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ATTITUDE CONTROL OF SPACECRAFT

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Introduction

This work is based on two reports to the European Space Agency [3], [4], which are principally concerned with the attitude control problem for spacecraft. Indeed, many of the missing details and other aspects of the problem can be found in these references. The equations describing the problem are basically those of a rotating rigid body with extra terms describing the effect of the control.

We shall represent the kinematic equations in two ways. If we require a global description of the problem, as is usually the case in this paper, we use the rotation matrix $R \in SO(3)$ which defines the transformation between an inertially fixed set of orthonormal axes e_1, e_2, e_3 denoted by I, and a set of orthonormal axes r_1, r_2, r_3 denoted by Q, of the same orientation, and fixed in the spacecraft:

$$Re_i = r_i, \quad i = 1, 2, 3.$$

We may now express the evolution of R by the matrix equation

$$\dot{R} = S(w)R \tag{1}$$

where w is the angular velocity of the spacecraft relative to Q, and $S(w)$ is the matrix defined by

$$S(w) = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix}.$$

Here $w = w_1r_1 + w_2r_2 + w_3r_3$, and $S(q)b = b \times q$ is just the usual cross product of vectors in $\mathbf{R}^3$.

Alternatively, the attitude can be described locally by the Euler angles φ, η, ψ, the consecutive rotations of the spacecraft about the axes r_1, r_2 and r_3, respectively. We have

$$\frac{d}{dt}\begin{bmatrix}\varphi\\ \eta\\ \psi\end{bmatrix} = \begin{bmatrix}\cos\eta & 0 & \sin\eta\\ \sin\eta\tan\varphi & 1 & -\cos\eta\tan\varphi\\ -\sin\eta\sin\varphi & 0 & \cos\eta\sec\varphi\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}. \tag{2}$$

We derive the dynamic equations by considering a momentum balance. In the case of control by gas jets we have the equations

$$Jw = Rh, \qquad R\dot{h} = \sum_{i=1}^{r} b_i u_i \tag{3}$$

where J is the inertia matrix of the spacecraft, h is the angular momentum of the spacecraft relative to I, and b_i are the axes about which the corresponding control torque $u_i/\|b_i\|$ is applied. r is of course the number of control torques. By combining equations (1) and (3) we obtain the closed set of equations

$$J\dot{w} = S(w)Jw + \sum_{i=1}^{r} b_i u_i \quad (Jw = Rh). \tag{4}$$

As expected, if we set $u_i = 0$, $i = 1, \ldots, r$ then we obtain $J\dot{w} = S(w)Jw$ which are simply the Euler equations for a rigid body.

In the case of control by momentum wheels we have the equations

$$\sum_{i=1}^{r} J_i^w(w + w_i^w) + Jw = Rh, \qquad \dot{h} = 0,$$

$$J_i^w(w + w_i^w) = Rh_i, \qquad b_i' R\dot{h}_i = -\|b_i\|^2 u_i$$

where J is the inertia matrix of the spacecraft without wheels, J_i^w is the inertia matrix of the ith wheel, h is the total angular momentum of the system relative to I, and h_i is the angular momentum of the ith wheel relative to I. w_i^w is the angular velocity of the ith wheel relative to the spacecraft, and b_i represents the axis about which the wheel spins, $u_i/\|b_i\|$ being the torque applied by the motor driving the ith wheel to the spacecraft. Throughout the paper $\|b\|^2 = b'b$ is the Euclidean norm.

To obtain a reasonable set of equations we make some additional assumptions, namely that b_i is a principal axis for the ith wheel, that is b_i is an eigenvector of J_i^w and that the ith wheel is symmetric about the axis b_i, that is the eigenvalues of J_i^w corresponding to eigenvectors perpendicular to b_i are equal. We also make some additional definitions by setting $J_{si}^w = b_i b_i' j_i^w/\|b_i\|^2$ where j_i^w is the moment of inertia of the ith wheel about its spin axis b_i, and $J^* = J + \sum_{i=1}^{r}(J_i^w - J_{si}^w)$. Since $J_i^w - J_{si}^w$ is positive semidefinite, we see that J^* is positive definite.

It now follows by differentiating the expression $J_i^w(w + w_i^w) = Rh_i$ and taking the scalar product with b_i, that

$$-u_i b_i = J_{si}^w(\dot{w} + \dot{w}_i^w).$$

It now follows that by setting $v = \sum_{i=1}^{r} J_{si}^w(w + w_i^w)$ we obtain the following set of equations:

$$J^*\dot{w} = S(w)\,(J^* w + v) + \sum_{i=1}^{r} b_i u_i \qquad (J^* w + v = Rh), \tag{5}$$

$$\dot{v} = -\sum_{i=1}^{r} b_i u_i. \tag{6}$$

Equations (5) and (6) yield a closed set of equations in the state variables v and w, whereas equations (1) and (5) yield a closed set of equations in the state variables R and w.

In this paper we are concerned with deriving conditions under which equations (1) and (4) are controllable, and equations (1) and (5) are controllable. The conditions will obviously depend on r, the number of control torques.

We prove controllability by applying the theorem proved by Bonnard in [2]. This states that an analytic system $\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$, $|u_i| \leqslant M_i$, on a manifold M, in which f is a Poisson stable vector field, is controllable if and only if it is accessible. Here f is Poisson stable on M if there exists a dense set of Poisson stable initial states on M, defined by: For any $T > 0$ and any neighbourhood U of p, there exist t_1 and $t_2 > T$ such that $\gamma_{t_1}(p) \subset U$ and $\gamma_{-t_2}(p) \subset U$. Here $(t, x) \mapsto \gamma_t(x)$ represents the flow of the complete vector field f.

The system is said to be *accessible* if the Lie algebra L generated by $f, g_1, \ldots, g_m$ is transitive on M, that is,

$$L(x) = \{X(x);\ X \in L\} = T_x M, \qquad \forall x \in M,$$

where $T_x M$ is the tangent space to M at x. We will denote the Lie bracket of two vector fields f and g by $[f, g]$.

In Section 2 of the paper we will prove Poisson stability of the vector field defining the free (uncontrolled) system in (1) and (4) and also (1) and (5). With this result, and Bonnard's theorem it is sufficient to find conditions for accessibility of the systems, in order to demonstrate the same conditions for controllability. This will be done in Section 3. In

Section 4 we shall consider the controllability conditions in more detail, and in Section 5 describe the implications for local controllability and stabilization algorithms, in the case of control by gas jets yielding two control torques.

2. Poisson stability

We prove Poisson stability by first showing that the equations

$$
\begin{aligned}
J\dot{w} &= S(w)(Jw+v), \qquad w(0) = w_0, \\
\dot{v} &= 0, \quad v(0) = v_0
\end{aligned}
\tag{7}
$$

have trajectories which are periodic for a dense set of initial states irrespective of v_0. We may apply this result to both equations (4) and (5) by setting $v_0 = 0$ in (4) and $J = J^*$ in (5). Clearly, initial states for which the resulting trajectories are periodic are Poisson stable.

In equation (7) w is constrained by the equations

$$\|Jw+v\| = H = \text{constant}, \qquad w'Jw = 2T = \text{constant}.$$

Of course, H is just the magnitude of the total angular momentum, and T is the kinetic energy. Thus Jw lies on the intersection of a sphere of radius H and centre $Jw = -v$, and an ellipse centred at the origin. This intersection is a union of closed curve Γ and isolated points. The motion is therefore periodic unless the intersection is a closed curve Γ, and the velocity vector w vanishes somewhere on Γ but not identically on Γ. We need therefore only characterize these situations. Clearly we can discount the cases where J is a multiple of the identity and H or T vanishes.

Now $\dot{w}$ vanishes when $\lambda w = (Jw+v)$ for some $\lambda \neq 0$, or $(I\lambda - J)w = v$. Noting that H and T are constant along solutions, we see that this imposes constraints on the possible relative values of T and H. In fact, in [4] it is shown that

$$H^2 = \lambda 2T + v'v + v'(\lambda J^{-1} - I)^{-1} v \tag{8}$$

if λ is not an eigenvalue of J, and

$$H^2 = \lambda 2T + v_P' v_P + v_P'(\lambda J^{-1} - I)_P^{-1} v_P \tag{9}$$

if λ is an eigenvalue of J. Here $(\lambda J^{-1} - I)_P$ is the restriction of $(\lambda J^{-1} - I)$ to P, the subspace spanned by eigenvectors perpendicular to the eigenspace corresponding to λ, and v_P is the component of v lying in P.

Since $v = v_0$ is constant in (7), we see that equations (9) and (8) restrict x_0 to lie on some surface in $\boldsymbol{R}^3$, irrespective of the value of v_0. In particular we see that the initial states x_0 for which solutions of (7) are not periodic, form a dense set in $\boldsymbol{R}^3$.

In fact, in the case $v = v_0 = 0$, relating to the case of a free rigid body we have $H^2 = 2T\lambda$ where λ must be an intermediate principal inertia of J. This defines the nonperiodic solutions of Euler's equation.

Having considered the free solutions of (4) and (5) and (6), we must look at the implications for the full set of free equations which we may write as

$$\begin{aligned} &\dot{R} = S(w)R, \quad J\dot{w} = S(w)Rh, \quad (Jw+v = Rh) \\ &\dot{h} = 0, \qquad \dot{v} = 0. \end{aligned} \tag{10}$$

The relation $Jw+v = Rh$ determines R up to rotations about the axis $Jw+v$, which has constant magnitude $\|Jw+v\| = \|h\| = H$, and therefore, as in Synge and Griffith [6] the equation $\dot{R} = S(w)R$ reduces to one of the form $\dot{\theta}_1 = f(w, v)$, where θ_1 is the angle in radians of the rotation about $Jw+v$. If w is periodic, of period a say, we have

$$\theta_1(t_1+a)-\theta_1(t_2+a) = \theta_1(t_1)-\theta_1(t_2)$$

for all t_1 and t_2. Hence setting $\theta_1(a)-\theta(0) = 2\pi\beta$ we have $\theta_1(na)-\theta_1(0) = 2\pi n\beta$ and so if β is a rational number (including $\beta = 0$), the motion in (10) will be periodic. If β is not rational the motion will not be periodic. However in either case, we may parameterize w, which evolves along a closed curve (neglecting the periodic constant solutions), by a suitable function θ_2 of the arc length along Γ to obtain

$$\dot{\theta}_2 = 2\pi/a \mod 2\pi, \qquad \dot{\theta}_1 = 2\pi\beta/a \mod 2\pi.$$

These equations which represent those of (10) evidently evolve on a winding line of a 2-torus; the motion is again clearly Poisson stable. Combining this result with the previous one gives

THEOREM 1. *The equations*

$$\begin{aligned} &\dot{R} = S(w)R, \quad J\dot{w} = S(w)Rh \quad (Jw+v = Rh) \\ &\dot{h} = 0, \qquad \dot{v} = 0 \end{aligned}$$

represent a Poisson stable vector field on the state space $SO(3)\times \mathbf{R}^3$.

3. Controllability

In this section we first give necessary and sufficient conditions for accessibility of equations (1) and (4)

$$\dot{w} = J^{-1}S(w)Jw+\sum_{i=1}^{r} J^{-1}b_i u_i, \qquad \dot{R} = S(w)R$$

which we shall write as

$$\dot{x} = F(x)+\sum_{i=1}^{r} u_i G_i.$$

We shall write $f(x)$ for $J^{-1}S(w)Jw$, and g_i for $J^{-1}b_i$. Further we will always assume that the set $\{b_1, \dots, b_r\}$ is linearly independent, and in particular $r \leqslant 3$. We therefore divide the investigation into the cases $r = 1, 2$ and 3.

Case 1. $r = 3$. In this case the vector fields G_1, G_2 and G_3,

$$[F, G_i](x) = -\begin{bmatrix} \dfrac{\partial f}{\partial x}(x) g_i \\ S(g_i)R \end{bmatrix}, \quad i = 1, 2, \text{ or } 3$$

clearly span $T_x(\mathrm{SO}(3)\times \boldsymbol{R}^3)$ for all $x \in \mathrm{SO}(3)\times\boldsymbol{R}^3$ and so the system is always accessible. Combining this result with Theorem 1 and Bonnard's theorem yields the result:

THEOREM 2. *The system*

$$\dot{R} = S(w)R,$$

$$J\dot{w} = S(w)Jw + b_1u_1 + b_2u_2 + b_3u_3, \qquad |u_i| \leqslant M_i$$

is always controllable.

Case 2. $r = 2$. In this case we need to consider the vector fields G_1, G_2,

$$[F, G_i](x) = -\begin{bmatrix} \dfrac{\partial f}{\partial x}(x) g_i \\ S(g_i)R \end{bmatrix}$$

and also

$$\bar{G}_3(\gamma_1, \gamma_2) = \big[[F, (\gamma_1 G_1 + \gamma_2 G_2)], (\gamma_1 G_1 + \gamma_2 G_2)\big] = \begin{bmatrix} \bar{g}_3(\gamma_1, \gamma_2) \\ 0 \end{bmatrix}$$

where

$$\bar{g}_3(\gamma_1, \gamma_2) = 2J^{-1}S(\gamma_1 g_1 + \gamma_2 g_2)J(\gamma_1 g_1 + \gamma_2 g_2),$$

$$[F, \bar{G}_3(\gamma_1, \gamma_2)](x) = -\begin{bmatrix} \dfrac{\partial f}{\partial x}(x)\bar{g}_3(\gamma_1, \gamma_2) \\ S\big(\bar{g}_3(\gamma_1, \gamma_2)\big)R \end{bmatrix}.$$

Let P be the subspace of $\boldsymbol{R}^3$ spanned by b_1 and b_2 and consider the condition

$$\boldsymbol{R}^3 = \mathrm{Span}\big\{b_1, b_2, S(x)J^{-1}x;\ x \in P = \mathrm{Span}\{b_1, b_2\}\big\}. \tag{11}$$

Clearly, if this assumption holds then

$$\boldsymbol{R}^3 = \mathrm{Span}\{g_1, g_2, \bar{g}_3(\gamma_1, \gamma_2), \gamma_1, \gamma_2 \in \boldsymbol{R}\}$$

and one then sees that the vector fields G_1, G_2,

$$\{\bar{G}_3(\gamma_1, \gamma_2);\ \gamma_1, \gamma_2 \in \boldsymbol{R}\}, \quad [F, G_1](x), \quad [F, G_2](x)$$

and

$$\{[F, \bar{G}_3(\gamma_1, \gamma_2)](x);\ \gamma_1, \gamma_2 \in \boldsymbol{R}\}$$

span $T_x(\mathrm{SO}(3)\times\boldsymbol{R}^3)$ for all $x \in \mathrm{SO}(3)\times\boldsymbol{R}^3$, and so the system is accessible.

We must now show that condition (11) is also necessary for accessibility. To do this we make the substitution $Jw = \gamma_1 b_1 + \gamma_2 b_2 + \gamma_3 S(b_2) b_1$ into the equation $J\dot{w} = S(w)Jw + b_1 u_1 + b_2 u_2$ to obtain an equation of the form

$$\begin{aligned} \dot{\gamma}_1 &= f_1(\gamma_1, \gamma_2, \gamma_3) + u_1, \\ \dot{\gamma}_2 &= f_2(\gamma_1, \gamma_2, \gamma_3) + u_2, \\ \dot{\gamma}_3 &= \gamma_3 f_3(\gamma_1, \gamma_2) + f_4(\gamma_1, \gamma_2)/\|S(b_2)b_1\|^2 \end{aligned} \tag{12}$$

where

$$f_4(\gamma_1, \gamma_2) = b_1' S(b_2) S(\gamma_1 b_1 + \gamma_2 b_2) J^{-1}(\gamma_1 b_1 + \gamma_2 b_2).$$

Clearly f_4 is not identically zero if and only if condition (11) holds. If f_4 is identically zero then we see that system (12) cannot be accessible since its maximal integral manifolds are

$$\boldsymbol{R}^3 \cap \{\gamma_3 = 0\}, \quad \boldsymbol{R}^3 \cap \{\gamma_3 < 0) \text{ and } \boldsymbol{R}^3 \cap \{\gamma_3 > 0\}.$$

Moreover, if (12) is not accessible then neither are equations (1) and (4), $r = 2$. On the other hand, these equations are accessible if (11) holds. By combining this with Theorem 1 and Bonnard's result we have

THEOREM 3. *The system*

$$\dot{R} = S(w)R,$$

$$J\dot{w} = S(w)Jw + b_1 u_1 + b_2 u_2, \qquad |u_i| \leqslant M_i$$

is controllable if and only if

$$\boldsymbol{R}^3 = \mathrm{Span}\{b_1, b_2, S(x)J^{-1}x;\ x \in \mathrm{Span}\{b_1, b_2\}\}.$$

Case 3. $r = 1$. In this case we need to consider the vector fields G_1,

$$[F, G_1](x) = -\begin{bmatrix} \dfrac{\partial f}{\partial x}(x) g_1 \\ S(g_1)R \end{bmatrix}, \quad [[F, G_1], G_1] = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2}(x)\ (g_1, g_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{g}_2 \\ 0 \end{bmatrix} = \bar{G}_2$$

where

$$\bar{g}_3 = 2J^{-1}S(g_1)Jg_1,$$

$$[F, (\gamma_1 G_1 + \gamma_2 \bar{G}_2)](x) = -\begin{bmatrix} \frac{\partial f}{\partial x}(x)(\gamma_1 g_1 + \gamma_2 \bar{g}_2) \\ S(\gamma_1 g_1 + \gamma_2 \bar{g}_2)R \end{bmatrix},$$

$$[[F, (\gamma_1 G_1 + \gamma_2 \bar{G}_2)], (\gamma_1 G_1 + \gamma_2 \bar{G}_2)] = \begin{bmatrix} \bar{g}_4(\gamma_1, \gamma_2) \\ 0 \end{bmatrix} = \bar{G}_4(\gamma_1, \gamma_2)$$

where

$$\bar{g}_4(\gamma_1, \gamma_2) = 2J^{-1}S\big((\gamma_1 g_1 + \gamma_2 \bar{g}_2)\big)J(\gamma_1 g_1 + \gamma_2 \bar{g}_2),$$

$$[F, \bar{G}_4(\gamma_1, \gamma_2)](x) = -\begin{bmatrix} \frac{\partial f}{\partial x}(x)\bar{g}_4(\gamma_1, \gamma_2) \\ S\big(\bar{g}_4(\gamma_1, \gamma_2)\big)R \end{bmatrix}.$$

Consider the condition

$$\boldsymbol{R}^3 = \operatorname{Span}\big\{b_1,\ S(b_1)J^{-1}b_1,\ S(x)J^{-1}x;\ x \in \operatorname{Span}\{b_1, S(b_1)J^{-1}b_1\}\big\}. \tag{13}$$

Clearly, if this assumption holds then

$$\boldsymbol{R}^3 = \operatorname{Span}\{g_1, g_2, \bar{g}_4(\gamma_1, \gamma_2);\ \gamma_1, \gamma_2 \in \boldsymbol{R}\}$$

and one sees that the vector fields $G_1, \bar{G}_2$,

$$\{\bar{G}_4(\gamma_1, \gamma_2);\ \gamma_1, \gamma_2 \in \boldsymbol{R}\}, \quad [F, G_1](x), \quad [F, \bar{G}_2](x)$$

and

$$\{[F, \bar{G}_4(\gamma_1, \gamma_2)](x), \gamma_1, \gamma_2 \in \boldsymbol{R}\}$$

span $T_x\big(\mathrm{SO}(3) \times \boldsymbol{R}^3\big)$ for all $x \in \mathrm{SO}(3) \times \boldsymbol{R}^3$, and so the system is accessible.

We must now show that condition (13) is also necessary for accessibility. As before we make a substitution

$$Jw = \gamma_1 b_1 + \gamma_2 S(b_1)J^{-1}b_1 + \gamma_3 S(b_1)S(b_1)J^{-1}b_1$$

in the equation $J\dot{w} = S(w)Jw + b_1 u_1$ to obtain an equation of the form

$$\begin{aligned} \dot{\gamma}_1 &= f_1(\gamma_1, \gamma_2, \gamma_3) + u_1, \\ \dot{\gamma}_2 &= f_2(\gamma_1, \gamma_2, \gamma_3), \\ \dot{\gamma}_3 &= \gamma_3 f_3(\gamma_1, \gamma_2) + f_4(\gamma_1, \gamma_2)/\|S(b_1)S(b_1)J^{-1}b_1\|^2 \end{aligned} \tag{14}$$

where

$$f_4(\gamma_1, \gamma_2) = -b_1' J^{-1}S(b_1)S(b_1)S\big(\gamma_1 b_1 + \gamma_2 S(b_1)J^{-1}b_1\big)J^{-1}\big(\gamma_1 b_1 + \\ + \gamma_2 S(b_1)J^{-1}b_1\big).$$

Clearly, f_4 is not identically zero if and only if condition (13) holds. If f_4 is identically zero then as in case (2) we see that system (14) cannot be accessible. In particular, system (1) and (4), $r = 1$, cannot be accessible either. If f_4 is not identically zero then we know that system (1) and (4), $r = 1$, is accessible. By combining this result with Theorem 1 and Bonnard's theorem we have

THEOREM 4. *The system*

$$\dot{R} = S(w)R,$$

$$J\dot{w} = S(w)Jw + b_1u_1, \qquad |u_1| \leqslant M_1$$

is controllable if and only if

$$R^3 = \mathrm{Span}\{b_1, S(b_1)J^{-1}b_1, S(x)J^{-1}x;\ x \in \mathrm{Span}\{b_1, S(b_1)J_4^{-1}b_1\}\}.$$

We note that the system cannot be controllable if $S(b_1)J^{-1}b_1 = 0$, which is the case if b_1 is an eigenvector for J. That is, a necessary condition for controllability is that b_1 is not a principal axis of J, which is physically obvious.

We now consider the case of control by momentum wheels, described in equations (1) and (5) and (6):

$$\dot{w} = J^{-1}S(w)Rh + \sum_{i=1}^{r} J^{-1}b_iu_i, \qquad \dot{R} = S(w)R \qquad (Jw + v = Rh)$$

$$\dot{h} = 0, \qquad \dot{v} = -\sum_{i=1}^{r} b_iu_i.$$

We first note that if $r < 3$ then we may take a vector $c \neq 0$, perpendicular to b_1 and b_2 so that $d/dt\, c'v = 0$, and hence $c'v$ = constant. In particular, $c'Rh = c'Jw + c'v$, which represents a constraint on the evolution of the state variables R and w. Thus the system cannot be controllable if $r < 3$. We therefore deal only with the case $r = 3$, although some comments are made about the structure of the maximal analytic surfaces in the case $r < 3$ in [1]. However, the computation in case (1) for control by gas jet goes through in exactly the same way, on setting

$$F(x) = \begin{bmatrix} J^{-1}S(w)Rh \\ S(w)R \end{bmatrix}, \qquad G_i = \begin{bmatrix} J^{-1}b_i \\ 0 \end{bmatrix}, \qquad i = 1, 2, 3.$$

We can therefore combine these results and those of Bonnard to obtain

THEOREM 5. *The system*

$$J\dot{w} = S(w)Rh + \sum_{i=1}^{r} b_i u_i,$$

$$\dot{R} = S(w)R \qquad \left(Rh = Jw + v, \quad \dot{v} = -\sum_{i=1}^{r} b_i u_i, \quad \dot{h} = 0\right)$$

is not controllable or accessible if $r < 3$, *and is controllable with* $|u_i| \leqslant M_i$ *if* $r = 3$.

4. Analysis of controllability conditions

In this section we give alternative descriptions of the conditions for controllability, in the case of control by gas jets, with one or two control torques. It is clear that the condition for lack of controllability in each case is equivalent to

$$c_1' S(c_2) S(\gamma_1 c_1 + \gamma_2 c_2) J^{-1}(\gamma_1 c_1 + \gamma_2 c_2) = 0$$

for arbitrary γ_1 and γ_2 and arbitrary vectors c_1 and c_2 in some two-dimensional subspace of $\boldsymbol{R}^3$. By expanding and recombining this expression we obtain the equivalent condition:

$$A\,\|\gamma_1 c_1 + \gamma_2 c_2\| = \|J^{-1/2}(\gamma_1 c_1 + \gamma_2 c_2)\|, \qquad \gamma_1, \gamma_2 \in \boldsymbol{R} \tag{15}$$

where

$$A^2 = c_2' J^{-1} c_2 / c_2' c_2 = c_1' J^{-1} c_1 / c_1' c_1 .$$

Clearly this shows that $J^{-1/2}$ is a transformation which scales lengths by a constant A, but leaves angles between vectors constant. We can apply (15) directly to the conditions for controllability as obtained in the previous section to obtain:

The system (1) and (4), $r = 2$, fails to be controllable if and only if

$$\exists A \neq 0 \text{ such that } A\,\|\gamma_1 b_1 + \gamma_2 b_2\| = \|J^{-1/2}(\gamma_1 b_1 + \gamma_2 b_2)\|, \qquad \gamma_1, \gamma_2 \in \boldsymbol{R}. \tag{16}$$

The system (1) and (4), $r = 1$, fails to be controllable if and only if

$$\exists A \neq 0 \text{ such that } A\,\|\gamma_1 b_1 + \gamma_2 S(b_1) J^{-1} b_1\| = \left\|J^{-1/2}\left(\gamma_1 b_1 + \gamma_2 S(b_1) J^{-1} b_1\right)\right\|, \ \gamma_1, \gamma_2 \in \boldsymbol{R}. \tag{17}$$

We note that conditions (17), (16) for lack of controllability, or equivalently (13), (11) ensuring controllability, are also applicable, in exactly the same sense, to system (4), $r = 1$, and 2, respectively. That is, once controllability of the velocity equation has been established, controllability of the full attitude equations follows automatically. In particular, the controllability (or more precisely, accessibility) problem for system (4), $r = 1$, was solved in Baillieul [1]. However, the conditions derived there are not directly comparable with those in (17) or (13), and were not applicable to the full equations (4) and (1). The equivalence of the two sets of conditions is demonstrated in [4]. We summarize them here.

We let r_1, r_2, r_3 be a set of principal axes for J with corresponding principal inertias j_1, j_2 and j_3. We let

$$A = \begin{vmatrix} r_3'b_2 & r_3'b_1 \\ r_2'b_2 & r_2'b_1 \end{vmatrix}, \quad B = \begin{vmatrix} r_3'b_2 & r_3'b_1 \\ r_1'b_2 & r_1'b_1 \end{vmatrix}, \quad C = \begin{vmatrix} r_1'b_2 & r_1'b_1 \\ r_2'b_2 & r_2'b_1 \end{vmatrix}$$

where $|\ |$ denotes determinant.

THEOREM 6 (Crouch [4]). *The system* (1) *and* (4), $r = 2$, *fails to be controllable if and only if any of the following equalities hold:*

$$\text{(i)} \quad \frac{C^2}{B^2}(j_1 - j_2) = (j_3 - j_1)(j_2/j_3), \qquad A = 0,\ C \neq 0,\ B \neq 0,$$

$$\text{(ii)} \quad \frac{A^2}{C^2}(j_2 - j_3) = (j_1 - j_2)(j_3/j_1), \qquad B = 0,\ A \neq 0,\ C \neq 0,$$

$$\text{(18)} \quad \text{(iii)} \quad \frac{B^2}{A^2}(j_3 - j_1) = (j_2 - j_3)(j_1/j_2), \qquad C = 0,\ B \neq 0,\ A \neq 0,$$

$$\text{(iv)} \quad j_1 = j_2, \qquad A^2 + B^2 = 0,\ C \neq 0,$$

$$\text{(v)} \quad j_1 = j_3, \qquad A^2 + C^2 = 0,\ B \neq 0,$$

$$\text{(vi)} \quad j_2 = j_3, \qquad B^2 + C^2 = 0,\ A \neq 0.$$

Clearly, we may now obtain conditions under which system (1) and (4), $r = 1$, fails to be controllable by substituting $S(b_1)J^{-1}b_1$ for b_2 into this theorem. We note that conditions (18) (iv), (v) and (vi) then imply that the subspace $P = \text{Span}\{b_1, S(b_1)J^{-1}b_1\}$ concides with a subspace spanned by two principal axes of J. This implies that $J^{-1}b_1$ has to be a principal axis, which is impossible if the system is to be controllable since as we have seen b_1 cannot be a principal axis in the case $r = 1$. It follows that conditions (18) (i), (ii) and (iii) may be replaced by the condition that any two of A, B and C vanish. We also note that if none of A, B or C vanish the system is automatically controllable, for $r = 1$ or $r = 2$. This is equivalent to the statement that if the torque axes b_1 and b_2 are in general position then the system is controllable.

Setting $J^{-1}b_1 = r_1p_1+r_2p_2+r_3p_3$ and $a_1 = (j_2-j_3)/j_1$, $a_2 = (j_3-j_1)/j_2$, $a_3 = (j_1-j_2)/j_3$, we can show the following

THEOREM 7 (Crouch [4]). *The system* (1) *and* (4), $r = 1$, *is controllable if and only if the following inequalities of Baillieul* [1] *hold:*

$$a_3p_2^2 \neq a_2p_3^2, \quad a_3p_1^2 \neq a_1p_3^2, \quad a_1p_2^2 \neq a_2p_1^2.$$

5. Practical applications

In this section we consider only the system

$$J\dot{w} = S(w)Jw+b_1u_1+b_2u_2, \quad \dot{R} = S(w)R. \tag{19}$$

In practical applications it is often desired to stabilize the motion of the spacecraft about one in which it is spinning on principal axis r_3 say with a desired constant angular velocity $\|w_d\|$. We include in this case the situation where we are required to stabilize the motion about $w = 0$ and a particular desired attitude. If we consider the error variable $w_e = w-w_d$ where $w_d = r_3\|w_d\|$ then we may obtain (see [4])

$$J\dot{w}_e = S(w_e)Jw_e+S(w_d)(J-j_3I)w_e+b_1u_1+b_2u_2. \tag{20}$$

Now we may represent the angular position in a neighbourhood of the equilibrium motion by equation (2) written as

$$\dot{\theta} = A(\theta)w$$

where $\theta' = (\varphi, \eta, \psi) = (\theta_1, \theta_2, \theta_3)$ represents the vector of Euler angles about axes r_1, r_2 and r_3, respectively. (We have noted that $A(\theta)$ does not depend on ψ.) Now $A(\theta) = I+\bar{A}(\theta)$ where $\bar{A}(\theta)|_{\theta=0} = 0$. If $w_d = 0$ we may now linearize equation (2) to obtain

$$\dot{\theta} = w_e. \tag{21}$$

If $w_d \neq 0$ we note that we are only interested in θ_1 and θ_2 and so we define a projection π by

$$q = a_1r_1+a_2r_2+a_3r_3 \mapsto \pi a = a_1r_1+a_2r_2.$$

Now we may write $\dot{\theta} = A(\theta)w$, as $\theta = w_e+w_d+\bar{A}(\theta)(w_e+w_d)$, so that the linearization becomes $\dot{\theta} = w_e+w_d$, and upon applying π we obtain (noting $\pi w_d = 0$)

$$(\pi\dot{\theta}) = \pi w_e. \tag{22}$$

Now it is clear that the linearization of (20) is

$$J\dot{w}_e = S(w_d)(J-j_3I)w_e+b_1u_1+b_2u_2 \tag{23}$$

so that the linear stabilization problem reduces to driving either (21) and (23), or (22) and (23) to the zero state. Clearly if $w_d = 0$ then system (23) is not controllable.

We now show that even if $w_d \neq 0$ systems (22) and (23) are not controllable. If P is the subspace spanned by r_1 and r_2 then $S(w_d)(J - j_3 I)$ maps P into itself. Let $\bar{A}$ be its restriction to P and let $\bar{B}$ be the matrix $(\pi b_1, \pi b_2)$. The controllability rank condition applied to (22) and (23) now shows that a necessary condition for controllability is that the rank of the matrix

$$C = \begin{bmatrix} \bar{A}\bar{B} & \bar{A}^2\bar{B} & \bar{A}^3\bar{B} \\ \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} \end{bmatrix}$$

is at least 3. However, $\bar{A}^2 = \mu I_2 = -\|w_d\|^2 j_1 j_2 I_2$ and $\bar{A}$ is nonsingular. Thus there exists a matrix T such that $\bar{A}^2\bar{B} = \mu\bar{B} = (\bar{A}\bar{B})T$, $\bar{A}\bar{B} = \bar{B}T$ and $\bar{A}^3\bar{B} = (\bar{A}^2\bar{B})T$. It follows that the matrix C has at most rank 2.

THEOREM 8. *The attitude control equations* (19), *linearized about* $w = 0$, *or* $w = \|w_d\| r_3$, *and any orientation, are never controllable.*

From this result it is clear that if one is to construct an algorithm to stabilize the system about these equilibrium states, the full nonlinear model must be used. The aim of any such stabilization algorithm will be to drive the current state in the direction of the equilibrium position, at any point in a neighbourhood of the equilibrium position. If we consider only the velocity equations, we may resort to the description given in equation (12). It is clear that such a stabilization algorithm is possible only if the term $F_1(\gamma_1, \gamma_2)$ takes opposite signs as γ_1 and γ_2 take values in a neighbourhood of zero. As we have seen, controllability is equivalent to the nonvanishing of $f_4(\gamma_1, \gamma_2)$. In fact, local controllability about the equilibrium states is equivalent to controllability and is in turn equivalent to $f_4(\gamma_1, \gamma_2)$ taking opposite signs. It is this property of local controllability which makes stabilization possible.

THEOREM 9 (Crouch [4]). *For system* (19) *the following statements are equivalent:*

(i) *Controllability.*

(ii) *Conditions* (16) *or* (11) ($f_4(\gamma_1, \gamma_2)$ *in* (12) *does not vanish*).

(iii) *Local controllability about* $w = 0$ *or* $w = \|w_d\| r_3$ *and any desired orientation.*

(iv) *The bilinear form* $f_4(\gamma_1, \gamma_2)$ *in* (12) *takes positive and negative values.*

In [4] a stabilization algorithm has been constructed using a method of Hermes [5].

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