POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCLXXI

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Closed ideals in algebras of smooth functions

WARSZAWA 1998

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Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
Publishing House of the Warsaw University of Technology
ul. Polna 50, 00-644 Warszawa

PRINTED IN POLAND

ISSN 0012-3862

CONTENTS

Acknowledgements. The author is grateful to E. M. Dyn'kin (Technion-I.I.T.) and John Wermer (Brown University) for valuable discussions.

1991 *Mathematics Subject Classification*: Primary 46E25, 46E35, 46H10, 46J10, 46J15, 46J20; Secondary 26A16, 41A10.

Key words and phrases: spectral synthesis of ideals, problem of spectral approximation, regular Banach algebra, point derivation, extension theorem, trace of a function, Sobolev space, Lipschitz space, Zygmund space.

Received 9.10.1996; revised version 20.2.1997.

Abstract

A topological algebra admits spectral synthesis of ideals (SSI) if every closed ideal in this algebra is an intersection of closed primary ideals. According to classical results this is the case for algebras of continuous, several times continuously differentiable, and Lipschitz functions. New examples (and counterexamples) of function algebras that admit or fail to have SSI are presented. It is shown that the Sobolev algebra $W_p^l(\mathbb{R}^n)$, $1 \le p < \infty$, has the property of SSI for and only for $n = 1$ and $2 \le n < p$. It is also proved that every algebra $C^m \operatorname{Lip} \varphi$ in one variable admits SSI. A unified approach to SSI based on an abstract spectral synthesis theorem for a class of Banach algebras (called D-algebras) defined in terms of point derivations and consisting of functions with "order of smoothness" not greater than 1 is discussed. Within this framework, theorems on SSI for Zygmund algebras Λ_φ in one and two variables not imbedded in C^1 as well as for their separable counterparts λ_φ are obtained. The fact that a Zygmund algebra Λ_φ is a D-algebra is equivalent to a special extension theorem of independent interest which leads to a solution of the spectral approximation problem for the algebras Λ_φ in the cases mentioned above. Closed primary ideals and point derivations in arbitrary Zygmund algebras are described.

Introduction

A topological algebra admits spectral synthesis of ideals (SSI) if every closed proper ideal in this algebra is an intersection of closed primary ideals. Assertions of this type go back to classical algebraic works of E. Noether [38] and E. Lasker [29] on primary decomposition of ideals in Noetherian rings. Later, theorems on SSI were discovered for a number of algebras of smooth functions (for more extensive discussion, see Section 1).

As shown by G. E. Shilov [43], *every* proper ideal in a regular semisimple commutative Banach algebra is an intersection of primary ideals. This shows that the collection of *all* primary ideals (not necessarily closed) of a regular functional algebra is excessively ample. Therefore, it is reasonable to confine oneself to *closed* primary ideals. Accordingly, every ideal which is an intersection of closed primary ideals is a *closed* ideal.

Due to the influence of classical works of L. Schwartz [40], [41], A. Beurling [3], [4], P. Malliavin [32]–[34], G. E. Shilov [43] and others, problems of spectral synthesis in harmonic analysis attracted a lot of interest and gave rise to plenty of publications from the late 40's to the middle 70's; see [2], [26], Chapter 10, [28], Chapter 5, and references therein.

By contrast, in the nonharmonic setting, the problem of SSI is much less studied, probably because it was commonly believed that all classical "local" algebras of smooth functions enjoy SSI though it is usually not easy to prove. Amazingly, this is not true even for some Sobolev algebras $W_p^l(\mathbb{R}^n)$ (see Section 2), the algebra $W_2^2(\mathbb{R}^2)$ providing apparently the simplest known "natural" counterexample to SSI.

The circle of questions discussed in this paper is closely related to the *problem of spectral approximation*, which consists in describing, for a linear topological space $\mathcal{A}$ of functions defined on a topological space X, the closure J_F of the set of functions in $\mathcal{A}$ vanishing in a neighborhood of a given closed set $F \subset X$. For any function in J_F, its derivatives, whatever they are, vanish in an appropriate sense on F. This renders the problem of spectral approximation for many natural classes of functions too restrictive. Indeed, it is not required in this problem that $\mathcal{A}$ is an algebra with respect to pointwise multiplication. However, if this is the case, a more general approach developed by G. E. Shilov [43] and based on the property of SSI is available, so that the problem of spectral approximation reduces to characterizing minimal closed ideals with a given cospectrum.

Stimulated by Malliavin's theorem on the lack of spectral synthesis in Wiener algebras for noncompact abelian groups [32]–[34], the problem of spectral approximation was originally studied mainly in the framework of harmonic analysis. However, profound results are also possible in the nonharmonic setting. As a most striking example, one can mention the solution of the spectral approximation problem for Sobolev spaces [23], [24].

The proofs of theorems on SSI are very individual, as a rule. An attempt at a unified approach which enables treating a number of essentially different cases from a single point of view is discussed in Section 4. This approach is materialized in the concept of a D-algebra, and a theorem claiming that all D-algebras admit SSI (see [20], [22]) is stated (see Theorem 4.1). The class of D-algebras is defined in terms of point derivations and embraces a number of algebras of functions with the "order of smoothness" ≤ 1, among them algebras C^1 of continuously differentiable functions on a closed cube in $\mathbb{R}^n$ and algebras $\mathrm{Lip}(X,\varrho)$ of Lipschitz functions on a compact metric space (X,ϱ). In these cases, the theorem on SSI for D-algebras simplifies and unifies the proofs of known results [53], [44], [31], [51], [17]. Furthermore, Theorem 4.1 is crucial for obtaining in Section 5 new results on SSI for algebras Λ_φ of functions on $[-1,1]^n$ satisfying the Zygmund condition $|f(x+h)-2f(x)+f(x-h)|\leq C\varphi(|h|)$.

The Zygmund space Λ_φ is a generalization of the classical space Λ which was introduced by A. Zygmund [54] and corresponds to the case $n=1$, $\varphi(t)=t$. The space Λ and its multivariate analogues appear in a very natural way in many problems of analysis as the limiting space in the Lipschitz scale $\mathrm{Lip}\,\alpha$, $0<\alpha<1$, rather than $\mathrm{Lip}\,1$ (cf. [54]). Despite the inclusions $\mathrm{Lip}\,1\subset\Lambda\subset\mathrm{Lip}\,\psi$ with $\psi(t)=t\log(2/t)$, analytic properties of Lipschitz and Zygmund functions are significantly different, as exemplified by the almost everywhere differentiability of functions in $\mathrm{Lip}\,1$ versus the existence in Λ of nowhere differentiable functions (see e.g. [48], Chapter 5, Section 4), and also by the fact that every real Lipschitz space $\mathrm{Lip}\,\varphi$ is a lattice as opposed to the Zygmund space Λ_φ which fails to be a lattice unless it coincides with $\mathrm{Lip}\,\varphi$. As a result, Zygmund spaces are much more difficult to handle.

The main contribution of this work consists of theorems on SSI for the Sobolev algebras $W_p^l(\mathbb{R}^n)$ (Theorem 2.1), for the algebras $C^m\,\mathrm{Lip}\,\varphi$ in one variable (Theorem 3.1), and for the Zygmund algebras Λ_φ and λ_φ in one and two variables with *any* majorant φ such that

$$\int_0^1 \frac{\varphi(t)}{t^2}\,dt=\infty, \tag{0}$$

or equivalently for which Λ_φ is not imbedded in C^1 (Theorem 5.2).

Our proof of Theorem 5.2 depends on two major assertions: on an abstract spectral synthesis theorem (Theorem 4.1) for D-algebras [20], [22] and on a special extension theorem (Theorem 5.1) for algebras Λ_φ with $n=1,2$ and φ subject to condition (0). The latter theorem claims that these algebras have an extension property (Ext) (see Section 5.4) which is equivalent to their being D-algebras, as stated in Proposition 5.5.1. We show that whenever (Ext) holds for an algebra Λ_φ with a majorant φ satisfying (0), the spectral approximation problem for Λ_φ has a "natural" solution (Theorem 5.3), and that every closed ideal in the corresponding "small" Zygmund algebra λ_φ is completely determined by its cospectrum (Proposition 5.5.2).

The proof of Theorem 5.1 is based on explicit descriptions of traces of Zygmund functions on an arbitrary closed subset of $\mathbb{R}^n$, $n=1,2$. In the "seminormed" setting, such descriptions were obtained in [45]. To establish the extension property (Ext), we need "normed" versions of these results with a control of equivalence constants; see Propositions 5.2.9 and 5.2.11. In the case $n = 1$, $\varphi(t) = t$, Theorem 5.1 was proved via a different method in [22].

For $n > 2$, an intrinsic description of traces of the Zygmund spaces Λ_φ is unknown. Moreover, should such a description exist it would be insurmountably difficult [46].

Nevertheless, for the algebras Λ_φ with majorant φ subject to (0) and *satisfying certain extra regularity conditions*, the extension property (Ext) holds also for $n > 2$ (cf. [11]). The proof of this result is based on the techniques of quasiharmonic extension of smooth functions ([9], [10]) that cannot be extended to arbitrary majorants. Accordingly, under these restrictions, this provides affirmative solution to the SSI problem for "big" and "small" Zygmund algebras, and leads to a standard description of the sets J_F in "big" Zygmund algebras [11].

In Section 1, we give main definitions related to SSI and present basic examples of algebras with the property of SSI.

A theorem providing complete solution to the problem of SSI for Sobolev algebras is a matter of our concern in Section 2. This result was announced in [18].

In Section 3, we establish the property of SSI for algebras $C^m \operatorname{Lip}\varphi$ of C^m-functions in one variable with the derivative of order m satisfying the Lipschitz condition with respect to a given *arbitrary* majorant φ. A natural conjecture is that this holds true for any number of variables. However, the multivariate case is genuinely harder, and the proof of the SSI in this case is not available to date.

The concept of D-algebras is discussed in detail in Section 4. We show that the algebras C^1 and $\operatorname{Lip}(X, \varrho)$ are D-algebras, while the algebras $C^1 \operatorname{Lip}\varphi$ are not in this class.

In Section 5, we focus our attention on Zygmund algebras. Some basic properties of Zygmund spaces are presented in Section 5.1. Their discussion is continued in Section 5.2 with an emphasis on extension theorems, characterization of traces, and approximations. The structure of closed primary ideals in algebras Λ_φ is described in Section 5.3, whereas Section 5.4 is devoted to description, properties, and applications of point derivations. In a simpler particular case $\varphi(t) = t$, closed primary ideals and point derivations in the Zygmund algebra Λ were characterized in [19]. Section 5.5 deals with the extension property (Ext) for Zygmund spaces. Here, we formulate Theorem 5.1 and derive Theorems 5.2 and 5.3 as its corollaries. The proof of Theorem 5.1 is given in Section 5.6.

In the Appendix, we provide the reader with complete self-contained proofs of some crucial facts about Zygmund algebras which are too technical to be included in our main discussion in Section 5.

All algebras studied below are supposed to be real. However, all results can be carried over almost word-for-word to the complex case.

1. Main definitions and basic examples

Let X be a locally compact Hausdorff space, $C(X)$ be the space of all continuous functions on X, and let $\mathcal{A}$ be a linear subspace in $C(X)$ which is supposed to be a topological algebra with respect to pointwise multiplication of functions. We assume that the algebra $\mathcal{A}$ is Shilov regular, that is, for every closed subset F of X and for each point $x \in X\setminus F$, there exists a function $f \in \mathcal{A}$ such that f vanishes on F and $f(x) = 1$. Also, it is supposed that the space of maximal ideals of $\mathcal{A}$ coincides with X, i.e., every maximal ideal in $\mathcal{A}$ is of the form $M_x = \{f \in \mathcal{A} : f(x) = 0\}$ for some point $x \in X$. If X is compact then $\mathcal{A}$ is assumed to contain the unity function.

An *ideal* I in $\mathcal{A}$ is a linear subspace of $\mathcal{A}$ such that $fg \in I$ whenever $f \in I$ and $g \in \mathcal{A}$. For every ideal I in $\mathcal{A}$, we define a closed subset in X, $\sigma(I) := \bigcap\{f^{-1}(0) : f \in I\}$, called the *cospectrum* of I. An ideal I is *primary* at a point $x \in X$ if $\sigma(I) = \{x\}$. In other words, a primary ideal is one contained in exactly one maximal ideal.

For each closed subset F in X, we define the set M_F of all functions in $\mathcal{A}$ vanishing on F, and the closure J_F of the set of functions in $\mathcal{A}$ vanishing in a neighborhood (depending on a function) of the set F. It follows from the above assumptions on $\mathcal{A}$ (see e.g. [15], Section 36) that M_F and J_F are the maximal and the minimal closed ideals in $\mathcal{A}$ with cospectrum F, respectively. In particular, M_x is the maximal (closed) primary ideal at x and J_x is the minimal closed primary ideal at x. Thus, for every closed ideal I in $\mathcal{A}$ with cospectrum F, one has $J_F \subset I \subset M_F$.

The *primary component* I_x of an ideal I at a point $x \in \sigma(I)$ is defined to be the smallest closed primary ideal at x containing I. It is easily seen that

$$I_x = \mathrm{clos}_{\mathcal{A}}(I + J_x). \tag{1.1}$$

We say that the algebra $\mathcal{A}$ admits SSI (notation: $\mathcal{A} \in$ Synt) if for every closed proper ideal I in $\mathcal{A}$,

$$I = \bigcap\{I_x : x \in \sigma(I)\}. \tag{1.2}$$

Equivalently, $\mathcal{A} \in$ Synt if every closed proper ideal in $\mathcal{A}$ is an intersection of closed primary ideals.

In the simplest case when $J_x = M_x$ for all $x \in X$, (1.2) acquires the form

$$I = M_F, \qquad \text{where } F = \sigma(I). \tag{1.3}$$

This means that every closed ideal in $\mathcal{A}$ is completely determined by its cospectrum, and in this case we write $\mathcal{A} \in$ synt.

In fact, the ideal M_F has property (1.2) for every nonempty closed subset F of X.

The following algebras of smooth functions were known to admit SSI.

1. The algebra $C(X)$ of all continuous functions f on a compact Hausdorff space X supplied with the norm $\|f\|_X := \sup\{|f(x)| : x \in X\}$. The fact that $C(X) \in$ synt was first established by M. Stone [49]. A simpler proof along the lines of the theory of Banach algebras was suggested by G. E. Shilov [43], while for a proof based on a duality argument the reader is referred e.g. to [13], Section 4.10.6. In like manner, (1.3) holds for the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X vanishing at "infinity" (see [25], Appendix C, Theorem 30).

2. The "small" Lipschitz algebra $\mathrm{lip}(X, \varrho)$ consisting of functions f on a compact metric space (X, ϱ) with the finite norm

$$\|f\|_{X,\varrho} := \max\{\|f\|_X, |f|_{X,\varrho}\},$$

where

$$|f|_{X,\varrho} := \sup\left\{\frac{|f(x)-f(y)|}{\varrho(x,y)} : x,y\in X,\ x\neq y\right\},$$

and satisfying the condition

$$\lim_{\varrho(x,y)\to 0}\frac{f(x)-f(y)}{\varrho(x,y)} = 0. \tag{1.4}$$

As shown in [42], $\operatorname{lip}(X,\varrho)\in$ synt.

3. The "big" Lipschitz algebra $\operatorname{Lip}(X,\varrho)$ of *all* functions on X with the finite norm $\|\cdot\|_{X,\varrho}$. Note that $\operatorname{lip}(X,\varrho)$ is a closed separable subalgebra in $\operatorname{Lip}(X,\varrho)$. The spectral synthesis theorem for the algebras $\operatorname{Lip}(X,\varrho)$ is due to L. Waelbroeck [51]. In the particular case of a compact subset of $\mathbb{R}^n$ with the metric $\varrho(x,y)=|x-y|^\alpha$, $0<\alpha\le 1$, this fact was established by G. Glaeser [17]. The proofs of these results are appreciably harder as compared to the case of "small" Lipschitz algebras, mainly due to the fact that in "big" Lipschitz algebras for every cluster point $x\in X$ the quotient space M_x/J_x is infinite-dimensional (and even nonseparable), while for "small" Lipschitz algebras this space is trivial.

4. The algebra $C^m(\Omega)$ of m times continuously differentiable functions on an open subset Ω of $\mathbb{R}^n$ with the topology of uniform convergence of functions and their derivatives up to order m on compact subsets of Ω. The classical theorem of H. Whitney [53] states that $C^m(\Omega)\in$ Synt. In fact, the same is also true for Banach algebras $C^m(Q)$ of C^m-functions on a closed cube Q in $\mathbb{R}^n$ (for $m=1$ and arbitrary n this was established in [44], and for $n=1$ with arbitrary m—in [43]) as well as for Banach algebras $C_0^m(\mathbb{R}^n)$ of C^m-functions on $\mathbb{R}^n$ vanishing at infinity together with their derivatives of order $\le m$. A simpler proof of Whitney's theorem and its extension to the case $m=\infty$ can be found in [31]. A similar argument yields the theorem on SSI for the algebras $C^m\operatorname{lip}\varphi(\Omega)$ of functions in $C^m(\Omega)$ with the derivatives of order m in $\operatorname{lip}\varphi(K)$ for every compact subset K of Ω (cf. [18]). Here φ is a nondecreasing function on $\mathbb{R}_+$ such that $\varphi(0)=\varphi(0+)=0$, $\varphi(t)>0$ for $t>0$, and $\lim_{t\to 0+}\varphi(t)/t=\infty$.

Let A_n^m be the algebra of polynomials of degree at most m with multiplication defined as the truncation of the usual product to degree m. For $x\in\Omega$, the mapping $\pi_x : f\to T_x^m f$, where $T_x^m f$ is the Taylor polynomial of order m for the function f at the point x, identifies the quotient algebra $C^m(\Omega)/J_x$ with A_n^m. Hence, for *every* ideal I in $C^m(\Omega)$, $\pi_x^{-1}(\pi_x(I)) = I + J_x$ is the smallest *closed* primary ideal in $C^m(\Omega)$ at the point x containing I, that is,

$$I_x = I + J_x \tag{1.5}$$

(compare with (1.1)). Also, the minimal closed primary ideal of the algebra $\mathcal{A}=C^m(\Omega)$ at a point $x\in\Omega$ has the following representation [53], [31]:

$$J_x = \operatorname{clos}_{\mathcal{A}} M_x^{m+1} = \{f\in\mathcal{A} : D^\alpha f(x)=0,\ |\alpha|\le m\}. \tag{1.6}$$

Relation (1.5) leads to an equivalent formulation of the property of SSI for the algebras $\mathcal{A}=C^m(\Omega)$ (see [53], [31]):

Let I be a closed ideal in $\mathcal{A}$ and $f \in \mathcal{A}$. Suppose that for every point $x \in \Omega$ there is a function $g_x \in I$ such that $T_x^m g_x = T_x^m f$. Then $f \in I$.

5. The non-quasi-analytic Denjoy–Carleman classes $C(M_n)$ of functions in one variable with regular weights $\{M_n\}_{n=0}^{\infty}$; see [7] for details.

Examples 1–5 reveal a remarkable universality of the property of SSI in a wide range of smoothness and topological assumptions on function algebras.

2. Closed ideals in Sobolev algebras

2.0. Notation. We denote by $|x|$ the Euclidean norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and put $B_r := \{x \in \mathbb{R}^n : |x| \le r\}$. A *cube* Q in $\mathbb{R}^n$ is a set of the form $Q(c, d) := \{x \in \mathbb{R}^n : |x_i - c_i| \le d,\ 1 \le i \le n\}$, and for a cube $Q = Q(c, d)$ we write $c_Q := c$ and $d_Q := d$. The space $\mathbb{R}^{n-1}$ is identified with $\{x \in \mathbb{R}^n : x_1 = 0\}$. For a multiindex $\alpha \in \mathbb{Z}_+^n$ we set $\alpha! := \alpha_1! \ldots \alpha_n!$ and $x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The symbols supp, ∇, and D^α stand, respectively, for the support, the gradient, and the derivative of order $|\alpha| := \alpha_1 + \ldots + \alpha_n$ (*) (the derivatives of first order will also be denoted by D_i, $i = 1, \ldots, n$). Denote by

$$T_x^m f(x) := \sum_{|\alpha| \le m} \frac{D^\alpha f(x)}{\alpha!} (y - x)^\alpha$$

the Taylor polynomial for a function f of order m at a point x. The notation $\overline{U}$ will be used for the closure of a subset U in $\mathbb{R}^n$. Let finally mes := mes_n be the Lebesgue measure in $\mathbb{R}^n$.

For a measurable subset S in $\mathbb{R}^n$ and for $1 \le p \le \infty$ we denote by $L_p(S)$ the space of all measurable functions f on S with the finite norm

$$\|f\|_{p,S} := \Big(\int_S |f(x)|^p \, dx\Big)^{1/p}$$

(if $p = \infty$ we set $\|f\|_{\infty,S} := \operatorname{ess\,sup}_{x \in S} |f(x)|$). In the case $S = \mathbb{R}^n$ we shall simply write L_p and $\|\cdot\|_p$.

We will use the following notation for spaces of functions:

- $C(S)$—the space of all continuous functions on a subset S in $\mathbb{R}^n$ (if $S = \mathbb{R}^n$ we write $C(\mathbb{R}^n) =: C$);
- C_0^m—the space of all m times continuously differentiable functions on $\mathbb{R}^n$ vanishing at infinity together with their derivatives up to order m;
- $\mathcal{D}$—the space of C^∞-functions on $\mathbb{R}^n$ with compact support;
- W_p^l, $l \in \mathbb{N}$, $1 \le p < \infty$—the Sobolev space of functions f on $\mathbb{R}^n$ with the generalized derivatives $D^\alpha f \in L_p$, $|\alpha| \le l$; the space W_p^l is supplied with the norm

$$\|f\| := \|f\|_{p,l} := \sum_{|\alpha| \le l} \|D^\alpha f\|_p.$$

(*) To distinguish this norm from the Euclidean one with the same notation, we use for multiindices Greek letters only.

Throughout Section 2, the letter A denotes positive constants (which may be different even in the same chain of estimates) that may depend only on p, l and n.

2.1. Preliminary observations and results. The following continuous imbeddings are classical [47], [14]:

(2.1.1) $W_p^l \subset C$ if $p > n/l$ or $p = n/l = 1$;

(2.1.2) $W_p^l \subset L_r,\ p \le r < \infty$ if $p = n/l > 1$;

(2.1.3) $W_p^l \subset L_s,\ s^{-1} = p^{-1} - l/n$ if $1 \le p < n/l$.

It follows from (2.1.1) that in the cases $p > n/l$ and $p = n/l = 1$,

$$\|f\|_\infty \le A\|f\|, \qquad f \in W_p^l. \tag{2.2}$$

This estimate and the density of the space $\mathcal{D}$ in W_p^l imply that in the case (2.1.1), W_p^l is continuously imbedded in C_0.

Also, we note that $W_p^l(\mathbb{R}^n) \subset L_p(\mathbb{R}^{n-1})$ [14] in the sense that

$$\|f\|_{p,\mathbb{R}^{n-1}} \le A\|f\|, \qquad f \in \mathcal{D}. \tag{2.3}$$

It is known that in (and only in) the case (2.1.1) Sobolev spaces are algebras with respect to pointwise multiplication. Usually, this fact is derived from the theory of multipliers for Sobolev spaces [50], [36]. A direct proof based solely on the imbeddings (2.1.1)–(2.1.3) is presented below (for $p > n/l$, a proof of the sufficiency part can be found in [1], Theorem 5.23).

PROPOSITION 2.1.1. *The Sobolev space* W_p^l *is an algebra with respect to pointwise multiplication of functions iff* $p > n/l$ *or* $p = n/l = 1$.

PROOF. *Necessity.* Suppose that $p<n/l$ or $p = n/l > 1$. We show that the space W_p^l is not an algebra.

If $p < n/l$, take λ such that $(n/p - l)/2 \le \lambda < n/p - l$. Let g be a C^∞-function that vanishes outside $B_{1/2}$ and equals 1 in a neighborhood of the origin. Set $f(x) := |x|^{-\lambda}$. It is easy to see that $D^\alpha(|x|^{-\lambda}) = P_\alpha(x)|x|^{-(\lambda+2|\alpha|)}$, where P_α is a homogeneous polynomial of order $|\alpha|$. Hence, in view of $(\lambda + l)p < n$, we have $f \in W_p^l$, while $(2\lambda + l)p \ge n$ implies $f^2 \notin W_p^l$. Therefore, W_p^l fails to be an algebra.

If $p = n/l > 1$, choose λ such that $1/(2q) \le \lambda < 1/q$, and set $f(x) := \ln^\lambda(1/|x|)g(x)$. Note that for $|\alpha| \ge 1$,

$$D^\alpha\left(\ln^\lambda \frac{1}{|x|}\right) = |x|^{-2|\alpha|} \sum_{k=1}^{|\alpha|} P_{\alpha,k}(x) \ln^{\lambda-k} \frac{1}{|x|},$$

where all nonzero $P_{\alpha,k}$ are homogeneous polynomials of degree $|\alpha|$, and $P_{\alpha,1} \ne 0$. Hence, in view of $(\lambda - 1)p < -1$, we have $f \in W_p^l$, while $(2\lambda - 1)p \ge 1$ implies $f^2 \notin W_p^l$. Thus, in the case under study, W_p^l is not an algebra either.

Sufficiency. Suppose $p > n/l$ or $p = n/l = 1$. To check that W_p^l is an algebra, it suffices to establish the following inequality:

$$\|fg\| \le A\|f\| \cdot \|g\|, \qquad f, g \in \mathcal{D}.$$

To this end, we will show that for $|\alpha| + |\beta| \le l,\ |\alpha| \le |\beta|$,

$$\|D^\alpha f \cdot D^\beta g\|_p \le A\|f\| \cdot \|g\|, \quad f, g \in \mathcal{D}. \tag{2.4}$$

If $p > n/(l - |\alpha|)$ or $p = n/(l - |\alpha|) = 1$ then by (2.2),

$$\|D^\alpha f \cdot D^\beta g\|_p \le \|D^\alpha f\|_\infty \|D^\beta g\|_p \le A\|D^\alpha f\|_{p,l-|\alpha|}\|g\| \le A\|f\| \cdot \|g\|.$$

If $1 < p = n/(l - |\alpha|) = n/(l - |\beta|)$ then using (2.1.2) with $r = 2p$ we have, by the Cauchy–Schwarz inequality,

$$\|D^\alpha f \cdot D^\beta g\|_p \le \|D^\alpha f\|_{2p}\|D^\beta g\|_{2p} \le A\|D^\alpha f\|_{p,l-|\alpha|}\|g\|_{p,l-|\beta|} \le A\|f\| \cdot \|g\|.$$

Now consider the case $1 < p = n/(l - |\alpha|) < n/(l - |\beta|)$. We set $1/s = 1/p - (l - |\beta|)/n$ and $r = n/(l - |\beta|)$ to obtain $p/r + p/s = 1$ with $p < r < \infty$. Applying (2.1.2) and (2.1.3) we have, using the Hölder inequality,

$$\|D^\alpha f \cdot D^\beta g\|_p \le \|D^\alpha f\|_r \|D^\beta g\|_s \le A\|D^\alpha f\|_{p,l-|\alpha|}\|g\|_{p,l-|\beta|} \le A\|f\| \cdot \|g\|.$$

In the remaining case $p < n/(l - |\alpha|) \le n/(l - |\beta|)$ we put $1/r_1 = 1/p - (l - |\alpha|)/n$ and $1/r_2 = 1/p - (l - |\beta|)/n$ and observe that $p/r_1 + p/r_2 \le 2 - pl/n \le 1$. Hence, there are numbers t_1 and t_2 such that $p < t_1 \le r_1$, $p < t_2 \le r_2$ and $p/t_1 + p/t_2 = 1$. Note that by (2.1.3),

$$\|D^\alpha f\|_{t_1} \le \|D^\alpha f\|_p + \|D^\alpha f\|_{r_1} \le A\|f\|,$$

and similarly $\|D^\beta g\|_{t_2} \le A\|g\|$. Therefore, again by the Hölder inequality,

$$\|D^\alpha f \cdot D^\beta g\|_p \le \|D^\alpha f\|_{t_1}\|D^\beta g\|_{t_2} \le A\|f\| \cdot \|g\|,$$

which completes the proof of the estimate (2.4) and thereby of Proposition 2.1.1.

Remark. Let $p > n/l$ or $p = n/l = 1$. It follows from Proposition 2.1.1 and from the imbedding $W_p^l \subset C_0$ that the Sobolev space W_p^l is a Banach algebra whose space of maximal ideals coincides with $\mathbb{R}^n$.

The next assertion will be used in what follows.

Proposition 2.1.2. *Let S be a measurable subset of $\mathbb{R}^n$ and $f \in W_p^l \cap C$. Suppose that $f|S \equiv 0$, and define $S_0 := \{x \in S : \nabla f(x) = 0\}$. Then*

$$\operatorname{mes}(S \setminus S_0) = 0.$$

Proof. Set $S_i := \{x \in S : D_i f(x) = 0\},\ i = 1, \dots, n$. It suffices to show that $\operatorname{mes}(S \setminus S_i) = 0$ for all i. Let, for example, $i = 1$. Write $x \in \mathbb{R}^n$ in the form $x = (t, y)$, where $t \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Denote by S_a the one-dimensional section of the set S by the line $y = a,\ a \in \mathbb{R}^{n-1}$. Let S' be the (measurable) subset in $\mathbb{R}^{n-1}$ of all points $y \in \mathbb{R}^{n-1}$ for which S_y is a nonempty measurable subset in $\mathbb{R}$. Observe that for almost all points $y \in \mathbb{R}^{n-1}$ the function $g_y(t) := f(t, y)$ belongs to $W_p^l(\mathbb{R}) \cap C(\mathbb{R})$ and its generalized derivative h_y coincides a.e. on $\mathbb{R}$ with $D_1 f|\mathbb{R} \times \{y\}$. Fix any such point y. It is well known that for almost all $t \in \mathbb{R}$ the classical derivative $g_y'(t)$ of the function g_y exists and equals $h_y(t)$. Since $f(t, y) = 0$ for all $t \in S_y$, we have $g_y' = 0$ a.e. on S_y (namely, at all limit points of S_y for which g_y' exists). Hence

$$\int_S |D_1 f(x)|\,dx = \int_{S'}\int_{S_y} |h_y(t)|\,dt\,dy = \int_{S'}\int_{S_y} |g'_y(t)|\,dt\,dy = 0,$$

i.e., $D_1 f(x) = 0$ for almost all $x \in S$, and Proposition 2.1.2 follows.

Proposition 2.1.3. *Suppose that $n = 1 \le p < \infty$ or $2 \le n < p < \infty$. Then for every cube $Q = Q(c,d)$ in $\mathbb{R}^n$ and for every function $f \in W_p^l$ such that $f(c) = 0$,*

$$\int_Q |f(x)|^p\,dx \le Ad^p \sum_{i=1}^n \int_Q |D_i f(x)|^p\,dx. \tag{2.5}$$

Proof. In the case $n = p = 1$ this can be checked straightforwardly. Observe that in the remaining cases we get $n/p < 1$. Assume for simplicity that $c = 0$. We have

$$f(x) = \sum_{i=1}^n x_i \int_0^1 D_i f(tx)\,dt,$$

hence, by the Minkowski inequality for integrals,

$$\|f\|_{p,Q} \le d \sum_{i=1}^n \int_0^1 \|D_i f(tx)\|_{p,Q}\,dt = d \sum_{i=1}^n \int_0^1 t^{-n/p} \|D_i f\|_{p,tQ}\,dt \le Ad \sum_{i=1}^n \|D_i f\|_{p,Q},$$

which implies (2.5).

2.2. Closed primary ideals. We assume hereafter that $p > n/l$ or $p = n/l = 1$. Let m be the maximal integer for which $W_p^l \subset C_0^m$. It follows from (2.1.1) that $m = l - n$ for $p = 1$ and m is determined by the condition $l - n/p - 1 \le m < l - n/p$ for $p > 1$.

Recall that in the algebra C_0^m the minimal closed primary ideal J_x at a point $x \in \mathbb{R}^n$ has the form (1.6). We will show that the same is true for Sobolev algebras.

Proposition 2.2.1. *Let W_p^l be a Sobolev algebra and m be defined as above. Then*

$$J_x = \{f \in W_p^l : D^\alpha f(x) = 0,\ |\alpha| \le m\}, \qquad x \in \mathbb{R}^n.$$

Proof. The set H_x on the right-hand side is a closed primary ideal at the point x, hence $J_x \subset H_x$. When checking the reverse inclusion we assume without loss of generality that $x = 0$. Also, it suffices to show that $f \in H_0$ implies $f \in J_0$ for functions $f \in \mathcal{D}$ only. For, suppose this is true, take any function $f \in H_0$, and fix $\varepsilon > 0$. There is a function $g \in \mathcal{D}$ with $\|g - f\| \le \varepsilon$. From the continuity of the imbedding $W_p^l \subset C_0^m$ we conclude that

$$\max_{|\alpha| \le m} |D^\alpha g(0)| = \max_{|\alpha| \le m} |D^\alpha g(0) - D^\alpha f(0)| \le A\|g - f\| \le A\varepsilon.$$

Let $\omega \in \mathcal{D}$ be a function supported in B_1 which equals 1 in a neighborhood of the origin. For the function $h := g - (T_0^m g)\omega$ we have $h \in H_0 \cap \mathcal{D}$ and $\|h - g\| \le A\varepsilon$, hence $\|h - f\| \le A\varepsilon$. By our hypothesis, $h \in J_0$. Therefore, $f \in J_0$ via arbitrariness of ε.

We start with the case $p \ne n/(l - m - 1)$. Then $\delta := m + n/p + 1 - l > 0$. We will show that for any function $f \in H_0 \cap \mathcal{D}$ and for all $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha| + |\beta| \le l$,

$$\varepsilon^{-|\beta|}\Big(\int_{|x|\le\varepsilon} |D^\alpha f(x)|^p\,dx\Big)^{1/p} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.6}$$

Set $M := \sup\{|D^\gamma f(x)| : x \in \mathbb{R}^n,\ |\gamma| \le l\}$. If $|\alpha| > m$ then

$$\varepsilon^{-|\beta|}\Big(\int\limits_{|x|\le\varepsilon} |D^\alpha f(x)|^p\,dx\Big)^{1/p} \le AM\varepsilon^{-|\beta|+n/p} = AM\varepsilon^{-|\beta|+\delta+l-m-1} \le AM\varepsilon^{\delta+l-|\alpha|-|\beta|} \le AM\varepsilon^{\delta}.$$

Let now $|\alpha| \le m$. In this case we note that $D^{\alpha+\gamma}f(0) = 0$ for $|\gamma| \le m - |\alpha|$ and use the identity

$$D^\alpha f(x) = (m - |\alpha| + 1) \sum_{|\gamma|=m-|\alpha|+1} \frac{x^\gamma}{\gamma!} \int\limits_0^1 D^{\alpha+\gamma} f(tx)(1-t)^{m-|\alpha|}\,dt$$

to obtain

$$\varepsilon^{-|\beta|}\Big(\int\limits_{|x|\le\varepsilon} |D^\alpha f(x)|^p\,dx\Big)^{1/p} \le AM\varepsilon^{-|\beta|+m-|\alpha|+1+n/p} = AM\varepsilon^{\delta+l-|\alpha|-|\beta|} \le AM\varepsilon^{\delta}.$$

Thus in both cases (2.6) holds.

For $0 < \varepsilon \le 1$, set $\omega_\varepsilon(x) := \omega(x/\varepsilon)$, where ω is the function defined above. It follows from (2.6) that for $|\alpha| + |\beta| \le l$ one has $\|D^\alpha f \cdot D^\beta \omega_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$, hence $\|f\omega_\varepsilon\| \to 0$ as $\varepsilon \to 0$, and in combination with $f - f\omega_\varepsilon \in J_0$ this yields $f \in J_0$.

Now we turn to the more delicate case $p = n/(l - m - 1)$. Observe that in this case $1 < p < \infty$ and $\delta = 0$, so that the above estimates lead only to the weaker conclusion

$$\varepsilon^{-|\beta|}\Big(\int\limits_{|x|\le\varepsilon} |D^\alpha f(x)|^p dx\Big)^{1/p} \le AM.$$

Therefore, $\|f\omega_\varepsilon\| \le AM$ for $0 < \varepsilon \le 1$.

Denote by T the set of pairs of multiindices (α, β) such that $|\alpha| + |\beta| \le l$, and let r be the number of elements in T. Consider the Banach space $Y := (L_p)^r$ supplied with the norm

$$\|F\|_Y := \max\{\|F_{\alpha,\beta}\|_p : (\alpha, \beta) \in T\}, \qquad F = (F_{\alpha,\beta})_{(\alpha,\beta)\in T} \in Y.$$

Take a sequence $\varepsilon_k \to 0$ as $k \to \infty$, and define $F_k \in Y$ by setting

$$F_k := (D^\alpha f \cdot D^\beta \omega_{\varepsilon_k})_{(\alpha,\beta)\in T}, \qquad k \in \mathbb{N}.$$

As was shown earlier, $\|F_k\|_Y \le AM$ for all k. It follows from the reflexivity of Y that there exist a subsequence $\{k_i\}_{i\in\mathbb{N}}$ and $F \in Y$ such that $F_{k_i} \to F$ in the weak topology on Y. For every $k \in \mathbb{N}$, $\operatorname{supp} F_k \subset B_{\varepsilon_k}$, hence $F = 0$. Therefore, there is a sequence of convex linear combinations

$$G_j = \sum_{i=1}^{n_j} \lambda_{ij} F_{k_i}, \qquad \lambda_{ij} \ge 0,\ \sum_{i=1}^{n_j} \lambda_{ij} = 1,$$

for which $G_j \to 0$ in the norm of Y. Now put $\omega_j := \sum_{i=1}^{n_j} \lambda_{ij}\omega_{\varepsilon_{k_i}}$. For $(\alpha, \beta) \in T$, we have $(G_j)_{\alpha,\beta} = D^\alpha f \cdot D^\beta \omega_j$, hence $\|D^\alpha f \cdot D^\beta \omega_j\|_p \to 0$ as $j \to \infty$.

As a result, we see that in the case $p = n/(l - m - 1)$ there is a sequence $\{\omega_j\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ (depending on f) such that for every j, $\omega_j \equiv 1$ in a neighborhood of zero and $\|f\omega_j\| \to 0$ as $j \to \infty$. Thus, $f \in J_0$, and the proof of Proposition 2.2.1 is now complete.

Remark 1. In the algebra $\mathcal{A} = W_p^l$ we have

$$J_x = \operatorname{clos}_{\mathcal{A}} M_x^{m+1}$$

for every $x \in \mathbb{R}^n$ (compare with (1.6)).

Remark 2. The primary component I_x of an ideal I in W_p^l at a point $x \in \sigma(I)$ has the representation (1.5).

2.3. Spectral synthesis of ideals. The main result of Section 2 is contained in the following theorem. The proof of its sufficiency part is based on the same idea as the proof due to B. Malgrange [31] of Whitney's theorem on SSI for the algebras $C^m(\mathbb{R}^n)$ (see [53]).

Theorem 2.1. *$W_p^l \in \mathrm{Synt}$ iff $m = l-1$, i.e. iff $n = 1$ or $2 \le n < p$.*

Proof. *Necessity.* Suppose that $0 \le m \le l-2$; hence $n \ge 2$, $1 \le p \le n$, and $l \ge 2$. We will show that $W_p^l \notin \mathrm{Synt}$.

For every compact set K in $\mathbb{R}^n$, we define its p-capacity by

$$\operatorname{cap}_p(K) := \inf\{\|\varphi\|_{p,1}^p : \varphi \in \mathcal{D},\ \varphi \ge 1 \text{ on } K\}.$$

Let E be a compact subset in $\mathbb{R}^{n-1}$ such that $\operatorname{mes}_{n-1}(E) > 0$. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi \ge 1$ on E then, in view of (2.3),

$$\|\varphi\|_{p,1}^p \ge A \int\limits_{\mathbb{R}^{n-1}} |\varphi(x)|^p\,dx \ge A \int\limits_E \varphi(x)^p\,dx \ge A\operatorname{mes}_{n-1}(E).$$

Therefore, $\operatorname{cap}_p(E) \ge A\operatorname{mes}_{n-1}(E) > 0$.

Take $I = J_E$; then I is a closed ideal in W_p^l, $\sigma(I) = E$, and $I_x = J_x$ for all $x \in E$. Let $f \in \mathcal{D}(\mathbb{R}^n)$ be a function such that $f(x) = x_1^{l-1}/(l-1)!$ in a neighborhood (with respect to $\mathbb{R}^n$) of the set E. Observe that $D^\alpha f | E \equiv 0$ for $|\alpha| \le l-2$; hence by Proposition 2.2.1, $f \in \bigcap\{I_x : x \in \sigma(I)\}$.

We claim that $f \notin I$. In fact, should $f \in I$, then there would exist a sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ of functions vanishing in certain neighborhoods of E (relative to $\mathbb{R}^n$) so that $g_k \to f$ in W_p^l as $k \to \infty$. For functions $\varphi_k := (\partial^{l-1}/\partial x_1^{l-1})(f - g_k)$, we have $\varphi_k \to 0$ in W_p^1 as $k \to \infty$. From $\varphi_k \in \mathcal{D}$ and $\varphi_k | E \equiv 1$ we conclude that $\operatorname{cap}_p(E) \le \|\varphi_k\|_{p,1}^p$, hence $\operatorname{cap}_p(E) = 0$. The contradiction obtained shows that $f \notin I$. Thus, property (1.1) is missed for the ideal I, that is, $W_p^l \notin \mathrm{Synt}$.

Sufficiency. Suppose that $W_p^l \subset C_0^{l-1}$. Let I be a proper closed ideal in W_p^l. For $x \in \mathbb{R}^n \setminus \sigma(I)$, set $I_x := W_p^l$. We need to show that $I = \bigcap\{I_x : x \in \mathbb{R}^n\}$. In fact, we only have to prove that the ideal $\bigcap I_x$ on the right-hand side is contained in I.

Define $F := \sigma(I)$; $L := \dim A_n^{l-1}$; $F_j := \{x \in \mathbb{R}^n : \operatorname{codim} I_x \le j\}$, $j = 0, 1, \ldots, L$; $E_0 := F_0$, $E_j := F_j \setminus F_{j-1}$, $j = 1, \ldots, L$. Clearly, all sets F_j are closed in $\mathbb{R}^n$.

We will prove the following statement.

Lemma 2.3.1. *Let $f \in \bigcap I_x$, and let E and K be compact subsets of $\mathbb{R}^n$ such that $\operatorname{supp} f \subset E$ and $K \subset E_j \cap E$ for some j. Then for every $\varepsilon > 0$ there exist functions $\Phi \in \mathcal{D}$ and $g \in I$ so that $\Phi \equiv 1$ in a neighborhood of K and $\|\Phi f - g\| < \varepsilon$.*

Proof. If $j = L$ then $K \cap F = \emptyset$. In this case take Φ to be a function in $\mathcal{D}$ that vanishes in a neighborhood of F and equals 1 in a neighborhood of K, and set $g := \Phi f$. Since $\Phi \in J_F \subset I$, we have $g \in I$. Thus, for $j = L$, the claim of Lemma 2.3.1 is satisfied.

Let now $0 \le j < L$, in which case $K \subset F$. Observe that $\operatorname{codim} I_x = j$ for $x \in K$. Assuming that $j > 0$ we find for any point $x \in K$ its open neighborhood U_x and functions $v_1, \dots, v_j \in I$ such that $\{T_y^{l-1} v_k\}_{k=1}^j$ is a basis of W_p^l / J_y for all $y \in \overline{U}_x \cap K$. Thus,

$$D^\alpha f(y) = \sum_{k=1}^{j} c_k(y) D^\alpha v_k(y), \qquad y \in \overline{U}_x \cap K,\ |\alpha| \le l-1, \tag{2.7}$$

where $c_1, \dots, c_j \in C(\overline{U}_x \cap K)$. In fact, the functions $c_1, \dots, c_j$ may be thought of as extended to continuous functions on K. Selecting from the covering $\{U_x\}_{x \in K}$ of the compact K a finite subcovering and "glueing" the corresponding expansions (2.7) with the help of an appropriate partition of unity, we find functions $f_1, \dots, f_r \in I$ and $\lambda_1, \dots, \lambda_r \in C(K)$ such that

$$D^\alpha f(x) = \sum_{k=1}^{r} \lambda_k(x) D^\alpha f_k(x), \qquad x \in K,\ |\alpha| \le l-1. \tag{2.8}$$

Note that in the case $j = 0$, we have $I_x = J_x$ for all $x \in K$, hence (2.8) is also valid with $r = 1$ and $\lambda_1 \equiv 0,\ f_1 \equiv 0$.

For $a \in K$, set $f_a := \sum_{k=1}^r \lambda_k(a) f_k$ and $h_a := f - f_a$. Obviously, $f_a \in I$ and $T_a^{l-1} h_a = 0$.

Partition $\mathbb{R}^n$ into equal cubes of edgelength d with disjoint interiors. Replacing every cube by the open concentric cube of edgelength $2d$ we get an open covering $\{U_i\}$ of $\mathbb{R}^n$. Let $\{\phi_i\}$ be a partition of unity such that $\phi_i \in \mathcal{D}$, $\operatorname{supp} \phi_i \subset U_i$, and $\|D^\alpha \phi_i\|_\infty \le A d^{-|\alpha|}$ for all i and $|\alpha| \le l$. For every i, define $S_i := \{k : U_k \cap U_i \ne \emptyset\},\ V_i := \bigcup\{U_k : k \in S_i\}$, and observe that V_i is the cube of edgelength $4d$ concentric with U_i. Set also $S := \{i : U_i \cap K \ne \emptyset\},\ U := \bigcup\{U_i : i \in S\},\ V := \bigcup\{V_i : i \in S\}$, and $\Phi := \sum_{i \in S} \phi_i$. For every i, choose a point $a_i \in U_i \cap K$, and define $g := \sum_{i \in S} f_{a_i} \phi_i$. We readily see that $\Phi \in \mathcal{D},\ \Phi \equiv 1$ in a neighborhood of K, and $g \in I$.

We are going to show now that for every $\varepsilon > 0$ there is $d > 0$ such that

$$\|\Phi f - g\| = \Big\| \sum_{i \in S} h_{a_i} \phi_i \Big\| < \varepsilon.$$

Using the Leibniz formula we have, for $|\alpha| \le l$,

$$\begin{aligned} \int_{\mathbb{R}^n} |D^\alpha(\Phi f - g)|^p\, dx &= \int_U \Big| D^\alpha \Big(\sum_{i \in S} h_{a_i} \phi_i \Big) \Big|^p dx \\ &\le A \sum_{i \in S} \int_{U_i} \sum_{k \in S_i} \sum_{\beta \le \alpha} |D^\beta h_{a_k}|^p d^{(|\beta| - |\alpha|)p}\, dx \\ &\le A \sum_{i \in S} \sum_{k \in S_i} \sum_{\beta \le \alpha} d^{(|\beta| - |\alpha|)p} \int_{V_i} |D^\beta h_{a_k}|^p\, dx. \end{aligned}$$

Note that for every $k \in S_i$, we have $a_k \in V_i$ and $D^\gamma h_{a_k}(a_k) = 0,\ |\gamma| \le l-1$. Iterating estimate (2.5) and taking into account the finiteness of multiplicity of the coverings $\{U_i\}$

and $\{V_i\}$, we get

$$\int\limits_{\mathbb{R}^n} |D^\alpha(\Phi f - g)|^p\,dx \le Ad^{(l-|\alpha|)p} \sum_{i\in S}\sum_{k\in S_i}\sum_{|\gamma|=l}\int\limits_{V_i} |D^\gamma h_{a_k}|^p\,dx$$
$$\le Ad^{(l-|\alpha|)p}\sum_{i\in S}\int\limits_{V_i}\Psi\,dx \le Ad^{(l-|\alpha|)p}\int\limits_{V}\Psi\,dx,$$

where

$$\Psi := \sum_{|\gamma|=l}\Big(|D^\gamma f| + M\sum_{j=1}^{r}|D^\gamma f_k|\Big)^p$$

with $M := \sup\{|\lambda_k(x)| : x\in K,\ 1\le k\le r\}$. Note that $\mathrm{mes}(V\setminus K)\to 0$ as $d\to 0$, hence for sufficiently small d,

$$\|\Phi f - g\| < \varepsilon + A\Big(\int\limits_K \Psi\,dx\Big)^{1/p}.$$

Observe that the functions $f, f_1, \dots, f_r$ vanish on K. Applying Proposition 2.1.2 to successive derivatives of these functions we conclude that $\Psi(x) = 0$ for almost all $x\in K$. Therefore, $\|\Phi f - g\| < \varepsilon$, and Lemma 2.3.1 follows.

To complete the proof of the theorem, we have to derive from Lemma 2.3.1 the required inclusion $\bigcap I_x \subset I$.

Let $h \in \bigcap I_x$ and $\delta > 0$. Choose a function $\eta\in\mathcal{D}$ for which $\|h - h\eta\| < \delta/2$. Set $f := h\eta$ and $E := \operatorname{supp} f$. We show by induction that for every $j = 0, 1, \dots, L$ the following assertion holds:

(P_j) For every $\varepsilon > 0$, there exist functions $\Phi_j\in\mathcal{D}$ and $g_j\in I$ such that $\Phi_j\equiv 1$ in a neighborhood of $F_j\cap E$ and $\|\Phi_j f - g_j\| < \varepsilon$.

Applying Lemma 2.3.1 to the set $K = E_0\cap E$ we obtain assertion (P_0). Suppose that for some $j = 1, \dots, L$ assertion (P_{j-1}) is valid. This provides us with functions $\Phi_{j-1}\in\mathcal{D}$ and $g_{j-1}\in I$ such that $\Phi_{j-1}\equiv 1$ on an open set $U\supset F_{j-1}\cap E$ and $\|\Phi_{j-1}f - g_{j-1}\| < \varepsilon/2$. Now set $K := (F_j\cap E)\setminus U$. Obviously, $K\subset E_j\cap E$. Applying Lemma 2.3.1 to the compact sets E and K and to the function $(1-\Phi_{j-1})f\in\bigcap I_x$, we find a function $\widehat{\Phi}_j\in\mathcal{D}$ which equals 1 on an open set $V\supset K$ and a function $\widehat{g}_j\in I$ such that $\|\widehat{\Phi}_j(1-\Phi_{j-1})f - \widehat{g}_j\| < \varepsilon/2$. We set $\Phi_j := \Phi_{j-1} + \widehat{\Phi}_j - \Phi_{j-1}\widehat{\Phi}_j$ and $g_j := g_{j-1} + \widehat{g}_j$ to check easily that $\Phi_j\in\mathcal{D}$, $\Phi_j\equiv 1$ in an open set $U\cup V\supset F_j\cap E$ and $g_j\in I$. It follows from the identity $\Phi_j f - g_j = [\widehat{\Phi}_j(1-\Phi_{j-1})f - \widehat{g}_j] + (\Phi_{j-1}f - g_{j-1})$ that $\|\Phi_j f - g_j\| < \varepsilon$. Thus, assertion (P_j) holds.

Assertion (P_L) with $\varepsilon = \delta/2$ yields $\|\Phi_L f - g_L\| < \delta$, where $\Phi_L\in\mathcal{D}$, $\Phi_L\equiv 1$ in a neighborhood of $E = \operatorname{supp} f$, and $g_L\in I$. In view of $\Phi_L f = f$ this implies $\|h - g_L\| < \delta$. Recalling that the ideal I is closed we conclude via arbitrariness of δ that $h\in I$.

Theorem 2.1 is proved.

Remark 1. Theorem 2.1 implies that every Sobolev algebra with $l = 1$ admits SSI.

Remark 2. In the univariate case, the property of SSI in Sobolev algebras can be established by a simpler argument using local contractions. A theorem on SSI for Sobolev algebras of periodic functions on the line for $p = 2$ was stated in [39].

Remark 3. For any open set Ω in $\mathbb{R}^n$, Theorem 2.1 holds true for Banach algebras $(W_p^l)^0(\Omega)$ defined as the closure of $\mathcal{D}(\Omega)$ in $W_p^l(\Omega)$.

Remark 4. It follows from Theorem 2.1 that if $n = 1$ or $2 \le n < p$ then for every closed set F in $\mathbb{R}^n$,

$$J_F = \{f \in W_p^l(\mathbb{R}^n) : D^\alpha f(x) = 0,\ x \in F,\ |\alpha| \le l - 1\}.$$

This solves the problem of spectral approximation in the case $m = l - 1$; see also [8]. For arbitrary Sobolev spaces with $1 < p < \infty$ this problem was settled in [23], [24].

Remark 5. Suppose that $m = 0$. If $n \ge 2$ and $l \ge 2$ then by Theorem 2.1 there exist closed subsets F in $\mathbb{R}^n$ such that for the Sobolev algebra $W_p^l(\mathbb{R}^n)$ one has $J_F \ne I_F$. It follows from the proof of Theorem 2.1 that this is the case for any compact subset F in $\mathbb{R}^n$ contained in an $(n-1)$-dimensional hyperplane and having there positive Lebesgue measure.

Remark 6. For $p = \infty$, the Sobolev space $W_\infty^l(\mathbb{R}^n)$ coincides with $C^{l-1}\operatorname{Lip}1(\mathbb{R}^n)$ and is obviously a Banach algebra whose space of maximal ideals is, however, larger than $\mathbb{R}^n$. For a similar algebra defined on a closed cube Q in $\mathbb{R}^n$ this difficulty does not arise. In the next section, it will be derived from a more general result that $W_\infty^l(Q) \in \mathrm{Synt}$ for $n = 1$.

3. Spectral synthesis of ideals in the algebras $C^m \operatorname{Lip}\varphi$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\varphi(0) = \varphi(0+) = 0,\ \varphi(t) > 0$ for $t > 0$, and $\varphi(t) = 1$ for $t \ge 1$. Denote by $C^m \operatorname{Lip}\varphi$ the space of all C^m-functions f on $[0,1]$ with the finite norm $\|f\| := \max\{\|f\|_m, |f^{(m)}|_\varphi\}$, where $\|f\|_m := \max\{\|f^{(k)}\|_{[0,1]} : 0 \le k \le m\}$, and

$$|g|_\varphi := \left\{\sup \frac{|g(x) - g(y)|}{\varphi(|x-y|)} : x, y \in [0,1],\ x \ne y\right\}.$$

It can (and will) be assumed without loss of generality that the function $\varphi(t)/t$ is nonincreasing for $t > 0$. It is easily seen that $C^m \operatorname{Lip}\varphi$ is a Banach algebra which satisfies all requirements of Section 1.

In Section 3, the letter A denotes various positive constants that may depend only on m.

The following two well-known elementary statements will be used in the proof of the spectral synthesis theorem for the algebras $C^m \operatorname{Lip}\varphi$.

Proposition 3.1. *Let F be a compact subset of a metric space X, and let $\{U_1, \ldots, U_p\}$ be a finite covering of F by open sets U_i, $1 \le i \le p$. Then there exists $\delta > 0$ such that every subset E of X for which $E \cap F \ne \emptyset$ and $\operatorname{diam} E \le \delta$ is contained in at least one of the sets U_i, $1 \le i \le p$.*

Proposition 3.2. *Let F be a subset of an interval $[a,b]$ and F' be the set of its cluster points. Then for every function $f \in C^m([a,b])$ such that $f|F \equiv 0$, we have $f^{(k)}|F' \equiv 0$, $1 \le k \le m$.*

We now state the main result of Section 3.

THEOREM 3.1. $C^m \operatorname{Lip} \varphi \in \operatorname{Synt}$.

PROOF. Let I be a proper closed ideal in $\mathcal{A} := C^m \operatorname{Lip} \varphi$ with cospectrum F. We need to show that $\bigcap\{I_x : x \in F\} \subset I$. To this end, take $f \in \bigcap\{I_x : x \in F\}$ and fix $\varepsilon > 0$. In view of (1.1) for every $x \in F$ there exist functions $g_x \in I$, $h_x \in J_x$, and $r_x \in \mathcal{A}$ such that $f = g_x + h_x + r_x$, $h_x \equiv 0$ in an open neighborhood U_x of the point x, and $\|r_x\| \le \varepsilon$. From the covering $\{U_x\}_{x\in F}$ of F we extract a finite subcovering $\{U_1, \dots, U_p\}$ and take the corresponding functions g_i, h_i, and r_i, $1 \le i \le p$. By Proposition 3.1 there is $\delta \in (0, 1]$ such that every set $E \subset [0, 1]$ with $\operatorname{diam} E \le \delta$ and $E \cap F \ne \emptyset$ is contained in some U_i, $1 \le i \le p$.

For $d > 0$ consider the covering of $\mathbb{R}$ by the intervals

$$V_n := ((n-1)d, (n+1)d), \qquad n \in \mathbb{Z}.$$

Set $W_n := V_{n-1} \cup V_n \cup V_{n+1} = ((n-2)d, (n+2)d)$, $n \in \mathbb{Z}$, and choose d so that $\operatorname{diam} W_n = 4d \le \delta$. Let $\{\psi_n\}_{n\in\mathbb{Z}}$ be a partition of unity with the properties $\psi_n \in C^{m+1}(\mathbb{R})$, $\operatorname{supp} \psi_n \subset [(n-3/4)d, (n+3/4)d]$, and $\|\psi_n^{(k)}\|_\infty \le Ad^{-k}$, $0 \le k \le m+1$, for all n.

Denote by F_0 the set of all isolated points of F, and put

$$\mathcal{N} := \{n \in \mathbb{Z} : V_n \cap [0, 1] \ne \emptyset\}, \qquad N_0 := \{n \in \mathcal{N} : V_n \cap F = \emptyset\},$$
$$N_1 := \{n \in \mathcal{N} : V_n \cap F' = \emptyset,\ V_n \cap F_0 \ne \emptyset\}, \qquad N_2 := \{n \in \mathcal{N} : V_n \cap F' \ne \emptyset\}.$$

Now set $f_i := \sum_{n\in N_i} f\psi_i$, $i = 0, 1, 2$. Then $f = f_0 + f_1 + f_2$.

Obviously, f_0 vanishes in a neighborhood of the set F, therefore $f_0 \in I$.

Further, we claim that $f_1 \in I$. To prove this, we have to show that the function f_1 locally belongs to I at any point $x \in [0, 1]$ (i.e., coincides with a function from I in some neighborhood of x). This is trivial for $x \notin F$. Next, f_1 vanishes in a neighborhood of F', hence it locally belongs to I at any point of F'. Finally, for $x \in F_0$, we use the fact that $f_1 \in I_x$ to conclude (see, e.g. [43], p. 52) that f_1 locally belongs to I at x.

Next we seek to prove that $f_2 \in I$. If $n \in N_2$ then by Proposition 3.1, $V_n \subset W_n \subset U_{i_n}$ for some i_n, $1 \le i_n \le p$. Hence

$$f_2 = \sum_{n\in N_2} (g_{i_n} + h_{i_n} + r_{i_n})\psi_n = \sum_{n\in N_2} g_{i_n}\psi_n + \sum_{n\in N_2} r_{i_n}\psi_n = g + r,$$

where $g := \sum_{n\in N_2} g_{i_n}\psi_n \in I$ and $r := \sum_{n\in N_2} r_{i_n}\psi_n$. It follows from $f = g_{i_n} + h_{i_n} + r_{i_n}$ that $r_{i_n}|F \cap U_{i_n} \equiv 0$, therefore $r|F \equiv 0$.

We will show that $\|r\|$ is small. Observe that for every $n \in N_2$ there is a point $x_n \in F' \cap V_n$. Then Proposition 3.2 yields $r^{(k)}(x_n) = 0$, $0 \le k \le m$. For $x \in V_n$, we have

$$|r^{(m)}(x)| = |r^{(m)}(x) - r^{(m)}(x_n)| \le |r^{(m)}|_\varphi \varphi(|x - x_n|) \le |r^{(m)}|_\varphi \varphi(2d) \le A|r^{(m)}|_\varphi.$$

Similarly, using the formula for the Taylor remainder we obtain

$$|r^{(k)}(x)| \le Ad^{m-k}|r^{(m)}|_\varphi, \qquad 0 \le k \le m-1.$$

Also, for $x \notin V := \bigcup\{V_n : n \in N_2\}$, we have $r^{(k)}(x) = 0$, $0 \le k \le m$. Thus,

$$\|r\|_m \le A|r^{(m)}|_\varphi. \tag{3.1}$$

To estimate $|r^{(m)}|_\varphi$, set $\Delta := |r^{(m)}(x) - r^{(m)}(y)|$, and consider the following cases.

1. If $x, y \notin V$ then $\Delta = 0$.

2. Suppose that only one of the points x, y belongs to V, for example, $x \in V,\ y \notin V$. If $x \notin \operatorname{supp}\psi_n$ for all $n \in N_2$ then obviously $\Delta = 0$. Let now $x \in \operatorname{supp}\psi_n$ for some $n \in N_2$ (note that there may exist not more than two such numbers). Since $y \notin V_n$, we have $|x-y| \geq d/4$. It follows from the Leibniz formula that

$$|(r_{i_n}\psi_n)^{(m)}(x)| \leq A\sum_{k=0}^{m}|r_{i_n}^{(k)}(x)|d^{k-m}. \tag{3.2}$$

Applying Proposition 3.2 to the function r_{i_n} vanishing on the set $F\cap V_n$, and repeating the argument which resulted in estimate (3.1), we readily see that for $0 \leq k \leq m$,

$$|r_{i_n}^{(k)}(x)| \leq Ad^{m-k}\varphi(2d)|r_{i_n}^{(m)}|_\varphi \leq Ad^{m-k}\varphi(d)\|r_{i_n}\| \leq Ad^{m-k}\varphi(d)\varepsilon. \tag{3.3}$$

Combining this with (3.2) we obtain

$$\Delta \leq |r^{(m)}(x)| = A\varphi(d)\varepsilon \leq A\varepsilon\varphi(|x-y|).$$

3. Suppose that the points x and y are both in V but do not belong simultaneously to any set $V_n,\ n \in N_2$. Then $|x-y| \geq d/2$, and as in the previous case we have

$$\Delta \leq |r^{(m)}(x)| + |r^{(m)}(y)| \leq A\varepsilon\varphi(d) \leq A\varepsilon\varphi(|x-y|).$$

4. Suppose finally that $x, y \in V_{n_0}$ for a certain $n_0 \in N_2$, in which case $|x-y| \leq 2d$. Let $n \in N_2$ be one of those (not more than three) numbers for which at least one of the points x, y belongs to V_n. By the Leibniz formula

$$\begin{aligned}|(r_{i_n}\psi_n)^{(m)}(x) - (r_{i_n}\psi_n)^{(m)}(y)| \leq A\Big[&\sum_{k=0}^{m}|r_{i_n}^{(k)}(x)|\cdot|\psi_n^{(m-k)}(x) - \psi_n^{(m-k)}(y)| \\ &+ \sum_{k=0}^{m}|r_{i_n}^{(k)}(x) - r_{i_n}^{(k)}(y)|d^{k-m}\Big].\end{aligned}$$

Denote the first and the second sum by S_1 and S_2, respectively.

To estimate S_1, observe that $r_{i_n}|F\cap W_n \equiv 0$ and $W_n \cap F' \neq \emptyset$. Applying Proposition 3.2 to the function r_{i_n} we obtain (3.3). Next,

$$|\psi_n^{(m-k)}(x) - \psi_n^{(m-k)}(y)| \leq |x-y|\cdot\|\psi_n^{(m-k+1)}\|_\infty \leq Ad^{k-m+1}|x-y|.$$

Thus in view of (3.3) and upon recalling that the function $\varphi(t)/t$ is nonincreasing we get

$$S_1 \leq A\varepsilon|x-y|\varphi(d)/d \leq A\varepsilon\varphi(|x-y|).$$

It remains to estimate S_2. For $0 \leq k \leq m-1$,

$$|r_{i_n}^{(k)}(x) - r_{i_n}^{(k)}(y)| = |x-y|\cdot|r_{i_n}^{k+1}(z)|$$

with some $z \in [x, y]$. Invoking (3.3) we have

$$|r_{i_n}^{(k)}(x) - r_{i_n}^{(k)}(y)| \leq A\varepsilon d^{m-k-1}\varphi(d)|x-y| \leq A\varepsilon d^{m-k}\varphi(|x-y|),$$

which yields $S_2 \leq A\varepsilon\varphi(|x-y|)$.

Thus, in case 4 as well as in all other cases $\Delta \leq A\varepsilon\varphi(|x-y|)$. Hence, $|r^{(m)}|_\varphi \leq A\varepsilon$, and so, due to (3.1), $\|r\| \leq A\varepsilon$.

Recalling that I is a *closed* ideal we conclude from the representation $f_2 = g + r$, where $g \in I$, that $f_2 \in I$. Therefore, $f = f_0 + f_1 + f_2 \in I$. The proof of Theorem 3.1 is complete.

REMARK 1. The proof of Theorem 3.1 depends heavily on Proposition 3.2 and thus cannot be carried over to the multivariate case.

REMARK 2. In like manner, one can prove a theorem on SSI for the algebra $C^m \operatorname{Lip}\varphi(\mathbb{R})$, which consists of C^m-functions on $\mathbb{R}$ whose mth derivatives are in $\operatorname{Lip}\varphi([a,b])$ for every interval $[a,b]$, and is supplied with an appropriate family of seminorms.

4. D-algebras

Let $\mathcal{A}$ be a Shilov regular Banach algebra of continuous functions on a compact Hausdorff space X. We are assuming in addition that it has the following inversion property:

$$\text{If } f \in \mathcal{A} \text{ and } f(x) \neq 0 \text{ for all } x \in X \text{ then } 1/f \in \mathcal{A}. \tag{4.1}$$

Hence (see e.g. [43]) the space of maximal ideals of the algebra $\mathcal{A}$ coincides with X.

DEFINITION 1. A bounded linear functional $D \in \mathcal{A}^*$ is called a *point derivation* of the algebra $\mathcal{A}$ at a point $x \in X$ if

$$D(fg) = f(x)Dg + g(x)Df \qquad \text{for all } f, g \in \mathcal{A}. \tag{4.2}$$

Let $\mathcal{D}_x$ be the linear space of all point derivations of $\mathcal{A}$ at a point $x \in X$. It follows from the above definition that $\mathcal{D}_x = (\mathbf{1} \cup M_x^2)^{\perp}$. Also, due to the regularity of $\mathcal{A}$, we have $D(J_x) = \{0\}$ for all $D \in \mathcal{D}_x$. Hence, for every isolated point $x \in X$ we have $\mathcal{D}_x = \{0\}$.

For a closed subset $F \subset X$, define

$$K_F := \{D : D \in \mathcal{D}_x,\ x \in F,\ \|D\| \leq 1\}, \tag{4.3}$$

where $\|\cdot\|$ stands hereafter for the usual norm on $\mathcal{A}^*$. Obviously, the set K_F is compact in the weak* topology on $\mathcal{A}^*$.

With each function $f \in \mathcal{A}$ we associate a function $\widehat{f} \in C(K_F)$ by setting

$$\widehat{f}(D) := Df, \qquad D \in K_F. \tag{4.4}$$

This formula determines a linear mapping $d_F : \mathcal{A}/J_F \to C(K_F)$ which is an analogue of the classic Gel'fand transform. We readily see that

$$\|\widehat{f}\|_{K_F} \leq \|\dot{f}\|_F, \qquad f \in \mathcal{A}, \tag{4.5}$$

where $\|\cdot\|_{K_F}$ is the supremum norm on $C(K_F)$ and $\|\dot{f}\|_F$ is the quotient norm of the element $\dot{f} \in \mathcal{A}/J_F$ corresponding to f.

Now we are going to define the class of D-algebras.

DEFINITION 2. An algebra $\mathcal{A}$ as above is called a *D-algebra* if for every closed subset F in X there is a constant $A(F) \geq 0$ such that

$$\|\dot{f}\|_F \leq A(F)\|\widehat{f}\|_{K_F} \qquad \text{for all } f \in M_F. \tag{4.6}$$

Remark 1. Equivalently, the class of D-algebras can be characterized by the condition that the mapping d_F is an injection onto a closed subspace of $C(K_F)$.

Remark 2. It follows from (4.6) that

$$J_F = \mathrm{clos}_{\mathcal{A}} M_F^2 \tag{4.7}$$

for every closed subset F of X. In particular, $J_x = \mathrm{clos}_{\mathcal{A}} M_x^2$ for every $x \in X$. This means that every D-algebra consists of functions whose "order of smoothness" is less than 2. It is intriguing that in all known examples the order of smoothness of functions in a D-algebra is ≤ 1; see below.

Remark 3. Condition (4.6) implies that in a D-algebra for every closed subset F of X one has $J_F = \bigcap_{x\in F} J_x$, which means that minimal closed ideals with any given cospectrum have the spectral synthesis property (1.2).

Remark 4. Condition (4.6) is a kind of extension theorem which implies in particular that the "trace" of a function from a D-algebra on a closed set $F \subset X$ is completely determined by the values of the function and its point derivations at points of F.

The following result is of prime importance in the context of this paper.

Theorem 4.1. *Every D-algebra admits SSI.*

For the proof of this result, we refer the reader to [20], [22].

All algebras satisfying condition (1.3) fail to have nonzero point derivations and thus provide trivial examples of D-algebras. This is the case for the algebras $C(X)$, $\mathrm{lip}(X, \varrho)$, and $W_p^1(\mathbb{R}^n)$ with $n = 1 \le p < \infty$ and $2 \le n < p < \infty$. A number of nontrivial examples of classical algebras which are (or fail to be) D-algebras are discussed in detail below.

(A) Algebras C^1. Let $C^1(Q)$ be the algebra of C^1-functions f on a closed cube Q in $\mathbb{R}^n$ supplied with the norm $\|f\| := \max\{\|f\|_Q, \| \, |\nabla f| \, \|_Q\}$. In this algebra,

$$J_x = \{f \in C^1(Q) : f(x) = 0,\ \nabla f(x) = 0\}, \qquad x \in Q, \tag{4.8}$$

and, in a more general way, for every closed set $F \subset Q$,

$$J_F = \{f \in C^1(Q) : f|F \equiv 0,\ \nabla f|F \equiv 0\}.$$

It follows from (4.8) that any point derivation of the algebra $C^1(Q)$ at a point $x \in Q$ has the form $f \mapsto \nabla f(x)\nu$ for some vector $\nu \in \mathbb{R}^n$, i.e. is a directional derivative.

We claim that $C^1(Q)$ is a D-algebra.

To check condition (4.6), take any closed set $F \subset Q$ and note that for any function $f \in C^1(Q)$ vanishing on F the norm $\|\widehat{f}\|_{K_F}$ equals $M := \| \, |\nabla f| \, \|_F$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(x)| \le \varepsilon \quad \text{and} \quad |\nabla f(x)| \le M + \varepsilon, \qquad x \in F_\delta, \tag{4.9}$$

where $F_\delta := \{x \in Q : d(x, F) \le \delta\}$ and $d(x, F)$ is the Euclidean distance from x to F. From

$$\nabla f(x) - \nabla f(y) \to 0 \quad \text{as } x - y \to 0, \qquad x, y \in F_\delta, \tag{4.10}$$

and

$$(4.11)\qquad f(x)-f(y)-\nabla f(y)(x-y)=(x-y)\int_0^1[\nabla f(y+t(x-y))-\nabla f(y)]\,dt,$$

it follows that

$$(4.12)\qquad \frac{f(x)-f(y)-\nabla f(y)(x-y)}{|x-y|}\to 0 \quad \text{as } x-y\to 0,\qquad x,y\in F_\delta.$$

In view of conditions (4.9), (4.10), and (4.12) one can apply the Whitney extension theorem [52], [31], providing a function $\widetilde{f}\in C^1(Q)$ such that $\widetilde{f}=f$ on F_δ and $\|\widetilde{f}\|\le A(M+\varepsilon)$, where A stands in Section 4 for a constant that may depend only on n. Since $\widetilde{f}-f\in J_F$, we have $\|\dot{f}\|_F\le\|\widetilde{f}\|\le A(M+\varepsilon)$, and (4.6) follows via arbitrariness of ε.

(B) Lipschitz algebras. Every real Lipschitz function defined on a subset of a metric space (X,ϱ) can be extended to a function from $\mathrm{Lip}(X,\varrho)$ with the same Lipschitz norm; see [37], [42]. This leads us right away to the equality

$$(4.13)\qquad \|\dot{f}\|_F=\max\left\{\|f\|_F,\ \limsup_{x,y\to F}\frac{|f(x)-f(y)|}{\varrho(x,y)}\right\},\qquad f\in\mathrm{Lip}(X,\varrho)$$

(see [51]), where the notation $x\to F$ means $\varrho(x,F)\to 0$. In particular,

$$J_F=\left\{f\in\mathrm{Lip}(X,\varrho): f|F\equiv 0,\ \lim_{x,y\to F}\frac{f(x)-f(y)}{\varrho(x,y)}=0\right\};$$

see [42].

For a cluster point $x\in X$, denote by Φ_x the set of all weak* limits of linear functionals on $\mathrm{Lip}(X,\varrho)$ of the form $f\mapsto(f(a)-f(b))/\varrho(a,b)$ as $a,b\to x$. It is easily seen that every functional in Φ_x is a point derivation at x (moreover, as shown in [42], $\mathcal{D}_x=V(\Phi_x)$, where $V(\cdot)$ stands for the weak* closure of the linear span). Together with (4.13) this shows that, for real Lipschitz algebras, inequality (4.6) is satisfied with $A(F)=1$ and hence in view of (4.5) turns into equality.

Thus, every Lipschitz algebra $\mathrm{Lip}(X,\varrho)$ on a compact metric space is a D-algebra.

Point derivations play an important role in Lipschitz algebras [42]. One of their remarkable extremal properties is stated below.

PROPOSITION 4.1. *Let D_1 and D_2 be point derivations of a real Lipschitz algebra* $\mathrm{Lip}(X,\varrho)$ *at two distinct points $x_1,x_2\in X$, respectively. Then*

$$\|D_1+D_2\|=\|D_1\|+\|D_2\|.$$

PROOF. We need to show that $\|D_1\|+\|D_2\|\le\|D_1+D_2\|$. Fix $\varepsilon>0$ and pick functions $f_1,f_2\in\mathrm{Lip}(X,\varrho)$ such that $\|f_i\|\le 1$ and $D_if_i\ge\|D_i\|-\varepsilon,\ i=1,2$. For $\delta\in(0,1/2)$, set $B_i:=B(x_i,\delta\varrho(x_1,x_2))$, where $B(x,d)$ is the closed ball in X of radius d and center x. Clearly, $B_1\cap B_2=\emptyset$. Put $g_i:=f_i-f_i(x_i)$ and define a function g on $F:=B_1\cup B_2$ by $g|B_i=g_i|B_i,\ i=1,2$. An easy calculation shows that

$$|g|_{F,\varrho}\le\max\left\{1,\frac{2\delta}{1-2\delta}\right\}.$$

Choose δ small enough to satisfy $2\delta/(1-2\delta) \le 1$ and $\delta\varrho(x_1, x_2) \le 1$; then $\|g\|_{F,\varrho} \le 1$. We extend functions g_i, $i = 1, 2$, to the whole space X preserving their Lipschitz norms (and notation) to get

$$\begin{aligned}\|D_1\| + \|D_2\| &\le D_1 f_1 + D_2 f_2 + 2\varepsilon = D_1 g_1 + D_2 g_2 + 2\varepsilon \\ &= (D_1 + D_2)g + 2\varepsilon \le \|D_1 + D_2\| + 2\varepsilon,\end{aligned}$$

and Proposition 4.1 follows.

(C) Algebras $C^1 \operatorname{Lip}\varphi$. Let φ be any nondecreasing function on $\mathbb{R}_+$ such that $\varphi(0) = \varphi(0+) = 0$ and $\varphi(t)/t \to \infty$ as $t \to 0$. Consider the Banach algebra $\mathcal{A} = C^1 \operatorname{Lip}\varphi$ of C^1-functions f on a cube Q in $\mathbb{R}^n$ with the finite norm

$$\|f\| := \max\{\|f\|_Q, \|\,|\nabla f|\,\|_Q, |\nabla f|_\varphi\},$$

where for a function $g : Q \to \mathbb{R}^n$ we set

$$|g|_\varphi := \sup\left\{\frac{|g(x) - g(y)|}{\varphi(|x-y|)} : x, y \in Q,\ x \ne y\right\}.$$

We are going to describe closed primary ideals and point derivations of the algebra $C^1 \operatorname{Lip}\varphi$. As a result of this study, we shall derive that this algebra *is not* a D-algebra.

For $x \in Q$, denote by $\|\cdot\|_x$ the quotient norm on $\mathcal{A}/J_x$, and set

$$N_x(f) := \max\left\{|f(x)|, |\nabla f(x)|, \limsup_{a,b\to x} \frac{|\nabla f(a) - \nabla f(b)|}{\varphi(|a-b|)}\right\}, \qquad f \in \mathcal{A}.$$

Claim 1. We have

$$A\|\dot f\|_x \le N_x(f) \le \|\dot f\|_x, \qquad f \in \mathcal{A}. \tag{4.14}$$

In fact, for every function $g \in \mathcal{A}$ which coincides with f in a neighborhood of the point x, we have $N_x(f) = N_x(g) \le \|g\|$, which yields the second inequality in (4.14). To verify the first inequality, take $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|f(y)| \le N_x(f) + \varepsilon, \qquad |\nabla f(y)| \le N_x(f) + \varepsilon,$$

and

$$|\nabla f(y) - \nabla f(z)| \le [N_x(f) + \varepsilon]\varphi(|y-z|)$$

whenever $y, z \in Q$ and $|y - x| \le \delta, |z - x| \le \delta$. In view of (4.11) for such y and z we also have

$$|f(z) - f(y) - \nabla f(y)(z-y)| \le [N_x(f) + \varepsilon]|z-y|\varphi(|y-z|).$$

Applying the Whitney–Glaeser extension theorem (see [16], [31] or [48], Chapter 6) to the set $F := Q(x,\delta) \cap Q$ we find a function $\widetilde f \in \mathcal{A}$ such that $\widetilde f|F = f|F$ and $\|\widetilde f\| \le A[N_x(f) + \varepsilon]$. Since $\widetilde f - f \in J_x$, one has $\|\dot f\|_x \le \|\widetilde f\| \le A[N_x(f) + \varepsilon]$. We finish the proof of Claim 1 by letting $\varepsilon \to 0$.

From (4.14) we deduce that

$$J_x = \left\{f \in \mathcal{A} : f(x) = 0,\ \nabla f(x) = 0,\ \lim_{a,b\to x} \frac{\nabla f(a) - \nabla f(b)}{\varphi(|a-b|)} = 0\right\}. \tag{4.15}$$

It follows from the condition $\lim_{t\to 0} \varphi(t)/t = \infty$ that any function in the ideal M_x^2 belongs to the right-hand side of (4.15). Hence, $\operatorname{clos}_{\mathcal{A}} M_x^2 \subset J_x$. Recall that J_x is the *minimal* closed primary ideal at x, therefore $J_x \subset \operatorname{clos}_{\mathcal{A}} M_x^2$. Thereby we have proved the equality

$$J_x = \operatorname{clos}_{\mathcal{A}} M_x^2. \tag{4.16}$$

REMARK. For any regular commutative semisimple real Banach algebra $\mathcal{A}$ with unity, (4.16) implies

$$J_x^{\perp} = \mathcal{D}_x + \{\lambda\delta_x\}_{\lambda\in\mathbb{R}}. \tag{4.17}$$

Indeed, the right-hand side of (4.17) is contained in $J_x^{\perp}$. To see the converse, take any functional $\psi \in J_x^{\perp}$. Using (4.16) we have

$$\psi - \psi(\mathbf{1})\delta_x \in (M_x^2 \cup \mathbf{1})^{\perp} = \mathcal{D}_x.$$

Therefore, $\psi \in \mathcal{D}_x + \{\lambda\delta_x\}_{\lambda\in\mathbb{R}}$.

To state the next claim, define

$$\|\widehat{f}\|_x := \|\widehat{f}\|_{K_x} = \sup\{Df : D \in \mathcal{D}_x,\ \|D\| \le 1\}.$$

CLAIM 2. $N_x(f) \le \|\widehat{f}\|_x \le AN_x(f)$ for $f \in M_x$.

To show this, observe that for $f \in M_x$,

$$N_x(f) = \max\left\{|\nabla f(x)|, \limsup_{a,b\to x} \frac{|\nabla f(a) - \nabla f(b)|}{\varphi(|a-b|)}\right\}.$$

Any functional $f \mapsto D^{e_i} f(x),\ i = 1, \dots, n$, is in fact a point derivation at x with norm ≤ 1. Now put

$$L_x(f) := \limsup_{a,b\to x} \frac{|\nabla f(a) - \nabla f(b)|}{\varphi(|a-b|)},$$

and take sequences of points $\{a_k\}, \{b_k\},\ a_k \ne b_k$, and vectors $\{\eta_k\},\ k \in \mathbb{N}$, such that $|\eta_k| = 1$ and

$$\frac{|D^{\eta_k} f(a_k) - D^{\eta_k} f(b_k)|}{\varphi(|a_k - b_k|)} \to L_x(f)$$

as $k \to \infty$. For every k,

$$\psi_k(f) := \frac{D^{\eta_k} f(a_k) - D^{\eta_k} f(b_k)}{\varphi(|a_k - b_k|)}, \qquad f \in \mathcal{A},$$

is a linear continuous functional on $\mathcal{A}$ with norm ≤ 1. By compactness of the closed unit ball of $\mathcal{A}^*$ in the weak* topology there is a subnet $\{k_\alpha\} \subset \mathbb{N}$ and a functional $\psi \in \mathcal{A}^*$ with $\|\psi\| \le 1$ such that $\lim_\alpha \psi_{k_\alpha}(f) = \psi(f)$ for every $f \in \mathcal{A}$. Using the identity

$$\psi_k(fg) = f(a_k)\psi_k(g) + g(a_k)\psi_k(f) + D^{\eta_k} g(b_k)\frac{f(b_k) - f(a_k)}{\varphi(|a_k - b_k|)} + D^{\eta_k} f(b_k)\frac{g(b_k) - g(a_k)}{\varphi(|a_k - b_k|)}$$

and the fact that $\varphi(t)/t \to \infty$ as $t \to 0$, we find that ψ satisfies (4.2) and hence is a point derivation at x. Therefore, $N_x(f) \le \|\widehat{f}\|_x$.

To prove the reverse inequality, recall that $D(J_x) = \{0\}$ for every $D \in \mathcal{D}_x$. Using the well-known dual representation of the quotient norm and Claim 1 we have

$$\|\widehat{f}\|_x \le \sup\{\lambda(f) : \lambda \in J_x^{\perp},\ \|\lambda\| \le 1\} = \|f\|_x \le AN_x(f), \qquad f \in \mathcal{A},$$

which completes the proof.

PROPOSITION 4.2. *The algebra $C^1 \operatorname{Lip}\varphi$ is not a D-algebra.*

PROOF. Clearly, it suffices to show this in the case $n = 1$ and $Q = [0,1]$.

Suppose, on the contrary, that $\mathcal{A} := C^1 \operatorname{Lip}\varphi([0,1])$ is a D-algebra. Take $F := \{0\} \cup \{a_k : k \in \mathbb{N}\}$, where $\{a_k\}_{k\in\mathbb{N}} \subset [0,1]$ is any positive sequence monotonically tending to zero. Fix k and choose a function $f_k \in \mathcal{A}$ which coincides with the function $x \mapsto x - a_k$ in a neighborhood of the point a_k and vanishes in a neighborhood of $F \setminus \{a_k\}$. Then $f_k \in M_F$ and $N_{a_k}(f_k) = |f_k'(a_k)| = 1$. Invoking Claim 2 one has

$$\|\widehat{f}_k\|_{K_F} = \sup_{x\in F} \|\widehat{f}_k\|_x = \|\widehat{f}_k\|_{a_k} \le A.$$

By the definition (4.6) of D-algebras there should exist a constant $A(F)$ such that for all $f \in \mathcal{A}$ vanishing on F,

$$\|\dot{f}\|_F \le A(F)\|\widehat{f}\|_{K_F}.$$

Therefore,

$$\frac{1}{\varphi(a_k)} = \frac{|f_k'(a_k) - f_k'(0)|}{\varphi(a_k)} \le \|\dot{f}_k\|_F \le A(F)\cdot A$$

for all k, which for $k \to \infty$ leads to a contradiction.

(D) Zygmund algebras. The most interesting new examples of D-algebras are found, however, among Zygmund algebras Λ_φ not contained in C^1, or equivalently with φ satisfying condition (0). These algebras are discussed thoroughly in Section 5.

5. Zygmund algebras

5.1. Basic properties. The following notation will be used throughout Section 5. For $x \in \mathbb{R}^n$, we put $|x| := \max_{1\le i\le n} |x_i|$, whereas $\|x\|$ will stand for the Euclidean norm of x. Recall that a cube in $\mathbb{R}^n$ is a set of the form $Q(c,d) = \{x \in \mathbb{R}^n : |x - c| \le d\}$, where $c = c_Q$ is the center of the cube and $d = d_Q$ is half of its edgelength.

For a function f defined on a convex set F in $\mathbb{R}^n$ and for all admissible x and h, we put

$$\Delta_h^1 f(x) := f(x+h) - f(x), \qquad \Delta_h^2 f(x) := f(x+h) - 2f(x) + f(x-h),$$

and denote by

$$\omega_2(f;F;t) := \sup\{|\Delta_h^2 f(x)| : x \pm h \in F,\ |h| \le t\}, \qquad t \ge 0,$$

the second modulus of continuity of f on F.

The letter A will stand for various positive constants possibly depending on n.

Let $\mathcal{P}_1$ be the set of all polynomials in n variables of degree not greater than 1. For a bounded function f defined on a set F in $\mathbb{R}^n$ we denote by

$$E_1(f;F) := \inf\{\|P - f\|_F : P \in \mathcal{P}_1\}$$

the best uniform approximation to f of order 1, and by $P(f;F)$ any polynomial of degree ≤ 1 such that $\|P(f;F) - f\|_F = E_1(f;F)$.

It is well known [5] that for every cube Q in $\mathbb{R}^n$,

$$E_1(f;Q) \le A\omega_2(f;Q;d_Q) \tag{5.1}$$

(the reverse inequality $\omega_2(f;Q;d_Q) \le 4E_1(f;Q)$ is obvious).

We define the *Zygmund space* $\Lambda_\varphi = \Lambda_\varphi(Q_0)$ to be the set of all bounded functions f on the cube $Q_0 := [-1,1]^n$ satisfying, for all admissible x and h, the following Zygmund condition:

$$|\Delta_h^2 f(x)| \le C\varphi(|h|) \tag{5.2}$$

with some constant C. Here, φ is a given nondecreasing function on $\mathbb{R}_+$ such that $\varphi(0) = \varphi(0+) = 0$ and $\varphi(t) > 0$ for $t > 0$. Also, it will be assumed without loss of generality that $\varphi(t) = 1$ for $t \ge 1$ and that

$$\frac{\varphi(t)}{t^2} \text{ is nonincreasing for } t > 0. \tag{5.3}$$

Every function φ with these properties will be referred to as a *majorant.*

Condition (5.3) implies that

$$\varphi(kt) \ge k^2\varphi(t), \quad k \ge 1,$$

and

$$\varphi(t) \ge t^2, \quad 0 \le t \le 1. \tag{5.4}$$

These inequalities will be systematically (sometimes tacitly) used in the sequel.

The space Λ_φ is supplied with the norm

$$\|f\|_{\Lambda_\varphi} := \max\{\|f\|_{Q_0}, |f|_{\Lambda_\varphi}\},$$

where $|f|_{\Lambda_\varphi} := \inf C$ over all C involved in (5.2).

We define the "small" *Zygmund space* λ_φ as the set of all functions $f \in \Lambda_\varphi$ satisfying the condition

$$\lim_{t\to 0} \frac{\omega_2(f;Q_0;t)}{\varphi(t)} = 0. \tag{5.5}$$

A standard argument shows that λ_φ is a closed separable linear subspace of Λ_φ which in the case $\lim_{t\to 0}\varphi(t)/t^2 = \infty$ (in other words, for all majorants except those equivalent to t^2) can be identified with the closure of $C^\infty(Q_0)$ in Λ_φ.

The following properties of majorants will be needed in what follows.

Proposition 5.1.1. *For any majorant φ, the following relations hold*:

$$s\int_s^3 \frac{\varphi(u)}{u^2}\,du \le At\int_t^3 \frac{\varphi(u)}{u^2}\,du, \quad 0 < s \le t \le 2; \tag{5.6.1}$$

$$s\int_s^{2s} \frac{\varphi(u)}{u^2}\,du \le At\int_t^{2t} \frac{\varphi(u)}{u^2}\,du, \quad 0 < s \le t \le 1; \tag{5.6.2}$$

$$\lim_{t\to 0} t\int_t^3 \frac{\varphi(u)}{u^2}\,du = 0; \tag{5.6.3}$$

(5.6.4)
$$\left[t\int_t^1 \frac{\varphi(u)}{u^2}\,du\right]^2 \le A\varphi(t), \quad 0<t\le 1/2.$$

If $\lim_{t\to 0}\varphi(t)/t^2=\infty$ *then*

(5.6.5)
$$\lim_{t\to 0}\frac{1}{\varphi(t)}\left[t\int_t^1 \frac{\varphi(u)}{u^2}\,du\right]^2 = 0.$$

Proof. The first two relations depend only on the boundedness of φ and property (5.3), and can be easily checked by appropriate linear changes of variables in the integrals standing on the left-hand sides.

To show (5.6.3) take $\varepsilon>0$ and choose $\delta>0$ such that $\varphi(\delta)\le\varepsilon$. For every $t\le\delta$ we have, in view of monotonicity of φ,

$$t\int_t^\delta \frac{\varphi(u)}{u^2}\,du \le t\varphi(\delta)\left(\frac{1}{t}-\frac{1}{\delta}\right)\le \varphi(\delta)\le\varepsilon.$$

Therefore, for sufficiently small t,

$$t\int_t^3 \frac{\varphi(u)}{u^2}\,du \le \varepsilon + t\int_\delta^3 \frac{\varphi(u)}{u^2}\,du \le 2\varepsilon,$$

and (5.6.3) follows.

To prove (5.6.4) we set $\omega(t):=\varphi(t)/t^2$ and note that what we need to show is

(5.7)
$$\left[\int_t^1 \omega(u)\,du\right]^2 \le A\omega(t), \quad 0<t\le\frac{1}{2}.$$

Define

$$\omega^*(u):=\sum_{k=1}^\infty \omega(2^{-k})\chi_{(2^{-k},2^{-k+1}]}(u), \quad 0\le u\le 1,$$

where χ_E is the characteristic function of a set E. Observe that the function ω is nonincreasing while $\omega(t)t^2$ is nondecreasing, hence $\omega\le\omega^*\le 4\omega$. Therefore, it suffices to check (5.4) for $t=2^{-n}$, $n\in\mathbb{N}$, and for functions of the form

(5.8)
$$\omega=\sum_{k=1}^\infty a_k\chi_{(2^{-k},2^{-k+1}]} \quad \text{with } 1\le\frac{a_{k+1}}{a_k}\le 4.$$

Setting $\tau_k:=a_{k+1}/(2a_k)$, $k\in\mathbb{N}$, we see that under these assumptions (5.7) becomes

(5.9)
$$\varrho(T):=\frac{(1+\tau_1+\tau_1\tau_2+\ldots+\tau_1\cdot\ldots\cdot\tau_n)^2}{2^n\tau_1\cdot\ldots\cdot\tau_n}\le A,$$

where $T:=(\tau_1,\tau_2,\ldots,\tau_n)$ and $1/2\le\tau_k\le 2$, $k=1,\ldots,n$.

As a function of $\tau=\tau_k$, the other arguments being fixed, $\varrho(T)$ has the form $\varrho_k(\tau)=\alpha_k/\tau+\beta_k+\gamma_k\tau$, where α_k, β_k, and γ_k are positive constants independent of τ. The function $\varrho_k(\tau)$ is convex for $\tau>0$, hence its maximum on the interval $[1/2,2]$ is either $\varrho_k(1/2)$ or $\varrho_k(2)$. Thus we may restrict ourselves in (5.9) to vectors T whose components are 1/2 or 2.

Let $T^*=(\tau_1^*,\ldots,\tau_n^*)$ be a vector maximizing ϱ. We claim that there exists m, $0\le m\le n$, such that $\tau_i^*=2$ for $1\le i\le m$ and $\tau_i^*=1/2$ for $m+1\le i\le n$. For, if not, then for some $k,\ 1 \le k \le n-1$, we would have $\tau_k^* = 1/2$ and $\tau_{k+1}^* = 2$. Now interchanging τ_k^* and τ_{k+1}^* we see that an alteration on the left-hand side of (5.9) occurs only in the $(k+1)$th item $\tau_1^* \cdot \ldots \cdot \tau_{k-1}^*/2$ in the numerator, which becomes $2\tau_1^* \cdot \ldots \cdot \tau_{k-1}^*$. Thus for the new vector $T' := (\tau_1^*, \ldots \tau_{k-1}^*, \tau_{k+1}^*, \tau_k^*, \ldots, \tau_n^*)$ we have $\varrho(T') > \varrho(T^*)$, which is a contradiction.

It remains to note that for every m,

$$\varrho(T^*) = \frac{[1 + \ldots + 2^m + 2^{m-1} + \ldots + 2^{m-(n-m)}]^2}{2^n 2^m 2^{-(n-m)}} < \frac{(2^{m+1} + 2^m)^2}{2^{2m}} = 9,$$

and the proof of (5.6.4) is finished.

Turning to (5.6.5) observe that we only need to establish the following relation:

$$\lim_{n\to\infty} \frac{1}{\omega(2^{-n})} \Big[\int_{2^{-n}}^{1} \omega(t)\, dt \Big]^2 = 0$$

for functions ω as in (5.8) with $a_k = \omega(2^{-k})$. Given $r \in \mathbb{N}$, we have by (5.9), for all $n > r$,

$$\frac{1}{\omega(2^{-n})} \Big[\int_{2^{-n}}^{2^{-r+1}} \omega(t)\, dt \Big]^2 = \omega(2^{-r})(2^{-r})^2 \varrho(T) = \varphi(2^{-r})\varrho(T) \le A\varphi(2^{-r}),$$

where $T = (\tau_r, \ldots, \tau_{n-1})$. Therefore, taking into account that $\varphi(t) \to 0$ as $t \to 0$ we can find (and fix) r such that

$$\frac{1}{\omega(2^{-n})} \Big[\int_{2^{-n}}^{2^{-r+1}} \omega(t)\, dt \Big]^2$$

is small enough *for all* $n > r$. Next, the condition $\omega(t) = \varphi(t)/t^2 \to \infty$ as $t \to 0$ implies that for all sufficiently large n,

$$\frac{1}{\omega(2^{-n})} \Big[\int_{2^{-r+1}}^{1} \omega(t)\, dt \Big]^2$$

will also be as small as desired.

Proposition 5.1.1 is proved.

We now state an important inequality due to Marchaud [35]: for every function f defined on an interval $L = [a-d, a+d]$ and for all $x,\ h$ such that $x \pm h \in L$ and $h \ne 0$,

$$|\Delta_h^1 f(x)| \le A|h| \Big[\int_{|h|}^{d} \frac{\omega_2(f; L; t)}{t^2}\, dt + \frac{\|f\|_L}{d} \Big]. \tag{5.10}$$

COROLLARY 1. *Let $Q = Q(a,d)$ be a cube in $\mathbb{R}^n$ and f be a function on Q. Suppose that $F \subset Q$ and $f|F \equiv 0$. Then for every $x \in Q$,*

$$|f(x)| \le Ad(x,F) \Big[\int_{d(x,F)}^{Ad} \frac{\omega_2(f; Q; t)}{t^2}\, dt + \frac{\|f\|_Q}{d} \Big]. \tag{5.11}$$

COROLLARY 2. *If $f \in \Lambda_\varphi(Q_0)$ then for any $x, y \in Q_0,\ x \ne y$, we have*

$$\frac{|f(y) - f(x)|}{|y-x|} \le A\|f\|_{\Lambda_\varphi} \int_{|y-x|}^{3} \frac{\varphi(t)}{t^2}\, dt. \tag{5.12}$$

In fact, by (5.10),

$$\frac{|f(y)-f(x)|}{|y-x|} \le A\Bigg[\int\limits_{\|y-x\|}^{A} \frac{\omega_2(f;Q_0;t)}{t^2}dt + \|f\|_{Q_0}\Bigg] \le A\|f\|_{\Lambda_\varphi} \int\limits_{|y-x|}^{3} \frac{\varphi(t)}{t^2}\,dt.$$

From (5.10) we conclude, invoking (5.6.3), that $\Lambda_\varphi \subset C$. Moreover, as follows from the definition (5.2) of the Zygmund space and from (5.10),

$$\operatorname{Lip}\varphi \subset \Lambda_\varphi \subset \operatorname{Lip}\psi,$$

where

$$\psi(t) := t\int\limits_t^3 \frac{\varphi(u)}{u^2}\,dt, \quad 0 < t \le 1, \quad \text{and} \quad \psi(0) := 0. \tag{5.13}$$

Note also that $\Lambda_\varphi = \operatorname{Lip}\varphi$ if and only if

$$t\int\limits_t^3 \frac{\varphi(u)}{u^2}\,du \le C\varphi(t), \qquad 0 < t \le 1, \tag{5.14}$$

for some constant C depending on φ. In particular, this is the case for $\varphi(t) = t^\alpha$, $0 < \alpha < 1$.

In fact, if a majorant φ satisfies (5.14) then due to (5.10), $\Lambda_\varphi \subset \operatorname{Lip}\varphi$, and hence these spaces coincide.

To prove the converse, suppose that $\Lambda_\varphi \subset \operatorname{Lip}\varphi$. Then this is true indeed for $n = 1$. Take the odd extension to the interval $[-1, 1]$ of the function ψ on $[0, 1]$ defined by (5.13), for which we preserve the same notation. As shown in the next section (see Proposition 5.2.2), $\psi \in \Lambda_\varphi([-1, 1])$. Hence by our hypothesis $\psi \in \operatorname{Lip}\varphi([-1, 1])$, which implies (5.14).

Proposition 5.1.2. *For every n and φ, the space Λ_φ is a Banach algebra with respect to pointwise multiplication.*

Proof. Applying (5.10), (5.4) and (5.6.4) to the identity

$$\Delta_h^2(fg)(x) = \Delta_h^2 f(x)g(x+h) + 2\Delta_h^1 f(x-h)\Delta_h^1 g(x) + f(x-h)\Delta_h^2 g(x), \tag{5.15}$$

we obtain for all $f, g \in \Lambda_\varphi$ the following inequality:

$$\|fg\|_{\Lambda_\varphi} \le A\|f\|_{\Lambda_\varphi}\|g\|_{\Lambda_\varphi}.$$

Therefore, Λ_φ is a Banach algebra.

Clearly, the Zygmund algebra Λ_φ meets all requirements from Section 1 and has the inversion property (4.1).

Proposition 5.1.3. *The condition*

$$\int\limits_0^1 \frac{\varphi(t)}{t^2}\,dt < \infty \tag{5.16}$$

is necessary and sufficient for the imbedding $\Lambda_\varphi \subset C^1$. Moreover, the latter inclusion automatically implies $\Lambda_\varphi \subset C^1 \operatorname{Lip}\gamma$ with

$$\gamma(t) := \int\limits_0^t \frac{\varphi(s)}{s^2}\,ds.$$

PROOF. We start with the following observation.

If $f \in \Lambda_\varphi([a,b])$ then for all $x, y, u, v \in [a,b]$ such that $0 < |y-x| \le |v-u|$, we have

$$\left|\frac{f(y)-f(x)}{y-x} - \frac{f(v)-f(u)}{v-u}\right| \le A|f|_{\Lambda_\varphi} \int_{|y-x|}^{2|v-u|+\max\{|u-x|,|v-y|\}} \frac{\varphi(t)}{t^2}\,dt \tag{5.17}$$

(for a proof of (5.17), see Proposition A2.1 in the Appendix and Remark 2 after it). Furthermore, setting here $y := x+h,\ u := x,\ v := x-h$ we conclude that for every bounded function f on $[a,b]$, (5.17) implies $f \in \Lambda_\varphi([a,b])$.

Now suppose (5.16) holds. Then by (5.10) any function from Λ_φ belongs to $\mathrm{Lip}\,1$, and hence its partial derivatives exist almost everywhere. If in (5.17) we let $y \to x$ and $v \to u$, we conclude that these partial derivatives are in $\mathrm{Lip}\,\gamma$ and thus $\Lambda_\varphi \subset C^1\,\mathrm{Lip}\,\gamma \subset C^1$.

Conversely, if $\Lambda_\varphi \subset C^1$ then by considering the function $\psi \in \Lambda_\varphi([-1,1])$ defined above we obtain (5.16).

Proposition 5.1.3 is proved.

It is worth noting that for majorants φ satisfying the growth condition (5.16), we have

$$\Lambda_\varphi = C^1\,\mathrm{Lip}\,\gamma \Leftrightarrow t\int_0^t \frac{\varphi(u)}{u^2}\,du \le C\varphi(t), \qquad 0 \le t \le 1. \tag{5.18}$$

In particular, if $\lim_{t\to 0}\varphi(t)/t^2 < \infty$ then $\Lambda_\varphi = C^1\,\mathrm{Lip}\,1$, while the space λ_φ is trivial, i.e., contains only constants and linear functions.

To show (5.18), assume first that the regularity condition in (5.18) is satisfied. Take a function $f \in C^1\,\mathrm{Lip}\,\gamma$ and fix x, h such that $x \pm h \in Q_0$ and $h \neq 0$. The function $g(t) := f(x + th/|h|)$, $t \in [-1,1]$, obviously belongs to $C^1\,\mathrm{Lip}\,\gamma([-1,1])$, hence

$$\begin{aligned}|\Delta_h^2 f(x)| &= \Big|\int_0^{|h|} [g'(x+t) - g'(x-t)]\,dt\Big| \\ &\le M\int_0^{|h|}\int_0^{2t} \frac{\varphi(s)}{s^2}\,ds\,dt \le M|h|\int_0^{2|h|}\frac{\varphi(s)}{s^2}\,ds \le AM\varphi(|h|)\end{aligned}$$

with some constant M, i.e., $f \in \Lambda_\varphi$. Therefore, $C^1\,\mathrm{Lip}\,\gamma \subset \Lambda_\varphi$, which means, owing to Proposition 5.1.3, that $\Lambda_\varphi = C^1\,\mathrm{Lip}\,\gamma$.

Now suppose that these two spaces coincide for some n. Then, in fact, this is true for $n = 1$. Clearly, $\gamma \in \mathrm{Lip}\,\gamma([-1,1])$, hence the function

$$f(t) := \int_0^{|t|} \gamma(s)\,ds = \int_0^{|t|} \frac{\varphi(s)}{s^2}(|t|-s)\,ds, \qquad |t| \le 1,$$

is in $C^1\,\mathrm{Lip}\,\gamma([-1,1])$ and therefore, due to our assumption, in $\Lambda_\varphi([-1,1])$. In particular,

$$f(t) = \tfrac{1}{2}\Delta_t^2 f(0) \le C\varphi(t), \qquad 0 \le t \le 1.$$

Now using (5.3) we have

$$t\int_0^t \frac{\varphi(s)}{s^2}\,ds \le A\int_0^{t/2} \frac{\varphi(s)}{s^2}(t-s)\,ds \le A\int_0^t \frac{\varphi(s)}{s^2}(t-s)\,ds, \quad 0 \le t \le 1,$$

and the required regularity condition in (5.18) follows upon comparing the last three strings of formulas.

5.2. Extensions, approximations, and traces. Collected in Section 5.2 are results of analytic nature that will be crucial in our study of closed ideals and point derivations in Zygmund algebras.

We start with the following simple observation (see e.g. [12]).

PROPOSITION 5.2.1. *Let f be a bounded function on an interval $L = [0, d]$ such that $f(0) = 0$, and let $\widetilde{f}$ be its odd extension to $\widetilde{L} := [-d, d]$. Then*

$$\omega_2(\widetilde{f}; \widetilde{L}; t) \leq 5\omega_2(f; L; t), \qquad t \geq 0.$$

PROOF. When estimating $\Delta_h^2 \widetilde{f}(x)$ we may assume that $x, x + h \in [0, d]$, $h > 0$, and $x - h \in [-d, 0)$. The required inequality follows from the identities

$$\begin{aligned}\Delta_h^2 \widetilde{f}(x) &= f(x+h) - 2f(x) - f(h-x)\\ &= [f(h+x) - 2f(h) + f(h-x)]\\ &\quad - 2[f(h-x) - 2f(h/2) + f(x)] + 2[f(h) - 2f(h/2) + f(0)]\\ &= \Delta_x^2 f(h) - 2\Delta_y^2 f(h/2) + 2\Delta_{h/2}^2 f(h/2)\end{aligned}$$

upon observing that all points involved in the last three second differences belong to $[0, d]$ and that $y := |x - h/2| \leq h/2$.

COROLLARY. *Every bounded function f on an interval $L = [c-d, c+d]$ can be extended to a function $\widetilde{f}$ on $\widetilde{L} := [c - 2d, c + 2d]$ in such a way that*

$$\|\widetilde{f}\|_{\widetilde{L}} \leq A\|f\|_L \quad \text{and} \quad \omega_2(\widetilde{f}; \widetilde{L}) \leq A\omega_2(f; L).$$

PROOF. Set $g(t) := f(c-d+t) - f(c-d)$, $t \in [0, d]$, and let $\widetilde{g}$ be the odd extension of g to $[-d, d]$. Then $\widetilde{f}(x) := f(c-d) + \widetilde{g}(x-c+d)$ is an extension of f to $[c-2d, c+d]$. Applying the same argument to the right end of the interval $[c - d, c + d]$ and using Proposition 5.2.1 we obtain the desired extension of the function f to the interval $[c - 2d, c + 2d]$.

A multivariate analogue of the corollary to Proposition 5.2.1 is contained in

PROPOSITION 5.2.2. *Let $Q = Q(c, d)$ be a cube in $\mathbb{R}^n$ and f be a bounded function on Q. There is a function $\widetilde{f}$ on $\widetilde{Q} := Q(c, 2d)$ such that $\widetilde{f}|Q = f$,*

$$\|\widetilde{f}\|_{\widetilde{Q}} \leq A\|f\|_Q, \quad \text{and} \quad \omega_2(\widetilde{f}; \widetilde{Q}; t) \leq A\omega_2(f; Q; t), \qquad t \geq 0.$$

This is a particular case of Proposition A1.2 in the Appendix.

In the sequel, it will be assumed when necessary that a function $f \in \Lambda_\varphi(Q)$ is extended to a function $\widetilde{f} \in \Lambda_\varphi(\widetilde{Q})$ such that $\|\widetilde{f}\|_{\Lambda_\varphi(\widetilde{Q})} \leq A\|f\|_{\Lambda_\varphi(Q)}$.

In the next proposition, ψ stands for the odd extension of the function defined in (5.13) to the interval $[-1, 1]$.

PROPOSITION 5.2.3. $\|\psi\|_{\Lambda_\varphi([-1,1])} \leq A$.

PROOF. By Proposition 5.2.1, we have to show that $\psi_{\Lambda_\varphi([0,1])} \le A$. Note that for $0 < x \le 1$,

$$x \int_x^3 \frac{\varphi(t)}{t^2}\, dt \le x\varphi(1)\left(\frac{1}{x} - \frac{1}{3}\right) \le 1,$$

therefore $\|\psi\|_{[0,1]} \le 1$.

When estimating $\Delta_h^2 \psi(x)$ we may assume that $h > 0$, which implies $h < x$. According to (5.15),

$$\Delta_h^2 \psi(x) = -2h \int_x^{x+h} \frac{\varphi(t)}{t^2}\, dt + (x-h)\left[\int_{x-h}^{x} \frac{\varphi(t)}{t^2}\, dt - \int_x^{x+h} \frac{\varphi(t)}{t^2}\, dt\right].$$

For the first term, I_1, we have, using property (5.3) of the majorant φ,

$$|I_1| = 2h \int_x^{x+h} \frac{\varphi(t)}{t^2}\, dt \le 2h^2 \frac{\varphi(x)}{x^2} \le 2\varphi(h).$$

Next, by monotonicity of φ,

$$\begin{aligned} I_2 &:= (x-h)\left[\int_{x-h}^{x} \frac{\varphi(t)}{t^2}\, dt - \int_x^{x+h} \frac{\varphi(t)}{t^2}\, dt\right] \\ &\le (x-h)\varphi(x)\left[\left(\frac{1}{x-h} - \frac{1}{x}\right) - \left(\frac{1}{x} - \frac{1}{x+h}\right)\right] \\ &= 2h^2 \frac{\varphi(x)}{x(x+h)} \le 2h^2 \frac{\varphi(x)}{x^2} \le 2\varphi(h). \end{aligned}$$

Thus, $|\psi|_{\Lambda_\varphi([-1,1])} \le 4$, and the proof is complete.

The following extension result is of prime importance for identifying closed primary ideals in Zygmund algebras not imbedded in C^1.

PROPOSITION 5.2.4. *Suppose that* $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$. *For every* $\delta \in (0, 1/2]$, *there is a function* $g_\delta \in \Lambda_\varphi([-1,1])$ *such that* $g_\delta(x) = x,\ |x| \le \delta$, *and*

$$\|g_\delta\|_{\Lambda_\varphi} \to 0 \quad \text{as } \delta \to 0.$$

The proof of Proposition 5.2.4 in full generality will result from Proposition 5.2.9. It is intriguing that for majorants φ satisfying the following extra regularity condition:

$$\int_0^t \frac{\varphi(s)}{s}\, ds \le A\varphi(t), \qquad 0 \le t \le 1, \tag{5.19}$$

the required extension of the identical function can be defined by a simple explicit formula. For $\varphi(t) = t$, another constructively defined extension with the same properties was found in [19]; see also [22].

PROPOSITION 5.2.5. *Suppose that a majorant* φ *satisfies the conditions* (5.19) *and* $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$. *Let* $0 < \delta \le 1/2$ *and* g_δ *be an odd function on* $[-1,1]$, *defined by* $g_\delta(x) := x,\ 0 \le x \le \delta$, *and*

$$g_\delta(x) = A(\delta)\Big[x\int_x^1 \frac{\varphi(t)}{t^2}\,dt + \int_\delta^x \frac{\varphi(t)}{t}\,dt\Big], \quad \delta < x \le 1,$$

where $A(\delta) := 1/\int_\delta^1[\varphi(t)/t^2]\,dt$. *Then*

$$\|g_\delta\|_{\Lambda_\varphi} \le A\cdot A(\delta). \tag{5.20}$$

Proof. We readily see that $g_\delta'(x) = 1$ for $0 \le x \le \delta$ and

$$g_\delta'(x) = A(\delta)\int_x^1 [\varphi(t)/t^2]\,dt$$

for $\delta < x \le 1$. Hence, g_δ is an increasing function, and using (5.19) with $t = 1$ we find that

$$\|g_\delta\|_{[0,1]} = g_\delta(1) = A(\delta)\int_\delta^1 \frac{\varphi(t)}{t}\,dt \le A\cdot A(\delta).$$

Next, due to Proposition 5.2.1, we only need to estimate $|g_\delta|_{\Lambda_\varphi[0,1]}$. Observe that $g_\delta''(x) = 0$ for $0 \le x \le \delta$ and $g_\delta''(x) = -A(\delta)\varphi(x)/x^2$ for $\delta < x \le 1$. For $x \pm h \in [0,1]$, we have

$$|\Delta_h^2 g_\delta(x)| = \Big|\int_0^h\int_{x-t}^{x+t} g_\delta''(s)\,ds\,dt\Big| \le A(\delta)\int_0^h\int_{x-t}^{x+t}\frac{\varphi(s)}{s^2}\,ds\,dt.$$

Since $h \le x$ we obtain, in view of (5.3),

$$\int_0^h\int_x^{x+t}\frac{\varphi(s)}{s^2}\,ds\,dt \le \frac{1}{2}h^2\frac{\varphi(x)}{x^2} \le \frac{1}{2}\varphi(h).$$

It remains to estimate the quantity

$$I := \int_0^h\int_{x-t}^{x}\frac{\varphi(s)}{s^2}\,ds\,dt.$$

We distinguish the following two cases.

Case 1: $x \ge 2h$. Then

$$I \le \frac{1}{2}h^2\frac{\varphi(x-h)}{(x-h)^2} \le \frac{1}{2}\varphi(h).$$

Case 2: $x < 2h$. In this case we use condition (5.19) to find that

$$I = \int_{x-h}^{x}[h-(x-s)]\frac{\varphi(s)}{s^2}\,ds \le \int_0^{2h}\frac{\varphi(s)}{s}\,ds \le A\varphi(2h) \le A\varphi(h).$$

This shows that $|g_\delta|_{\Lambda_\varphi} \le A\cdot A(\delta)$, as required.

Remark. Let φ be an *arbitrary* majorant. For *any* function g on $[-1,1]$ extending the identical function from the interval $[-\delta,\delta]$ we have

$$\|g\|_{\Lambda_\varphi} \ge A\Big/\int_\delta^1\frac{\varphi(t)}{t^2}\,dt.$$

Therefore, the bound in (5.20) is sharp.

To check the above lower estimate, observe that if $g(1) \ge 1/2$ then

$$\|g\|_{\Lambda_\varphi} \ge \|g\|_{[-1,1]} \ge g(1) \ge \frac{1}{2} \ge A\Big/\int_\delta^1 \frac{\varphi(t)}{t^2}\,dt.$$

Now if $g(1) \le 1/2$ then setting $x = u = 0,\ y = \delta$ and $v = 1$ in (5.17), we have

$$\|g\|_{\Lambda_\varphi} \ge |g|_{\Lambda_\varphi} \ge A\left|\frac{g(\delta)-g(0)}{\delta-0} - \frac{g(1)-g(0)}{1-0}\right|\Big/\int_\delta^3 \frac{\varphi(t)}{t^2}\,dt \ge A\Big/\int_\delta^1 \frac{\varphi(t)}{t^2}\,dt.$$

The latter remark shows that a function f from the Zygmund space $\Lambda_\varphi(L)$ on an interval $L \subset [-1,1]$ cannot in general be extended to a function $\widetilde{f}$ in the Zygmund space Λ_φ on the whole interval $[-1,1]$ in such a way that $\|\widetilde{f}\|_{\Lambda_\varphi} \le C\|f\|_{\Lambda_\varphi(L)}$ with a constant C independent of f and L. To have this property, one needs to renorm the space $\Lambda_\varphi(L)$. This is done in Proposition 5.2.9 below. Moreover, the description of the trace space found there applies to *any* closed subset of $[-1,1]$, and the property of extension uniformity mentioned above holds in this more general case as well.

In the next proposition we assume that the function $f \in \Lambda_\varphi(Q_0)$ is extended to the cube $2Q_0$ with Zygmund norm $\le A\|f\|_{\Lambda_\varphi}$. Recall that $E_1(f;Q)$ stands for the best polynomial approximation of order 1 to f on a cube Q.

PROPOSITION 5.2.6. *Let $f \in \Lambda_\varphi(Q_0)$, $x \in Q_0$, and $Q := Q(x,d)$ with $0 < d \le 1/2$. Suppose that for some constant $M \ge 0$,*

$$|\Delta_h^2 f(y)| \le M\varphi(|h|) \quad \text{for all } y,h \text{ with } y \pm h \in Q(x,2d).$$

Let $P \in \mathcal{P}_1$ be a polynomial satisfying $\|f-P\|_Q \le AE_1(f;Q)$. Also, let ω be a C^2-function such that $0 \le \omega \le 1$, $\operatorname{supp}\omega \subset Q$, $\omega \equiv 1$ on $Q(x,d/2)$, and $\|D^\alpha\omega\|_\infty \le Ad^{-|\alpha|}$, $1 \le |\alpha| \le 2$. Then

$$\|(f-P)\omega\|_{\Lambda_\varphi} \le AM.$$

PROOF. Set $g := (f-P)\omega$ and observe that, by (5.1),

$$\|g\|_{Q_0} \le \|f-P\|_Q \le AE_1(f;Q) \le A\omega_2(f;Q;d) \le AM\varphi(d) \le AM.$$

When estimating $\Delta_h^2 g(y)$ for $y \pm h \in Q_0$ we consider the following cases.

CASE 1: $|h| \le d/4,\ |y-x| \le d/4$. Then $|\Delta_h^2 g(y)| = |\Delta_h^2 f(y)| \le M\varphi(|h|)$.

CASE 2: $|h| \le d/4,\ d/4 < |y-x| \le 5d/4$. Applying (5.15) we have

$$\begin{aligned}\Delta_h^2 g(y) &= \Delta_h^2 f(y)\omega(y+h) + 2\Delta_h^1(f-P)(y-h)\Delta_h^1\omega(y) + (f-P)(y-h)\Delta_h^2\omega(y)\\ &= I_1 + I_2 + I_3.\end{aligned}$$

Obviously, $|I_1| \le |\Delta_h^2 f(y)| \le M\varphi(|h|)$. Next, using (5.10), (5.1), and (5.3) we obtain

$$\begin{aligned}|I_2| &= 2|\Delta_h^1(f-P)(y-h)|\cdot|\Delta_h^1\omega(y)| \le A\frac{|h|^2}{d}\left[\int_{\|h\|}^{Ad} \frac{\omega_2(f;Q;d)}{t^2}\,dt + \frac{\|f-P\|_Q}{d}\right]\\ &\le AM\frac{|h|^2}{d}\left[\int_{|h|}^{d} \frac{\varphi(t)}{t^2}dt + \frac{\varphi(d)}{d}\right] \le AM|h|^2\left[\frac{\varphi(|h|)}{|h|^2} + \frac{\varphi(d)}{d^2}\right] \le AM\varphi(|h|).\end{aligned}$$

Finally,

$$|I_3| \le A\|f-P\|_Q|h|^2/d^2 \le AM\varphi(d)|h|^2/d^2 \le AM\varphi(|h|).$$

CASE 3: $|h| \le d/4,\ |y-x| \ge 5d/4$. In this case $\Delta_h^2 g(y) = 0$.

CASE 4: $|h| > d/4$. Then $|\Delta_h^2 g(y)| \le 4\|g\|_{Q_0} \le AM\varphi(d) \le AM\varphi(|h|)$.

Summarizing we conclude that $\|(f-P)\omega\|_{\Lambda_\varphi} \le AM$, as required.

PROPOSITION 5.2.7. *Suppose that* $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$. *Let* F *be a closed subset of* Q_0, *and let* f *be a function in* Λ_φ *such that* $f|F \equiv 0$. *Define*

$$M := \limsup_{d(x,F)\to 0, h\to 0} \frac{|\Delta_h^2 f(x)|}{\varphi(|h|)}.$$

Then

$$\limsup_{d(x,F)\to 0} \frac{|f(x)|}{d(x,F)\int_{d(x,F)}^3 [\varphi(t)/t^2]\,dt} \le AM. \tag{5.21}$$

PROOF. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|\Delta_h^2 f(x)| \le (M+\varepsilon)\varphi(|h|)$ for all $x,\ h$ with $x \pm h \in Q_0,\ d(x,F) \le \delta$, and $|h| \le \delta$. Take a point $x \in Q_0$ such that $0 < d(x,F) \le \delta$, and let a be a point of F with $d(x,F) = \|x-a\|$. Setting $Q := Q(a,\delta)$ and applying inequality (5.11) we have

$$\begin{aligned} |f(x)| &\le Ad(x,F)\Bigg[\int_{d(x,F)}^{\delta} \frac{\omega_2(f;Q;t)}{t^2}\,dt + \frac{\|f\|_Q}{\delta}\Bigg] \\ &\le Ad(x,F)\Bigg[(M+\varepsilon)\int_{d(x,F)}^{3} \frac{\varphi(t)}{t^2}\,dt + \frac{\|f\|_{Q_0}}{\delta}\Bigg] \\ &\le Ad(x,F)\int_{d(x,F)}^{3} \frac{\varphi(t)}{t^2}\,dt\Bigg[M+\varepsilon + \frac{\|f\|_{Q_0}}{\delta\int_{d(x,F)}^3 [\varphi(t)/t^2]\,dt}\Bigg]. \end{aligned}$$

Our conclusion (5.21) now follows from (0) in view of arbitrariness of ε.

The next result is an analogue of Lemma 4 in [45], which deals with the case $n = 2$; compare also with Proposition 2 in [46].

PROPOSITION 5.2.8. *Let* $Q = Q(c,d)$ *be a cube in* $\mathbb{R}^n$ *and* f *be a bounded function on* Q. *Let* $x,y,z \in Q$ *and* $0 < d_1 := |y-x| \le |z-x| := d_2$. *Suppose that* $a_1,\dots,a_{n-1}$ *are points* (*in* Q) *such that* $|a_i - (x+y)/2| = d_1/2$, $1 \le i \le n-1$, *and* $(a_i - x)(a_j - x) = 0$, $1 \le i < j \le n-1$. *Similarly, let points* $b_1,\dots,b_{n-1}$ (*in* Q) *satisfy* $|b_i - (x+z)/2| = d_2/2$, $1 \le i \le n-1$, *and* $(b_i - x)(b_j - x) = 0$, $1 \le i < j \le n-1$. *Denote by* $P_{xy}(f) \in \mathcal{P}_1$ *the polynomial interpolating* f *at the points* $x,y,a_1,\dots,a_{n-1}$ *and by* $P_{xz}(f) \in \mathcal{P}_1$ *the polynomial interpolating* f *at the points* $x,z,b_1,\dots,b_{n-1}$. *Then*

$$\|P_{xy}(f) - P_{xz}(f)\|_Q \le Ad\int_{d_1}^{2d_2} \frac{\omega_2(f;Q;t)}{t^2}\,dt. \tag{5.22}$$

PROOF. Take $r \in \mathbb{N}$ such that $2^{r-1}d_1 \le d_2 < 2^r d_1$, and set $Q_j := Q(x, 2^{j-1}d_1)$, $1 \le j \le r,\ Q_{r+1} := Q(x,d_2)$. For every j, define $P_j := P(f;Q_j) - P(f;Q_j)(x) + f(x)$

(recall that $P(f;S)$ is a polynomial of best uniform approximation of order 1 to f on a set S). We put $\omega(t) := \omega_2(f;Q;t)$ to have, by (5.1),

$$\|P_j - f\|_{Q_j} \le 2\|P(f;Q_j) - f\|_{Q_j} \le A\omega(2^{j-1}d_1), \qquad 1 \le j \le r+1.$$

Next, for every $g \in L_\infty(Q_1)$,

$$P_{xy}(g)(u) = \frac{g(x)+g(y)}{2} + \frac{g(y)-g(x)}{d_1^2}(y-x)\left(u - \frac{x+y}{2}\right) + \frac{4}{d_1^2}\sum_{i=1}^{n-1}\left[g(a_i) - \frac{g(x)+g(y)}{2}\right]\left(a_i - \frac{x+y}{2}\right)\left(u - \frac{x+y}{2}\right). \tag{5.23}$$

Hence,

$$\|P_{xy}(g)\|_{Q_1} \le A\|g\|_{Q_1}, \qquad g \in L_\infty(Q_1). \tag{5.24}$$

Observe now that for every cube $Q(a,d)$ and for each polynomial $P \in \mathcal{P}_1$ such that $P(a) = 0$ we have

$$\|P\|_{Q(a,\lambda d)} = \lambda\|P\|_{Q(a,d)}, \qquad \lambda \ge 0.$$

Therefore, using (5.1) and (5.24) we obtain

$$\begin{aligned}\|P_{xy}(f) - P_1\|_Q &\le \|P_{xy}(f) - P_1\|_{Q(x,2d)} = \frac{2d}{d_1}\|P_{xy}(f) - P_1\|_{Q_1} \\ &= \frac{2d}{d_1}\|P_{xy}(f - P_1)\|_{Q_1} \le A\frac{d}{d_1}\|f - P_1\|_{Q_1} \le Ad\frac{\omega(d_1)}{d_1},\end{aligned}$$

and in like manner

$$\|P_{xz}(f) - P_{r+1}\|_Q \le Ad\frac{\omega(d_2)}{d_2}.$$

Further, for $j = 1, \ldots, r$ we have

$$\begin{aligned}\|P_j - P_{j+1}\|_Q &\le \|P_j - P_{j+1}\|_{Q(x,2d)} \le \frac{Ad}{2^j d_1}\|P_j - P_{j+1}\|_{Q_j} \\ &\le \frac{Ad}{2^j d_1}[\|P_j - f\|_{Q_j} + \|P_{j+1} - f\|_{Q_{j+1}}] \\ &\le \frac{Ad}{2^j d_1}[\omega(2^{j-1}d_1) + \omega(2^j d_1)] \le Ad\frac{\omega(2^j d_1)}{2^j d_1}.\end{aligned}$$

Combining the above estimates we finally get

$$\begin{aligned}\|P_{xy}(f) - P_{xz}(f)\|_Q &\le \|P_{xy}(f) - P_1\|_Q + \sum_{j=1}^{r}\|P_j - P_{j+1}\|_Q + \|P_{xz}(f) - P_{r+1}\|_Q \\ &\le Ad\left[\frac{\omega(d_1)}{d_1} + \sum_{j=1}^{r}\frac{\omega(2^j d_1)}{2^j d_1} + \frac{\omega(d_2)}{d_2}\right] \le Ad\sum_{j=1}^{r}\frac{\omega(2^j d_1)}{2^j d_1} \\ &\le Ad\sum_{j=1}^{r}\int_{2^{j-1}d_1}^{2^j d_1}\frac{\omega(t)}{t^2}\,dt = Ad\int_{d_1}^{2^r d_1}\frac{\omega(t)}{t^2}\,dt \le Ad\int_{d_1}^{2d_2}\frac{\omega(t)}{t^2}\,dt.\end{aligned}$$

The proof of Proposition 5.2.8 is finished.

Corollary. *If* $f \in \Lambda_\varphi$ *then the vector field* $t_{xy} := \nabla P_{xy}(f)$, $x, y \in Q_0$, $x \ne y$, *satisfies the following conditions*:

(a) $t_{xy}(y-x) = f(y) - f(x)$;
(b) $t_{yx} = t_{xy}$;
(c) $|t_{xy}| \le A\|f\|_{\Lambda_\varphi} \int_{|y-x|}^{3} [\varphi(t)/t^2]\,dt$;
(d) *if* $|y-x| \le |z-x|$ *then*

$$|t_{xy} - t_{xz}| \le A|f|_{\Lambda_\varphi} \int\limits_{|y-x|}^{2|z-x|} \frac{\varphi(t)}{t^2}\,dt.$$

Proof. Condition (a) expresses the interpolation property of $P_{xy}(f)$, equality (b) follows from $P_{yx}(f) = P_{xy}(f)$, relation (c) results from (5.23) and (5.12), whereas (d) is derived from (5.22).

Let F be a closed subset in Q_0. We introduce the trace space

$$\Lambda_\varphi(F) := \{f|F : f \in \Lambda_\varphi\},$$

and supply it with the norm

$$\|f\|_{\Lambda_\varphi(F)} := \inf\{\|\widetilde{f}\|_{\Lambda_\varphi} : \widetilde{f}|F = f\}, \qquad f \in \Lambda_\varphi(F),$$

which coincides with the quotient norm on Λ_φ/M_F. In contrast to the Lipschitz space $\operatorname{Lip}\varphi(F)$, "intrinsic" descriptions of the Zygmund spaces $\Lambda_\varphi(F)$ are known only for $n = 1, 2$ (see [45], [46]). They are formulated in Propositions 5.2.9 and 5.2.11 below, while a more general but less constructive description valid for any n is given in Proposition 5.2.10. For proofs of Propositions 5.2.10 and 5.2.11 the reader is referred to the Appendix.

Proposition 5.2.9 ($n = 1$). *Let F be a closed subset of $[-1,1]$, and f be a function on F. Then $f \in \Lambda_\varphi(F)$ iff for some $M \ge 0$ the following conditions are satisfied*:

(i) $|f(x)| \le M$ *for all* $x \in F$;
(ii) *for all* $x, y \in F$, $x \ne y$,

$$\left|\frac{f(y)-f(x)}{y-x}\right| \le M \int\limits_{|y-x|}^{3} \frac{\varphi(t)}{t^2}\,dt;$$

(iii) *for all* $x, y, z \in F$ *such that* $0 < |y-x| \le |z-x|$,

$$\left|\frac{f(y)-f(x)}{y-x} - \frac{f(z)-f(x)}{z-x}\right| \le M \int\limits_{|y-x|}^{2|z-x|} \frac{\varphi(t)}{t^2}\,dt.$$

Also,

$$A_1\|f\|_{\Lambda_\varphi(F)} \le \inf M \le A_2\|f\|_{\Lambda_\varphi(F)}, \qquad f \in \Lambda_\varphi(F), \tag{5.25}$$

with absolute constants A_1 and A_2.

Remark 1. For an individual function $f \in L_\infty(F)$, condition (iii) is necessary and sufficient for $f \in \Lambda_\varphi(F)$, and $\inf M$ in (iii) is equivalent to the seminorm $|f|_{\Lambda_\varphi}$ (cf. [45]). The necessity of condition (iii) is an immediate consequence of (5.22). In fact, for $n = 1$ the interpolation polynomial P_{xy} defined in Proposition 5.2.8 is simply

$$P_{xy}(f)(u) = f(x) + \frac{f(y)-f(x)}{y-x}(u-x),$$

and therefore

$$\|P_{xy}(f)-P_{xz}(f)\|_Q = d\left|\frac{f(y)-f(x)}{y-x}-\frac{f(z)-f(x)}{z-x}\right|.$$

Remark 2. Condition (iii) is equivalent to (5.17) (see Proposition A2.1 in the Appendix and remarks to it).

Remark 3. For $0<\delta\le 1/2$, the identical function on $[-\delta,\delta]$ satisfies conditions (i)–(iii) of Proposition 5.2.9 with $M=\max\{\delta, 1/\int_\delta^3[\varphi(t)/t^2]dt\}$. In view of monotonicity of φ and by (5.4),

$$\int_\delta^3 \frac{\varphi(t)}{t^2}\,dt \ge \varphi(\delta)\left(\frac{1}{\delta}-\frac{1}{3}\right) \ge \frac{A\varphi(\delta)}{\delta} \ge A\delta,$$

hence $M\le A\cdot A(\delta)$ (recall that $A(\delta):=1/\int_\delta^1[\varphi(t)/t^2]\,dt$). The sufficiency part of Proposition 5.2.9, with $F=[-\delta,\delta]$, applied to $f(x)=x,\ |x|\le\delta$, provides a function $g_\delta\in\Lambda_\varphi$ such that $\|g_\delta\|_{\Lambda_\varphi}\le A\cdot A(\delta)$. This justifies Proposition 5.2.4.

Proposition 5.2.10 (arbitrary n). *Let F be a closed subset of $[-1,1]^n$, and f be a function on F. Then $f\in\Lambda_\varphi(F)$ iff for some $M\ge 0$ the following conditions are satisfied*:

(i) $|f(x)|\le M$ *for all* $x\in F$;

(ii) *for every pair of points $x,y\in F$, $x\ne y$, there exists a vector $t_{xy}\in\mathbb{R}^n$ such that*

(a) $t_{xy}(y-x)=f(y)-f(x)$;

(b) $t_{yx}=t_{xy}$;

(c) $|t_{xy}|\le M\int_{|y-x|}^3[\varphi(t)/t^2]\,dt$;

(d) *if* $0<|y-x|\le|z-x|$ *then*

$$|t_{xy}-t_{xz}|\le M\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt.$$

Also,

$$A_1\|f\|_{\Lambda_\varphi(F)}\le \inf M\le A_2\|f\|_{\Lambda_\varphi(F)},\qquad f\in\Lambda_\varphi(F),$$

where the constants A_1 and A_2 may depend only on n.

A "seminormed" version of Proposition 5.2.10 was stated in a slightly different form in [46], Proposition 2.

For $n=1$, Proposition 5.2.9 is derived as an immediate consequence of Proposition 5.2.10. In the case $n=2$, an additional effort is needed (see Section 3 of the Appendix) to obtain the following constructive description of the space $\Lambda_\varphi(F)$.

Proposition 5.2.11 ($n=2$). *Let F be a closed subset of $[-1,1]^2$ not lying in a line, and let f be a function on F. In order that $f\in\Lambda_\varphi(F)$ it is necessary and sufficient that for some $M\ge0$ the following conditions are satisfied*:

(i) $|f(x)|\le M$ *for all* $x\in F$;

(ii) *for any two triples of points $u=\{x,y,z\}$ and $u'=\{x',y',z'\}$ in F, neither belonging to a line, and such that $0<|y-x|\le|z-x|$, $0<|y'-x'|\le|z'-x'|$, and*

$|y-x| \le |y'-x'|$, we have

$$|t_u - t_{u'}| \le M\Bigg[\frac{1}{\sin\alpha_u}\int\limits_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt + \frac{1}{\sin\alpha_{u'}}\int\limits_{|y'-x'|}^{2|z'-x'|}\frac{\varphi(t)}{t^2}\,dt + \int\limits_{|y-x|}^{2|y'-x'|+\max\{|x'-x|,|y'-y|\}}\frac{\varphi(t)}{t^2}\,dt\Bigg],$$

where

$$t_u := \frac{f(y)-f(x)}{\|y-x\|}e_{xy} + \frac{1}{\sin\alpha_u}\left[\frac{f(y)-f(x)}{\|y-x\|} - \frac{f(z)-f(x)}{\|z-x\|}\cos\alpha_u\right]e_{xy}^{\perp},$$

$e_{xy} := (y-x)/\|y-x\|$, α_u is the angle between the vectors e_{xy} and e_{xz}, and $e^{\perp}$ stands for the vector orthogonal to e such that the basis $(e, e^{\perp})$ has standard orientation.

Also,

$$A_1(F)\|f\|_{\Lambda_\varphi(F)} \le \inf M \le A_2\|f\|_{\Lambda_\varphi(F)}, \qquad f \in \Lambda_\varphi(F). \tag{5.26}$$

Here, $A_1(F) \le AS(F)/d(F)$, where $d(F)$ is the Euclidean diameter of F, $S(F)$ is the area of the convex hull of F, and A_2 is an absolute constant.

Remark. In fact, Proposition 5.2.11 holds true for *any* cube Q in $\mathbb{R}^2$ in place of $[-1,1]^n$ but the constant $A_1(F)$ in (5.26) will depend also on $\varphi(d_Q)$. For an individual function $f \in L_\infty(F)$, condition (ii) is necessary and sufficient for $f \in \Lambda_\varphi(F)$ and $\inf M$ in (ii) is equivalent to $|f|_{\Lambda_\varphi}$ with *absolute* constants ([45]; see also the proof of Proposition 5.2.11 in Section 3 of the Appendix).

5.3. Closed primary ideals. We start with showing that relation (4.7) holds in the Zygmund algebra Λ_φ with any majorant φ (even if Λ_φ is not a D-algebra) except for the "critical" case when $\varphi(t)$ is equivalent to t^2.

Proposition 5.3.1. *Suppose that $\lim_{t\to 0}\varphi(t)/t^2 = \infty$. Then for every closed set $F \subset Q_0$,*

$$J_F = \mathrm{clos}_{\Lambda_\varphi} M_F^2.$$

Proof. The right-hand side of this equality is a closed ideal in Λ_φ with cospectrum F and hence contains J_F. To prove the converse, it suffices to show that $f^2 \in J_F$ for every $f \in M_F$.

For $\delta > 0$, let ω_δ be a C^2-function such that $0 \le \omega_\delta \le 1$, $\omega_\delta \equiv 1$ on F_δ, ω_δ vanishes outside $F_{2\delta}$, and $\|D^\alpha\omega_\delta\|_\infty \le A\delta^{-|\alpha|}$, $1 \le |\alpha| \le 2$ (such a function ω_δ can be obtained by regularization of the characteristic function of the set $F_{3\delta/2}$).

We claim that if $f \in M_F$ then $\|f^2\omega_\delta\|_{\Lambda_\varphi} \to 0$ as $\delta \to 0$. In view of $f^2 - f^2\omega_\delta \in J_F$ this will lead us to the required inclusion $\mathrm{clos}_{\Lambda_\varphi} M_F^2 \subset J_F$.

To prove our claim, note first that $\|f^2\omega_\delta\|_{Q_0} \le \|f\|_{F_{2\delta}}^2 \to 0$ as $\delta \to 0$. Next, for $x \pm h \in Q_0$, define $\Delta(x,h) := |\Delta_h^2(f^2\omega_\delta)(x)|$. To estimate this quantity, consider the following cases.

CASE 1: $\|h\| \le \delta/2$, $d(x, F) \le \delta/2$. Using (5.11) and (5.7) we have

$$\Delta(x,h) = |\Delta_h^2 f^2(x)| \le |\Delta_h^2 f(x)| \cdot [|f(x+h)| + |f(x-h)|] + 2|\Delta_h^1 f(x-h)| \cdot |\Delta_h^1 f(x)|$$
$$\le A|f|_{\Lambda_\varphi}\|f\|_{F_\delta}\varphi(|h|) + A|h|^2\Big[\int_{|h|}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big]^2.$$

CASE 2: $\|h\| \le \delta/2$, $\delta/2 < d(x,F) \le 5\delta/2$. In this case, by (5.15),

$$\Delta(x,h) \le |\Delta_h^2 f^2(x)|\omega_\delta(x+h) + 2|\Delta_h^1 f^2(x-h)| \cdot |\Delta_h^1 \omega_\delta(x)|$$
$$+ |f^2(x-h)| \cdot |\Delta_h^2 \omega_\delta(x)| = I_1 + I_2 + I_3.$$

The first item, I_1, can be estimated as in Case 1 (with $F_{3\delta}$ in place of F_δ). Further, we apply (5.10), (5.11), and (5.6.1) to obtain a bound for I_2 as follows:

$$|I_2| \le A\frac{|h|}{\delta}|\Delta_h^1 f(x-h)| \cdot [|f(x-h)| + |f(x)|]$$
$$\le A|h|^2\Big[\int_{|h|}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big]\Big[\int_{\delta}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big] \le A|h|^2\Big[\int_{|h|}^1 \frac{\varphi(t)}{t^2}dt + \|f\|_{Q_0}\Big]^2.$$

Finally, combining (5.11) and (5.6.1) we have

$$I_3 \le A|h|^2\Big[\int_{|h|}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big]^2.$$

CASE 3: $\|h\| \le \delta/2$, $d(x,F) > 5\delta/2$. Then $\Delta(x,h) = 0$.

CASE 4: $\|h\| > \delta/2$. Taking account of (5.11) and (5.6.1) we find that

$$\Delta(x,h) \le A\|f^2\|_{F_{2\delta}} \le A\delta^2\Big[\int_{2\delta}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big]^2 \le A|h|^2\Big[\int_{|h|}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big]^2.$$

Summarizing the above estimates we see that

$$\Delta(x,h) \le A|f|_{\Lambda_\varphi}\|f\|_{F_{3\delta}}\varphi(|h|) + A|h|^2\Big[\int_{|h|}^1 \frac{\varphi(t)}{t^2}\,dt + \|f\|_{Q_0}\Big]^2.$$

Therefore, due to $\lim_{t\to 0}\varphi(t)/t^2 = \infty$ and in view of (5.6.5),

$$|f^2\omega_\delta|_{\Lambda_\varphi} = \sup\Big\{\frac{|\Delta(x,h)|}{\varphi(|h|)} : x \pm h \in Q_0\Big\} \to 0 \quad \text{as } \delta \to 0.$$

COROLLARY 1. *For every* $x \in Q_0$, $J_x = \operatorname{clos}_{\Lambda_\varphi} M_x^2$.

COROLLARY 2. *A closed linear subspace* I *of* Λ_φ *is a closed primary ideal in* Λ_φ *if and only if* $J_x \subset I \subset M_x$.

PROOF. This is true for any commutative semisimple regular unital Banach algebra $\mathcal{A}$ in which $J_x = \operatorname{clos}_{\mathcal{A}} M_x^2$. In fact, we have to show the "only if" part of the statement. Suppose $J_x \subset I \subset M_x$; we check that I is an ideal. Let $f \in I$ and $g \in \Lambda_\varphi$. Then

$$fg = f[g - g(x)] + g(x)f \in M_x^2 + I \subset J_x + I = I.$$

It follows from Corollary 2 that in order to describe all closed primary ideals of the algebra Λ_φ we have to characterize minimal closed primary ideals J_x. Zygmund functions can be locally approximated by linear functions. Accordingly, we start with the following assertion.

PROPOSITION 5.3.2. *If* $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$ *then the identical function* $h(x) = x$ *belongs to the minimal closed primary ideal* J_0 *of the algebra* $\Lambda_\varphi([-1,1])$.

PROOF. Proposition 5.2.4 provides an extension g_δ of the identical function from $[-\delta,\delta]$ to $[-1,1]$. Note that $h - g_\delta$ vanishes in the δ-neighborhood of 0 and that $\|g_\delta\|_{\Lambda_\varphi} \to 0$ as $\delta \to 0$. Therefore, $h \in J_0$.

REMARK. If $\int_0^1 [\varphi(t)/t^2]\,dt < \infty$ then by Proposition 5.1.3, $\Lambda_\varphi \subset C^1$, hence for every function $f \in J_0$ we have $f'(0) = 0$. Therefore, in the case in question the identical function *does not* belong to J_0.

PROPOSITION 5.3.3. (a) *Suppose that* $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$. *Then the minimal closed primary ideal of the algebra* Λ_φ *at a point* $x \in Q_0$ *has the form*

$$J_x = \left\{ f \in \Lambda_\varphi : f(x) = 0,\ \lim_{y \to x, h \to 0} \frac{\Delta_h^2 f(y)}{\varphi(|h|)} = 0 \right\}. \tag{5.27.1}$$

(b) *If* $\int_0^1 [\varphi(t)/t^2]\,dt < \infty$ *then*

$$J_x = \left\{ f \in \Lambda_\varphi : f(x) = 0,\ \nabla f(x) = 0,\ \lim_{y \to x, h \to 0} \frac{\Delta_h^2 f(y)}{\varphi(|h|)} = 0 \right\}. \tag{5.27.2}$$

PROOF. The right-hand side of (5.27.1) is a closed linear subspace in Λ_φ, which we denote by H_x. The inclusion $J_x \subset H_x$ is obvious. To show the converse, take $f \in H_x$ and fix $\varepsilon > 0$. There exists $d \in (0, 1/2]$ such that

$$|\Delta_h^2 f(y)| \le \varepsilon \varphi(|h|) \qquad \text{for all } y,\ h \text{ with } y \pm h \in Q := Q(x, 2d). \tag{5.28}$$

Set $P := P(f;Q) - P(f;Q)(x)$; then $\|f - P\|_Q \le 2E_1(f;Q)$. Applying Proposition 5.2.6 we obtain for the function $g := (f - P)\omega$ the following estimate: $\|g\|_{\Lambda_\varphi} \le A\varepsilon$ (for the definion of ω, also see Proposition 5.2.6).

The polynomial P has the form $P(y) = \sum_{i=1}^n c_i(y_i - x_i)$. It follows from Proposition 5.3.2 that for every $i = 1, \dots, n$ the function $y \mapsto y_i - x_i$ belongs to the ideal J_x. Therefore, $P \in J_x$. From the representation $f = g + (f - f\omega) + P\omega$, where the last two terms are in J_x, and the norm of the first is not greater than $A\varepsilon$, we derive that $f \in J_x$.

To prove formula (5.27.2), it again suffices to show that its right-hand side is contained in J_x. Take $\varepsilon > 0$ and choose $d \in (0, 1]$ such that

$$|f(y)| \le \varepsilon \quad \text{and} \quad \frac{|f(y) - f(z)|}{\|y - z\|} \le \varepsilon \quad \text{for all} \quad y, z \in Q := Q(x, d),\ y \ne z,$$

and (5.28) holds. Therefore, for all $u, v \in Q,\ u \ne v$, by (5.23) and (5.12) we have

$$|\nabla P_{uv}(f)| \le A\varepsilon \le A\varepsilon \int_{|v-u|}^{3} \frac{\varphi(t)}{t^2}\,dt,$$

where $P_{uv}(f)$ is the interpolating polynomial defined in Proposition 5.2.8. Applying the Corollary to Proposition 5.2.8 we see that for $M = A\varepsilon$ all conditions of Proposition 5.2.10 are satisfied. Consequently, there exists a function $g \in \Lambda_\varphi$ such that $g|Q = f|Q$ and $\|g\|_{\Lambda_\varphi} \le A\varepsilon$. Since $g - f \in J_x$, we conclude that $f \in J_x$.

REMARK 1. For $f \in \Lambda_\varphi$, set

$$N_x(f) := \max\left\{|f(x)|, \limsup_{y\to x, h\to 0} \frac{|\Delta_h^2 f(y)|}{\varphi(|h|)}\right\}$$

if $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$, and

$$N_x(f) := \max\left\{|f(x)|, |\nabla f(x)|, \limsup_{y\to x, h\to 0} \frac{|\Delta_h^2 f(y)|}{\varphi(|h|)}\right\}$$

if $\int_0^1 [\varphi(t)/t^2]\, dt < \infty$. If follows from Proposition 5.3.2 that N_x determines a norm on Λ_φ/J_x, and an obvious modification of its proof leads us to the conclusion that this norm is equivalent to the quotient norm $\|\cdot\|_x$ on Λ_φ/J_x:

$$N_x \le \|\cdot\|_x \le AN_x.$$

REMARK 2. Exactly as in Proposition 4.2, we derive from (5.27.2) that in the case $\int_0^1 [\varphi(t)/t^2]\, dt < \infty$ the Zygmund algebra Λ_φ *is not* a D-algebra.

5.4. Point derivations. For $y \pm h \in Q_0$ and $h \ne 0$, we set

$$\psi_{y,h}(f) := \frac{\Delta_h^2 f(y)}{\varphi(|h|)}, \qquad f \in \Lambda_\varphi.$$

It is easily seen that $\psi_{y,h}$ is a linear functional on Λ_φ with norm ≤ 1. For $x \in Q_0$, denote by Ψ_x the set of all weak* limits of functionals $\psi_{y,h}$ as $y \to x$ and $h \to 0$. It follows from (5.15), (5.10), and (5.6.5) that in the case $\lim_{t\to 0} \varphi(t)/t^2 = \infty$ every such limit is a point derivation of Λ_φ at the point x. Thus, $\Psi_x \subset \mathcal{D}_x$. The structure of the space $\mathcal{D}_x$ of all point derivations of the Zygmund algebra Λ_φ at $x \in Q_0$ is described in the following statement.

PROPOSITION 5.4.1. *Suppose that* $\lim_{t\to 0} \varphi(t)/t^2 = \infty$.

(a) *If* $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$ *then for every* $x \in Q_0$,

$$\mathcal{D}_x = V(\Psi_x),$$

where $V(\ldots)$ *stands for the weak* closure of the linear span.*

(b) *If* $\int_0^1 [\varphi(t)/t^2]\, dt < \infty$ *then*

$$\mathcal{D}_x = V(\Psi_x \cup \{D^\eta : \eta \in S^{n-1}\}),$$

where D^η *is the directional derivative along a vector* η *and* S^{n-1} *is the unit sphere of* $\mathbb{R}^n$.

PROOF. In case (a) we only need to check the inclusion $\mathcal{D}_x \subset V(\Psi_x)$. To this end, we will show that ${}^\perp\Psi_x \subset {}^\perp\mathcal{D}_x$, where ${}^\perp L \subset \Lambda_\varphi$ is the annihilator of a linear subspace L in Λ_φ^*.

Suppose the converse; then there is a function $f \in {}^\perp\Psi_x \setminus {}^\perp\mathcal{D}_x$. In fact, we may assume that $f(x) = 0$. Recall that in every commutative Banach algebra with unity, $\mathcal{D}_x = (M_x^2 \cup \mathbf{1})^\perp$. We use Corollary 1 to Proposition 5.3.1 to obtain ${}^\perp\mathcal{D}_x = \{t\mathbf{1}\}_{t\in\mathbb{R}} + J_x$, see (4.17). Therefore, $f \notin J_x$. By Proposition 5.3.3(a) there exist $\varepsilon > 0$ and sequences $\{x_m\} \subset Q_0$ and $\{h_m\} \subset \mathbb{R}^n \setminus \{0\}$ with the properties

$$x_m \pm h_m \in Q_0, \quad |x_m - x| \le \frac{1}{m}, \quad |h_m| \le \frac{1}{m}, \quad \frac{|\Delta_{h_m}^2 f(x_m)|}{\varphi(|h_m|)} \ge \varepsilon, \qquad m \in \mathbb{N}.$$

Put $\psi_m := \psi_{x_m,h_m}$, $m \in \mathbb{N}$. Since $\|\psi_m\| \le 1$ for all m and in view of compactness of the closed unit ball of the space Λ_φ^* in the weak* topology, we conclude that the set $\{\psi_m : m \in \mathbb{N}\}$ has a cluster point ψ. By the definition of Ψ_x we have $\psi \in \Psi_x$, hence $\psi \in \mathcal{D}_x$. But $|\psi_m(f)| \ge \varepsilon$ implies $|\psi(f)| \ge \varepsilon$, which means that $f \notin {}^\perp\Psi_x$. The contradiction with the choice of f shows that ${}^\perp\Psi_x \subset {}^\perp\mathcal{D}_x$. This completes the proof of (a).

In case (b), the proof is essentially the same with reference to part (b) of Proposition 5.3.3.

An alternative approach to describing the subset Ψ_x of the set of point derivations in Λ_φ is based on making use of the Stone–Čech compactification of the topological space

$$\widetilde{Q}_0 := \{(x,h) : x \in Q_0,\ h \in \mathbb{R}^n \setminus \{0\},\ x \pm h \in Q_0\},$$

which will be denoted by K. For $x \in Q_0$, let K_x be the set of all limits in K of nets $\{(x_\alpha, h_\alpha)\} \subset \widetilde{Q}_0$ such that $x_\alpha \to x$ and $h_\alpha \to 0$.

Proposition 5.4.2. $K \setminus \widetilde{Q}_0 = \bigcup\{K_x : x \in Q_0\}$, *where all sets* K_x *are nonempty and disjoint.*

Proof. Let $\xi \in K \setminus \widetilde{Q}_0$. Since $\widetilde{Q}_0$ is dense in K, there is a net $\{(x_\alpha, h_\alpha)\} \subset \widetilde{Q}_0$ such that $(x_\alpha, h_\alpha) \to \xi$. Passing if necessary to a subnet we may assume that $x_\alpha \to x$ for some $x \in Q_0$. Then $h_\alpha \to 0$. For, if not, then for some subnet $\beta = \alpha_\beta$ we would have $h_\beta \to h \ne 0$, and along with $x_\beta \to x$ this would lead us to the conclusion that $\xi = \lim_\beta (x_\beta,\ h_\beta) = (x,h) \in \widetilde{Q}_0$, which contradicts the choice of ξ. Thus, $\xi \in K_x$, and therefore, $K \setminus \widetilde{Q}_0 = \bigcup\{K_x : x \in Q_0\}$.

To show that K_x is nonempty, pick for every $m \in \mathbb{N}$ points $x_m \in Q_0$ and $h_m \in \mathbb{R}^n \setminus \{0\}$ so that $x_m \pm h_m \in Q_0$, $|x_m - x| \le 1/m$, and $|h_m| \le 1/m$. It follows from the compactness of K that there is a subnet $\{\alpha\}$ of $\mathbb{N}$ such that $(x_\alpha, h_\alpha) \to \xi$ for some $\xi \in K$. Clearly, $\xi \in K_x$, and hence $K_x \ne \emptyset$.

Now let x and y be two distinct points of Q_0. Suppose that K_x and K_y contain a common element ξ. Then there are nets $\{(x_\alpha, h_\alpha)\}$ and $\{(y_\beta, e_\beta)\}$ in $\widetilde{Q}_0$ with the properties $(x_\alpha, h_\alpha) \to \xi$ as $x_\alpha \to x$, $h_\alpha \to 0$, and $(y_\beta, e_\beta) \to \xi$ as $y_\beta \to y$, $e_\beta \to 0$. Set

$$g(z,h) := \frac{|z-x|}{|y-x|}, \qquad (z,h) \in \widetilde{Q}_0.$$

The function g is continuous and bounded on $\widetilde{Q}_0$, hence it can be extended to a (unique) continuous function $\widetilde{g}$ on K. We have $g(x_\alpha, h_\alpha) \to 0$, therefore $\widetilde{g}(\xi) = 0$. On the other hand, $g(y_\beta, e_\beta) \to 1$, hence $\widetilde{g}(\xi) = 1$. This contradiction shows that $K_x \cap K_y = \emptyset$.

Proposition 5.4.2 is proved.

With every function $f \in \Lambda_\varphi(Q_0)$, we associate a (uniquely defined) function $\widetilde{f} \in C(K)$ which is the extension to K of the mapping

$$(y,h) \mapsto \frac{\Delta_h^2 f(y)}{\varphi(|h|)}, \qquad (y,h) \in \widetilde{Q}_0. \tag{5.29}$$

For $\xi \in K_x$, define the functional $\theta_\xi(f) := \widetilde{f}(\xi)$. A standard topological argument leads us to

Proposition 5.4.3. $\Psi_x = \{\theta_\xi : \xi \in K_x\}$, $x \in Q_0$.

We will show next that for the Zygmund algebras Λ_φ with φ subject to condition (0), the subspaces $\mathcal{D}_x$ for different x cannot be "too close" (compare with Proposition 4.1).

PROPOSITION 5.4.4. *Suppose that* $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$. *Let* D_1 *and* D_2 *be point derivations of the algebra* Λ_φ *at two distinct points* x_1 *and* x_2 *in* Q_0, *respectively. Then* $\|D_1\| + \|D_2\| \le A\|D_1 + D_2\|$.

PROOF. Take $\delta \in (0, 1/2)$ such that the cubes $Q_i := Q(x_i, 2\delta)$, $i = 1, 2$, are disjoint. Given $\varepsilon > 0$, choose functions $f_i \in \Lambda_\varphi(2Q_0)$ with $\|f_i\|_{\Lambda_\varphi} \le 1$, $D_i f_i \ge \|D_i\| - \varepsilon$, and set $P_i := P(f_i; Q_i)$, $i = 1, 2$. Proposition 5.2.6 with $M = 1$ provides us with smooth functions ω_i which equal 1 in a neighborhood of x_i and vanish outside Q_i and are such that for the functions $g_i := (f_i - P_i)\omega_i$ we have $\|g_i\|_{\Lambda_\varphi} \le A$, $i = 1, 2$. By Proposition 5.3.2 any linear function $\sum_{j=1}^n c_j(y_j - x_j)$ belongs to J_x. Therefore, $g_i - f_i \in \{t\mathbf{1}\}_{t\in\mathbb{R}} + J_{x_i}$, whence $D_i g_i = D_i f_i$, $i = 1, 2$. Observe also that $D_1 g_2 = D_2 g_1 = 0$. Setting $g := g_1 + g_2$ we finally have

$$\begin{aligned}\|D_1\| + \|D_2\| - 2\varepsilon &\le D_1 f_1 + D_2 f_2 = D_1 g_1 + D_2 g_2 \\ &= (D_1 + D_2)g \le \|D_1 + D_2\|\,\|g\|_{\Lambda_\varphi} \le A\|D_1 + D_2\|,\end{aligned}$$

and by letting $\varepsilon \to 0$ we obtain the required inequality.

A number of important properties of Zygmund algebras can be expressed in terms of point derivations, as exemplified by the following three results whose counterparts for Lipschitz algebras are presented in [42]. In the first statement, $\mathcal{D}$ stands for the set of all point derivations of the algebra Λ_φ.

PROPOSITION 5.4.5. *Suppose that* $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$. *Then*

$$\lambda_\varphi = {}^\perp\mathcal{D} := \{f \in \Lambda_\varphi : Df = 0 \textit{ for all } D \in \mathcal{D}\}.$$

PROOF. This is an immediate consequence of Proposition 5.4.1(a) and of the definition (5.5) of the "small" Zygmund space.

COROLLARY. $(\Lambda_\varphi/\lambda_\varphi)^*$ *is canonically isometrically isomorphic to* $\mathcal{D}$.

The following proposition provides a convenient criterion of weak convergence of sequences in the space Λ_φ.

PROPOSITION 5.4.6. *Suppose that* $\lim_{t\to 0} \varphi(t)/t^2 = \infty$. *A sequence of functions* $\{f_m\}_{m\in\mathbb{N}} \subset \Lambda_\varphi$ *converges to* $f \in \Lambda_\varphi$ *in the weak topology iff it satisfies the following conditions*:

(i) *the set* $\{f_m : m \in \mathbb{N}\}$ *is bounded in* Λ_φ;
(ii) $f_m(x) \to f(x)$ *as* $m \to \infty$ *for every* $x \in Q_0$;
(iii) $Df_m \to Df$ *as* $m \to \infty$ *for every* $D \in \mathcal{D}$.

PROOF. A proof is required for the sufficiency part only. Suppose conditions (i)–(iii) are satisfied. Denote by X the disjoint union $Q_0 \cup K$, where K is defined above as the Stone–Čech compactification of $\widetilde{Q}_0$. Introduce a mapping $\pi : \Lambda_\varphi \to C(X)$ by setting $(\pi f)|Q_0 := f$ and $(\pi f)|K := \widetilde{f}$, $f \in \Lambda_\varphi$, where $\widetilde{f}$ is the extension to K of the function (5.29). Indeed, π is an isometry.

Take a functional $\eta \in \Lambda_\varphi^*$. Then $\eta \circ \pi^{-1}$ is a bounded linear functional on the linear subspace $\pi(\Lambda_\varphi) \subset C(X)$. By the Hahn–Banach theorem it can be extended to a bounded linear functional on $C(X)$. Hence there exist finite Borel measures $\mu \in M(Q_0)$ and $\nu \in M(K)$ such that

$$\eta(g) = \int_{Q_0} g d\mu + \int_K \widetilde{g}\, d\nu, \qquad g \in \Lambda_\varphi.$$

If $(x,h) \in \widetilde{Q}_0$ then, due to condition (ii),

$$\widetilde{f}_m(x,h) = \frac{\Delta_h^2 f_m(x)}{\varphi(|h|)} \to \frac{\Delta_h^2 f(x)}{\varphi(|h|)} = \widetilde{f}(x,h) \qquad \text{as } m \to \infty.$$

Further, if $\xi \in K \setminus \widetilde{Q}_0$ then, by Proposition 5.4.3 and owing to (iii),

$$\widetilde{f}_m(\xi) = \theta_\xi(f_m) \to \theta_\xi(f) = \widetilde{f}(\xi) \qquad \text{as } m \to \infty.$$

In view of condition (i) we can use the Lebesgue theorem on dominated convergence to have, for $m \to \infty$,

$$\eta(f_m) \to \int_{Q_0} f\, d\mu + \int_K \widetilde{f}\, d\nu = \eta(f).$$

Therefore, $f_m \to f$ weakly in Λ_φ as $m \to \infty$, and Proposition 5.4.6 follows.

Our last application concerns the correspondence between the set $\mathcal{I}_x$ of all closed primary ideals of the algebra Λ_φ at a point $x \in Q_0$ and the set $\mathcal{L}_x$ of all weak* closed linear subspaces of $\mathcal{D}_x$. The proof of the following statement is based on the formula ${}^\perp\mathcal{D}_x = \{t1\}_{t\in\mathbb{R}} + J_x$.

Proposition 5.4.7. *Suppose that* $\lim_{t\to 0} \varphi(t)/t^2 = \infty$. *The mapping*

$$I \mapsto L_I := \mathcal{D}_x \cap I^\perp, \qquad I \in \mathcal{I}_x,$$

establishes a one-to-one correspondence between $\mathcal{I}_x$ *and* $\mathcal{L}_x$. *The reverse correspondence is given by the mapping*

$$L \mapsto I_L := M_x \cap^\perp L, \quad L \in \mathcal{L}_x.$$

Corollary. *If* I *is an ideal in* Λ_φ *then its primary component at a point* $x \in \sigma(I)$ *coincides with the ideal* I_L *for* $L = \mathcal{D}_x \cap I^\perp$.

5.5. An extension property and spectral synthesis. We formulate the extension property for Zygmund spaces as follows:

(Ext) For every closed subset $F \subset Q_0$, for each function $f \in \Lambda_\varphi$ vanishing on F, and for every $\varepsilon > 0$, there exist $\varrho > 0$ and a function $g \in \Lambda_\varphi$ such that $g|F_\varrho \equiv f|F_\varrho$ and $\|g\|_{\Lambda_\varphi} \le C[N_F(f) + \varepsilon]$, where

$$N_F(f) := \limsup_{d(x,F)\to 0,\, h\to 0} \frac{|\Delta_h^2 f(x)|}{\varphi(|h|)},$$

and C is a positive constant independent of f, ε, and ϱ.

Proposition 5.5.1. *Suppose that* $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$. *Then* Λ_φ *is a D-algebra iff it has property* (Ext).

PROOF. *Sufficiency.* Suppose Λ_φ satisfies condition (Ext). Take any function $f \in \Lambda_\varphi$ vanishing on a closed set $F \subset Q_0$. By the definition of $N_F(f)$, there are sequences $\{x_m\} \subset F$ and $\{h_m\} \subset \mathbb{R}^n \setminus \{0\}$, $m \in \mathbb{N}$, such that $d(x_m, F) \to 0$, $h_m \to 0$, and $|\Delta^2_{h_m} f(x_m)|/\varphi(|h_m|) \to N_F(f)$ as $m \to \infty$. We may assume, in fact, that $x_m \to x$ as $m \to \infty$ for some point $x \in F$. For every m, the mapping $f \mapsto \Delta^2_{h_m} f(x_m)/\varphi(|h_m|)$ defines a bounded linear functional ψ_m on Λ_φ with $\|\psi_m\| \le 1$. Hence, there is a subnet $\{m_\alpha\}$ of $\mathbb{N}$ and a functional $\psi \in \Lambda_\varphi^*$ with $\|\psi\| \le 1$ such that $\psi_m \to \psi$ in the weak* topology on Λ_φ^*. It follows from our assumption (0) that $\lim_{t\to 0} \varphi(t)/t^2 = \infty$. We argue as at the beginning of Section 5.4 to find that ψ is a point derivation of Λ_φ at the point x. Therefore (see (4.3) and (4.4)),

$$N_F(f) = \lim_\alpha |\psi_{m_\alpha}(f)| = |\psi(f)| = |\widehat{f}(\psi)| \le \|\widehat{f}\|_{K_F}.$$

Let g be the function provided for a given $\varepsilon > 0$ by (Ext). Since $g - f \in J_F$, we have $\|\dot{f}\|_F \le \|g\|_{\Lambda_\varphi} \le C[N_F(f) + \varepsilon]$. Hence, due to arbitrariness of ε,

$$\|\dot{f}\|_F \le C N_F(f) \le C\|\widehat{f}\|_{K_F},$$

where C may depend only on F, n, and φ. Thus, Λ_φ is a D-algebra.

Necessity. Suppose Λ_φ is a D-algebra. By Remark 1 to Proposition 5.3.3 we have, for every $x \in F$, $D \in \mathcal{D}_x$ with $\|D\| \le 1$, and $f \in M_F$,

$$|Df| \le \|\dot{f}\|_x \le AN_x(f) \le AN_F(f),$$

hence $\|\widehat{f}\|_{K_F} \le AN_F(f)$. Combining this with the definition (4.6) of a D-algebra we see that $\|\dot{f}\|_F \le CN_F(f)$, where $C := A \cdot A(F)$. This shows that for every $\varepsilon > 0$ there exists a function $k \in \Lambda_\varphi$ vanishing in a ϱ-neighborhood of the set F and such that $\|f + k\|_{\Lambda_\varphi} \le C[N_F(f) + \varepsilon]$. Therefore, $g := f + k$ is the function required in (Ext).

Proposition 5.5.1 is proved.

PROPOSITION 5.5.2. *Let* $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$. *Suppose that* (Ext) *holds. Then* $\lambda_\varphi \in$ synt.

PROOF. To establish property (1.3) for the algebra λ_φ we have to show that $M_F \subset J_F$ for every closed subset F of Q_0. Take $f \in M_F$ and fix $\varepsilon > 0$. We may think of f as being extended to a function in $\lambda_\varphi(2Q_0)$ with norm $\le A\|f\|_{\Lambda_\varphi}$ (see Proposition 5.2.2 for substantiation). Preserving the notation f for this function, we have $N_F(f) = 0$. Hence by (Ext) there is a number $\varrho > 0$ and a function $g \in \Lambda_\varphi(2Q_0)$ such that $g|F_\varrho \equiv f|F_\varrho$ and $\|g\|_{\Lambda_\varphi(2Q_0)} \le \varepsilon$. For the function $k := g - f$, we have $k|F_\varrho \equiv 0$ and $\|f - k\|_{\Lambda_\varphi(2Q_0)} \le \varepsilon$.

Let w be a C^∞-function with the properties $w \ge 0$, $\operatorname{supp} w \subset Q_0$, and $\int_{\mathbb{R}^n} w(x)\, dx = 1$. Set $w_\delta(x) := \delta^{-n} w(x/\delta)$, $\delta \in (0, 1]$. For $u \in \Lambda_\varphi(2Q_0)$, consider the mollified functions

$$u_\delta(x) := \int_{\mathbb{R}^n} u(x - t) w_\delta(t)\, dt.$$

It is easy to see that $u_\delta \in C^\infty(Q_0) \subset \lambda_\varphi(Q_0)$ and $\|u_\delta\|_{\Lambda_\varphi} \le \|u\|_{\Lambda_\varphi}$.

Since $f \in \lambda_\varphi(2Q_0)$, there exists $\sigma \in (0, 1]$ such that $|\Delta^2_h f(x)| \le \varepsilon\varphi(|h|)$ for all $x \in 2Q_0$ and $|h| \le \sigma$ with $x \pm h \in 2Q_0$. Therefore, for *any* $\delta \in (0, 1]$,

$$|\Delta^2_h (f - f_\delta)(x)| \le 2\varepsilon\varphi(|h|), \quad x \in Q_0,\ |h| \le \sigma. \tag{5.30}$$

Further, it follows from the continuity of f that there is $\tau \in (0,1]$ such that for all $\delta \in (0,\tau]$ we have $\|f - f_\delta\|_{Q_0} \le \varepsilon\varphi(\sigma)$. Hence, for *any* $|h| > \sigma$ and $x \in Q_0$,

$$|\Delta_h^2(f - f_\delta)(x)| \le 4\|f - f_\delta\|_{Q_0} \le 4\varepsilon\varphi(\sigma) \le 4\varepsilon\varphi(|h|), \qquad \delta \in (0,\tau]. \tag{5.31}$$

Combining (5.30) and (5.31) we see that $\|f - f_\delta\|_{\Lambda_\varphi} \le 4\varepsilon$ whenever $\delta \in (0,\tau]$ and therefore, for such δ,

$$\|f - k_\delta\|_{\Lambda_\varphi} \le \|f - f_\delta\|_{\Lambda_\varphi} + \|f_\delta - k_\delta\|_{\Lambda_\varphi} \le 4\varepsilon + \|f - k\|_{\Lambda_\varphi} \le 5\varepsilon.$$

Observe finally that if $0 < \delta < \min\{\varrho, \tau\}$ then the function k_δ vanishes in a neighborhood of the set F. Therefore, $f \in J_F$, which completes the proof.

We are now in a position to state the main results of Section 5.

THEOREM 5.1. *For $n = 1, 2$ and for any majorant φ such that $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$, the algebra Λ_φ possesses property* (Ext).

The proof of Theorem 5.1 will be given in Section 5.6.

Our next result is a combination of Theorem 4.1 with Theorem 5.1, Proposition 5.5.1, and Proposition 5.5.2.

THEOREM 5.2. *Suppose that $n = 1, 2$ and $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$. Then $\Lambda_\varphi \in$* Synt *and $\lambda_\varphi \in$* synt.

As another corollary to Theorem 5.1 we obtain the following solution to the spectral approximation problem for Zygmund algebras not imbedded in C^1.

THEOREM 5.3. *Suppose that $n = 1, 2$ and $\int_0^1 [\varphi(t)/t^2]\,dt = \infty$. Then for any closed subset F of Q_0, the quotient norm on M_F/J_F is equivalent to N_F*:

$$N_F(f) \le \|\dot{f}\|_F \le C N_F(f), \qquad f \in M_F,$$

where C is the constant specified in (Ext).

In particular,

$$J_F = \left\{ f \in \Lambda_\varphi : f|F \equiv 0 \text{ and } \lim_{d(x,F)\to 0, h\to 0} \frac{\Delta_h^2 f(x)}{\varphi(|h|)} = 0 \right\}.$$

It is shown in [11] that if a majorant φ satisfies along with (0) the following two extra regularity conditions:

$$\int_0^t \frac{\varphi(s)}{s}\,ds \le C\varphi(t), \qquad 0 \le t \le 1,$$

$$t^2 \int_t^1 \frac{\varphi(s)}{s^3}\,ds \le C\varphi(t), \qquad 0 < t \le 1,$$

then Theorems 5.1–5.3 are true also for $n > 2$.

5.6. Proof of Theorem 5.1. First, we are going to check (Ext) for $n = 1$ proceeding from Proposition 5.2.9. Then we will combine the constructive part of our univariate argument with Proposition 5.2.11 to obtain (Ext) in the case $n = 2$.

5.6.1. $n = 1$. Let F be a closed subset of $[-1, 1]$ and f be a function in Λ_φ vanishing on F. By the definition of the number $N_F(f)$ and in view of Proposition 5.2.7, given $\varepsilon > 0$ one can find $\delta \in (0, 1]$ such that $\|f\|_{F_\delta} \le \varepsilon$,

$$|\Delta_h^2 f(x)| \le [N_F(f) + \varepsilon]\varphi(|h|) \quad \text{for } d(x, F) \le 2\delta,\ |h| \le \delta, \tag{5.32}$$

and

$$|f(x)| \le A[N_F(f) + \varepsilon] d(x, F) \int_{d(x,F)}^{3} \frac{\varphi(t)}{t^2}\, dt \quad \text{for } d(x, F) \le 2\delta. \tag{5.33}$$

Since $\int_0^1 [\varphi(t)/t^2]\, dt = \infty$, there is $\tau \in (0, \delta]$ for which

$$\int_{2\delta}^{3} \frac{\varphi(t)}{t^2}\, dt \le \int_{\tau}^{2\delta} \frac{\varphi(t)}{t^2}\, dt. \tag{5.34}$$

By (5.6.3) one can pick a $\varrho \in (0, \tau]$ such that

$$\varrho \int_{\varrho}^{3} \frac{\varphi(t)}{t^2}\, dt \le \tau \int_{\tau}^{2\tau} \frac{\varphi(t)}{t^2}\, dt. \tag{5.35}$$

We claim that the number ϱ thus defined is the one required in (Ext). To show this, we need to check conditions (i)–(iii) of Proposition 5.2.9 with $F = F_\varrho$ and with a constant $M \le AC$, where $C := N_F(f) + \varepsilon$.

Estimate (i) is trivially satisfied.

To prove (ii), consider the following two cases.

Case 1: $|y - x| \le \delta$. Let L be an interval of length δ such that $x, y \in L \subset [-1, 1]$. Since $\varrho \le \delta$, we use (5.33) and (5.6.1) to obtain

$$\|f\|_L \le AC\delta \int_{2\delta}^{3} \frac{\varphi(t)}{t^2}\, dt. \tag{5.36}$$

Now invoking inequality (5.10) and taking account of (5.32) and (5.36) we have

$$\begin{aligned}\left|\frac{f(y) - f(x)}{y - x}\right| &\le A\left[\int_{|y-x|}^{\delta} \frac{\omega_2(f; L; t)}{t^2}\, dt + \frac{\|f\|_L}{\delta}\right] \\ &\le AC\left[\int_{|y-x|}^{\delta} \frac{\varphi(t)}{t^2}\, dt + \int_{\delta}^{3} \frac{\varphi(t)}{t^2}\, dt\right] = AC \int_{|y-x|}^{3} \frac{\varphi(t)}{t^2}\, dt.\end{aligned}$$

Case 2: $|y - x| > \delta$. In this case, using (5.33) and (5.6.1) we argue as follows:

$$|f(y) - f(x)| \le |f(y)| + |f(x)| \le AC\varrho \int_{\varrho}^{3} \frac{\varphi(t)}{t^2}\, dt \le AC|y - x| \int_{|y-x|}^{3} \frac{\varphi(t)}{t^2}\, dt.$$

Checking condition (iii) splits into the following three cases.

Case 1: $|z-x| \le \delta$. Let L be an interval of length $\le 2\delta$ such that $x, y, z \in L \subset [-1,1]$. We apply Proposition 5.2.8 and (5.22) (see also Remark 1 to Proposition 5.2.9) to obtain

$$\left|\frac{f(z)-f(x)}{z-x} - \frac{f(y)-f(x)}{y-x}\right| \le AC \int_{|y-x|}^{2|z-x|} \frac{\varphi(t)}{t^2}\,dt.$$

Case 2: $|z-x| > \delta$, $|y-x| > \tau$. Using consecutively (5.33), (5.6.1), (5.35), and finally (5.6.2) we have

$$\begin{aligned}|f(y)-f(x)| &\le |f(y)|+|f(x)| \le AC\varrho\int_{\varrho}^{3}\frac{\varphi(t)}{t^2}\,dt \le AC\tau\int_{\tau}^{2\tau}\frac{\varphi(t)}{t^2}\,dt \\ &\le AC|y-x|\int_{|y-x|}^{2|y-x|}\frac{\varphi(t)}{t^2}\,dt \le AC|y-x|\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt,\end{aligned}$$

and similarly,

$$|f(z)-f(x)| \le AC|z-x|\int_{|z-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt \le AC|z-x|\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt,$$

which implies (iii).

Case 3: $|z-x| > \delta$, $|y-x| \le \tau$. Using condition (ii) established above and (5.34) we find that

$$\begin{aligned}\left|\frac{f(y)-f(x)}{y-x}\right| &\le AC\int_{|y-x|}^{3}\frac{\varphi(t)}{t^2}\,dt = AC\left[\int_{|y-x|}^{2\delta}\frac{\varphi(t)}{t^2}\,dt + \int_{2\delta}^{3}\frac{\varphi(t)}{t^2}\,dt\right] \\ &\le AC\left[\int_{|y-x|}^{2\delta}\frac{\varphi(t)}{t^2}\,dt + \int_{\tau}^{2\delta}\frac{\varphi(t)}{t^2}\,dt\right] \\ &\le AC\int_{|y-x|}^{2\delta}\frac{\varphi(t)}{t^2}\,dt \le AC\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt.\end{aligned} \tag{5.37}$$

Also, as shown in Case 2,

$$\left|\frac{f(z)-f(x)}{z-x}\right| \le AC\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt.$$

Thus, conditions (i)–(iii) of Proposition 5.2.9 are satisfied, and property (Ext) for $n=1$ follows.

5.6.2. $n=2$. Let F be a subset of $[-1,1]^2$ not lying in a line and f be a function in Λ_φ vanishing on F. Set $A(F) := d(F)/S(F)$. For a given $\varepsilon > 0$, we choose $\delta > 0$ for which $\|f\|_{F_{2\delta}} \le \varepsilon$,

$$|\Delta_h^2 f(x)| \le [N_F(f)+\varepsilon]\varphi(|h|) \quad \text{for } d(x,F) \le 4\delta,\ |h| \le 2\delta, \tag{5.38}$$

and condition (5.33) is met (all distances are taken in the norm $|\cdot|$). Next, we pick

$\tau \in (0, \delta]$ subject to (5.34) and $\varrho \in (0, \tau]$ satisfying (5.35) so that

$$\frac{d(F_\varrho)}{S(F_\varrho)} \le 2A(F). \tag{5.39}$$

We only have to show that, for the function f and the set F_ϱ, condition (ii) of Proposition 5.2.11 is satisfied with a constant $\le AC$, where $C := N_F(f) + \varepsilon$. For, if so, then applying the sufficiency part of Proposition 5.2.11, (5.26), and (5.39) we will find a function $g \in \Lambda_\varphi$ extending f from F_ϱ and such that $\|g\|_{\Lambda_\varphi} \le A \cdot A(F)[N_F(f) + \varepsilon]$, as required in (Ext).

Let $u = (x, y, z)$ and $u' = (x', y', z')$ be two triples of points in F_ϱ, neither lying in a line, and such that $0 < |y - x| \le |z - x|$, $0 < |y' - x'| \le |z' - x'|$, and $|y - x| \le |y' - x'|$. We will show that

$$|t_u - t_{u'}| \le AC\left[\frac{1}{\sin\alpha}\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt + \frac{1}{\sin\alpha'}\int_{|y'-x'|}^{2|z'-x'|}\frac{\varphi(t)}{t^2}\,dt + \int_{|y-x|}^{2|y'-x'|+d}\frac{\varphi(t)}{t^2}\,dt\right], \tag{5.40}$$

where $\alpha := \alpha_u$, $\alpha' := \alpha_{u'}$, $d := \max\{|x' - x|, |y' - y|\}$, and

$$t_u := \frac{f(y) - f(x)}{\|y - x\|}e_{xy} + \frac{1}{\sin\alpha}\left[\frac{f(z) - f(x)}{\|z - x\|} - \frac{f(y) - f(x)}{\|y - x\|}\cos\alpha\right]e_{xy}^{\perp}; \tag{5.41}$$

see Proposition 5.2.11 for further notation.

Denote by R the right-hand side of (5.40). When proving (5.40) we consider the following cases.

Case 1: $|z - x| \le \delta$, $|z' - x'| \le \delta$, and $|x' - x| \le 2\delta$. In this case $\{x, y, z\}, \{x', y', z'\} \subset Q(x, 3\delta)$. Applying to the cube $Q(x, 3\delta) \subset F_{4\delta}$ the sufficiency part of Proposition 5.2.11 and the Remark to it, and taking account of (5.38) we obtain (5.40).

Case 2: $|z - x| \le \delta$, $|z' - x'| \le \delta$, and $|x' - x| > 2\delta$. If $|y - x| > \tau$ then, as shown in the case $n = 1$,

$$\frac{|f(y) - f(x)|}{\|y - x\|} + \frac{|f(z) - f(x)|}{\|z - x\|} \le AC\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt. \tag{5.42}$$

Therefore (see (5.41)), $|t_u| \le R$. Since $|y' - x'| \ge |y - x|$, the same argument yields $|t_{u'}| \le R$, and (5.40) follows.

Suppose now that $|y - x| \le \tau$. Take a point $\widehat{y}$ such that $\widehat{y} - x$ is orthogonal to $z - x$ and $|\widehat{y} - x| = |y - x|$. As already shown in Case 1,

$$\begin{aligned} |t_{xyz} - t_{x\widehat{y}z}| &\le AC\left[\frac{1}{\sin\alpha}\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt + \int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt + \int_{|y-x|}^{4|y-x|}\frac{\varphi(t)}{t^2}\,dt\right] \\ &\le \frac{AC}{\sin\alpha}\int_{|y-x|}^{2|z-x|}\frac{\varphi(t)}{t^2}\,dt \le R. \end{aligned} \tag{5.43}$$

Further, recalling that $|y - x| \ge \tau$ and $d \ge 2\delta$ we have, owing to (5.41), (5.33), and (5.37),

$$|t_{x\widehat{y}z}| \le \frac{|f(\widehat{y})-f(x)|}{\|\widehat{y}-x\|} + \frac{|f(z)-f(x)|}{\|y-x\|} \le AC\int_{|y-x|}^{3} \frac{\varphi(t)}{t^2}\,dt \tag{5.44}$$
$$\le AC\int_{|y-x|}^{2\delta} \frac{\varphi(t)}{t^2}\,dt \le AC\int_{|y-x|}^{2|y'-x'|+d} \frac{\varphi(t)}{t^2}dt \le R.$$

From (5.43) and (5.44) we conclude that $|t_u| \le R$. Now, depending on whether $|y'-x'|$ is $\le \tau$ or $> \tau$ we apply to u' either (5.43)–(5.44) or (5.42) to have in both cases $|t_{u'}| \le R$. Thus, in Case 2, (5.40) holds.

Case 3: $|z-x| \le \delta,\ |z'-x'| > \delta$. If $|y-x| > \tau$ then, as shown in (5.42), $|t_u| \le R$. Let now $|y-x| \le \tau$. Following (5.44) we have

$$|t_{x\widehat{y}z}| \le AC\int_{|y-x|}^{2\delta} \frac{\varphi(t)}{t^2}\,dt = AC\left[\int_{|y-x|}^{|y'-x'|} \frac{\varphi(t)}{t^2}\,dt + \int_{|y'-x'|}^{2\delta} \frac{\varphi(t)}{t^2}\,dt\right]$$
$$\le AC\left[\int_{|y-x|}^{2|y'-x'|+d} \frac{\varphi(t)}{t^2}\,dt + \int_{|y'-x'|}^{2|z'-x'|} \frac{\varphi(t)}{t^2}\,dt\right] \le R,$$

and together with (5.43) this yields $|t_u| \le R$.

In like manner, if $|y'-x'| > \tau$ then we know already that $|t_{u'}| \le R$, while in the case $|y'-x'| \le \tau$ the same conclusion can be drawn by applying (5.43) to the triple u' and upon observing that, according to (5.44),

$$|t_{x'\widehat{y}'z'}| \le AC\int_{|y'-x'|}^{2\delta} \frac{\varphi(t)}{t^2}\,dt \le AC\int_{|y'-x'|}^{2|z'-x'|} \frac{\varphi(t)}{t^2}\,dt \le R. \tag{5.45}$$

Case 4: $|z-x| > \delta,\ |z'-x'| \le \delta$. This case is symmetric to Case 3.

Case 5: $|z-x| > \delta,\ |z'-x'| > \delta$. If $|y-x| > \tau$ then $|t_u| \le R$. Now, if $|y-x| \le \tau$ then we obtain the same estimate by combining (5.43) with (5.45) applied to the triple u. In fact, $|t_{u'}| \le R$ is also true.

Thus, in all cases relation (5.40) is valid.

The proof of Theorem 5.1 for $n = 2$ is now complete.

Appendix

The purpose of the Appendix is to provide the reader with complete self-contained proofs of Propositions 5.2.2, 5.2.10, 5.2.11, and of relation (5.17).

We proceed from a very general but nonconstructive description of traces of the generalized Lipschitz spaces $\Lambda^k_\varphi,\ k \ge 2$, determined by differences of order k, on arbitrary closed sets $F \subset Q_0$ (see Theorem A1 in Section 1 below). This description involves polynomials of degree $\le k-1$ associated with closed cubes centered at F. For power majorants, this description appeared in [27], and for general majorants in the "seminormed" setting in [6]. In our proof of the "normed" version we follow the approach of [6] with some simplifications. Proposition 5.2.2 is obtained as a simple corollary to Theorem A1.

In the subsequent discussion, we confine ourselves to the Zygmund spaces Λ_φ ($k=2$). In this case the above collection of cubes can be reduced to that consisting of cubes with a point of the set F on the boundary. A process of such reduction was suggested in [45]. The same can be achieved by means of simple explicit formulas; see our proof of Theorem A1 and also [21]. This leads us to a proof of Proposition 5.2.10. Next, we reformulate condition (ii)(d) of Proposition 5.2.10 (see Proposition A2.1 below) to obtain (5.17). The case of two variables is treated in Section 3 where the method of [45] is used to extend the trace description to the "normed" case with the dependence of the norm of the extension operator on geometric properties of the set $F \subset [-1,1]^2$ required in Proposition 5.2.11.

1. Traces of generalized Lipschitz spaces. We introduce the following notation (compare with Section 5.1).

For an integer k and for a function f on $\mathbb{R}^n$, we define the kth difference at a point x with step h by $\Delta_h^k f(x) := \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x+ih)$. The kth modulus of continuity of f on a cube Q is

$$\omega_k(f;Q;t) := \sup\{|\Delta_h^k f(x)| : x, x+kh \in Q,\ |h| \le t\}, \qquad t \ge 0.$$

Recall that $|x| := \max_{1\le i\le n} |x_i|$ and note that all distances in the sequel, if not stated otherwise, are taken in this norm.

Denote by $\mathcal{P}_{k-1}$ the set of all polynomials in n variables of degree $\le k-1$. For a set F in $\mathbb{R}^n$ and for a bounded function f on F, we define the best uniform polynomial approximation of order $k-1$ by

$$E_{k-1}(f;Q) := \inf\{\|f-P\|_F : P \in \mathcal{P}_{k-1}\}.$$

A polynomial for which this infimum is attained will be denoted by $P_{k-1}(f;F)$. As shown in [5],

$$E_{k-1}(f;Q) \le A\omega_k(f;Q;d_Q) \tag{A1}$$

(the reverse inequality $\omega_k(f;Q;d_Q) \le 2^k E_{k-1}(f;Q)$ trivially holds). Similarly, if Q' is a cube centered at a point of Q then

$$E_1(f;Q'\cap Q) \le A\omega_2(f;Q'\cap Q;d_{Q'}). \tag{A2}$$

We will need two elementary properties of polynomials. The notation Q^t, where $Q = Q(c,d)$ and $t>0$, will be used hereafter for the cube $Q(c,td)$.

Proposition A1.1. *For $P \in \mathcal{P}_m$ and for a cube $Q = Q(c,d)$, we have*

$$\|P\|_{Q^t} \le A(m,n)t^m \|P\|_Q, \qquad t \ge 1. \tag{A3}$$

Also, if $Q' = Q(c',d')$, where $c' \in Q$ and $d' \le \lambda d$, then

$$\|P\|_{Q'} \le A(m,n,\lambda)\|P\|_{Q'\cap Q}. \tag{A4}$$

Proof. We check (A3) by induction on m. For $m=0$ this is trivial. Let now $m \ge 1$ and suppose that (A3) holds for all $P \in \mathcal{P}_{m-1}$. For $P \in \mathcal{P}_m$, we have

$$P(u) = P(c) + \sum_{i=1}^n (u_i - c_i) \int_0^1 \frac{\partial}{\partial x_i} P(c + s(u-c))\, ds.$$

This implies, via the induction hypothesis, that

$$\|P\|_{Q^t} \le |P(c)| + ntd \max_{1\le i\le n} \left\| \frac{\partial}{\partial x_i} P \right\|_{Q^t} \le \|P\|_Q + nA(m-1,n)t^m d \max_{1\le i\le n} \left\| \frac{\partial}{\partial x_i} P \right\|_Q,$$

and (A3) follows by using the Markov inequality $\|\frac{\partial}{\partial x_i} P\|_Q \le md^{-1}\|P\|_Q$.

To show (A4), consider the following two cases.

CASE (a). If $d' > 2d$ then $Q \subset Q' \subset Q(c,(\lambda+1)d)$. Therefore, by (A3),

$$\|P\|_{Q'} \le \|P\|_{Q(c,(\lambda+1)d)} \le A(m,n)(\lambda+1)^m \|P\|_Q = A(m,n,\lambda)\|P\|_{Q'\cap Q}.$$

CASE (b). Let $d' \le 2d$. Then there is a cube $Q'' = Q(c'',d'')$ with vertex c' and with $d'' := \min\{d,d'\}$ such that $Q'' \subset Q' \cap Q$. It is easily seen that $Q' \subset Q(c'',5d'')$, hence, by (A3),

$$\|P\|_{Q'} \le \|P\|_{Q(c'',5d'')} \le A(m,n)\|P\|_{Q''} \le A(m,n)\|P\|_{Q'\cap Q}.$$

Thus, in both cases (A4) holds. This proves Proposition A1.1.

We define the space $\Lambda^k_\varphi := \Lambda^k_\varphi(Q_0)$ as the set of all bounded functions on $Q_0 := [-1,1]^n$ satisfying for some constant $M \ge 0$ and for all admissible x and h the following generalized Lipschitz condition: $|\Delta^k_h f(x)| \le M\varphi(|h|)$. Here φ is any given nondecreasing function on $\mathbb{R}_+$ such that $\varphi(0) = \varphi(0+) = 0$ and $\varphi(t) > 0$ for $t > 0$. Also, we will assume without loss of generality that $\varphi(t) = 1$ for $t \ge 1$ and that the function $\varphi(t)/t^k$ is nonincreasing. The latter implies that

$$\varphi(\lambda t) \le \lambda^k \varphi(t), \qquad \lambda \ge 1.$$

This inequality will be systematically used in the sequel.

The space Λ^k_φ is endowed with the norm $\|f\|_{k,\varphi} := \max\{\|f\|_{Q_0}, |f|_{k,\varphi}\}$, where $|f|_{k,\varphi} :=$ inf M over all constants M involved in the definition of the space Λ^k_φ.

If F is a closed subset of Q_0 then we define in a usual way the trace space $\Lambda^k_\varphi(F) := \{f|F : f \in \Lambda^k_\varphi\}$, and supply it with the norm

$$\|f\|_{k,\varphi;F} := \inf\{\|\widetilde{f}\|_{k,\varphi} : \widetilde{f}|F = f\}, \qquad f \in \Lambda^k_\varphi(F).$$

For a closed set $F \subset Q_0$, we denote by $\mathcal{K}_F$ the collection of all cubes $Q(c,d)$ with $c \in F$ and $d \le A_0$, where A_0 is an absolute constant (we may take, for example, $A_0 = 66$).

Throughout the Appendix, the letter A (with or without an index) stands for a positive constant which may depend only on n and k.

THEOREM A1. *Let F be a closed subset of Q_0 and f be a function on F. Then $f \in \Lambda^k_\varphi(F)$ iff there is a mapping that associates with every cube $Q \in \mathcal{K}_F$ a polynomial $P_Q \in \mathcal{P}_{k-1}$ such that for some constant $M \ge 0$ the following conditions are satisfied*:

(i) *$P_Q(c_Q) = f(c_Q)$ for all $Q \in \mathcal{K}_F$;*
(ii) *$\|P_Q\|_Q \le M$ for all $Q \in \mathcal{K}_F$;*
(iii) *$\|P_{Q'} - P_Q\|_{Q'} \le M\varphi(d_Q)$ for all $Q', Q \in \mathcal{K}_F$ such that $Q' \subset Q$.*

Also,

$$A_1\|f\|_{k,\varphi;F} \le \inf M \le A_2\|f\|_{k,\varphi;F}, \qquad f \in \Lambda^k_\varphi(F).$$

PROOF. *Necessity.* Suppose that $f \in \Lambda^k_\varphi(F)$ and put $C := 2\|f\|_{k,\varphi;F}$. There exists a function $\widetilde{f} \in \Lambda^k_\varphi$ such that $\widetilde{f}|F = f$ and $\|\widetilde{f}\|_{k,\varphi} \le C$. For $Q \in \mathcal{K}_F$, set $P^0_Q := P_{k-1}(\widetilde{f}; Q \cap Q_0)$ and $P_Q := P^0_Q - P^0_Q(c_Q) + f(c_Q)$. We claim that the collection $\{P_Q\}_{Q\in\mathcal{K}_F}$ of polynomials meets conditions (i)–(iii) with $M \le AC$.

First, condition (i) is obviously satisfied. Next, we check (ii). By Proposition A1.1 we have $\|P_Q\|_Q \le A\|P_Q\|_{Q\cap Q_0}$. Combining this with

$$\|P_Q\|_{Q\cap Q_0} \le 2\|P^0_Q\|_{Q\cap Q_0} + |f(c_Q)| \le 2\|P^0_Q - \widetilde{f}\|_{Q\cap Q_0} + 3\|\widetilde{f}\|_{Q\cap Q_0} \le 5\|\widetilde{f}\|_{Q\cap Q_0},$$

we obtain $\|P_Q\|_Q \le AC$. Passing to (iii), take a pair of cubes $Q', Q \in \mathcal{K}_F$ with $Q' \subset Q$. Invoking (A4) and (A2) we have

$$\begin{aligned}
\|P_{Q'} - P_Q\|_{Q'} &\le \|P^0_{Q'} - P^0_Q\|_{Q'} + |P^0_{Q'}(c'_Q) - \widetilde{f}(c_{Q'})| + |P^0_Q(c_Q) - \widetilde{f}(c_Q)| \\
&\le A\|P^0_{Q'} - P^0_Q\|_{Q'\cap Q_0} + \|P^0_{Q'} - \widetilde{f}\|_{Q'\cap Q_0} + \|P^0_Q - \widetilde{f}\|_{Q\cap Q_0} \\
&\le A(\|P^0_{Q'} - \widetilde{f}\|_{Q'\cap Q_0} + \|P^0_Q - \widetilde{f}\|_{Q\cap Q_0}) \\
&\le A(\|P^0_Q - \widetilde{f}\|_{Q'\cap Q_0} + \|P^0_Q - \widetilde{f}\|_{Q\cap Q_0}) \\
&\le A\|P^0_Q - \widetilde{f}\|_{Q\cap Q_0} \le A\omega_k(\widetilde{f}; Q\cap Q_0; d_Q) \le AC\varphi(d_Q).
\end{aligned}$$

Thus, conditions (i)–(iii) are satisfied with a constant $M \le A\|f\|_{k,\varphi;F}$.

Sufficiency. Let W_F be a *Whitney decomposition* of $\mathbb{R}^n \setminus F$ into a family of closed cubes with disjoint interiors. An argument similar to that in [48], Chapter 6, Section 1, shows that the cubes constituting W_F can be chosen to meet the following requirements:

(a) $2d_Q \le \operatorname{dist}(Q,F) \le 8d_Q$ for all $Q \in W_F$;
(b) if $Q_1,\ Q_2 \in W_F$ and $Q_1 \cap Q_2 \ne \emptyset$ then $1/4 \le d_{Q_1}/d_{Q_2} \le 4$;
(c) $Q^t \cap F = \emptyset$ for every $Q \in W_F$ and $0 < t < 3$;
(d) if $Q_1, Q_2 \in W_F$ and $Q_1 \cap Q_2 = \emptyset$ then for $1 < t < 4/3$ we have $Q^t_1 \cap Q^t_2 = \emptyset$;
(e) the covering W_F is locally finite with multiplicity $< 6^n$.

For $Q \in W_F$, define $Q^* := Q^{4/3}$, and construct a smooth partition of unity by C^∞-functions $\{\psi_Q\}_{Q\in W_F}$ such that $\psi_Q \ge 0$, $\operatorname{supp}\psi_Q \subset Q^*$, and $\|D^\alpha\psi_Q\|_\infty \le Ad_Q^{-|\alpha|}$ for $1 \le |\alpha| \le k$.

For $Q \in W_F$, define $\widehat{Q} := Q(a_Q, d_Q)$, where a_Q is a point of F closest to Q (in the norm $|\cdot|$). The *Whitney extension* of the function f is defined by $\widetilde{f}(x) = f(x)$ for $x \in F$ and

$$\widetilde{f}(x) = \sum_{Q\in W_F} P_{\widehat{Q}}(x)\psi_Q(x) \quad \text{for } x \in \mathbb{R}^n \setminus F. \tag{A5}$$

We seek to prove that $\widetilde{f} \in \Lambda^k_\varphi$ and $\|\widetilde{f}\|_{k,\varphi} \le AM$.

First, note that the sum in (A5) may be restricted to the subfamily W'_F of cubes $Q \in W_F$ for which Q^* intersects Q_0, and a simple geometric argument based on property (a) shows that for every such $Q \in W'_F$ we have $d_Q \le 6/5$.

Further, it is easily seen that for every cube $Q \in W_F$, $Q^* \subset \widehat{Q}^{A_1}$ with $A_1 = 31/3$. Hence, taking into account inequality (A3) and condition (ii) we have, for every $Q \in W'_F$,

$$\|P_{\widehat{Q}}\|_{Q^*} \le \|P_{\widehat{Q}}\|_{\widehat{Q}^{A_1}} \le A\|P_{\widehat{Q}}\|_{\widehat{Q}} \le AM.$$

This implies $\|\widetilde{f}\|_{Q_0\setminus F} \le AM$. Besides this, owing to (i) and (ii) we have $\|f\|_F \le M$. Therefore, $\|\widetilde{f}\|_{Q_0} \le AM$.

Now we will show that $|\widetilde{f}|_{k,\varphi} \le AM$. According to (A1) it is sufficient to check that $E_{k-1}(\widetilde{f};K) \le AM\varphi(d_K)$ for every cube $K \subset Q_0$.

Suppose first that $c_K \in F$. For every cube $Q \in W_F$, we have $\operatorname{dist}(Q^*, F) \ge (5/3)d_Q$. Hence, if $Q^* \cap K \ne \emptyset$ then $d_Q \le (3/5)d_K$. Using simple geometric considerations we derive from this that $\widehat{Q} \subset K^{A_2}$ with $A_2 = 39/5$. Now set $\widetilde{K} := K^{A_2}$. We claim that $\|\widetilde{f} - P_{\widetilde{K}}\|_K \le AM\varphi(d_K)$.

For $x \in F \cap K$, we put $Q' = \{x\}$ and $Q = \widetilde{K}$ in (iii) to obtain $|\widetilde{f}(x) - P_{\widetilde{K}}(x)| \le M\varphi(d_{\widetilde{K}}) \le AM\varphi(d_K)$.

Let now $x \in K \setminus F$. We have

$$\widetilde{f}(x) - P_{\widetilde{K}}(x) = \sum_{Q\in W_F'} [P_{\widehat{Q}}(x) - P_{\widetilde{K}}(x)]\psi_Q(x). \tag{A6}$$

For every cube $Q \in W_F'$ with $Q^* \cap K \ne \emptyset$, we find using (A3) and (iii) that

$$\|P_{\widehat{Q}} - P_{\widetilde{K}}\|_{Q^*} \le \|P_{\widehat{Q}} - P_{\widetilde{K}}\|_{\widehat{Q}^{A_1}} \le A\|P_{\widehat{Q}} - P_{\widetilde{K}}\|_{\widehat{Q}} \le AM\varphi(d_K).$$

Therefore, we conclude from (A6) that $\|\widetilde{f} - P_{\widetilde{K}}\|_{K\setminus F} \le AM\varphi(d_K)$. Thus, in the case $c_K \in F$,

$$E_{k-1}(\widetilde{f};K) \le \|\widetilde{f} - P_{\widetilde{K}}\|_K \le AM\varphi(d_K). \tag{A7}$$

Suppose now that $c_K \notin F$ and fix a cube $Q \in W_F'$ containing the point c_K. For the cube Q, we have two possibilities.

Case 1: K is not contained in Q^*. In this case we obviously have $d_K \ge d_Q/3$. Also, using property (a) it is easy to see that $K \subset \widehat{K}^{A_3}$ with $A_3 = 32$. We set $K' := \widehat{K}^{A_3}$ and upon observing that $c_{K'} \in F$ we make use of (A7) to obtain

$$E_{k-1}(\widetilde{f};K) \le E_{k-1}(\widetilde{f};K') \le AM\varphi(d_K') \le AM\varphi(d_K).$$

Case 2: $K \subset Q^*$. For a polynomial $P \in \mathcal{P}_{k-1}$ (that will be chosen later) we put

$$g(x) := \widetilde{f}(x) - P(x) = \sum_{Q'\in W_F'} [P_{\widehat{Q}'}(x) - P(x)]\psi_{Q'}(x), \qquad x \in \mathbb{R}^n \setminus F. \tag{A8}$$

Let

$$T(x) := \sum_{|\alpha|\le k-1} D^\alpha g(c_K)\frac{(x-c_K)^\alpha}{\alpha!}$$

be the Taylor polynomial for the function g of order $k-1$ at the point c_K (note that $\widetilde{f} \in C^\infty(\mathbb{R}^n \setminus F)$). A standard estimate for the Taylor remainder gives

$$\|g - T\|_K \le Ad_K^k \max_{|\alpha|=k} \|D^\alpha g\|_{Q^*}. \tag{A9}$$

Using properties (d) and (e) of the collection W_F we find that for every $x \in \operatorname{int} Q^*$, the sum in (A8) ranges over $< 6^n$ cubes $Q' \in W_F'$ for which $Q' \cap Q \ne \emptyset$. For every such cube Q' and for $|\alpha| = k$, we have, using the Leibniz formula, the Markov inequality, and property (c) of the Whitney cubes,

$$\|D^\alpha[(P_{\widehat{Q}'} - P)\psi_{Q'}]\|_{Q^*} \le A \max_{\beta\le\alpha}\{d_{Q'}^{|\beta|-k}\|D^\beta(P_{\widehat{Q}'} - P)\|_{Q^*}\}$$
$$\le A d_{Q^*}^{-|\beta|} d_{Q'}^{|\beta|-k}\|P_{\widehat{Q}'} - P\|_{Q^*} \le A d_Q^{-k}\|P_{\widehat{Q}'} - P\|_Q.$$

Hence, in view of (A8),

$$\text{(A10)}\qquad \|D^\alpha g\|_{Q^*} \le A d_Q^{-k} \max_{Q'\cap Q\neq\emptyset} \|P_{\widehat{Q}'} - P\|_Q, \qquad |\alpha| = k.$$

Note that for any two cubes $Q, Q' \in W_F$ with $Q \cap Q' \neq \emptyset$ we have $Q \subset (\widehat{Q}')^{A_4}$ and $\widehat{Q}' \subset \widehat{Q}^{A_5}$, where one can take $A_4 = 18$ and $A_5 = 54$. We define $P := P_{\widehat{Q}^{A_5}}$ to obtain, by Proposition A1 and in view of condition (iii),

$$\text{(A11)}\qquad \|P_{\widehat{Q}'} - P\|_Q \le \|P_{\widehat{Q}'} - P\|_{(\widehat{Q}')^{A_4}}$$
$$\le A\|P_{\widehat{Q}'} - P_{\widehat{Q}^{A_5}}\|_{\widehat{Q}'} \le AM\varphi(A_5 d_Q) \le AM\varphi(d_Q).$$

Hence by (A10) we have $\|D^\alpha g\|_{Q^*} \le A d_Q^{-k}\varphi(d_Q)$. Therefore, recalling that $g = \widetilde{f} - P$ and taking account of (A9) we obtain the estimate

$$E_{k-1}(\widetilde{f}; K) \le \|\widetilde{f} - P - T\|_K = \|g - T\|_K \le AM(d_K/d_Q)^k\varphi(d_Q).$$

In our case, $d_K \le d_{Q^*} = (4/3)d_Q$, hence $\varphi(d_Q)/d_Q^k \le A\varphi(d_K)/d_K^k$. Continuing (A11) we finally have

$$\text{(A12)}\qquad E_{k-1}(\widetilde{f}; K) \le AM\varphi(d_K).$$

Observe that for an appropriate choice of the constant A_0 in the definition of the family $\mathcal{K}_F$ all cubes involved in the proof belong to $\mathcal{K}_F$, and therefore (A12) holds for every cube $K \subset Q_0$. Hence, owing to (A1), $|\widetilde{f}|_{k,\varphi} \le AM$.

Thus, $f \in \Lambda_\varphi^k(F)$ and $\|f\|_{k,\varphi;F} \le AM$, as required.

Remark. Theorem A1 is valid for any cube Q in place of Q_0 with $\mathcal{K}_F$ defined to consist of all cubes $Q(c,d)$ with $c \in F$ and $d \le A_0 d_Q$. Furthermore, Theorem A1 is also true for $Q = \mathbb{R}^n$ and thus provides a description of the trace of the space $\Lambda_\varphi^k(\mathbb{R}^n)$ to *any* closed subset F in $\mathbb{R}^n$. If F is bounded then every function $f \in \Lambda_\varphi^k(F)$ can be extended to a function in $\Lambda_\varphi^k(\mathbb{R}^n)$ with compact support.

Proposition A1.2. *Let $Q = Q(c,d)$ be a cube in $\mathbb{R}^n$. Every bounded function f on Q can be extended to a function $\widetilde{f}$ on $\widetilde{Q} := Q(c,\widetilde{d})$ with $\widetilde{d} \le Ad$ such that*

$$\|\widetilde{f}\|_{\widetilde{Q}} \le A\|f\|_Q \quad \textit{and} \quad \omega_k(\widetilde{f}; \widetilde{Q}; t) \le A\omega_k(f; Q; t), \quad t \ge 0.$$

Proof. Consider the space $\Lambda_\varphi^k(Q)$ with $\varphi := \omega_k(f; Q)$. Define $\mathcal{K}_Q$ as in the remark to Theorem A1. For $K \in \mathcal{K}_Q$, let

$$P_K := P_{k-1}(f; K \cap Q) - P_{k-1}(f; K \cap Q)(c_K) + f(c_K).$$

A word-for-word repetition of the proof of the necessity part of Theorem A1 shows that these polynomials satisfy conditions (i), (ii) with a constant $M_1 \le A\|f\|_Q$, and also condition (iii) with a constant $M_2 \le A$. Now applying the sufficiency part of Theorem A1 and observing that the extension $\widetilde{f}$ satisfies $\|\widetilde{f}\|_{\widetilde{Q}} \le AM_1$ and $|\widetilde{f}|_{k,\varphi} \le AM_2$ we obtain the required conclusion.

Proposition 5.2.2 is a particular case of Proposition A1.2 with $k = 2$.

2. Traces of Zygmund spaces. In the case $k = 2$, a more constructive version of Theorem A1 is possible. This is Proposition 5.2.10, which we are going to prove now.

PROOF OF PROPOSITION 5.2.10. *Necessity.* Set $M := \|f\|_{k,\varphi}$. Then, for such M, condition (i) is obviously satisfied. For $x, y \in F,\ x \neq y$, define $t_{xy} := \nabla P_{xy}$, where P_{xy} is the interpolation polynomial described in Proposition 5.2.8. According to the Corollary to that proposition, conditions (a)–(d) in (ii) are also met.

Sufficiency. Suppose a function f on F meets conditions (i) and (ii). By Theorem A1 we need to show that for every cube $Q = Q(c,d) \in \mathcal{K}_F$ there exists a polynomial $P_Q(u) = f(c) + t_Q(u - c)$ that satisfies conditions (ii) and (iii) of Theorem A1.

We introduce a few geometric characteristics of the mutual position of the sets F and Q. Fix a point $a_Q \in Q \cap F$, put $r_Q := \max_{x\in Q\cap F} |x - a_Q|$, and denote by x_Q a point in $Q \cap F$ for which $|x_Q - a_Q| = r_Q$. Clearly,

$$\tfrac{1}{2}\operatorname{diam}(Q\cap F) \le r_Q \le \operatorname{diam}(Q \cap F) \le 2. \tag{A13.1}$$

If F is not contained in Q then define also $\varrho_Q := \max\{r_Q, \inf_{y\in F\setminus Q} |y - a_Q|\}$, and denote by y_Q a point in F for which $\varrho_Q = |y_Q - a_Q|$. Observe that

$$r_Q \le \varrho_Q \le \operatorname{diam} F \le 2. \tag{A13.2}$$

We define the required vector field $t_Q,\ Q \in \mathcal{K}_F$, as follows:

CASE 1. If $d = 0$ then $t_Q := 0$.

CASE 2. If $d \neq 0$ but $Q \cap F = \{c\}$ then $t_Q := t_{cy_Q}$.

CASE 3. If $Q \cap F \neq \{c\}$ and $Q \cap F \neq F$ then $t_Q := \theta_Q t_{a_Q x_Q} + (1 - \theta_Q)t_{a_Q y_Q}$, where $\theta_Q := \int_{d_Q}^{2\varrho_Q} [\varphi(t)/t^2]\,dt / \int_{r_Q}^{2\varrho_Q} [\varphi(t)/t^2]\,dt$.

CASE 4. If $F \subset Q$ then $t_Q := \gamma_Q t_{a_Q x_Q}$, where $\gamma_Q := \int_{d_Q}^{3} [\varphi(t)/t^2]\,dt / \int_{r_Q}^{3} [\varphi(t)/t^2]\,dt$.

In the sequel, the magnitudes defined above will be written without the index Q, while those related to a cube Q' will be marked with the symbol "$'$".

For the sake of simplicity we will use the notation $\Phi(a,b) := \int_a^b [\varphi(t)/t^2]\,dt$. Note the following obvious properties of the function Φ, for our further reference:

$$\Phi(a,b) \le a^{-1}\phi(b), \qquad 0 < a \le b; \tag{A14.1}$$

$$|\Phi(a,b)| \le \Phi(\min\{a,b\}, \max\{a,b\}); \tag{A14.2}$$

$$\text{if } 0 < a_2 \le a_1 \le b_1 \le b_2 \text{ then } \Phi(a_1,b_1) \le \Phi(a_2,b_2). \tag{A14.3}$$

We will check condition (ii) of Theorem A1. Since $\|P\|_Q \le |f(c)| + Ad|t|$ and $|f(c)| \le M$, we only have to show that $d|t| \le AM$.

In Case 1, $t = 0$. Next, in Case 2, $r = 0$ and $\varrho = |y - c| \ge d$, hence by (ii)(c) and (A14.1) we have $d|t| \le Md\Phi(|y - c|, 3) \le M$.

Now consider Case 3. Note that in this case $\varrho \ge d/3$. For, if $\operatorname{diam}(F \cap Q) < 2d/3$ then for all $z \in F \setminus Q,\ |z - a| \ge |z - c| - |c - a| \ge d - 2d/3 = d/3$. Otherwise, i.e. for $\operatorname{diam}(F \cap Q) \ge 2d/3$, by (A13.1) and (A13.2) we again have $\varrho \ge \operatorname{diam}(F \cap Q)/2 \ge d/3$. Taking into account condition (ii)(c), (A14.1), and the above definitions of $t := t_Q$ we find that, in Case 3,

$$d|t| \le d(|t_{ay}| + |\theta| \cdot |t_{ax} - t_{ay}|) \le Md[\Phi(\varrho, 3) + |\Phi(d, 2\varrho)|]$$
$$\le M[d/\varrho + d\Phi(2d/3, 4)] \le AM.$$

Finally, in Case 4, by (A14.1) we have $d|t| \le Md|\Phi(d,3)| \le AM$.

We now pass to checking condition (iii) of Theorem A1. We write

$$(P_Q - P_{Q'})(u) = [f(c') - f(c) - t(c' - c)] + (t' - t)(u - c'),$$

therefore

$$\|P_Q - P_{Q'}\|_{Q'} \le |f(c') - f(c) - t(c' - c)| + Ad'|t' - t| := I_1 + AI_2.$$

Our first step consists in estimating $I_1 := |f(c') - f(c) - t(c' - c)|$.

In Cases 1 and 2, $c' = c$, hence $I_1 = 0$. While considering cases 3 and 4 one may assume that $c' \ne c$. We claim that

$$\text{(A15)} \qquad |c' - c| \cdot |t_{cc'} - t_{ax}| \le AM\varphi(d).$$

To show this, observe that in view of (A13.1), $|x - a| = r \ge |c' - c|/2$. Hence if $|x - c| \ge |c' - c|/4$ then due to (ii)(b), (ii)(d), and to relations (A14),

$$|t_{cc'} - t_{ax}| \le |t_{cc'} - t_{cx}| + |t_{cx} - t_{ax}| \le M\Phi(|c' - c|/4, 2d) \le AM\frac{\varphi(d)}{|c' - c|}.$$

Else, we have $|a - c| \ge |c' - c|/4$, and for similar reasons

$$|t_{cc'} - t_{ax}| \le |t_{cc'} - t_{ca}| + |t_{ac} - t_{ax}| \le AM\frac{\varphi(d)}{|c' - c|},$$

which completes the proof of (A15).

By (ii)(a) we have $f(c') - f(c) - t(c' - c) = (t_{cc'} - t)(c' - c)$, therefore in Case 3 we make use of (ii)(b), (ii)(d), (A14), and (A15) to find that

$$I_1 \le A|t_{cc'} - t| \cdot |c' - c| \le A|c' - c| \cdot [|t_{cc'} - t_{ax}| + |1 - \theta| \cdot |t_{ax} - t_{ay}|]$$
$$\le AM[\varphi(d) + |c' - c| \cdot |\Phi(r, d)|] \le AM[\varphi(d) + |c' - c|\Phi(|c' - c|/2, 2d)] \le AM\varphi(d).$$

In like manner, in Case 4,

$$I_1 \le A|c' - c| \cdot [|t_{cc'} - t_{ax}| + |1 - \gamma| \cdot |t_{ax}|] \le AM[\varphi(d) + |c' - c| \cdot |\Phi(r, d)|] \le AM\varphi(d).$$

When estimating $I_2 := d'|t' - t|$ we may assume that $d' > 0$. If $F \cap Q = \{c\}$ then $t = t'$, i.e. $I_2 = 0$. In the case $F \cap Q' = \{c'\}$ we have

$$d' \le |y' - c'| \le \operatorname{diam}(F \cap Q) \le 2|x - a|,$$

hence repeating the above argument with y' in place of c we obtain the estimate

$$I_2 = d'|t' - t| \le |y' - c'| \cdot |t_{y'c'} - t| \le AM\varphi(d).$$

Thus we are left with the combinations of Cases 3 and 4 for the cubes Q and Q'.

If both of them fall into Case 3 and $F \cap Q' = F \cap Q$ then $a' = a$, $x' = x$, $y' = y$, $r' = r$, and $\varrho' = \varrho$. Therefore, recalling our definitions of t_Q and invoking (ii)(d) and (A13.1) we have

$$I_2 = d'|\theta' - \theta| \cdot |t_{ax} - t_{ay}| \le Md'\Phi(d', d) \le M\varphi(d).$$

Next, if $F \cap Q' \neq F \cap Q$ then

$$|t' - t| \leq |\theta'| \cdot |t_{a'x'} - t_{a'y'}| + |t_{a'y'} - t_{ax}| + |1-\theta| \cdot |t_{ax} - t_{ay}| := I_{21} + I_{22} + I_{23}.$$

Since $d'/3 \leq \varrho' \leq 2d$, one has, using relations (A14),

$$I_{21} \leq M|\Phi(d', 2\varrho')| \leq M\Phi(2d'/3, 2d) \leq AM\varphi(d)/d'.$$

Further, in the case under study there is a point $z \in (F \cap Q) \backslash Q'$, hence $2r \geq \operatorname{diam}(F \cap Q) \geq |z - c'| \geq d'$. Therefore,

$$I_{23} \leq M|\Phi(r, d)| \leq M|\Phi(d'/2, 2d)| \leq AM\varphi(d)/d'. \tag{A16}$$

To estimate I_{22}, observe that $r = |x - a| \geq d'/2$, hence either $|x - a'| \geq d'/4$, in which case upon recalling that $|y' - a'| = \varrho' \geq d'/3$ we have

$$I_{22} \leq |t_{a'y'} - t_{a'x}| + |t_{xa'} - t_{xa}| \leq 2M\Phi(d'/4, 2d) \leq AM\varphi(d)/d',$$

or $|a - a'| \geq d'/4$, and in the latter case we get in quite a similar way the same estimate. Thus, $|t' - t| \leq AM\varphi(d)/d'$, whence $I_2 \leq AM\varphi(d)$.

Suppose now that $F \cap Q'$ pertains to Case 3 and $F \cap Q$ belongs to Case 4. We have

$$|t' - t| \leq |\theta'| \cdot |t_{a'x'} - t_{a'y'}| + |t_{a'y'} - t_{a'x}| + |1-\gamma| \cdot |t_{ax}| := I_{21} + I_{22} + I'_{23}.$$

The terms I_{21} and I_{22} are estimated exactly as above whereas for I'_{23} we apply estimates (A16).

Let finally Q' fall into Case 4; then the same is necessarily true for Q. This implies $r' = r$, and therefore

$$I_2 = d'|\gamma' - \gamma| \cdot |t_{ax}| \leq Md'\Phi(d', d) \leq M\varphi(d).$$

Scrutinizing the above estimates we conclude that in all cases $I_1 + I_2 \leq AM\varphi(d)$. Consequently, condition (iii) of Theorem A1 is satisfied.

By Theorem A1, there exists a function $\widetilde{f} \in \Lambda_\varphi$ such that $\widetilde{f}|F = f$ and $\|\widetilde{f}\|_{\Lambda_\varphi} \leq AM$. The proof is finished.

Our next objective is to prove estimate (5.17). It will be derived as a corollary to the following multivariate statement.

For a subset F in $\mathbb{R}^n$, we let hereafter $\widetilde{F} := \{(x, y) : x, y \in F,\ x \neq y\}$.

Proposition A2.1. *Let F be a subset of $\mathbb{R}^n$, and let t_{xy}, $(x, y) \in \widetilde{F}$, be a vector field such that $t_{xy} = t_{yx}$ for all $(x, y) \in \widetilde{F}$ and*

$$|t_{xy} - t_{xz}| \leq M\Phi(|y - x|,\ 2|z - x|) \tag{A17}$$

for all $x, y, z \in F$ with $0 < |y - x| \leq |z - x|$. Then for all pairs $u = (x, y)$ and $u' = (x', y')$ in $\widetilde{F}$ such that $0 < |y - x| \leq |y' - x'|$, we have

$$|t_u - t_{u'}| \leq AM\varrho(u, u'), \tag{A18}$$

where

$$\varrho(u, u') := \Phi(|y - x|, 2|y' - x'| + |u' - u|) \tag{A19}$$

and $|u' - u| := \max\{|x' - x|, |y' - y|\}$.

PROOF. Suppose, for example, that $|u'-u| = |x'-x|$. We have either $|x'-x| \geq |y-x|/2$ or $|x'-y| \geq |y-x|/2$. In the first case, using the inequality

$$\Phi(\lambda a, \lambda b) \leq \max\{\lambda, \lambda^{-1}\}\Phi(a,b), \quad 0 < a \leq b, \qquad \lambda > 0, \tag{A20}$$

we find that

$$\begin{aligned}
|t_u - t_{u'}| &\leq |t_{xy} - t_{xx'}| + |t_{x'x} - t_{x'y'}| \\
&\leq M\Phi(|y-x|/2, 2\max\{|y-x|, |x'-x|\}) \\
&\quad + M\Phi(|y-x|/2, 2\max\{|y'-x'|, |x'-x|\}) \\
&\leq 2M\Phi(|y-x|/2, 2\max\{|y'-x'|, |x'-x|\}) = 2M\Phi(|y-x|/2, |y-x|) \\
&\quad + 2M\Phi(|y-x|, 2|y-x|) + 2M\Phi(2|y-x|, 2\max\{|y'-x'|, |x'-x|\}) \\
&\leq 6M\Phi(|y-x|, 2|y-x|) + 4M\Phi(|y-x|, \max\{|y'-x'|, |x'-x|\}) \\
&\leq 10M\Phi(|y-x|, 2|y'-x'| + |x'-x|).
\end{aligned}$$

If $|x'-y| \geq |y-x|/2$ then via a similar argument we obtain

$$\begin{aligned}
|t_u - t_{u'}| &\leq |t_{xy} - t_{x'y}| + |t_{x'y} - t_{x'y'}| \\
&\leq 6M\Phi(|y-x|, 2|y-x|) + 4M\Phi(|y-x|, \max\{|y'-x'|, |x'-y|\}) \\
&\leq 10M\Phi(|y-x|, 2|y'-x'| + |x'-x|).
\end{aligned}$$

Thus, $|t_u - t_{u'}| \leq 10Mr(u,u')$.

REMARK 1. Conversely, inequality (A18) which holds with a constant C implies (A17) with a constant $\leq AC$. In fact, if $0 < |y-x| \leq |z-x|$ then setting $u = (x,y)$, $u' = (x,z)$ in (A18) and (A19), we have

$$\begin{aligned}
|t_{xy} - t_{xz}| &\leq C\Phi(|y-x|, 2|z-x| + |z-y|) \\
&\leq C\Phi(|y-x|, 2|z-x|) + C\Phi(2|z-x|, 4|z-x|) \\
&\leq 3C\Phi(|y-x|, 2|z-x|).
\end{aligned}$$

Thus, for symmetric vector fields, conditions (A17) and (A18) are equivalent.

REMARK 2. Let $n = 1$, $F = [-1,1]$, and $f \in \Lambda_\varphi$. By the Corollary to Proposition 5.2.8, $t_{xy} = (f(y)-f(x))/(y-x)$ satisfies condition (A17). Therefore, (A18) is also true. This justifies (5.17).

Observe that (A18) is a Lipschitz condition with respect to the function ϱ on $\widetilde{F}$ defined by (A17), $\varrho(u',u) := \varrho(u,u')$ if $u' \neq u$, and $\varrho(u,u') := 0$ if $u' = u$.

PROPOSITION A2.2. *ϱ is a metric on $\widetilde{F}$.*

PROOF. We only need to check the triangle inequality. Let $u_i = (x_i, y_i)$, $i = 1,2,3$, be three points in $\widetilde{F}$. We will show that

$$\varrho(u_1,u_3) \leq \varrho(u_1,u_2) + \varrho(u_2,u_3). \tag{A21}$$

Set $r_i = |y_i - x_i|$, $i = 1,2,3$. We may assume that $r_1 \leq r_3$. Define $d_1 := |u_2 - u_1|$, $d_2 := |u_3 - u_1|$, and $d_3 := |u_3 - u_2|$; then $d_2 - d_1 \leq d_3 \leq d_1 + d_2$. Consider the following cases:

CASE 1: $r_2 \leq r_1$. If $d_2 \leq d_3$ then (A21) follows from the estimate

$$\varrho(u_1,u_3) = \Phi(r_1, 2r_3 + d_2) \leq \Phi(r_2, 2r_3 + d_3) = \varrho(u_2,u_3).$$

Let now $d_2 > d_3$. Then by (A14.3) we have

$$\begin{aligned}\varrho(u_1,u_3) &= \Phi(r_1, 2r_3+d_3) + \Phi(2r_3+d_3, 2r_3+d_2)\\ &\le \Phi(r_2, 2r_3+d_3) + \Phi(r_2, 2r_1+d_2) = \varrho(u_2,u_3) + \varrho(u_1,u_2).\end{aligned}$$

CASE 2: $r_1 < r_2 \le r_3$. If $2r_3 + d_2 \le 2r_2 + d_1$ then

$$\varrho(u_1,u_3) \le \Phi(r_1, 2r_2+d_1) = \varrho(u_1,u_2),$$

which implies (A21), whereas in the opposite case we have, by (A14.3),

$$\begin{aligned}\varrho(u_1,u_3) &= \Phi(r_1, 2r_2+d_1) + \Phi(2r_2+d_1, 2r_3+d_2) \le \Phi(r_1, 2r_2+d_1) + \Phi(2r_2, 2r_3+d_3)\\ &\le \Phi(r_1, 2r_2+d_1) + \Phi(r_2, 2r_3+d_3) = \varrho(u_1,u_2) + \varrho(u_2,u_3).\end{aligned}$$

CASE 3: $r_2 > r_3$. Again, on the basis of (A14.3),

$$\begin{aligned}\varrho(u_1,u_3) &= \Phi(r_1, 2r_3+d_2) \le \Phi(r_1, 2r_3+d_1+d_3)\\ &= \Phi(r_1, 2r_3+d_1) + \Phi(2r_3+d_1, 2r_3+d_1+d_3)\\ &\le \Phi(r_1, 2r_2+d_1) + \Phi(r_3, 2r_2+d_3) = \varrho(u_1,u_2) + \varrho(u_2,u_3),\end{aligned}$$

and (A21) follows.

3. Proof of Proposition 5.2.11. We start with recalling the following well-known result which goes back to the ideas of E. J. McShane [37] and is a ramification of the Helly theorem; see also [30] and [45].

PROPOSITION A3.1. *Let* $Q_\alpha = Q(c_\alpha, d_\alpha), \alpha \in \aleph$, *be a family of cubes in* $\mathbb{R}^n$, *and* ϱ *be a semimetric on* $\aleph$. *Suppose that for every* $\alpha, \alpha' \in \aleph$ *there are points* $a_{\alpha,\alpha'} \in Q_\alpha$ *and* $a_{\alpha',\alpha} \in Q_{\alpha'}$ *such that* $|a_{\alpha,\alpha'} - a_{\alpha',\alpha}| \le \varrho(\alpha,\alpha')$. *Then for every* $\alpha \in \aleph$, *one can choose a point* $a_\alpha \in Q_\alpha$ *such that* $|a_\alpha - a_{\alpha'}| \le \varrho(\alpha,\alpha')$ *for all* $\alpha, \alpha' \in \aleph$.

PROOF. It suffices to prove the proposition for $n = 1$. We claim that

$$a_\alpha := \inf\{c_{\alpha'} + d_{\alpha'} + \varrho(\alpha,\alpha') : \alpha' \in \aleph\}, \qquad \alpha \in \aleph,$$

is the required collection of points. First, $a_\alpha \le c_\alpha + d_\alpha$. Next, for every $\alpha' \in \aleph$,

$$c_{\alpha'} + d_{\alpha'} + \varrho(\alpha,\alpha') \ge a_{\alpha',\alpha} + \varrho(\alpha,\alpha') \ge a_{\alpha,\alpha'} \ge c_\alpha - d_\alpha,$$

hence $a_\alpha \ge c_\alpha - d_\alpha$. Therefore, $a_\alpha \in Q_\alpha$ for all α. Further, for every $\beta \in \aleph$,

$$a_\alpha \le c_\beta + d_\beta + \varrho(\alpha,\beta) \le c_\beta + d_\beta + \varrho(\alpha',\beta) + \varrho(\alpha,\alpha').$$

Taking infimum over $\beta \in \aleph$ we obtain the inequality $a_\alpha \le a_{\alpha'} + \varrho(\alpha,\alpha')$, whence for symmetry reason we conclude that $|a_\alpha - a_{\alpha'}| \le \varrho(\alpha,\alpha')$.

Now we are prepared to prove Proposition 5.2.11.

PROOF OF PROPOSITION 5.2.11. *Necessity.* Suppose F is a closed subset of $[-1,1]^2$ and $f \in \Lambda_\varphi(F)$. For $C := \|f\|_{\Lambda_\varphi(F)}$, there exists a function in the space $\Lambda_\varphi := \Lambda_\varphi([-1,1]^2)$ whose restriction to the set F is f and the norm in the space Λ_φ is not larger than $2C$. For this function, we retain the notation f.

We will show that f satisfies conditions (i) and (ii) of Proposition 5.2.11. The first of them is trivially valid with $M = 2C$. Now we prove (ii). For a triple $u = \{x, y, z\}$ of distinct points of F not lying in a line, denote by P_u the (unique) polynomial of degree

≤ 1 that interpolates f at these three points. It is easy to see that $P_u(s)=f(x)+t_u(s-x)$, where $t_u=t_{xyz}$ is the vector involved in condition (ii) of Proposition 5.2.11.

Let P_{xy} and P_{xz} be the polynomials interpolating f at points x,y and x,z, respectively, as defined in Proposition 5.2.8. We write these polynomials in the form $P_{xy}(s)=f(x)+t_{xy}(s-x)$, $P_{xz}(s)=f(x)+t_{xz}(s-x)$. Assuming that $0<|y-x|\le|z-x|$ we have by Proposition 5.2.8 the following estimate:

$$\|t_{xy}-t_{xz}\|\le A\int\limits_{|y-x|}^{2|z-x|}\frac{\omega_2(f;t)}{t^2}\,dt\le AC\Phi(|y-x|,2|z-x|).$$

The lines $t(y-x)=f(y)-f(x)$ and $t(z-x)=f(z)-f(x)$ in $\mathbb{R}^2$ form the angle $\min\{\alpha_u,\pi-\alpha_u\}$, where α_u is the angle between the vectors e_{xy} and e_{xz} (see condition (ii) of Proposition 5.2.11), pass through the points t_{xy} and t_{xz}, respectively, and intersect at the point t_{xyz}. Denote by d the Euclidean distance from the point t_{xy} to the line $t(z-x)=f(z)-f(x)$. Then

$$\|t_{xy}-t_{xyz}\|=\frac{d}{\sin\alpha_u}\le\frac{\|t_{xy}-t_{xz}\|}{\sin\alpha_u}\le\frac{AC}{\sin\alpha_u}\Phi(|y-x|,2|z-x|).$$

Now using Proposition A2.1 we conclude that for two triples u, u' of points of F as in condition (ii) of Proposition 5.2.11,

$$\begin{aligned}|t_u-t_{u'}|&\le|t_{xyz}-t_{xy}|+|t_{xy}-t_{x'y'}|+|t_{x'y'}-t_{x'y'z'}|\\&\le AC\biggl[\frac{\Phi(|y-x|,2|z-x|)}{\sin\alpha_u}+\Phi(|y-x|,2|y'-x'|+\max\{|x'-x|,|y'-y|\})\\&\quad+\frac{\Phi(|y'-x'|,2|z'-x'|)}{\sin\alpha_{u'}}\biggr].\end{aligned}$$

Thus, conditions (i) and (ii) hold with a constant $M\le AC$, and this also proves the right-hand estimate in (5.26).

Sufficiency. Let F be a closed subset of $[-1,1]^2$ not contained in a line, and let f be a function on F that meets conditions (i) and (ii) of Proposition 5.2.11 with a constant M. We may (and will) assume that $M=1$. Due to Proposition 5.2.10 we only have to show that there exists a vector field t_{xy}, $(x,y)\in\widetilde{F}$, which satisfies conditions (ii)(a)–(d) of that proposition with a constant $\le Ad(F)/S(F)$ (the reader is reminded that $d(F)$ is the Euclidean diameter of F and $S(F)$ is the area of the convex hull of F).

Let $\mathcal{V}$ be the set of all ordered triples $v=(x,y,z)$ of distinct points of F not lying in a line. Define a semimetric d on $\mathcal{V}$ by $d(v,v'):=\varrho((x,y),(x',y'))$, where $v=(x,y,z)$, $v'=(x',y',z')$, and ϱ is the metric on $\widetilde{F}$ defined above. To every $v\in\mathcal{V}$ we relate the cube $Q_v:=Q(t_v,d_v)$ with $d_v:=r(v)/\sin\alpha_v$, where

$$r(v):=\Phi(\min\{|y-x|,|z-x|\},2\max\{|y-x|,|z-x|\}),\qquad v=(x,y,z).$$

In these terms, condition (ii) of Proposition 5.2.11 can be rewritten in the form

$$|t_v-t_{v'}|\le d_v+d_{v'}+d(v,v'),\qquad v,v'\in\mathcal{V}.$$

Geometrically, this means that there are points $a_{vv'}\in Q_v$ and $a_{v'v}\in Q_{v'}$ such that $|a_{vv'}-a_{v'v}|\le d(v,v')$. Applying Proposition A3.1 we infer that for every $v\in\mathcal{V}$ there exists a point $a_v \in Q_v$ such that $|a_v - a_{v'}| \le d(v, v')$ for all $v, v' \in \mathcal{V}$, i.e. that

$$|a_{xyz} - a_{x'y'z'}| \le \varrho((x,y),(x',y')).$$

Setting here $x' = x$ and $y' = y$ we see that $a_{xyz} = a_{xyz'}$ for all $z, z' \in F$ with $(x,y,z),(x,y,z') \in \mathcal{V}$, that is, a_{xyz} is independent of z and will be denoted in the sequel by a_{xy}. Thus, the previous inequality assumes the form

$$\text{(A22)} \qquad |a_{xy} - a_{x'y'}| \le \varrho((x,y),(x',y')), \qquad (x,y),(x',y') \in \widetilde{F}.$$

In particular, by Remark 1 to Proposition A2.1,

$$|a_{xy} - a_{xz}| \le Ar(v)$$

for every $v = (x,y,z) \in \mathcal{V}$. Also, (A22) implies that

$$|a_{yx} - a_{zx}| \le \varrho((y,x),(z,x)) = \varrho((x,y),(x,z)) \le Ar(v).$$

Since $a_{xy} \in Q_v$, we have

$$|a_{xy} - t_{xyz}| \le \frac{r(v)}{\sin\alpha_v}, \quad v = (x,y,z).$$

Hence, by (A22),

$$\begin{aligned}|a_{yx} - t_{xyz}| &\le |a_{yx} - a_{xy}| + |a_{xy} - t_{xyz}| \le \Phi(|y-x|, 3|y-x|) + \frac{r(v)}{\sin\alpha_v} \\ &\le 2\Phi(|y-x|, 2|y-x|) + \frac{r(v)}{\sin\alpha_v} \le \frac{3r(v)}{\sin\alpha_v}.\end{aligned}$$

Therefore, setting $b_{xy} := (a_{xy} + a_{yx})/2$, $(x,y) \in \widetilde{F}$, we see that there is a *symmetric* vector field b satisfying condition (ii)(d) of Proposition 5.2.10 and such that

$$\text{(A23)} \qquad |b_{xy} - t_{xyz}| \le \frac{Ar(v)}{\sin\alpha_v}, \qquad v = (x,y,z).$$

To comply with condition (ii)(a) of Proposition 5.2.10, we set $t_{xy} := \mathrm{Pr}_{xy} b_{xy}$, $(x,y) \in \widetilde{F}$, where Pr_{xy} stands for the orthogonal projection onto the line $t(y-x) = f(y) - f(x)$. Observe that t_{xy}, $(x,y) \in \widetilde{F}$, is a symmetric vector field satisfying the condition (ii)(a). We claim that the condition (ii)(d) also holds. In fact, if $0 < |y-x| \le |z-x|$ then

$$\begin{aligned}|t_{xy} - t_{xz}| &\le \|t_{xy} - t_{xz}\| = \|\mathrm{Pr}_{xy} b_{xy} - \mathrm{Pr}_{xz} b_{xz}\| \\ &\le \|\mathrm{Pr}_{xz} b_{xy} - \mathrm{Pr}_{xz} b_{xz}\| + \|\mathrm{Pr}_{xy} b_{xy} - \mathrm{Pr}_{xz} b_{xy}\| \\ &\le \|b_{xy} - b_{xz}\| + I \le A\Phi(|y-x|, 2|z-x|) + I,\end{aligned}$$

where $I := \|\mathrm{Pr}_{xy} b_{xy} - \mathrm{Pr}_{xz} b_{xy}\|$. To estimate I, put $p := \mathrm{Pr}_{xy} b_{xy}$ and $q := \mathrm{Pr}_{xz} b_{xy}$. A simple geometric argument together with (A23) shows that

$$\begin{aligned}I = \|p - q\| &\le A \max\{\|b_{xy} - p\|, \|b_{xy} - q\|\} \sin\alpha_v \\ &\le A\|b_{xy} - t_v\| \sin\alpha_v \le A\Phi(|y-x|, 2|z-x|).\end{aligned}$$

Therefore, for the vector field t_{xy}, $(x,y) \in \widetilde{F}$, the condition (ii)(d) is met.

Finally, we have to check condition (ii)(b) of Proposition 5.2.10, i.e. that

$$|t_{xy}| \le A\frac{d(F)}{S(F)}\Phi(|y-x|, 3), \qquad (x,y) \in \widetilde{F}.$$

Take points x_0, z_0 in F such that $d(F) = \|z_0 - x_0\|$. Let H be the height of a rectangle with base $[x_0, z_0]$ containing F and having the minimal area, which we denote by S. There is a point $y_0 \in F$ whose Euclidean distance from the interval $[x_0, z_0]$ is not less than $H/2$. We may assume in fact that $\|y_0 - x_0\| \geq d(F)/2$. Let α_0 be the angle between the vectors $y_0 - x_0$ and $z_0 - x_0$. Then

$$2\|z_0 - x_0\| \, \|y_0 - x_0\| \sin\alpha_0 \geq S \geq S(F),$$

hence $\sin\alpha_0 \geq S(F)/[2d^2(F)]$.

Note also that both $|f(y_0) - f(x_0)|/\|y_0 - x_0\|$ and $|f(z_0) - f(x_0)|/\|z_0 - x_0\|$ do not exceed $A/d(F)$. Therefore, upon observing that $S(F) \leq d^2(F)$, we have for

$$t_{v_0} = \frac{f(y_0) - f(x_0)}{\|y_0 - x_0\|} e_{x_0 y_0} + \frac{1}{\sin\alpha_0}\left[\frac{f(y_0) - f(x_0)}{\|y_0 - x_0\|} - \frac{f(z_0) - f(x_0)}{\|z_0 - x_0\|}\cos\alpha_0\right] e^{\perp}_{x_0 y_0}$$

with $v_0 := (x_0, y_0, z_0)$ the estimate $|t_{v_0}| \leq Ad(F)/S(F)$. Next, in view of (A23) and (A14.1) we conclude that

$$\begin{aligned}|t_{x_0 z_0} - t_{v_0}| &\leq \|t_{x_0 z_0} - t_{v_0}\| \leq \|b_{x_0 z_0} - t_{v_0}\| \leq A\frac{d^2(F)}{S(F)} r(v_0) \\ &= A\frac{d^2(F)}{S(F)} \Phi(|y_0 - x_0|, 2|z_0 - x_0|) \leq A\frac{d(F)}{S(F)} \phi(d(F)) \leq A\frac{d(F)}{S(F)}.\end{aligned}$$

Therefore,

$$|t_{x_0 z_0}| \leq |t_{x_0 z_0} - t_{v_0}| + |t_{v_0}| \leq A\frac{d(F)}{S(F)}.$$

Keeping in mind that $d(F) \geq AS(F)$ we obtain for $(x, y) \in \widetilde{F}$ the required estimate

$$\begin{aligned}|t_{xy}| &\leq |t_{xy} - t_{x_0 z_0}| + |t_{x_0 z_0}| \\ &\leq A\Phi(|y - x|, 2|z_0 - x_0|) + A\frac{d(F)}{S(F)} \leq A\frac{d(F)}{S(F)} \Phi(|y - x|, 3).\end{aligned}$$

Thus, the vector field t_u, $u \in \widetilde{F}$, satisfies conditions (ii)(a)–(ii)(d) of Proposition 5.2.10 with a constant specified in (5.26). Applying the sufficiency part of Proposition 5.2.10 we finalize the proof of Proposition 5.2.11.

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