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UNIQUENESS AND REGULARITY PROPERTIES OF LINEAR OPERATORS AND THEIR APPLICATIONS

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1. Introduction

In this lecture we shall give results on uniqueness in the Cauchy problem and propagation of singularities contained in parameters for linear differential operators. But let us begin with some applications of these properties. Consider e.g. the so-called von Kármán equations

(1) $$\Delta^2 u - b(u, v) = f,$$

(2) $$\Delta^2 v + b(u, u) = 0,$$

used in the theory of elasticity, where $\Delta^2 = (\partial_x^2 + \partial_y^2)^2$ is the biharmonic operator and $b(u, v)$ is the bilinear form $\partial_x^2 u \partial_y^2 v - 2\partial_{xy}^2 u \partial_{xy}^2 v + \partial_y^2 u \partial_x^2 v$. It is known (see e.g. [7]) that the buckled equilibrium states of elastic plates can be well described by these equations under the boundary conditions of Dirichlet type:

(3) $$u|_\Gamma = \partial_\nu u|_\Gamma = 0, \quad v|_\Gamma = v_0, \quad \partial_\nu v|_\Gamma = v_1,$$

where Γ is the contour of a bounded simply connected domain Ω which is the projection of the plate onto the horizontal plane $\{(x, y)\}$; ∂_ν is the derivative in the normal direction of Γ. In the von Kármán problem (1), (2), (3), a solution $(u, v) \in (H_0^2(\Omega) \cap H^4(\Omega)) \times H^3(\Omega)$ has to be found under the following assumptions: $f \in L^2(\Omega)$, $v_0 \in H^{5/2}(\Gamma)$, $v_1 \in H^{3/2}(\Gamma)$, where H^s, H_0^s are the Sobolev spaces (see e.g. [1], [11]), $L^2 = H^0$. The function $u = u(x, y)$ e.g. has a physical interpretation as the vertical deflection of the plate at the point (x, y). An essential reason to have constant coefficients in equations (1), (2) is the hypothesis that the plate is homogeneous and isotropic (see [6]). So if we disregard these restrictions, we can expect variable coefficients. Then we can formulate the hypothesis:

There exist buckled states of inhomogeneous and anisotropic plates well described by elliptic equations of the following type:

$$\sum_{|\alpha|=4} a_\alpha(x, y)\,\partial^\alpha_{(x,y)} u + \sum_{|\beta'|,|\beta''|=2} b_{\beta',\beta''}(x, y)\,\partial^{\beta'}_{(x,y)} u\,\partial^{\beta''}_{(x,y)} v = f(x, y), \tag{4}$$

$$\sum_{|\alpha|=4} c_\alpha(x, y)\,\partial^\alpha_{(x,y)} v + \sum_{|\beta'|,|\beta''|=2} d_{\beta',\beta''}(x, y)\,\partial^{\beta'}_{(x,y)} u\,\partial^{\beta''}_{(x,y)} v = 0, \tag{5}$$

where $\alpha = (\alpha^1, \alpha^2)$, $\beta = (\beta^1, \beta^2)$ are multiindices.

Now let us consider the following practical question: Suppose we have a tensed plate which cannot be seen directly. Can we establish whether a given part of the plate is horizontal – by suitable measurements of a little segment only? And what kind of data will it be better to measure? If we can measure the Cauchy data a positive answer can be given for functions f vanishing on an open subset of Ω. The nonlinearity in (1) e.g. does not play any role in our considerations because for a given v equation (4) is linear with respect to u.

2. Properties of system (4), (5) following from uniqueness in the linear case

Let us consider a differential operator $P = \sum_{|\alpha| \leq m} a_\alpha(x)\,\partial^\alpha_x$, $x \in \mathbf{R}^n$, with continuous coefficients in a neighbourhood of $x = x_0$ for $|\alpha| = m$ (and a_α are e.g. locally bounded for $|\alpha| < m$). The roots $z = \lambda(x, \xi, \theta)$ of the equation $\sum_{|\alpha|=m} a_\alpha(x)(\xi + z\theta)^\alpha = 0$ are called *characteristic roots* of P (or of the equation $Pu = f$) in the direction θ at the point x, where $\xi, \theta \in \mathbf{R}^n \setminus 0$ and ξ is not parallel to θ.

THEOREM 1. *Let $f = 0$ in a domain $\omega \subset \Omega$ (f can be $\neq 0$ in $\Omega \setminus \omega$). Suppose the coefficients $b_{\beta',\beta''}$ are locally bounded in ω and the characteristic roots $\lambda(x, y, (\xi, \eta), \theta)$ of (4) are at most double and belong to $C^{1+\delta}(\omega)$, $0 < \delta < 1$, for every fixed (ξ, η), $\theta \in \mathbf{R}^2 \setminus 0$, (ξ, η) not parallel to θ. Let us have a solution $(u, v) \in H^4(\omega) \times H^4(\omega)$ of (4), (5) such that $u|_\gamma = \mathrm{const}$, $\partial^j_\nu u|_\gamma = 0$, $j = 1, 2, 3$, where $\gamma \subset \omega$ is a C^∞-curve. Then $u = \mathrm{const}$ in ω. An analogous statement is true for v.*

Proof. We shall give below a uniqueness theorem from which it will not be difficult to deduce the following uniqueness property for the operator in (4):

(UP) For an arbitrary given $v \in H^4$ and a sufficiently smooth curve $\gamma' \subset \omega$ we have $u = \mathrm{const}$ in a neighbourhood of $(x, y) \in \gamma'$, for every $u \in H^4$ satisfying (4) and the Cauchy data $u|_{\gamma'} = \mathrm{const}$, $\partial^j_\nu u|_{\gamma'} = 0$ ($j = 1, 2, 3$) near (x, y).

Take an arbitrary compact $\bar{\omega}_0 \subset \omega$, where ω_0 is an open connected set. Then by property (UP) it is easy to see in addition the following fact: If u

$= \text{const}$ near a point $(x, y) \in \omega_0$, then $u = \text{const}$ in the circle $C_{\varrho_0}(x, y)$ with center (x, y) and radius equal to ϱ_0, where $\varrho_0 = \text{dist}(\omega_0, \partial\omega)$ does not depend on (x, y). Now apply property (UP) near a point of γ and join this point with an arbitrary point $(x, y) \in \omega_0$ by a curve $\varkappa$. Then by a finite number of steps using circles $C_{\varrho_0}(x_j, y_j)$, $(x_j, y_j) \in \varkappa$, $j = 1, 2, \ldots, N$, we get $u(x, y) = u|_\gamma = \text{const}$.

For the next theorem we need the hypothesis:

(D) The Dirichlet problem (4), (5), (3) has a unique solution $(u, v) \in H^{s_1}(\Omega) \times H^{s_2}(\Omega)$ for suitable $s_1 \geqslant 4$, $s_2 > 3$, when the coefficients a_α, c_α, $b_{\beta',\beta''}$, $d_{\beta',\beta''}$, the right-hand side f, the data v_0, v_1 and the contour Γ of Ω are smooth enough, and $|v_0|$, $|v_1|$ are sufficiently small.

Note that Ciarlet and Rabier have proved in [7] that the von Kármán problem (1), (2), (3) has a unique solution $(u, v) \in \left(H_0^2(\Omega) \cap H^4(\Omega)\right) \times H^3(\Omega)$ when Γ is sufficiently smooth, $f \in L^2(\Omega)$, $v_0 \in H^{5/2}(\Gamma)$, $v_1 \in H^{3/2}(\Gamma)$ and $|v_0|$, $|v_1|$ are small enough.

THEOREM 2 (A maximum principle for the level lines). *Suppose the system* (4), (5) *satisfies hypothesis* (D) *for every domain* $\omega' \subset \Omega$ *with sufficiently smooth contour,* $f = 0$ *in a domain* $\omega \subset \Omega$ *and the characteristic roots of* (4) *satisfy the assumptions in Theorem* 1. *Assume that for a solution* (u, v) *of* (4), (5), u *has a closed locally extremal level line* $\gamma \subset \omega$ *which is smooth enough. Then* $u = \text{const}$ *in* ω. *A similar statement is true for* v.

Proof. First we recall that a line γ is a *level line* for u if $u|_\gamma = \text{const}$; a level line is *locally extremal* if $u|_{\gamma'} \leqslant u|_\gamma$ (or $u|_{\gamma'} \geqslant u|_\gamma$) for any other level line γ' close enough to γ. Now it is clear that $\partial_\nu u|_\gamma = 0$. Then it is easy to see that from hypothesis (D) it follows that $u = \text{const}$ in ω', where $\partial\omega' = \gamma$. Now Theorem 2 is an obvious consequence of Theorem 1.

3. Uniqueness in the Cauchy problem for linear operators

As we have seen the uniqueness for the Cauchy problem in the case of linear operators is essential for certain properties of equations similar to (1), (2). Consider a linear operator of the type

$$P(t, x, \partial_t, \partial_x) = \sum_{|\alpha| \leqslant m} a_\alpha(t, x)\, \partial^\alpha_{(t,x)}, \tag{6}$$

where $(t, x) = (t, x^1, \ldots, x^m)$ varies in a neighbourhood of $(0, x_0)$. We shall use an arbitrary conic neighbourhood $U_\eta = \left(-\delta(\eta), \delta(\eta)\right) \times \omega_{x_0,\eta} \times \left\{\left|\xi/|\xi| - \eta/|\eta|\right| < \varepsilon(\eta)\right\}$ of $(0, x_0, \eta)$, where $\delta(\eta)$, $\varepsilon(\eta)$ are positive numbers and $\omega_{x_0,\eta}$ is a neighbourhood of x_0 depending on $\eta \in \boldsymbol{R}^n \setminus 0$. In the next assumptions all the derivatives are Schwartz ones. The membership of functions $p(t, x, \xi)$ (as functions of (x, ξ) for a fixed t) to the Zygmund–Calderón classes $S^{\mu,\infty,r}_{x_0,\eta}$ is uniform for almost every t in $(-\delta(\eta), \delta(\eta))$; the elements of $S^{\mu,\infty,r}_{x_0,\eta}$ are restrictions to conic neighbourhoods of (x_0, η) of functions belonging to the traditional Zygmund–Calderón classes (see e.g. [6], [14]). The requirements on the characteristic roots $\lambda(t, x, \xi)$ of P (in the direction $\theta = (1, 0, \ldots, 0)$ normal to $\{t = 0\}$) are the following:

(a) If $\operatorname{Im}\lambda = 0$ in U_η then at least one of the conditions (7), (8) holds:

(7) $$\partial_x \partial_\xi^\alpha \lambda \in L^\infty(U_\eta), \quad \forall\alpha\colon |\alpha| \leqslant 1, \quad \forall\beta\colon |\beta| \leqslant n+1+|\alpha|$$

($f \in L^\infty(G) \Leftrightarrow |f| < +\infty$ almost everywhere in G);

(8) $$\lambda \in S^{0,\infty,1}_{x_0,\eta}.$$

(b) When $\operatorname{Im}\lambda \neq 0$ at $(0, x_0, \eta)$ we require one of the conditions:

(9) $\partial_x^\beta \partial_\xi^\alpha q \in L^\infty(U_\eta)$, $\forall\alpha$: $|\alpha| \leqslant 2$ when $q = \operatorname{Im}\lambda$ and $|\alpha| \leqslant 1$ when $q = \operatorname{Re}\lambda$, $\forall\beta$: $|\beta| \leqslant n+5-|\alpha|$, and at least one of the derivatives $\partial_t \partial_x^\beta \operatorname{Im}\lambda$ ($|\beta| \leqslant n+1$), $\partial_t \partial_\xi^\alpha \operatorname{Im}\lambda$ ($|\alpha| \leqslant 2n$) belongs to $L^\infty(U_\eta)$;

(10) $\lambda \in S^{\mu,\infty,1}_{x_0,\eta}$, $1 < \mu$, at least one of $\partial_t \partial_x^\beta \operatorname{Im}\lambda$ ($|\beta| \leqslant n+1$), $\partial_t \partial_\xi^\alpha \operatorname{Im}\lambda$ ($|\alpha| \leqslant 2n$) belongs to $L^\infty(U_\eta)$.

(c) If $\operatorname{Im}\lambda(0, x_0, \eta) = 0$ but $\operatorname{Im}\lambda \not\equiv 0$ in U_η let one of the Nirenberg conditions (11), (12) be true:

(11) $\operatorname{Im}\lambda \geqslant 0$ in U_η and $\operatorname{Im}\lambda$ satisfies (13) below;

(12) $$\partial_t \operatorname{Im}\lambda \leqslant \sum_{|\alpha|=1} (\partial_\xi^\alpha \operatorname{Re}\lambda \cdot \partial_x^\alpha \operatorname{Im}\lambda - \partial_x^\alpha \operatorname{Re}\lambda \cdot \partial_\xi^\alpha \operatorname{Im}\lambda)$$

almost everywhere in U_η and $\partial_t \operatorname{Im}\lambda$, $\partial_\xi^\alpha \lambda$, $\partial_x^\beta \lambda$ satisfy (13), $\forall\alpha, \beta$: $|\alpha|, |\beta| = 1$:

(13) $$\partial_x^\gamma \partial_\xi^\alpha p \in L^\infty(U_\eta), \quad \forall\alpha, \gamma\colon |\alpha| \leqslant 2, |\gamma| \leqslant n+3-|\alpha|$$

($p = \operatorname{Im}\lambda$, $\partial_t \operatorname{Im}\lambda$, $\partial_\xi^\alpha \lambda$, $\partial_x^\beta \lambda$).

Now we have the following general result (valid not only for elliptic operators):

THEOREM 3. *Assume that: near* $(0, x_0)$ *the coefficients* a_α *of* P *are at least continuous* $\forall\alpha$: $|\alpha| = m$, *and* $a_\alpha \in L^\infty$, $\forall\alpha$: $|\alpha| < m$; $\forall\eta \in \mathbf{R}^n \backslash 0$ *the characteristic roots* $\lambda(0, x_0, \eta)$ *are at most double, the real ones are simple and* $\lambda(t, x, \xi)$ *satisfy the requirements* (a), (b), (c). *If* $u \in H^m(O)$ *is a solution of the equation* $Pu = 0$ *such that* $(O_0 \cap \operatorname{supp} u) \backslash \{(0, x_0)\} \subset \{t > 0\}$ *for some neighbourhoods* O, O_0 *of* $(0, x_0)$, *then* $u = 0$ *in a whole neighbourhood of* $(0, x_0)$.

Notes on the proof

A traditional tool (see e.g. [5], [10], [12], [13], [15], etc.) to prove a uniqueness assertion are the Carleman estimates – inequalities e.g. in the form:

$$(14)\qquad \int_0^T e^{k(T-t)^2}\sum_{|\alpha|<m}\|\partial^\alpha_{(x,t)}\varphi\|^2\,dt \leqslant C(k^{-1}+T^2)\int_0^T e^{k(T-t)^2}\|P\varphi\|^2\,dt,$$

$\forall\varphi\in C_0^\infty(\boldsymbol{R}^{n+1})$, $\operatorname{supp}\varphi\subset O_{x_0}\times[0, T]$, k^{-1}, T small enough; $O_{x_0}\subset\boldsymbol{R}^n$ is a neighbourhood of x_0; $\|f\|^2=\int|f(x)|^2dx$. A basic method for obtaining estimates like (14) is due to Calderón. The Calderón scheme consists in several steps (see e.g. [15]):

(1) A reduction of the operator P to a system $D_t-M(t, x, D_x)$ with the same characteristic roots as those of P and obtaining the estimate (14) by a Carleman estimate for D_t-M ($D_t=-i\partial_t$, $D_{x_j}=-i\partial_{x_j}$).

(2) Estimating D_t-M by a corresponding inequality for a system $D_t-N(t, x, D_x)$, where the principal symbol of $N(t, x, D_x)$ is a canonical form $\mathscr{N}(t, x, \xi)$ of the matrix $\mathscr{M}(t, x, \xi)$, the principal symbol of M.

(3) Obtaining the estimate for $D_t-N(t, x, D_x)$ by proving Carleman inequalities for operators of the type

$$D_t-\lambda(t, x, D_x)\qquad\text{or}\qquad D_t-\begin{pmatrix}\lambda_1(t, x, D_x) & c(t, x, D_x)\\ 0 & \lambda_2(t, x, D_x)\end{pmatrix},$$

where $\lambda(t, x, D_x)$, $\lambda_j(t, x, D_x)$ have principal symbols equal to $\lambda(t, x, \xi)$, $\lambda_j(t, x, \xi)$ which are characteristic roots of $\mathscr{M}(t, x, \xi)$ and $\lambda_1(0, x_0, \eta)=\lambda_2(0, x_0, \eta)$ is a double root for some $\eta\in\boldsymbol{R}^n\backslash 0$.

But to apply the Calderón scheme in our case we have to solve two basic problems.

The first problem: If $\mathscr{M}$ is a smooth matrix, is there a regular and at least continuous matrix $\mathscr{C}_\eta(t, x, \xi)$ such that we have

$$(15)\qquad \mathscr{C}_\eta\mathscr{M}\mathscr{C}_\eta^{-1}=\mathscr{N}_\eta\quad\text{in a conic neighbourhood of } (0, x_0, \eta)$$

for a given $\eta\in\boldsymbol{R}^n\backslash 0$, where $\mathscr{N}_\eta=\mathscr{N}_\eta(t, x, \xi)$ has a block diagonal form

$$\mathscr{N}_\eta=\begin{pmatrix}\mathscr{B}_1 & & \\ & \ddots & \\ & & \mathscr{B}_l\end{pmatrix}\ ?$$

(All elements outside the blocks $\mathscr{B}_j$ are equal to zero). The question is not trivial because $\mathscr{M}$ can have eigenvalues of variable multiplicity. Then (as is not difficult to see) $\mathscr{N}_\eta(t, x, \xi)$ cannot be the Jordan form of $\mathscr{M}(t, x, \xi)$, because it is not even continuous. There is no answer to this question in any known paper using the Calderón scheme. But a positive answer can be given using a canonical form of Arnold ([3], [4]) for a matrix depending on parameters. When $\mathscr{M}(t, x, \xi)$ has at most double eigenvalues the result of Arnold yields for the blocks of $\mathscr{N}_\eta$ that $\mathscr{B}_j=\lambda_j(t, x, \xi)$ or

$$\mathscr{B}_j=\begin{pmatrix}\lambda_j(0, x_0, \eta/|\eta|)|\xi| & |\xi|\\ \alpha(t, x, \xi) & \lambda_j(0, x_0, \eta/|\eta|)|\xi|+\beta(t, x, \xi)\end{pmatrix}$$

in a conic neighbourhood of $(0, x_0, \eta)$, where $\lambda_j(t, x, \xi)$ is an eigenvalue of $\mathcal{M}(t, x, \xi)$, $\alpha(t, x, \xi)$ and $\beta(t, x, \xi)$ are homogeneous functions of ξ of degree 1 as smooth as $\mathcal{M}(t, x, \xi)$, and $\alpha(0, x_0, \eta) = \beta(0, x_0, \eta) = 0$.

The second problem: Let $\lambda(t, x, D_x)$ be a pseudodifferential operator with principal symbol $\lambda(t, x, \xi)$ which does not belong to C^∞ in all variables. Note that the eigenvalues $\lambda(t, x, \xi)$ of $\mathcal{M}(t, x, \xi)$ may not belong to C^∞ even when $\mathcal{M}(t, x, \xi) \in C^\infty$. Then the traditional C^∞-theory of pseudodifferential operators cannot be used. Now the problem is to make certain operations possible for operators whose symbols are not very smooth. A positive answer to this problem can also be given under conditions like those in (a), (b), (c). Note that such restrictions on the smoothness of the symbols lead generally to operators different from those considered in the known papers in the field (see e.g. [8]).

Remark. In Theorem 3 the direction $\theta = (1, 0, \ldots, 0)$ is assumed to be noncharacteristic for P at the point $(0, x_0)$.

EXAMPLES.

1. $\partial_t^4 + (1 + a_2(t))^2 \partial_x^4 + 2(1 - a_2(t)) \partial_x^2 \partial_t^2, \quad a_2(t) = \begin{cases} t^2, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$

(This operator is a "deformation" of the biharmonic operator near the point $(t, x) = (0, 0)$.)

2. $\left[\partial_t^2 - (1 + a_2(t))(\partial_x^2 + \partial_y^2) - (x^{2l} + y^{2l}) \partial_y^2\right]^2 + 4(\partial_x^2 + \partial_y^2) \partial_t^2,$

where l is a large enough integer.

For the operators in Example 1 ($n = 1$) and Example 2 ($n = 2$) Theorem 3 is true. These operators have nonsmooth characteristic roots of variable multiplicity. The roots in Example 1 are equal to

$$\tau_{1,2} = -i\xi \pm \sqrt{a_2(t)}\,\xi, \qquad \tau_{3,4} = i\xi \pm \sqrt{a_2(t)}\,\xi,$$

because we have

$$\tau^4 + (1 + a_2)^2 \xi^4 + 2(1 - a_2) \xi^2 \tau^2 = [\tau^2 + 2i\xi\tau - (1 + a_2)\xi^2][\tau^2 - 2i\xi\tau - (1 + a_2)\xi^2],$$

and those in Example 2 are equal to

$$\tau_{1,2} = -i\sqrt{\xi^2 + \eta^2} \pm \sqrt{a_2(t)(\xi^2 + \eta^2) + (x^{2l} + y^{2l})\eta^2},$$

$$\tau_{3,4} = i\sqrt{\xi^2 + \eta^2} \pm \sqrt{a_2(t)(\xi^2 + \eta^2) + (x^{2l} + y^{2l})\eta^2},$$

because we have

$$[\tau^2 - (1 + a_2)(\xi^2 + \eta^2) - (x^{2l} + y^{2l})\eta^2]^2 + 4(\xi^2 + \eta^2)\tau^2$$

$$= [\tau^2 - 2i\sqrt{\xi^2+\eta^2}\,\tau - (1+a_2)(\xi^2+\eta^2) - (x^{2l}+y^{2l})\eta^2]$$
$$\cdot [\tau^2 + 2i\sqrt{\xi^2+\eta^2}\,\tau - (1+a_2)(\xi^2+\eta^2) - (x^{2l}+y^{2l})\eta^2].$$

At the end of this section I would like to underline that recently many authors have obtained interesting uniqueness results: see e.g. Egorov's paper [9]. Very useful articles are the survey of Alinhac [2] and the course of Zuily [17].

4. Propagation of singularities contained in parameters

Let us consider an elliptic operator with respect to x

$$P(x, y, \partial_x) = \sum_{|\alpha| \leq m} a_\alpha(x, y)\,\partial_x^\alpha$$

with C^∞-coefficients in a neighbourhood $\omega_{x_0} \times O_{y_0}$ of (x_0, y_0), where $x = (x^1, x^2)$ (for the sake of simplicity) and $y = (y^1, \ldots, y^l)$ are parameters.

THEOREM 4. *Suppose P has in any direction at most double characteristic roots satisfying the smoothness requirements of Theorem 3 uniformly with respect to y when (x, y) varies in $\omega_{x_0} \times O_{y_0}$. Then if $Pu \in C^\infty(\omega_{x_0} \times O_{y_0})$ and $(x_0, y_0) \in \operatorname{sing\,supp} u$, then $(x, y_0) \in \operatorname{sing\,supp} u$, $\forall x \in \omega_{x_0}$.*

The proof uses a technique of Sjöstrand [16] based on a method of Hörmander and Carleman estimates. An easy consequence of this assertion is the following

PROPOSITION. *Let the coefficients of the system (4), (5) depend on parameters $\gamma = (\gamma_1, \ldots, \gamma_l)$ and belong to $C^\infty(\Omega \times G)$, $\gamma \in G$. On the characteristic roots of (4) we impose requirements analogous to those of the above theorem. Then if $(x_0, y_0, \gamma_0) \in \operatorname{sing\,supp} u \setminus \operatorname{sing\,supp} v$ and $f \in C^\infty$ near (x_0, y_0, γ_0), we have $(x, y, \gamma_0) \in \operatorname{sing\,supp} u$, $\forall (x, y)$ in a neighbourhood of (x_0, y_0). A similar assertion is valid for v.*

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