr{S})$. Then we have $h(a_R(v))$

$= \|a_R(v)\| \neq V$. On account of I, 3.1, $\mathfrak{A}(\mathscr{S})$ can be extended to a semi-Boolean algebra $\mathfrak{B}$. Let g be a monomorphism from $\mathfrak{A}(\mathscr{S})$ to $\mathfrak{B}$. Then $g \circ h(a_R(v)) \neq V$, V being the greatest element of $\mathfrak{B}$. By the completeness theorem for H–B logic we infer that is not in $C(\emptyset)$. ∎

We add that the propositional fragment of H–B logic is equal to the propositional calculus of IL, i.e. if $\forall, \exists, \dot{-}, \ulcorner$ do not appear in a then a is a propositional tautology of H–B logic iff a is a propositional tautology of IL.

Now we observe that Kripke models for D-logic are of the same type as for H–B logic, i.e. the domains must be constant. Of course we must leave out conditions which characterize the operations $\dot{-}$ and $\ulcorner$.

It is not difficult to see that we can adapt the methods and the results from the preceding parts and apply them to the examination of DI logic. Moreover, many proofs are easier for this case because we need not introduce "dual" notions, such as, for example, "formally rejected", etc.

So we have

3.7. *Let $\mathscr{T}$ be a D-theory and let $\mathfrak{M}_k$ be a k-structure with constant domains. Then there exists an algebraic structure $\mathfrak{M}_a$ such that $\mathfrak{M}_a$ is an a-model for $\mathscr{T}$ if $\mathfrak{M}_k$ is a k-model for $\mathscr{T}$.* ∎

3.8. *Let $\mathscr{T}$ be a D-theory and let $\mathfrak{M}_a$ be an algebraic structure for $\mathscr{T}$. Then there exists a k-structure $\mathfrak{M}_k$ for $\mathscr{T}$ such that $\mathfrak{M}_k$ is a k-model for $\mathscr{T}$ if $\mathfrak{M}_a$ is a model for $\mathscr{T}$.*

We can prove these lemmas in the same way as in the case of H–B theory. ∎

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List of symbols

List of definitions