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Notes on integral transformations

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WROCLAWSKA DRUKARNIA NAUKOWA

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1. Introduction

One of the objectives of the theory of integral transformations is to single out special properties of linear transformations in vector spaces of measurable functions, which are of the form

$$u \to Ku, \qquad (Ku)(x) = \int_Y k(x, y)\, u(y)\, dy, \qquad x \in X.$$

The peculiarities of the situation stem from the nature of spaces of measurable functions as opposed to abstract vector space and from the special form of the transformations considered, as opposed to arbitrary linear transformations.

In the above paragraph (X, dx), (Y, dy) are measure spaces, $k(x, y)$ is a measurable function on $(X \times Y, dxdy)$ which is referred to as kernel of K.

$L^0(X)$, $L^0(Y)$ denote the spaces of all scalar valued, measurable, finite a.e. functions on X and respectively on Y.

The natural domain of K is defined by

$$D_K = \{u \in L^0(Y)\colon \int |k(x, y)|\, |u(y)|\, dy < \infty \text{ a.e.}\}.$$

Then $K\colon D_K \subset L^0(Y) \to L^0(X)$ is a linear transformation. $L^0(X)$ and $L^0(Y)$ are topological vector spaces with the topology of convergence in measure.

It is of interest to study properties of K in the following cases.

K considered as a transformation of D_K into $L^0(X)$.

K considered as an (unbounded) transformation from $L^0(Y)$ into $L^0(X)$.

K considered as a transformation from a topological subspace of $L^0(Y)$ into a topological subspace of $L^0(X)$.

Integral transformations are among the most common ones in Analysis and various aspects of their theory are by now classical. A recent renewal of interest in more abstract parts of the theory is documented by the lecture notes [K] and the monograph [HS].

The aim of this paper is to outline the part of the theory which can be developed with minimal assumptions on the kernel k and on the measure spaces X and Y. The point of view we adopt here is that of [ASz].

We did not make an attempt either to develop all topics described to the very end or to include all possible topics. The choice of material and its arrangement follows the courses given by the author at the University of Nice in Fall 1979 and at the University of Warsaw in Spring 1980.

The contents of the paper is organized as follows.

Section 2 is devoted to preliminaries and to some results about spaces of measurable functions, needed in the sequel.

In Section 3 we discuss the proper (or natural) domain D_K of an integral transformation with its natural topology. It is perhaps of interest that the graph topology of K on D_K is not a suitable one, however the graph topology of the sublinear transformation $u \to \int |k(x, y)| |u(y)| dy$ seems to be appropriate to study questions of continuity.

In Section 4 we consider various aspects of K as an unbounded transformation from $L^0(Y)$ into $L^0(X)$.

In Section 5 we deal with extensions by continuity of integral transformations to topological spaces of measurable functions. An example of the situation considered here is the Fourier transform in L^2. In this context the category of solid spaces seems to be particularly suitable — in this category there exists for every integral transformation K with a nontrivial domain, a maximal (in a suitable sense) solid space to which K can be extended by continuity. It is impossible to abandon entirely the hypothesis of solidity without imposing unreasonable restrictions on the kernel k.

In Section 6 we present some mostly known results about compactness of integral transformations; some effort has been made here to avoid imposing restrictions on the absolute values of the kernel k.

In Section 7 we collected in a rather sketchy form various results that did not fit into the preceding sections yet seemed necessary to give a rounded up description of the theory.

Section 8 is devoted to bibliographical remarks which were entirely kept out from the text and to some comments.

One of the conclusions that could be drawn from these notes is that it is expedient to study integral transformations in the context of spaces which are a priori not assumed to be locally convex (for instance $L^0(X)$ is not locally convex unless X is purely atomic). Indeed, we do not know if the natural domain D_K and the extended domain D_K (see Section 5) are locally convex in general, even though this is the case in many concrete examples.

Another conclusion is that even though the theory of integral transformations is relatively painless in the category of solid spaces, it would be of considerable interest to relax the latter hypothesis perhaps at the expense of imposition of some reasonable restrictions on K. Very little is known about unbounded integral transformations (e.g., in L^2) and part of the difficulties lies in the fact that their domains are not in general solid.

Throughout this paper we have made an effort to avoid imposing the condition of σ-finiteness of the measure space, unless we could not do without it. The reward is the possibility of including the treatment of the Fourier transform on locally compact groups in Section 5.

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University of Nice and of the Polish Academy of Sciences in Warsaw. It is the author's pleasant duty to thank both institutes for their hospitality.

2. Spaces of measurable functions

2.1. Throughout this paper (X, dx), (Y, dy) are measure spaces. Without stating this explicitly at each instance we consider only measurable subsets of X or Y. We use the symbol χ_E to denote the characteristic function of E. The measure of a subset $E \subset X$ $(E \subset Y)$ is noted by $|E|$. $\mathscr{F}(X)$ denotes the family of all subsets of X of finite measure and $\mathscr{F}_\sigma(X)$ the family of all subsets of X which are countable unions of sets in $\mathscr{F}(X)$ (i.e. are σ-finite). $\mathscr{F}(Y)$ and $\mathscr{F}_\sigma(Y)$ are defined in the same way. The term "almost everywhere" on X (a.e.) will be understood as almost everywhere on every set in $\mathscr{F}(X)$ (or equivalently on any set in $\mathscr{F}_\sigma(X)$); this convention is relevant only in the case when X is not σ-finite.

We use the adjective "non-atomic" to designate spaces which are not purely atomic and the adjective "divisible" to designate spaces that contain no atoms.

We denote by $L^0(X)$ the space of all equivalence classes of real or complex valued functions defined and finite a.e. on X, which are measurable on any subset in $\mathscr{F}(X)$. The equivalence relation is of course the equality a.e. $L^0(X)$ is a vector space with operations of pointwise addition and multiplication by scalars.

The term "vector subspace of $L^0(X)$" requires no explanation. If V is such subspace then we say on occasions that V is a vector space algebraically contained in $L^0(X)$. Vector subspaces of $L^0(X)$ are also referred to as vector spaces of measurable functions (on X).

The following comments are of significance only in the case when X is not σ-finite.

If $f \in L^0(X)$ and $E \subset X$ then the restriction $f|_E$ of f to E belongs to $L^0(E)$. The restrictions $\{f|_E\}_{E \in \mathscr{F}(X)}$ satisfy the condition $(f|_E)_{E \cap E'} = (f|_{E'})_{E \cap E'}$ a.e. for $E, E' \in \mathscr{F}(X)$. Conversely, any family $\{f_E\}_{E \in \mathscr{F}(X)}$ such that $f_E \in L^0(E)$ and $f_E|_{E \cap E'} = f_{E'}|_{E \cap E'}$ a.e. for any $E, E' \in \mathscr{F}(X)$ gives rise in an obvious way to a function $f \in L^0(X)$.

The sets in $\mathscr{F}(X)$ are partially ordered by inclusion: we write $E' \subset E$ provided $|(E \cup E') \Delta E| = 0$ where Δ denotes the symmetric difference. With this understanding we can consider $L^0(X)$ as an inductive limit of the spaces $L^0(E)$, $E \in \mathscr{F}(X)$.

2.2. The topology of $L^0(X)$. The natural topology of $L^0(X)$ is that of convergence in measure on all subsets of $\mathscr{F}(X)$ (it is worth noting that the topology of convergence in measure on X need not be a vector topology).

This topology can be defined by means of the following family of pseudometrics (or pseudonorms):

$$\varrho_E(f) = \varrho_E(f, 0) = |E|^{-1} \int_E (1+|f(x)|)^{-1} |f(x)|\, dx, \qquad E \in \mathscr{F}(X).$$

It is easy to see using the concluding remarks of subsection 2.1 that $L^0(X)$ is complete.

In the case when X is σ-finite, $X = \bigcup_n X_n$, $X_n \in \mathscr{F}(X)$, $X_n \subset X_{n+1}$, the topology of $L^0(X)$ can be defined by the sequence of pseudometrics ϱ_{X_n} and hence by a single metric. For instance we can define

$$\varrho_{(X_n)}(f) = \sum_{n=1}^{\infty} 2^{-n} \varrho_{X_n}(f).$$

Another way of writing down a metric of this kind is

$$\varrho_X(f) = \int_X (1+|f(x)|)^{-1} |f(x)|\, \varphi(x)\, dx$$

where $\varphi > 0$ a.e., $\int \varphi\, dx = 1$. Existence of a function φ with these properties is equivalent to σ-finiteness of X.

We remark that the topology of $L^0(X)$ is locally convex if and only if X is purely atomic.

2.3. Topological subspaces of $L^0(X)$. Solid spaces. If V is a topological vector space of measurable functions on X, then we write $V \underset{c}{\subset} L^0(X)$ to indicate that the inclusion mapping is continuous.

A subset $S \subset L^0(X)$ is *solid* if the conditions $u \in S$, $v \in L^0(X)$, $|v| \leqslant |u|$ a.e. imply that $v \in S$. A vector topology on a space of measurable functions is solid if it has a base consisting of solid neighborhoods of the origin. From now on by a solid topological vector space we shall mean a solid space with a solid topology. For example the spaces $L^p(X)$, $0 \leqslant p \leqslant \infty$, are solid, but the space of continuous functions on a topological space X is not solid.

2.3.1. THEOREM. *If A is a solid metric space of measurable functions on X, then $A \underset{c}{\subset} L^0(X)$. In other words the algebraic inclusion $A \subset L^0(X)$ implies a continuous inclusion.*

Proof. Assuming the contrary one can find a neighborhood U of the origin in $L^0(X)$ and a sequence $u_n \underset{A}{\to} 0$ with the property that $u_n \notin U$, $n = 1, 2, \ldots$ We may assume that U is of the form $U = \{u \in L^0(X)\colon |\{x \in E\colon |u(x)| > \alpha\}| < \alpha\}$ for some $E \in \mathscr{F}(X)$ and $\alpha > 0$. The condition $u_n \notin U$ amounts to saying that with $E_n = \{x \in E\colon |u_n(x)| > \alpha\}$ we have $|E_n| \geqslant \alpha$. Let $E'_n = \bigcup_{m=n}^{\infty} E_m$, $E' = \bigcap E'_n$, $E''_n = \bigcup_{m=n}^{m_n} E_m$, $E'' = \bigcap E''_n$ where the increasing sequence $\{m_n\}$, $m_n \geqslant n$ is so chosen that $|E'_n \sim E''_n| < 2^{-n-1}\alpha$. Then $E' \sim E'' = \bigcap E'_n \sim \bigcap E''_n \subset \bigcup (E'_n \sim E''_n)$ and $|E' \sim E''| < \alpha/2$, and it follows that

$|E''| \geqslant \alpha/2$. Denote by χ_m the characteristic function of E''. Then for every n, $\alpha\chi(x) \leqslant \sum_{m=n}^{m_n} |u_m(x)|$ a.e. and the hypothesis that A is solid implies that $\chi \in A$ and that

$$\varrho_A(\alpha\chi) \leqslant \varrho_A\Big(\sum_{m=n}^{m_n} |u_m|\Big) \leqslant 2^{-n-1}, \qquad n = 1, 2, \ldots$$

This implies that $\alpha\chi = 0$ contradicting the inequality $|E''| \geqslant \alpha/2$. ∎

2.3.2. *If A is a solid metrizable vector space, then there is a metric ϱ on A with the following properties:*

(i) $\varrho(u) = \varrho(|u|)$ *for every* $u \in A$,

(ii) $\varrho(u) \leqslant \varrho(v)$ *whenever* $|u| \leqslant |v|$ *a.e.*

Proof. Let ϱ' be any metric defining the topology on A. If $u \in A$ and $|v| \leqslant |u|$ a.e., then there is a solid neighborhood of 0, U, such that $U \subset \{w \in A\colon \varrho'(w) \leqslant \varrho'(u)\}$. Also there is an integer m such that $(1/m)\{w \in A\colon \varrho'(w) \leqslant \varrho'(v)\} \subset U$. If follows that $(1/m)v \in U$ and $\varrho'(v) \leqslant m\varrho'(u)$ and that $\sup\{\varrho'(v)\colon |v| \leqslant |u| \text{ a.e.}\} := \varrho(u) < \infty$ for every $u \in A$. We will prove that ϱ is a metric on A with the desired properties. Clearly, $\varrho'(u) \leqslant \varrho(u) = \varrho(|u|)$ and $\varrho(u) \leqslant \varrho(w)$ if $|u| \leqslant |w|$ a.e. To prove that ϱ satisfies the triangle inequality, we note that if $|v| \leqslant g_1 + g_2$, $g_i \geqslant 0$, then

$$v = \frac{g_1}{g_1+g_2}v + \frac{g_2}{g_1+g_2}v := v_1 + v_2 \qquad (= 0 \text{ if } g_1 + g_2 = 0)$$

where $|v_1| \leqslant g_1$, $|v_2| \leqslant g_2$ and $\varrho(u_1+u_2) = \sup\{\varrho'(v)\colon |v| \leqslant |u_1+u_2|\} \leqslant \sup\{\varrho'(v)\colon |v| \leqslant |u_1|+|u_2|\} = \sup\{\varrho'(v_1+v_2)\colon |v_1| \leqslant |u_1|,\ |v_2| \leqslant |u_2|\} \leqslant \varrho(u_1) + \varrho(u_2)$.

It remains to verify that $\varrho'(u_n) \to 0$ implies that $\varrho(u_n) \to 0$ for any sequence $\{u_n\} \subset A$. Assuming the contrary we find a sequence $\{u_n\} \subset A$ and $\alpha > 0$ such that $\varrho'(u_n) \to 0$ and $\varrho(u_n) \geqslant \alpha$. By the definition of the metric ϱ we then can find $\{v_n\}$ such that $|v_n| \leqslant |u_n|$ a.e. and $\varrho'(v_n) \geqslant \alpha/2$ which contradicts the hypothesis that the topology of A is solid. ∎

We observe that if ϱ' is a norm, then so is ϱ.

From now on when dealing with solid metric (or normed) spaces we shall without loss of generality assume that given metrics (or norms) satisfy (i) and (ii).

2.3.3. Let $A \subset L^0(X)$ be a vector space of measurable function. A subset $E \subset X$ is an unfriendly set for A if $f|_E = 0$ for every $f \in A$.

PROPOSITION. *Assume that X is σ-finite and let A be a vector space of measurable functions on X. Then*

(i) *there exists a maximal and unique up to sets of measure* 0 *unfriendly set for A which we denote X_A;*

(ii) *if A is a complete metric space, then there exists $v \in A$ such that* $\{x\colon v(x) = 0\} = X_A$.

Proof. The proof of (i) depends on the following lemma:

If X is σ-finite and if $u, v \in L^0(X)$, then, except for an at most countable set of values of $\xi \in \mathbf{R}$, the sets $E = \{x\colon |u(x)|+|v(x)| > 0\}$ and $E_\xi = \{x\colon u(x)+\xi v(x) \neq 0\}$ differ by a set of measure 0.

This follows from the fact that the sets $E \cap (X \sim E_\xi)$ are disjoint for different values of ξ and since X is σ-finite, at most countably many of them can have positive measure.

Let $\varphi \in L^0(X)$, $\varphi > 0$, $\int_X \varphi\, dx = 1$ and define $\mu(E) = \int_E \varphi\, dx$ for $E \subset X$. Denote $\alpha = \sup\{\mu(\{x\colon u(x) \neq 0\})\colon u \in A\}$ and let $u_n \in A$ be any sequence with the property that $\mu(\{x\colon u_n(x) \neq 0\}) \to \alpha$; using the lemma we construct by induction a sequence $v_n \in A$ of the form

$$v_n = u_1 + \xi_2 u_2 + \ldots + \xi_n u_n$$

with the property that

$$X_n = \{x\colon |u_1(x)| + \ldots + |u_n(x)| > 0\} = \{x\colon v_n(x) \neq 0\}.$$

Then $\mu(X_n) = \alpha_n \nearrow \alpha$. Let $X' = \bigcup X_n$ and $X_A = X \sim X'$. We claim that X_A has the desired property.

If $u \in A$ and $\{x \in X_A\colon u(x) \neq 0\}$ is of positive measure, then for n sufficiently large, using the lemma, we can find a coefficient $\xi \in \mathbf{R}$ such that $\mu(\{x\colon u_n(x)+\xi u(x) \neq 0\}) > \alpha$, contrary to the choice of α. It follows that $u|_{X_A} = 0$ for every $u \in A$.

If $X'_A \supset X_A$ is any set with the property that $u|_{X'_A} = 0$ for every $u \in A$, then $|X'_A \sim X_A| = 0$, otherwise $\mu(\{x\colon u(x) \neq 0\}) \leqslant \mu(X') - \mu(X'_A \sim X_A) = \alpha - \mu(X'_A \sim X_A)$, which proves the maximality and uniqueness properties.

(ii) With the same notations as in the proof of (i) we choose the coefficients $\{\xi_n\}$ so as to insure that the sequence $\{v_n\}$ converges in A and that its limit v does not vanish on X'. This is done by induction as follows. Choose $\{M_n\}$, $\{\lambda_n\}$-sequences of positive numbers in such a way that $\mu(\{x\colon |u_n(x)| > M_n\}) \leqslant 2^{-n}$, $\varrho(\xi u_n) \leqslant 2^{-n}$ for all ξ such that $|\xi| \leqslant \lambda_n$, where ϱ denotes the metric of A. Assuming that $\xi_1, \ldots, \xi_n$, $\eta_1, \ldots, \eta_{n-1}$ are already chosen, $\xi_1 = 1$, $\eta_1 \leqslant 1$, pick $\eta_n > 0$, such that $\eta_n \leqslant \frac{1}{2}\eta_{n-1}$ and $\mu(\{x\colon |v_n(x)| > \eta_{n-1}\}) > \alpha_n - 2^{-n}$ and use the lemma to find ξ_{n+1}, $0 < \xi_{n+1} \leqslant \min(\lambda_{n+1}, (2M_{n+1})^{-1}\eta_n)$ such that $\{x\colon v_n(x) + \xi_{n+1} u_{n+1}(x) \neq 0\} = X_n$ as in the proof of (i).

Let $v_{n+1} = v_n + \xi_{n+1} u_{n+1}$. Then the sequence $\{v_n\}$ is convergent in A and the series $\lim v_n = v = \sum \xi_n u_n$ is convergent absolutely a.e. Also, for every m we have

$$|v(x)| \geqslant |v_m(x)| - \sum_{n=m+1}^{\infty} \xi_n |u_n(x)| \geqslant \eta_m - \tfrac{1}{2} \sum_{n=m+1}^{\infty} \eta_n \geqslant \tfrac{1}{2}\eta_m$$

on

$$X_m \cap \{x\colon |v_m(x)| > \eta_m\} \cap \{|x\colon |u_1(x)| \leqslant M_l,\ l = m+1, m+2, \ldots\}.$$

The latter set differs from X' by a set of μ-measure not exceeding $\alpha - \alpha_m + 2^{-m} + 2^{-m}$ and it follows that $v(x) \neq 0$ a.e. on X'. ∎

In most of applications we can replace X by $X \sim X_A$ and identify A with the space of restrictions $u|X \sim X_A$, $u \in A$, and can thus consider the case when A has no unfriendly sets.

For a solid topological vector space A of measurable functions on X, we denote by A_a the subspace of A of all functions $u \in A$ with the property $\chi_{E_n} u \to 0$ for every sequence $E_n \subset X$ such that $E_n \searrow \emptyset$, the latter notation meaning that E_n is decreasing and that $|(\bigcap E_n) \cap E| = 0$ for every $E \in \mathscr{F}(X)$. In the case when $|X| < \infty$ an equivalent requirement is $\lim_{|E| \to 0} \chi_E u = 0$.

It is easy to check that A_a is a solid closed subspace of A. Also, if $X_n \nearrow X$ is any sequence then $\chi_{X_n} u \to u$ for every $u \in A_a$.

If $A = L^p(X)$ and $0 < p < \infty$ then $A_a = A$, however $A_a = \{0\}$ if $p = \infty$ and X is non-atomic or contains infinitely many atoms.

2.3.4. PROPOSITION. *Suppose that X is σ-finite and that A is a complete solid vector metric space of measurable functions on X. Then a set $C \subset A_a$ is compact in A if and only if*

(i) *C is compact in $L^0(X)$,*

(ii) *for every sequence $E_n \searrow \emptyset$ of subsets of X the limit* $\lim \varrho_A(\chi_{E_n} u) = 0$ *is uniform in $u \in C$.*

Proof. The necessity of the condition (i) follows from Theorem 2.3.1, the necessity of (ii) is obvious.

If $\{u_n\}$ is any sequence in C then by (i) there exists $u \in C$ and a subsequence of $\{u_n\}$ denoted again by $\{u_n\}$ such that $u_n \xrightarrow[L^0(X)]{} u$. Selecting if necessary another subsequence (using σ-finiteness of X) we can assume that $u_n \to u$ a.e. We will show now that $u_n \to u$ in A.

To this effect we may assume that A has no unfriendly sets and by 2.3.2 we can find $\varphi \in A$ such that $\varphi > 0$ a.e. Let

$$X_{nk} = \left\{x \in X\colon \sup_{m \geqslant n} |u_m(x) - u(x)| \geqslant \frac{1}{k}\varphi(x)\right\},$$

then for every fixed k $X_{nk} \searrow \emptyset$ and for $\varepsilon > 0$ we can find by condition (ii) an index n_k such that $\varrho(\chi_{nk} v) < \varepsilon/3$ for every $n \geqslant n_k$ and $v \in C$, where χ_{nk} denotes the characteristic function of X_{nk}. We can now write

$$\begin{aligned}\varrho(u_n - u) &\leqslant \varrho(\chi_{nk}(u_n - u)) + \varrho((1-\chi_{nk})(u_n - u))\\ &\leqslant \varrho(\chi_{nk} u_n) + \varrho(\chi_{nk} u) + \varrho((1-\chi_{nk})(u_n - u))\\ &\leqslant \varrho(\chi_{nk} u_n) + \varrho(\chi_{nk} u) + \varrho\left(\frac{1}{k}\varphi\right)\end{aligned}$$

(see 2.3.2).

We now choose k so that $\varrho\left(\frac{1}{k}\varphi\right) < \varepsilon/3$, then for $n \geqslant n_k\, \varrho(u_n-u) < \varepsilon$. ∎

2.3.5. PROPOSITION (monotone convergence theorem). *Let $A \subset L^0(X)$ be a solid vector metric space. Then $u \in A_a$ if and only if for every sequence $v_n \geqslant 0$ such that $|u(x)| \geqslant v_1(x) \geqslant v_2(x) \geqslant \ldots \geqslant v_n(x) \to 0$ a.e. we have $v_n \underset{A}{\to} 0$.*

Proof. If u satisfies the condition and $E_n \searrow \emptyset$ then $v_n = \chi_{E_n}|u|$ satisfies $v_n \leqslant |u|$, $v_n \geqslant 0$ a.e. and $\chi_{E_n} u \underset{A}{\to} 0$. Hence $u \in A_a$ and the condition is sufficient. To prove the necessity, suppose that $u \in A_a$ and let v_n be any sequence with the indicated properties. Let $E_{mn} = \left\{x\colon v_n(x) \leqslant \frac{1}{m}|u(x)|\right\}$ and χ_{mn} be the characteristic function of E_{mn}. Then $(X \sim E_{mn}) \searrow \emptyset$ for $n \to \infty$ and $(1-\chi_{mn})u \underset{A}{\to} 0$ for $n \to \infty$. We have

$$\varrho(v_n) \leqslant \varrho\left(\frac{1}{m}\chi_{mn}u\right) + \varrho\big((1-\chi_{mn})u\big) \leqslant \varrho\left(\frac{1}{m}u\right) + \varrho\big((1-\chi_{mn})u\big).$$

For every $\varepsilon > 0$ there is m such that $\varrho\left(\frac{1}{m}u\right) < \varepsilon/2$ and for m fixed we can find n such that $\varrho\big((1-\chi_{mn})u\big) < \varepsilon/2$. ∎

2.3.6 (dominated convergence theorem). *With the hypotheses of 2.3.5 $u \in A_a$ if and only if for every sequence $\{u_n\} \subset A$ such that $|u_n(x)| \leqslant |u(x)|$ a.e. and $u_n(x) \to u(x)$ a.e. we have $u_n \underset{A}{\to} u$.*

Proof. If $E_n \searrow \emptyset$ then $u_n = (1-\chi_{E_n})u$ has the property indicated in the statement and it follows that the condition is sufficient. Suppose that $u \in A_a$ and that $\{u_n\}$ has the indicated properties. Then $v_n(x) = \sup\{|u_m(x) - u(x)|\colon m \geqslant n\}$ satisfies $|v_n(x)| \leqslant 2u(x)|$ a.e. and $v_n \searrow 0$. By 2.3.5 $v_n \underset{A}{\to} 0$ and the condition is necessary. ∎

2.4. Solid Banach spaces.

2.4.1. Associated spaces. Let $A \subset L^0(X)$ be a solid Banach space with the norm $\|\ \|$. The associated space A' is defined by

$$A' = \{v \in L^0(X)\colon \textstyle\int |uv|\,dx < \infty \text{ for all } u \in A\}.$$

The associated norm $\|\ \|'$, is given by

$$\|v\|' = \sup\left\{\left|\textstyle\int uv\,dx\right|\colon u \in A,\ \|u\| \leqslant 1\right\}, \qquad v \in A'.$$

The norm $\|\ \|'$ satisfies the properties 2.3.2 (i), (ii) and A' with the norm $\|\ \|'$ is a solid Banach space. With the pairing $\langle u, v\rangle = \int_X uv\,dx$, A' can be identified isometrically with a closed subspace of the dual space A^* but in general $A' \neq A^*$; consider for instance $A = L^\infty(X)$.

2.4.2. THEOREM. *$(A')' = A$ and the norms $\| \ \|$ and $\| \ \|''$ on A are equivalent.*

2.4.3. We denote by $\sigma(A, A')$ the weak topology of A' on A.

THEOREM. *Suppose that $A \subset L^0(X)$ is a solid Banach space, that X is σ-finite and that $A'_a = A'$. Then every bounded sequence in A contains a subsequence convergent in $\sigma(A, A')$-topology to an element of A.*

2.5. Bounded subsets of $L^0(X)$. It is easily seen from the definition of the topology of $L^0(X)$ that a set $B \subset L^0(X)$ is bounded if and only if for every $E \in \mathscr{F}(X)$ and $\varepsilon > 0$ there exists $N_{\varepsilon,E} > 0$ such that $|\{x \in E: |u(x)| \geqslant N_{\varepsilon,E}\} < \varepsilon$.

2.5.1. LEMMA. *Suppose that $|X| < \infty$ and that $B \subset L^0(X)$ is a convex bounded set consisting of non-negative functions. Then for every $\varepsilon > 0$ there is a set $X_\varepsilon \subset X$ such that $|X \sim X_\varepsilon| < \varepsilon$ and there is a constant $N_\varepsilon > 0$ such that $\int_{X_\varepsilon} u\,dx \leqslant N_\varepsilon$ for every $u \in B$.*

Proof. For fixed $L > 0$, $u \in L^0(X)$ define $u_L(x) = u(x)$ if $|u(x)| \leqslant L$ and $u_L(x) = L \operatorname{sign} u$ if $|u(x)| > L$ and let $B_L = \{u_L: u \in B\}$. B_L is convex. Let $V = \{v \in L^0(X): v: X \to [0, 1], \int v\,dx \geqslant 1 - \frac{1}{2}\varepsilon\}$. Then V is a weakly compact convex set in $L^2(X)$ and the function $f(u, v) = \int uv\,dx$. $(u, v) \in BxV$ is continuous and convex in v for fixed u and concave in u for fixed v. By the minimax theorem of Ky Fan

$$\sup_{u \in B_L} \inf_{v \in V} f(u, v) = \inf_{v \in V} \sup_{u \in B_L} f(u, v).$$

Let $N > 0$ be such that $|\{x \in X: u_L(x) \geqslant N\}| \leqslant |\{x \in X: u(x) \geqslant N\}| < \varepsilon/2$ for every $u \in B$. Then choosing $v = \chi_{Y_\varepsilon} \in V$, $Y_\varepsilon = \{x \in X: |u(x)| < N\}$ we see that $\inf_{v \in V} f(u, v) < N$ for every fixed $u \in B_L$ and it follows that there is $v_L \in V$ such that $f(u, v) = \int uv_L\,dx \leqslant N$ for every $u \in B_L$. Observe that for $L' < L$ we have $\int uv_L\,dx \leqslant N$ for every $u \in B_{L'}$. Let $v \in V$ be a weak limit of $\{v_L\}$, then $\int uv\,dx \leqslant N$ for every $u \in \bigcup B_L$ and, by the monotone convergence theorem, for every $u \in B$. It suffices not to notice that $|\{x: v(x) \leqslant \frac{1}{2}\}| < \varepsilon$. ■

2.5.2. PROBLEM. Is the conclusion of the lemma valid without the hypothesis that B consists of nonnegative functions?

2.5.3. COROLLARY. *Let A be a solid Banach space, $A \subset L^0(X)$ and let X be of finite measure. Then for every $\varepsilon > 0$ there is a subset $X_\varepsilon \subset X$ such that $|X \sim X_\varepsilon| < \varepsilon$ and $A|_{X_\varepsilon} \subset L^1(X_\varepsilon)$.*

Proof. Since by Theorem 2.3.1 the inclusion $A \subset L^0(X)$ is continuous, the unit sphere of A is bounded in $L^0(X)$. Let $B = \{u \in A: u \geqslant 0, \|u\| \leqslant 1\}$. The result follows directly from Lemma 2.5.1 and the decomposition $u = u^+ - u^-$, $\|u^+\| \leqslant \|u\|$, $\|u^-\| \leqslant \|u\|$, u^+, $u^- \geqslant 0$, valid for every $u \in A$. ■

3. Proper domain of an integral transformation

3.1. Let X, Y be measure spaces and $k \in L^0(X \times Y)$. We consider the linear transformation from $L^0(Y)$ into $L^0(X)$ given by

$$Ku = \int k(x, y) u(y) dy.$$

We refer to k as the kernel of K and to K as the integral transformation with the kernel k. In the case when $X = Y$ we refer to K as an integral operator with kernel k.

3.2. Throughout the paper we will, without explaining it in every instance, use k (possibly with subscripts) to denote kernels and K (with corresponding subscripts) to denote the corresponding integral transformations or operators. The proper (natural) domain of K is defined by

$$D_K = \{u \in L^0(Y): \int |k(x, y)| |u(y)| dy < \infty \text{ a.e. and } \operatorname{supp} u \in \mathscr{F}_\sigma(Y)\},$$

where $\operatorname{supp} u = \{y: u(y) \neq 0\}$. The second condition guarantees that Ku is measurable for every $u \in D_K$; obviously this condition is satisfied automatically if Y is σ-finite. It is clear that D_K is a solid vector subspace of $L^0(Y)$. For $u \in D_K$ we write

$$|K| |u| = \int |k(x, y)| |u(y)| dy.$$

The symbol D_K is to be distinguished from $D(K)$ which will be used to denote the domain of K considered as a transformation between subspaces of $L^0(Y)$ and $L^0(X)$ (to be specified).

3.3. For $u \in D_K$, $F \in \mathscr{F}(Y)$, $E \in \mathscr{F}(X)$ we define $\varrho_{K,E,F,}(u) = \varrho_F(u) + \varrho_E(|K| |u|)$.

3.3.1. THEOREM. *With the topology defined by the family of pseudometrics $\varrho_{K,E,F}$ D_K is a sequentially complete solid topological subspace of $L^0(X)$ and the linear transformation $K: D_K \to L^0(X)$ is continuous.*

Proof. It is clear that the functions $u \to \varrho_{E,F,K}(u)$ are pseudometrics. The continuity of K follows from the inequalities $|Ku| \leqslant |K| |u|$ a.e. and $\varrho_E(|K| |u|) \leqslant \varrho_{K,E,F}(u)$ valid for every $u \in D_K$, $F \in \mathscr{F}(Y)$ and $E \in \mathscr{F}(X)$.

It is obvious that the topology of D_K defined above is solid.

It remains to verify that D_K is sequentially complete. Any Cauchy sequence $\{u_n\}$ in D_K is Cauchy in $L^0(X)$ and therefore there exists $u \in L^0(Y)$ such that $u_n \xrightarrow[L^0(Y)]{} u$. We will show now that $u \in D_K$ and that $u_n \xrightarrow[D_K]{} u$. By the definition of D_K the set $F = \bigcup \operatorname{supp} u_n$ is in $\mathscr{F}_\sigma(Y)$ and (see 2.2) $\{u_n\}$ is Cauchy with respect to any of the pseudometrics $\varrho_{K,E,F}(v) = \varrho_F(v) + \varrho_E(|K| |v|)$, $E \in \mathscr{F}_\sigma(X)$. This implies existence of a subsequence $\{u_n'\} \subset \{u_n\}$ with the property that $\sum_n \varrho_{K,E,F}(u_{n+1}' - u_n') < \infty$. It follows that the series $\sum_{n=1}^{\infty} |u_{n+1}' - u_n'|$

and $\sum_{n=1}^{\infty} |K||u'_{n+1}-u'_n|$ converge in $L^0(F)$ and $L^0(E)$, in particular they converge a.e. on E and respectively on F. The inequality $|u(x)| \leqslant |u'_1|+\sum_{n=1}^{\infty} |u'_{n+1}-u'_n|$ and the monotone convergence theorem show that

$$|K|\,|u| \leqslant |K|\big(|u'_1|+\sum_{n=1}^{\infty} |u'_{n+1}-u'_n|\big) = |K|\,|u'_1|+\sum_{n=1}^{\infty} |K|\,|u'_{n+1}-u'_n| < \infty \text{ a.e. on } E$$

and the same use of the inequality $|u-u'_m| \leqslant \sum_{n=m}^{\infty} |u'_{n+1}-u'_n|$ yields

$$|K|\,|u-u'_m| \xrightarrow[m\to\infty]{} 0 \text{ a.e. on } E, \quad \text{and} \quad \varrho_{E,F,K}(u-u'_n) \xrightarrow[m\to\infty]{} 0.$$

Since E is arbitrary $u \in D_K$ and $u_m \xrightarrow[D_K]{} u$. ∎

3.3.2. The vector topology defined on D_K by the family of pseudometrics in 3.3 is called the natural topology of D_K. This topology can also be defined by the family of pseudometrics

$$\varrho_{K,E,F}(\cdot) = \varrho_F(\cdot)+\varrho_E(|K|\,|\cdot|), \quad F \in \mathscr{F}_\sigma(Y),\ E \in \mathscr{F}_\sigma(X) \quad \text{(see 2.2).}$$

We also mention the following special cases. If X is σ-finite then the natural topology of D_K is given by the family of pseudometrics

$$\varrho_{K,F}(\cdot) = \varrho_F(\cdot)+\varrho_X(|K|\,|\cdot|), \quad F \in \mathscr{F}(Y).$$

If Y is σ-finite then the topology is given by the family of metrics

$$\varrho_{K,F}(\cdot) = \varrho_Y(\cdot)+\varrho_F(|K|\,|\cdot|), \quad E \in \mathscr{F}(X).$$

If X and Y are both σ-finite then the topology is given by the single metric

$$\varrho_K(\cdot) = \varrho_Y(\cdot)+\varrho_X(|K|\,|\cdot|).$$

3.3.3. It should be noted that the natural topology of D_K is the graph topology of the sublinear transformation $u \to |K|\,|u|$. The graph topology of K on D_K is in general weaker than the natural topology, and in general the graph topology need not be sequentially complete (see Th. 4.3.).

3.3.4. If $A \subset D_K$ is a topological subspace of $L^0(X)$, then we say that $K|_A$ (or K for brevity) is an integral transformation on A. In the case when $X = Y$, $A \subset D_K$ and $KA \subset A$, we say that K is an integral operator in A.

3.3.5. THEOREM. *Suppose that $A \underset{c}{\subset} L^0(Y)$ is a complete metric vector subspace of $L^0(X)$ and suppose that $A \underset{c}{\subset} D_K$. Then the integral transformation $K\colon A \to L^0(X)$ is continuous.*

Proof. Consider first the case when both X and Y are σ-finite. Then D_K is complete metric vector space and we can use the closed graph theorem to prove that the inclusion mapping $A \subset D_K$ is continuous. Indeed, if $u_n \xrightarrow[A]{} u$

and $u_n \underset{D_K}{\to} 0$, then $u_n \xrightarrow[L^0(X)]{} u$, since $A \underset{c}{\subset} L^0(X)$ and $u_n \xrightarrow[L^0(X)]{} 0$ since $D_K \underset{c}{\subset} L^0(X)$. It follows that $u = 0$. The claim follows now from the continuity of K: $D_K \to L^0(X)$.

In the general case we have to show that for every sequence $u_n \in A$ such that $u_n \underset{A}{\to} 0$ and for every $E \in \mathscr{F}(X)$ we have $\varrho_E(Ku_n) \to 0$; the convergence to 0 of $\varrho_F(u_n)$, $F \in \mathscr{F}(Y)$ follows from the continuity of the inclusion $A \subset L^0(Y)$. Since $u_n \in D_K$ the set $Y' = \bigcup \operatorname{supp} u_n$ is σ-finite and we can apply the preceding result with A replaced by $\overline{[\{u_n\}]}^A$ – the closure in A of the linear span of $\{u_n\}$, with Y replaced by Y' and with X replaced by E. ■

3.3.6. Corollary. *Suppose that A satisfies the hypotheses of Theorem 3.3.5 and $B \underset{c}{\subset} L^0(X)$ is a complete metric vector space. Suppose further that $KA \subset B$. Then K: $A \to B$ is continuous.*

Proof. We use again the closed graph theorem. If $u_n \in A$ $u_n \underset{A}{\to} 0$, $Ku_n \underset{B}{\to} v$ then by Theorem 3.3.5 $u_n \underset{D_K}{\to} 0$ and $Ku_n \xrightarrow[L^0(X)]{} 0$. Since $B \underset{c}{\subset} L^0(X)$, $Ku_n \xrightarrow[L^0(X)]{} v$ and $v = 0$.

3.3.7. Proposition. *Let K be an integral transformation. Then $D_K = (D_K)_a$.*

Proof. If $f \in D_K$, $E_n \searrow \emptyset$ then, for every $F \in \mathscr{F}(Y)$, $F \cap E_n \to 0$ and $\varrho_F(\chi_{E_n} f) \to 0$. Also $|K||\chi_{E_n} f| \to 0$ a.e. by the dominated convergence theorem and $\varrho_E(|K||\chi_{E_n} f|) \to 0$ for every $E \in \mathscr{F}(X)$. ■

3.3.8. Remark. If Y is non-atomic then there is no integral transformation with the property that $D_K = L^\infty(Y)$.

3.3.9. It is not clear to what extend the hypotheses that A, B in 3.3.5, 3.3.6 are complete metric spaces can be relaxed without changing the conclusions.

3.3.9. We do not know whether D_K with its natural topology is complete.

3.4. Examples.

(a) Let $X = [0, 2\pi]$, with the Lebesgue measure, $Y = \boldsymbol{Z}$ with the discrete measure $|\{y\}| = 1$, $y = 0, \pm 1, \ldots, k(x, y) = e^{ixy}$. Then $D_K = L^1(\boldsymbol{Z}) = l^1$.

(b) Let $X = \boldsymbol{Z}$, $Y = [0, 2\pi]$ with measure as above, $k(x, y) = \frac{1}{2\pi} e^{ixy}$. Then $D_K = L^1(0, 2\pi)$.

(c) Let $X = \boldsymbol{R} = Y$ with the Lebesgue measure and let $k(x, y) = \frac{1}{\sqrt{2\pi}} e^{-ixy}$. Then $D_K = L^1(\boldsymbol{R}^1)$.

(d) Let $X = \boldsymbol{R} = Y$, let $\varphi \in C(\boldsymbol{R})$ be a nonvanishing function with compact support and let $k(x, y) = \varphi(x - y)$. Then $D_K = L^1_{\text{loc}}(\boldsymbol{R}^1)$.

3.4.1. 3.4(d) shows that D_K with its natural topology need not be a Banach space. We do not know a kernel k with the property that D_K with its natural topology is not locally convex.

4. Integral transformations in L^0. Continuity, and closibility

4.1. Suppose that X, Y are σ-finite and let K be an integral transformation with kernel k.

By 2.3.3 there exists the maximal unfriendly set Y_K for D_K. K is called *nonsingular* if $Y_K = \emptyset$, K is *singular* if $Y_K = Y$ and K is *partly singular* if $|Y_K| > 0$, $|Y \sim Y_K| > 0$.

An example of a singular kernel is the Hilbert kernel $k(x, y) = 1/(x-y)$, $X = Y = \boldsymbol{R}^1$.

4.2. From now on in this section we assume that K is nonsingular. All the results can be interpreted for partly singular kernel by replacing Y by $Y \sim Y_K$.

We consider $K\colon D_K \subset L^0(Y) \to L^0(X)$ as an unbounded linear transformation and look for conditions on the kernel k in order that K be continuous, closed or closible.

Since Y is σ-finite, Y can be written in the form $Y = Y' \cup \{b_n\}$ where Y' is divisible (purely non-atomic) and $\{b_n\}$ is the set of all atoms of Y, which is at most countable. We consider the following conditions on the kernel k.

(A) $k(x, y) = 0$ a.e. on $X \times Y'$.

(B) Let $X_m = \{x \in X\colon k(x, b_n) \neq 0\}$. Then

$$|\limsup X_m| = \Big|\bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} X_m\Big| = 0.$$

(C) Let $B_m = \overline{[\{k(x, b_n)\colon n \geqslant m\}]}$ where $\overline{[\ ]}$ denotes the closure in $L^0(X)$ of the linear span.

4.3. THEOREM. *In order that* $K\colon L^0(Y) \to L^0(X)$ *be closible it is necessary and sufficient that* k *satisfy* (A) *and* (C).

4.4. THEOREM. *In order that* $K\colon L^0(Y) \to L^0(X)$ *be continuous it is necessary and sufficient that* k *satisfy* (A) *and* (B). *An equivalent condition is that* $D_K = L^0(Y)$.

Proof of Theorems 4.3, 4.4. Assume that (A) is satisfied. Then, with $|b_n|$ denoting the measure of the atom b_n we can write

$$Ku(x) = \sum u(b_n)|b_n|\, k(x, b_n).$$

If $u_n \xrightarrow[L^0(Y)]{} 0$, then for every m, $u_n(b_m) \xrightarrow[n\to\infty]{} 0$ and outside of the set $\limsup X_n$, $Ku_n \to 0$. Thus if (B) is satisfied K is continuous. Also (A) and (B) imply that $D_K = L^0(Y)$.

If $u_n \to 0$ and $Ku_n \to v$, then (still assuming (A)) there is a sequence of integers $m_j \to \infty$ such that $\sum_{s=m_n}^{\infty} u_n(b_s)|b_s|\, k(x, b_s) \to v$ and it follows that $v \in \bigcap_{m=1}^{\infty} B_m$. Thus, if (C) is satisfied, then K is closible.

Suppose now that K is closible, let $Y_1 = Y' \cap \{y\colon |\{x\colon k(x, y) \neq 0\}| > 0\}$ and suppose that $|Y_1| > 0$. Using the hypothesis that K is nonsingular, we can find a subset $Y_2 \subset Y$ such that $0 < |Y_2| < \infty$ and $\chi_{Y_2} \in D_K$. Then there exists a subset $X_1 \subset X$ such that

$$0 < \int_{X_1} \int_{Y_2} |k(x, y)|\, dy dx < \infty$$

and consequently a subset $C \subset X_1 \times Y_2$ such that

$$\iint_C k(x, y)\, dy dx \neq 0.$$

Since C can be approximated in measure by sets of the form $\bigcup E_j \times F_j$, $E_j \subset X_1$, $F_j \subset Y_2$, it follows that there exist $E \subset X_1$, $F \subset Y_2$ such that

$$\int_E \int_F k(x, y)\, dy dx \neq 0.$$

We may assume without a loss of generality that $|F| = 1$. Using the fact that F is divisible we construct a sequence of binary partitions of F

$$P_k = \{F_{i_1,\dots,i_k}\colon i_1, \dots, i_k = 0, 1\}, \qquad F = \bigcup \{F_{(i)} \in P_k\},$$

where $F_{i_1,\dots,i_k} = F_{i_1,\dots,i_k,0} \cup F_{i_1,\dots,i_k,1}$, $|F_{i_1,\dots,i_k}| = 2^{-k}$. Denote by $\chi_{(i)}$ the characteristic function of $F_{(i)}$ and let $\{I_{(i)}\}$ be the usual binary partition of $[0, 1]$. Since $\chi_E \in D_K$, $|K|\chi_F$ is defined and finite a.e., we may assume that this is the case for every $x \in E$. For $x \in E$ define

$$\Phi_x(I_{(i)}) = \int_{F_{(i)}} k(x, y)\, dy;$$

it is easily checked that Φ_x can be extended to an additive and absolutely continuous with respect to the Lebesgue measure set function on $[0, 1]$. If I' denotes the complement in I of the set of binary points, then for every $t \in I'$ there is a unique sequence $I_{(i)_n(t)}$, $(i)_n = (i_1, \dots, i_n)$, such that $I_{(i)_n(t)} \downarrow \{t\}$. By the Lebesgue theorem, for every $x \in E$ the limit $\lim_{n \to \infty} 2^n \Phi_x(I_{(i)_n(t)}) = v_t(x)$ exists for a.e. $t \in I'$, thus for a.e. $t \in I$. It follows that for a.e. $t \in I$ the above limit exists and is finite for a.e. $x \in E$, and for a.e. $t \in I$, $v_t(x) \in L^0(E) \subset L^0(X)$. We remark that for some $t_0 \in I'$, $v_{t_0}(x) \not\equiv 0$; in fact, if $v_t(x) = 0$ a.e. for every $t \in I$, *then*

$$\Phi(I_{(i)}) = \int_{I_{(i)}} v_t(x)\, dt = 0 \text{ for a.e. } x \in E$$

and

$$\int_E \int_F k(x, y)\, dy dx = 0$$

which is a contradiction. With t_0 chosen so that $v_{t_0}(x) \not\equiv 0$ we observe that $u_n = 2^n \chi_{(i)_n(t)} \xrightarrow[L^0(Y)]{} 0$ and $Ku_n \xrightarrow[L^0(X)]{} v_{t_0}$ showing that if K is closible, then K satisfies (A). If K is continuous, then K is closible and K satisfies (A).

Suppose now that $v \in \bigcap_{l_m} B_m$ and that K is closible. Then for every m there exists $v_m(x) = \sum_{l=k_m} c_l^m k(x, b_l)$ such that $v_m \xrightarrow[L^0(X)]{} v$ and $k_m \geqslant m$. Let $u_m(b_l) = c_l^m |b_l|^{-1}$ if $k_m \leqslant l \leqslant l_m$ and $u_m(b_l) = 0$ otherwise. Then $u_m \in D_K$, $u_m \xrightarrow[L^0(Y)]{} 0$ and $Ku_m \xrightarrow[L^0(X)]{} v$. Since K is closible, it follows that $v = 0$ and closibility of K implies (C).

Suppose now that K is continuous and that there is a set

$$F \subset \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} X_m, \quad \text{where} \quad X_m = \{x\colon\ k(x, b_m) \neq 0\},$$

such that $|F| > 0$. We can assume that $|F| < \infty$. For $N \geqslant m$ let

$$v_{m,N}(x) = k(x, b_m) + \sum_{l=m+1}^{N} \xi_{m,l}\, k(x, b_l)$$

where the coefficients $\xi_{m,l} \in \boldsymbol{R}$ are so chosen that

$$|F \cap \{x\colon\ v_{m,N}(x) \neq 0\}| = \left|\{x\colon\ \sum_{l=m}^{N} |k(x, b_l)| > 0\} \cap F\right|$$

(see the Lemma in 2.3.3). By the choice of F there exists for each m an index N_m such that

$$|F \sim \{x\colon\ v_{m,N_m}(x) \neq 0\}| \leqslant 2^{-m-1}|F|,$$

also for each m there is $\varepsilon_m > 0$ such that

$$\left|F \sim \{x\colon\ |v_{m,N_m}(x)| \geqslant \varepsilon_m\}\right| \leqslant 2^{-m}|F|.$$

Then the sequence $v_m = \varepsilon_m^{-1} v_m N_m$ does not converge to 0 in $L^0(X)$. On the other hand $v_m = Ku_m$ where $u_m(b_l) = (\varepsilon_m |b_l|)^{-1} \xi_{m,l}$ if $m \leqslant l \leqslant N_m$ and $u_m(b_l) = 0$ otherwise. Clearly $u_m \xrightarrow[L^0(Y)]{} 0$ and K is not continuous. This contradiction shows that continuity of K implies (B).

If $D_K = L^0(Y)$, then by the closed graph theorem the natural topology of D_K concides with the topology of $L^0(Y)$ (the inclusion $D_K \subset L^0(Y)$ being continuous) and by Theorem 3.3.1 $K\colon L^0(Y) \to L^0(X)$ is continuous. ∎

4.5. Theorems 4.3, 4.4 can be extended to the situation when X is not assumed σ-finite: Conditions (A), (B), (C) should be replaced by sets of corresponding conditions with X replaced by E, $E \in \mathscr{F}(X)$ (or $E \in \mathscr{F}_\sigma(X)$).

It is not clear how the Theorems 4.3, 4.4 should be stated in the case when Y is not σ-finite.

4.6. We do not know a condition on the kernel k characterizing the property that the corresponding integral transformation K is closed. There exist integral transformations which are closed but not continuous and integral transformations that are closible but not closed.

4.7.1. Let $\{r_n\}_{n=1}^{\infty}$ denote the sequence of Rademacher functions on $[0, 1]$. We recall the following inequality

$$[|E|-(2(|E|-|E|^2))^{1/2}]\sum_{n=1}^{N}|\alpha_k|^2 \leqslant \int_E\Big|\sum_{n=1}^{N}\alpha_k r_k(t)\Big|^2 dt$$
$$\leqslant [|E|+(2(|E|-|E|^2))^{1/2}]\sum_{n=1}^{N}|\alpha_k|^2$$

valid for every $E \subset [0, 1]$. The following is an immediate consequence: a sequence of Rademacher polynomials (i.e. of linear combinations of Rademacher functions) converges in $L^0([0, 1])$ if and only if it converges in $L^0([0, 1])$.

If F is a divisible measure space, $|F| = 1$, then by means of binary partitions P_k of F (see the proof of Theorems 4.3, 4.4) one can transfer the values of r_k from $[0, 1]$ onto F and thus define the sequence of Rademacher functions (sometime called generalized Rademacher functions) on F.

The inequality and the property stated above remain valid without change.

4.7.2. Let $X = [0, 1]$ and $Y = Z = \{0, \pm 1, \dots\}$, X with the Lebesgue measure, Z with the point measure $|\{n\}| = 1$ for $n = 0, \pm 1, \dots$ Consider $k(x, n) = r_n(x)$. Then k satisfies (A), also if $v \in \bigcap_{m=1}^{\infty} \overline{[\{r_n\colon n \geqslant m\}]}$, then v is the limit in $L^0(0, 1)$ of a sequence of Rademacher polynomials $v_n = \sum_{l=n}^{N_n} \xi_{l,n} r_l$. It follows that $v_n \xrightarrow[L^2(0,1)]{} v$ and ortho-normality of $\{r_l\}$ implies that $v = 0$. Hence k satisfies (B) and K is closible. It is easy to check that $D_K = l^1 = L^1(Z)$ and the domain of the closure of K is $l^2 = L^2(Z)$, therefore K is not closed.

4.7.3. Let $X = [0, 1]\times[0, 1]$ with the Lebesgue measure $Y = Z$ as in 4.7.2, $k((t, \tau), n) = \tau^n r_n(t)$, thus $Ku(t, \tau) = \sum \tau^n r_n(t) u(n)$.

Then $D_K = \{\{u(n)\}\colon \limsup |u(n)|^{1/n} \leqslant 1\}$. If $u_j \in D_K$, $u_j \to u$ and if $Ku_j \to v$, then there exists a sequence of integers n_j such that the sequence defined by $u_j'(n) = u_j(n)$, $n \leqslant n_j$, $u_j'(n) = 0$ otherwise, has the property $u_j' \to u$ and $Ku_j' \to v$. For a suitable subsequence of u_j' which we denote again by u_j', $Ku_j' \to v$ a.e. on $[0, 1]\times[0, 1]$, in particular, for almost every τ, $\sum_{n=1}^{n_j} \tau^n r_n(t) u_j(n)$ is convergent to $v(t, \tau)$ for almost every t. The properties of Rademacher polynomials explained in 4.7.1 imply that for almost every $\tau \in [0, 1]$, $\{\sum_{n=1}^{n_j} \tau^n r_n(t) u_j(n)\}$ is convergent in $L^2(0, 1)$; in particular $\{\tau^n u_j(n)\}$ is in l^2. Since $u_j(n) \xrightarrow[j\to\infty]{} u(n)$, it follows that $\sum \tau^{2n} |u(n)|^2 < \infty$ for a.e. $\tau \in [0, 1]$. This implies that $\lim |u(n)|^{1/n} \leqslant 1$ and we conclude that $u \in D_K$ and that $v = Ku$. It follows that K is closed. On the other hand K does not satisfy the condition (B) and K is not continuous.

5. Extensions by continuity. Compatibility problem

5.1. Let V, W be topological vector spaces and $K\colon D(K) \subset V \to W$ be a linear transformation. If $A \underset{c}{\subset} V$, then K can be extended to A by continuity provided that

(i) $\overline{A \cap D_K}^A = A$ (i.e. $A \cap D(K)$ is dense in A),

(ii) $K\colon A \cap D(K) \to W$ is continuous (in the A-topology on $D(K) \cap A$). If (i), (ii) are satisfied, then we denote by K_A the extension by continuity of K to A and by $\mathscr{C}_K$ the family of all topological vector subspaces $A \underset{c}{\subset} V$ satisfying (i) and (ii).

If $\mathscr{A}$ is any family topological vector subspaces of V, then we say that K has the compatibility property relative to $\mathscr{A}$ provided that for any $A, B \in \mathscr{A} \cap \mathscr{C}_K$ and any $u \in A \cap B$ we have $K_A u = K_B u$. In the case when $\mathscr{A} = \mathscr{C}_K$ the property is referred to as the universal compatibility property.

5.2. Proposition. *If K is closible, then K has the universal compatibility property.*

Proof. If $A, B \in \mathscr{C}_K$ and if $u \in A \cap B$, then there exist nets $u_\alpha \xrightarrow[A]{} u$, $v_\beta \to u$, $u_\alpha \in A \cap D(K)$, $v_\beta \in B \cap D_K$ and $K_A u = \lim K u_\alpha$, $K_B u = \lim K v_\beta$. Since the inclusions $A \subset V$, $B \subset V$ are continuous, the net $u_\alpha - v_\beta$ converges to 0 in V (with natural ordering of pairs (α, β)), and $K(u_\alpha - v_\beta) \xrightarrow[W]{} K_A u - K_B u$. Since K is closible, $K_A u - K_B u = 0$. ∎

It is not known if the converse of Proposition 5.2 is valid in the general case. The following special case includes integral transformations on σ-finite spaces; note that the condition (a) in the Proposition is the negation of closibility of K.

5.3. Proposition. *Suppose that V, W, $D(K)$ are complete vector metric spaces, that $D(K) \subset V$ and that $K\colon D(K) \to W$ is continuous. Suppose further that there exists a sequence $\{v_n\} \subset D(K)$ such that*

(a) $v_n \xrightarrow[V]{} 0$, $Kv_n \xrightarrow[D(K)]{} w \neq 0$.

(b) *For every formal series* $\sum a_n v_n$ *and for every sequence of its partial sums* $S_{n_k} = \sum_{n=1}^{n_k} a_n v_n$ *the conditions* $s_{n_k} \underset{V}{\rightharpoonup} v$, $v \in D_K$ *imply that* $s_{n_k} \to v$ *in* V.

Then there exist two Hilbert spaces H_1, $H_2 \in \mathscr{C}_K$ *and an element* $u \in H_1 \cap H_2$ *such that* $K_{H_1} u \neq K_{H_2} u$. *In particular* K *does not have the property of universal compatibility.*

Without loss of generality we can assume that the metrics ϱ_V, ϱ_W of V and W satisfy the condition

$$\varrho(\alpha u) \leqslant \varrho(u) \quad \text{for all } \alpha, |\alpha| \leqslant 1.$$

The proof of Proposition 5.3 depends on the following

LEMMA. *Let* $u_1, \ldots, u_n \in V$ *be linearly independent. Then there is a constant* $C > 0$ *depending only on* $u_1, \ldots, u_n$ *and such that*

$$C\Big(1 + \sum_{k=1}^{n} |\xi_k|\Big)^{-1} \sum_{k=1}^{n} |\xi_k| \leqslant \varrho_V\Big(\sum_{k=1}^{n} \xi_k u_k\Big)$$

for all scalars $\xi_1, \ldots, \xi_n$.

Proof. For $u = \sum_{k=1}^{n} \xi_k u_k$ let $\|u\|_1 = \sum |\xi_k|$. It is easy to see that

$$\inf\{\varrho_V(u)(1 + \|u\|_1)\|u\|_1^{-1} : \|u\|_1 \geqslant \tfrac{1}{2}\} = C_0 > 0.$$

For $0 < \|u\|_1 < \frac{1}{2}$ let $N =$ integral part of $\|u\|_1^{-1}$. Then $\frac{1}{2} \leqslant \|Nu\|_1 \leqslant 1$ and

$$\begin{aligned}(1 + \|u\|_1)^{-1}\|u\|_1 &\leqslant \|u\|_1 \leqslant 2N^{-1}(1 + \|Nu\|_1)^{-1}\|Nu\|_1 \\ &\leqslant 2N^{-1} C_0^{-1} \varrho_V(Nu) \leqslant 2C_0^{-1} \varrho_V(u),\end{aligned}$$

and the inequality holds with $C = \frac{1}{2} C_0$.

Proof of Proposition 5.3. By (a) $\{v_n\}$ contains a linearly independent subsequence, also (b) is inherited by any subsequence of $\{v_n\}$. Hence we may assume that $\{v_n\}$ is linearly independent. Define now the subsequence $\{u_s\}$ of $\{v_n\}$ as follows. Let $u_1 = v_1$; suppose that $u_1, \ldots, u_l$ are already chosen, $l \geqslant 1$, and let c_l be the largest constant C in the lemma. Let u_{l+1} be the first v_m following u_l in the sequence $\{v_n\}$, with the property that

$$\varrho_V(v_m) \leqslant (l+1)^{-1} c_l 2^{-l-1}, \quad \varrho_W(Kv_m - w) \leqslant \frac{2^{-l-1}}{l+1}.$$

Note that the sequence $\{c_l\}$ is non-increasing.

Let H be the Hilbert space of all sequences $\{\xi_k\}$ such that

$$\sum \frac{1}{k^2} |\xi_k|^2 = \|\{\xi_k\}\|^2 < \infty$$

and define for $\xi = \{\xi_k\} \in H$

$$T_1 \xi = \sum_{k=1}^{\infty} \xi_k (u_{2k} - u_{2k-1}), \qquad T_2 \xi = \xi_1 u_1 + \sum_{k=2}^{\infty} \xi_k (u_{2k-1} - u_{2k-2}).$$

Since $|\xi_k| \leqslant k$ for k sufficiently large and $\varrho_V(\xi_k(u_{2k} - u_{2k-1})) \leqslant k(\varrho_V(u_{2k}) + \varrho_V(u_{2k-1})) \leqslant c_{2k-2} 2^{-2k+2}$, it follows that the series $T_1 \xi$, $T_2 \xi$ are convergent in V and therefore define continuous transformations of H into V. If $T_1 \xi = 0$, then for every N

$$-\sum_{k=1}^{N} \xi_k (u_{2k} - u_{2k-1}) = \sum_{k=N+1}^{\infty} \xi_k (u_{2k} - u_{2k-1})$$

and if $|\xi_k| \leqslant k$ for $k \geqslant N$, we get

$$c_{2N}\Big(\sum_{k=1}^{N} |\xi_k|\Big)/\Big(1 + 2\sum_{k=1}^{N} |\xi_k|\Big) \leqslant \varrho_V\Big(\sum_{k=N+1}^{\infty} \xi_k (u_{2k} - u_{2k-1})\Big) \leqslant c_{2N} \sum_{k=N+1}^{\infty} 2^{-2k+2}$$

showing that $\xi = 0$.

Similarly $T_2 \xi = 0$ implies that $\xi = 0$ and T_1, T_2 are injective. It follows that $H_1 = T_1 H$, $H_2 = T_2 H$ with $\|T_i \xi\|_{H_i} = \|\xi\|$ are Hilbert spaces continuously contained in V.

By a similar argument we verify that the formal series obtained by applying K term by term to $T_i \xi$, $i = 1, 2$, are convergent in W and therefore define continuous transformations $K_i\colon H_i \to W$. We verify now that $H_i \in \mathscr{C}_K$ and that $K_i = K_{H_i}$. The vectors $\{u_{2k} - u_{2k-1}\}_{k=1}^{\infty}$ and u_1, $\{u_{2k} - u_{2k-1}\}_{k=1}^{\infty}$ form orthogonal bases in H_1 and H_2 and belong to $D(K)$, hence $D(K) \subset H_i$ is dense in H_i, $i = 1, 2$. It remains to check that for $v \in D(K) \cap H_i$, $K_i v = Kv$; here we use the condition (b). It suffices to consider the case when $i = 1$. Let $v = T_1 \xi \in D(K)$, $\xi \in H$. Then $v = \sum \xi_k (u_{2k} - u_{2k-1}) = \sum \eta_k u_k$ – the series converges in V. Since $v \in D(K)$, (b) implies that $\sum \eta_k u_k$ converges in $D(K)$. Since $K\colon D(K) \to W$ is continuous, $Kv = \sum \xi_k K(u_{2k} - u_{2k-1}) = K_1 v$. For $\xi = \{1, 1, \ldots\} \in H$ we have

$$T_1 \xi + T_2 \xi = \sum_{k=1}^{\infty} (u_{2k} - u_{2k-1}) + \sum_{k=2}^{\infty} (u_{2k-1} - u_{2k-2}) + u_1 = \lim_{m \to \infty} u_{2m} = 0,$$

implying that

$$v = T_1 \xi = T_2(-\xi) \in H_1 \cap H_2.$$

On the other hand

$$K_1 v - K_2 v = \lim_{m \to \infty} \Big(\sum_{k=1}^{m} K(u_{2k} - u_{2k-1}) + \sum_{k=2}^{m} K(u_{2k-1} - u_{2k}) + Ku_1\Big)$$

$$= \lim_{m \to \infty} Ku_{2m} = w \neq 0. \;\blacksquare$$

5.4. Let $V = L^0(Y)$, $W = L^0(X)$, X, Y be σ-finite and $K\colon D_K \subset L^0(Y) \to L^0(X)$ be an integral transformation, $D(K) = D_K$.

PROPOSITION. *If K is not closible, then there exists a sequence $\{v_n\} \subset D_K$ satisfying conditions* (a), (b) *of Proposition* 5.3.

Proof. If K is not closible, then K fails to satisfy one of the conditions (A), (C) of 4.2. If K does not satisfy (A), then as in the proof of Theorem 4.3 there exists a sequence $f_n = |E_n|^{-1}\chi_n$, $\chi_n = \chi_{E_n}$, where $\{E_n\}$ is a decreasing sequence of subsets of the divisible part of Y, $|E_n| \to 0$, $f_n \in D_K$ and $Kf \xrightarrow[L^0(X)]{} w \neq 0$.

For $m > n$ let $v_{n,m} = |E_n|^{-1}(\chi_n - \chi_m)$. By the dominated convergence theorem we get $\varrho_K(f_n - v_{n,m}) \underset{\to\infty}{\to} 0$ and we can define the sequence $\{v_n\}$ as follows: choose an increasing sequence of indices n_j such that $n_0 = 1$, $\varrho_K(f_{n_j} - v_{n_j,n_{j+1}}) \leqslant 2^{-j}$, $j = 0, 1, 2, \ldots$, and let $v_j = v_{n_j,n_{j+1}}$. Then v_j have mutually disjoint supports, $Kv_j \to w$ and (b) is easily checked.

If k satisfies (A) but does not satisfy (C), then there is $w \neq 0$, $w \in \bigcap_{m=1}^{\infty} \overline{[\{k(x, b_n)\colon n \geqslant m\}]}$. Then there exists an increasing sequence of indices n_j and sequences of scalars $\{\xi_l^j\}$ such that

$$w_j(x) = \sum_{l=n_j+1}^{n_{j+1}} \xi_l^j k(x, b_l)|b_l| \xrightarrow[L^0(X)]{} w.$$

With $v_j(b_l) = \xi_l^j$ for $n_j + 1 \leqslant l \leqslant n_{j+1}$ and $v_j(b_l) = 0$ otherwise, we have $Kv_j = w_j$, $v_j \to 0$ and the functions v_j have mutually disjoint supports. ∎

Propositions 5.3 and 5.4 yield the following

5.5. THEOREM. *If X, Y are σ-finite and if $K\colon D_K \subset L^0(Y) \to L^0(X)$ is an integral transformation, then K has the universal compatibility property if and only if K is closible.*

5.6. Compatibility in the class of solid spaces.

5.6.1. Let $K\colon D_K \subset L^0(Y) \to L^0(X)$ be an integral transformation. We do not assume σ-finiteness of X or Y. For $F \in \mathscr{F}(Y)$, $E \in \mathscr{F}(X)$ and $u \in L^0(Y)$ define

$$\tilde{\varrho}_{K,E,F}(u) = \varrho_F(u) + \sup\{\varrho_E(Kv)\colon v \in D_K, |v| \leqslant |u| \text{ a.e.}\}.$$

PROPOSITION. *For every $E \in \mathscr{F}(X)$, $F \in \mathscr{F}(Y)$ $\tilde{\varrho}_{K,E,F}$ is a pseudometric on $L^0(X)$. The family $\{\tilde{\varrho}_{K,E,F}\}_{E\in\mathscr{F}(X), F\in\mathscr{F}(Y)}$ defines a complete Hausdorff topology on $L^0(X)$ compatible with its structure of an additive group, which we refer to as the $\tilde{K}$-topology.*

Proof. We have to show that

$$d_{K,E}(u) = \sup\{\varrho_E(Kv)\colon v \in D_K, |v| \leqslant |u|\}$$

satisfies the triangle inequality and that the $\tilde{K}$-topology is complete.

If $u_1, u_2 \in L^0(Y)$ and $|v| \leqslant |u_1+u_2|$, then writing v in the form $v = v_1+v_2$ where $v_i = (|u_1|+|u_2|)^{-1}|u_i|v$ it follows readily that $|v_i| \leqslant |u_i|$ and that $u_i \in D_K$ if $v \in D_K$. The triangle inequality for $d_{K,E}$ follows easily.

If $\{u_\alpha\} \in L^0(Y)$ is a Cauchy net in $\tilde{K}$ topology, then $\{u_\alpha\}$ is a Cauchy net in the $L^0(Y)$-topology and there exists $u \in L^0(Y)$ such that $u_\alpha \xrightarrow[L^0(Y)]{} u$. It remains to show that $u_\alpha \xrightarrow[\tilde{K}]{} u$, i.e. $d_{K,E}(u_\alpha - u) \to 0$ for every $E \in \mathscr{F}(X)$. It suffices to consider the case when $\{u_\alpha\}$ is a sequence; we can also assume that $u = 0$. By the definition there exists sequence $\{v_\alpha\} \subset D_K$ such that $\varrho_E(Kv_\alpha) \leqslant d_{K,E}(u_\alpha) \leqslant \varrho_E(Kv_\alpha)+2^{-\alpha}$.

Let $F_\alpha = \{x\colon v_\alpha(x) \neq 0\}$ and $F = \bigcup F_\alpha$. Then $F \in \mathscr{F}_\sigma(Y)$ and

$$d_{K,E}(u_\alpha) \leqslant \varrho_E(Kv_\alpha)+2^{-\alpha} \leqslant d_{K,E}(\chi_F u_\alpha)+2^{-\alpha}.$$

Thus it suffices to show that $d_{K,E}(\chi_F u_\alpha) \to 0$. Note that for every $w \in L^0(Y)$, $d_{K,E}(\chi_F w) \leqslant d_{K,E}(w)$, and choose a subsequence of $\{u_\alpha\}$, denoted again by $\{u_\alpha\}$, such that

$$\sum_{\alpha=1}^{\infty} \varrho_F\big(\chi_F(u_{\alpha+1}-u_\alpha)\big) < \infty, \qquad \sum_{\alpha=1}^{\infty} d_{K,E}\big(\chi_F(u_{\alpha+1}-u_\alpha)\big) < \infty.$$

Then $\sum_{\alpha=1}^{\infty} \chi_F|u_{\alpha+1}-u_\alpha|$ is convergent a.e. and

$$\chi_F|u_\alpha| \leqslant \sum_{\beta=\alpha}^{\infty} \chi_F|u_{\beta+1}-u_\beta|.$$

If $f \in D_K$, $|f| \leqslant \chi_F|u_\alpha|$, then the functions

$$f_l = f|u_{l+1}-u_l|\Big(\sum_{n=\alpha}^{\infty}|u_{n+1}-u_n|\Big)^{-1}, \qquad l \geqslant \alpha$$

(with $f_l = 0$ if $\sum_{n=\alpha}^{\infty}|u_{n+1}-u_n| = 0$) satisfy the conditions $|f_l| \leqslant |u_{l+1}-u_l|$, $f = \sum f_l$ and $\sum |f_l| \in D_K$. By the dominated convergence theorem, $Kf = \sum_{n=\alpha}^{\infty} Kf_n$ and $\varrho_E(Kf_l) \leqslant d_{K,E}(u_{l+1}-u_l)$. It follows that

$$d_{K,E}(\chi_F u_\alpha) = \sup\{\varrho_E(Kf)\colon f \in D_K, |f| \leqslant \chi_F|u_\alpha|\} \leqslant \sum_{l=\alpha}^{\infty} d_{K,E}(u_{l+1}-u_l) \xrightarrow[\alpha\to\infty]{} 0. \blacksquare$$

5.6.2. For $u \in D_K$ we have (see 3.3)

(i) $$\tilde{\varrho}_{K,E,F}(u) \leqslant \varrho_{K,E,F}(u)$$

and

(ii) $$\varrho_E(Ku) \leqslant \tilde{\varrho}_{K,E,F}(u), \qquad E \in \mathscr{F}(X),\ F \in \mathscr{F}(Y).$$

(i) implies that the restriction to D_K of the $\tilde{K}$-topology is a vector topology and (ii) implies that K is continuous in $\tilde{K}$-topology.

It follows that the closure $\tilde{D}_K$ of D_K in the $\tilde{K}$-topology is a topological vector space continuously contained in $L^0(Y)$ and that $K: D_K \to L^0(X)$ can be extended to a continuous transformation from $\tilde{D}_K$ into $L^0(X)$ which we denote by $\tilde{K}$.

5.6.3. THEOREM. (i) *$\tilde{D}_K$ with the $\tilde{K}$-topology is solid.*

(ii) *If $A \in \mathscr{C}_K$ is solid, then $A \underset{c}{\subset} \tilde{D}_K$ and $K_A = \tilde{K}|_A$.*

Proof. It is clear that if $|v| \leqslant |u|$ a.e., then $\tilde{\varrho}_{K,E,F}(v) \leqslant \tilde{\varrho}_{K,E,F}(u)$. If $u \in \tilde{D}_K$ and $|v| \leqslant |u|$ a.e., then there exists a net $u_\alpha \in D_K$ s.t. $\tilde{\varrho}_{K,E,F}(u-u_\alpha) \to 0$ for every $E \in \mathscr{F}(X)$, $F \in \mathscr{F}(Y)$. Define $v_\alpha = u_\alpha v/u$, $v_\alpha = 0$ if $u = 0$. Then

$$|v-v_\alpha| = \left|\frac{v(u-u_\alpha)}{u}\right| \leqslant |u-u_\alpha| \text{ a.e.} \quad \text{and} \quad \varrho_{K,E,F}(v-v_\alpha) \to 0.$$

(ii) follows from the observation that if $A \in \mathscr{C}_K$ is a solid space, then the A-topology on $D_K \cap A$ is stronger than the $\tilde{K}$-topology. Since the transformation $K: D_K \cap A \to L^0(X)$ and the inclusion $A \subset L^0(Y)$ are both continuous, for every $E \in \mathscr{F}(X)$, $F \in \mathscr{F}(Y)$ and for every $\varepsilon > 0$ there is a solid neighborhood V of 0 in A such that $\varrho_E(Kv)+\varrho_F(u) < \varepsilon$ for every $u \in V$ and $v \in V \cap D_K$. If $u \in V \cap D_K$ and $|v| \leqslant |u|$ a.e., then $v \in V \cap D_K$ and $\varrho_E(Kv)+\varrho_F(u) < \varepsilon$. It follows that $u \in V \cap D_K$ implies $\tilde{\varrho}_{K,E,F}(u) \leqslant \varepsilon$ and the assertion follows.

5.6.4. PROPOSITION. $\tilde{D}_K = (\tilde{D}_K)_a$.

Proof. We have to show that for any sequence $E_n \searrow \emptyset$, for every $u \in \tilde{D}_K$, for every $E \in \mathscr{F}(X)$ and for every $F \in \mathscr{F}(Y)$, we have $\tilde{\varrho}_{K,E,F}(\chi_{E_n} u) \to 0$. For any $\varepsilon > 0$ choose $v \in D_K$ such that $\tilde{\varrho}_{K,E,F}(u-v) < \varepsilon$. Then

$$\tilde{\varrho}_{K,E,F}(\chi_{E_n} u) \leqslant \tilde{\varrho}_{K,E,F}(\chi_{E_n}(u-v))+\tilde{\varrho}_{K,E,F}(\chi_{E_n} v) < \varepsilon+\varrho_{K,E,F}(\chi_{E_n} v)$$

and Proposition 3.3.7 yields the conclusion. ∎

Theorem 5.6.3 implies that an integral transformation K has the compatibility property relative to the class of solid subspaces of $L^0(Y)$. Also $\tilde{D}_K$ is the maximal solid space in $\mathscr{C}_K$.

It is thus of some interest to determine explicitly $\tilde{D}_K$ for concrete kernels K.

5.7. Properties of $\tilde{D}_K$.

5.7.1. *If* Re $k(x, y)$, Im $k(x, y)$ *are of constant sign. e.g.* Re $k(x, y) \geqslant 0$, Im $k(x, y) \geqslant 0$ *a.e., then* $\tilde{D}_K = D_K$.

Proof. The condition implies that for $u \in D_K$

$$|K||u| = \int (k_1(x, y)^2+k_2(x, y)^2)^{1/2} |u(y)|\, dy$$
$$\leqslant \left|\int k_1(x, y)|u(y)|\, dy\right| + \left|\int k_2(x, y)|u(y)|\, dy\right| \leqslant \sqrt{2}\,\big|K|u|\big| \text{ a.e.}$$

where

$$k(x, y) = k_1(x, y)+ik_2(x, y)$$

and it follows that

$$d_{K,E}(u) = d_{K,E}(|u|) \geqslant \varrho_E(K|u|) \geqslant \tfrac{1}{2}\varrho_E(|K|\,|u|)$$

and it follows that the closure of D_K in the $\tilde{K}$-topology is D_K.

5.7.2. PROPOSITION. *Let* $\theta \in L^0(X)$, $\theta \neq 0$ *a.e. and let* $k'(x, y) = \theta(x)k(x, y)$. *Then* $D_K = D_{K'}$ *and* $\tilde{D}_K = \tilde{D}_{K'}$.

5.7.3. COROLLARY. *Suppose that for a.e.* $x \in X$ *the values of* $k(x, y)$ *satisfy* $\alpha(x) \leqslant \arg k(x, y) \leqslant \beta(x)$, *for a.e.* $y \in Y$, *where* $\alpha, \beta \in L^0(X)$ *and* $\beta(x) - \alpha(x) \leqslant \pi/2$ *a.e. Then* $\tilde{D}_K = D_K$.

This is an immediate consequence of 5.7.1 and 5.7.2 with $\theta(x) = e^{-i\alpha(x)}$.

5.7.4. PROPOSITION. (i) *Let* $Y_0 \subset Y$, $|Y_0| > 0$ *and* $k_0 = k|_{X \times Y_0}$, *where* $|$ *denotes the restriction. Then* $D_{K_0} = D_K|_{Y_0} = \{u|_{Y_0}\colon u \in D_K\}$.

(ii) *Let* $Y_1 \cup \ldots \cup Y_m = Y$ *be a partition of* Y, $|Y_j| > 0$, $j = 1, \ldots, m$, *and let* $k_i = k|_{X \times Y_i}$. *Then* $D_K = \sum \chi_i D_{K_i}$, $\tilde{D}_K = \sum \chi_i \tilde{D}_{K_i}$, $\chi_i = \chi_{Y_i}$, *with the understanding that* $\chi_i u = 0$ *outside* Y_i.

Proof. The statements concerning the natural domains are obvious. Since $(u-v)|_{Y_0} = u|_{Y_0} - v|_{Y_0}$, it is clear that if $v \in D_K$ is near to $u \in D_K$ in the $\tilde{K}$-topology, then the same is true for $u|_{Y_0}$ and $v|_{Y_0}$ in the $\tilde{K}_0$-topology. Also if $v \in D_{K_0}$ is near to $u \in \tilde{D}_{K_0}$ in the $\tilde{K}_0$-topology, then the same holds for $\chi_{Y_0} v$ and $\chi_{Y_0} u$ in the $\tilde{K}$-topology. This remark proves the remaining parts of (i) and (ii). ∎

In the next proposition we assume that K is nonsingular in the following sense (see 4.1). For every $F \in \mathscr{F}_\sigma(Y)$ there exists $\varphi \in D_K$ such that $\varphi > 0$ a.e. on F.

5.7.5. PROPOSITION. (i) *Let* $X_0 \subset X$, $|X_0| > 0$ *and* $k^0 = k|_{X_0 \times Y}$. *Then* $D_K \subset D_{K^0}$ *with dense inclusion and* $\tilde{D}_K \subset \tilde{D}_{K^0}$.

(ii) *If* $X_1 \cup X_2 \cup \ldots$ *is an at most denumerable partition of* X, $|X_i| > 0$, *then* $D_K = \bigcap_i D_{K^i}$ *and* $\tilde{D}_K = \bigcap_i \tilde{D}_{K^i}$.

Proof. (i) It is obvious that $D_K \subset D_{K^0}$. If $u \in D_{K^0}$, then $F = \{x\colon u(x) \neq 0\} \in \mathscr{F}_\sigma(Y)$ and there exists $\varphi \in D_K$ such that $\varphi > 0$ on F. Define for $n = 1, 2, \ldots$ $u_n = u$ if $|u| \leqslant n\varphi$ and $u_n = n\varphi \operatorname{sign} u$ if $|u| > n\varphi$. Then $u_n \to u$ in $L^0(Y)$ and a.e., also $u_n \in D_K$ and $|u_n - u| \leqslant u$. By 2.3.5 and 3.3.7 $u_n \to u$ in D_{K^0}.

To prove the inclusion $\tilde{D}_K \subset \tilde{D}_{K^0}$ observe that for $E \in \mathscr{F}(X)$ and $E_0 = E \cap X_0$ we have $\varrho_E(v) \leqslant |E_0|\,|E|^{-1}\varrho_{E_0}(v)$ if $|E_0| > 0$, $v \in L^0(X)$, which implies that $d_{K,E}(u) \geqslant |E_0|\,|E|^{-1} d_{K^0,E_0}(u)$ for $u \in L^0(Y)$. Hence the $\tilde{K}^0$-topology is weaker than the $\tilde{K}$-topology and the desired inclusion follows from the first part.

(ii) The first equality is obvious and the inclusion $\tilde{D}_K \subset \bigcap_{j=1}^{\infty} \tilde{D}_{K^j}$ follows from (i). If $u \in \bigcap \tilde{D}_{K^j}$, then for every $E \in \mathscr{F}(X)$ and $F \in \mathscr{F}(Y)$ and for every

$\varepsilon > 0$ we find $\{u_j\} \subset D_K$ such that $\varrho_F(u_j - u) + d_{K^j, E^j}(u_j - u) < \min(2^{-j-1}\varepsilon|E|, \varepsilon/4)$ for every j with $|E_j| > 0$, where $E_j = E \cap X_j$. Since $E = \bigcup E_j$, it follows that for some n, $\sum_{j=1}^{\infty} |E_j| < \frac{1}{4}\varepsilon|E|$. Let $v_n = f_n \operatorname{sign} u$ if $|u| \leqslant f_n$ and $v_n = u$ otherwise, where $f_n = \max\{|u_1|, \ldots, |u_n|\}$. Then $v_n \in D_K$, $|u_n - u| \leqslant |u_j - u|$, $j = 1, \ldots, n$ and $d_{K^j, E_j}(v_n - u) \leqslant d_{K^j, E_j}(u_j - u)$, $j = 1, \ldots, n$. Since $\varrho_E(w) = |E|^{-1} \sum |E_j| \varrho_{E_j}(w)$ with the understanding that $|E_j| \varrho_{E_j}(w) = 0$ if $|E_j| = 0$, we have $d_{K,E}(v) \leqslant |E|^{-1} \sum |E_j| d_{K^j, E_j}(v)$ and we can write

$$d_{K,E}(v_n - u) \leqslant |E|^{-1} \Big(\sum_{j=1}^{n} |E_j| d_{K^j, E_j}(v_n - u) + \sum_{j=n+1}^{\infty} |E_j| \Big)$$
$$\leqslant |E|^{-1} \sum_{j=1}^{n} 2^{-j-1} |E| \varepsilon + \tfrac{1}{4}\varepsilon \leqslant \tfrac{3}{4}\varepsilon.$$

Since $\varrho_F(v_n - u) \leqslant \varrho_F(u_j - u) < \frac{1}{4}\varepsilon$ for $j = 1, 2, \ldots, n$, it follows that $\tilde{\varrho}_{K,E,F}(v_n - u) < \varepsilon$. ∎

5.7.6. Corollary. *Suppose that X and Y are topological spaces with Borel measures, that Y is compact and that X satisfies the Lindelöf property, i.e. every open cover of X contains a denumerable subcover. Suppose that k is continuous and that $k(x, y) \neq 0$ everywhere on $X \times Y$. Then $\tilde{D}_K = D_K$.*

Proof. For every $(x, y) \in X \times Y$ there is an open neighborhood of (x, y) of the form $U_{x,y} \times V_{x,y}$ in which $|k(x, y) - k(x', y')| \leqslant 2^{-1/2}|k(x, y)|$. For fixed x the sets $\{V_{x,y}\}_{y \in Y}$ form an open cover of Y; let $\{V_{x, y_i(x)}\}$ be a finite subcover and let $U_x = \bigcap U_{x, y_i(x)}$, then $\{U_x\}_{x \in X}$ is an open cover of X and thus contains a subcover $\{U_{x_j}\}$ which is at most denumerable. We now write $X_j = U_{x_j} \sim \bigcup_{l<j} U_{x_l}$, $Y_i^j = V_{x_j y_i(x_j)} \sim \bigcup_{l<i} V_{x_j, y_l(x_j)}$. Then $\{X_j\}$ is an (at most denumerable) partition of X and for every j, $\{Y_i^j\}$ is a finite partition of Y. If $k_i^j = k|_{X_j \times Y_i^j}$, then by 5.7.3 $\tilde{D}_{K^j} = \bigcap \tilde{D}_{K_i^j}$. Also with $k^j = k|_{X_j \times Y}$ we have by 5.7.5 $\tilde{D}_K = \bigcap \tilde{D}_{K^j}$ and by 5.6.7 $\tilde{D}_{K^j} = \sum_i \chi_{Y_i^j} \tilde{D}_{K_i^j} = \sum_i \chi_{Y_i^j} D_{K_i^j} = D_{K^j}$. Hence $\tilde{D}_K = \bigcap_j \tilde{D}_{K^j} = \bigcap_j D_{K^j} = D_K$. ∎

If Y is a topological space and $A \subset L^0(Y)$, then we write $A_{\text{loc}} = \{u\colon \chi_C u \in A \text{ for every compact } C \subset Y\}$.

5.7.7. Corollary. *Suppose that X has the Lindelöf property, that Y is an arbitrary topological space, that k is continuous and that $k \neq 0$ everywhere on $X \times Y$. Then $\tilde{D}_K \subset D_{K\,\text{loc}}$.*

Proof. If $C \subset Y$ is compact, then with $k_C = k|_{X \times C}$ we have by 5.7.4 $\tilde{D}_K|_C = \tilde{D}_{K_C}$, $D_K|_C = D_{K_C}$ and by 5.7.3 $\tilde{D}_{K_C} = D_{K_C}$.

5.7.8. Proposition. *Suppose that $u \in \tilde{D}_K$ and that $\{y\colon u(y) \neq 0\} = F$ is σ-finite. If $F_n \nearrow F$, then $\chi_{F_n} u \to u$ in $\tilde{D}_K$, in particular $\tilde{K}\chi_{F_n} u \to \tilde{K}u$.*

Proof. Let $E \in \mathscr{F}(X)$ and $\varepsilon > 0$. Choose $v \in D_K$, $|v| \leqslant |u|$, such that $\varrho_F(u-v) + d_{K,F}(u-v) < \varepsilon/2$. Then

$$\tilde{\varrho}_{K,E,F}\big((1-\chi_{F_n})u\big) \leqslant \tilde{\varrho}_{K,E,F}\big((1-\chi_{F_n})(u-v)\big) + \tilde{\varrho}_{K,E,F}\big((1-\chi_{F_n})v\big)$$
$$\leqslant \varepsilon/2 + \varrho_{K,E,F}\big((1-F_n)v\big)$$

and by the dominated convergence theorem $\varrho_{E,F,K}\big(((1-\chi)_{F_n})v\big) \xrightarrow[n\to\infty]{} 0$. ∎

5.7.9. PROPOSITION. *Suppose that $u \in \tilde{D}_K$, $E \in \mathscr{F}(X)$ and suppose that for a sufficiently small $\varepsilon > 0$, $d_{K,E}(u) < \varepsilon^4$. Then for any finite or infinite sequence $\{Y_n\}$ of disjoint subsets $Y_n \subset Y$ and any sequence $\{u_n\} \subset D_K$ such that $|u_n| \leqslant \chi_{Y_n}|u|$ there exists $E_\varepsilon \subset E$ such that $|E \sim E_\varepsilon| < \varepsilon$ and $\sum |Ku_n(x)|^2 < \varepsilon$ for every $x \in E_\varepsilon$.*

Proof. Since $\{y\colon u_n(y) \neq 0\} \in \mathscr{F}_\sigma(Y)$, we may assume that $Y_n \in \mathscr{F}_\sigma(Y)$, hence $F = \bigcup Y_n \in \mathscr{F}_\sigma(Y)$. For $t \in [0, 1]$ consider the series $u(y, t) = \sum u_n(y) r_n(t)$ where $\{r_n(t)\}$ is the sequence of Rademacher functions on $[0, 1]$ (see 4.7.1). By Proposition 5.7.8 the series $\sum r_n(t) Ku_n$ is convergent in $L^0(Y)$ for almost every $t \in [0, 1]$ to a function $w(x, t)$ and

$$\varrho_E\big(w(x,t)\big) = |E|^{-1} \int_E |w(x,t)|\big(1+|w(x,t)|\big)^{-1}\,dx < \varepsilon^4.$$

It follows that

$$|E|^{-1} \int_0^1 \int_E |w(x,t)|\big(1+|w(x,t)|\big)^{-1}\,dx\,dt < \varepsilon^4.$$

Let $E'_\varepsilon = \{(x,t) \in E \times [0,1]\colon |w(x,t)| \geqslant \varepsilon\}$, then E'_ε is measurable and $|E|^{-1}|E'_\varepsilon|\varepsilon(1+\varepsilon)^{-1} < \varepsilon^4$, which implies that $|E'_\varepsilon| < \varepsilon^2$ provided that $(1+\varepsilon)|E|\varepsilon < 1$.

Let $E_\varepsilon = \{x \in E\colon |\{t \in [0,1]\colon (x,t) \notin E'_\varepsilon\}| \geqslant 1-\varepsilon\}$. Then

$$\varepsilon^2 \geqslant |E'_\varepsilon| = \int_E |\{t\colon (x,t) \in E'_\varepsilon\}|\,dx \geqslant \int_{E \sim E_\varepsilon} \varepsilon\,dx = \varepsilon|E \sim E_\varepsilon|$$

implying that $|E \sim E_\varepsilon| < \varepsilon$.

For every $x \in E_\varepsilon$, $\big|\{t\colon |w(x,t)| < \varepsilon\}\big| \geqslant 1-\varepsilon$ and an application of the inequality in 4.7.1 gives:

$$(1-\varepsilon-\sqrt{2\varepsilon})\sum |Ku_n(x)|^2 < \varepsilon^2$$

showing that for $x \in E_\varepsilon$, $\sum |Ku_n(x)|^2 < \varepsilon$, provided $\varepsilon \leqslant 1/8$. ∎

5.7.10. COROLLARY. *Suppose that $u \in \tilde{D}_K$ and that $\{u_n\}$, Y_n are as in Proposition 5.7.9. Then $\sum |Ku_n(x)|^2 < \infty$ a.e.*

Proof. Let F be as in the proof of 5.7.9, let $E \in \mathscr{F}(X)$ and let $\varepsilon > 0$. We can find $v \in D_K$ such that $\tilde{\varrho}_{K,E,F}(u-v) < \varepsilon^4$. Define $v_n = \min(|v|, |u_n|)\operatorname{sign} u_n$. Then $|u_n - v_n| \leqslant |u-v|\chi_{Y_n}$ and by Proposition 5.7.9

$$\sum |K(u_n - v_n)(x)|^2 < \varepsilon$$

outside of a set of measure less than ε. On the other hand

$$\sum_{n=1}^{\infty} |Kv_n|^2 \leqslant \Big(\sum_{n=1}^{\infty} |Kv_n|\Big)^2 \leqslant (|K|\,|v|)^2 < \infty \qquad \text{a.e.}$$

and

$$\Big(\sum |Ku_n|^2\Big)^{1/2} \leqslant \Big(\sum |K(u_n - v_n)|^2\Big)^{1/2} + |K|\,|v|. \ \blacksquare$$

5.8. Examples.

5.8.1. Let $G = Y$ be a compact group, let $G' = X$ be its group of characters, and denote $k(x, y) = \overline{x(y)}$ and let ${}^t k(y, x) = \overline{k(x, y)}$ be the transposed kernel. Thus

$$Ku(x) = \int_G \overline{x(y)}\, u(y)\, dy, \qquad {}^t Kv(y) = \int_\Gamma x(y)\, v(x)\, dx = \sum_{x \in G'} x(y)\, v(x).$$

It is seen immediately that $D_K = L^1(G)$; in fact, for any fixed $x_0 \in G'$ and with $k^0 = k|_{\{x_0\} \times G}$, we have $D_{K^0} = L^1(G)$. Also by 5.7.6 $\tilde{D}_{K^0} = D_{K^0}$, and $\tilde{D}_K \subset \tilde{D}_{K^0}$ implies that $\tilde{D}_K = L^1(G)$.

Similarly $D_{{}^t K} = L^1(G') = l^1(G')$. 5.7.10 implies that $\tilde{D}_{{}^t K} \subset l^2(G')$. Since the characters of G form an orthonormal system in $L^2(G)$ it follows that ${}^t K$ can be extended by continuity to $l^2(G')$ and by 5.6.3 (ii) $l^2(G') \subset \tilde{D}_{{}^t K}$.

5.8.2. Consider the set up as in 5.8.1 assuming this time that G is a locally compact abelian group.

A paving in G is any family of mutually disjoint translates of a relatively compact neighborhood of the identity element, i.e. a family of the form

$$\mathscr{P} = \{g_l + F\}, \quad (g_l + F) \cap (g_k + F) = \emptyset \quad \text{for} \quad k \neq l.$$

For fixed F we consider all pavings $\mathscr{P}$ consisting of translates of F and define for $1 \leqslant p,\ q \leqslant \infty$, $f \in L^q_{\mathrm{loc}}(G)$

$$\|f\|_{l^p(L^q)} = \sup\Big\{\Big(\sum \|f\|^p_{L^q(g_l + F)}\Big)^{1/p}\colon\ \{g_l + F\} \text{ a paving in } G\Big\}.$$

We denote by $l^p(L^q)(G)$ the (Banach) space of all functions in $L^q_{\mathrm{loc}}(G)$ with finite norm $\|\ \|_{l^p(L^q)}$.

A particular norm $\|\ \|_{l^p(L^q)}$ depends on the choice of the neighborhood F of the identity element and should be denoted by $\|\ \|_{l^p(L^q), F}$. It can be shown that different choices of F give rise to equivalent norms and the definition of $l^p(L^q)$ is independent of the choice of F.

Also, for $1 \leqslant p \leqslant \infty$, $l^p(L^p)$ is isomorphic to $L^p(G)$, i.e. consists of the same functions with equivalent norms.

5.8.3. THEOREM. *If* $k(x, y) = \overline{x(y)}$, $x \in G'$, $y \in G$, *then* $\tilde{D}_K = l^2(L^1)$.

Proof. It is obvious as in 5.8.1 that $D_K = L^1(G) = D_{K^0}$ where $k^0 = k|_{C \times G}$ where C is any compact set in G'. Corollary 5.7.7 implies that $\tilde{D}_K \subset L^1_{\mathrm{loc}}(G)$.

Let F be a compact neighborhood of the identity in G and $E \in \mathscr{F}(G')$, $|E| > 0$ be such that $|g'(g) - 1| < 1/2$ for every $g \in E$, $g' \in F$. Let $\varepsilon \leqslant 1/8$ be such that $(1+\varepsilon)\varepsilon|E| < 1$ and $\varepsilon < |E|$ (see 5.7.9). Suppose that $u \in \tilde{D}_K$; we can assume that $u \geqslant 0$. There exists $v \in L^1(G)$, $0 \leqslant v \leqslant u$, such that $d_K(u-v) < \varepsilon^4$. For a paving $\mathscr{P} = \{g_n + F\}$ we define $u_n = \chi_{g_n+F}(u-v) \in D_K$ and we apply Proposition 5.7.9 to get

$$\sum \Big| \int_{g_n+F} \overline{x(y)}\,(u(y)-v(y))\,dy\Big|^2 < \varepsilon \quad \text{for} \quad x \in E_\varepsilon \subset E,\ |E \sim E_\varepsilon| < \varepsilon,$$

which after changing the variable of integration becomes

$$\sum \Big| \int_E \overline{x(y)}\,(u(y-g_n)-v(y-g_n))\,dy\Big|^2 < \varepsilon.$$

Since for any $x \in E$ and $y \in F$ we have $|1 - \overline{x(y)}| < 1/2$

$$\begin{aligned}\sum \Big| \int_{g_n+F} (u(y)-v(y))\,dy\Big|^2 &= \sum \Big| \int_F (u(y-g_n)-v(y-g_n))\,dy\Big|^2 \\ &\leqslant 2\sum \Big| \int (1-\overline{x(y)})(u(y-g_n)-v(y-g_n))\,dy\Big|^2 + 2\varepsilon \\ &< \tfrac{1}{2}\sum \Big| \int_{g_n+F} (u(y)-v(y))\,dy\Big|^2 + 2\varepsilon\end{aligned}$$

and

$$\sum \Big| \int_{g_n+F} (u(y)-v(y))\,dy\Big|^2 < 4\varepsilon.$$

Hence

$$\begin{aligned}&\Big(\sum \Big| \int_{g_n+F} u(y)\,dy\Big|^2\Big)^{1/2} \\ &\qquad \leqslant \Big(\sum \Big| \int_{g_n+F} v(y)\,dy\Big|^2\Big)^{1/2} + \Big(\sum \Big| \int_{g_n+F} (u(y)-v(y))\,dy\Big|^2\Big)^{1/2} \\ &\qquad \leqslant \|v\|_{L^1(Y)} + 2\sqrt{\varepsilon}\end{aligned}$$

showing that $u \in l^2(L^1)$ (recall $u \geqslant 0$). We have shown that $\tilde{D}_K \subset l^2(L^1)$.

To prove the opposite inclusion we notice that $l^2(L^1)$ is a solid Banach space and that $L^1 \subset l^2(L^1)$ with dense inclusion. Thus by Theorem 5.6.3, the proof will be complete if we show that $K\colon L^1 \subset l^2(L^1) \to L^0(X)$ is continuous. To this effect we notice that $L^2 \cap L^1$ is dense in L^1 and that L^1 is dense in $l^2(L^1)$. We shall prove that for every compact $E \subset X$ $(= G')$ and for every symmetric compact neighborhood F of the origin in Y $(= G)$ there exists a constant C such that

$$\|Kf\|_{L^2(E)} \leqslant C\,\|f\|_{l^2(L^1),F} \quad \text{for every } f \in L^1 \cap L^2.$$

Let F_1 be a compact neighborhood of the origin in G such that

$F_1+F_1 \subset F$. Since $F+F$ is compact there exists $\tilde{g}_1, \ldots, \tilde{g}_N \in F+F$ such that $\{\tilde{g}_i+F\}_{i=1,\ldots,N}$ is a cover of $F+F$.

Denote by χ the characteristic function of F_1 and let $\varphi = \chi * \chi$. Writing $Kg = \hat{g}$ $(g \in L^1)$ we note that $\hat{\varphi} = |\hat{\chi}|^2 \geqslant 0$ and $\hat{\varphi}(0) = |F_1| > 0$. Since E is compact, E can be covered by a fixed finite number of sets of the form $x_i+\{x\colon \hat{\varphi}(x) > |F_1|/2\}$ and it suffices to find an estimate for the integral $\int \hat{\varphi}(x)|\hat{f}|^2\,dx$.

Using Plancherel's theorem we can write

$$\begin{aligned}\int \hat{\varphi}(x)|\hat{f}(x)|^2\,dx &= \int \hat{\varphi}(x)\overline{\hat{f}(x)}\,\hat{f}(x)\,dx = \int (\varphi * f)(y)\overline{f(y)}\,dy \\ &= \int\int \varphi(y-z)\,f(z)\,\overline{f(y)}\,dz\,dy \leqslant \iint_{\tilde{F}} |f(y)|\,|f(z)|\,dy\,dz,\end{aligned}$$

where $\tilde{F} = \{(y, z) \in G \times G\colon\ y-z \in F\}$. Thus we may assume that $f \geqslant 0$.

The last integral can be written in the form

$$I = \int_G f(y) \int_F f(y+z)\,dz\,dy.$$

Since $f(y) \int_F f(y+z)\,dz \in L^1$, and $l^1(L^1)$ and $L^1(G)$ are isomorphic (see 5.8.2), for every $\varepsilon > 0$ there exists a paving $\{g_l+F\}$ such that

$$I \leqslant C \sum \int_{g_l+F} f(y) \int_F f(y+z)\,dz\,dy + \varepsilon$$

where C is a constant depending only on F. Thus it is sufficient to find a bound for the last sum.

We observe that for $y \in g_l+F$,

$$\int_F f(y+z)\,dz \leqslant \int_{g_l+F+F} f(z)\,dz$$

and applying Cauchy–Schwartz inequality, we get

$$\begin{aligned}\sum \int_{g_l+F} f(y)dy \int_{g_l+F+F} f(z)\,dz &\leqslant \Big(\sum\Big(\int_{g_l+F} f(y)\,dy\Big)^2\Big)^{1/2}\Big(\sum_l\Big(\sum_i \int_{g_l+\tilde{g}_i+F} f(y)\,dy\Big)^2\Big)^{1/2} \\ &\leqslant \Big(\sum\Big(\int_{g_l+F} f(y)\,dy\Big)^2\Big)^{1/2} N^{1/2}\Big(\sum_i\sum_l\Big(\int_{g_l+\tilde{g}_i+F} f(y)\,dy\Big)^2\Big)^{1/2} \leqslant N\|f\|^2_{l^2(L^1),F}. \quad \blacksquare\end{aligned}$$

5.8.4. COROLLARY. *Denote by* ^ *the Fourier transform in the Schwartz space of tempered distributions* $\mathcal{S}'(\boldsymbol{R}^n)$. *We say that* $f \in \mathcal{S}'$ *is a function, provided that* $f \in L^1_{\mathrm{loc}}$ *and* $\int |f\varphi|\,dx < \infty$ *for every* $\varphi \in \mathcal{S}$; *we have then* $\langle f, \varphi\rangle = \int f\varphi\,dx$. *Consider the following property* (P) *of a function* $f \in \mathcal{S}'$.

(P) *for every* $g \in L^0$, $|g| \leqslant f \Rightarrow \hat{g} \in L^1_{\mathrm{loc}}$.

Then f *satisfies* (P) *if and only if* $f \in l^2(L^1)$.

Proof. We notice that $l^2(L^1) = \{u\colon \sum(\int_{I_m} |u|\,dx)^2 < \infty\}$ where $\{I_m\}$ is any covering of $\boldsymbol{R}^n$ with congruent rectangles. If $f \in l^2(L^1)$, then the condition $|g| \leq |f|$ implies that $g \in l^2(L^1)$ and by 5.8.3

$$\hat{g} = \tilde{K}g \in L^2_{\text{loc}}, \quad \big(k(x, y) = (2\pi)^{-n/2} e^{-ixy}\big)$$

and f satisfies (P). Conversely, suppose that f satisfies (P). We can assume that $f \geq 0$. Define

$$A_f = \{u \in L^0(\boldsymbol{R}^n): |u| \leq \alpha f \text{ a.e. for some } \alpha > 0\}.$$

A_f is a solid Banach space with the norm $\|u\|_f = \inf\{\alpha > 0: |u| \leq \alpha f\}$. We shall prove now that $\hat{}: A_f \to L^1_{\text{loc}}$ is continuous. Since by (P) $\hat{A}_f \subset L^1_{\text{loc}}$, it suffices to show that $u_n \xrightarrow[A_f]{} 0$, $\hat{u}_n \xrightarrow[L^1_{\text{loc}}]{} v$ imply that $v = 0$, and then apply the closed graph theorem. To verify the latter we observe that if $u_n \xrightarrow[A_f]{} 0$, then by the definition of $\| \ \|_f$,

$$|\langle u_n, \varphi\rangle| \leq (\|u_n\|_f + 2^{-n}) \int f|\varphi|\, dx \xrightarrow[n\to\infty]{} 0 \quad \text{for} \quad \text{every } \varphi \in \mathscr{S}.$$

On the other hand

$$\langle \hat{u}_n, \varphi\rangle \xrightarrow[n\to\infty]{} \langle v, \varphi\rangle \quad \text{for every } \varphi \in \mathscr{D} = C_0^\infty.$$

But

$$\langle \hat{u}_n, \varphi\rangle = \langle u_n, \hat{\varphi}\rangle \xrightarrow[n\to\infty]{} 0 \quad \text{for} \quad \varphi \in \mathscr{D} \text{ and } v = 0.$$

Recall that $D_K = L^1$; it is not clear a priori that $L^1 \cap A_f$ is dense in A_f; to overcome this inconvenience denote by $\tilde{A}_f$ the closure $\overline{L^1 \cap A_f}^{A_f}$. Since for $u \in L^1 \cap A_f$ $\hat{u} = Ku$, it follows from 5.6.3 that $\tilde{A}_f \subset \tilde{D}_K = l^2(L^1)$. If $\{I_m\}$ is any partition of $\boldsymbol{R}^n$ into congruent rectangles and $\{\alpha_m\}$ is any sequence, $\alpha_m \to 0$, then it is easily checked that $\sum \alpha_n \chi_{I_n} \zeta \in \tilde{A}_f$. Hence $\sum \alpha_n \chi_{I_n} f \in l^2(L^1)$, i.e. $\sum_n |\alpha_n|^2 (\int_{I_n} f\, dx)^2 < \infty$ for every sequence $\alpha_m \to 0$, and it follows that $\sum_n (\int_{I_n} f\, dx)^2 < \infty$ and $f \in l^2(L^2)$. ■

5.9. Transposed of an integral transformation.

5.9.1. Let $k(x, y)$ be a kernel on $X \times Y$. The transposed kernel is defined by ${}^t k(y, x) = k(x, y)$ and the corresponding transposed transformation ${}^t K: L^0(X) \to L^0(Y)$ is given by

$${}^t Kf(y) = \int_X k(x, y) f(x)\, dx.$$

It is of some interest to explain the relationship between the transformation ${}^t K$ and the adjoint of K.

5.9.2. Recall (5.7.5) that K is nonsingular if for every $F \in \mathscr{F}_\sigma(Y)$ there is $\varphi \in D_K$ such that $\varphi > 0$ a.e. on F.

PROPOSITION. *If Y is σ-finite and if K is nonsingular then $'K$ is nonsingular.*

Proof. If $E \in \mathscr{F}_\sigma(X)$ and $\varphi \in D_K$, $\varphi > 0$ a.e. on Y, then there exists $\psi \in L^0(X)$ such that $\psi > 0$ a.e. on E, $\psi = 0$ on $X \sim E$, and that $\int \psi(x) \int |k(x, y)| \varphi(y) dy\, dx < \infty$. By the Fubini theorem $\int \varphi(y) \int |k(x, y)| \times \times \psi(x) dx\, dy$ and $|'K| \psi < \infty$ a.e. on Y, i.e. $\psi \in D_{'K}$. ∎

Remark. If X, Y are both σ-finite, then K is nonsingular if and only if $'K$ is nonsingular. It would be interesting to see if the result remains true without the assumption of σ-finiteness.

5.9.3. PROPOSITION. *Suppose that Y is σ-finite, that $A \subset L^0(Y)$ is a solid metric space such that $A = A_a$ and that K is a nonsingular integral transformation. Then $D_K \cap A$ is dense in A.*

Proof. Let $\varphi \in D_K$, $\varphi > 0$ a.e. and define for $u \in A$, $u_n = u$ if $|u| \leqslant n\varphi$ and $u_n = n\varphi \operatorname{sign} u$ when $|u| \geqslant n\varphi$. Then

$$|u - u_n| \leqslant \chi_{Y_n}(|u| - n) \quad \text{where } Y_n = \{y\colon |u(y)| \geqslant n\varphi\}.$$

Also $|u_n| \leqslant u$, $|u_n| \leqslant n\varphi$ a.e. which implies that $u \in A \cap D_K$. Using the solidity of the distance we get $\varrho_A(u - u_n) \leqslant \varrho_A(\chi_{Y_n}|u|)$, and the hypothesis $A = A_a$ implies that $\varrho_A(u - u_n) \xrightarrow[n\to\infty]{} 0$. ∎

5.9.4. Remark. The result remains valid without the hypothesis that Y is σ-finite provided that the functions in A with σ-finite supports are dense in A (this is the case when $A = L^p(Y)$, $p < \infty$).

5.9.5. PROPOSITION. *Let A, B be solid Banach spaces, suppose that $A \subset \tilde{D}_K$, $B \subset L^0(X)$ and that $A = A_a$, $B' = B'_a$ and that $\tilde{K}A \subset B$, where K is a nonsingular integral transformation. Then $K_A = \tilde{K}|_A\colon A \to B$ is a continuous linear transformation. If X, Y are σ-finite, then there exists a subspace $S \subset D_{'K} \cap B'$ dense in B' and in $B' \cap D_{'K}$ and such that $K_A^* v = {'K}v$ for every $v \in S$.*

Proof. The first statement follows readily from the closed graph theorem. Let $\varphi \in D_K$, $\psi \in D_{'K}$ be such that $\varphi > 0$, $\psi > 0$ a.e. and that

$$\int\int |k(x, y)| \varphi(x) \psi(y) dx\, dy < \infty \qquad \text{(see 5.9.2)}.$$

Let

$$S' = \bigcup_{n=1}^{\infty} \{v \in B'\colon |v| \leqslant n\psi \text{ a.e.}\} \quad \text{and} \quad S = \bigcup_{n=1}^{\infty} \{u \in A\colon |u| \leqslant n\varphi \text{ a.e.}\}.$$

Then $S \subset D_K \cap A$, $S' \subset D_{'K} \cap B'$ and S is dense in A, and in $A \cap D_K$, S' is dense in B' and in $B' \cap D_{'K}$. For $u \in A$, $v \in B'$ we have $\int K_A uv\, dx = \int u K_A^* v\, dy$, in particular for $u \in S$, $v \in S'$ we can write, using Fubini's theorem,

$$\int u K_A^* v\, dy = \int Kuv\, dx = \int u\, {'K} v\, dy.$$

Since S is dense in A the conclusion follows. ∎

5.9.6. THEOREM. *Suppose that X, Y are σ-finite; let $A \subset L^0(Y)$, $B \subset L^0(X)$ be solid Banach spaces and let K be a nonsingular integral transformation such that $A \subset \tilde{D}_K$ and $\tilde{K}A \subset B$. Suppose further that $A = A_a$ and $B' = B'_a$. Then*

$$K_A^* B' \subset A', \quad B' \subset \tilde{D}_{{}^tK} \quad \text{and} \quad K_A^*|_{B'} = {}^t\tilde{K}|_{B'} = {}^tK_{B'}.$$

Proof. By 5.9.5 we have for $v \in S'$: $K_A^* v = {}^tKv$ and by the dominated convergence theorem the same relation prevails for $v \in D_{{}^tK} \cap B'$. Also for $v \in D_{{}^tK} \cap B'$ we have $\|{}^tKv\|_{A^*} \leqslant \|K_A\| \, \|v\|_{B'}$. The conclusion will follow from 5.6.3 provided we prove that for $v \in D_{{}^tK} \cap B'$, ${}^tKv \in A'$, i.e. $|\int u\,{}^tKv\,dy| < \infty$ for every $u \in A$. With S and S' as in 5.9.5 we have

$$\left|\int u\,{}^tKv\,dy\right| = \left|\int Kuv\,dx\right| \leqslant \|K_A\| \, \|u\|_A \, \|v\|_{B'} \quad \text{for} \quad u \in S,\ v \in S'.$$

By the density of S' in $D_{{}^tK} \cap B'$ the same inequality prevails for $v \in D_{{}^tK} \cap B'$ and $u \in S$. For $u \in A$ we can assume that $\operatorname{sgn} u = \operatorname{sgn} {}^tKv$ and find a sequence $u_n \underset{A}{\to} u$ such that $u_n \in S$, $\operatorname{sgn} u_n = \operatorname{sgn} u$ for $u_n \neq 0$ and $|u_n| \nearrow |u|$ a.e. Then by the monotone convergence theorem the inequality in question is established for every $u \in A$, $v \in B' \cap D_{{}^tK}$. ∎

5.9.7. Remark. The result of the theorem remains true if σ-finiteness of X, Y is replaced by the hypothesis on A and B' as in 5.9.4.

5.9.8. Remark. In the case when $A \subset D_K$, $KA \subset B$ and $B' \subset D_{{}^tK}$, where A, B are solid Banach spaces, we have $K^*|_{B'} = {}^tK|_{B'}$ where K^* is the adjoint of the transformation $K \colon A \to B$.

6. Compactness of integral transformations

Throughout this section we assume that X and Y are σ-finite and we shall not repeat this hypothesis in each statement.

Various result concerning compactness of integral transformations are consequences of the following two propositions.

6.1. PROPOSITION. *Let A be a solid Banach space, $A \subset L^0(Y)$, such that $A'_a = A'$ and let K be an integral transformation such that $A \subset D_K$, $KA \subset L^1(X)$, $L^\infty(X) \subset D_{{}^tK}$ and $|{}^tK| L^\infty(X) \subset A'$. Then $K \colon A \to L^1(X)$ is compact.*

Proof. Assume that $\{u_n\} \subset A$, $\|u_n\|_A \leqslant 1$. We have to show that $\{Ku_n\}$ contains a subsequence convergent in $L^1(X)$. Suppose that this is not the case. By 2.4.3 there is a subsequence $\{u'_n\} \subset \{u_n\}$ such that $u'_n \to u$ in $\sigma(A, A')$-topology, for some $u \in A$. The assumptions imply that with u_n replaced by

$\{u_n' - u\}$ we have $u_n \to 0$ in $\sigma(A, A')$-topology and $\|Ku_n\|_{L^1(X)} \geqslant \alpha$ for some fixed $\alpha > 0$. We can find then $w_n \in L^\infty(X)$, $\|w_n\|_{L^\infty} = 1$ such that

$$\int Ku_n w_n\, dx = \|Ku_n\| \geqslant \alpha.$$

We can assume that $w_n \to w$ weakly-$*$ in $L^\infty(X)$. The hypothesis that $L^\infty \subset D_{{}^tK}$ implies that $k(\cdot, y) \in L^1(X)$ for a.e. y and that ${}^tKw_n \to {}^tKw$ a.e. Since $|{}^tKw_n| \leqslant |{}^tK|\,|w_n| \leqslant |{}^tK|\, 1(\cdot) \in A'$ ($1(\cdot)$ denotes the function identically equal to 1), the dominated convergence theorem 2.3.6 implies that ${}^tKw_n \to {}^tKw$ in A'. Hence we can write

$$\begin{aligned}\left|\int {}^tKw_n u_n\, dy\right| &= \left|\int ({}^tKw_n - {}^tKw)\, u_n\, dy + \int {}^tKwu_n\, dy\right| \\ &\leqslant \|{}^tKw_n - {}^tKw\|_{A'} \|u_n\|_A + \left|\int {}^tKwu_n\, dy\right| \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

On the other hand, since $\int (|{}^tK|\,|w_n|)\,|u_n|\, dy < \infty$, we can apply the Fubini theorem to conclude that $\int {}^tKw_n u_n\, dy = \int w_n K u_n\, dx \geqslant \alpha$ which is a contradiction. ∎

6.1.1. Remark. The conclusion of 6.1 may be false if the hypothesis that $A' = A_a'$ is omitted.

With $X = Y = [0, 1]$ let $\{E_n\}$ be a sequence of disjoint subsets of $[0, 1]$, $|E_n| > 0$ and $\{\varphi_n\}$ an orthonormal uniformly bounded sequence in $L^2(0, 1)$, e.g. $\varphi_n(x) = e^{2\pi i n x}$. Define

$$k(x, y) = \sum_{n=1}^{\infty} \varphi_n(x)\chi_n(y), \qquad \text{where} \qquad \chi_n = \chi_{E_n}.$$

Then $\int |k(x, y)|\, dx$ is in $L^\infty(0, 1)$ which implies that $L^1(0, 1) \subset D_K$ and $K\colon L^1(0, 1) \to L^1(0, 1)$. If $u_n(y) = |E_n|^{-1}\chi_n(y)$, then $u_n \in L^1$, $\|u_n\|_{L^1} = 1$ and $Ku_n = \varphi_n$. But $\{\varphi_n\}$ does not contain any subsequence convergent in measure in particular in L^1 and $K\colon L^1 \to L^1$ is not compact.

6.2. Proposition. *Suppose that $|X| < \infty$, that $L^\infty(Y) \subset D_K$, where K is an integral transformation. Then for every $\varepsilon > 0$ there is a subset $X_\varepsilon \subset X$, such that $|X \sim X_\varepsilon| < \varepsilon$, $KL^\infty(Y)|_{X_\varepsilon} \subset L^\infty(X_\varepsilon)$ and $K\colon L^\infty(Y) \to L^\infty(X_\varepsilon)$ is compact.*

Proof. The hypothesis $L^\infty(Y) \subset D_K$ implies that $x \to k(x, \cdot)$ defines a vector valued measurable function with values in $L^1(Y)$. There is a sequence of simple vector valued functions

$$k_n(x, \cdot) = \sum_{l=1}^{l_n} \chi_{ln}(x) f_{ln}(\cdot),$$

where χ_{ln} are characteristic functions of subsets of X, such that $\|k_n(x, \cdot) - k(x, \cdot)\| \xrightarrow[L^1(Y)]{} 0$ a.e. There exists a set $X_\varepsilon' \subset X$ such that $|X \sim X_\varepsilon'| < \varepsilon/2$ such that $\|k(x, \cdot)\|_{L^1} \in L^\infty(X_\varepsilon')$, also, by Egorov's theorem there is X_ε'' such that $|X \sim X_\varepsilon''| < \varepsilon/2$ and $\|k_n(x, \cdot) - k(x, \cdot)\| \xrightarrow[L^1(Y)]{} 0$ uniformly on X_ε''. Since $\|K\|_{L^\infty(Y), L^\infty(X)} = \operatorname{essup}_X \|k(x, \cdot)\|_{L^1(Y)}$, it follows that with $X_\varepsilon = X_\varepsilon' \cap X_\varepsilon''$,

$u \to \chi_{X_\varepsilon} Ku$ transforms $L^\infty(Y)$ into $L^\infty(X_\varepsilon)$ and that $\chi_{X_\varepsilon} K$ is a uniform limit of transformations of finite rank. ∎

6.3. We give some examples of applications of 6.1.

6.3.1. THEOREM. *Let K be an integral transformation and $A \subset D_K$ be a solid Banach space such that $A'_a = A'$. Then $K\colon A \to L^0(X)$ is compact.*

Proof. Suppose first that $|X| < \infty$. Denote by A_1 the closed unit ball in A. Then the set $\{|K|\,|u|\colon u \in A_1\}$ is convex, bounded in $L^0(X)$ and consists of positive functions. By 2.5.1 we can find for every $\varepsilon > 0$ a subset $X_\varepsilon \subset X$, $|X \sim X_\varepsilon| < \varepsilon$ and a constant N_ε such that $\|\chi_{X_\varepsilon} Ku\|_{L^1(X_\varepsilon)} \leqslant N_\varepsilon$ for every $u \in A$ and it follows that $\chi_{X_\varepsilon} K\colon A \to L^1(X_\varepsilon)$. 6.1 implies that for every $\varepsilon < 0$, $\chi_{X_\varepsilon} KA_1$ is relatively compact in $L^1(X)$ and a fortiori in $L^0(X)$. Since for every $v \in L^0(X)$, $\varrho_X(v - \chi_E v) \leqslant |X|^{-1}|X \sim E|$, it follows that KA_1 is relatively compact in $L^0(X)$. If $X = \bigcup X_n$, $|X_n| < \infty$ and $\varrho_X(v) = \sum 2^{-n} \varrho_{X_n}(v)$, $v \in L^0(X)$, then for every ε we find $X_{n\varepsilon} \subset X_n$ such that $\chi_{X_{n\varepsilon}} KA_1$ is relatively compact in $L^1(X)$, and therefore in $L^0(X)$, and $|X_n \sim X_{n\varepsilon}| < \varepsilon$. Let $X_{m\varepsilon} = \bigcup_{l=1}^{m} X_{l\varepsilon}$. Then $\chi'_m KA_1$ is relatively compact in $L^0(X)$, $\chi'_m = \chi_{X_{m\varepsilon}}$ and $|X_l \sim X'_{m\varepsilon}| < \varepsilon$ for $l = 1, \ldots, m$. Also

$$\varrho_X(v - \chi'_m v) \leqslant \sum_{n=1}^{m} 2^{-n} |X_n|^{-1} \varepsilon + 2^{-m} \leqslant (|X_1|^{-1} + 1)\varepsilon$$

provided $2^{-m} < \varepsilon$ and KA_1 is relatively compact in $L^0(X)$. ∎

6.3.2. THEOREM. *Let K be an integral transformation and $A \subset L^0(Y)$, $B \subset L^0(X)$ be solid Banach spaces such that $A' = A'_a$, $A \subset D_K$ and $KA \subset B_a$. Then $K\colon A \to B$ is compact if and only if $\|\chi_{E_n} K\|_{A \to B} \xrightarrow[n\to\infty]{} 0$ for every sequence $E_n \downarrow \emptyset$ of subsets of X.*

Proof. If the image of the unit ball A_1 of A is (relatively) compact in B, then $\|\chi_{E_n} v\| \xrightarrow[n\to\infty]{} 0$ for every sequence $E_n \downarrow \emptyset$ uniformly with respect to $v \in KA_1$ which is equivalent to the condition in the statement. Hence the condition is necessary. The sufficiency follows readily from 6.3 and 2.3.4 applied to KA_1. ∎

6.3.3. THEOREM. *With K, A, B as in 6.3.2 suppose that $|K|A \subset B$. Then $K\colon A \to B$ is compact if and only if $\|K\chi_{F_n}\|_{A \to B} \xrightarrow[n\to\infty]{} 0$ for every sequence $F_n \downarrow \emptyset$ of subsets of Y.*

Proof. The result follows from 5.9.8 and 6.3.2 applied to $'K$ and from the theorem of Schauder (if necessary, one has to replace B by its closed subspace B_a). ∎

6.3.4. COROLLARY. *Suppose that A is as in 6.3.2, that $KA \subset L^p(X)$, that $|X| < \infty$ and that $p > 1$. Then for every q such that $1 \leqslant q < p$, $KA \subset L^q(X)$ and $K\colon A \to L^q$ is compact.*

Proof. For $v \in L^p$ and $E \subset X$ we can write $\|\chi_E v\|_{L^q} \leqslant \|v\|_{L^p} |E|^{1/q-1/p}$, which implies that the condition in 6.3.2 is satisfied. ■

6.3.5. COROLLARY (Hille Tamarkin operators). *Suppose that k satisfies the condition* $k(x,\cdot) \in L^{p'}$ *for a.e. x and* $\int \|k(x,\cdot)\|_{L^{p'}}^p dx < \infty$ *for some p, q,* $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, *where* $1/p' + 1/p = 1$. *Then* $K\colon L^p(Y) \to L^q(X)$ *is compact.*

Proof. By Hölder's inequality we have

$$\|\chi_E K\|_{L^p \to L^q} \leqslant \Big(\int_E \|k(x,\cdot)\|_{L^{p'}}^q dx\Big)^{1/q},$$

which implies that the condition in 6.3.2 is satisfied. ■

6.4. We consider next some consequences of 6.2.

6.4.1. PROPOSITION. *Suppose that* $|X|, |Y| < \infty$ *and that K is a nonsingular integral transformation. Then for every* $\varepsilon > 0$, $\delta > 0$ *there exist* $X_\varepsilon \subset X$, $Y_\delta \subset Y$ *such that* $|X \sim X_\varepsilon| < \varepsilon$, $|Y \sim Y_\delta| < \delta$ *and* $u \to \chi_{X_\varepsilon} K_{\chi_Y} \delta$ *is a compact transformation from* L^1 *into* L^1.

Proof. By the hypothesis that K is nonsingular we can find $X'_\varepsilon \subset X$ such that $\int_{X'_\varepsilon} |k(x, y)|\, dx < \infty$ a.e. and $|X \sim X_\varepsilon| < \varepsilon$. Hence there exists $Y'_\delta \subset Y$ and a constant N_δ such that $|Y \sim Y'_\delta| < \delta/2$ and $\int_{X_\varepsilon} |k(x, y)|\, dx \leqslant N_\delta$ for $y \in Y'_\delta$. If follows that ${}^t|K|\colon L^\infty(X_\varepsilon) \to L^\infty(Y'_\delta)$ and by 6.2 there exists $Y_\delta \subset Y'_\delta$ such that $u \in L^\infty(X_\varepsilon) \to {}^tKu|_{Y_\delta} \in L^\infty(Y_\delta)$ is compact and that $|Y'_\delta \sim Y_\delta| < \delta/2$. Schauder's theorem and 5.9.8 imply that $K\colon L^1(Y_\delta) \to L^1(X_\varepsilon)$ is compact. ■

6.4.2. Suppose that $A \subset L^0(Y)$ is a solid Banach space without unfriendly sets. Then there exists a function $f \in A'$ such that $f > 0$. Then $M_f\colon L^0(Y) \to L^0(Y)$ defined by $M_f u = fu$ is a linear isomorphism of $L^0(Y)$. Denote by A_f the image $M_f A$ with the norm $\|M_f u\|_{A_f} = \|u\|_A$. Then $(A_f)_a = (A_a)_f$ and if $C \subset A_a$ is a set with uniformly absolutely continuous norms, then $M_f C$ has the same property in A_f. Also $A_f \subset L^1(Y)$.

6.4.3. THEOREM. *Let* $A \subset L^0(Y)$ *be a solid Banach space and K be a nonsingular integral transformation. Suppose that* $A \subset \tilde{D}_K$ *and let C be a bounded subset of A of functions with uniformly absolutely continuous norms. Then* $\tilde{K}C$ *is relatively compact in* $L^0(X)$.

Proof. Observe first that the continuity of $\tilde{K}\colon A \to L^0(X)$ (5.6.1) and the properties of C imply that for every sequence $Y_n \nearrow Y$, $|Y_n| < \infty$, the limit $\lim_{n\to\infty} \varrho_X(\tilde{K}(1-\chi_{Y_n})u) = 0$ is uniform in $u \in C$. Hence it is sufficient to show existence of a sequence $Y_n \nearrow Y$ such that for every n the set $\tilde{K}\chi_{Y_n} C$ is relatively compact in $L^0(X)$. By 6.4.2 we can assume that $A \subset L^1$. Using σ-finiteness of X, Y and 6.4.1 we find sequences $X_n \nearrow X$, $Y_n \nearrow Y$, $|X_n|, |Y_n| < \infty$ such that for every n, $\chi_{Y_n} L^1(Y) \subset D_K$ and that for every $m \geqslant n$, $\chi_{X_m} K \chi_{Y_n} C$ is relatively compact in $L^1(X)$. It follows that for every n, $K\chi_{Y_n}C$ is relatively compact in $L^0(X)$ and that for every $u \in C$, $K\chi_{Y_n}u = \tilde{K}\chi_{Y_n}u$. ∎

6.4.4. Proposition. *Suppose that* $|X|$, $|Y| < \infty$, *that* K *is a nonsingular integral transformation and that* $1 < p < \infty$. *Then for every* $\varepsilon > 0$ *there exists* $X_\varepsilon \subset X$, $Y_\varepsilon \subset Y$ *such that* $|X \sim X_\varepsilon| < \varepsilon$, $|Y \sim Y_\varepsilon| < \varepsilon$ *and with* $k_\varepsilon(x, y) = \chi_{X_\varepsilon}k(x, y)\chi_{Y_\varepsilon}(y)$ *we have* $L^p(Y) \subset D_{K_\varepsilon}$, $K_\varepsilon L^p(Y) \subset L^p(X)$ *and* K_ε: $L^p(Y) \to L^p(X)$ *is compact.*

Proof. As in 6.4.1 we find $X'_\varepsilon \subset X$, $Y'_\varepsilon \subset Y$ such that $\int_{X'_\varepsilon} |k(x, y)|\,dx$ is bounded on Y'_ε and that $|X \sim X'_\varepsilon| < \varepsilon/2$ and $|Y \sim Y'_\varepsilon| < \varepsilon/2$. Then $\chi_{X'_\varepsilon}K\chi_{Y'_\varepsilon} = K'_\varepsilon$ has the property that $L^1(Y) \subset D_{K'_\varepsilon}$ and K'_ε: $L^1(Y) \to L^1(X)$. We next find $X_\varepsilon \subset X'_\varepsilon$, $Y_\varepsilon \subset Y'_\varepsilon$, $|X'_\varepsilon \sim X_\varepsilon| < \varepsilon$, $|Y'_\varepsilon \sim Y_\varepsilon| < \varepsilon$, such that $\int_Y |k(x, y)|\,dy$ is bounded on X_ε and $K_\varepsilon = \chi_{X_\varepsilon}K\chi_{Y_\varepsilon}$: $L^\infty(Y) \to L^\infty(X)$ is compact (6.2). The result follows then from the "compact interpolation" theorem. ∎

6.4.5. Remark. In the case when $X = Y$ the sets X_ε, Y_ε in 6.4.4 can be chosen so that $X_\varepsilon = Y_\varepsilon$.

The same remark is true concerning 6.4.1 with $\varepsilon = \delta$.

6.4.6. Corollary. *Suppose that* $X = Y$ *is non-atomic and let* $1 \leqslant p \leqslant \infty$. *Then there does not exist a nonsingular integral transformation* K *with the property that* $L^p \subset \tilde{D}_K$ *and* $\tilde{K}|_{L^p} =$ *identity on* L^p.

Proof. For a nonsingular K with the above property we could use 6.4.1, 6.4.4 or 6.2, depending on whether $p = 1$, $1 < p < \infty$, $p = \infty$, to find a divisible subset $X_1 \subset X$, $0 < |X_1| < \infty$, such that the operator K, with the kernel $k_1(x, y) = \chi_{X_1}(x)k(x, y)\chi_{X_1}(y)$ would have the property that $L^p \subset D_{K_1}$ and K_1: $L^p \to L^p$ is compact. It would follow then that the identity in $L^p(X_1)$ is compact which is impossible. ∎

6.4.7. We say that 0 is a *point of Weyl of an (in general unbounded) transformation* T: $D(T) \subset A \to B$, where A, B are Banach spaces, provided there is a bounded sequence $\{u_n\} \subset D(T)$ such that $\{u_n\}$ contains no convergent subsequence and $Tu_n \underset{B}{\to} 0$.

6.4.8. In the case when $A = B = H$ is a Hilbert space 0 is a point of Weyl of T if and only if $0 \in \sigma_{le}(T)$ where $\sigma_{le}(T)$ denotes the left essential spectrum of T. Moreover, if $0 \in \sigma_{le}(T)$, then there exists an orthonormal sequence $\{e_n\} \subset H$ such that $Te_n \underset{H}{\to} 0$.

6.4.9. $0 \in \sigma_{le}(T)$ if and only if $0 \in \sigma_{re}(T^*)$ where σ_{re} denotes the right essential spectrum.

6.4.10. Theorem. *If* X *is non-atomic, if* $A \subset L^0(Y)$, $B \subset L^0(X)$ *are solid Banach spaces such that* $A' = A'_a$ *and* B *has no unfriendly sets, if* K *is a* *nonsingular integral transformation such that $A \subset D_K$ and $KA \subset B$, then 0 is a point of Weyl of K^*: $B^* \to A^*$. In particular, K cannot have a continuous inverse.*

Proof. Let $X_0 \subset X$ be a divisible subset of X such that $0 < |X_0| < \infty$, $\chi_{X_0} \in B \cap B'$ and that $u \in A \to \chi_{X_0} Ku$ is compact from A into $L^1(X_0)$ (see 6.1 and the proof of 6.3.1). We can assume that $|X_0| = 1$. As in 4.7.1 we consider the sequence of generalized Rademacher functions $\{r_n\}$ on X_0 extended by 0 to $X \sim X_0$. $\{r_n\}$ does not contain any subsequence convergent in $L^0(X)$; on the other hand by theorem of Riemann–Lebesgue $r_n \to 0$ weakly-$*$ in $L^\infty(X)$. Since $(\chi_{X_0} K)^*$: $L^\infty \to A'$ is compact, $(\chi_{X_0} K)^* r_n \underset{A'}{\to} 0$. But $(\chi_{X_0} K)^* r_n = K^* r_n$. ∎

6.4.11. Proposition. *If K is an integral transformation $X = Y$ is non-atomic, $L^2(X) \subset D_K$ and $KL^2(X) \subset L^2(X)$, then $0 \in \sigma_{re}(K)$.*

Proof. The result follows readily from 6.4.7, 6.4.8. ∎

6.5. Examples. A convenient way of constructing a variety of linear transformations between spaces of measurable functions consists of choosing two sequences $f_n \in L^0(X)$, $g_n \in L^0(Y)$ and a sequence a_{nm} of scalars, and setting $Tu = \sum a_{nm}(u, g_m) f_n$ where $(u, g) = \int ug\,dy$. A "sufficient" condition for T to be an integral transformation is the convergence in $L^0(X \times Y)$ of the sum $\sum a_{nm} f_n(x) g_m(y)$.

6.5.1. Let $g_n \in L^2(0, 1)$ be an orthonormal complete sequence and let $f_n = |E_n|^{-1/2} \chi_{E_n}$ where $|E_n|$ is a sequence of disjoint subsets of $[0, 1]$, $|E_n| > 0$. Then the sum $k(x, y) = \sum f_n(x) g_n(y)$ contains for each x at most one nonzero term and $T = K$ is an integral operator in $L^2(0, 1)$. Evidently $T^* v = \sum (v, f_n) g_n$ and an easy calculation shows that $T^* Tu = u$ for every $u \in L^2$. It follows that $T = K$ is a right inverse of T^*, that $0 \notin \sigma_{re}(T^*)$ and by 6.4.9 T^* is not an integral operator. Nevertheless T^* is of the form $T^* = {}'\tilde{K}|_{L^2}$ (see 5.9.6) showing that 6.4.10 does not hold for extensions by continuity of integral transformations. The example also shows that a composition of an extension to L^2 of an integral operator with an integral operator need not be an extension to L^2 of an integral operator.

6.5.2. Let $\{\varphi_n\}$ be an orthonormal sequence in $L^2(0, 1)$ and $\{\lambda_n\}$ be any bounded sequence. Define a diagonal operator $T \in \mathscr{L}(L^2)$ by setting $Tu = \sum \lambda_n (u, \varphi_n) \varphi_n$.

If $\{\varphi_n\}$ is complete one can choose $\{\lambda_n\}$ in such a way that the spectrum $\sigma(T)$ is any given compact subset of C. If $\{\varphi_n\}$ is not complete then $\sigma(T)$ may be any compact set in C containing 0.

Choosing $\varphi_n = |E_n|^{-1/2} \chi_{E_n}$, where E_n is a disjoint sequence of subsets of $[0, 1]$, $|E_n| > 0$, it is easily seen that T is an integral operator. We conclude that for every compact $C \subset \boldsymbol{C}$ containing 0 there exists an integral operator K in L^2 such that $\sigma(K) = C$.

6.5.3. Let $f_n = \varphi_n$ where $\{\varphi_n\}$ is the Haar basis in $L^2(0, 1)$, i.e. $\{\varphi_n\} = \{\varphi_0, \varphi_1^1, \varphi_1^2, \varphi_2^1, \ldots, \varphi_2^4, \ldots\}$ where $\varphi_0 = 1$ $\varphi_n^j = 2^{n/2}$ in $(2^{-n}(j-1), 2^{-n}(j-1/2))$, $\varphi_n^j = -2^{n/2}$ in $(2^{-n}(j-1/2), 2^{-n}j)$, $\varphi_n^j = 0$ otherwise, $j = 1, 2, \ldots, 2^n$ and let $g_n(x) = e^{2\pi inx}$. Define $Tu = \sum \lambda_n(u, g_n)\varphi_n$ where $\lambda_n = 2^{-n/2}$. Then $T \in \mathscr{L}(L^2)$ and $\lambda_n \to 0$ implies that T is compact. However T is not an integral operator. This follows from the following obvious remark.

6.5.4. If $K\colon A \subset D_K \to L^0(X)$ where $A \supset L^\infty$, then there exists a function $\Lambda \in L^0(X)$, $\Lambda \geqslant 0$, such that $\forall u \in L^\infty$: $|Ku(x)| \leqslant \Lambda(x)\|u\|_{L^\infty}$.

The operator T in 6.5.3 does not satisfy the conclusion of 6.5.3 since $|Tg_n| = \lambda_n^{-1/2}$ on the set where $\varphi_n \neq 0$. It is possible that T is maybe of the form $T = \tilde{K}|_{L^2}$ where K is an integral transformation.

6.5.5. Let $X = Y = [0, 1]$, $f_n(x) = |E_n|^{-1/2}\chi_{E_n}(x)$ and $g_n = e^{2\pi inx}$ where $\{E_n\}$ is a sequence of disjoint subsets of $[0, 1]$, $|E_n| > 0$. $\sum |E_n|^{1/2} = \infty$. Then $Ku = \sum (u, g_n) f_n$ is an integral operator with the following properties:

(i) $L^2 \subset D_K$, $K\colon L^2 \to L^2$, $|K|\,L^2 \not\subset L^2$, $|K|\,L^\infty \subset L^2$.

(ii) $L^2 \not\subset D_{{}^tK}$.

(iii) $\chi_E K\colon L^2 \to L^1$ and $K^*\chi_E\colon L^\infty \to L^2$ are not compact for any $E \subset [0, 1]$, $|E| > 0$.

(iv) For no $E \subset [0, 1]$, $|E| > 0$, $\chi_E K^*$ can be extended to a bounded transformation from L^p to L^1, $1 \leqslant p < 2$.

(v) For every $E \subset [0, 1]$, $|E| > 0$, there is a set $S \subset L^2$ of functions with uniformly absolutely continuous norms such that the L^2-norm on $K\chi_E S$ is not uniformly absolutely continuous.

(vi) There is a constant $C > 0$ such that $\|\chi_E K\| = \|K^*\chi_E\| \geqslant C$ for every $v \cdot E$ such that $|E| > 0$.

(vii). (vi) holds true if $\|\chi_E K\| = \|\chi_E K\|_{L^2 \to L^2}$ is replaced by $\|\chi_E K\|_{L^\infty \to L^2}$ and $\|K^*\chi_E\|$ is replaced by $\|K^*\chi_E\|_{L^2 \to L^1}$.

6.5.6. With f_n as in 6.5.5 and $g_n = \varphi_n$ where φ_n are Haar functions $\lim\limits_{|E|+|F| \to 0} \|\chi_E K \chi_F\|$ is not zero.

7. Miscellaneous results and comments

7.1. Method of Schur.

7.1.1. We consider conditions on a kernel k in order that $L^p \subset D_K$ and $|K|\,L^p \subset L^q$, $1 \leqslant p, q \leqslant \infty$. By 3.3.5 the last inclusion implies the continuity of $|K|\colon L^p \to L^q$ and a fortiori of $K\colon L^p \to L^q$. The formulation of the question implies that we can assume that $k \geqslant 0$.

7.1.2. If $1 = q \leqslant p \leqslant \infty$ then $K\colon L^p \subset D_K \to L^q$ if and only if $\int k(x, y)\,dx \in L^{p'}(Y)$, $1/p + 1/p' = 1$.

7.1.3. If $1 \leqslant q \leqslant p = \infty$ then $K\colon L^p \subset D_K \to L^q$ if and only if $\int k(x, y)\,dy \in L^q$.

7.1.4. If $p = 1$, $q = \infty$ then $K\colon L^p \to L^q$ if and only if

$$\left|\int\int k(x, y)\,u(y)\,v(x)\,dx\,dy\right| \leqslant C\,\|u\|_{L^1}\,\|v\|_{L^1}.$$

In the case when $X = \boldsymbol{R}^n$, $Y = \boldsymbol{R}^m$ this is equivalent to $k \in L^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^m)$.

7.1.5. Let $A \subset L^0(Y)$, $B \subset L^0(X)$ be solid Banach spaces. Then $A \subset D_K$ and $K\colon A \to B$ if and only if

$$\int\Big(\int k(x, y)\,u(y)\,dy\Big)\,v(x)\,dx < \infty \qquad \text{for every } u \in A,\ v \in B',\ u,\ v \geqslant 0 \text{ a.e.}$$

7.1.6. THEOREM. *Suppose that* $1 \leqslant q \leqslant p < \infty$ *and that* $k = k(x, y) \geqslant 0$. *Then* $L^p \subset D_K$ *and* $KL^p \subset L^q$ *if and only if there exist functions* $\varphi \in L^0(Y)$, $\psi \in L^0(X)$, φ, $\psi > 0$ *a.e. and there exists a constant* $C > 0$ *such that*

(i) $K\varphi \leqslant C\psi^{q'/q}$,

(ii) ${}^tK\psi \leqslant C\varphi^{p/p'}$ *and*

(iii) $\int\int k(x, y)\,\varphi(y)\,\psi(x)\,dx\,dy \leqslant C$.

In the case when $p = q$ (iii) *is to be omitted.*

Proof. Sufficiency. If (i), (ii), (iii) are satisfied then for $u \in L^p$, $v \in L^{q'}$ we can write:

$$\int kuv\,dx = \int\int k^{1/p}\psi^{1/p}\varphi^{-1/p'}uk^{1/q'}\varphi^{1/q'}\psi^{-1/q}vk^{1/q-1/p}\psi^{1/q-1/p}\varphi^{1/q-1/p}\,dy\,dx$$

and applying Hölder's inequality with the exponents p, q', $(1/q - 1/p)^{-1}$ we get

$$\int kuv\,dx \leqslant \Big(\int {}^tK\psi\varphi^{-p/p'}u^p\,dy\Big)^{1/p}\Big(\int K\varphi\psi^{-q'/q}v^{q'}\,dx\Big)^{1/q'}\Big(\int\int k\varphi\psi\,dx\,dy\Big)^{1/q-1/p}$$
$$\leqslant C\,\|u\|_{L^p}\,\|v\|_{L^{q'}}.$$

The sufficiency follows from 7.1.5.

The proof of necessity is based on the following lemma.

LEMMA. *Let* B *be a Banach space and let* $P \subset B$ *be a convex cone which is strictly convex at the origin (i.e.* $\alpha u_1 + \beta u_2 = 0$, $u_1, u_2 \in P$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ *imply* $u_1 = u_2 = 0$) *and let* $S\colon P \to P$ *be a continuous mapping. Assume the following conditions*:

(i) *If* $\{u_n\} \subset P$, *if* $u_{n+1} - u_n \in P$ *and if* $\|u_n\| \leqslant M$ *for* $n = 1, 2, \ldots$, *then there exists* $u \in P$ *such that* $u_n \underset{B}{\to} u$.

(ii) *For* u, $v \in P$, *such that* $u - v \in P$ *we have* $Su - Sv \in P$.

(iii) *If* $\|u\| \leqslant 1$ *and if* $u \in P$, *then* $\|Su\| \leqslant 1$.

Then for every $\sigma > 0$ there exists $u \in P$ such that $(1+\sigma)u - Su \in P$ and $0 < \|u\| \leq 1$.

Proof of the lemma. Let $u_1 \in P$, $0 < \|u_1\| \leq (1+\sigma)^{-1}\sigma$ and let $u_n = u_1 + (1+\sigma)^{-1} Su_{n-1}$; $n = 2, 3, \ldots$ Then

$$\|u_n\| \leq \|u_1\| + (1+\sigma)^{-1} \|Su_{n-1}\| \leq (1+\sigma)^{-1}\sigma + (1+\sigma)^{-1} \|Su_{n-1}\|$$

and $\|u_n\| \leq 1$ for $n = 1, 2, \ldots$ Also $u_{n+1} - u_n = (1+\sigma)^{-1}(Su_n - Su_{n-1})$ and $u_2 - u_1 = (1+\sigma)^{-1} Su_1 \in P$ which implies that $u_{n+1} - u_n \in P$ for all n. It follows from (i) that $u_n \underset{B}{\to} u \in P$. Since S is continuous, $u = u_1 + (1+\sigma)^{-1} Su$ and $(1+\sigma)u - Su = (1+\sigma)$, $u \in P$. If $u = 0$, then $(1+\sigma)u_1 + Su = 0$ and strict convexity of P at 0 implies that $u_1 = 0$ contrary to the choice of u_1. Hence $u \neq 0$. ∎

Proof of necessity of 7.1.6. Let $B = L^p$, $P = \{u \in L^p \colon u \geq 0\}$, and define S by the formula

$$Su = \left(\|K\|^{-1} [{}^tKv_0 + C^{-1} K(u_0 + (1+\sigma)^{-1} u)]^{q/q'}\right)^{p'/p}$$

where $\|K\| = \|K\|_{L^p \to L^q}$, $u_0 \in L^p(Y)$, $\|u_0\|_{L^p} \leq (1+\sigma)^{-1}\sigma$, $u_0 > 0$ a.e., $v_0 \in L^{q'}$, $v_0 > 0$ a.e., $\|v_0\|_{L^{q'}} \leq (\|K\| + \varepsilon)^{-1}\varepsilon$ where for $\varepsilon > 0$ σ is so chosen that $\|K\|(1+\sigma)^{2p/p'} \leq \|K\| + \varepsilon = C$.

By inspection, $S\colon P \to P$ and S is continuous, by the monotone convergence theorem P satisfies (i). Also $Su \geq Sv$ if $u \geq v$ and S satisfies (ii). If $\|u\|_{L^p} \leq 1$, then

$$\|u_0 + (1+\sigma)^{-1} u\|_{L^p} \leq (1+\sigma)^{-1} + (1+\sigma)^{-1}\sigma = 1,$$
$$\left\|C^{-1} K(u_0 + (1+\sigma)^{-1} u)\right\|_{L^q} \leq \|K\| (\|K\| + \varepsilon)^{-1}$$

and it follows that S satisfies (iii).

By the lemma we find $u \in L^p$, $0 < \|u\| \leq 1$ such that $Su \leq (1+\sigma)u$. Let $\varphi = u_0 + (1+\sigma)^{-1} u \in L^p$, $\psi = (v_0 + C^{-1} K\varphi)^{q/q'}$. Then

$$\varphi \in L^p, \quad \|\varphi\|_{L^p} \leq 1, \quad \psi \in L^{q'}, \quad \|\psi\|_{L^{q'}} \leq 1$$

and

$$K\varphi \leq C\psi^{q/q'}, \quad {}^tK\psi \leq \|K\| \left((1+\sigma)u\right)^{p/p'} \leq \|K\| (1+\sigma)^{2p/p'} \varphi^{p/p'}$$

and

$$\int\int k(x, y)\varphi(y)\psi(x)\,dx\,dy = \int K\varphi(x)\psi(x)\,dx \leq \|K\|$$

and (i), (ii), (iii) follow with $C = \|K\| + \varepsilon$, $\varepsilon > 0$. ∎

7.1.7. Corollary. *Suppose that $k(x, y) \geq 0$, that $L^p \subset D_K$, and that $K(L^p) \subset L^q$ where $p > q \geq 1$. Then $K\colon L^p \to L^q$ is compact.*

Proof. It suffices to show (see 6.3.2) that $\|\chi_{E_n} K\|_{L^p \to L^q} \to 0$ for every sequence $E_n \subset X$, $E_n \searrow \emptyset$. With φ and ψ as in 7.1.6 we can write

$$\|\chi_{E_n} K\|_{L^p \to L^q} \leqslant \left(\int {}^t K\psi\varphi^{-p/p'}\, dy\right)^{1/p} \left(\int K\varphi\psi^{-q'/q}\, dx\right)^{1/q'} \left(\int\int \chi_{E_n} k\varphi\psi\, dx\, dy\right)^{1/q-1/p}$$

and the last factor converges to 0 for every sequence $E_n \downarrow \emptyset$. ∎

It is natural to conjecture that 7.1.7 remains valid without the hypothesis that $k \geqslant 0$.

7.2. Integral operators in L^2.

7.2.1. Let $A \underset{c}{\subset} L^0(Y)$, $B \underset{c}{\subset} L^0(X)$ be Banach spaces of measurable functions and let K be an integral transformation. Define $D_{K,A,B} = \{u \in A \cap D_K\colon Ku \in B\}$ and the corresponding transformation $K_{A,B}\colon A \cap D_K \to B$; we will omit the subscripts A, B in instances when their meaning is clear.

Similarly we can define $\tilde{D}_{K,A,B} = \{u \in A \cap \tilde{D}_K\colon \tilde{K}u \in B\}$.

There are following possibilities, which we formulate in the case of K; the formulation in the case of $\tilde{K}$ is similar.

(a) $A \subset D_K$, $KA \subset B$; this case is taken care of by 3.3.5, and in the case of $\tilde{D}_K$ by 5.6.3 provided A is solid.

(b) $D_{K,A,B}$ is closed in A. Then $K\colon D_{K,A,B} \cap A \to B$ is an unbounded transformation which is closed. In fact, if $u_n \in D_{K,A,B}$, $u_n \underset{A}{\to} u$, $Ku_n \underset{B}{\to} v$, then $u \in D_{K,A,B}$. As in 5.6.3 the inclusion $D_{K,A,B} \subset D_K$ is continuous and $K\colon D_{K,A,B} \to L^0(X)$ is continuous. It follows that $Ku_n \to Ku$ and $Ku = v$.

(c) Neither (a) nor (b) is satisfied. In this case nothing of interest can be said without additional assumptions about K, A and B.

7.2.2. We consider the case when $X = Y$ is non-atomic and $A = B = L^2$. Then by 6.4.9 $0 \in \sigma_{re}(K)$.

THEOREM. *Let $T \in \mathscr{L}(L^2)$ and suppose that $0 \in \sigma_{re}(T)$. Then there exists a unitary operator $U\colon L^2 \to L^2$ such that $U^* TU$ is an integral operator with a Carleman kernel.*

We recall that $k(x, y)$ is a Carleman kernel if $\int |k(x, y)|^2\, dy < \infty$ a.e.

7.2.3. An operator $T \in \mathscr{L}(L^2)$ is strongly integral if for every unitary operator U, $U^* TU$ is an integral operator as in 7.2.1(a).

THEOREM. *An operator K is strongly integral if and only if K is an integral operator with a kernel satisfying the condition of Hilbert–Schmidt $\int\int |k(x, y)|^2\, dx\, dy < \infty$.*

7.2.4. If K is an integral transformation with a Carleman kernel, then it is easy to see that $L^2 \subset D_K$ and that with $D_{K2} = \{u \in L^2\colon Ku \in L^2\}$, $K\colon D_{K2} \subset L^2 \to L^2$ is a closed operator. Indeed, by 3.3.5, $K\colon L^2 \to L^0$ is continuous and if $u_n \underset{L^2}{\to} u$, $Ku_n \underset{L^2}{\to} v$, then $Ku_n \underset{L^0}{\to} Ku$ and it follows that $v = Ku$ and that $u \in D_{K2}$.

It is easy to construct examples of Carleman operators K such that D_{K2} is not dense in L^2.

7.2.5. If K is an integral operator such that $'K$ is a Carleman operator, i.e. $\int |k(x, y)|^2\,dx < \infty$ a.e. then D_{K2} is dense in L^2 however K need not be closed.

7.2.6. An integral transformation K is bi-Carleman if both K and $'K$ are Carleman operators. It follows from 7.2.4, 7.2.5 that if K is a bi-Carleman transformation then K^* is densely defined and $(K^*)^* = K$.

PROPOSITION. *If K is a bi-Carleman transformation and Y is non-atomic then 0 is a Weyl point of K: $D_{K2} \subset L^2(Y) \to L^2(X)$.*

Proof. Since $\int |k(x, y)|^2\,dx < \infty$ a.e. there exists $F \subset Y$, $0 < |F| < \infty$ and a constant $M > 0$ such that F is divisible and that $\int |k(x, y)|^2\,dx \leqslant M$ on F. We may assume that $|F| = 1$. It follows that

$$\int \Big[\int_F |k(x, y)|\,dy\Big]^2 dx \leqslant \Big(\int_F \Big(\int |k(x, y)|^2\,dx\Big)^{1/2} dy\Big)^2 \leqslant M$$

and that for every $u \in L^0(Y)$ such that $|u| \leqslant \chi_F$ we have $u \in D_{|K|2} \subset D_{K2}$ and $|Ku| \leqslant |K|\,\chi_F$.

If $\{r_n\}$ is the sequence of generalized Rademacher functions on F, extended by 0 outside F, then $\{r_n\} \subset D_{K2}$ and the hypothesis $\int |k(x, y)|^2\,dy < \infty$ a.e. implies that $Kr_n(x) \xrightarrow[n\to\infty]{} 0$ a.e. Since $|Kr_n| \leqslant |K|\chi_F \in L^2$ the dominated convergence theorems allows us to conclude that $Kr_n \xrightarrow[L^2(X)]{} 0$. ∎

7.2.7. COROLLARY. *If $X = Y$ is non-atomic and K is a bi-Carleman operator then $0 \in \sigma_{le}(K_{L^2\to L^2}) \cap \sigma_{re}(K_{L^2\to L^2})$.*

Proof. The statement follows directly from 6.4.7.

7.2.8. Remark. If K: $D_{K2} \subset L^2(X) \to L^2(X)$ is a bi-Carleman operator, then K^*K is a self-adjoint operator in $L^2(X)$. On the other hand, the inequality of Cauchy–Schwartz implies that $k_1(x, y) = \int k(z, x)\,k(z, y)\,dz$ is well defined and an argument used in the proof of the proposition above can be used to show that the integral operator K_1 is nonsingular.

It would be of interest to explain the relation between the operator K^*K and the operator K_1. It is natural to expect that $D(K^*K) \subset \tilde{D}_{K_1}$ and that $K^*K = \tilde{K}_1|_{D(K^*K)}$.

7.2.9. An answer to the question in 7.2.8 is likely to imply that $0 \in \sigma_{le}(K^*K)$. Note that the latter property would imply Proposition 7.2.6; indeed, if $\{\varphi_n\} \subset D(K^*K)$ is an orthonormal sequence (see 6.4.6) such that $K^*K\varphi_n \xrightarrow[L^2]{} 0$, then $(K^*K\varphi_n, \varphi_n) \xrightarrow[n\to\infty]{} 0$ and $\|K\varphi_n\| \xrightarrow[n\to\infty]{} 0$.

8. Bibliographical remarks and comments

2.2. The topology in $L^0(X)$ considered here is weaker and the space is larger than those introduced in [DS]. The idea of considering measure spaces which need not be σ-finite is taken from [Sz 1].

2.3.1 is due to Luxemburg (lecture notes, unpublished) in the case of σ-finite spaces. The proof given here is taken from [Sz 2].

2.3.3. The notion of unfriendly sets is one to Luxemburg and Zaanen in the case of solid spaces, see [LZ 1] for references. The proposition is taken from [ASz].

2.3.4, 2.3.5, 2.3.6 are due to Luxemburg and Zaanen [LZ 2].

2.5.1 is due to Maurey [M]; the proof given here is based on an idea of Nikišyn [N]; for the minimax theorem of Ky-Fan see [KF].

3. The material here is an elaboration of § 4 of [ASz]. The first version of 3.3.5 appeared in [B], a similar result was obtained by Gribanov [Gr], see also [K] and [HS]. It was observed by Mr Boukhaffa that the result of [Gr] is a consequence of 3.3.5.

4 is taken from §5 of [ASz].

5. The material here is taken essentially from §9 of [ASz], with modifications needed to accomodate measure spaces which need not be σ-finite. The material concerning $l^p(L^q)$-spaces in 5.8.2 can be found in [BD 1], [BD 2]. 5.8.3 was proved in [Sz 3] in the case when G is compactly generated and in [BD 2], in the general case. The proof given here is an elaboration of [Sz 1].

5.8.4 is taken from [Sz 3]. Extension of integral transformations are treated in [K] under the name of partly integral transformations.

6 is taken from [K]. 6.3.2 is due to [LZ 2] where more restrictive hypotheses on k are imposed. For 6.3.4 with $p = q = 2$ see [HS]. For 6.4.3 see [Ga 2], [Sh]. 6.4.5 with $\tilde{D}_K$ replaced by D_K can be found in [K] and [HS] which also see for information concerning the essential spectra. The composition of two integral operators in L^2 need be an integral operator, see [HS]. The condition in 6.5.4 characterizes integral transformations, see [Bu], [Sc].

7.1.6. The sufficiency of the conditions here gives rise to the so-called method of Schur [HS]. For necessity (and the proof given here as well as for references see [Ga 1] and also [Ka]. Note that by an appropriate choice of

$\varphi \in L^p$ and $\psi \in L^{p'}$ it is possible to estimate $\|K\|_{L^p \to L^q}$ as closely as desired. However it may be impossible to compute the norm $\|K\|_{L^p \to L^q}$ exactly in this way, see [Ga 2]. The references for 7.2 are [K] and [HS]—7.2.2 and 7.2.3 are taken from there. The problem of spectral description of integral operators in L^2 dates back to [V]. For 7.2.6, 7.2.7 in the case of bounded operators in L^2 see [K]. In connection with 7.2.9 see also Th. II. 4.10 in [K].

A wealth of examples of integral transformations is to be found in [KZPS], [J], [HLP].

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